1 Sets and Propositions

1.1 : Sets

Important Points to Remember

1. **Set :** A collection of well defined objects is called set.
2. **Empty set :** A set which does not contain any element is called null set or empty set. It is denoted by ϕ.
3. **The cardinality of a set** is the number of elements in the set.
4. **Power set :** The power set of given set A is the set of all subsets of set A. It is denoted by P (A). If n has n elements then P (A) has 2^n elements.
5. **Finite set :** A set having finite number of elements is called finite set. Otherwise infiite set.
6. If A and B are any two sets then

   (a) $A \cup B = \{x \mid x \in A \text{ or } x \in B\}$,

   $A \cap B = \{x \mid x \in A \text{ and } x \in B\}$.

   (b) $A - B = \{x \mid x \in A \text{ but } x \notin B\}$

   (c) $A^c = U - A \ \{x \mid x \notin A \text{ and } x \in U\}$

   (d) $A \oplus B = (A \cup B) - (A \cap B)$ ... [Symmetric difference]
7. **Multiset :** A set in which elements are present more than once.

   The multiplicity of an element in a multiset is defined as the number of times the element appear in the set. It is denoted as μ (a).

   If A and B are multisets then,

   (a) $x \in A \cup B$ and $\mu(x) = \max\{\mu_A(x), \mu_B(x)\}$

   (b) $x \in A \cap B$ and $\mu(x) = \min\{\mu_A(x), \mu_B(x)\}$

   (c) $x \in A - B$ and $\mu(x) = \mu_A(x) - \mu_B(x)$ if $\mu(x) \geq 0$

   If difference is negative then $\mu(x) = 0$.

(d) $x \in A + B$ and $\mu(x) = \mu_A(x) + \mu_B(x)$

8. Inclusion exclusion theorem :

(a) For two sets : $|A \cup B| = |A| + |B| - |A \quad B|$

(b) For three sets : $|A \cup B \cup C| = |A| + |B| + |C|$
$- A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

9. List of basic identities of set theory :

(a) $A \cap \phi = \phi$, $A \cap A = A$, $A \cap U = A$, $A \cap A'$
$= A' \cap A = \phi$

(b) $A \cup \phi = A$, $A \cup A = A$, $A \cup U = \cup$, $A \cup A'$
$= A' \cup A = U$

(c) $(\overline{\overline{A}}) = A$, $A - B = A \cap \overline{B}$

(d) $A \cup B = B \cup A$, $A \cap B = B \cap A$

(e) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

(f) $A \cup (B \cup C) = (A \cup B) \cup C$
$A \cap (B \cap C) = (A \cap B) \cap C$

(g) $(\overline{A \cup B}) = \overline{A} \cap \overline{B}$
$(\overline{A \cap B}) = \overline{A} \cup \overline{B}$

(h) $A \cup (A \cap B) = A$, $A \cap (A \cup B) = A$

Q.1 If $A = \{x, y, \{x, z\} \phi\}$. Determine the following sets
i) $A - \{x, z\}$ ii) $\{\{x, z\} - A$ iii) $A - \{\{x, y\}\}$ iv) $\{x, z\} - A$
v) $A - P(A)$ vi) $\{x\} - A$ vii) $A - \{x\}$ viii) $A - \phi$
ix) $\phi - A$ x) $\{x, y, \phi\} - A$ **☞[SPPU : Dec.-08, Marks 4]**

Ans. : i) $A - \{x, z\} = \{x, y, \phi\}$

ii) $\{\{x, z\}\} - A = \phi$ iii) $A - \{\{x, y\}\} = A$

iv) $\{x, z\} - A = \{z\}$ v) $A - p(A) = \{x, y, \{x, z\}\}$

vi) $\{x\} - A = \phi$ vii) $A - \{x\} = \{y, \{x, z\}, \phi\}$

viii) $A - \phi = \{x, y, \{x, z\}\}$ ix) $\phi - A = \phi$

x) $\{x, y, \phi\} - A = \{\{x, z\}\}$

Q.2 Let A, B, C be three sets

i) Given that $A \cup B = A \cup C$, is it necessary that B = C ?

ii) Given that $A \cap B = A \cap C$, is it necessary that B = C ?

Ans. : i) No,

Let

$$A = \{x, y, z\}, \quad B = \{x\}, \quad C = \{z\}$$

$$A \cup B = A \quad \text{and} \quad A \cup C = A$$

$$\therefore \quad A \cup B = A \cup C = A \quad \text{But} \quad B \neq C$$

ii) No,

Let

$$A = \{x, y\}, \quad B = \{y, z, w\}, \quad C = \{y, p, q\}$$

$$A \cap B = \{y\} = B \cap C \quad \text{But} \quad B \neq C$$

Q.3 If ϕ is an empty set then find $p(\phi)$, $p(p(\phi))$ $p(p(p(\phi)))$

Ans. :

$$p(\phi) = \{\phi\}$$

$$p(p(\phi)) = \{\phi, \{\phi\}\}$$

$$p(p(p(\phi))) = \{\phi, \{\phi\}, \{\{\phi\}\}, \{\phi, \{\phi\}\}\}$$

1.2 : Combination of Sets

Q.4 Salad is made with combination of one or more eatables, how many different salads can be prepared from onion, carrot, cabbage and cucumber ?

[SPPU : Dec.-13, Marks 4]

Ans. : The number of different salads can be prepared from onion, carrot, cabbage and cucumber with combination of one or more eatables is $2^4 - 1 = 16 - 1 = 15$

Q.5 Explain the concepts of countably infinite set with example.

[SPPU : Dec.-14, Marks 4]

Ans. : A set is said to be countable if its all elements can be labelled as 1, 2, 3, 4, ... A set is said to be countably infinite

if, i) It is countable

ii) It has infinitely many elements i.e. It's cardinality is ∞.

For example

1) The set of natural numbers {1, 2, 3, ...} is countably infinite.

2) The set of integers is countably infinite.

3) The set of real numbers is infinite but not countable.

1.3 : Venn Diagram

Q.6 Draw Venn diagram and prove the expression. Also write the dual of each of the given statements.
i) $(A \cup B \cup C)^C = (A \cup C)^C \cap (A \cup B)^C$ ii) $(U \cap A) \cup (B \cap A) = A$

☞[SPPU : Dec.-11, Marks 6]

Ans. : Consider the following Venn diagrams.

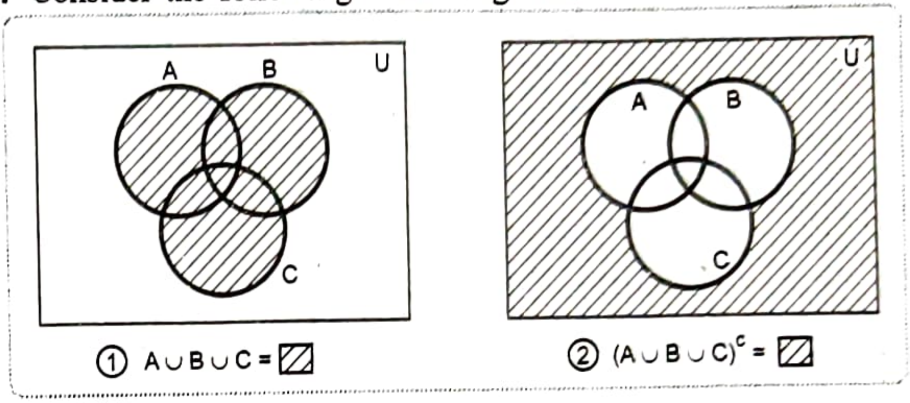


Fig. Q.6.1

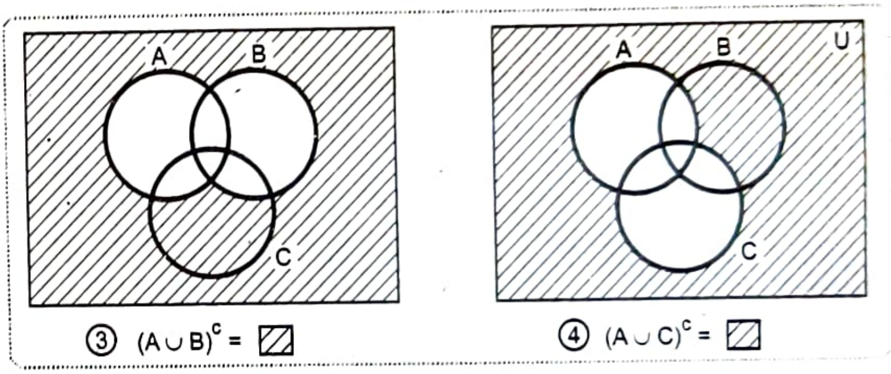


Fig. Q.6.2

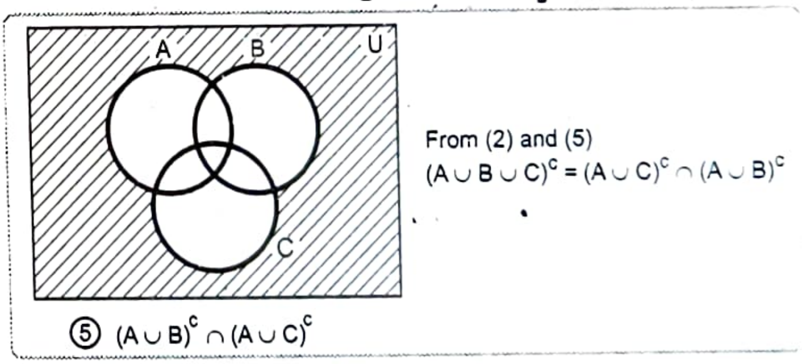


Fig. Q.6.3

ii) Consider the following Venn diagrams.

From Venn diagrams (3) and (4)

$$(U \cap A) \cup (B \cap A) = A$$

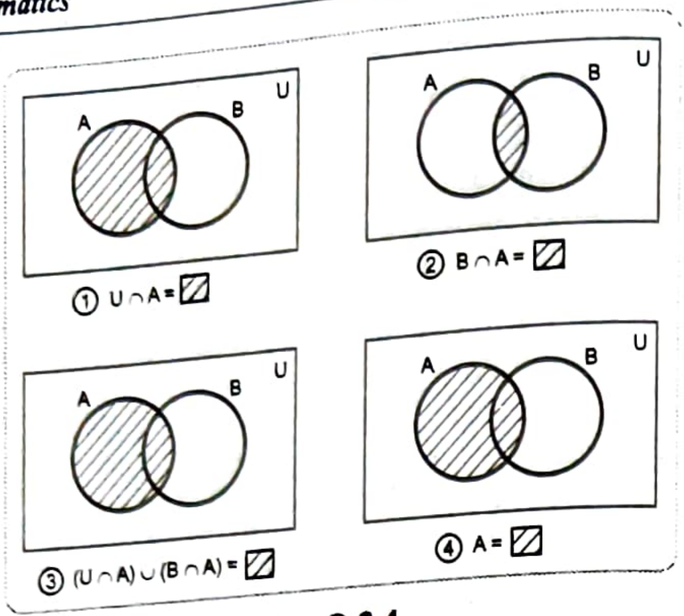
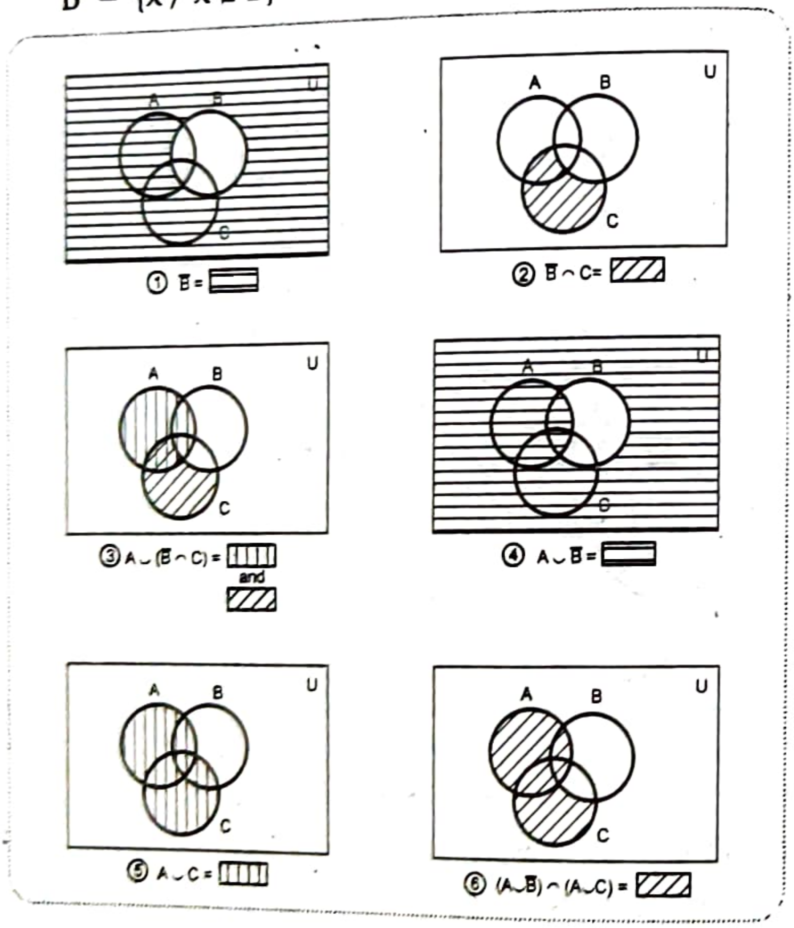


Fig. Q.6.4

Q.7 Using Venn diagram show that :
$A \cup (\overline{B} \cap C) = (A \cup \overline{B}) \cap (A \cup C)$ **☞[SPPU : May-05, Marks 4]**

Ans. : Consider the following Venn diagrams,

$$\overline{B} = \{x / x \notin B\}$$

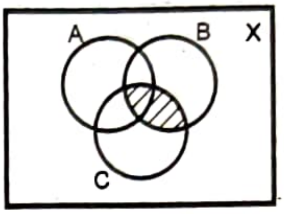


From (3) and (6),

$$A \cup (\overline{B} \cap C) = (A \cup \overline{B}) \cap (A \cup C)$$

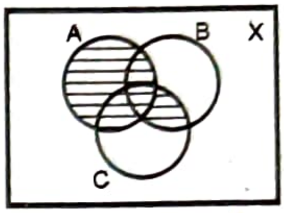
Q.8 Using Venn diagrams show that
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ ☞**[SPPU : May-08, Dec.-12, Marks 3]**

Ans. :



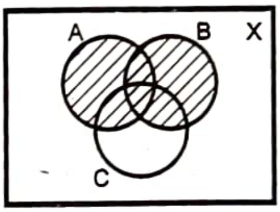
$B \cap C =$

(1)



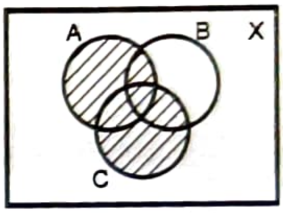
$A \cup (B \cap C) =$

(2)



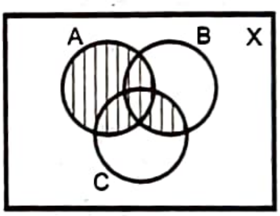
$A \cup B =$

(3)



$A \cup C =$

(4)



$(A \cup B) \cap (A \cup C) =$

(5)

From (2) and (5),

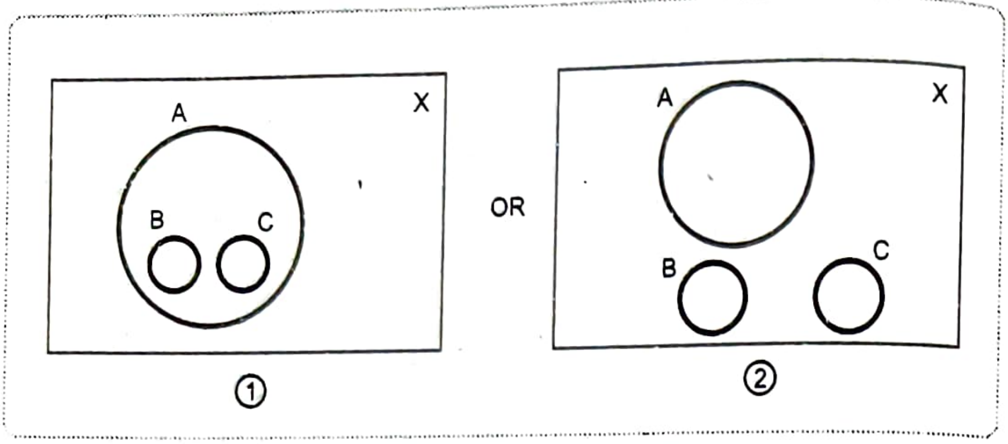
$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Q.9 Let A, B, C be sets. Under what conditions the following statements are true ?

i) $(A - B) \cup (A - C) = A$

ii) $(A - B) \cup (A - C) = \varnothing$ ☞**[SPPU : Dec.-06, 15, Marks 3]**

Ans. : i) $(A - B) \cup (A - C) = A$



If $B \subset A$ and $C \subset A$

then $(A - B) \cup (A - C) = A$

Or A, B, C are disjoint sets.

i.e. $A \cap B \cap C = \varnothing$

Then $(A - B) \cup (A - C) = A$

ii) $(A - B) \cup (A - C) = \varnothing$ is true.

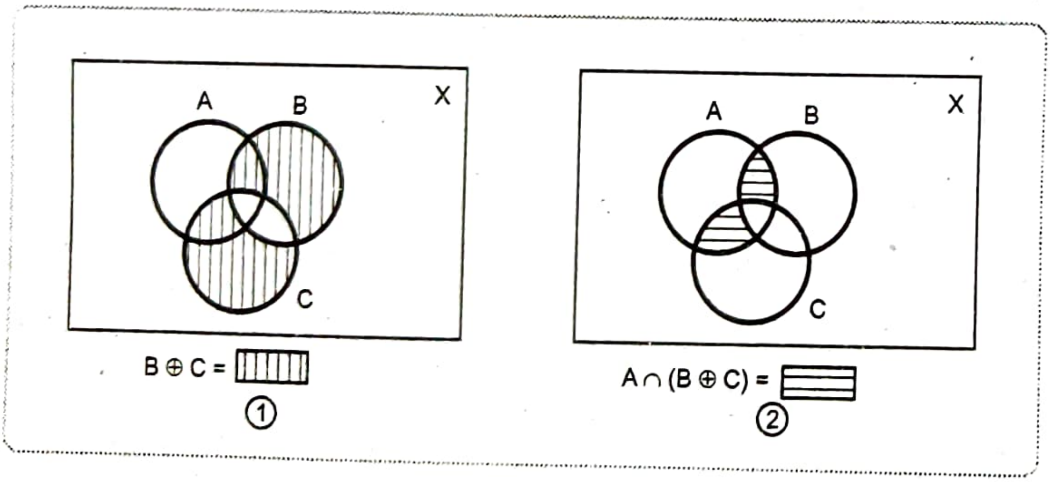
If A is empty set i.e. $A = \varnothing$

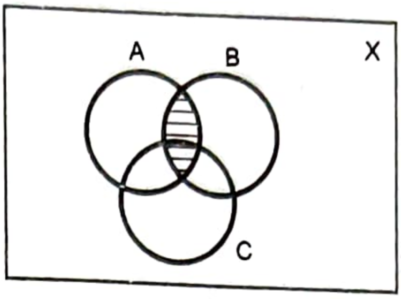
Q.10 Prove the following using Venn diagram.

$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$

[SPPU : May-08, 14, Dec.-12, Marks 3]

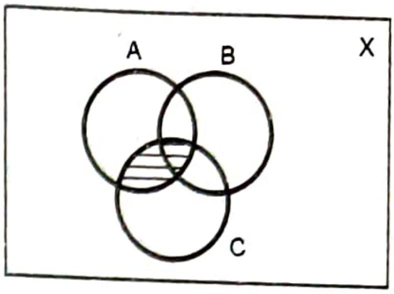
Ans. :





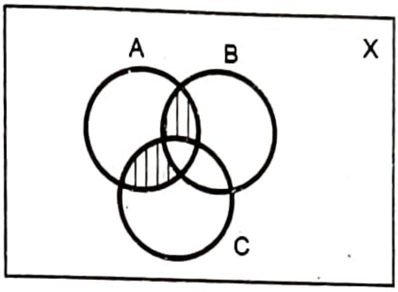
$A \cap B =$

③



$A \cap C =$

④



$(A \cap B) \oplus (A \cap C) =$

⑤

From ② and ⑤

$$A \cap (B \oplus C) = (A \cap B) \oplus (A \cap C)$$

1.4 : Principle of Inclusion and Exclusion

Q.11 State and prove the principle of inclusion and exclusion for sets.

Ans. :

Theorem : (Principle of Inclusion - Exclusion for 2 sets)

Let A and B be finite sets then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Proof : By venn diagram

$$A \cup B = (A - B) \cup B$$

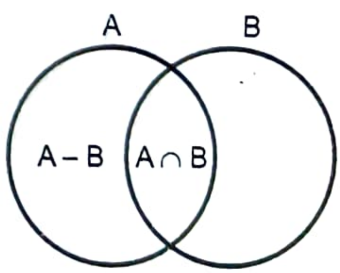


Fig. Q.11.1

As A–B and B are disjoint sets.

$$|A \cup B| = |A - B| + |B|$$
$$= |A| - |A \cap B| + |B|$$
$$|A \cup B| = |A| + |B| - |A \cap B|$$

Principle of Inclusion-Exclusion for three sets.

Theorem : Let A, B, C be finite sets. Then

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Proof Let $D = B \cup C$

$\therefore \quad |A \cup D| = |A| + |D| = |A \cap D|$

and $\quad |D| = |B \cup C| = |B| + |C| - |B \cap C|$

$$\therefore \quad |A \cup B \cup C| = |A \cup D| = |A| + |B| + |C| - |B \cap C| - |A \cap D|$$
$$= |A| + |B| + |C| - |B \cap C| - |A \cap (B \cup C)|$$
$$= |A| + |B| + |C| - |B \cap C| - |(A \cap B) \cup (A \cap C)|$$
$$= |A| + |B| + |C| - |B \cap C| - |A \cap B| - |A \cap C| + |A \cap B \cap C|$$
$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Theorem : Let A, B, C, D be finite sets then

$$|A \cup B \cup C \cup D| = |A| + |B| + |C| + |D| - |A \cap B| - |A \cap C| - |A \cap D|$$
$$- |B \cap C| - |B \cap D| - |C \cap D| + |A \cap B \cap C| + |A \cap B \cap D + |A \cap C \cap D$$
$$+ |B \cap C \cap D| - |A \cap B \cap C \cap D|$$

Q.12 Among the integers 1 to 300 find how many are not divisible by 3, not by 5. Find also how many are divisible by 3 but not by 7.

☞[SPPU : Dec.-08, Marks 6]

Ans. : Let A denotes the set of integers 1 to 300 divisible by 3.

B denotes the set of integers 1 to 300 divisible by 5.

C denotes the set of integers 1 to 300 divisible by 7.

$$|A| = \left[\frac{300}{3}\right] = 100, \ |B| = \left[\frac{300}{5}\right] = 60, \ |C| = \left[\frac{300}{7}\right] = 42,$$

$$|A \cap B| = \left[\frac{300}{3 \times 5}\right] = 20$$

Find $|\overline{A}\cap B|$ and $|A-C|$

We have $\overline{A}\cap\overline{B} = \overline{A\cup B} = U-(A\cup B)$

$$\therefore \quad |A\cup B| = |A|+|B|-|A\cap B| = 100 + 60 - 20 = 140$$

$$\therefore \quad |\overline{A}\cap\overline{B}| = |U|-|A\cup B| = 300 - 140 = \mathbf{160}$$

Hence 160 integers between 1 - 300 are not divisible by 3, not by 5.

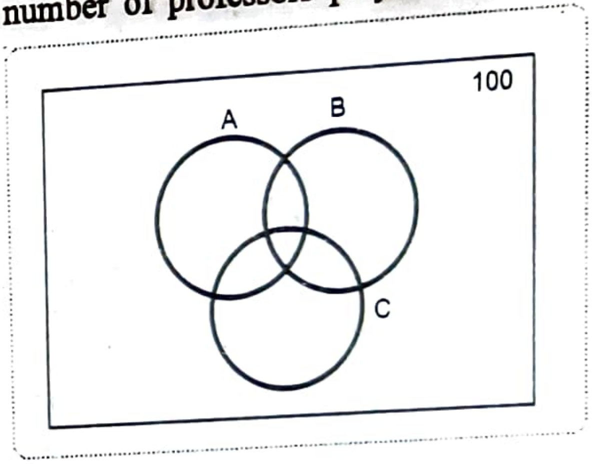
$$|A-C| = |A|-|A\cap C|, \qquad |A\cap C| = \left[\frac{300}{3\times 7}\right] = 14$$

$$|A-C| = 100 - 14 = 86$$

Hence, 86 integers between 1 - 300 are not divisible by 3 but not by 7.

Q.13 It is known that at the university 60 percent of the professors play tennis, 50 percent of them play bridge. 70 percent jog, 20 percent play tennis and bridge, 30 percent play tennis and jog, and 40 percent play bridge and jog. If some one claimed that 20 percent of the professors jog and play bridge and tennis, would you believe this claim ? Why ? **☞[SPPU : Dec.-13, Marks 6]**

Ans. : Let A, B, C, denotes the number of professors play tennis, bridge and jog respectively.



$$|A| = 60$$
$$|B| = 50$$
$$|C| = 70$$
$$|A\cap B| = 20$$
$$|A\cap C| = 30$$
$$|B\cap C| = 40$$
$$|A\cap B\cap C| = 20$$

$$|A\cup B\cup C| = |A|+|B|+|C|-[|A\cap B|+|A\cap C|+|B\cap C|]+|A\cap B\cap C|$$

$$= 60 + 50 + 70 - [20 + 30 + 40] + 20 = \mathbf{110}$$

which is not possible as $|A\cup B\cup C| \subset X$ and the number of elements in $|A\cup B\cup C|$ cannot exceed number of elements in the universal set X.

Q.14 Consider a set of integers 1 to 500. Find

i) How many of these numbers are divisible by 3 or 5 or by 11 ? **☞[SPPU : Dec.-14]**

ii) Also indicate how many are divisible by 3 or by 11 but not by all 3, 5 and 11.

iii) How many are divisible by 3 or 11 but not by 5 ?

☞[SPPU : May-05, Marks 6]

Ans. : Let A denote set of numbers 1 to 500 which are divisible by 3.

B denote set of numbers 1 to 500 which are divisible by 5.

C denote set of numbers 1 to 500 which are divisible by 11.

$|A|$ denotes cardinality of A similarly $|B|$ and $|C|$ denotes cardinality of B and C.

$$|A| = \left[\frac{500}{3}\right] = 166$$

$$|B| = \left[\frac{500}{5}\right] = 100$$

$$|C| = \left[\frac{500}{11}\right] = 45$$

$$|A \cap B| = \left[\frac{500}{3 \times 5}\right] = 33$$

$$|A \cap C| = \left[\frac{500}{3 \times 11}\right] = 15$$

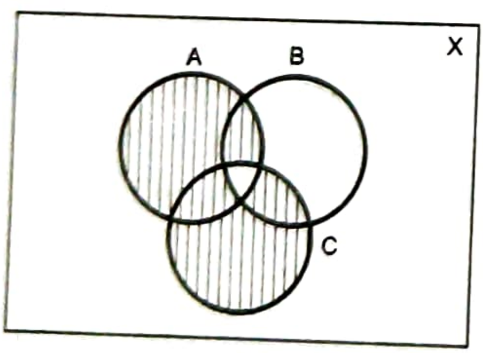
$$|B \cap C| = \left[\frac{500}{5 \times 11}\right] = 9$$

$$|A \cap B \cap C| = \left[\frac{500}{3 \times 5 \times 11}\right] = 3$$

i) $|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$

$$= 166 + 100 + 45 - [13 + 15 + 9] + 3 = \mathbf{257}$$

ii)



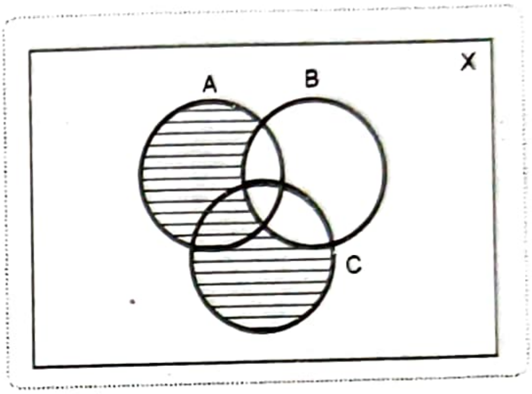
Number of integers divisible by 3 or by 11 but not by all 3, 5 and 11.

$$= |A \cup C| - |A \cap B \cap C|$$

$$= [|A| + |C| - |A \cap C|] - |A \cap B \cap C|$$

$$= 166 + 45 - 15 - 3 = \mathbf{193}$$

iii)



Number of integers divisible by 3 or 11 but not by 5.

$$= |A \cup B \cup C| - |B| = 257 - 100 = \mathbf{157}$$

Q.15 Out of the integers 1 to 1000.

i) How many of them are not divisible by 3, nor by 5, nor by 7 ?

ii) How many are not divisible by 5 and 7 but divisible by 3 ?

☞[SPPU : May-06, May-08, May-14, Marks 6]

Ans. : i) Let A denote the set of integers 1 to 1000 which are divisible by 3.

B denote set of integers 1 to 1000 which are divisible by 5.

and C denote set of integers 1 to 1000 which are divisible by 7.

$$|A| = \left[\frac{1000}{3}\right] = 333$$

$$|B| = \left[\frac{1000}{5}\right] = 200$$

$$|C| = \left[\frac{1000}{7}\right] = 142$$

$$|A \cap B| = \left[\frac{1000}{3 \times 5}\right] = 66$$

$$|A \cap C| = \left[\frac{1000}{3 \times 7}\right] = 47$$

$$|B \cap C| = \left[\frac{1000}{5 \times 7}\right] = 28$$

$$|A \cap B \cap C| = \left[\frac{1000}{3 \times 5 \times 7}\right] = 9$$

$$|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$$

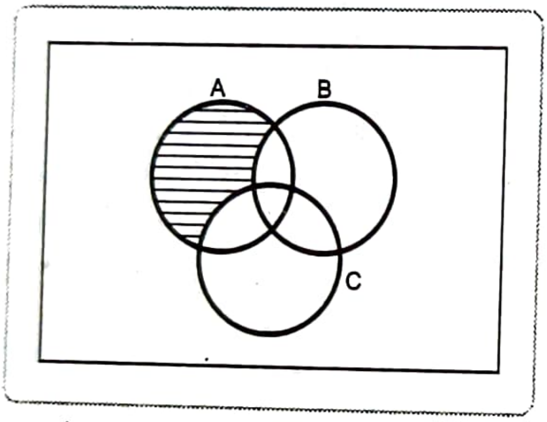
$$= 333 + 200 + 142 - [66 + 47 + 28] + 9 = 543$$

This show 543 numbers are divisible by 3 or 5 or 7. Hence numbers which are not divisible by 3, nor by 5, nor by 7.

$$= |A' \cap B' \cap C'| = |X| - |A \cup B \cup C|$$

$$= 1000 - 543 = \mathbf{457}$$

ii)



Number of integers divisible by 3 but not by 5 and not by 7 is $|A \cap B' \cap C'|$.

$$|A \cap B' \cap C'| = |A \cap (B \cup C)'| = |A| - [|A \cap B| + |A \cap C|] + |A \cap B \cap C|$$

$$= 333 - [66 + 47] + 9 = \mathbf{229}$$

Q.16 It was found that in the first year computer science class of 80 students, 50 knew COBOL, 55 'C' and 46 PASCAL. It was also know that 37 knew 'C' and COBOL, 28 'C' and PASCAL and 25 PASCAL and COBOL. 7 students however knew none of the languages. Find

i) How many knew all the three languages ?

ii) How many knew exactly two languages ?
iii) How many knew exactly one language ?

☞[SPPU : May-15, Dec.-15, Marks 4]

Ans. : Let A denote the set of students who know COBOL.
B denote the set of students who know 'C'.

and C denote the set of students who know PASCAL.

and X denote universal set.

Then $|X| = 80, \; |A| = 50, \; |B| = 55, \; |C| = 46$

$|A \cap B| = 37, |B \cap C| = 28, |A \cap C| = 25$

$$|A' \cap B' \cap C'| = 7$$

Hence $\quad |(A \cup B \cup C)'| = 7$

Also $\quad |A \cup B \cup C| = |X| - |(A \cup B \cup C)'|$

Hence $\quad |A \cup B \cup C| = 80 - 7 = 73$

i) $|A \cup B \cup C| = |A| + |B| + |C| - [|A \cap B| + |A \cap C| + |B \cap C|] + |A \cap B \cap C|$

$73 = 50 + 55 + 46 - [37 + 28 + 25] + |A \cap B \cap C|$

$\Rightarrow \quad |A \cap B \cap C| = \mathbf{12}$

ii) Number of students who know only COBOL and 'C' but not PASCAL.

$$= |A \cap B| - |A \cap B \cap C|$$

$$= 37 - 12 = \mathbf{25} \qquad \text{... (Q.16.1)}$$

Number of students who know only COBOL and PASCAL but not 'C'.

$$= |A \cap C| - |A \cap B \cap C|$$

$$= 25 - 12 = \mathbf{13} \qquad \text{... (Q.16.2)}$$

Number of students who know only 'C' and PASCAL but not COBOL.

$$= |B \cap C| - |A \cap B \cap C|$$

$$= 28 - 12 = \mathbf{16} \qquad \text{... (Q.16.3)}$$

Hence number of students who know exactly two languages.

$$= 25 + 13 + 16 = \mathbf{54}$$

iii) Number of students who know only COBOL but not 'C' and not PASCAL.

$$= |A| - [|A \cap B| + |A \cap C|] + |A \cap B \cap C|$$

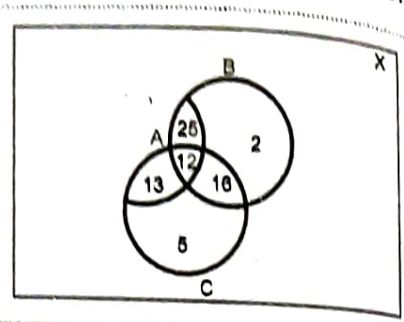
$$= 50 - [37 + 25] + 12 = \mathbf{0} \qquad \text{... (Q.16.4)}$$

Number of students who know only 'C' but not COBOL and not PASCAL.

$$= |B|-[|A \cap B|+|B \cap C|]+|A \cap B \cap C|$$

$$= 55 - [37 + 28] + 12 = \mathbf{2} \quad \text{... (Q.16.5)}$$

Number of students who know only PASCAL but not COBOL and not 'C'.



$$= |C|-[|A \cap C|+|B \cap C|]+|A \cap B \cap C|$$

$$= 46 - [25 + 28] + 12 = \mathbf{5} \quad \text{... (Q.16.6)}$$

Hence number of students who know only one language = 0 + 2 + 5 = **7**

Q.17 Find the multiset to solve equation

$\boldsymbol{A \cap [1, 2, 2, 3, 4] = [1, 2, 3, 4]}$

$\boldsymbol{A \cup [1, 2, 2, 3] = [1, 1, 2, 2, 3, 3, 4]}$

Ans. : Maximum multiplicity of each element is as follows

$$\mu(1) = 2, \quad \mu(2) = 2, \quad \mu(3) = 2, \quad \mu(4) = 1$$

and minimum multiplicity of each element in A is as follows

$$\mu(1) = 1, \quad \mu(2) = 1, \quad \mu(3) = 1, \quad \mu(4) = 1$$

Therefore, $A = [1, 2, 3, 4]$ or $A = [1, 1, 2, 2, 3, 3, 4]$

Q.18 Explain examples of multisets with its significance.

☞[SPPU : Dec.-13, Marks 4]

Ans. : 1) Multisets are used to denote roots of the polynomial.

$\therefore \quad x^3 + 3x^2 + 3x + 1 = 0$

Roots are −1, −1, −1

$$\therefore \quad A = [-1, -1, -1]$$

2) Multisets are used to denote prime factors of every non-negative integer.

e.g. prime factors of 72 are $2 \times 2 \times 2 \times 3 \times 3$

$$\therefore \quad A = [2, 2, 2, 3, 3]$$

3) Multisets are used to denote zeros or poles of analytic functions.

4) In Computer Science, multisets are applied in a variety of search and sort procedure.

1.5 : First Principle of Mathematical Induction Statement

Let P(n) be a statement involving a natural number $n \geq n_0$ such that,

1) If P(n) is true for $n = n_0$ where $n_0 \in N$ and

2) Assume that P(k) is true for $k \geq n_0$

We prove P(k+1) is also true,

Then P(n) is true for all natural numbers $n \geq n_0$.

Step 1 is called as the basis of induction.

Step 2 is called as the induction step.

1.6 : Second Principle of Mathematical Induction Statement

Let P(n) be a statement involving a natural number $n \geq n_0$ such that,

1) If P(n) is true for $n = n_0$ where $n_0 \in N$ and

2) Assume that P (n) is true for $n_0 < n \leq k$ i.e. $P(n_0 + 1)(n_0 + 2)$............P(k) are true.

we prove that P(k + 1) is true,

Then P(n) is true for all natural numbers $n \geq n_0$.

Q.19 Prove by mathematical induction for n ≥ 1.

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n(n+1) = \frac{n(n+1)(n+2)}{3}$$

[SPPU : May-05, Marks 6]

Ans. : Let P(n) the given statement

1. Basis of induction

For $n_0 = 1$ L.H.S. = 1.2 = 2

$$\text{R.H.S.} = \frac{1(2)(3)}{3} = 2 \Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Hence P(1) is true.

2. Induction step

Assume that, P(k) is true

i.e. $$1\cdot 2+2\cdot 3+3\cdot 4+\ldots+k(k+1) = \frac{k(k+1)(k+2)}{3} \quad \ldots (Q.19.1)$$

Then we have

$$[1\cdot 2+2\cdot 3+3\cdot 4+\ldots+k(k+1)]+(k+1)(k+2)$$
$$= \frac{k(k+1)(k+2)}{3}+(k+1)(k+2) \quad \ldots \text{(Using Q.19.1)}$$
$$= (k+1)(k+2)\left[\frac{k}{3}+1\right]$$
$$= \frac{(k+1)(k+2)(k+3)}{3}$$

Hence assuming P(k) is true, P(k + 1), is also true. Therefore by mathematical induction P(n) is true for all $n \geq 1$.

Q.20 Show by induction that , $n \geq 1$

$$1^2+3^2+5^2+..(2n-1)^2 = \frac{n(2n-1)(2n+1)}{3}$$ **☞[SPPU : May-14, Marks 4]**

Ans. : Let P(n) be the given statement,

1. Basis of induction :

For n = 1, L.H.S. $= 1^2 = 1$,

R.H.S. $= \frac{1(1)(3)}{3} = 1$

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true

2. Induction step : Assume that P(k) is true.

i.e. $$1^2+3^2+5^2+\ldots+(2k-1)^2 = \frac{k(2k-1)(2k+1)}{3} \quad \ldots (Q.20.1)$$

Hence

$$[1^2+3^2+5^2+\ldots+(2k-1)^2]+(2k+1)^2 = \frac{k(2k-1)(2k+1)}{3}+(2k+1)^2$$

... (Using Q.20.1)

$$= \frac{(2k+1)}{3}[2k^2-k+3(2k+1)]$$

$$= \frac{(2k+1)}{3}[2k^2 + 5k + 3]$$

$$= \frac{(2k+1)}{3}[2k^2 + 2k + 3k + 3]$$

$$= \frac{(2k+1)}{3}[(2k+3)(k+1)]$$

$$= \frac{(k+1)(2k+1)(2k+3)}{3}$$

$$= \frac{(k+1)[2(k+1)-1][2(k+1)+1]}{3}$$

Hence assuming P(k) is true P(k + 1) is also true. Therefore by mathematical induction P(n) in true for all $n \geq 1$.

Q.21 Show that $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2 (n+1)^2}{4}$

☞[SPPU : Dec.-12, Marks 4]

Ans. : Let P(n) be the given statement,

1. Basis of induction :

For n = 1, L.H.S. = 1,

$$\text{R.H.S.} = \frac{1(1+1)^2}{4} = 1$$

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e. $1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2 (k+1)^2}{4}$ ... (Q.21.1)

Then we have

$(1^3 + 2^3 + 3^3 + \ldots + k^3) + (k+1)^3 = (1+2+3+\ldots+k)^2 + (k+1)^3$

(Using Q.21.1)

$$= \left(\frac{k(k+1)}{2}\right)^2 + (k+1)^3 = (k+1)^2\left[\frac{k^2}{4} + k + 1\right]$$

$$= (k+1)^2\left[\frac{k^2+4k+4}{4}\right] = \frac{(k+1)^2\,(k+2)^2}{4}$$

$$= \left(\frac{(k+1)(k+2)}{2}\right)^2 = \frac{(k+1)^2\,(k+2)^2}{4}$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore by mathematical induction P(n) is true for all $n \geq 1$.

Q.22 Show that $\dfrac{1^2}{1\cdot 3} + \dfrac{2^2}{3\cdot 5} + \ldots + \dfrac{n^2}{(2n-1)\,(2n+1)} = \dfrac{n\,(n+1)}{2(2n+1)}$

Ans. : Let P(n) be the given statement,

1. Basis of induction : For n = 1

We have, $\quad \text{L.H.S.} = \dfrac{1}{1.3} = \dfrac{1}{3}$

$$\text{R.H.S.} = \frac{1\,(2)}{1(3)} = \frac{1}{3}$$

$$\Rightarrow \text{L.H.S.} = \text{R.H.S.}$$

Hence P(1) is true.

2. Induction step : Assume that P(k) is true

i.e. $\dfrac{1^2}{1\cdot 3} + \dfrac{2^2}{3\cdot 5} + \ldots \dfrac{k^2}{(2k-1)\,(2k+1)} = \dfrac{k\,(k+1)}{2\,(2k+1)}$ ... (Q.

Then we have,

$$\left[\frac{1^2}{1\cdot 3} + \frac{2^2}{3\cdot 5} + \ldots + \frac{k^2}{(2k-1)\,(2k+1)}\right] + \frac{(k+1)^2}{[2\,(k+1)-1][2\,(k+1)+1]}$$

$$= \frac{k\,(k+1)}{2\,(2k+1)} + \frac{(k+1)^2}{(2k+1)\,(2k+3)} \quad \text{... (Using Q.22.}$$

$$= \frac{(k+1)}{(2k+1)}\left[\frac{k(2k+3)+2\,(k+1)}{2\,(2k+3)}\right]$$

$$= \frac{(k+1)}{2k+1}\left[\frac{2k^2+5k+2}{2\,(2k+3)}\right]$$

$$= \frac{(k+1)}{(2k+1)}\left[\frac{2k^2+4k+k+2}{2(2k+3)}\right]$$

$$= \frac{(k+1)}{(2k+1)}\left[\frac{2k(k+2)+1(k+2)}{2(2k+3)}\right]$$

$$= \frac{(k+1)(k+2)}{2(2k+3)} = \frac{(k+1)[(k+1)+1]}{2[2(k+1)+1]}$$

$\therefore$ P(k + 1) is true.

Therefore by mathematical induction P(n) is true for all $n \geq 1$.

Q.23 Prove by induction for $n \geq 0$. $1+a+a^2+\ldots+a^n = \dfrac{1-a^{n+1}}{1-a}$

[SPPU : Dec.-10, Marks 4]

Ans. : Let P(n) be the given statement,

1. Basis of induction

For n = 0, L.H.S. = 1, R.H.S. = $\dfrac{1-a}{1-a} = 1$

For n = 1, L.H.S. = 1 + a, R.H.S. = $\dfrac{1-a^2}{1-a} = 1+a$

$\therefore$ For n = 0, 1, L.H.S. = R.H.S.

Hence P(0), P(1) are true.

2. Induction step : Assume that P(k) is true

$$\therefore \quad 1+a+a^2+\ldots+a^k = \frac{1-a^{k+1}}{1-a} \quad \ldots \text{(Q.23.1)}$$

Consider,

$$1+a+a^2+\ldots+a^k+a^{k+1} = \frac{1-a^{k+1}}{1-a}+a^{k+1} \quad \ldots \text{(Using Q.23.1)}$$

$$= \frac{1-a^{k+1}+(1-a)a^{k+1}}{1-a}$$

$$= \frac{1-a^{k+1}+a^{k+1}-a^{k+2}}{1-a} = \frac{1-a^{k+2}}{1-a}$$

Hence P(k + 1) is true.

Therefore by the mathematical induction P(n) is true for all $n \geq 0$.

Q.24 Use mathematical induction to show that $n(n^2 - 1)$ is divisib by 24. Where n is any odd positive number.

☞[SPPU : Dec.-14, Marks

Ans. : If $n(n^2 - 1) = n^3 - n$ is divisible by 24.

Then $\qquad n^3 - n = 24(m)$ where m is any positive integral.

Let P(n) be the given statement,

1. Induction step :

For n = 1,

$n(n^2 - 1) = 0$ which is divisible by 24.

For n = 3, $n(n^2 - 1) = 24$ which is divisible by 24.

∴ P(1) and P(3) is true.

2. Induction step : Assume that P(k) is true.

i.e. $\qquad k(k^2 - 1) = k^3 - k$ is divisible by 24.

∴ $\qquad k(k^2 - 1) = k^3 - k = 24(m_0),\ m_0 \leftarrow z \qquad$ ... (Q.24.1

Consider

$$(k+1)[(k+1)^2 - 1] = (k+1)^3 - (k+1) = k^3 + 3k^2 + 3k + 1 - k - 1$$

$$= k^3 + 3k^2 + 2k = (k^3 - k) + 3k^2 + 3k$$

... (Using Q.24.1)

$$= 24m_0 + 3k(k+1)$$

(As k(k + 1) is multiple of 8 for k odd positive integer and $k \geq 3$)

$$= 24m_0 + 3(8m_1) = 24(m_0 + m_1)$$

$$= 24m_2 \quad (\because m_0 + m_1 = m_2)$$

∴ P(k + 1) is true.

∴ By mathematical induction P(n) is true for all n odd positive number.

Q.25 Show that $n^4 - 4n^2$ is divisible by 3 for all $n \geq 2$.

☞[SPPU : Dec.-15, Marks 4]

Ans. : Let P(n) be the given statement,

1. Basis of induction

For $n = 2$

$$2^4 - 4(2^2) = 16 - 16 = 0 \text{ is divisible by 3 as 0 is divisible}$$

by every number

$\therefore$ P(2) is true.

2. Induction step : Assume that P(k) is true

i.e. $k^4 - 4k^2$ is divisible by 3

Then we have,

$$(k+1)^4 - 4(k+1)^2 = k^4 + 4k^3 + 6k^2 + 4k + 1 - 4(k^2 + 2k + 1)$$

$$= (k^4 - 4k^2) + 4(k^3 + 2k) + 6k^2 + 12k - 3$$

$k^4 - 4k^2$ is divisible by 3.

$k^3 + 2k$ is divisible by 3

Also $6k^2 + 12k - 3 = 3(2k^2 + 4k - 1)$ is divisible by 3.

Hence $(k+1)^4 - 4(k+1)^2$ is divisible by 3.

Hence assuming P(k) is true.

P(k + 1) in also true. Therefore P(n) is true for $n \geq 2$.

Q.26 Prove the statement is true using mathematical induction : $n^3 + 2n$ is divisible by 3 for all $n \geq 1$. **☞[SPPU : Dec.-16]**

Ans. : Let P(n) be the given statement

1) Basis of induction : For $n = 1$, $1^3 + 2(1) = 3$ is divisible by 3.

$\therefore$ P(1) is true.

2) Induction step : Assume that P(k) is true.

i.e. $k^3 + 2k$ is divisible by 3

$$\therefore \quad k^3 + 2k = 3k_0 \quad \ldots \text{(Q.26.1)}$$

We have,

$$(k+1)^3 + 2(k+1) = k^3 + 3k^2 + 3k + 1 + 2k + 2$$

$$= (k^3 + 2k) + 3k^2 + 3k + 3$$

$$= (k^3 + 2k) + 3(k^2 + k + 1)$$

$$= 3k_0 + 3(k^2 + k + 1) = 3(k_0 + k^2 + k + 1)$$

$\therefore$ $(k+1)^3 + 2(k+1)$ is divisible by 3

$\therefore$ P(k +1) is true.

Hence, by mathematical induction P(n) is true for all $n \geq 1$.

Q.27 Using mathematical induction, prove that

$$1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n-1} n^2 = (-1)^{n-1} \frac{n(n+1)}{2}$$

[SPPU : Dec.-04, Marks...]

Ans. : Let P(n) be the given statement,

1. Basis of induction

For n = 1 L.H.S = 1, R.H.S. = 1

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that P(k) in true.

i.e. $1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1} k^2 = (-1)^{k-1} \frac{k(k+1)}{2}$ ... (Q.27...

Then we have,

$[1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{k-1} k^2] + (-1)^k (k+1)^2$ ... (Using Q.27...

$$= (-1)^{k-1} \frac{k(k+1)}{2} + (-1)^k (k+1)^2$$

$$= (-1)^k (k+1)\left[-\frac{k}{2} + (k+1)\right]$$

$$= (-1)^k (k+1)\left[\frac{-k + 2k + 2}{2}\right]$$

$$= (-1)^k \frac{(k+1)(k+2)}{2}$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) in true for all $n \geq 1$.

Q.28 Prove by mathematical induction that for $n \geq 1$:
$1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + n \cdot n! = (n+1)! - 1$

[SPPU : May-08, 15, Marks 6]

Ans. : Let P(n) be the given statement,

1. Basis of induction

For $n = 1$, L.H.S $= 1$, R.H.S. $= 1$

$\Rightarrow$ L.H.S. = R.H.S.

Hence P(1) is true.

2. Induction step : Assume that, P(k) is true. ... (Q.28.1)

i.e. $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + k \cdot k! = (k+1)! - 1$

Then we have, ... (Using Q.28.1)

$$[1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \ldots + k \cdot k!] + (k+1) \cdot (k+1)!$$
$$= [(k+1)! - 1] + (k+1) \cdot (k+1)!$$
$$= (k+1)! + (k+1) \cdot (k+1)! - 1$$
$$= (k+1)!\ [k+1-1] - 1 = (k-2)\ (k-1)! - 1$$
$$= (k+2)! - 1$$

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) is true for $n \geq 1$.

Q.29 Prove that for any positive integer n the number $n^5 - n$ is divisible by 5. **[SPPU : Dec.-08, Marks 6]**

Ans. : Let P(n) be the given statement,

1. Basis of induction :

For $n = 1$, $1^5 - 1 = 0$ is divisible by 5.

As 0 is divisible by every number.

Hence P(1) is true.

2. Induction step : Assume that, P(k) is true.

i.e. $k^5 - k$ is divisible by 5

Then we have

$(k+1)^5 - (k+1) =$

$(k^5 + {}^5C_1 k^4 + {}^5C_2 k^3 + {}^5C_3 k^2 + {}^5C_4 k + {}^5C_5) - (k+1)$

$$= k^5 + 5k^4 + 10k^3 + 10k^2 + 5k + 1 - k - 1$$

$$= (k^5 - k) + 5[k^4 + 2k^3 + 2k^2 + k]$$

$k^5 - k$ is divisible by 5.

and $5(k^4 + 2k^3 + 2k^2 + k)$ is divisible by 5.

Hence $(k+1)^5 - (k+1)$ is divisible by 5.

Hence assuming P(k) is true. P(k + 1) is also true. Therefore P(n) is true for $n \geq 1$.

Q.30 Prove that $8^n - 3^n$ is a multiple of 5 by mathematical induction for $n \geq 1$. ☞[SPPU : May-06, 07, Dec.-13, Marks 6]

Ans. : Let P(n) be the given statement,

1. Basis of induction :

For n = 1 $\quad 8^1 - 3^1 = 5 = 5 \cdot 1$

Obviously a multiple of 5.

$\therefore$ P(1) is true.

2. Induction step : Assume that, P(k) in true.

i.e. $8^k - 3^k$ is multiple of 5 say 5 r

i.e. $\quad 8^k - 3^k = 5r \quad$ ... (Q.30.1)

where r is an integer

Then we have,

$$8^{k+1} - 3^{k+1} = 8^k \cdot 8 - 3^k \cdot 3 = 8^k \cdot (5+3) - 3^k \cdot 3$$

$$= 8^k \cdot 5 + (8^k \cdot 3 - 3^k \cdot 3) = 8^k \cdot 5 + 3(8^k - 3^k)$$

Obviously $8^k \cdot 5$ is multiple of 5 and also $8^k - 3^k$ is multiple of 5.

Therefore, $8^{k+1} - 3^{k+1}$ is multiple of 5.

Hence assuming P(k) is true, P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.

Q.31 Show that the sum of the cubes of three consecutive natural number is divisible by 9. ☞[SPPU : Dec.-06, Marks 6]

Ans. : Let n, n + 1, n + 2 be three consecutive natural numbers.

We have to show that $n^3 + (n+1)^3 + (n+2)^3$ is divisible by 9.

Let P(n) be the above statement,

1. Basis of induction : For n = 1

$$1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36 \quad \text{which is divisible by 9.}$$

$\therefore$ P(1) is true.

2. Induction step : Assume that P(k) is true.

i.e. $k^3 + (k+1)^3 + (k+2)^3$ in divisible by 9.

Then we have,

$$(k+1)^3 + (k+2)^3 + (k+3)^3 = [(k+1)^3 + (k+2)^3] + [k^3 + {}^3C_1 k^2 (3) + {}^3C_2 k(3)^2 + {}^3C_3 (3^3)]$$

$$= (k+1)^3 + (k+2)^3 + k^3 + [9k^2 + 27k + 27]$$

$$= [k^3 + (k+1)^3 + (k+2)^3] + 9[k^2 + 3k + 3]$$

$k^3 + (k+1)^3 + (k+2)^3$ is divisible by 9 and $9(k^3 + 3k + 3)$ is divisible by 9.

$\Rightarrow (k+1)^3 + (k+2)^3 + (k+3)^3$ is divisible by 9.

Hence assuming P(k) in true, P(k + 1) is also true. Therefore P(n) is true for all $n \geq 1$.

Q.32 Using mathematical induction prove that

$$3 + 3.5 + 3.5^2 + \ldots + 3.5^n \cdot \left(\frac{5^{n+1} - 1}{4}\right).$$

For non-negative number n. **[SPPU : Dec.-11, Marks 6]**

Ans. : Cancelling 3 from the both sides of given statement, we get,

$$1 + 5 + 5^2 + \ldots + 5^n = \frac{5^{n+1} - 1}{5 - 1} \quad \ldots \text{(Q.32.1)}$$

Let P(n) be the above statement.

To prove this refer Q.23 for a = 5.

1.7 : Logic

Important Points to Remember

1. **Statement (Propositions)** : A statement or propositions is declarative sentence which is either true or false but not both. These two values 'True' and 'False' are denoted by 'T' (or 1) and 'F' (or 0) respectively.
2. **Logical connectives** : The words or symbols which are used to form compound statements are called connectives. There are five logical connectively i.e. negation, conjunction, disjunction, conditional and biconditional.

(i) **Negation (NOT)** : If p is any statement then the negation of p is denoted by ~p or ¬P and it is a statement read as 'not p'. It's truth table is

p	~p
T	F
F	T

(ii) **Conjunction (AND)** : If p and q are two statements then the conjunction of p and q is the compound statement 'p and q' and it is denoted by $p \wedge q$. The compound statement is true if both p and q are true, otherwise false.

(iii) **Disjunction (OR)** : If p and q are two statements then the disjunction of p and q is the compound statement 'p OR q' and it is denoted by $P \vee q$. The compound statement $p \vee q$ is false if both p and q are false otherwise true.

(iv) **Conditional statement** *(If ......then)* : If p and q are two statements then the conditional statement of p and q is the compound statement. 'If p then q' and it is denoted by $p \Rightarrow q$. The compound statement $P \Rightarrow q$ is false if p is true and q is false otherwise true.

(v) **Biconditional statement** : *(If and only if)* : If p and q are two statements then the biconditional statement of p and q is the compound 'p if and only if q' and it is denoted by $p \Leftrightarrow q$. The statement $p \Leftrightarrow$ is true if both p and q are true or both are false, otherwise false.

Truth table for all above types is

p	q	$p \wedge q$	$p \vee q$	$p \Rightarrow q$	$p \Leftrightarrow q$
T	T	T	T	T	T
T	F	F	T	F	F
F	T	F	T	T	F
F	F	F	F	T	T

3. If $p \rightarrow q$ is the conditional statement then

(i) $q \rightarrow p$ is called its converse statement

(ii) $\sim p \rightarrow \sim q$ is called inverse statement.

(iii) $\sim q \rightarrow \sim p$ is called its contrapositive statement.

4. Tautology : A statement which is true for all possible values of its propositional variables is called *tautology.*

5. Contradiction : A statement which is false for all possible values of its propositional variables is called *contradiction.* It is true negation of tautology.

6. Contingency : A statement which is neither tautology nor contradiction is called *contingency.*

7. Predicates : An assertion that contains one or more variables is called predicates. It's truth value is predicated after assigning truth values to its variables.

8. Quantifiers :

(i) The existential quantifiers tell us that for some objects, given statement is true. It is denoted by $\exists$. This symbol ($\exists$) read as *'there exist'* or 'for some'.

(ii) The universal quantifiers tell us that all objects possess the property. It is denoted by $\forall$ and read as 'for all' or for each or for every.

(iii) Negation :

Statement	Negation
1. $\forall x\ [p(x)]$	$\exists x \exists n\ [\sim p(n)]$
2. $\exists x\ (\sim p(x))$	$\forall n\ (p(n))$
3. $\forall x\ (\sim p(x))$	$\exists n\ (p(n))$
4. $\exists x\ p(x)$	$\forall n\ [\sim p(n)]$

Q.33 Let p denote the statement, "The material is interesting", denote the statement, "The exercises are challenging", and r denote the statement, "The course is enjoyable".

Write the following statements in symbolic form :

i)The material is interesting and exercises are challenging.

ii) The material is interesting means the exercises are challenging and conversely.

iii)Either the material is interesting or the exercises are not challenging but not both.

iv) If the material is not interesting and exercises are not challenging then the course is not enjoyable.

v) The material is uninteresting, the exercises are not challenging and the course is not enjoyable.

☞[SPPU : Dec.-06, May-08, Marks

Ans. : i) $p \wedge q$ ii) $(p \to q) \wedge (q \to p)$

iii) $p \oplus \sim q$ iv) $(\sim p \wedge \sim q) \to \sim r$ v) $\sim p \wedge \sim q \wedge \sim r$

Q.34 Express following statement in propositional form :

i) There are many clouds in the sky but it did not rain.

ii) I will get first class if and only if I study well and score above 80 in mathematics.

iii) Computers are cheap but softwares are costly.

iv) It is very hot and humid or Ramesh is having heart problem.

v) In small restaurants the food is good and service is poor.

vi)If I finish my submission before 5.00 in the evening and it is not very hot I will go and play a game of hockey.

☞[SPPU : May-05, Marks 6]

Ans. : i) p : There are many clouds in the sky

q : It rain

$\therefore \quad p \wedge \sim q$

ii) p : I will get first class

q : I study well

r : Score above 80 in mathematics

$\therefore \quad p \leftrightarrow (q \wedge r)$

iii) p : Computers are cheap

q : Softwares are costly

$\therefore \quad p \wedge q$

iv) p : It is very hot

q : It is very humid

r : Ramesh is having heart problem

$\therefore \quad (p \wedge q) \vee r$

v) p : In small restaurant food in good

q : Service is poor

$\therefore \quad p \wedge q$

vi) p : I finish my submission before 5:00 p.m.

q : It is very hot

r : I will go

s : I will play a game of hockey

$\therefore \quad (p \wedge \sim q) \rightarrow (r \wedge s)$

Q.35 Express the contrapositive, converse and inverse forms of the following statement if $3 < b$ and $1 + 1 = 2$, then $\sin \frac{\pi}{3} = \frac{1}{2}$.

[SPPU : May-07, Marks 6]

Ans. : p : $3 < b$

q : $1 + 1 = 2$

r : $\sin \frac{\pi}{3} = \frac{1}{2}$

Symbolic form : $p \wedge q \rightarrow r$

Contrapositive : $(\sim r \rightarrow \sim (p \wedge q))$

i.e. $\sim r \rightarrow (\sim p \vee \sim q)$

i.e. if $\sin\frac{\pi}{3} \neq \frac{1}{2}$ then $3 \geq b$ or $1+1 \neq 2$

Converse : $r \rightarrow (p \wedge q)$

i.e. if $\sin\frac{\pi}{3} = \frac{1}{2}$ then $3 < b$ and $1 + 1 = 2$

Inverse : $\sim(p \wedge q) \rightarrow \sim r$

i.e. $(\sim p \vee \sim q) \rightarrow \sim r$ i.e. if $3 \geq b$ or $1+1 \neq 2$ then $\sin\frac{\pi}{3} \neq \frac{1}{2}$

Q.36 Show that $p \wedge \sim p$ is a contradiction and $\sim(p \wedge \sim p)$ is tautology.

Ans. : We construct truth table for $\sim(p \wedge \sim p)$

p	~p	$p \wedge \sim p$	$\sim(p \wedge \sim p)$
T	F	F	T
F	T	F	T

As $p \wedge \sim p$ is always false. Hence $p \wedge \sim p$ is a contradiction. As $\sim(p \wedge \sim p)$ is always true, Hence $\sim(p \wedge \sim p)$ is a tautology.

Q.37 Determine whether each of the following statement formula is a tautology, contradiction or contingency.

i) $(p \wedge q) \wedge \sim(p \vee q)$

ii) $(p \rightarrow q) \leftrightarrow (q \vee \sim p)$

iii) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

iv) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ **[SPPU : Dec.-12, Marks 6]**

Ans. : i) Consider the truth table

p	q	$p \wedge q$	$p \vee q$	$\sim(p \vee q)$	$(p \wedge q) \wedge \sim(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Hence $(p \wedge q) \wedge \sim(p \vee q)$ is a contradiction

ii) Consider the truthtable

p	q	$\sim p$	$p \to q$	$q \vee \sim p$	$(p \to q) \leftrightarrow (q \vee \sim p)$
T	T	F	T	T	T
T	F	F	F	F	T
F	T	T	T	T	T
F	F	T	T	T	T

As $(p \to q) \leftrightarrow (q \vee \sim p)$ is always true. Hence it is a tautology.

iii) Consider the truth table

p	q	r	$p \to q$	$p \to r$	$q \to r$	$p \to (q \to r)$	$(p \to q) \to (p \to r)$	$[p \to (q \to r)] \to [(p \to q) \to (p \to r)]$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Hence $[(p \to (q \to r)] \to [(p \to q) \to (p \to r)]$ is a tautology.

iv) Consider the truth table

p	q	r	$p \to q$	$q \to r$	$(p \to q) \wedge (q \to r)$	$p \to r$	$[(p \to q) \wedge (q \to r)] \to (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T

				T	T	T	T
F	T	T	T	F	F	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Hence given statement formula is a tautology.

Q.38 Show that $(p \rightarrow q) \wedge \sim q \rightarrow \sim p$ is a tautology without using truth table.

Ans. : We know that $p \rightarrow q$ is true if p is true and q is also true.

$\therefore$ We need only to show that $p \rightarrow q$ and $\sim q$ both are true imply $\sim p$ is true.

As the truth value of $\sim q$ is T, the truth value of q is F. And as $p \rightarrow q$ is true, this means that p is false ($\because F \rightarrow F$ is true)

$\therefore$ The truth value of p is T. Hence the proof.

Q.39 Prove that $[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow (q \vee s)$ is a tautology.

[SPPU : Dec.-10, Marks 4]

Ans. : Consider truth table

p	q	r	s	$p \rightarrow q$	$r \rightarrow s$	$p \vee r$	$(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)$ (I)	$q \vee s$	$I \rightarrow (q \vee s)$
T	T	T	T	T	T	T	T	T	T
T	T	T	F	T	F	T	F	T	T
T	T	F	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T	T	T
T	F	T	T	F	T	T	F	T	T
T	F	T	F	F	F	T	F	F	T
T	F	F	T	F	T	T	F	T	T
T	F	F	F	F	T	T	F	F	T
F	T	T	T	T	T	T	T	T	T
F	T	T	F	T	F	T	F	T	T
F	T	F	T	T	T	F	F	T	T
F	T	F	F	T	T	F	F	T	T

F	F	T	T	T	T	T	T	T	T
F	F	T	F	T	F	T	F	F	T
F	F	F	T	T	T	F	F	T	T
F	F	F	F	T	T	F	F	F	T

Hence given statement formula is a tautology.

Q.40 Prove by truth table $p \rightarrow (Q \vee R) \equiv (P \rightarrow Q) \vee (P \rightarrow R)$.

[SPPU : Dec.-12]

Ans. : Consider the truth table

				I			II
P	Q	R	$Q \vee R$	$P \rightarrow (Q \vee R)$	$P \rightarrow Q$	$P \rightarrow R$	$(P \rightarrow Q) \vee (P \rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

From truth table

$$p \rightarrow (Q \vee R) \equiv (P \rightarrow Q) \vee (P \rightarrow R)$$

Q.41 Prove by constructing the truth table $p \rightarrow (q \vee r) \equiv (p \rightarrow q) \vee (p \rightarrow r)$.

[SPPU : Dec.-12, Marks 3]

Ans. : Consider truth table

p	q	r	$q \vee r$	$p \rightarrow (q \vee r)$	$p \rightarrow q$	$p \rightarrow r$	$(p \rightarrow q) \vee (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F

F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

In the columns of $p \to (q \vee r)$ and $(p \to q) \vee (p \to r)$, truth values are same for all possible choices of truth values of p, q and r. Hence

$$p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$$

Q.42 Prove that $p \leftrightarrow q \equiv (p \to q) \wedge (q \to p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$.

Ans. : Consider the truth table

p	q	~ p	~ q	p → q	q → p	(p → q) ∧(q → p)	p ↔ q	~ p∨ q	~ q ∨ p	(~ p∨ q) ∧(~ q∨p)
T	T	F	F	T	T	T	T	T	T	T
T	F	F	T	F	T	F	F	F	T	F
F	T	T	F	T	F	F	F	T	F	F
F	F	T	T	T	T	T	T	T	T	T

From above table

$$(p \leftrightarrow q) \equiv (p \to q) \wedge (q \to p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$$

Q.43

Sr. No.	*p*	*q*	*Conjunction*	p∧q
1.	*3 > 2 (T)*	*5 is prime (T)*	*3 > 2 and 5 is prime*	*T*
2.	*Delhi is in India (T)*	*3 + 30 = 30 (F)*	*Delhi is in India and 3 + 30 = 30*	*F*
3. Construct a truth table for the conjunction of n < 20 and n > 5, n ∈ N				

Ans. : When n < 20 and n > 5 are true then the conjunction "n < 20 and n > 5" is true.

The truth table is as follows :

$p : n < 20$	$q : n > 5$	$p \wedge q : n < 20$ and $n > 5$ i.e. $5 < n < 20$
T (n = 15)	T (n = 15)	T ($\because 5 < 15 < 20$)
T (n = 2)	F (n = 2)	F ($\because n < 5$)
F (n = 30)	T (n = 30)	F ($\because n > 20$)
F (n = 50)	F ---	F (There does not exist any number s.t. $n > 20$ and $n < 5$)

Q.44 Express following statement in propositional form :

i) There are many clouds in the sky but it did not rain.

ii) I will get first class if and only if I study well and score above 80 in mathematics.

iii) Computers are cheap but softwares are costly.

iv) It is very hot and humid or Ramesh is having heart problem.

v) In small restaurants the food is good and service is poor.

vi) If I finish my submission before 5.00 in the evening and it is not very hot I will go and play a game of hockey.

[SPPU : May-05, Marks 6]

Ans. : i) p : There are many clouds in the sky

q : It rain

$\therefore p \wedge \sim q$

ii) p : I will get first class

q : I study well

r : Score above 80 in mathematics

$\therefore \quad p \leftrightarrow (q \wedge r)$

iii) p : Computers are cheap

q : Softwares are costly

$\therefore \quad p \wedge q$

iv) p : It is very hot

q : It is very humid

r : Ramesh is having heart problem

$\therefore$ $(p \wedge q) \vee r$

v) p : In small restaurant food in good

q : Service is poor

$\therefore$ $p \wedge q$

vi) p : I finish my submission before 5:00 p.m.

q : It is very hot

r : I will go

s : I will play a game of hockey

$\therefore$ $(p \wedge \sim q) \rightarrow (r \wedge s)$

Q.45 Express the contrapositive, converse and inverse forms of the following statement if 3 < b and 1 + 1 = 2, then $\sin \frac{\pi}{3} = \frac{1}{2}$.

[SPPU : May-07, Marks 6]

Ans. : $p : 3 < b$

$q : 1 + 1 = 2$

$r : \sin \frac{\pi}{3} = \frac{1}{2}$

Symbolic form : $p \wedge q \rightarrow r$

Contrapositive : $(\sim r \rightarrow \sim (p \wedge q))$

i.e. $\sim r \rightarrow (\sim p \vee \sim q)$

i.e. if $\sin \frac{\pi}{3} \neq \frac{1}{2}$ then $3 \geq b$ or $1+1 \neq 2$

Converse : $r \rightarrow (p \wedge q)$

i.e. if $\sin \frac{\pi}{3} = \frac{1}{2}$ then $3 < b$ and $1 + 1 = 2$

Inverse : $\sim (p \wedge q) \rightarrow \sim r$

i.e. $(\sim p \vee \sim q) \rightarrow \sim r$

i.e. if $3 \geq b$ or $1+1 \neq 2$ then $\sin \frac{\pi}{3} \neq \frac{1}{2}$

Q.46 Express the contrapositive, converse, inverse and negation forms of the conditional statement given below.
'If x is rational, then x is real'. ☞[SPPU : Dec.-06, Marks 4]

Ans. : Let p : x is rational
q : is real

Symbolic form : $p \rightarrow q$

Contrapositive : $(\sim q \rightarrow \sim p)$

If x is not real, then x is not rational

Converse : $(q \rightarrow p)$

If x is real then x is rational

Inverse - $(\sim p \rightarrow q)$

If x is not rational, then x is not real

Negation : $\sim(p \rightarrow q) \equiv \sim(p \vee \sim q) \equiv \sim p \wedge \sim\sim q \equiv \sim p \wedge q$

Q.47 Determine whether each of the following statement formula is a tautology, contradiction or contingency.

i) $(p \wedge q) \wedge \sim (p \vee q)$ ii) $(p \rightarrow q) \leftrightarrow (q \vee \sim p)$

iii) $(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))$

iv) $[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$ ☞[SPPU : Dec.-12, Marks 6]

Ans. : i) Consider the truth table

p	q	$p \wedge q$	$p \vee q$	$\sim(p \vee q)$	$(p \wedge q) \wedge \sim(p \vee q)$
T	T	T	T	F	F
T	F	F	T	F	F
F	T	F	T	F	F
F	F	F	F	T	F

Hence $(p \wedge q) \wedge \sim(p \vee q)$ is a contradiction

ii) Consider the truthtable

p	q	$\sim p$	$p \rightarrow q$	$q \vee \sim p$	$(p \rightarrow q) \leftrightarrow (q \vee \sim p)$
T	T	F	T	T	T
T	F	F	F	F	T

F	T	T	T	T	T
F	F	T	T	T	T

As $(p \rightarrow q) \leftrightarrow (q \vee \sim p)$ is always true. Hence it is a tautology.

iii) Consider the truth table

p	q	r	$p \rightarrow q$	$p \rightarrow r$	$q \rightarrow r$	$p \rightarrow (q \rightarrow r)$	$(p \rightarrow q) \rightarrow (p \rightarrow r)$	$[p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$
T	T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F	T
T	F	T	F	T	T	T	T	T
T	F	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T	T
F	T	F	T	T	F	T	T	T
F	F	T	T	T	T	T	T	T
F	F	F	T	T	T	T	T	T

Hence $[(p \rightarrow (q \rightarrow r)] \rightarrow [(p \rightarrow q) \rightarrow (p \rightarrow r)]$ is a tautology.

iv) Consider the truth table

p	q	r	$p \rightarrow q$	$q \rightarrow r$	$(p \rightarrow q) \wedge (q \rightarrow r)$	$p \rightarrow r$	$[(p \rightarrow q) \wedge (q \rightarrow r)] \rightarrow (p \rightarrow r)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	T
T	F	T	F	T	F	T	T
T	F	F	F	T	F	F	T
F	T	T	T	T	T	T	T
F	T	F	T	F	F	T	T
F	F	T	T	T	T	T	T
F	F	F	T	T	T	T	T

Hence given statement formula is a tautology.

Q.48 Prove that $[(p \rightarrow q) \wedge (r \rightarrow s) \wedge (p \vee r)] \rightarrow (q \vee s)$ is a tautology.

[SPPU : Dec.-10, Marks 4]

ns. : Consider truth table

	q	r	s	$p \to q$	$r \to s$	$p \vee r$	$(p \to q) \wedge (r \to s) \wedge (p \vee r)$ (I)	$q \vee s$	$I \to (q \vee s)$
[illegible]	T	T	T	T	T	T	T	T	T
[illegible]	T	T	F	T	F	T	F	T	T
[illegible]	T	F	T	T	T	T	T	T	T
[illegible]	T	F	F	T	T	T	T	T	T
[illegible]	F	T	T	F	T	T	F	T	T
T	F	T	F	F	F	T	F	F	T
T	F	F	T	F	T	T	F	T	T
T	F	F	F	F	T	T	F	F	T
F	T	T	T	T	T	T	T	T	T
F	T	T	F	T	F	T	F	T	T
F	T	F	T	T	T	F	F	T	T
F	T	F	F	T	T	F	F	T	T
F	F	T	T	T	T	T	T	T	T
F	F	T	F	T	F	T	F	F	T
F	F	F	T	T	T	F	F	T	T
F	F	F	F	T	T	F	F	F	T

Hence given statement formula is a tautology.

Q.49 Prove by truth table $p \to (Q \vee R) \equiv (P \to Q) \vee (P \to R)$.

[SPPU : Dec.-12]

Ans. : Consider the truth table

				I			II
P	Q	R	$Q \vee R$	$P \to (Q \vee R)$	$P \to Q$	$P \to R$	$(P \to Q) \vee (P \to R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

From truth table

$$p \to (Q \vee R) \equiv (P \to Q) \vee (P \to R)$$

Q.50 Prove by constructing the truth table
$p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$. **[SPPU : Dec.-12, Marks 3]**

Ans. : Consider truth table

p	q	r	$q \vee r$	$p \to (q \vee r)$	$p \to q$	$p \to r$	$(p \to q) \vee (p \to r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	T	T
F	F	T	T	T	T	T	T
F	F	F	F	T	T	T	T

In the columns of $p \to (q \vee r)$ and $(p \to q) \vee (p \to r)$, truth values are same for all possible choices of truth values of p, q and r. Hence

$$p \to (q \vee r) \equiv (p \to q) \vee (p \to r)$$

1.8 : Logical Identities

Important Points to Remember

Sr. No.	Name of identity	Identity
1.	Idempotence of $\vee$	$p \equiv p \vee p$
2.	Idempotence of $\wedge$	$p \equiv p \wedge p$
3.	Commutativity of $\vee$	$p \vee q \equiv q \vee p$
4.	Commutativity of $\wedge$	$p \wedge q \equiv q \wedge p$
5.	Associativity of $\vee$	$p \vee (q \vee r) \equiv (p \vee q) \vee r$
6.	Associativity of $\wedge$	$p \wedge (q \wedge r) \equiv (p \wedge q) \wedge r$

7.	Distributivity of $\wedge$ over $\vee$	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$
8.	Distributivity of $\vee$ over $\wedge$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
9.	Double negation	$p \equiv \sim(\sim p)$
10.	De Morgan's laws	$\sim(p \vee q) \equiv \sim p \wedge \sim q$
11.	De Morgan's laws	$\sim(p \wedge p) \equiv \sim p \wedge \sim q$
12.	Tautology	$p \vee \sim q \equiv$ Tautology
13.	Contradiction	$p \wedge \sim p \equiv$ Contradiction
14.	Absorption laws	$p \vee (p \wedge q) \equiv p$
15.	Absorption laws	$p \wedge (p \vee q) \equiv p$

Q.51 De Morgan's laws i) $\sim(p \vee q) \equiv \sim p \wedge \sim q$ ii) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

Solution : i) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Consider the truth table

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

From the table, truth values of $\sim(p \vee q)$ and $\sim p \wedge \sim q$ are same for each choice of p and q

Hence $\sim(p \vee q) \equiv \sim p \wedge \sim q$

ii) $\sim(p \vee q) \equiv \sim p \vee \sim q$

Consider the table

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T

F	T	T	F	F	T	T
F	F	T	T	F	T	T

From the table, truth values of $\sim(p \wedge q)$ and $(\sim p \vee \sim q)$ are same for each choice of p and q.

Hence $\sim(p \wedge q) \equiv (\sim p \vee \sim q)$

Q.52 De Morgan's laws i) $\sim(p \vee q) \equiv \sim p \wedge \sim q$ ii) $\sim(p \wedge q) \equiv \sim p \vee \sim q$

Ans. : i) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

Consider the truth table

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

From the table, truth values of $\sim(p \vee q)$ and $\sim p \wedge \sim q$ are same for each choice of p and q

Hence $\sim(p \vee q) \equiv \sim p \wedge \sim q$

ii) $\sim(p \vee q) \equiv \sim p \vee \sim q$

Consider the table

p	q	$\sim p$	$\sim q$	$p \wedge q$	$\sim(p \wedge q)$	$\sim p \vee \sim q$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

From the table, truth values of $\sim(p \wedge q)$ and $(\sim p \vee \sim q)$ are same for each choice of p and q.

Hence $\sim(p \wedge q) \equiv (\sim p \vee \sim q)$

1.9 : Normal Forms

Disjunctive Normal Form (dnf)

A statement formula which consists of a disjunction $(\vee)$ of fundamental conjunctions or elementary product $(\wedge)$. It is abbreviated as dnf.

A disjunctive normal form of a given formula is constructed as follows :

1) Replace '$\rightarrow$' or '$\leftrightarrow$' by using logical connectives $\wedge, \vee$ & $\sim$. $p \rightarrow q \equiv \sim p \vee q$, $p \leftrightarrow q \equiv (\sim p \vee q) \wedge (\sim q \vee p)$
2) Use De Morgan's laws to eliminate '~' before sums or products.
3) Apply distributive laws repeatedly and eliminate product of variables to obtain the required normal form.

Examples :

1) $(p \wedge q) \vee (q) \vee (\sim q \wedge p)$
2) $(p \wedge q \wedge r) \vee (p \wedge r) \vee (p \wedge q) \vee (p \wedge r)$
3) $(p \wedge r) \vee (p \wedge q)$
4) $(\sim p \wedge r) \vee (\sim q \wedge r) \vee (\sim r)$

All above examples are in disjunctive normal form.

Conjunctive Normal Form (cnf)

A statement formula which consists of a conjunction of the fundamental disjunctions $(\vee)$. It is denoted as cnf.

Examples :

1) $p \wedge (p \vee q)$
2) $(p \vee q) \wedge (\sim p \vee q) \wedge (\sim q)$
3) $(p \vee q \vee r) \wedge (\sim p \vee r) \wedge (p \vee \sim q \vee r)$

All above examples are in conjunctive normal form.

Principal Normal Form

Let p and q be two statement variables, then $p \wedge q$, $p \wedge \sim q$, $\sim p \wedge q$, $\sim p \wedge \sim q$ are called minterms of p and q. They are also called Boolean conjunctives of p and q. The number of minterms with n variables is 2^n. None of the minterms should contain both a variable and it's negation. $\therefore$ $p \wedge \sim p$ is not minterm.

The dual of minterm is called a maxterm.

$\therefore$ For two statement variables p and q, maxterms are $p \vee q$, $\sim p \vee q$, $p \vee \sim q$ and $\sim p \vee \sim q$.

I) Principal disjunctive normal form

A statement formula which consists of a disjunction of minterms only is called the principal disjunctive normal form.

e.g.

1) $(p \vee q) \wedge (\sim p \wedge \sim q)$, $(p \wedge q \wedge r) \vee (\sim p \wedge q \wedge r)$ are principal cnf.
2) $p \vee (p \wedge q)$, $(p \wedge q) \vee (\sim p \wedge q) \vee (\sim q)$, $(p \wedge q \wedge r) \vee (p \wedge q)$ are not principal dnf.

II) Principal conjuctive normal form

A statement formula which consists of a conjunction of maxterms only is called the principle conjunctive normal form.

e.g.

1) $(p \vee q) \wedge (\sim p \vee q)$, $(p \vee \sim q) \wedge (\sim p \vee \sim q)$ are principal cnf.
2) $(p \vee q) \wedge (\sim p)$ is cnf but not principal cnf.

Q.53 Obtain the conjunctive normal form and disjunctive normal form of the following formulae given below :

i) $p \wedge (p \rightarrow q)$ ii) $\sim (p \vee q) \rightleftarrows (p \wedge q)$ **[SPPU : Dec.-12, Marks 4]**

Ans. : i) $p \wedge (p \rightarrow q) \equiv p \wedge (\sim p \vee q) \sim \text{cnf}$

$$\equiv (p \wedge \sim p) \vee (p \wedge q)$$

$$\equiv F \vee (p \vee q)$$

$$\equiv (p \wedge q) \quad \text{... Definition a single conjunctive}$$

ii) $\sim (p \vee q) \rightleftarrows (p \wedge q) \equiv (\sim\sim (p \vee q) \vee (p \wedge q)) \wedge (\sim (p \wedge q) \vee \sim (p \vee q))$

$$\equiv ((p \vee q) \vee (p \wedge q) \wedge ((\sim p \vee \sim q) \vee (\sim p \wedge \sim q)$$

$$\equiv (p \vee q) \wedge ((\sim p \vee \sim q \vee \sim p) \wedge (\sim p \vee \sim q \vee \sim q))$$

$$\equiv (p \vee q) \wedge (\sim p \vee \sim q) \wedge (\sim p \vee \sim q)$$

$$\equiv (p \vee q) \wedge (\sim p \vee \sim q) \quad \text{... cnf}$$

Further

$$(p \vee q) \wedge (\sim p \vee \sim q) \equiv ((p \vee q) \wedge \sim p) \vee ((p \vee q) \wedge \sim q)$$

$$\equiv (p \wedge \sim q) \vee (q \wedge \sim p) \vee ((p \wedge \sim q) \vee (q \wedge \sim q)$$

$$\equiv F \vee (q \wedge \sim p) \vee (p \wedge \sim q) \vee F$$

$$\equiv (p \wedge \sim p) \vee (p \wedge \sim q) \quad \text{... Definition}$$

Q.54 Find the conjunctive normal form and disjunctive normal form for the following : i) $(p \vee \bar{q}) \rightarrow q$ ii) $p \leftrightarrow (\bar{p} \vee \bar{q})$

[SPPU : Dec.-06, 14]

Ans. : i) $(p \vee \bar{q}) \rightarrow q \equiv \overline{(p \vee \bar{q})} \vee q \equiv (\bar{p} \wedge \bar{\bar{q}}) \vee q$

$$\equiv (\bar{p} \wedge q) \vee q \quad \text{... Definition}$$

$$(\bar{p} \wedge q) \vee q \equiv (\bar{p} \vee q) \wedge (q \vee q)$$

$$\equiv (\sim p \vee q) \wedge q \quad \text{... cnf}$$

$$p \times (\bar{p} \vee \bar{q}) \equiv (\bar{p} \vee (\bar{p} \vee \bar{q})) \wedge (\overline{(\bar{p} \vee \bar{q})} \vee p)$$

$$\equiv (\bar{p} \vee \bar{p} \vee \bar{q}) \wedge ((\bar{\bar{p}} \wedge \bar{\bar{q}}) \vee p)$$

$$\equiv (\bar{p} \vee \bar{q}) \wedge (p \vee p) \wedge (q \vee p)$$

$$\equiv (\bar{p} \vee \bar{q}) \wedge p \wedge (q \vee p) \quad \text{... cnf}$$

$$\equiv ((\bar{p} \wedge p) \vee (\bar{q} \wedge p)) \wedge (q \vee p)$$

$$\equiv (F \vee (\bar{q} \wedge p)) \wedge (q \vee p) \equiv (\bar{q} \wedge p) \wedge (q \vee p)$$

$$\equiv (\bar{q} \wedge p \wedge q) \vee (\bar{q} \wedge p \wedge p) \equiv (F \wedge p) \vee (\bar{q} \wedge p)$$

$$\equiv F \vee (\bar{q} \wedge p) \equiv (\bar{q} \wedge p) \quad \text{... Definition}$$

Q.55 Find the conjunctive and disjunctive normal forms for the following without using truth table
i) $(p \rightarrow q) \wedge (q \rightarrow p)$ ii) $((p \wedge (p \rightarrow q)) \rightarrow q)$

[SPPU : Dec.-04, 10, 14, Marks 4]

Ans. : i) $(p \rightarrow q) \wedge (q \rightarrow p) \equiv (\sim p \vee q) \wedge (\sim q \vee p)$

Further, using the distributive law on the above cnf we have

$$((\sim p \vee q) \wedge \sim q) \vee ((\sim p \vee q) \wedge p) = (\sim p \wedge \sim q) \vee (q \wedge \sim q) \vee (\sim p \wedge p) \vee (q \wedge p)$$

$$= (\sim p \wedge \sim q) \vee (q \wedge p) \quad \text{... dnf}$$

$$(\because \ p \wedge \sim p \equiv q \wedge \sim q)$$

ii) $(p \wedge (p \rightarrow q)) \rightarrow q \equiv \sim (p \wedge (\sim p \vee q)) \vee q \equiv \sim p \vee \sim (\sim p \vee q) \vee q$

$$\equiv \sim p \vee (p \wedge \sim q) \vee q$$

Q.56 Obtain cnf of each of the following

i) $p \wedge (p \rightarrow q)$ ii) $q \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q)$

[SPPU : Dec.-08, Marks

Ans. : i) $p \wedge (p \rightarrow q) \equiv (p \wedge (\sim p \vee q)) \equiv (p \wedge \sim p) \vee (p \wedge q)$

$\equiv \vee (p \wedge q) \equiv (p \wedge q)$ ... cnf

ii) $q \vee (p \wedge \sim q) \vee (\sim p \wedge \sim q) \equiv ((q \vee p) \wedge (q \vee \sim q) \vee (\sim p \vee \sim q)$

$\equiv (q \vee p) \wedge T \vee (\sim p \wedge \sim q)$

$\equiv (q \vee p) \vee (\sim p \wedge \sim q)$

$\equiv (q \vee p \vee \sim p) \wedge (q \wedge p \vee \sim q)$

$\equiv (q \vee T) \wedge (p \vee q \vee \sim q)$

$\equiv T \wedge (p \vee T) \equiv T \wedge T$

$\equiv T \equiv (p \vee \sim p)$

which is the required cnf (single disjunct)

Q.57 Find DNF of $((p \rightarrow q) \wedge (q \rightarrow p)) \vee p$

[SPPU : Dec.-14, Marks 4

Ans. : $((p \rightarrow q) \wedge (q \rightarrow p) \vee p \equiv [(\sim p \vee q) \wedge (\sim q \vee p)] \vee p$

$(\because p \rightarrow q \equiv - \sim p \vee q)$

$\equiv [p \vee (\sim p \vee q)] \wedge [p \vee (\sim q \vee p)]$

$\equiv [(p \vee \sim p) \vee q] \wedge [p \vee p \vee \sim p]$ $(\because p \vee p = p$

$p \vee \sim p = T)$

$\equiv (T \vee q) \wedge (p \vee \sim q) \equiv T \wedge (p \vee \sim q)$

$\equiv p \vee \sim q$ which is the required DNF

Q.58 Obtain the cnf : $(\sim p \wedge q \wedge r) \vee (p \wedge q)$

[SPPU : Dec.-12, Marks 4

Ans. : $(\sim p \wedge q \wedge r) \vee (p \wedge q) \equiv (p \wedge q) \vee (\sim p \wedge q \wedge r)$

$\equiv (p \vee (\sim p \wedge q \wedge r)] \wedge [q \vee (\sim p \wedge q \wedge r)]$

$\equiv [(p \vee \sim p) \wedge (p \vee q)] \wedge [p \vee r)] \wedge [(q \vee \sim p) \wedge (q \vee q) \wedge (q \vee r)]$

$\equiv [T \wedge (p \vee q) \wedge (p \vee r)] \wedge [(q \vee \sim p) \wedge q \wedge (q \vee r)]$

$\equiv (p \vee q) \wedge (p \vee r) \wedge (q \vee \sim p) \wedge q$ which is the required cnf

Q.59 Find dnf by using truth table $(p \leftrightarrow (q \vee r)) \rightarrow \sim p$

Ans. : Consider the truth table

p	q	r	~p	$q \vee r$	$p \leftrightarrow (q \vee r)$	$[p \leftrightarrow (q \vee r)] \rightarrow \sim p$
T	T	T	F	T	T	F
T	T	F	F	T	T	F
T	F	T	F	T	T	F
T	F	F	F	F	F	T ← 1
F	T	T	T	T	F	T ← 2
F	T	F	T	T	F	T ← 3
F	F	T	T	T	F	T ← 4
F	F	F	T	F	T	T ← 5

Consider only 'T' from last column and choose corresponding values of 'T' from p, q and r. For the first marked row 1 corresponding p is T, q is F and r is F. So take $p \wedge \sim q \wedge \sim r$ or $p \wedge q' \wedge r'$

For 2[nd] $T \rightarrow \sim p \wedge q \wedge r$, 3[rd] $T \rightarrow \sim p \wedge q \wedge \sim r$.

4[th] $T \rightarrow \sim p \wedge q \wedge r$, and 5[th] $T \rightarrow \sim p \wedge \sim q \wedge \sim r$

Hence the logically equivalent dnf form is

$$[p \leftrightarrow (q \vee r)] \rightarrow \sim p \equiv (p \wedge \sim q \wedge r) \vee (\sim p \wedge q \wedge r) \vee (\sim p \wedge q \wedge \sim r) \vee (\sim p \wedge \sim q \wedge r) \vee (\sim p \wedge \sim q \wedge \sim r)$$

1.10 : Law of Detachment (or Modus Ponens)

Whenever the statements p and $p \rightarrow q$ are accepted as true, then we must accept the statement q as true. This rule is represented in the following form

$$\begin{array}{l} p \rightarrow q \\ \underline{\quad p \quad} \\ \therefore q \end{array}$$

The assertions above the horizontal line are called premises or hypothesis. And the assertion below the line is called the conclusion.

This rule constitutes a valid argument as $(p \rightarrow q) \wedge p \rightarrow q$ is a tautology.

The truth table is as follows

p	q	$p \rightarrow q$	$(p \rightarrow q) \wedge p$	$[(p \rightarrow q) \wedge p] \rightarrow q$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	T
F	F	T	F	T

This form of valid argument is called the law of detachment as conclusion q is detached from premise $p \rightarrow q$ and p.

It is also called as the law of direct inference.

Example

If Sushma gets a first class with distinction in B.E. then she will get a good job easily.

Let p : Sushma gets a first class with distinction in B.E.

q : She will get a good job easily.

The inference rule is
$$\begin{array}{l} p \rightarrow q \\ \underline{\quad p \quad} \\ \therefore q \end{array}$$

Hence this form of argument is valid.

Modus Tollen (Law of Contrapositive)

Modus Tollen is a rule of denying. It can be stated as "If $p \rightarrow q$ is true and q is false, then p is false." This is represented in the following form.

$$\begin{array}{l} p \rightarrow q \\ \underline{\quad \sim p \quad} \\ \therefore \sim q \end{array}$$

Above argument is valid as $(p \rightarrow q) \wedge \sim q \rightarrow \sim p$ is a tautology. In above example is $b \rightarrow q$ and $\sim q$: Sushma will not get a good job easily then $\sim p$: She has not a first class with distinction in B.E.

Disjunctive Syllogism

This rule states that "If $p \vee q$ is true and p is false then q is true."

It is represented in the following form as

$$\begin{array}{l} p \vee q \\ \underline{\sim p} \\ \therefore\ q \end{array}$$

This argument is valid as $(p \vee q) \wedge \sim p \rightarrow q$ is a tautology.

Hypothetical Syllogism

It is also known as the transitive rule.

It can be stated as follows

"If $(p \rightarrow q)$ and $(q \rightarrow r)$ are true then $p \rightarrow r$ is true."

This rule is presented in the following form.

$$\begin{array}{l} p \rightarrow q \\ \underline{q \rightarrow r} \\ \therefore\ p \rightarrow r \end{array}$$

This argument is valid as $(p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$ is a tautology.

Q.60 Determine whether the argument is valid or not. If I try hard and I have talent then I will become a musician. If I become a musician, then I will be happy. Therefore if I will not be happy then I did not try hard or I do not have talent.

[SPPU : May-10, Marks 4]

Ans. : Let p : I try hard

q : I have talent

r : I will become musician

s : I will be happy

It's symbolic form is as follows

$$\begin{array}{l} s_1 : (p \wedge q) \rightarrow r \\ \underline{s_2 : \quad r \rightarrow s} \\ s_3 : \sim s \rightarrow \sim p \vee \sim q \end{array}$$

To check validity of this statement, one way is to use truth table or pr... logically.

Suppose assignment is invalid. This means that for some assignment truth values s_1 is T, s_2 is T but s_3 is F. s_3 will have truth value F if ... is T and $\sim p \vee \sim q$ is F

i.e. s is F and ~ p is F and ~ q is F

i.e. s is F, and p is T and q is T

As s_2 is T, the truth values of r and s both are F. Since s_1 is T, r is ... implies either p or q is F. This is contradiction since by assumption b... p and q are true.

Hence given statement is valid.

Q.61 Test the validity of the argument. If a person is poor, he ... unhappy. If a person is unhappy, he dies young. Therefore po... person dies young.

☞[SPPU : Dec.-09, Marks...

Ans. : Let p : Person is poor

q : Person is unhappy

r : Person dies young

In symbolic form argument is

$$\begin{array}{l} s_1 : p \rightarrow q \\ \underline{s_2 : q \rightarrow r} \\ s : p \rightarrow r \end{array}$$

The above argument is the rule of hypothetical syllogism. Hence it ... valid.

Q.62 Determine the validity of the argument

s_1 : All my friends are musicians

s_2 : John is my friend

s_3 : None of my neighbours are musicians

S : John is not my neighbour

☞[SPPU : May-0...

Ans. : Let p : All my friends are musicians

q : John is my friend

r : My neighbours are musicians

s : John is my neighbour

$s_1 : p$

$s_2 : q$

$s_3 : \sim r$

In symbolic form

$$\begin{array}{c} p \\ q \\ \sim r \\ \hline \sim s \end{array}$$

As all my friends are musicians and John is my friend $\Rightarrow$ John is musician.

$\therefore$ $p \wedge q \rightarrow$ John is musician

$p \wedge q \wedge \sim r \rightarrow$ John is musician and my neighbours are not musicians

$\therefore$ $p \wedge q \wedge \sim r \rightarrow$ John is not my neighbour

$p \wedge q \wedge \sim r \rightarrow \sim s$ is true.

Therefore given argument is valid.

1.11 : Quantifiers

Important Points to Remember

Consider the statement $\forall x p(x)$. It's negation is "It is not the case that for all x, p(x) is true". This means that for some x = a, p(a) is not true or $\exists$ x s.t. $\sim$ p(x) is true.

Hence the negation of $\forall x p(x)$ is logically equivalent to $\exists x [\sim p(x)]$.

Sr. No.	Statement	Negation
1.	$\forall x p(x)$	$\exists x [\sim p(x)]$
2.	$\exists x [\sim p(x)]$	$\forall x p(x)$
3.	$\forall x [\sim p(x)]$	$\exists x\, p(x)$
4.	$\exists x\, p(x)$	$\forall x [\sim p(x)]$

I) Equivalence involving quantifiers

1) Distributivity of $\exists$ over $\vee$

$$\exists x[p(x) \vee Q(x)] \equiv \exists x p(x) \vee \exists x Q(x)$$

$$\exists x[p \vee Q(x)] \equiv p \vee (\exists x Q(x))$$

2) Distributivity of $\forall$ over $\wedge$

$$\forall x[p(x) \wedge Q(x)] \equiv \forall x p(x) \wedge \forall x Q(x)$$

$$\forall x[p \wedge Q(x)] \equiv p \wedge (\forall x Q(x))$$

3) $\exists x[p \wedge Q(x)] \equiv p \wedge [\exists x Q(x)]$

4) $\forall x[p \vee Q(x)] \equiv p \wedge [\forall x Q(x)]$

5) $\sim[\exists x p(x)] \equiv \forall x[\sim p(x)]$

6) $\sim[\forall x p(x)] \equiv \exists x[\sim p(x)]$

7) $\forall x p(x) \Rightarrow \exists x p(x)$

8) $\forall x p(x) \vee \forall x Q(x) \Rightarrow \forall x(p(x) \vee Q(x)$

9) $\exists x(p(x) \wedge Q(x)) \Rightarrow \exists x p(x) \wedge \exists x Q(x)$

II) Rules of Inference for addition and deletion of quantifiers

1) Rule 1 : Universal Instantiation

$\dfrac{\forall x p(x)}{\therefore\ p(k)}$, k is some element of the universe

2) Rule 2 : Existential Instantiation

$\dfrac{\exists x p(x)}{\therefore\ p(k)}$, k is some element for which p(k) is true.

3) Rule 3 : Universal Generalization

$\dfrac{px}{\forall x p(x)}$

4) Rule 4 : Existential Generalization

$\dfrac{p(k)}{\therefore \exists x p(x)}$

k is some element of the universe

III)

Sr. No.	Quantifiers	Expression
1.	$\exists x \forall y\, p(x,y)$	There exists a value of x such that for all values of y, p(x, y) is true.
2.	$\forall y\, \exists x p(x,y)$	For each value of y, there exists x such that p(x, y) is true.
3.	$\exists x \exists y\, p(x,y)$	There exist value of x and value of y such that p(x, y) is true.
4.	$\forall x\, \forall y\, p(x,y)$	For all values of x and y. p(x, y) is true.

Q.63 Represent the arguments using quantifiers and find its correctness. All students in this class understand logic. Ganesh is a student in this class. Therefore Ganesh understands logic.

☞[SPPU : Dec.-11, Marks 4]

Ans. : Let C(x) : x is a student in this class

L(x) : x understands logic

In symbolic form

$$\begin{array}{l} \forall x(C(x) \rightarrow L(x)) \\ \quad C(a) \\ \hline \therefore \quad L(a) \end{array}$$

Here a means Ganesh

This is Modus Ponen

Therefore this argument is valid.

Q.64 Let p(x) : x is even

Q(x) : x is a prime number

R(x, y) : x + y is even

a) Using the information given above write the following sentences in symbolic form.

i) Every integer is an odd integer

ii) Every integer is even or prime

iii) The sum of any two integers is an odd integer.

[SPPU : Dec.-10, Marks 4]

Ans. : i) $\forall x[\sim p(x)]$ ii) $\forall x[p(x) \vee Q(x)]$ iii) $\forall x \forall y [\sim R(x, y)]$

Q.65 Write an english sentence for each of the symbolic statement given below

i) $\forall x (\sim Q(x))$ ii) $\exists y(\sim p(y))$ iii) $\sim[\exists x (p(x) \wedge Q(x))]$

[SPPU : Dec.-10, Marks 4]

Ans. : i) All integers are not prime numbers.

ii) At least one integer is not even.

iii) It is not the case that there exists an integer which is even and prime.

Q.66 Rewrite the following statements using quantifier variables and predicate symbols.

i) All birds can fly ii) Not all birds can fly

iii) Some men are genius iv) Some numbers are not rational

v) There is a student who likes Maths but not Hindi

vi) Each integer is either even or odd **[SPPU : Dec.-08, Marks 4]**

Ans. : i) Let $B(x)$: x is a bird

$F(x)$: x can fly

Then the statement can be written as

$\forall x[B(x) \rightarrow F(x)]$

ii) $\exists x[B(x) \wedge \sim F(x)]$

iii) Let $M(x)$: x is a man

$G(x)$: x is a genius

The statement in symbolic form as $\exists x[M(x) \wedge G(x)]$

iv) Let $N(x)$: x is a number

$R(x)$: x is rational

The statement in symbolic form as $\exists x[N(x) \wedge \sim R(x)]$ or $\sim[\forall x(N(x) \rightarrow R(x)]$

v) Let $S(x)$: x is a student

$M(x)$: x likes Maths

$H(x)$: x likes Hindi

$\therefore$ The statement in symbolic form as $\exists x[S(x) \wedge M(x) \wedge \sim H(x)]$

vi) Let $I(x)$: x is an integer

$E(x)$: x is even

$O(x)$: x is odd

The statement in symbolic form as $\forall x[I(x) \rightarrow E(x) \vee O(x)]$

Q.67 Negate each of the following statements

i) $\forall x, |x| = x$ ii) $\exists x, x^2 = x$

[SPPU : Dec.-09, 15, May-15, Marks 4]

Ans. : i) $\exists x, |x| \neq x$ ii) $\forall x, x^2 \neq x$

Q.68 Negate the following

i) If there is a riot, then someone is killed.

ii) It is day light and all the people are arisen.

[SPPU : May-15, Dec.-15, Marks 4]

Ans. : i) It is not the case that if there is a riot then someone is killed.

ii) It is not the case that it is day light and all the people are arisen.

OR

i) Let p : There is a riot

q : Someone is killed

Given statement is $p \rightarrow q$

Hence $\sim(p \rightarrow q) \equiv \sim(\sim p \vee q) \equiv p \wedge \sim q$

$\equiv$ There is a riot and someone is not killed.

ii) Let p : It is a day light

q : All the people are arisen

Given statement is $p \wedge q$

Hence $\sim(p \wedge q) = \sim p \vee \sim q$

Hence either it is not a day light or all the people are not arisen.

END...

Unit II

2 Combinatorics and Discrete Probability

2.1 : Basic Counting Principles

Important Points to Remember

1. Sum rule (principle of disjunctive counting)

We know that, if $S = A \cup B$ and A and B are disjoint sets i.e. $A \cap B = \phi$ then

$$|S| = |A| + |B|$$

i.e. A and B are disjoint partitions of S.

Now we can extend this logic to state sum rule.

Sum rule : If one experiment E_1 has n_1 possible outcomes and another experiment E_2 has n_2 possible outcomes and E_1 and E_2 are disjoint (exclusive) then there are $n_1 + n_2$ possible outcomes when E_1 or E_2 take place.

Product rule (The principle of sequential counting)

We have, if A and B are non empty sets and $|A| = n$, $|B| = n$. then the number of elements in the cartesian product of A and B is equal to $n \times m$

i.e. $|A \times B| = n \times m$

Now extend this analogy for the product rule.

Product rule : If one experiment E_1 has n_1 possible outcomes and another experiment E_2 has n_2 possible outcomes then there are $n_1 \cdot n_2$ possible outcomes when the sequence of experiment E_1 first followed by E_2.

Permutations : An arrangement in a sequence of elements of a set is called a permutation of elements.

Depending upon the nature of arrangements, there are three types of permutations.

Type I) : Permutations when all objects are distinct : A permutation of n objects taken r at a time is an arrangement of r objects out of n objects where $r \le n$.

It is called r - permutations or r - arrangements and denoted by P (n, r) or ${}^nP_r$.

e.g. 1) Consider the three letters a, b, c. The arrangements of the letters a, b, c taken two at a time are ab, ba, ac, ca, bc, cb.

$\therefore$ The number of 2 - arrangements are 6. i.e. the number of permutations of 3 symbols taken two at a time = ${}^3P_2 = 6$.

Therefore as discussed above, the first place in the sequence can be filled up in n - ways, the second place in (n – 1) ways, the third place in (n – 2) ways and proceeding in this manner the r^{th} place can be filled up in $n - (r - 1) = n - r + 1$ ways.

$$\text{Hence } {}^nP_r = n \cdot (n-1) \cdot (n-2) \ldots (n-(r-1))$$

$$= \frac{n \cdot (n-1)(n-2) \ldots (n-r+1) \times (n-r)!}{(n-r)!}$$

$${}^nP_r = \frac{n!}{(n-r)!}\ ;\ 0 \le r \le n$$

Properties : 1) $${}^nP_n = \frac{n!}{(n-n)!} = \frac{n!}{0!} = n!$$

2) $${}^nP_1 = \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n$$

$${}^nP_2 = \frac{n!}{(n-2)!} = \frac{n(n-1)[(n-2)!]}{(n-2)!} = n(n-1)$$

$${}^nP_3 = n(n-1)(n-2) \text{ and so on.}$$

3) $0! = 1$

Q.1 Given that A = {1, 2, 3, 4, 5, 6}, find the number of permutations of A taken

i) 2 at a time ii) 3 at a time iii) 4 at a time iv) 5 at a time v) 6 at a time.

Ans. : We have A = {1, 2, 3, 4, 5, 6}

$\therefore \qquad |A| = 6$

i) The permutation at 6 letters taken 2 at a time is $^{6}P_2 = \frac{6!}{(6-2)!} = \frac{6}{4}$

$$= \frac{6\times 5\times 4!}{4!} = 30$$

ii) The permutation of 6 letters taken 3 at a time is $^{6}P_3 = \frac{6!}{(6-3)!}$

$$= \frac{6\times 5\times 4\times 3!}{3!} = 120$$

Similarly,

iii) $$^{6}P_4 = \frac{6!}{(6-2)!} = \frac{6!}{4!} = 30$$

iv) $$^{6}P_5 = \frac{6!}{(6-5)!} = \frac{6!}{1!} = 6\times 5\times 4\times 3\times 2\times 1 = 720$$

v) $$^{6}P_6 = \frac{6!}{(6-6)!} = \frac{6!}{0!} = 6! = 720$$

Q.2 How many four digit numbers can be formed out of digits 1, 3, ...9. if i) No repetition is permitted.
ii) How many of these will be greater than 3000.

Ans. : i) The number of ways of selecting 4 digits out of 9 digits is

$$^{9}P_4 = \frac{9!}{(9-4)!} = \frac{9\times 8\times 7\times 6\times 5!}{5!} = 3024$$

ii) There is a restriction that the 4 digit numbers so formed must greater than 3000.

Therefore the thousandth position can be filled with numbers 3, 4, 5, 6, 8, 9 i.e. the thousandth place can be selected in 7 different ways.

Now out of remaining 8 digits, hundredth position can be filled in different ways, Tenth place can be filled in 7 different ways and u place can be filled in 6 different ways.

Thus the total number of 4 digit numbers greater than 3000 can formed in

$7\times 8\times 7\times 6 = 2352$ ways.

Q.3 i) Suppose repetitions are not permittable, how many four dig numbers can be formed from six digits 1, 2, 3, 5, 7, 8 ? ii) Ho many of such numbers are less than 4000 ? iii) How many in (i) ar

even ? iv) How many in (ii) are odd ?

v) How many in (i) contain both 3 and 5.

vi) How many in (i) are divisible by 10. **☞[SPPU : Dec.-05]**

Ans. : i) Out of 6 numbers, 4 digit numbers can be formed in ${}^6P_4$ ways.

$$\therefore \quad \text{Number of ways} = \frac{6!}{2!} = 360$$

ii) The four digit numbers which are less than 4000 are the numbers in which first digit is 1, 2 or 3 i.e. 1st digit can be chosen in 3 ways, 2nd digit can be any one of the remaining 5 digits. 3rd digit can be any of the remaining 4 digit and the 4th digit is any one of the remaining 3 digits.

Hence the total number of ways = $3 \times 5 \times 4 \times 3 = 180$.

iii) Those numbers ending in 2 or 8 are even numbers. Hence the last digit (4th digit) can be chosen in 2 ways (the number 2 or 8). The first digit can be chosen in any one of the remaining 5 digits, 2nd in any of the 4 digits and 3rd in any of the 3 digits. Hence the total number of ways

$$= (5 \times 4 \times 3) \times 2$$

TH	H	Ten	Unit place
5	4	3	2

= 120 ways

iv) The numbers less than 4000 and are odd. The numbers ending with 1, 3, 5 or 7 are odd. The 4 digit numbers ending in 1 and less than 4000 should begin with eithor 2 or 3.

Then there are $2 \times 4 \times 3 \times 1 = 24$ such numbers. Similarly the number ending in 3 are 24. However the number ending with 5 or 7 are

TH UnitPlace

$$\overset{1}{3} \times 4 \times 3 \times \overset{1}{2} = 72 \text{ ways.}$$

Hence the total number which are odd and less than 4000 are $24 + 24 + 72 = 120$

v) The digit 3 can occupy any of the 4 positions and the remaining 3 positions will be occupied by the digit 5. Hence the number of ways in which two positions are occupied by 3 and 5 will be 4×3 i.e. 12.

Now the remaining two positions will be filled by the remaining 4 numbers i.e. 1, 2, 7 and 8.

Hence out of remaining two positions one position can be occupi 4 different ways and the remaining position will be occupie 3 different ways.

Hence total number of 4 digit numbers in which both 3 and 5 are pr $= 12 \times 4 \times 3 = 144$.

vi) Not even a single number is divisible by 10 as there is no zer unit's place.

Q.4 A menu card in a restaurant displays four soups, five m courses, three desserts and 5 beverages. How many different me can a customer select if, i) He selects one item from each gro without omission.

ii) He chooses to omit the beverages, but selects one each from other groups.

iii) He choose to omit the desserts but decides to take a beverage a one item each from the remaining groups.

Ans. : i) The customer can select the soup in 4 ways, the main course 5 ways, the dessert in 3 ways and beverages in 5 ways.

Hence by product rule, the number of ways in which he can select item each, without omission is $4 \times 5 \times 3 \times 5 = 300$.

ii) The number of ways in which he omit beverages $= 4 \times 5 \times 3 =$ ways.

iii) The number of ways in which he omit desserts but he takes all oth items $= 4 \times 5 \times 5 = 100$ ways.

Q.5 10 different M_1 books, 3 different M_2 books, 5 different M books and 7 different D.S. books are to be arranged on a shelf. How many different arrangements are possible if

i) The books in each subject must all be together

ii) Only M_3 books must be together.

Ans. : i) M_1 books can be arranged among themselves in 10 ! ways, th M_2 books in 3 ! ways, M_3 books in 5 ! ways and D.S. books in 7 ways.

Hence the total number of arrangements $= 4! \cdot 10! \cdot 3! \cdot 5! \cdot 7!$

ii) Consider the 5 M_3 books as a single book. Then there are 21 books which can be arranged in 2 ! ways. In each of these arrangements the M_3 books can be arranged among themselves in 5! ways.

Hence the number of arrangements in $5! \cdot 2!$

Q.6 2 mathematics papers and 5 other papers are to be arranged at an examination. Find the total number of ways if,
i) Mathematics papers are consecutive. ii) Mathematics papers are not consecutive.

Ans. : i) Both mathematics papers (M_1 and M_2) are together, consider both M_1 and M_2 as single paper.

These two papers among themselves can be arranged in 2! ways.

Now 6 papers (as M_1 and M_2 is considered as single paper) can be arranged in 6! ways.

Hence total number of arrangements = $2! \cdot 6!$

ii) If M_1 or M_2 are not consecutive than they are to be arranged between the 4 gaps or at the 2 ends.

M_1 or M_2	$\square$	M_1 or M_2	$\square$	M_1 or M_2	$\square$	M_1 or M_2	$\square$	M_1 or M_2	$\square$	M_1 or M_2

Where $\square$ denotes other papers

Hence there are 6 places where mathematics papers can be arranged. Therefore, 2 mathematics papers can be arranged in 6 places in ${}^6P_2$ ways. Five other papers can be arranged among themselves in 5! ways.

Therefore total number of arrangements

$$= 5! \cdot {}^6P_2 = 5! \cdot 6 \cdot 5 = (120) \cdot (30) = 3600$$

Q.7 How many permutations can be made out of the letter of word "COMPUTER" ? How many of these i) begin with C ii) end with R iii) begin with C and end with R iv) C and R occupy the end places

Ans. : There are 8 letters in the word "COMPUTER" and all are distinct

∴ The total number of permutations of these letters is 8! = 40320.

i) Permutations begin with C.

The first position can be filled in only one way i.e. C and the remaining 7 letters can be arranged in 7! ways.

∴ The total no. of permutations begining with C = $1 \times 7! = 5040$

ii) Permutations end with R :

The Last position can be filled in only one way and the remaining 7 letters can arranged in 7! ways.

∴ The total no. of permutations ending with R be = $7! \times 1 = 5040$

iii) Permutation begin with C and end with R :

The first position can be filled in only one way i.e. C and the end position also can be filled in only one way i.e. R and the remaining 6 letters can be arranged in 6! ways.

$\therefore$ The required no. of permutations $= 1 \times 6! \times 1 = 720$

iv) Permutation in which C and R occupy end places :

C and R occupy end positions in 2 ! ways i.e. CR or RC and the remaining 6 letters can be arranged in 6! ways.

$\therefore$ The total no. of required permutations $= 2! \times 6! = 1440$

A) Permutations with restrictions :

1) The number of permutations of n different objects taken r at a time in which p particular objects do not occur is ${}^{(n-p)}P_r$.

2) The number of permutations of n different objects taken r at a time in which p particular objects are present is ${}^{(n-p)}P_{r-p} \times {}^{r}P_p$.

Q.8 Show that the number of injective functions from a set with r elements to a set with n elements is ${}^{n}P_r$; $r \leq n$.

Ans. : Let A and B be two sets with $|A| = r$ and $|B| = n$.

$A = \{a_1, a_2, \ldots a_r\}$ and $B = \{b_1, b_2, b_3 \ldots b_r, \ldots b_n\}$

Let $f : A \to B$ be an injective function

Hence by product rule, the number of injective functions from A to B is $n(n-1)\ (n-2) \ldots (n-r+1) = {}^{n}P_r$

Q.9 Find the number of permutations that can be made out of the letters i) MISSISSIPPI ii) ASSASSINATION. **[SPPU : Dec.-14]**

Ans. : i) There are 11 letters in the word out of which S, I, P, M are distinct.

S appears 4 times, I appears 4 times

P appears 2 times, M appears 1 time

$\therefore$ The required no. of permutations $= \dfrac{11\,!}{4!\ 4!\ 2!\ 1!} = 34650$

ii) There are 13 letters of which A, I, N, S, T and O are different.

A appears 3 times, I appears 2 times N appears 2 times,

S appears 4 times T appears 1 time, O appears 1 time

$\therefore$ The required no. of permutations $= \dfrac{13\,!}{3!\ 2!\ 2!\ 4!\ 1!\ 1!} = 10810800$

Q.10 How many ways can the letters in the word "PIONEER" be arranged so that the two E's are always together.

Ans. : The word 'PIONEER' has two E's and remaining 5 letters are distinct. These distinct five letters can be arranged in 5 ways and for each such arrangement two E's can occupy any of the six remaining places. Hence the required no. of permutations are

$$6 \times 5! = 6! = 720.$$

Q.11 How many seven digit numbers can be formed using digits 1, 7, 2, 7, 6, 7, 6 ?

Ans. : There are 7 digits out of which 7 is repeated 3 times, 6 is repeated twice, appears 1 time and 2 appear 1 time.

$\therefore$ The total no. of permutation $= \dfrac{7\,!}{3! \times 2! \times 1! \times 1!} = 420$

Q.12 How many 4 digits numbers can be formed by using the digits 2, 4, 6, 8 when repetition of digits are allowed ?

Ans. : We have 4 digits numbers.

No. of ways of filling unit's place = 4

No. of ways of filling ten's place = 4

No. of ways of filling hundred's place = 4

No. of ways of filling thousand's place = 4

Therefore the total number of 4 digits numbers $= 4 \times 4 \times 4 \times 4 = 4^4 = 256$

Q.13 How many 4 digits even numbers can be formed by using the digits 1, 3, 4, 6, 8 when repetition of digits are allowed ?

Ans. : We have 3 even numbers and 2 odd numbers. Therefore,

The no. of ways of filling unit's place = 3

The no. of ways of filling ten's place = 5

The no. of ways of filling hundred's place = 5

The no. of ways of filling thousand's place = 5

Thus, the total no. of required 4 digits numbers = $3 \times 5^3 = 3 \times 125 = 375$

Q.14 In how many ways can 5 software projects be alloted to 6 final year students when all of five projects are not alloted to the same student.

Ans. : We have 5 projects and 6 students. Each project can be alloted in 6 ways.

Thus, the number of ways of alloting 5 projects = $6 \times 6 \times 6 \times 6 \times 6 = 6^5$

The number of ways in which all 5 projects are allotted to same student = 6.

Therefore, total number of ways to allocate 5 projects to 6 students = $6^5 - 6 = 7770$

Q.15 A bit is either 0 or 1. A byte is a sequence of 8 bits. Find i) Number of bytes ii) Number of bytes that begin with 11 and end with 11.

Ans. : i) Total number of byte is $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 2 = 2^8 = 256$.

ii) As the first two and last two bits are fixed i.e. 11 the remaining bits in the sequence are either 0 or 1.

$\therefore$ The required no. of total bytes = $2^4 = 16$.

Q.16 Prove that the number of circular permutations of n different objects is (n – 1) !

Ans. : Let us consider that k be the number of required permutations.

For each such circular permutation of k, there are n corresponding linear permutations. We can start from every object of n objects in the circular permutation. Thus for k circular permutations, we have kn linear permutations.

Therefore $k \cdot n = n! \Rightarrow k = \frac{n!}{n} = (n-1)!$ Hence the proof.

Q.17 How many ways can these letters A, B, C, D, E and F be arranged in a circle ?

Ans. : There are six letters. Hence the no. of ways to arrange these six letters in a circle is $(6-1)! = 5! = 120$.

Q.18 In how many waye 10 programmers can s.t. on a round table to discuss the project. So that project leader and a particular programmer always sit together ?

Ans. : There are 10 programmers. But project leader and particular programmer always sit together. So both become a single unit and hence there are

$(10 - 2 + 1) = 9$ remins

Thus these 9 units can be arreanged on round table in $(9-1)! = 8!$ ways.

The two programmer i.e. project leader and particular programmer can be arranged in 2! ways.

Therefore the required no. of ways $= 2! \times 8! = 80640$.

Q.19 Determine the number of ways in which 5 software engineers and 6 electronics engineers can be sitted at a round table so that no two software engineers can sit together.

Ans. : There are 6 electronics engineers that can be arranged round a table in $(6-1)!$ ways. There are 5 software engineers and they are not to sit together, so there are six places for software engineers and can be placed in 6! ways as shown in Fig. Q.19.1

Therefore the required no. of ways

$= (6-1)! \times 6! = 5! \times 6! = 120 \times 720 = 86400$

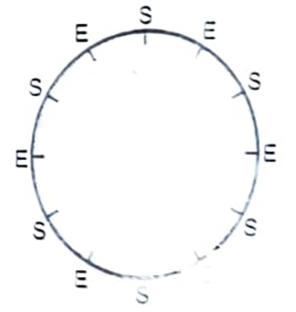


Fig. Q.19.1

2.2 : Combinations

Important Points to Remember

A combination is a selection of some or all objects from a set of given objects where the order of the objects does not matter. In this context, we used mainly two words "Selection" and "arrangement". In selection, order of objects is immaterial i.e. selection is a set. But in arrangement, the order of objects is important it is not a set. Arrangement is a n-tupple. Arrangement is associated with permutation selection with combination.

I) Definition : The number of combinations of n different objects taken r at a time is given by ${}^{n}C_{r}$ and defined as

$$^{n}C_{r} = \frac{n!}{r!\,(n-r)!} \quad ; \; r \leq n$$

Properties :

1) $$^{n}C_{r} = \frac{n!}{r!(n-r)!} = \frac{^{n}P_{r}}{r!}$$

2) $$^{n}C_{n} = \frac{n!}{n!\,(n-n)!} = 1$$

3) $$^{n}C_{0} = \frac{n!}{0!\,(n-0)!} = \frac{n!}{n!} = 1$$

4) $$^{n}C_{1} = \frac{n!}{1!\,(n-1)!} = n \;,\; ^{n}C_{2} = \frac{n!}{2!\,(n-2)!} = \frac{n\,(n-1)}{2}$$

$$^{n}C_{2} = \frac{n\,(n-1)\,(n-2)}{3!} \;,\; ^{n}C_{4} = \frac{n\,(n-1)\,(n-2)\,(n-3)}{4!}$$

Q.20 Find the value of n if i) ${}^{n}C_{n-2} = 10$ and ii) ${}^{25}C_{n+2} = {}^{25}C_{2n-1}$

Ans. : i) We have $^{n}C_{n-2} = 10 \Rightarrow \dfrac{n!}{(n-2)!\,(n-(n-2))!} = 10$

$$\therefore \quad \frac{n\,(n-1)\,(n-2)!}{(n-2)!\,(2!)} = 10$$

$$n\,(n-1) = 10 \times 2 = 20$$

$$n^2 - n - 20 = 0$$

$$\Rightarrow \quad (n-5)\,(n+4) = 0$$

$n = 5$ or $n = -4$ as $n = -4$ is not possible

$$\boxed{n = 5}$$

ii) we know that $^{n}C_{r} = {}^{n}C_{n-2}$

Now $^{25}C_{n-2} = {}^{25}C_{n-2}$

$\Rightarrow$ Either $n + 2 = 2n - 1$ or $(n + 2) + (2n - 1) = 25$

$\therefore$ either $\quad n = 3$ or $3n = 25 - 1 = 24$

$\therefore \quad n = 3 \quad$ or $\quad n = 8$

$\therefore \quad n = 3, 8.$

Q.21 How many 16 bit strings are there containing exactly five 0's ?

Ans. : Each string of 16 bit has 16 digits. A 16-bit string having exactly five 0's is determined if we tell which bit are 0's. So, here order is immaterial.

$\therefore$ This can be done in ${}^{16}C_5$ ways.

$\therefore$ The total number of 16-bit strings is

$$ {}^{16}C_5 = \frac{16!}{5! \times (16-5)!} = \frac{16 \times 15 \times 14 \times 13 \times 12 \times 11!}{5 \times 4 \times 3 \times 2 \times 1 \times 11!} = 4368 $$

Q.22 In how many ways can 30 late admitted students be assigned to three practical batches A, B, C if A can accomodate 10 students, B - 15 students and C - 5 students only ?

Ans. : The batch A can accomodate 10 students out of 30.

$\therefore$ The batch A can be assigned 10 students in ${}^{30}C_{10}$ ways.

then batch B can be assigned 15 students in ${}^{20}C_{15}$ ways.

then batch C can be assigned 5 students in ${}^{5}C_5$ ways.

Therefore by the product rule, the total no of ways of assigned students is

$$ {}^{30}c_{10} \times {}^{20}c_{15} \times {}^{5}c_5 = \frac{30!}{10!\,(20!)} \times \frac{20!}{15! \times 5!} \times 1 = \frac{30!}{10!\,15!\,5!} $$

Q.23 How many ways can we select a software development group of 1 project leader, 15 programmers and 6 data entry operators from a group of 5 project leaders 20 programmers and 25 data entry operators.

Ans. : One project leader can be selected from 5 project leaders in ${}^{5}C_1 = 5$ ways.

15 programmers can be selected from 20 programmers in ${}^{20}C_{15}$ ways.

6 data entry operators can be selected from 25 data entry operators in ${}^{25}C_6$ ways.

Therefore the total number of ways to select the software development group is

$^5C_1 \times {}^{20}C_{15} \times {}^{25}C_4 = 96101544000.$

Q.24 From 10 programmers in how many ways can 5 be selected when i) A particular programmer is included everytime
ii) A particular programmer is not included at all time.

Ans. : The number of ways to select 5 programmers from 10 is $^{10}C_5 = \frac{10!}{5!\,5!} = \frac{10\times 9\times 8\times 7\times 6}{5\times 4\times 3\times 2\times 1} = 252$

i) When a particular programmer is included every time then the remaining $5-1=4$ programmers can be selected from the remaining $10-1=9$ programmers. This can be done in $^9C_4$

$$\therefore \quad {}^9C_4 = \frac{9!}{4!\,5!} = \frac{9\times 8\times 7\times 5}{4\times 3\times 2} = 126$$

ii) When a particular programmer is not included at all then the 5 programmers can be selected from the remaining $10-1$ $=9$ programmers.

This can be done in $^9C_5$ ways.

$$\therefore \quad {}^9C_5 = \frac{9!}{4!\,5!} = 126$$

Q.25 A committee of 5 people is to be formed from a group of 4 men and 7 women. How many possible committees can be formed if at least 3 women are on the committee ? **[SPPU : May-14]**

Ans. : If at least three women are on commitee, it means commitee with 3 women or 4 women or 5 women.

i) 3 women can be selected in $^7C_3$ ways.

2 men can be selected in $^4C_2$ ways.

The no. of ways this can be done is $^7C_3 \times {}^4C_2 = 210$ ways.

ii) 4 women and 1 man can be selected in $^7C_4$ and $^4C_1$ ways respectively.

$\therefore$ The number of ways to form a commitee is $^7c_4 \times {}^4c_1 = 140$ ways.

iii) 5 women can be selected in $^7C_5$ ways.

$\therefore$ The no. of ways to form a committee is $^7C_5 = 21$ ways

Hence the total no. of ways a committee can be formed with at least 3 women is $210 + 140 + 21 = 371$.

Q.26 How many automobile license plates can be made if each plate contains two different letters followed by three different digits. Solve the problem if the first digit can not be zero. [SPPU : May-06]

Ans. : The first position is a letter and can be selected from 26 letters in

$$^{26}C_1 = 26 \text{ ways.}$$

The second position is a letter and can be selected from 26 − 1 = 25 letters in

$$^{25}C_1 = 25 \text{ ways.}$$

For digits :

1) The first digit can be selected from 10 digits in $^{10}C_1 = 10$ ways.

2) The second digit can be selected from 9 digits in $^9C_1 = 9$ ways

3) The third digit can be selected from 8 digits in $^8C_1 = 8$ ways.

Therefore the total number of license plates

$$= 26 \times 25 \times 10 \times 9 \times 8 = 468000$$

Now, in license plate, the first digit can not be 0 then the first position can selected from 9 digits in $^9C_1$ ways.

The second digit can be zero, but one digit is already selected for the first position. Hence the second digit can be selected in $^9C_1$ ways. The third digit can be selected in $^8C_1$ ways.

Hence the total no. of required license plates are

$26 \times 25 \times 9 \times 9 \times 8 = 421200$.

Q.27 In the discrete structure paper, there are 10 questions. In how many ways can an examiner select five questions in all the first question is compulsory.

Ans. : The first question is compulsory, so the examiner has to select 4 questions from the remaining 9 questions.

$\therefore$ The number of ways to select five questions is

$$1 \times {}^9C_4 = \frac{9!}{4!\,5!} = \frac{9 \times 8 \times 7 \times 6}{4 \times 3 \times 2} = 126 \text{ ways.}$$

Q.28 Determine the number of triangles that are formed by selecting points from a set of 12 points out of which 5 are colinear.

Ans. : By using 12 points, the number of triangles formed is ${}^{12}C_3$.

As five points are colinear i.e. lie on same line, they do not form any triangle. Thus ${}^5C_3$ triangles are lost.

$\therefore$ The total number of triangles produced is

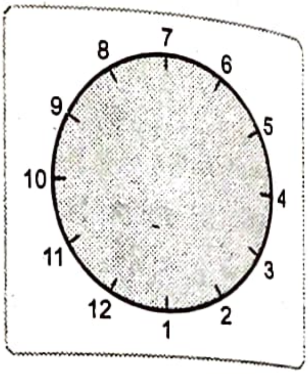
$${}^{12}C_3 - {}^5C_3 = \frac{12!}{3! \times 9!} - \frac{5!}{3!\ 2!} = \frac{12 \times 11 \times 10}{3 \times 2} - \frac{5 \times 4}{2} = 220 - 10 = 210$$

Q.29 How many lines can be drawn through 12 points on a circle and line passes through exactly two points ?

Ans. : As all points on the circle are not colinear, thus no line will lost.

The total no. of lines drawn through 12 points is

$${}^{12}C_2 = \frac{12\,!}{2!\ 10!} = \frac{12 \times 11}{2 \times 1} = 66$$



Q.30 Determine the number of diagonals that can be drawn by joining the nodes of octgon.

Ans. : The number of lines that can be drawn by 2 points out of 8 points of octagon is ${}^8C_2 = 28$. Out of these 28 lines, 8 are the sides of the octagon.

$\therefore$ The number of diagonals $= 28 - 8 = 20$.

Q.31 In a box, there are 40 floppy disks of which 4 are defective. Determine

i) In how many ways we can select five floppy disks ?

ii) In how many ways we can select five non defective floppy disks ?

iii) In how many ways we can select five floppy disks containing exactly three defective disks ?

iv) In how many ways we can select five floppy disks containing at least 1 defective disk ?

Ans. : i) There are 40 floppy disks out of which we have to select 5 floppy disks in ${}^{40}C_5$ ways.

$${}^{40}C_5 = \frac{40\,!}{5!\times(40-5)\,!} = \frac{40\times 39\times 38\times 37\times 36}{5\times 4\times 3\times 2\times 1} = 658008$$

ii) There are 40 – 4 = 36 nondefective floppy disks out of which we have to select 5. This can be done in ${}^{36}C_5$ ways.

$$\therefore \quad {}^{36}C_5 = \frac{36\,!}{5!\times(31)\,!} = \frac{36\times 35\times 34\times 33\times 32}{5\times 4\times 3\times 2\times 1} = 376992$$

iii) To select exactly three defective floppy disks out of 4 disks, we have ${}^{4}C_3$ ways and the remaining two floppy disks can be selected from 36 disks in ${}^{36}C_2$ ways.

Therefore, the required no. of ways = ${}^{4}C_3 \times {}^{36}C_2$

$$= \frac{4!}{3!\times 1!}\times\frac{36!}{2!\ 34!} = 4\times\frac{36\times 35}{2} = 2520$$

iv) There are 4 defective floppy disks out of which at least one must be selected. We know that the total number of ways to select 5 disks from 40 disks is ${}^{40}C_5$.

Also the number of ways to select 5 floppy disks with no defective is ${}^{36}C_5$ way.

Therefore the required no. of ways

$$= {}^{40}C_5 - {}^{36}C_5 = 658008 - 376992 = 281016$$

Q.32 How many 4 combinations of {1, 2, 3, 4, 5, 6} are there with unlimited repetition ?

Ans. : We have r = 4, n = 6

$\therefore$ The number of 4 combinations of {1, 2, 3, 4, 5, 6} are

$$C\,(6+4-1,\,4) = C\,(9,\,4) = \frac{9!}{4!\ 5!} = \frac{9\times 8\times 7\times 6\times 5!}{4\times 3\times 2\times 1\times 5!} = 126.$$

Q.33 Find the number of 3-combinations of $\{\infty\cdot a_1, \infty\cdot a_2, \infty\cdot a_3, \infty\cdot a_4\}$.

Ans. : We have n = 4, r = 3

$\therefore$ The number of 3-combinations of the given set is

$$C\,(4+3-1,\,3) = C\,(6,\,3) = \frac{6!}{3!\ 3!} = \frac{6\times 5\times 4\times 3!}{3\times 2\times 3!} = 20$$

Q.34 The number of non negative integer solutions to $x_1 + x_2 + x_3 + x_4 = 20$

Ans. : we have r = 20, n = 4

∴ The number of non negative integer solutions

$$= C(4 + 20 - 1, 20) = C(23, 20) = \frac{23!}{20! \times 3!} = \frac{23 \times 22 \times 21}{3 \times 2} = 1771$$

Q.35 The number of ways of placing 8 similar balls in 5 numbere 'boxes.

Ans. : The number of ways of placing r = 8 similar balls in n = 5 box is

$C(5 + 8 - 1, 8) = c(12, 8) = 495.$

Q.36 Find the number of binary numbers with six 1's and 4 zero's.

Ans. : The number of binary numbers with 6 one's and 4

$$\text{zero's} = C(6 + 4, 4) = C(10, 4) = \frac{10 \times 9 \times 8 \times 7 \times 6!}{4 \times 3 \times 2 \times 1 \times 6!} = 210$$

Q.37 How many integral solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 16$ where each $x_i \geq 2$?

Ans. : Let $x_i = y_i + 2$ where $y_i \geq 0$.

we have $x_1 + x_2 + x_3 + x_4 + x_5 = 16$

iff $y_i + 2 + y_2 + 2 + y_3 + 2 + y_4 + 2 + y_5 + 2 = 16$

Iff $y_1 + y_2 + y_3 + y_4 + y_5 = 16 - 10 = 6$

Thus the number of integral solutions of given equation is the same as the number of integral solutions of $y_1 + y_2 + y_3 + y_4 + y_5 = 6$

There are C (5 – 1 + 6, 6) = C (10, 6) such solutions

$$\therefore \quad C(10, 6) = \frac{10!}{6! \times 4!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{4 \times 3 \times 2 \times 1 \times 6!} = 210$$

Q.38 How many integral solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 30$ where $x_1 \geq 2, x_2 \geq 3, x_3 \geq 4, x_4 \geq 2, x_5 \geq 0$.

Ans. : Let $x_i = y_1 + 2,\ x_2 = y_2 + 3,\ x_3 = y_3 + 4,\ x_4 = y_4 + 2$
$x_5 = y_5 + 0$

$$\therefore \quad x_1 + x_2 + x_3 + x_4 + x_5 = 30$$

$$\Rightarrow \quad y_1 + 2 + y_2 + 3 + y_3 + 4 + y_4 + 2 + y_5 = 30$$

$$y_1 + y_2 + y_3 + y_4 + y_5 = 19$$

$\therefore$ The required number of integral solutions are

$$C(5-1+19, 19) = C(23, 19) = \frac{23 \times 22 \times 21 \times 20}{4 \times 3 \times 2 \times 1} = 8855$$

Theorem 1 : The number of integer solutions to $a_1 + a_2 + a_3 + \ldots + a_n = r$ when $a_1 \geq b_1, a_2 \geq b_2, a_3 \geq b_3, \ldots a_n \geq b_n$ is $C(n + r - 1 - b_1 - b_2 - b_3 - \ldots b_n, r - b_1 - b_2 - b_3 - \ldots - b_n)$

Theorem 2 : The number of ways to select r things from n categories with b total restrictions on the r things is $C(n + r - 1 - b, r - b)$

Theorem 3 : The number of ways to select r things from n categories with atleast 1 thing from each category is $C(r - 1, r - n)$ $(\because b = n)$

2.3 : Generation of Permutations and Combinations

Important Points to Remember

I) Generation of permutations : Suppose we want to generate n! permutations of n distinct objects. For n = 1, 2, 3, it is simple but when n is large it is difficult to keep track of what we have written and make sure that we shall write down all permutations with no repetition or omissions.

An interesting problem is to find a systematic procedure for generating all n! permutations of a set with n distinct elements.

Suppose from the initial permutation 1, 2, 3, ... n by using the next permutation procedure repeatedly we shall obtain all the permutations of 1, 2, 3, ..., n. The last permutation is n, (n – 1), (n – 2), ... 4, 3, 2, 1.

Procedure for next permutation :

Step 1 : Given a permutation $a_1, a_2, a_3, \ldots a_n$ of 1, 2, 3, ..., n

Step 2 : Scan from right to left $(L \leftarrow R)$. Find the first m such that $a_m < a_{m+1}$

Step 3 : $\alpha = \min \{a_k \mid k = m+1, m+2, m+3, \ldots n, a_k > a_m\}$

Step 4 : The next permutation is

$a_1\ a_2\ a_3 \ldots a_{m-1}, \alpha, x, x, x \ldots$

where x, x, x ... are the remaining numbers arranged in the increasing order.

Algorithms : I) Algorithm for generating the next Permutation in Lexicographic order :

Next permutation $[(a_1, a_2, \ldots, a_n)]$: Permutation of {1, 2, 3, ...n} not equal to (n, (n – 1) (n – 2) ... 3 2 1)]

$i := n - 1$

while $a_i > a_k$

$k := k - 1$ {a_k is the smallest integer greater than a_i to the right of a_i}

interchange a_i and a_k

$r := n$

$s := j + 1$

while $r > s$

begin

interchange a_r and a_s

$r := r - 1$

$s = s + 1$

end

{This puts the tail end of permutation after the i^{th} position in increasing order}

Q.39 Let n = 6 and given permutation is 125364 find the next permutation.

Ans. : Step 1 : Given permutation is 125364

Step 2 : Scan from right to left and find the first m such that.

$a_m < a_{m+1}$

Now start with 5^{th} position ; As $6 > 4$, $m \neq 5$

As $3 < 6$ and position of 3 is 4^{th} $\therefore m = 4$.

Step 3 : $\alpha = \min \{a_k | a_k > a_m\} = \min \{6, 4\} = 4$

Step 4 : Replace element a_m by α i.e. 3 by 4, keep previous elements as its i.e. 125 and write all remaining elements in increasing order i.e. 36.

$\therefore$ The next permutation is 125436.

Q.40 Find the next two permutations of 125436

Ans. : i) Given that 1254 $\underset{\uparrow}{3}$ 6

$\therefore$ $m = 5$ and $\alpha = \min\{6\} = 6$

$\therefore$ The next permutation is 125463.

ii) Find the next permutation of 125 $\underset{\uparrow}{4}$ 63

$m = 4, \alpha = \min\{6\} = 6$

$\therefore$ The next permutation is 125634.

Q.41 Generate all permutations for n = 3 by next permutation method.

Ans. : Consider the following table to generate all permutations for n = 3.

Sr. No.	Given permutation	Next permutation
1)	1 $\underset{\uparrow}{2}$ 3 $m = 2, \alpha = 3$	1 3 2
2)	$\underset{\uparrow}{1}$ 3 2 $m = 1, \alpha = 2$	2 1 3
3)	2 $\underset{\uparrow}{1}$ 3 $m = 2, \alpha = 3$	2 3 1
4)	$\underset{\uparrow}{2}$ 3 1 $m = 1, \alpha = 3$	3 1 2
5)	3 $\underset{\uparrow}{1}$ 2 $m = 2, \alpha = 2$	3 2 1 Last permutation.

$\therefore$ The set of all permutations for n = 3 is

{(1 2 3) (1 3 2) (2 1 3) (2 3 1) (3 1 2) (3 2 1)}

Procedure to generate subsets of {1, 2, 3, ... n}

Let $\{a_1, a_2, a_3, \ldots a_k\}$ be a subset of size k of $\{1, 2, 3, \ldots n\}$ with $a_1 < a_2 < a_3 < \ldots < a_k$.

The maximum possible value of a_k is n

The maximum possible value of $a_{k-1} = n - 1$

In general the maximum possible value of a_i is $n - k + i$. Consider the subset $\{1, 2, 3, \ldots k - 1, k\}$ If $k \neq n$, its maximum value then increase k by 1, so that the next subset $\{1, 2, 3, \ldots k - 1, k + 1\}$ is generated. We continue this procedure till we reached to $\{1, 2, 3 \ldots (k - 1), n\}$. Now repeat the procedure for $k - 1$, if $k - 1 \neq n - 1$ then increase it by 1 and continue this process with $k - 1$ till we reached to $\{1, 2, 3, \ldots (k - 2) (n - 1), n\}$ Then move to $(k - 2)$ and repeat the same process. In this manner, moving from right to left we finally reach to an element a_j such that a_j can be increased to a_{j+1} but no a_j with $i > j$ can be increased which means that at some stage a_i is equal to its maximum value $n-k+i$. This procedure terminates when a_1 reaches to its maximum value.

II) Algorithm for generating the next r-combination in lexicographic order.

Next r-combination $(\{a_1, a_2, a_3 \ldots a_r\})$: proper subset of $\{1, 2, 3, \ldots n\}$ not equal to $\{n - r + 1, n - r + 2, \ldots n\}$: with $a_1 < a_2 < a_3 < \ldots a_r)$

$i ; = r$

while $a_i = n - r + i$

$i : = i - 1$

$a_i = a_2 + 1$

for $j = i + 1$ to r

$a_j = a_i + j - i$

Q.42 Generate all subsets of size 4 of {1, 2, 3, 4, 5, 6}.

Ans. : Let us begin with $\{1, 2, 3, 4\}$. We know that for any subset $\{a_1, a_2, a_3, a_4\}$ with $a_1 < a_2 < a_3 < a_4$ the maximum possible value of a_4 is 6, a_3 is 5, a_2 is 4 and a_1 is 3.

i) For $a_4 = 4$:

Hence increasing 4 by 1 we obtain a subset $\{1, 2, 3, 5\}$. Since a_4 has not still reached to 6.

$\therefore$ Again increase 5 by 1, we get $\{1, 2, 3, 6\}$.

ii) For $a_3 = 3$:

The maximum value of a_3 is 5.

∴ Increase a_3 successively by 1 still, we reach to 5.

∴ {1, 2, 4, 5} , {1, 2, 4 6} , {1, 2, 5, 6}

iii) For $a_2 = 2$: The maximum value of a_2 is 4

∴ We get (1, 3, 4, 5}, {1, 3, 4, 6}, {1, 3, 5, 6}, {1, 3, 4, 6}

iv) For $a_1 = 1$: Maximum value of a_1 is 3.

∴ We get {2, 3, 4, 5}, {2, 3, 4, 6}, {2, 3, 5, 6}, {2, 4, 5, 6},
{3, 4, 5, 6}

Thus we obtain the following 15 subsets

{ {1, 2, 3, 4} {1, 2, 3, 5} {1, 2, 3, 6}
{1, 2, 4, 5} {1, 2, 4, 6} {1, 2, 5, 6}
{1, 3, 4, 5} {1, 3, 4, 6}, {1, 3, 5, 6} {1, 4, 5, 6}
{2, 3, 4, 5}, {2, 3, 4, 6}, {2, 3, 5, 6} {2, 4, 5, 6} {3, 4, 5, 6} }

2.4 : Discrete Probability

Important definitions

I) **Trial :** The performance of an experiment is called a trial.

II) **Experiment :** It is a well defined process that leads to a well defined outcome.

e.g. Tossing a coin is an experiment. It may consist of one or more trials.

III) **Sample Space :** The set of all possible outcomes of an experiment is called sample of that experiment. It is denoted by S.

IV) **Event :** Any subset of a sample space is called an event.

e.g. i) If $S = \{1, 2, 3, 4, 5, 6\}$ then $E_1 = \{1\}$, $E_2 = \{1, 3, 5\}$

$E_3 = \{2, 4, 6\}$, $E_5 = \phi$ $E_6 = S$ are events of S.

V) **Complement of Event :** The set of all outcomes which are in sample space but not in an event is called the complement of the event. The complement of an event E is denoted by $\bar{E}$

e.g. In above example

$\bar{E}_1 = \{2, 3, 4, 5, 6\}$, $\bar{E}_2 = \{2, 4, 6\}$

$\bar{E}_3 = \{1, 3, 5\}$, $\bar{E}_6 = \phi$ are complement events

VI) **Equally Likely Event :** Events are said to be equally likely if one of them can not be expected to occur in preference to others.

e.g. If a coin is tossed then outcomes H and K are equally likely.

VII) **Mutually Exclusive or Disjoint Events :** Two events are said to mutually exclusive or incompatible when both can not happ simultaneously in a single trial or in other words the occurrence anyone of them precludes the occurrence of the other. e.g. If a coin tossed then it will be either head or tail both cannot be up at the sa time.

VIII) **Exhaustive Events :** When a sample space is, distributed down i some, mutually exclusive events such that their union forms the sam space it self, then such events are called exhaustive event.

IX) **Independent Events :** Two events A and B are said to independent events if the occurrence of or non occurrence of one does affect the occurrence or non occurrence of the other.

e.g. A coin is tossed trice, outcome of the second throw is independent the outcome of the first throw. Events which are not independent, called dependent events.

X) **Favourable Events or Cases :** The number of cases favourable to event in a trial is the number of outcomes entail the happening of event.

XI) **Discrete Sample Space :** A sample space which has finite countably infinite number of sample points is called a discrete samp space.

e.g. $S_1 = \{H, T\}, S_2 = \{1, 2, 3, 4, 5, 6\}, S_3 = \{1, 2, 3, ...\}$

are discrete sample space.

XII) **Equiprobable Sample Space :** A sample space is said to equiprobable sample if every element of a sample space has an equ chance of occurrence.

2.5 : Probability

Let S be an equiprobable sample space and A be any event of S. Th probability of happening of A is denoted by P(A) and defined as

$$P(A) = \frac{\text{Number of favourable outcomes}}{\text{Total number of outcomes}}$$

$$= \frac{\text{Number of elements in A}}{\text{Total number of elements in S}} = \frac{n(A)}{n(S)}$$

The probability of non happing of A is $P(\overline{A}) = \frac{n(\overline{A})}{n(S)}$

Note :

1) $0 \le P(A) \le 1$

2) $P(A) + P(\overline{A}) = 1$

3) Two events A and B are said to be independent if $P(A \cap B) = P(A) \cdot P(B)$

Before solving examples, consider the following basic information.

A)

Sr. No.	Statement Corresponding to Event	Set Notation of Event
1)	Event a does not occur	$\overline{A}$
2)	At least one of A or B occur	$A \cup B$
3)	Both events A and B occur	$A \cap B$
4)	Neither event A nor B occur	$\overline{A} \cap \overline{B} = \overline{(A \cup B)}$
5)	Event A occurs or B occurs but not both	$(A \cap \overline{B}) \cup (\overline{A} \cap B)$ $= A \oplus B$
6)	Not more than one event occurs	$\overline{(A \cup B)} = (\overline{A} \cup \overline{B})$
7)	A and B are mutually exclusively	$A \cap B = \phi$

B)

Sr.No.	Experiment	Sample Space
1)	Tossing two coins	{HH, HT, TH, TT}
2)	Tossing 3 coins	{HHH, HHT, HTH, HTT, THT, THH, TTH, TTT}
3)	Die is rolled	{1, 2, 3, 4, 5, 6}

4)	Two dice are rolled	{(1, 1) (1, 2) (1, 3) (1, 4) (1, 5) (1, 6) (2, 1) (2, 2) (2, 3) (2, 4) (2, 5) (2, 6) (3, 1) (3, 2) (3, 3) (3, 4) (3, 5) (3, 6) (4, 1) (4, 2) (4, 3) (4, 4) (4, 5) (4, 6) (5, 1) (5, 2) (5, 3) (5, 4) (5, 5) (5, 6) (6, 1) (6, 2) (6, 3) (6, 4) (6, 5) (6, 6)}

C) Theorems on Probability :

a) Addition theorem

1) If A and B are mutually exclusive events then Prob (A or B)

i.e. $P(A \cup B) = P(A) + P(B)$

If A, B, C are mutually exclusively then

$P(A \cup B \cup C) = P(A) + P(B) + P(C)$

2) When the events are not mutually exclusively then the probability that atleast one of the two events A and B will occur is

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

i.e. $P(A \text{ or } B) = P(A) + P(B) - P(A \,\&\, B)$

In case of three events

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(A \cap C) - P(B \cap C) + P(A \cap B \cap C)$$

b) Multiplication theorem : If A and B are two independent events then the probability that both will occur is equal to the product of the individual probabilities.

$$P(A \,\&\, B) = P(A) \times P(B) \quad \text{i.e.} \quad P(A \cap B) = P(A) \times P(B)$$

Similarly $P(A, B \,\&\, C) = P(A) \times P(B) \times P(C)$

Q.43 What is the probability that a leap year will contain 53 Mondays ?

Ans. : A leap year has 366 days.

This contains complete 52 weeks and two more days. These two days may take following combinations.

(i) Monday - Tuesday (ii) Tuesday - Wednesday (iii) Wednesday - Thursday (iv) Thursday - Friday (v) Friday - Saturday (vi) Saturday - Sunday and (vii) Sunday - Monday.

Out of these 7 combinations only 2 contain Monday.

$\therefore$ Required probability $= \frac{2}{7}$

Q.44 Prof. X and Madam Y appear for an interview for two posts. The probability of Prof. X's selection is $\frac{1}{7}$ and that of Madam Y's selection is $\frac{1}{5}$. Find the probability that only one of them is selected. What is probability that at least one of them is selected.

Ans. : $P(X) = \frac{1}{7}$ $\qquad$ $P(Y) = \frac{1}{5}$

$P(\overline{X}) = 1 - \frac{1}{7} = \frac{6}{7}$ $\qquad$ $P(\overline{Y}) = 1 - \frac{1}{5} = \frac{4}{5}$

i) As only one of this is selected, so there are two possibilities.

Case A : If X is selected & Y is not selected.

Case B : If Y is selected & X is not selected.

$$\therefore \quad P(A) = P(X) \times P(\overline{Y}) = \frac{1}{7} \times \frac{4}{5} = \frac{4}{35}$$

$$P(B) = P(\overline{Y}) \times P(\overline{Y}) = \frac{6}{7} \times \frac{1}{5} = \frac{6}{35}$$

$\therefore$ Required probability

$$P(A \text{ or } B) = P(A) + P(B) = \frac{4}{35} \times \frac{6}{35} = \frac{10}{35} = \frac{2}{7}$$

ii) Probability that none is selected $= P(\overline{X}) \times P(\overline{Y}) = \frac{6}{7} \times \frac{4}{5} = \frac{24}{35}$

$$\therefore \quad P(\text{at least one is selected}) = 1 - \frac{24}{35} = \frac{11}{35}$$

Q.45 A can hit the target 1 out of 4 times. B can hit the target 2 of 3 times. C can hit the target 2 out of 3 times. Find the probab that at least two hit the target.

Ans. : $P(A) = \frac{1}{4}$ $\qquad P(\overline{A}) = 1-\frac{1}{4} = \frac{3}{4}$

$P(B) = \frac{2}{3}$ $\qquad P(\overline{B}) = 1-\frac{2}{3} = \frac{1}{3}$

$P(C) = \frac{3}{4}$ $\qquad P(\overline{C}) = 1-\frac{3}{4} = \frac{1}{4}$

$$\text{Required probability} = P(A \cap B \cap \overline{C}) + P(A \cap \overline{B} \cap C) + P(\overline{A} \cap B \cap C) + P(A \cap B \cap C)$$

$$= P(A) \cdot P(B) \cdot P(\overline{C}) + P(A) \cap (\overline{B}) \cdot P(C) + P(\overline{A}) \cdot P(B) \cdot P(C) + P(A) \cdot P(B) \cdot P(C)$$

$$= \frac{1}{4}\cdot\frac{2}{3}\cdot\frac{1}{4}+\frac{1}{4}\cdot\frac{1}{3}\cdot\frac{3}{4}+\frac{3}{4}\cdot\frac{2}{3}\cdot\frac{3}{4}+\frac{1}{4}\cdot\frac{2}{3}\cdot\frac{3}{4} = 29/48$$

Q.46 A class consist of 80 students, 25 of them are girls and 55 boy 10 of them are rich and remaining poor. 20 of them are fa complexioned. What is the probability of selecting a fa complexioned rich girl ?

Ans. : Probability of fair $= \frac{20}{80} = \frac{1}{4}$

Probability of rich $= \frac{10}{80} = \frac{1}{8}$

Probability of girl $= \frac{25}{80} = \frac{5}{16}$

$\therefore$ Probability of rich fair complexioned girl $= \frac{1}{4}\times\frac{1}{8}\times\frac{5}{16} = \frac{5}{512}$

Q.47 A problem of statistics is given to 5 students. A, B, C, D and E Their chances of solving the problem are 1/2, 1/3, 1/4, 1/5 and 1/6 What is the probability that problem is solved.

Ans. : Probability that problem is solved means the probability th atleast one of the student solve it.

$$P(A) = \frac{1}{2} \Rightarrow P(\overline{A}) = 1-\frac{1}{2} = \frac{1}{2}$$

$$P(B) = \frac{1}{3} \Rightarrow P(\overline{B}) = 1-\frac{1}{3} = \frac{2}{3}$$

$$P(C) = \frac{1}{4} \Rightarrow P(\overline{C}) = 1-\frac{1}{4} = \frac{3}{4}$$

$$P(D) = \frac{1}{5} \Rightarrow P(\overline{D}) = 1-\frac{1}{5} = \frac{4}{5}$$

$$P(E) = \frac{1}{6} \Rightarrow P(\overline{E}) = 1-\frac{1}{6} = \frac{5}{6}$$

$\therefore$ Probability that problem is not solved.

$$= P(\overline{A}) \times P(\overline{B}) \times P(\overline{C}) \times P(\overline{D}) \times P(\overline{E})$$

$\therefore$ Probability that problem is solved.

$$= 1 - P(\overline{A}) \times P(\overline{B}) \times P(\overline{C}) \times P(\overline{D}) \times P(\overline{E}) = 1-\frac{1}{6} = \frac{5}{6}$$

Q.48 A bag contains 3 red and 5 black balls and a 2^{nd} bag contains 6 red and 4 black balls. A ball is drawn from each bag. Find the probability that 1) one is red and other is black. 2) both are red 3) both are black. **[SPPU : Dec.-16]**

Ans. : For 1^{st} bag

$$P(r_1) = \frac{3}{8} \quad \text{and} \quad P(b_1) = \frac{5}{8}$$

For 2^{nd} bag.

$$P(r_2) = \frac{6}{10} \quad \text{and} \quad P(b_2) = \frac{4}{10}$$

1) A ball is drawn from each bag 1 red from 1^{st} and 1 black from 2^{nd} or 1 black from 1^{st} and 1 red from 2^{nd}.

First case : $P(r_1 \text{ and } b_2) = P(r_1) \times P(b_2) = \frac{3}{8} \times \frac{4}{10} = \frac{12}{80}$

Second case : $P(b_1 \text{ and } r_2) = \frac{5}{8} \times \frac{6}{10} = \frac{30}{80}$

$$\text{Required probability} = \frac{12}{80} + \frac{30}{80} = \frac{42}{80}$$

Mutually exclusive case.

2) Both balls are red

$$P(r_1 \text{ and } r_2) = P(r_1) \cdot P(r_2) = \frac{3}{8} \cdot \frac{6}{10} = \frac{18}{80} = \frac{9}{40}$$

3) Both balls are black

$$P(b_1 \text{ and } b_2) = P(b_1) \cdot P(b_2) = \frac{5}{8} \cdot \frac{4}{10} = \frac{20}{80} = \frac{1}{4}$$

Q.49 If 3 out of 20 tubes are defective and 4 of them are random chosen for inspection then what is the probability that only one the defective tubes will be selected.

Ans. : Out of 3 defective 1 can be chosen in ${}^3C_1$ ways and out 17 non-defective 3 can be chosen in ${}^{17}C_3$ ways and out of 20 tubes 4 can be chosen in ${}^{20}C_4$ ways.

$$\therefore \quad \text{Required probability} = \frac{{}^3C_1 \times {}^{17}C_3}{{}^{20}C_4} = \frac{2040}{4845} = 0.42$$

Q.50 A is one of the eight horses entered for a race and is to ridden by one of the two jockeys B and C. It is 2 to 1 that rides A, in which case all the horses are equally likely to win, wi rider C, A's chance is doubled.

1) Find the probability that A wins.

2) What are odds against, A's winning ?

Ans. : 1) A can win in the following two mutually exclusive cases.

i) B rides A and A wins.

ii) C rides A and A wins.

$$P(i) = \frac{2}{3} \times \frac{1}{8} = \frac{1}{12}$$

$$P(ii) = \frac{1}{3} \times \frac{2}{8} = \frac{1}{12}$$

$$\text{Probability of A winning} = P(i) + P(ii) = \frac{1}{12} + \frac{1}{12} = \frac{1}{6}$$

$$\text{2) Probability of A's losing} = 1 - \frac{1}{6} = \frac{5}{6}$$

Hence odds against A's winning are $\frac{5}{6}; \frac{1}{6}$, i.e. 5 : 1

Q.51 The probability that a contractor will get a plumbing contra is $\frac{2}{3}$ and the probability that he will not get an electric contract is

If the probability of geting at least one contract is $\frac{4}{5}$. What is th probability that he will get both the contracts. ☞[SPPU : Dec.-

Ans. : P(A) = Probability that a contractor will get a plumbing contract

$$= \frac{2}{3}$$

P(B) = Probability that a contractor will get an electric contract

$$= 1 - \frac{5}{9} = \frac{4}{9}$$

Given that,

$$P(A \cup B) = \frac{4}{5}$$

Find $P(A \cap B)$

We have,

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$= \frac{2}{3} + \frac{4}{9} - \frac{4}{5} = \frac{6+4}{9} - \frac{4}{5} = \frac{10}{9} - \frac{4}{5} = \frac{50-36}{45} = \frac{14}{45}$$

2.6 : Conditional Probability

Let A and B be two events of a sample space S. The event of occurrence of B under assumption that A has already occurred is called as conditional event. It is denoted by B/A. It's probability is P(B/A).

$$\therefore \quad P(B/A) = \frac{P(A \cap B)}{P(A)}$$

Similarly $$P(B/A) = \frac{P(A \cap B)}{P(B)}$$

Remarks

1) $$P(A \cap B) = P(A) \cdot P(B/A) = P(A) \cdot P(A/B)$$

2) If an event B is depends on how A has taken place and C depends on how A and B have taken place then

$$P(A \cap B \cap C) = P(A) \cdot P((B/A) \cdot P(C/A \cap B)$$

3) Two events A and B are said to be independent.

If $$P(A \cap B) = P(A) \cdot P(B)$$

Therefore $P(A/B) = P(A)$ in case of independent events.

Q.52 Find the probability of drawing a card of spades on each of two consecutive draws from a well shuffled pack of cards without replacement of the card.

Ans. : Let A be the event of drawing the first card spade. Let B be event of drawing the second spade card. Then $P(A) = \frac{13}{52}$

and $$P(B/A) = \frac{12}{51} \text{ (as one Spade card is drawn)}$$

Hence $$P(A \cap B) = P(A) \cdot P(B/A) = \frac{13}{52} \cdot \frac{12}{51}$$

Q.53 A bag contains 6 white and 9 black balls. Four balls are drawn at a time. Find the probability for the first draw to give and white and the second to give a black balls in the following cases.

i) The balls are replaced before the second draw.

ii) The balls are not replaced before the second draw.

Ans. : The experiment of drawing 4 balls from a bag of 6 white and 9 black can be done in 15_{C4} ways.

Let A be the event that the first drawing gives. A white balls and B be the event that the second drawing gives A black balls.

i) The event A consist of ${}^6C_4$ points.

$\therefore$ $$P(A) = \frac{{}^6C_4}{{}^{15}C_4}$$

Now whatever be the result of the first experiment, as the balls are replaced before the second draw.

Therefore A and B are independent events.

and $$P(B) = \frac{{}^9C_4}{{}^{15}C_4}$$

$\therefore$ $$P(A \cap B) = P(A \cdot B) = \frac{{}^6C_4}{{}^{15}C_4} \cdot \frac{{}^9C_4}{{}^{15}C_4} = \frac{6}{5926}$$

ii) Here $$P(A) = \frac{{}^6C_4}{{}^{15}C_4}$$

Hence $$P(A \cap B) = P(A) \cdot P(B/A) = \frac{{}^6C_4}{{}^{15}C_4} \cdot \frac{{}^9C_4}{{}^{11}C_4} = \frac{3}{715}$$

2.7 : Baye's Thoerem

If $E_1, E_2, \ldots, E_n$ are mutually disjoint events with $P(E_i) \neq 0 \forall_i$. Then for any event A which is a subset of $\bigcup_{i=1}^{n} E_i$ then

$$P(E_i/A) = \frac{P(E_i)\ P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\ P(A/E_i)}$$

1) For $n = 2$, i.e. E_1, E_2

$$\therefore \quad P(E_1/A) = \frac{P(E_1)\ P(A/E_i)}{P(E_1)\ P(A/E_1) + P(E_2)\ P(A/E_2)}$$

2) For $n = 3$, i.e. E_1, E_2, E_3

$$\therefore \quad P(E_2/A) = \frac{P(E_2)\ P(A/E_2)}{P(E_1)\ P(A/E_1) + P(E_2)\ P(A/E_2) + P(E_3)\ P(A/E_3)}$$

Here the probabilities $P(E_1), P(E_2), \ldots, P(E_n)$ are called priori probabilities. The probabilities $P(E_i/A)$ are called likelihoods and the probabilities $P(E_i/A)$ are called posteriori probabilities.

Q.54 A software company selects marketing professionals on the basis of an aptitude test. Past experience indicates that only 75 % of the candidates were found satisfactory in actual marketing 80 % had passed the aptitude test. Only 20 % of those found unsatisfactory has passed the test. Given that the candidate passed the test what is the probability that he would be found satisfactory.

Ans. : Let A be the event that the candidate passed the aptitude test and B be the event that he would be found satisfactory. Find P(B/A). Consider B and $\overline{B}$.

$$\therefore \quad P(B/A) = \frac{P(A/B) \cdot P(A/B)}{P(A/B) \cdot P(B) + P(A/\overline{B})\ P(A/\overline{B})}$$

$$P(B) = 0.75, \qquad P(\overline{B}) = 0.25$$

$$P(A/B) = 0.80, \qquad P(A/\overline{B}) = 0.25$$

$$P(B/A) = \frac{0.75 \times 0.8}{(0.75)(0.8) + (0.25)(0.2)} = \mathbf{0.923}$$

Q.55 The contents of urns I, II, III are as follows respectively.
I → 1 white, 2 black, 3 red balls
II → 2 white, 1 black, 1 red balls
III → 4 white, 5 black, 3 red balls
One urn is chosen at random and two balls are drawn. They happen to be white and red.
What is the probability that they come from urn I, II or III ?

Ans. : If E_1, E_2, E_3 denote the events that the urn I, II and III are chosen respectively.

Let A be the event that the two balls taken from the selected urn are white or red.

Then $$P(E_1) = P(E_2) = P(E_3) = \frac{1}{3}$$

$$P(A/E_1) = \frac{{}^1C_1 \cdot {}^3C_1}{{}^6C_2} = \frac{3}{15} = \frac{1}{5}$$

$$P(A/E_2) = \frac{{}^2C_2 \cdot {}^1C_1}{{}^4C_2} = \frac{2}{6} = \frac{1}{3}$$

$$P(A/E_3) = \frac{{}^4C_1 \cdot {}^3C_1}{{}^{12}C_2} = \frac{4.3}{\frac{12\times 11}{2}} = \frac{2}{11}$$

Hence

$$P(E_1/A) = \frac{P(E_i)\, P(A/E_i)}{\sum_{i=1}^{n} P(E_i)\, P(A/E_i)} = \frac{\frac{1}{3}\cdot\frac{1}{5}}{\frac{1}{3}\cdot\frac{1}{5}+\frac{1}{3}\frac{1}{3}+\frac{1}{3}\cdot\frac{2}{11}}$$

$$= \frac{\frac{1}{5}}{\frac{1}{5}+\frac{1}{3}+\frac{2}{11}} = \frac{1}{5}\times\frac{165}{118} = \frac{33}{118}$$

$$P(E_2/A) = \frac{\frac{1}{3}\cdot\frac{1}{3}}{\frac{1}{3}\times\frac{1}{5}+\frac{1}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{2}{11}} = \frac{\frac{1}{3}}{\frac{118}{165}} = \frac{1}{3}\cdot\frac{165}{118} = \frac{55}{118}$$

$$P(E_3/A) = \frac{\frac{1}{3}\cdot\frac{2}{11}}{\frac{1}{3}\cdot\frac{1}{5}+\frac{1}{3}\cdot\frac{1}{3}+\frac{1}{3}\cdot\frac{2}{11}} = \frac{\frac{2}{11}}{\frac{118}{165}} = \frac{30}{118}$$

2.8 : Information and Mutual Information

I) Let A be an event, with probability of occurrence P(A).

Then the information contained in a statement regarding the occurrence of A is given by the formula,

$$I(A) = -\log P(A)$$

Where $\qquad 0 \leq P(A) \leq 1$

Hence log P(A) is a non-positive number therefore – log P(A) is always positive.

The numerical value of information in above equation depends upon what logarithmic base is used.

If 2 is used as base of logarithm, then $\log_2 2 = 1$ is called a binary digit (bit).

If 10 is used as base of logarithm, then $\log_{10} 10 = 1$ is called a decimal digit or decit.

Generally, we use the base 2 to express the value of the information.

II) Mutual Information

If A and B are two events. Then mutual information between A and B is the amount of information concerning the occurrence of A that is contained in a statement asserting the occurrence of event B.

It is denoted by I(A, B)

$$I(A, B) = -\log P(A) + -\log P(A/B)$$

and $\qquad I(B, A) = -\log P(B) + \log P(B/A)$

Also $\qquad P(B/A) = \dfrac{P(A \cap B)}{P(A)}$

$\therefore \qquad I(B, A) = \dfrac{P(A \cap B)}{P(A)}$

$\therefore \qquad I(B, A) = -\log P(B) + \log [P(A \cap B)] - \log P(A)$

$$= -\log P(A) + \log\left(\frac{P(A \cap B)}{P(B)}\right)$$

$$= -\log P(A) + \log P(A/B)$$

$$\therefore \quad I(A, B) = I(B, A)$$

Q.56 Given a binary communication channel when A = input and B = output. Let P(A) = 0.4, P(B/A) = 0.9, $P(B'/A') = 0.6$. Find the mutual information. i) Between A and B ii) Between A and B'

Ans. : i) $I(A, B) = -\log P(A) + \log P(A/B)$

$$P(A, B) = \frac{P(A \cap B)}{P(B)}$$

Hence we have to find $P(A \cap B)$ $P(B)$

Now $\quad P(B/A) = 0.9$

$$P(B/A) \cdot P(A) = P(A \cap B)$$

$$\text{Hence } \frac{P(A \cap B)}{P(B)} = P(B/A) = 0.9$$

$$P(A \cap B) = 0.9 \times 0.4$$

$$\text{Now } P(A' \cap B') = 0.6 \times P(A')$$

$$= 0.6 \times (1 - 0.4) = 0.36$$

$$\therefore \quad P[(A \cap B)'] = 0.36$$

$$\therefore \quad P(A \cup B) = 1 - 0.36 = 0.64$$

$$\therefore \quad P(A) + P(B) - P(A \cap B) = 0.64$$

$$P(B) = 0.64 - 0.4 + 0.36 = 0.6$$

$$\text{Hence} \quad P(A/B) = \frac{0.36}{0.60} = 0.6$$

$$\therefore \quad I(A, B) = -\log 0.4 + \log 0.6 = \log 3 - \log 2$$

$$= 1.585 - 1 = 0.585 \text{ bits}$$

ii) $\quad I(A, B') = -\log P(A) + P(A/B')$

$$P(A/B') = \frac{P(A \cap B')}{P(B')}$$

$$A = (A \cap B) \cup (A \cap B')$$

$$P(A) = P(A \cap B) + P(A \cap B')$$

$$P(A \cap B') = P(A) - P(A \cap B) = 0.4 - 0.36 = 0.04$$

$$\therefore \quad P(A/B') = \frac{0.04}{1-0.6} = \frac{0.04}{0.4} = 0.1$$

$\therefore$ $I(A, B') = -\log 0.4 + \log 0.1 = -\log 4 = -2$ bits.

Q.57 A box contains 6 red, 4 while 4, 5 black balls. A person draws 4 balls from the box at random. Find the probability that among the balls drawn there is at least one ball of each colour ?

[SPPU : May-14]

Ans. : Total number of ball in a box = 6 + 4 + 5 = 15

4 balls can be selected from the box in ${}^{15}C_4$ ways.

i.e. ${}^{15}C_4 = 1365$

Let E be the required event. Event E can take place.

Red balls	White balls	Black balls	Number of ways
2	1	1	$1 = {}^6C_2 \times {}^4C_1 \times {}^5C_1$
1	2	1	$1 = {}^6C_1 \times {}^4C_2 \times {}^5C_1$
1	1	2	$2 = {}^6C_1 \times {}^4C_1 \times {}^5C_2$

$\therefore$ The total number of favourable cases

$$= 6C_2 \times 4C_1 \times 5C_1 \times 6C_1 \times 4C_2 \times 5C_1 + 6C_2 \times 4C_1 \times 5C_2$$

$$= 15 \times 4 \times 5 + 6 \times 6 \times 5 + 6 \times 4 \times 10$$

$$= 300 + 180 + 240 = 720$$

$\therefore$ The required probability $= \dfrac{720}{1365} = 0.5274$

Q.58 How many numbers of 7 digits can be formed with the numbers 0, 2, 2, 5, 6, 6, 6 How many of them are even ?

[SPPU : May-14]

Ans. : The number of seven digit numbers can be formed from digits 0, 2, 2, 5, 6, 6, 6 taking 7 at a time in

$${}^7P_7 = 7! = 5040 \text{ ways.}$$

But some of them begin with zero.

∴ The number of numbers which begin with 0 is

$$= {}^6P_6 = 6! = 720$$

∴ The total number of 7 digit numbers = 5040 – 720 = 4320

Even Numbers : To get even number the unit place can be filled w the digits 0, 2, 2, 6, 6, 6 only in 6 ways.

And the remaining six places can be filled in

$$= {}^6P_6 = 6! = 720 \text{ ways.}$$

= The total number of even numbers are = 6 × 720 = 43

But some of numbers begin with zero.

∴ The required 7 digit numbers are = 4320 – 720 = **3600**

Q.59 Two cards are drawn at random from an ordinary deck of 5 cards. Find the probability that

i) Both are spades ii) One is spade and one is heart

[SPPU : May-1

Ans. : Two cards are drawn from 52 cards in ${}^{52}C_2$ ways.

∴ The total number of elements in a sample space = ${}^{52}C_2$ = 1326

i) Both cards are spade : There are 13 spade cards and 2 cards a drawn from 13 cards in ${}^{13}C_2$ ways.

$$\therefore \text{ The probability that both cards are spade} = \frac{{}^{13}C_2}{{}^{52}C_2} = \frac{78}{1326} = 0.0588$$

ii) One card is spade and one card is heart : One spade card drawn in ${}^{13}C_1$ ways and one heart is drawn in ${}^{13}C_1$ ways.

∴ The number of favourable ways = ${}^{13}C_1 \times {}^{13}C_1$ = 13 × 13 = 169

$$\therefore \text{ The required probability} = \frac{169}{1326} = 0.12745$$

Q.60 Three students A, B and C are swimming in the race A and B have same probability of winning and each each is twice as likely to win as C. Find the probability that :

i) B wins ii) C wins iii) B or C wins **[SPPU : May-15**

Ans. : From the given data, Let x be the probability of C to win.

$\therefore$ The probability of A and B are 2x, 2x respectively.

$$\therefore \quad x + 2x + 2x = 1 \Rightarrow x = \frac{1}{5}$$

$$\therefore \quad P(A) = \frac{2}{5}, \quad P(B) = \frac{2}{5}, \quad P(C) = \frac{1}{5}$$

$$\therefore \quad P(\overline{A}) = \frac{3}{5}, \quad P(\overline{B}) = \frac{3}{5}, \quad P(\overline{C}) = \frac{4}{5}$$

i) The probability that B wins game

$$= P(\overline{A} \cap B \cap \overline{C}) = P(\overline{A}) \cdot P(\overline{B}) \cdot P(\overline{C})$$

$$= \frac{3}{5} \cdot \frac{2}{5} \cdot \frac{4}{5} = \frac{24}{125} = 0.192$$

ii) The probability that C wins the game

$$= P(\overline{A} \cap B \cap \overline{C}) = P(\overline{A}) \cdot P(\overline{B}) \cdot P(\overline{C})$$

$$= \frac{3}{5} \cdot \frac{3}{5} \cdot \frac{1}{5} = \frac{9}{125} = 0.072$$

iii) The probability that B or C wins the game

$$P(\overline{A} \cap B \cap \overline{C}) + P(\overline{A} \cap B \cap C) = \frac{24}{125} + \frac{9}{125} = \frac{33}{125} = 0.264$$

Q.61 A box contains 6 white and 6 black balls. Find the number of ways 4 balls can be drawn from the box if i) Two must be white ii) All of them must be of same color ? **☞[SPPU : Dec.-14]**

Ans. : i) Two white balls can be selected in 6_{C_2} ways. The remaining 2 balls can be selected in (6 + 4 + 2) = 10_{C_2} ways. Hence the number of ways to make necessary selection is

$$6_{C_2} \cdot 10_{C_2} = \frac{6 \times 5}{2} \cdot \frac{10 \times 9}{2} = 675 \text{ ways}$$

ii) All of them must have the same colour implies that the balls must be all white or all black balls. The number of ways in which exactly one of these combinations can be done is

$$6_{C_4} + 6_{C_4} = 15 + 15 = 30 \text{ ways}$$

Q.62 Out of 5 males and 6 females, a committee of 5 is to be formed. Find the number of ways in which it can be formed so that among the person chosen in the committee there are

i) Exactly 3 male and 2 female.

ii) At least 2 males and one female. **[SPPU : Dec.-14]**

Ans. : i) The number of ways to form a committee of exactly 3 males and 2 females.

$$= {}^5C_3 \cdot {}^6C_2 = \frac{5\times4}{2}\cdot\frac{6\times5}{2} = 150 \text{ ways}$$

ii) The number of ways to form a committee of at least 2 males and one female is

$$= {}^5C_3 \cdot {}^6C_2 \cdot {}^5C_3 \cdot {}^6C_2 + {}^5C_4 \cdot {}^6C_1$$

$$= (10)\,(20) + 10.15 + 5.6$$

$$= 200 + 150 + 30 = 380 \text{ ways}$$

END...

Unit III

3 Graph Theory

3.1 : Basic Terminologies

Q.1 Define graphs with examples.

Ans. : Graphs : A graph is an ordered pair (V(G), E(G)) where

i) V(G) is non empty finite set of elements known as vertices or nodes. V(G) is called the vertex set.

ii) E(G) is a family of unordered pairs (not necessarily distinct) of elements of v, known as edges or arc or branches of G. E(G) is known as edge set.

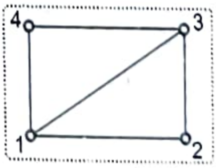
Graphs are so named because they can be represented diagrammatically in the plane.

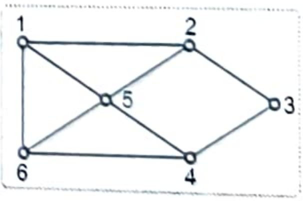
It is denoted by V(G, E).

a) Each vertex of G is represented by a point or small circle in the plane. In practical examples vertex set may be the set of states or cities or objects etc.

b) Every edge is represented by a continuous curve or straight line segment. Edges may be the route among states or cities or relation among objects etc. Diagrams of road maps, electrical circuits, chemical compounds, job scheduling family trees, all have two objects common namely vertices and edges.

Let us consider the following examples of graphs with V(G) and E(G).

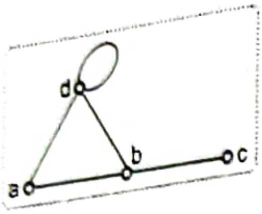
1) 

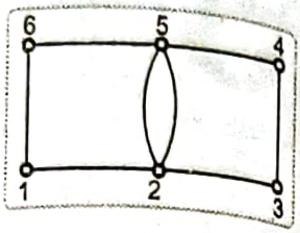
2) 

$V(G_1) = \{1, 2, 3, 4\}$ $\qquad$ $V(G_2) = \{1, 2, 3, 4, 5, 6\}$

$E(G_1) = \{(1, 2), (1, 3) (1, 4)$
$(2, 3), (3, 4)\}$

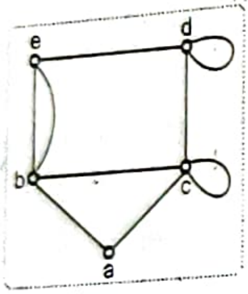
$E(G_2) = \{(1, 2), (1, 5), (1, 6),$
$(2, 5), (2, 3) (3, 4), (4, 5),$
$(4, 6), (5, 6)\}$

3) 

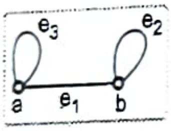
4) 

$V(G_3) = \{a, b, c, d\}$
$E(G_3) = \{(a, b), (a, d) (b, c)$
$(b, d), (d, d)\}$

$V(G_2) = \{1, 2, 3, 4, 5, 6\}$
$E(G_4) = \{(1, 2), (1, 6), (2, 3),$
$(2,5), (2, 5), (3, 4),$
$(4, 5), (5, 6)\}$

5) 

$V(G_5) = \{a, b, c, d\}$
$E(G_5) = \{(a, b), (a, c), (b, e) (b, e), (b, c)$
$(c, c) (c, d) (d, d), (d, e)\}$

6) 

$V(G_6) = \{a, b\}$
$E(G_6) = \{a, b) (a, a) (b, b) = \{e_1, e_2, e_3\}$

i) If x and y are two vertices of a graph G and unordered pair $\{x, y\} = (x, y) = e$ is an edge then we say that edge e joins x and y or e is incident to both vertices x and y.

In this case, vertices x and y are said to be incident one e.g. In example (1), e = (2, 3) ∴ e is incident at 2 and 3 and vertices 2, 3 are one incident on e = (2, 3).

ii) Two vertices x and y are said to be adjacent to each other if the pair (x, y) is an edge of G.

If $e = (x, y)$ is an edge of G then x and y are said to be end vertices of e and we can say that e is incident at x and y.

iii) Two edges e_1 and e_2 are said to be adjacent if they have a common vertex i.e. If e_1 and e_2 are adjacent then $e_1 = \{x, y\}$ and $e_2 = \{y, z\}$.

iv) An edge joining a vertex to itself is called a loop. E.g. In example (5) there are 2 loops (c, c) and (d, d).

v) A pair of vertices of a graph is joined by two or more edges, such edges are called as multiple or parallel edges.

In example (4) (2, 5) (2, 5) are multiple edges.

3.2 : Types of Graphs

Q.2 Explain different types of graphs.

1) **Multigraph :** A graph in which a pair of vertices is joined by two or more edges is called a multigraph or multiple graph.

i.e. A graph having multiple edges is called a multigraph. In examples (1), (2), (3), (6), graphs are not multigraphs and graphs in examples (4), (5) are multigraphs.

2) **Pseudograph :** A graph having loops but no multiple edges is called a Pseudograph.

Graphs in examples (3), (5) and (6) are pseudographs. A graph having only loops is called a Haary graph. For example :

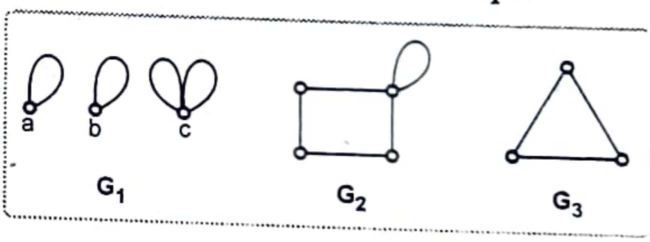
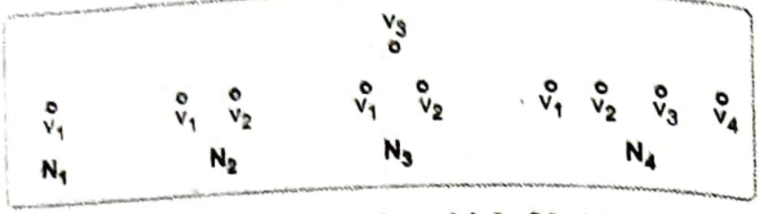


Fig. Q.2.1

Graph G is a pseudograph is well as Haary graph. Graph G_2 is Pseudograph but not Haary graph. Graph G_3 is neither Pseudo not Haary graphs.

3) **Simple graph :** A graph without loops and multiple edges is called a simple graph. Graphs in examples (1) and (2) are simple graphs. Graphs in examples (3), (4), (5), (6) are not simple graphs.

4) **Null graph :** A graph G (V, E) is said to be null graph if E is a empty set. Null graph on n vertices is denoted by N_n.



5) **Finite graph :** A graph G (V, E) in which V (x) and E (x) are finite sets is called a finite graph. Otherwise infinite graph.

6) **Directed graph :** A graph G (V, E) is said to be directed graph if the elements of E are an ordered pairs of vertices. E.g.

$E = \{(a, b), (b, c), (a, c)\}$

Here $(a, c) \neq (c, a)$, $(c, a) \notin E(G)$.

A graph which is not directed is called Non-directed graph or graph.

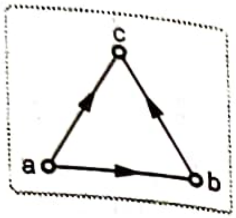


Fig. Q.2.2 Directed graph

7) **Weighted graph :** A graph G (V, E) in which some weight is assigned to every edge of G, is called weighted graph.

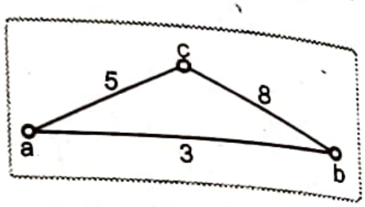


Fig. Q.2.3 Weighted graph

8) **Degree of a vertex :**

a) In a directed graph G the number of edges ending at vertex v is called the indegree of v. It is denoted by deg $G^{+}(v)$ or $d^{+}(v)$

b) **Outdegree :** In a directed graph G, the number of edges beginning at vertex v is called the outdegree of v. It is denoted by deg $G^{-}(v)$ or $d^{-}(v)$.

c) The number of edges incident at a vertex v of a graph G with self loops counted twice is called the degree of the vertex v. It is denoted by d(v). A vertex of degree one is called pendent vertex. A vertex of degree zero is called isolated vertex. An edge incident at pendent vertex is called pendent edge.

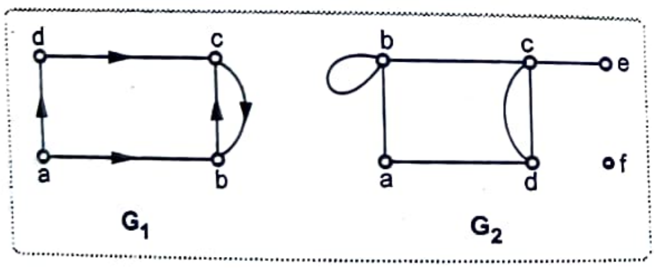


Fig. Q.2.4

e.g.

In graph G_1,

Vertices	Indegree	Outdegree
a	0	2
b	2	1
c	2	1
d	1	1

In graph G_2,

$$d(a) = 2 \;,\; d(b) = 2 + 2 = 4 \;,\; d(c) = 4 \;,\; d(e) = 1$$

$$d(d) = 3 \;,\; d(f) = 0$$

$\therefore$ f is an isolated vertex. e is a pendent vertex. An edge {c, e} is a pendent edge.

9) **Order and size of graph :** The number of vertices in a finite graph G is called the order of G. The number of edges in a finite graph G is called size of the graph. A graph of order n and size m is called (n, m) graph.

If G is a (p, q) graph then G has p vertices and q edges.

10) **Degree sequence of a graph :**

Let G be a graph with vertex set $V = \{v_1, v_2, v_3, \ldots v_n\}$ and $d_i = \deg(v_i)$ then the sequence $(d_1, d_2, d_3, \ldots d_n)$ in any order is called the degree sequence of G.

Note : 1) Vertices of G are ordered so that degree sequence is monotonically increasing.

2) Two graphs with same degree sequence are called to be degree equivalent. e.g.

$d(v_1) = 4$, $d(v_2) = 3$,

$d(v_3) = 2$, $d(v_5) = 5$,

$d(v_6) = 3)$, $d(v_4) = 1$

$\therefore$ Its degree seq. is (4, 3, 2, 1, 5, 3)

By relabelling vertices we may write degree sequence as (1, 2, 3, 3, 4, 5).

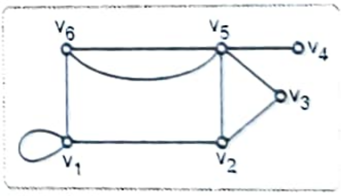


Fig. Q.2.5

Q.3 Handshaking lemma : Let G (V, E) be any graph then $\sum_{v \in V} d(v) = 2q$ where q denotes the number of edges of G.

Ans. : Proof : Let us argue by induction on q. Suppose $q = 0$ i.e. G has no edge i.e. E is an empty set. So $d(v) = 0, \forall v \in V$.

$\therefore \sum d(v) = 2q = 0$.

Let G be a graph with $q > 0$ edges. Choose any edge $e = \{u, v\}$ of G. Consider the graph G_1 obtained from G as follows :

i) The vertex set of G_1 is same as the vertex set of G i.e. $V(G_1) = V(G) = V = p$

ii) The edges of G_1 are all edges of G except e.

In other words, G_1 is obtained from G by deleting the edge e.

$\therefore$x is a (p, q − 1) graph.

$\therefore$ By induction principle, result is true for q − 1 edges

i.e. $$\sum_{x \in V} d(x) = 2(q-1) \qquad \cdots (Q.3.1)$$

The degree of a vertex x other than u or v in G_1 is same as that at G. And the degree of u in G_1 is one less than the degree of u in G.

i.e. $$d_{G_1}(u) = d_G(u) - 1$$

Similarly $$d_{G_1}(v) = d_G(v) - 1$$

Hence equation (Q.3.1) becomes

$$\sum_{\substack{x \in V \\ x \neq u, v}} d(x) + d_{G_1}(u) + d_{G_1}(v) = 2(q-1)$$

$$\sum_{\substack{x \in V \\ x \neq u, v}} d(x) + d_{G_1}(u) - 1 + d_G(v) - 1 = 2(q-2)$$

$\therefore$ $$\sum_{x \in V} d(x) = 2q \text{ Hence the proof.}$$

The result is so named because it implies that if several people shake hands, the total number of hands shaken must be even as two hands are involved in one handshake.

Note : If $\sum_{v \in V} d(v) =$ Odd number then there does not exist any graph with this degree sequence.

Q.4 Explain matrix representation of a graph with suitable examples.

[SPPU : Dec.-09, 10, May-10]

Ans. : 1. Adjacency Matrix : Let G be a graph with n vertices and no parallel edges. The adjacency matrix of x is denoted by

$$A(G) = [a_{ij}]_{n \times n} \text{ and defined as}$$

$$a_{ij} = 1 \text{ if } v_i \text{ and } v_j \text{ are adjacent}$$

$$= 0 \text{ if } v_i \text{ and } v_j \text{ are not adjacent.}$$

Note : i) A (G) is asymmetric - binary matrix.

ii) The principal diagonal entries are all zeros if G has no loops.

iii) The i^{th} row sum = i^{th} column sum = $d(v_i)$

e.g. 1) The adjacent matrices of the following graphs are

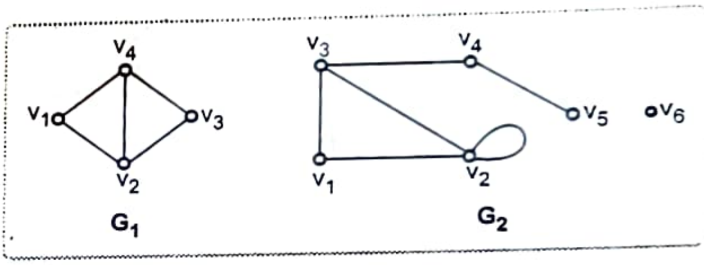


Fig. Q.4.1

$$A(G_1) = \begin{array}{c c} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}_{4 \times 4} \end{array}, \quad A(G_2) = \begin{array}{c c} & \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6 \times 6} \end{array}$$

The adjacency matrix for a multigraph G is a n × n Matrix $A(G) = \left[a_{ij}\right]_{n \times n}$ where

$$a_{ij} = \text{Number of edges joining } v_i \text{ and } v_j$$

The adjacency matrix of the following graph is

$$A(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & v_4 & v_5 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 2 & 1 \end{bmatrix} \end{array}$$

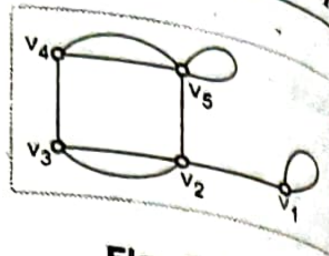


Fig. Q.4.2

2. Incidence Matrix : Let G be a graph with n vertices and m edg... without self loops. The incidence matrix is denoted by X (G) or I (G) a... defined as

$$X(G) = [x_{ij}]_{n \times m} \text{ where}$$

$$x_{ij} = 1 \text{ if } j^{th} \text{ edge is incident on } i^{th} \text{ vertex } v_i.$$

$$= 0 \text{ otherwise.}$$

X (G) is a n × m matrix whose n rows correspond to the n vertices a... m columns correspond to m edges. The graph and it's incidence mat... are given below

$$X(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{5 \times 7} \end{array}$$

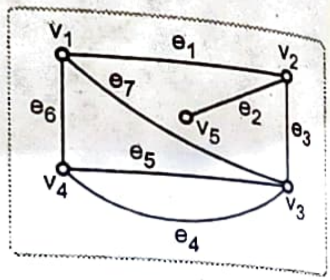


Fig. Q.4.3

Properties of Incidence Matrix

1) It contains only 0 and 1.
2) Each column in the incidence matrix has exactly two 1's appearing... that column.
3) The sum of elements in a row is equal to the degree of... corresponding vertex.
4) Two identical columns correspond to the parallel edges in graph.
5) A row with all 0's represents an isolated vertex.
6) A row with single 1 represents a pendent vertex.

The incidence matrix of a graph with loop is given as follows :

$$X(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix}_{3 \times 3} \end{array}$$

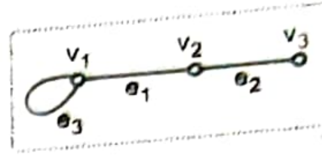


Fig. Q.4.4

3. Adjacency Matrix of a Diagraph (Directed graph)

Let G be a directed graph with n vertices and without parallel edges. The adjacency matrix is denoted by

Where, $A(D) = [a_{ij}]_{n \times n}$

a_{ij} = 1 if there is an edge directed from v_i to v_j

= 0 otherwise

In network flow, adjacency matrix is also known as connection matrix or transition matrix. The adjancency matrix and diagraph are given below.

$$A(D) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6 \times 6} \end{array}$$

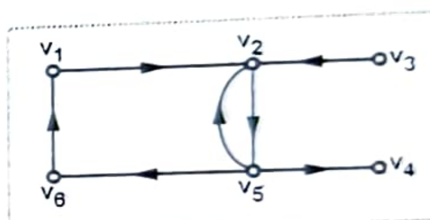


Fig. Q.4.5

4. Incidence Matrix of Diagraph : The incidence matrix of a diagraph with n vertices m edges and no self loops is a matrix.

$X(G) = [x_{ij}]_{n \times m}$ where

x_{ij} = 1 if j^{th} edge e_j is incident out of i^{th} vertex v_i.

= −1 if j^{th} edge e_j is incident into of vertex v_i.

= 0 otherwise.

A graph and its incidence matrix are given below :

$$X(G) = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{bmatrix}_{6\times 6} \end{array}$$

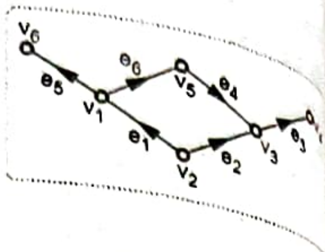


Fig. Q.4.6

Note : i) Sum of elements in each column of incidence matrix is zero.

Q.5 Show that in a graph the number of vertices of odd degree is even.

Ans. : Let G be a (p – q) graph.

By handshaking lemma $\sum_{v_i \in V} d(v_i) = 2q$

Now separate out vertices of even degree and odd degree

$$\therefore \quad \sum_{v_i \in V} d(v_i) = \sum_{\substack{x \in V \\ \text{even degree}}} d(x) + \sum_{\substack{y \in V \\ \text{odd degree}}} d(y) = 2q$$

$$\therefore \quad \sum_{\substack{v_i \in V \\ \text{odd degree}}} d(v_i) = 2q - \sum_{\substack{x \in V \\ \text{even degree}}} d(x) = \text{Even number}$$

$\therefore$ The sum of vertices of odd degree is even.

Hence the number of vertices of odd degree is even.

Q.6 Show that the maximum number of edges in a simple graph with n vertices is $\frac{n(n-1)}{2}$.

Ans. : Let G be a graph with n vertices m edges

$\therefore$ By handshaking lemma

$$\sum_{v \in V} d(v) = 2m \rightarrow$$

Let $x \in V$ $\therefore x$ must be adjacent to remaining $(x - 1)$ vertices

$\therefore d(x) = n - 1.\ \forall x \in V$

$\therefore$ Equation (1) $\Rightarrow (n - 1) + (n - 1) + \ldots$ n times $= 2m$

$$n(n - 1) = 2m$$

$$m = \frac{n(n-1)}{2}$$

Hence the maximum number of edges in any simple graph with n vertices is $\frac{n(n-1)}{2}$

Q.7 Determine the number of edges in a graph with 6 nodes, 2 of degree 4 and 4 of degree 2. Draw to such graphs. [SPPU : Dec.-09]

Ans. : Let G be the required graph with 6 nodes and m edges.

$\therefore$ By handshaking lemma

$$\sum_{v \in G} d(v) = 2m$$

$$d(v_1) + d(v_2) + d(v_3) + d(v_4) + d(v_5) + l(v_6) = 2m$$

$$4 + 4 + 2 + 2 + 2 + 2 = 2m$$

$$2m = 16$$

$$m = 8$$

Hence 8 edges are required.

Two such graphs are given below.

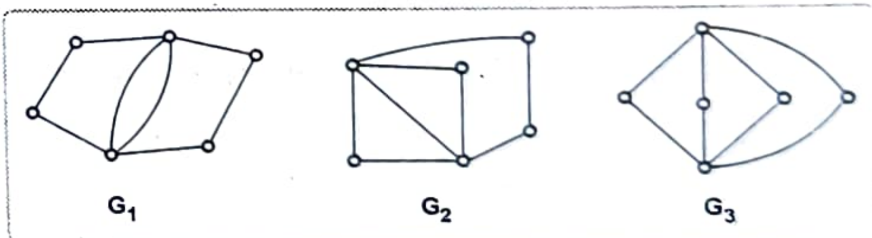


Fig. Q.7.1

Q.8 Is it possible to construct a graph with 12 nodes such that 2 of the nodes have degree 3 and the remaining have degree 4.

[SPPU : Dec.-10]

Ans. : Let G be the required graph with 12 vertices.

By handshaking lemma.

$$\sum_{v \in V(G)} d(v) = 2m$$

$$(3+3)+(4+4+4+4+4+4+4+4+4+4) = 2m$$

$$6+40 = 2m$$

$$\Rightarrow \quad m = 23$$

$\therefore$ It is possible to construct such graph.

Q.9 Is graph exist for the degree sequence 4, 4, 3, 3, 2, 2, 1.

Ans. : Now apply handshaking lemma

$$\sum d(v) = 2m = \text{Even}$$

$$4+4+3+3+2+2+1 = \text{Even}$$

$$19 = \text{Even which is impossible}$$

$\therefore$ Such graph does not exist.

Q.10 How many simple labelled graphs with n vertices are there ?

[SPPU : May-10]

Ans. : We know that a simple graph with n vertices has maximum possible number of edges $\frac{n(n-1)}{2} = m$ (say).

To construct a simple graph with e edges and n vertices, can be done in $\binom{m}{e}$ ways.

i.e. ${}^{m}C_{e}$ ways where $m = \frac{n(n-1)}{2}$ and $0 \le e \le m$

Hence the total number of ways to construct such graphs is given by

$$\binom{m}{0}+\binom{m}{1}+\binom{m}{2}+\ldots+\binom{m}{m} = 2^{m} = 2^{\left(\frac{n(n-1)}{2}\right)} \text{ ways.}$$

Q.11 Show that a simple graph of order 4 and size 7 does not exist.

Ans. : Let G be a simple graph with 4 vertices.

Then G has at most $\frac{n(n-1)}{2} = \frac{4 \times 3}{2} = 6$ edges.

But given that G has 7 edges which is contradiction.

$\therefore$ there can not be a simple graph with 4 vertices and 7 edges.

Q.12 Explain i) Regular graph ii) Complete graph iii) Bipartite graph iv) Complete bipartite graph.

Ans. : I) Regular Graph : A graph G is said to be r-regular graph if every vertex of G has degree r.

i) Regular graph of degree zero is called null graph.

ii) A regular graph of degree 3 is called cubic graph.

e.g.

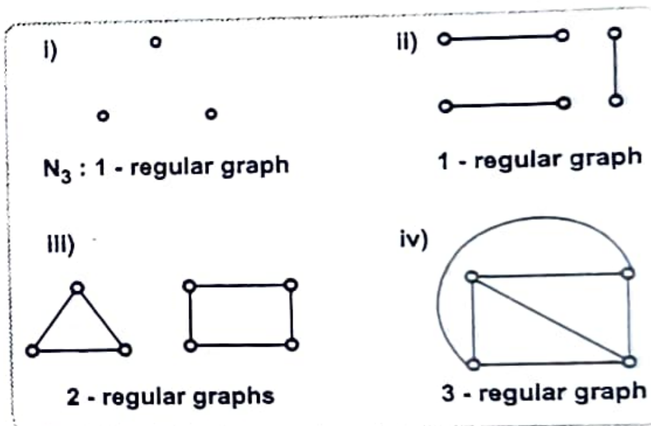


Fig. Q.12.1

II) Complete Graph : A simple graph G in which every pair of distinct vertices are adjacent is called a complete graph. If G is a complete graph on n vertices then it is denoted by K_n.

In a complete graph, there is an edge between every pair of distinct vertices.

In graph K_n, every vertex is adjacent to remaining $n - 1$ vertices so degree of each vertex is $n - 1$.

Thus K_n is a $(n - 1)$ - regular graph.

K_n has exactly $\dfrac{n(n-1)}{2}$ edges.

Consider the following examples :

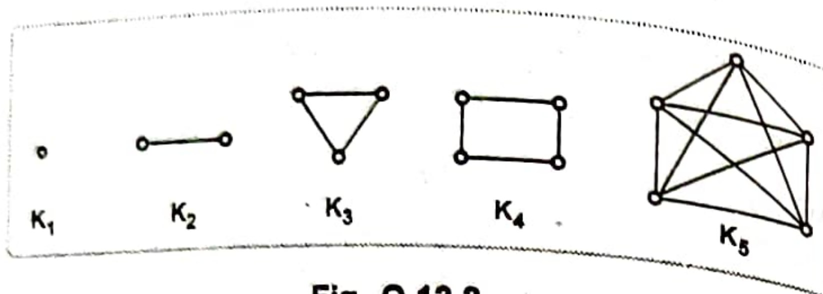


Fig. Q.12.2

III) Bipartite Graph : A graph G (v, E) is said to be bipartite graph if its vertex set can be partitioned into two disjoint subsets say v_1 and v_2 such that $v_1 \cup v_2 = v$ and

$v_1 \cap v_2 = \phi$ and every edge of G joins a vertex of v_1 to a vertex of v_2. In Bipartite graph, vertices of v_1 should not be adjacent. It is free from loops.

Following graphs are bipartite graphs

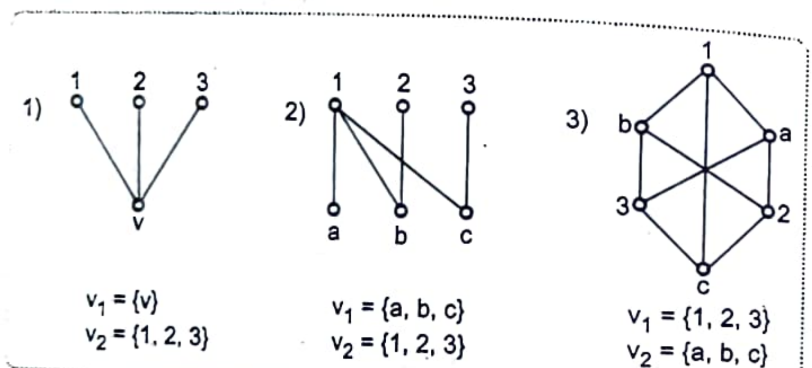


Fig. Q.12.3

IV) Complete Bipartite graph : A bipartite graph G (v, E), $v_1 \cup v_2 = v$ and $v_1 \cap v_2 = \phi$ is said to be complete Bipartite graph if each vertex of V_i is joined to every vertex of v_2 by a unique edge.

If $|v_1| = m$, $|v_2| = n$, then the complete bipartite graph $G(v_1 \cup v_2, E)$ is denoted by $K_{m,n}$

Examples :

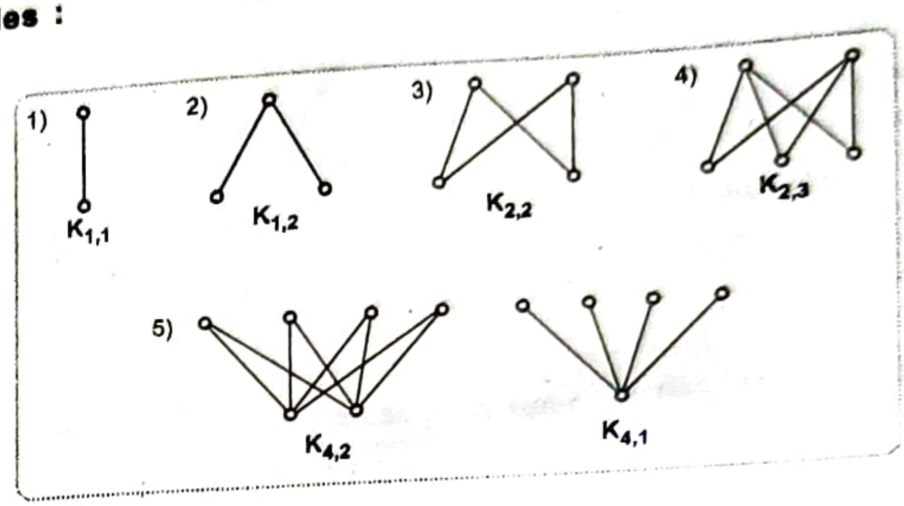


Fig. Q.12.4

The graph $K_{1,n}$ is called as star graph.

Q.13 Is there exist any complete bipartite graph with 7 vertices and 14 edges ?

Ans. : First find all possible bipartitions of 7. They are 6 + 1, 5 + 2, 4 + 3.

We know that, if $G(v_1 \cup v_2, E)$ is a bipartite graph then the number edges in G is equal to $|v_1| \cdot |v_2|$

i.e. $|E| = |v_1| \cdot |v_2|$

Here $|E| = 14$ But 6.1 = 6, 5.2 = 10, 4.3 = 12

Therefore the complete bipartite graphs with 7 vertices has 6 or 10 or 12 edges only.

Therefore any complete bipartite graph with 7 vertices and 14 edges.

Q.14 Explain isomorphism of graphs with examples.

[SPPU : Dec.-12]

Ans. : In real life we come across so many similar objects or figures with respect to size, shape or orientation. Similarly there are a few concepts in graph theory which deal with the similarity of graphs w.r.t. number of vertices or number of edges, number of regions and so on. Among all such similarities the most important one is an isomorphism of graphs.

Definition : Let G_1 (V_1, E_1) and G_2 (V_2, E_2) be two graphs. G_1 and G_2 are said to be isomorphic graphs if

i) There exists a bijective function $\phi : V_1 \rightarrow V_2$

ii) There exists a bijective function $\psi : E_1 \rightarrow E_2$ such that $e = (x,$ ... an edge in G_1 iff $(\phi(x), \phi(y))$ is an edge in G_2.

The pair of functions ϕ and ψ is called an isomorphism of G_1 and G_2 ... is denoted by $G_1 \equiv G_2$.

Suppose two graphs G_1 (V_1, E_1) and G_2 (V_2, E_2) are isomorphic gr... Then it is clear that

i) $|V_1| = |V_2|$ i.e. G_1 and G_2 must have same number of vertices.

ii) $|E_1| = |E_2|$ i.e. G_1 and G_2 must have same number of edges.

iii) G_1 and G_2 must have an equal number of vertices with the s... degree.

iv) G_1 and G_2 must have an equal number of loops.

v) G_1 and G_2 must have same number of pendent.

vi) G_1 and G_2 must have same number of pendent edges.

vii) If u and v are adjacent in G_1 then the corresponding vertices in ... are also adjacent.

In general it is easier to prove two graphs are not isomorphic by prov... that any one of the above property fails.

Consider the following example

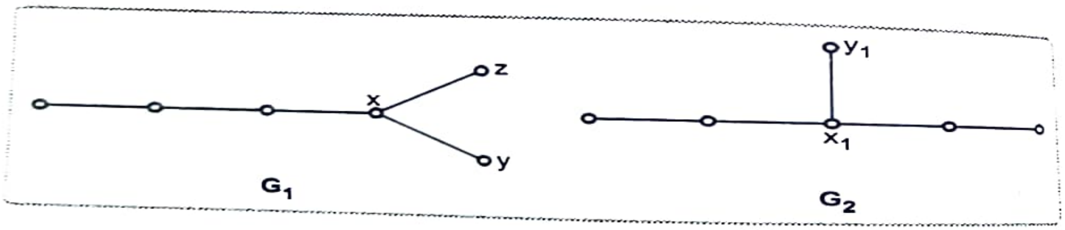


Fig. Q.14.1

These graphs have i) The same number of vertices.

ii) The same number of edges.

iii) An equal number of vertices of degree k.

These conditions are necessary for two graphs to be isomorphic bu... sufficient.

raph G_1, d (x) = 3, d (y) = 1, d (z) = 1 and x and y are adjacent to ex x.

raph G_2, d (x_1) = 3, d (y_1) = 1

re is only one pendent vertex adjacent to x_1. Hence adjacency is not served. Therefore G_1 is not isomorphic to G_2 i.e. $G_1 \not\equiv G_2$.

te : Isomorphism of graphs is an equivalence relation.

amples :

15 Draw all isomorphic graphs on vertices 2 and 3.

ns. : i) For 3 vertices.

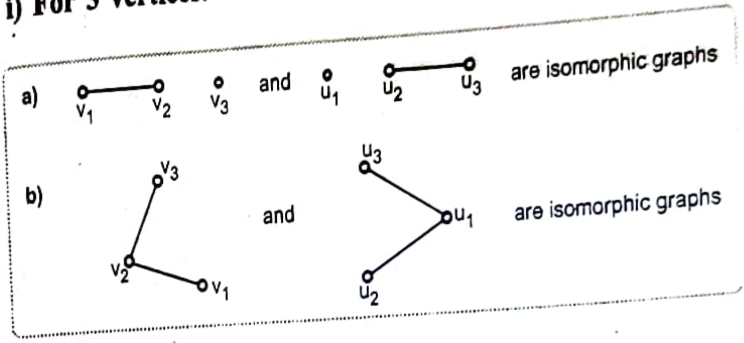


Fig. Q.15.1

i) For two vertices.

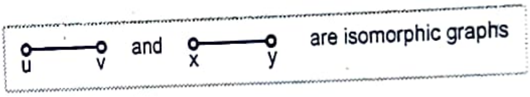


Fig. Q.15.1 (a)

Q.16 Draw all non isomorphic graphs on 2 and 3 and 4 vertices.

Ans. : All non-isomorphic graphs on 2 vertices are

Fig. Q.16.1

All non isomorphic graphs on 3 vertices.

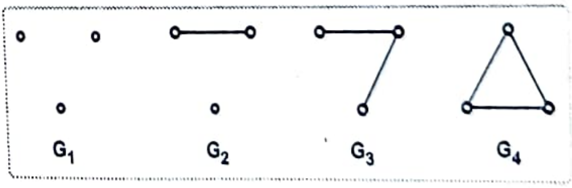


Fig. Q.16.1 (a)

All non isomorphic graphs on 4 vertices.

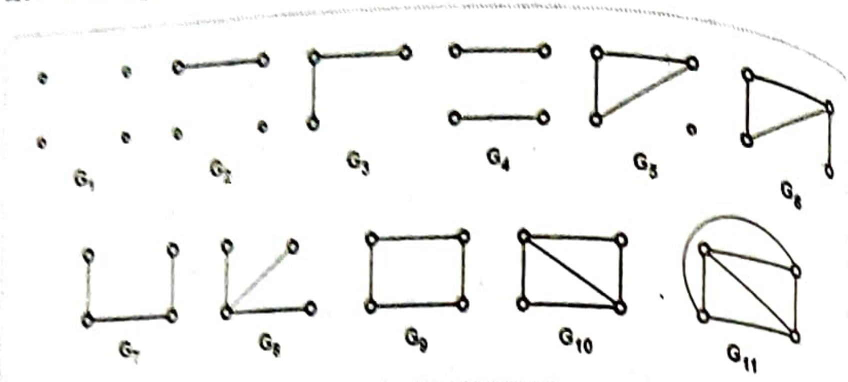


Fig. Q.16.1 (b)

Q.17 Draw all non isomorphic graphs on 5 vertices and 5 edges.

Ans. : The following are non isomorphic graphs with 5 vertices and edges.

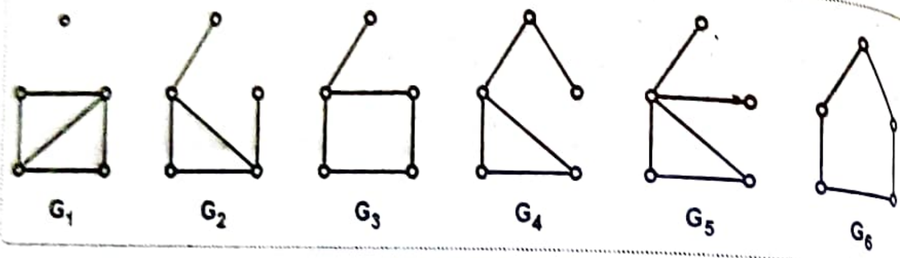


Fig. Q.17.1

Q.18 Find whether the following pairs of graphs are isomorphic or not.

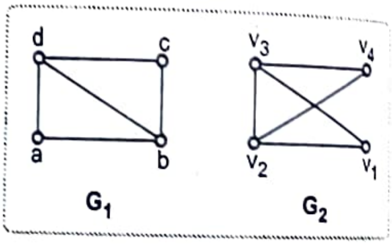


Fig. Q.18.1

Ans. : i) Both the graphs have 4 vertices and 5 edges.

Both have 2 vertices of degree 3 and 2 vertices of degree 2.

$\therefore \phi : \{a, b, c, d\} \rightarrow \{v_1, v_2, v_3, v_4\}$ such that

$$a \rightarrow v_1$$
$$c \rightarrow v_4$$
$$b \rightarrow v_2$$
$$d \rightarrow v_3$$

ϕ preserves adjacency and non-adjacency of vertices.

$\psi \to E_1 \to E_2$ Is bijective.

$\therefore$ G_1 is isomorphic to Graph G_2.

ii)

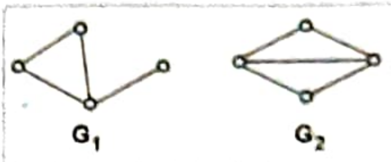


Fig. Q.18.1 (a)

As G_1 has 4 edges and G_2 has 5 edges, G_1 and G_2 are not isomorphic graphs.

iii) G_1 And G_2 are not isomorphic graphs because in G_1 vertices v_1 and v_3 of 4 degree are non adjacent while in G_2, the vertices x and y of degree 4 are adjacent.

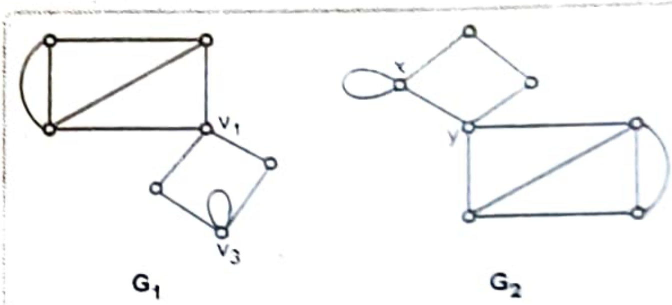


Fig. Q.18.1 (b)

Q.19 Identify whether the given graphs are isomorphic or not.

[SPPU : Dec.-12]

Ans. : In graph G_1, there are 2 vertices of degree 3. But in G_2, there is only one vertex of degree 3. So G_1 and G_2 are not isomorphic graphs.

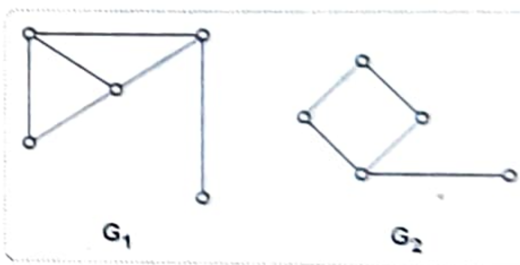


Fig. Q.19.1

ii)

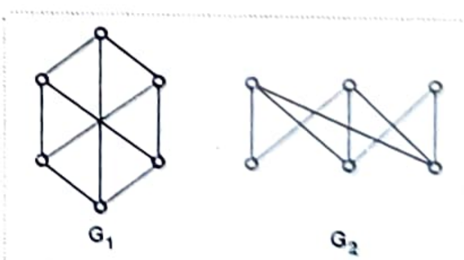


Fig. Q.19.1 (a)

Graph G_1 has 9 edges and G_2 has 8 edges.

Therefore G_1 and G_2 are not isomorphic graphs.

Q.20 Show that the following graphs are isomorphic.

[SPPU : May

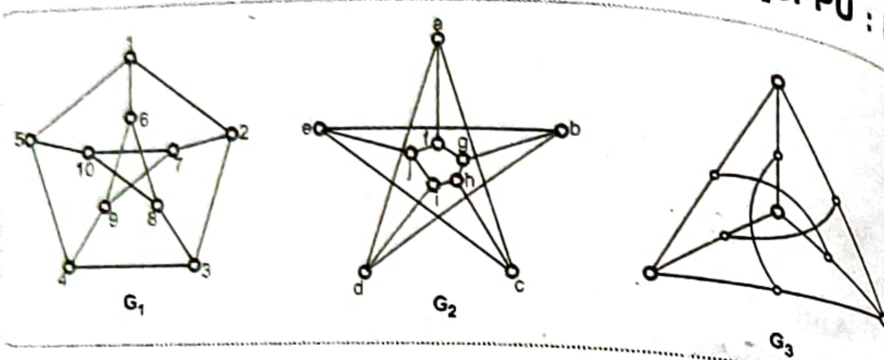


Fig. Q.20.1

Ans. : All graphs G_1, G_2 and G_3 have 10 vertices and 15 edges. All these graphs are 3-regular graphs. Also they preserve adjacen Hence all graphs are isomorphic. Isomorphism is given by

1 → f 2 → g 3 → h 4 → 5 → j 6 → a

7 → b 8 → c 9 → d 10 → e

In the similar way, we can show that G_1 and G_3 are isomorphic graphs

Q.21 Are the graphs isomorphic ? Why ?

Ans. : Given graphs G_1 and G_2 have 8 vertices and 10 edges.

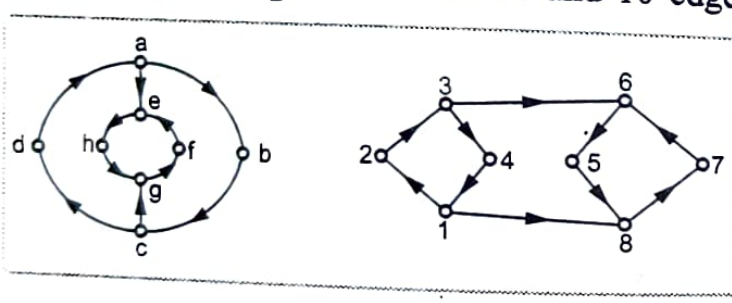


Fig. Q.21.1

Both the graphs have 4 vertices of degree 2 and 4 vertices of degree Also the adjacency is preserved.

$\phi : v(G_1) \rightarrow v(G_2)$ is defined as

a → 1, b → 2, c → 3, d → 4, e → 8, f → 5, g → 6, h → 7,

ϕ is bijective.

$\therefore$ G_1 and G_2 are isomorphic graphs.

Q.22 Explain how to obtain new graphs from old graphs with examples. **[SPPU : May-07, Dec.-09, 12]**

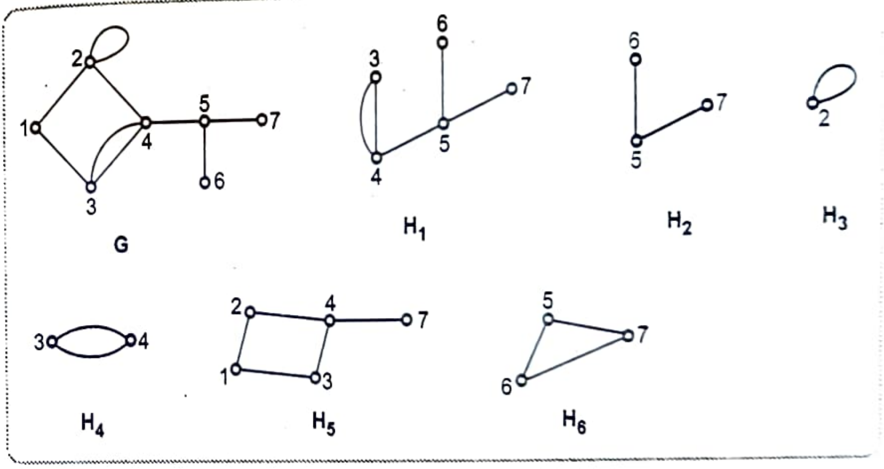
Ans. : A good mathematical theory must contain sufficient number of models and examples. Moreover it must have methods to generate new objects from old ones.

In this section we derive new graphs from old graphs.

1) **Subgraphs :** Let G (V, E) be any graph. A graph H (V_1, E_1) is said to be subgraph of G if $V_1 \subseteq V$ and $E_1 \subseteq E$.

We also say that G is a supergraph of H.

e.g.

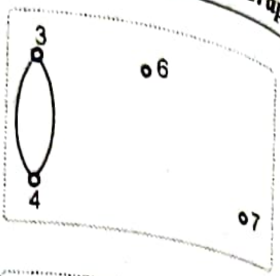


Graphs H_1, H_2, H_3 and H_4 are subgraphs of G. But graphs H_5 and H_6 are not subgraph as $(4, 7) \in E(H_5)$ but $(4, 7) \notin E(G)$ and $(6, 7) \in E(H_6)$ but $(6, 7) \notin E(G)$.

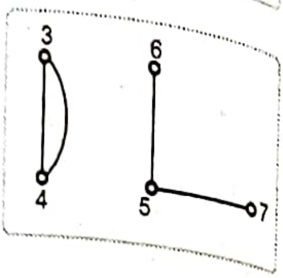
Properties :

1) Each graph is a subgraph of itself.
2) A subgraph of a subgraph of a graph G is a subgraph of G.
3) A graph G - {v} is a subgraph of G which is obtained from G by removing the vertex $v \in G$ and also the edges which are incident at v.
4) If $e \in (G)$ then G - e is a subgraph of G obtained from G by deleting the edge e.

In above example H_1 - {5} is



and H_1 - (4, 5) is given by



2) **Edge Disjoint Subgraphs :** Two subgraphs H_1 and H_2 of the graph are said to be edge disjoint subgraphs of a graph G if there is no e common between H_1 and H_2 i.e. $E(H_1) \cap E(H_2) = \phi$ It may h common vertex.

3) **Vertex Disjoint Subgraphs :** Two subgraphs H_1 and H_2 of the gr G are said to be vertex disjoint subgraphs if there is no vertex comm between H_1 and H_2 i.e. $V(H_1) \cap V(H_2) = \phi$.

Note : 1) All vertex disjoint subgraphs are edge disjoint subgraphs.

4) **Spanning Subgraph :** Let G (V, E) be any graph. A subgraph H of graph G is said to be spanning subgraph if V (G) = V (H).

Example : Let G be the following graph :

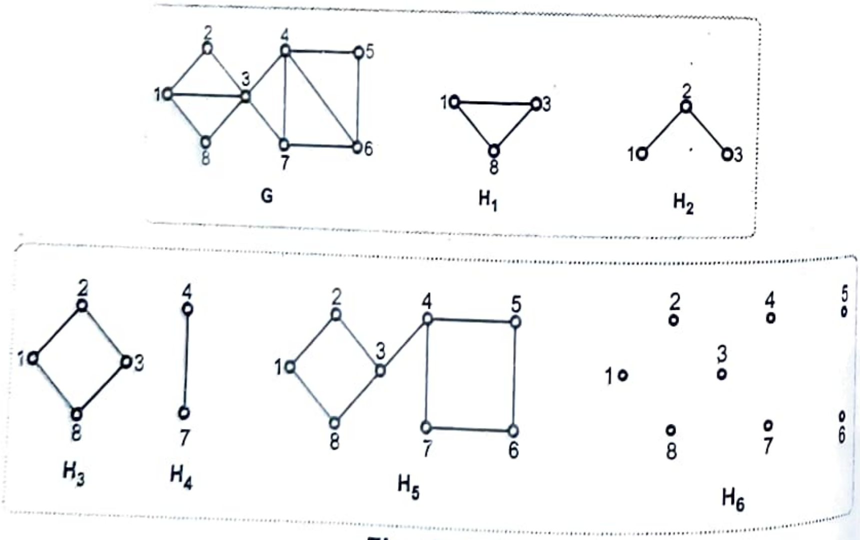


Fig. Q.22.1

Graphs H_1, H_2, ... H_6 are subgraphs of G.

H_1 and H_2 are edge disjoint subgraphs but not vertex disjoint subgraphs.

H_3 and H_4 are vertex disjoint subgraphs as well as edge disjoint subgraphs.

Subgraphs H_5 and H_6 are spanning subgraphs of G as

$V(H_5) = V(H_6) = V(G)$.

5) **Factors of a Graph :** Let G be any graph. A k-factor of a graph G is defined to be a spanning subgraph of the graph with the degree of each of its vertex being K. i.e. K-factor is a K-regular graph.

When G has a 1-factor, say G_1, if the number of vertices are even and edges of G are point disjoint.

In particular, K_{2n+1} can not have a 1-factor but K_{2n} can have 1-factor of graph.

Example 1)

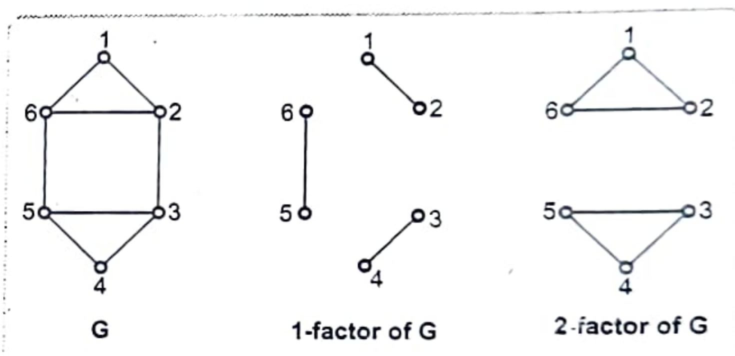


Fig. Q.22.2

Example 2)

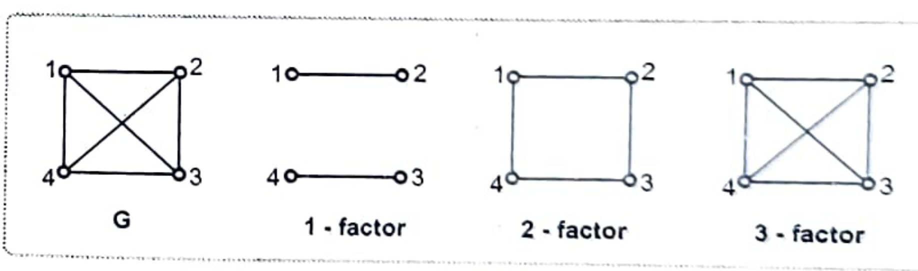


Fig. Q.22.3

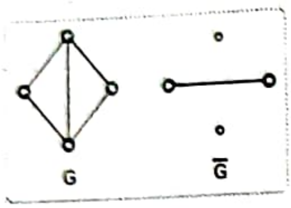
6) **Complement of a Graph :** Let G be a simple graph. The complement of G is denoted by $\overline{G}$ is the graph whose vertex set is the same as the

vertex set of G and in which two vertices are adjacent if and only if they are not adjacent in G.

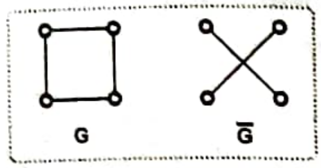
A graph is said to be self complementary graph if it is isomorphic to its complement.

e.g.

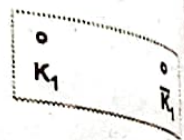
i)



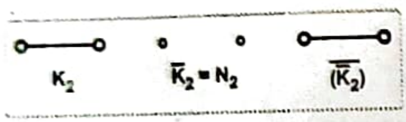
ii)



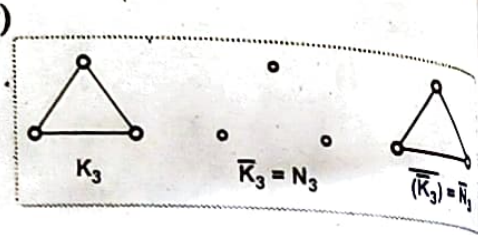
iii)



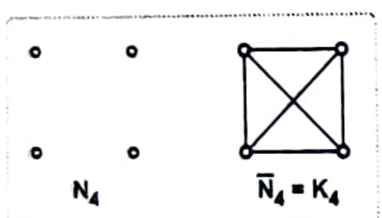
iv)



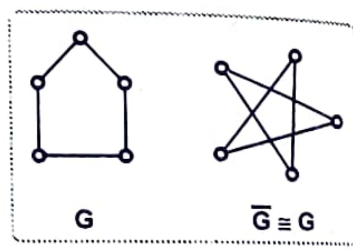
v)



vi)



vii)



G is isomorphic to $\overline{G}$. $\therefore$ G is self complementary graph.

Note :

1) For any graph G, $\overline{(\overline{G})} = G$

2) The complement of the null graph on n vertices is the complete graph K_n on n vertices and vice versa.

3) K_1 is self complementary graph.

Examples :

Q.23 For the following graphs, determine whether H (V', E') is a subgraph of G where

i) $V' = \{A, B, C\}$, $E' = \{(A, B), (A, F)\}$

ii) $V' = \{B, C, D\}$, $E' = \{(B, C), (B, D)\}$

iii) $V' = \{A, B, C, D\}$,

$E' = \{(A, C)\}$

[SPPU : May-07, Dec.09]

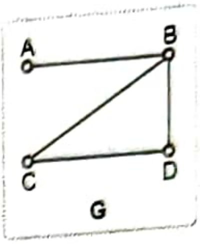


Fig. Q.23.1

Ans. : i) H is not a subgraph of G because $F \in V(H)$ but $F \notin V(G)$, so $V(H) \not\subset V(G)$

ii) Here $V' \subset V(G)$, $E' \subset E(G)$, so $H(V', E')$ is a subgraph of G.

iii) Here $V' \subset v(G)$, but $E' \not\subset E(G)$ Therefore $H(V', E')$ is not a subgraph of G.

Q.24 Draw all self complementary graphs on 5 vertices.

[SPPU : Dec.-12]

Ans. : The following graphs are self complementary graphs on 5 vertices.

Here $\overline{G_1} = G_2$ and $\overline{G_2} = G_1$

$\therefore$ G_1 as well as G_2 are self complementary graphs.

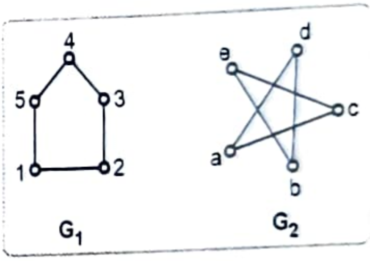


Fig. Q.24.1

Q.25 Explain operations on graphs.

Ans. : We define some standard operations of graphs like intersection, union, Ringsum etc.

A) Intersection of Two Graphs : The intersection of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is a graph G (V, E) whose vertex set is $V = V_1 \cap V_2$ and edge set is $E = E_1 \cap E_2$. The intersection of G_1 and G_2 is denoted by $G_1 \cap G_2$.

e.g.

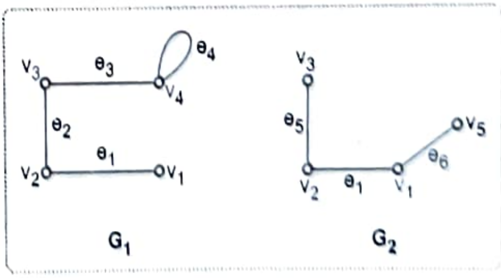


Fig. Q.25.1

$V_1 = \{v_1, v_2, v_3, v_4\}$ $V_2 = \{v_1, v_2, v_3, v_5\}$

$E_1 = \{e_1, e_2, e_3, e_4\}$ $E_2 = \{e_1, e_5, e_6\}$

Therefore $G = G_1 \cap G_2$ (v, E) where

$V = V_1 \cap V_2 = \{v_1, v_2, v_3\}$,

$E = E_1 \cap E_2 = \{e_1\}$

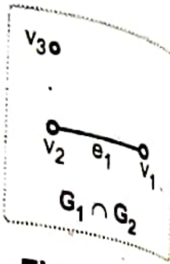


Fig. Q.25.2

B) Union of Two Graphs : Let $G_1(V_1, E_1)$, $G_2(V_2, E_2)$ be graphs. The union of G_1 and G_2 is denoted by $G_1 \cup G_2 = G$ (v, E) it is a graph whose vertex set is

$V = V_1 \cup V_2$ and Edge set is

$E = E_1 \cup E_2$

Consider the graphs G_1 and G_2 as shown in above example :

The union of G_1 and G_2 is given by G (v, E)

where $V = V_1 \cup V_2 = \{v_1, v_2, v_3, v_4, v_5\}$

$V = V_1 \cup V_2 = \{e_1, e_2, e_3, e_4, e_5, e_6\}$

Note : Both graphs G_1 and G_2 are subgraphs of $G_1 \cup G_2$.

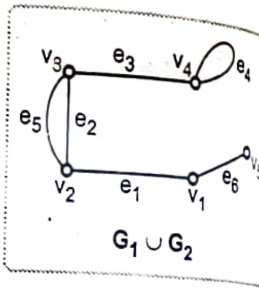


Fig. Q.25.3

C) The Ring Sum of Two Graphs : The ring sum of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is denoted by $G = G_1 \oplus G_2$ (V, E) whose vertex set is $V = V_1 \cup V_2$ and the edge set consists of those edges which are either in E_1 or in E_2 but not in both i.e. $E = (E_1 \cup E_2) - (E_1 \cap E_2)$

The ring sum of above graphs G_1 and G_2 is given by $G(V, E) = G_1 \oplus G_2$

$V = \{v_1, v_2, v_3, v_4, v_5\} = V_1 \cup V_2$

$E = (E_1 \cup E_2) - (E_1 \cap E_2)$

$= \{e_2, e_3, e_4, e_5, e_6\}$

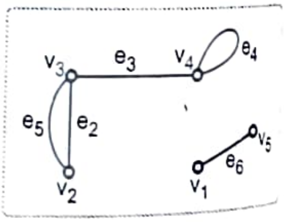


Fig. Q.25.4

D) Sum of Two Graphs : The sum of two vertex disjoint graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is denoted by $G_1 + G_2 = G$ (V, E) is defined as the graph whose vertex set is $V(G_1 \cup G_2)$ and consisting of edges which are in G_1 or G_2 together with the edges obtained by joining each vertex of G_1 to each vertex of G_2. Thus $G_1 + G_2$ is nothing but the graph $G_1 \cup G_2$ in which each vertex of G_1 is joined to each vertex of G_2 by an edge.

e.g. If

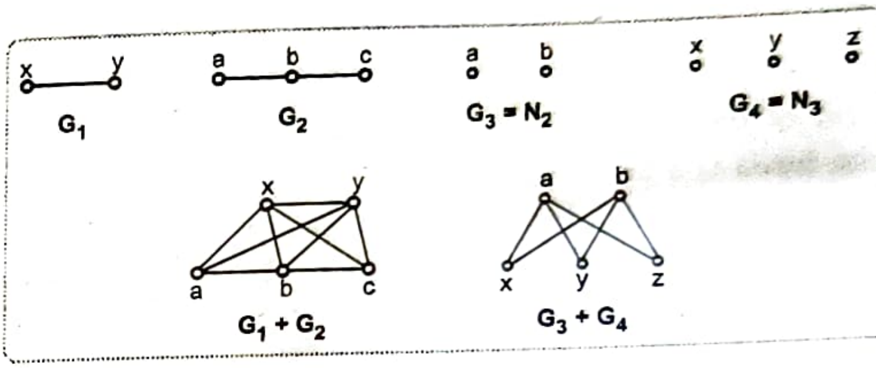


Fig. Q.25.5

Note : The sum $N_m + N_n$ of null graphs is nothing but the complete bipartite graph $K_{m,n}$.

E) Product of Two Graphs : Let G_1 (V_1, E_1) and G_2 (V_2, E_2) be two vertex disjoint graphs then the product of G_1 and G_2 is denoted by $G_1 \times G_2 = G$ (V, E) is a graph whose vertex set is $V = V_1 \times V_2$ and two edges (x_1, x_2) and (y_1, y_2) are adjacent if $x_1 = y_1$ and x_2 is adjacent to y_2 in G_2 or $x_2 = y_2$ and x_1 is adjacent to y_1 in G_1.

e.g. If

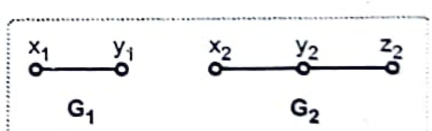


Fig. Q.25.6

Then $G_1 \times G_2$ is given below :

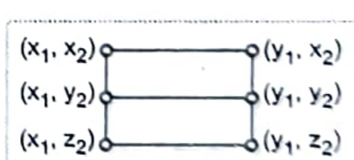


Fig. Q.25.7

F) Decomposition : A graph G is said to have been decomposed into two subgraphs H and K if $H \cup K = G$ and $H \cap K =$ Null graph i.e. each edge of G occurs either in H or in K but not in both. But vertices may occur in both. In this context isolated vertices are not considered.

e.g. The decomposition of G into H and K is given below :

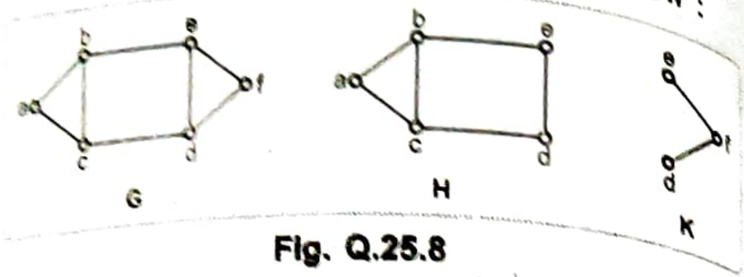


Fig. Q.25.8

G) Fusion of vertices : A pair of vertices a and b in a graph said to be fused if a and b are replaced by a single new vertex say that every edge that was incident on either a or b or both is incid the new vertex c. The fusion of two vertices do not change the num edges but reduced number of vertices by 1.

e.g.

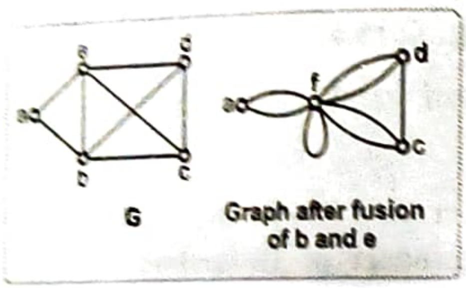


Fig. Q.25.9

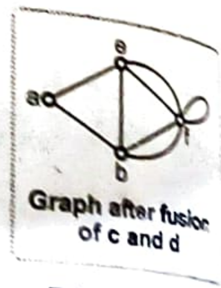


Fig. Q.25.10

Q.26 Paths and circuits with examples.

1) Path : An alternating sequence of vertices and e $v_0 - e_1 - e_2 - e_3 - \ldots v_{n-1} - e_n - v_n$ beginning are ending vertices in which each edge is incident with the two vertices immedia preceding if and following it is called a path.

The vertices v_0 and v_n are called terminal vertices and $v_1, v_2, \ldots v_{n-1}$ are called its interior vertices.

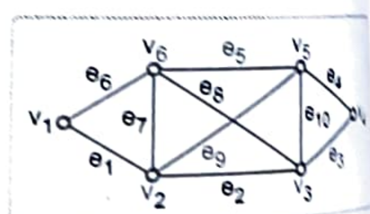


Fig. Q.26.1

e.g. Let G be the following graph.

Following are some examples of path

i) $v_1 - e_1 - v_2 - v_3$

ii) $v_2 - e_2 - v_3 - e_3 - v_4 - e_4 - v_5 - e_{10} - v_3 - e_2 - v_2$

iii) $v_6 - e_5 - v_5 - e_{10} - v_3 - e_8 - v_6$

iv) $v_1 - e_6 - v_6$

here are so many paths between every distinct pair of vertices of given raph G. Depending upon the nature of terminal vertices, there are two ypes at path.

path in which terminal vertices are equal is called a **closed path**. A losed path is known as circuit. A path in which terminal vertices are istinct, is called an **open path.**

n above examples, paths in (i) and (iv) are open paths and (ii) and iii) are closed paths.

1. Simple Path : A path in a graph G is said to be a simple path if he edges do not repeat in the path. Vertices may be repeated.

e.g. i) $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5$ is a simple path.

ii) $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5 - e_9 - v_2 - e_7 - v_6$ is a simple path in which v_3 is repeated.

iii) $v_3 - e_3 - v_4 - e_4 - v_5 - e_{10} - v_3 - e_8 - v_6$ is a simple path in which v_3 is repeated.

iv) $v_3 - e_3 - v_4 - e_4 - v_5 - e_{10} - v_3 - e_3 - v_4$ is not a simple path as an edge e_3 is repeated.

2. Elementary Path : A path in a graph G is said to be elementary path if vertices do not repeat in the path. Every elementary path is a simple path.

e.g. i) $v_1 - e_1 - v_2 - e_2 - v_3$ is an elementary path.

ii) $v_1 - e_1 - v_2 - e_2 - v_3 - e_8 - e_6 - e_7 - v_2$ is not an elementary path. But it is simple path.

3. Simple and Elementary Circuits : A closed path is known as circuit.

A simple path which is closed is called a simple circuit of graph.

In other words, A circuit in a graph G is said to be simple circuit if all edges of a circuit are distinct.

A circuit in a graph G is said to be elementary circuit if all vertices of a circuit are distinct except the terminal vertices i.e. the first and last vertices. The number of edges in any circuit (or path) is called the length of the circuit (or path).

In above graph G.

e.g. i) $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5 - e_9 - v_2 - e_1 - v_1$ is a circuit with e_1 repeated twice and v_2 is also repeated twice.

ii) $v_1 - e_1 - v_2 - e_2 - v_3 - e_{10} - v_5 - e_9 - v_2 - e_7 - v_6 - e_6 - v_1$ simple circuit but not elementary circuit as v_2 is repeated.

iii) $v_1 - e_1 - v_2 - e_7 - v_6 - e_6 - v_1$ is an elementary circuit.

Q.27 Define connected and disconnected graphs.

Ans. : A graph G is said to be connected graph if there exists a between every pair of vertices. A graph which is not connected is c the disconnected graph.

A disconnected graph consists of two or more parts called componen blocks if each of which is connected and there is no path between vertices if they belong to different components.

Q.28 Explain edge and vertex connectivity.

Ans. : Edge Connectivity : A set of edges of a connected graph whose removal disconnects G is called a disconnecting set of G. A cu is defined as a minimal disconnecting set i.e. A minimal set of ed whose removal from G gives a disconnected graph is called a cutset.

If a cutset of a graph contains only one edge, then that edge is called as an isthmus or bridge. The number of edges in the smallest cutset of G is called the edge connectivity of G. It is denoted by λ (G).

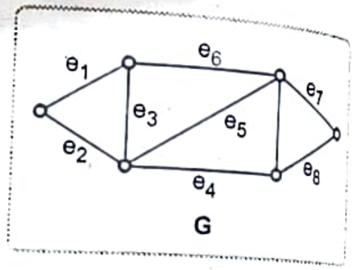


Fig. Q.28.1

e.g. Consider the following graph G.

Cutsets of G are as follows :

i) $\{e_4, e_5, e_6\}$, ii) $\{e_1, e_3, e_6\}$, iii) $\{e_1, e_2\}$,

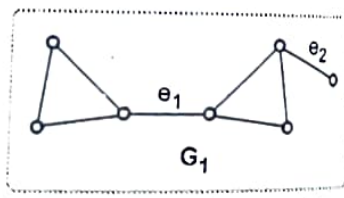


Fig. Q.28.2

A set $\{e_1, e_2, e_3\}$ is not a cutset because its subset $\{e_1, e_2\}$ is a cutset. The edge connectivity of graph G is 2. i.e. $\lambda (G) = 2$.

Consider the following graph G_1.

Graph G_1 has edge connectivity 1 as $G_1 - \{e_1\}$ is a disconnected graph. $\therefore$ e_1 is an isthmus or Bridge.

$G - e_2$ is also disconnected graph. $\therefore$ e_2 is also isthmus.

2) Vertex Connectivity : The vertex connectivity k(G) of a simple connected graph G is defined as the smallest number of vertices whose removal disconnects the graph.

In graph G, the sets $\{v_2, v_5, v_4\}$, $\{v_2, v_3, v_4, v_5\}$, $\{v_2, v_3\}$ disconnect graph G. The smallest set is $\{v_2, v_3\}$

$\therefore$ k (G) = 2.

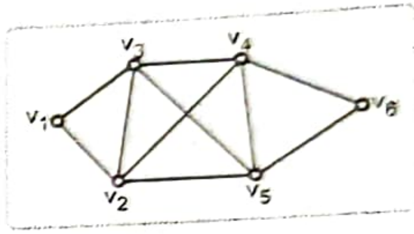


Fig. Q.28.3

1) A graph G is said to be k-connected if its vertex connectivity is k.
2) A graph G is said to be seperable graph if its vertex connectivity is one.
3) A vertex v of a connected graph G is said to be cut vertex if G − {v} is a disconnected graph.
4) $k(G) \leq \lambda(G) \leq \delta$

i.e. vertex connectivity ≤ edge connectivity ≤ minimum degree of a vertex in G and

$$\lambda(G) \leq \left[\frac{2e}{n}\right]$$

where e = Number of edges in G. n = Number of vertices in G.

Q.29 Explain shortest path algorithm and Dijkstra's algorithm.

☞[SPPU : May-05, 07, 14, 15, Dec.-06, 07, 12, 13, 14, 15]

Ans. : Suppose there is associated to each edge e of a graph a real number w (e). w (e) is called the weight of e. A weighted graph is a graph in which each edge has a weight. The weight of graph G is the sum of weight of all edges of G. Weighted graph has many applications in communication networks. Given a railway network connecting several cities, determine a shortest route between two cities.

We consider the weighted graph where the vertices are the towns, rail roads are the edges and the weight represent the distance between directly linked cities. Therefore weights are non negative integers. The problem is to find a path of minimum weight connectivity two given cities. Of course, this is possible theoretically. One has to list all paths, find their weights and select minimum one. But for large networks (large number of vertices and edges) this may not be efficient. So we required different method to take such problems. The algorithm was found by Dijkshtra (1959) and is known as Dijkshtra's algorithm.

A) Dijkshtra's algorithm to find the shortest path from the vertex a to vertex z of a graph G. Let G (v, E) be a simple graph and a, z ∈ V.

Suppose L(x) is the label of the vertex which represents the length of shortest path from the vertex a. W_{ij} = Weight of an edge $e_{ij} = (v_i, v_j)$

Consider the following steps :

Step 1 : Let P be the set of those vertices which have permanent labels and T = set of all vertices of G.

$$\text{Set } L(a) = 0,\ L(x) = \infty\ ;\ \forall\, x \in T \text{ and } x \neq a$$

$$P = \phi \text{ and } T = v.$$

Step 2 : Select the vertex v in T which has the smallest label. This label is called the permanent label of v. Also set P as P ∪ {v} and T as T - {v}.

If v = z then L (z) is the length of the shortest path from the vertex a to z and stop the procedure.

Step 3 : If v ≠ z, then revise the labels of the vertices of T. i.e. The vertices which do not have permanent labels.

The new label of x in T is given by

$$L(x) = \min \{\text{old } L(x),\ L(v) + w(v, x)\}$$

where w (v, x) is the weight of the edge joining v and x. If there is no edge joining v and x then take w (v, x) = ∞.

Step 4 : Repeat the steps 2 and 3 until z gets the permanent label.

Examples :

Q.30 Use Dijkstra's algorithm to find the shortest path between a and z.

[SPPU : May-05, 14, 8 Marks, Dec.-06, 6 Marks]

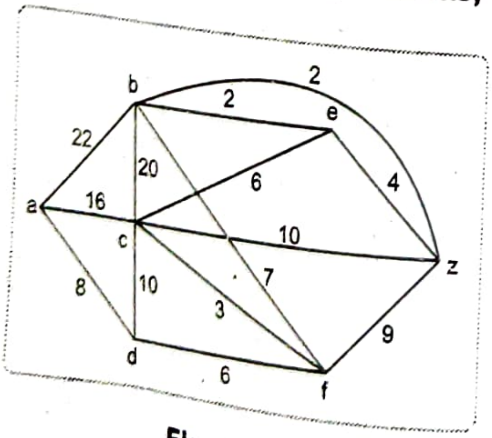


Fig. Q.30.1

Ans. :

Step 1 :
$$P = \varnothing, \quad T = \{a, b, c, d, e, f, z\}$$
$$L\{a\} = 0$$
$$L\{x\} = \infty, \quad \forall\ x \in T, \quad x \neq a$$

Step 2 :
v = a, the permanent label of a is 0
$$P = \{a\}, \quad T = \{b, c, d, e, f, z\}$$
$$L\{b\} = \min\{\text{old } L(b), L(a) + w(a, b)\}$$
$$= \min\{\infty, 0 + 22\} = 22$$
$$L\{c\} = \min\{\infty, 0 + 16\} = 16$$
$$L\{d\} = \min\{\infty, 0 + 8\} = 8$$
$$L\{e\} = \min\{\infty, 0 + \infty\} = \infty$$
$$L\{f\} = \min\{\infty, 0 + \infty\} = \infty$$
$$L\{z\} = \min\{\infty, 0 + \infty\} = \infty$$
$\therefore$ $L\{d\}$ = 8 is the minimum label.

Step 3 :
v = d, the permanent label of d is 8.
$$P = \{a, d\}, \quad T = \{b, c, e, f, z\}$$
$$L\{b\} = \min\{\text{old } L(b), L(d) + w(d, b)\}$$
$$= \min\{22, 8 + \infty\} = 22$$
$$L\{c\} = \min\{16, 8 + 10\} = 16$$
$$L\{e\} = \min\{\infty, 8 + \infty\} = \infty$$
$$L\{f\} = \min\{\infty, 8 + 6\} = 14$$
$$L\{z\} = \min\{\infty, 8 + \infty\} = \infty$$
$\therefore$ $L\{f\}$ = 14 is the minimum label.

Step 4 :
v = f, the permanent label of f is 14.
$$P = \{a, d, f\}, \quad T = \{b, c, e, z\}$$
$$L\{b\} = \min\{\text{old } L(b), L(f) + w(b, f)\}$$
$$= \min\{22, 14 + 7\} = 21$$
$$L\{c\} = \min\{16, 14 + 3\} = 16$$
$$L\{e\} = \min\{\infty, 14 + \infty\} = \infty$$

$$L\{z\} = \min\{\infty, 14 + 9\} = 23$$

$\therefore$ $L\{c\} = 16$ is the minimum label.

Step 5 : $v = c$, the permanent lable of c is 16.

$P = \{a, d, f, c\}, \quad T = \{b, e, z\}$

$$L\{b\} = \min\{\text{old } L(b), L(f) + w(f, b)\}$$
$$= \min\{21, 16 + 20\} = 21$$
$$L\{e\} = \min\{\infty, 16 + 6\} = 22$$
$$L\{z\} = \min\{23, 16 + 10\} = 23$$

$\therefore$ $L\{b\} = 2$ is the minimum label.

Step 6 : $v = b$, the permanent label of b is 21.

$P = \{a, d, f, c, b\}, \quad T = \{e, z\}$

$$L\{e\} = \min\{\text{old } L(e), L(b) + w(e, b)\}$$
$$= \min\{22, 21 + 2\} = 22$$
$$L\{z\} = \min\{23, 21 + 2\} = 23$$

$\therefore$ $L\{e\} = 22$ is the minimum label.

Step 7 : $v = e$, the permanent label of e is 22.

$P = \{a, d, f, c, b, e\}, \quad T = \{z\}$

$$L(z) = \min\{\text{old } L(z), L(e) + w(e, z)\}$$
$$= \min\{23, 22 + 4\} = 23 \text{ which is the minimum label}$$

Step 8 : $v = z$, the permanent label of z is 23.

Hence the length of the shortest path from a to z is 23.

The shortest path is adfz or adfbz.

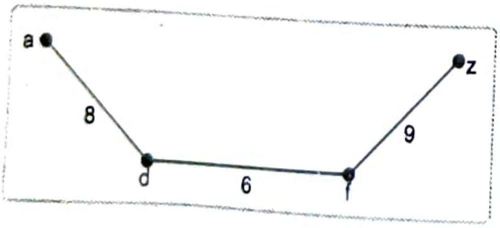


Fig. Q.30.1 (a)

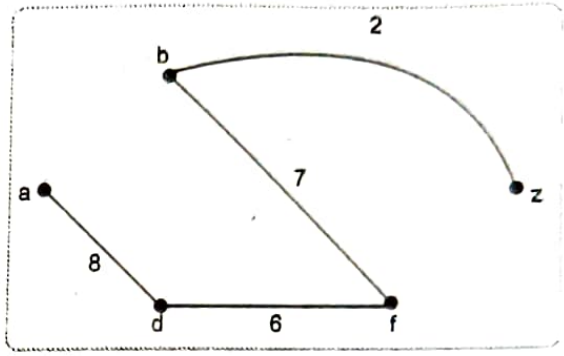


Fig. Q.30.1 (b)

Q.31 Find the shortest path from a-z in the given graph using Dijkstra's algorithm. [SPPU : May-07, Dec.-07, 15]

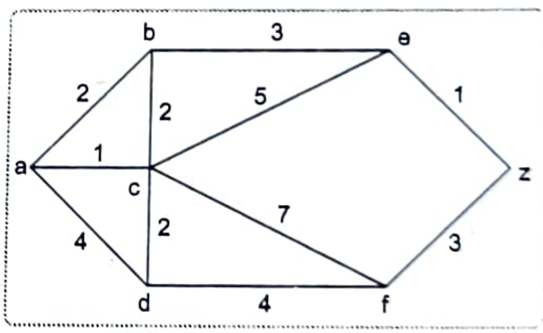


Fig. Q.31.1

Ans. : Step 1 :

Set $P = \varnothing$, $T = \{a, b, c, d, e, f, z\}$

$L\{a\} = 0$

$L\{x\} = \infty, \ \forall x \in T, \ x \neq a$

Step 2 : $v = a$, the permanent label of a is 0.

$P = \{a\}, \quad T = \{b, c, d, e, f, z\}$

$L\{b\} = \min\{\text{old } L(b), L(a) + w(a, b)\}$

$= \min\{\infty, 0 + 2\} = 2$

$L\{c\} = \min\{\infty, 0 + 1\} = 1 \quad L\{d\} = \min\{\infty, 0 + 4\} = 4$

$L\{e\} = \min\{\infty, 0 + \infty\} = \infty \quad L\{f\} = \min\{\infty, 0 + \infty\} = \infty$

$L\{z\} = \min\{\infty, 0 + \infty\} = \infty \ \therefore L\{c\} = 1$ is the minimum label.

Step 3 : v = c, the permanent label of c is 1.

P = {a, c}, T = {b, d, e, f, z}

L {b} = min {2, 1 + 2} = 2   L {d} = min {4, 1 + 2} = 3

L {e} = min {∞, 1 + 5} = 6   L {f} = min {∞, 1 + 7} = 8

L {z} = min {∞, 1 + ∞} = ∞ ∴ L {b} = 2 is the minimum label.

Step 4 : v = b, the permanent label of b is 2.

P = {a, c, b}, T = {d, e, f, z}

L {d} = min {3, 2 + ∞} = 3   L {e} = min {6, 2 + 3} = 5

L {f} = min {8, 2 + ∞} = 8   L {z} = min {∞, 2 + ∞} = ∞

∴ L {d} = 3 is the minimum label.

Step 5 : v = d, the permanent label of d is 3.

P = {a, c, b, d}, T = {e, f, z}

L {e} = min {5, 3 + ∞} = 5   L {f} = min {8, 3 + 4} = 7

L {z} = min {∞, 3 + ∞} = ∞ ∴ L {e} = 5 is the minimum label.

Step 6 : v = e, the permanent label of e is 5.

P = {a, c, b, d, e}, T = {f, z}

L {f} = min {7, 5 + ∞} = 7   L {z} = min {∞, 5 + 1} = 6

∴ L {z} = 6 is the minimum label.

Step 7 : v = z, the permanent label of z is 6.

Hence the length of shortest path from a to z is 6.

The shortest path is a b e z.

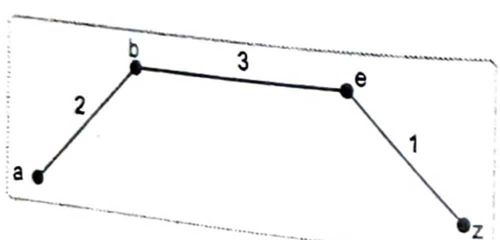


Fig. Q.31.1 (a)

Q.32 Find the shortest path between a-z for the given graph : using Dijkstra's algorithm.

[SPPU : Dec.-12, 13, 14, May-15, 8 Marks]

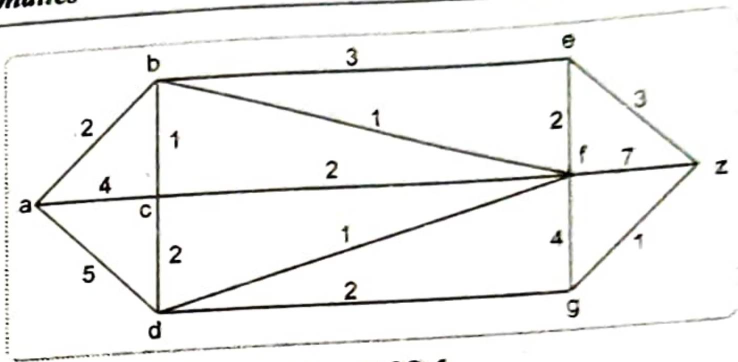


Fig. Q.32.1

Ans. :

Step 1 : Set $P = \varnothing$, $T = \{a, b, c, d, e, f, g, z\}$

$L\{a\} = 0$

$L\{x\} = \infty$, $\forall x \in T$, $x \neq a$.

Step 2 : $v = a$, the permanent label of a is 0.

$P = \{a\}$, $T = \{b, c, d, e, f, g, z\}$

$L\{b\} = \min\{\text{old } L(b), L(a) + w(a, b)\} = \min\{\infty, 0 + 2\} = 2$

$L\{c\} = \min\{\infty, 0 + 4\} = 4$ $L\{d\} = \min\{\infty, 0 + 5\} = 5$

$L\{e\} = L\{f\} = L\{g\} = L\{z\} = \infty$

$\therefore$ $L\{b\} = 2$ is the minimum label. The permanent label of b is 2.

Step 3 : $v = b$

$P = \{a, b\}$, $T = \{c, d, e, f, g, z\}$

$L\{c\} = \min\{L(c), L(b) + w\{b, c\}\} = \min\{4, 2 + 1\} = 3$

$L\{d\} = \min\{5, 2 + \infty\} = 5$ $L\{e\} = \min\{\infty, 2 + 3\} = 5$

$L\{f\} = \min\{\infty, 2 + 1\} = 3$ $L\{g\} = L\{z\} = \infty$

$\therefore$ $L\{c\} = L\{f\} = 3$ are the minimum labels.

Step 4 : $v = c$ or f

Let $v = f$, permanent label of f is 3.

$P = \{a, b, f\}$, $T = \{c, d, e, g, z\}$

$L\{c\} = \min\{3, 3 + 2\} = 3$ $L\{d\} = \min\{5, 3 + 1\} = 4$

$L\{e\} = \min\{5, 3 + 2\} = 5$ $L\{g\} = \min\{\infty, 3 + 4\} = 7$

$L\{z\} = \min\{\infty, 3 + 7\} = 10$

$\therefore$ $L\{c\} = 3$ is the minimum label.

Step 5 : v = c, permanent label of c = 3.

P = {a, b, f, c}, T = {d, e, g, z}

L {d} = min {4, 3 + 2} = 4 L {e} = min {5, 3 + ∞} = 5

L {g} = min {7, 3 + ∞} = 7 L {z} = min {10, 3 + ∞} = 10

∴ L {d} = 4 is the minimum label.

Step 6 : v = d, permanent label of d = 4.

P = {a, b, f, c, d}, T = {e, g, z}

L {e} = min {5, 4 + ∞} = 5 L {g} = min {7, 4 + 2} = 6

L {z} = min {10, 4 + ∞} = 10 ∴ L {e} = 5 is the minimum label.

Step 7 : v = e, permanent label of e is 5.

P = {a, b, f, c, d, e}, T = {g, z}

L {g} = min {6, 5 + ∞} = 6 L {z} = min {10, 5 + 3} = 8

∴ L {g} = 6 is the minimum label.

Step 8 : v = g, permanent label of g is 6.

P = {a, b, f, c, d, e, g}, T = {z}

L {z} = min {8, 6 + 1} = 7 which is the minimum label.

Step 9 : v = z, permanent label of z is 7.

Hence the length of shortest path from a to z is 7.

The shortest path is a b f d g z.

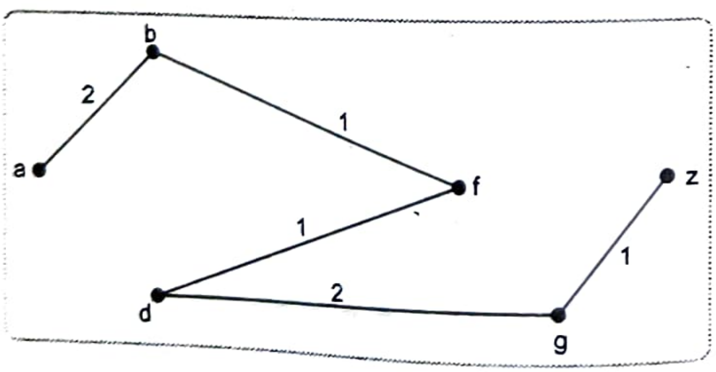


Fig. Q.32.2

Q.33 Define Eulerian path and circuit.

Ans. : A path is called an Eulerian path if every edge of graph G appears exactly once in the path.

A circuit of a graph which contains every edge of graph exactly once is called the Eulerian circuit.

A graph which has an Eulerian circuit is called as Eulerian graph.

The problem of find Eulerian path is the same as the problem of drawing a network without lifting the pencil off the paper and without retracing an edge.

Consider the following graphs :

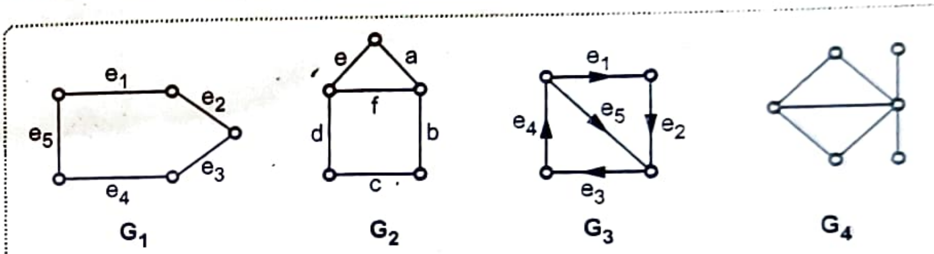


Fig. Q.33.1

In graph G_1, Eulerian circuit is $e - e_2 - e_3 - e_4 - e_5 - e_1$

$\therefore G_1$ is an Eulerian graph.

In graph G_2, Eulerian circuit does not exist.

$\therefore G_2$ is not Eulerian graph.

In graph G_3, Eulerian path is $e_1 \rightarrow e_2 \rightarrow e_3 \rightarrow e_4 \rightarrow e_4 \rightarrow e_5$ but G_3 does not have any Eulerian circuit.

$\therefore G_3$ is not Eulerian graph.

G_4 is also not an Eulerian graph.

The existence of Eulerian paths and circuits in a graph depends upon the degree of vertices.

Theorem 1 : An undirected graph possesses an Eulerian path iff it is connected and has either zero or two vertices of odd degree.

Theorem 2 : An undirected graph possesses an Eulerian circuit iff it is connected and its vertices are all of even degree.

Theorem 3 : A directed graph possesses an Eulerian circuit iff it is connected and incoming degree of every vertex is equal to its outgoing degree.

Examples :

Q.34 a) Find under what conditions $K_{m,n}$ the complete bipartite graph will have an Eulerian circuit.
b) What is the complement of $K_{m,n}$? ☞[SPPU : Dec.-09]

Ans. : a) In $K_{m,n}$ consider the following cases.

Case 1 : m = n and both m and n are even :

In this case, degree of each vertex is even, Hence by theorem 1, $K_{m,n}$ will have an Eulerian circuit. For example $K_{1,2}$ and $K_{4,4}$

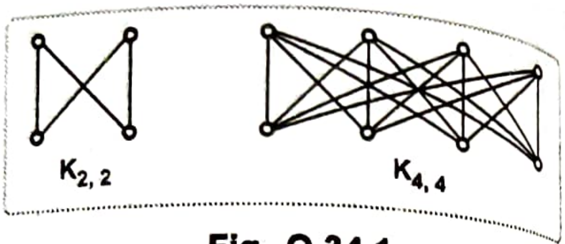


Fig. Q.34.1

Case 2 : If m = n and m, n are odd :

In this case degree of each vertex is odd. Hence Eulerian circuit will not exist.

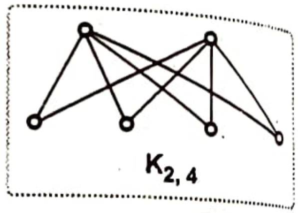


Fig. Q.34.2

Case 3 : If m ≠ n but m and n are even :

In this case, degree of each vertex is even. So there exists an Eulerian circuit.

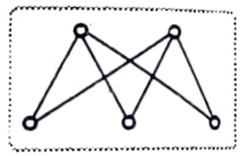


Fig. Q.34.3

Case 4 : If m ≠ n and either m is odd or n is odd or both are odd : Then graph will have vertices of odd degree. Hence Eulerian circuit does not exist. e.g. K_2, 3.

b) The complement of $K_{m,n}$ is the two graphs K_m and K_n.

Q.35 Define Hamiltonian graphs. ☞[SPPU : Dec.-04, 09, 10, 12, 15]

Ans. : In this section, we introduce a class of graphs which possess a strinking similarity to Eulerian graphs.

We will now define Hamiltonian path and circuits of the connected graph.

A path in a connected graph G is called a Hamiltonian path if it contains every vertex of G exactly once.

A circuit in a connected graph G is called a Hamiltonian circuit if it contains every vertex of G exactly once.

A graph which has a Hamiltonian circuit is called a Hamiltonian Graph.

Consider the following graphs :

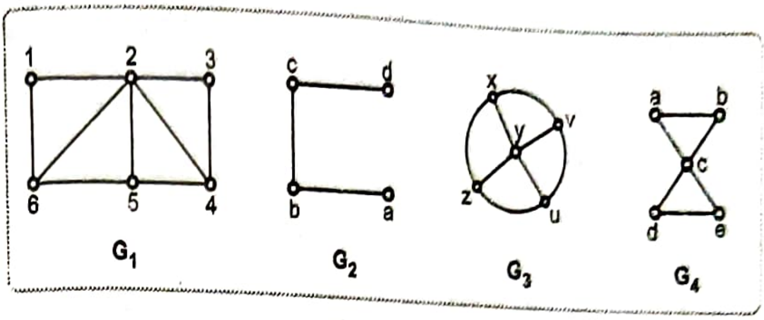


Fig. Q.35.1

In graph G_1, Hamiltonian circuit is 1-2-3-4-5-6-1

$\therefore G_1$ is a Hamiltonian graph.

In graph G_2, Hamiltonian path is a-b-c-d but Hamiltonian circuit does not exist.

$\therefore G_2$ is not Hamiltonian graph.

In graph G_3, Hamiltonian circuit is x-y-z-u-v-x

$\therefore G_3$ is Hamiltonian graph but it is not Eulerian.

In graph G_4, Hamiltonian path is a-b-c-d-e but Hamiltonian circuit does not exist.

$\therefore G_4$ is not Hamiltonian but Eulerian graph.

Important theorems :

Theorem 1 : Let G be a simple connected graph on n vertices. If the sum of the degree of each pair of vertices in G is $(n - 1)$ or large then there exists a Hamiltonian path in G.

Theorem 2 : If G (v, E) is a simple connected graph on n vertices and $d(v) = \frac{n}{2}$; $\forall v \in V$ then G will contain a Hamiltonian circuit.

This condition is sufficient condition but not necessary.

Theorem 3 : Let G (v, E) be a connected simple graph. If G has a Hamiltonian circuit then for every proper non empty subsets of v, the components in the graph G-S is less than or equal to the number of vertices in S.

Theorem 4 : A Hamiltonian graph contains no cut vertices and hence is 2-connected.

Q.36 Show that the complete bipartite graph $K_{m, n}$ is Hamiltonian for m = n and for m ≠ n, $K_{m, n}$ is not Hamiltonian graph.

Ans. : In a complete bipartite graph $K_{m, n}$ for m = n i.e. $K_{n, n}$, degree of each vertex is n.

Therefore $d(v) \geq \frac{n}{2}$ for all $v \in v(K_{n, n})$

By theorem 2, G contains a Hamiltonian circuit.

Hence $K_{n, n}$ is a Hamiltonian graph.

If m ≠ n, Let V_1 and V_2 be the partitions of the vertex set of $k_{m, n}$ where $|v_1| = m$ and $|v_2| = n$. Without loss of generality assume that m < n.

The graph $K_{m, n} - V_1$ is a null graph on n vertices.

Hence it is a disconnected graph with n components.

Therefore the number of components in $K_{m, n} - V_1 = n$ which is greater than the number of vertices in V_1

Hence by theorem 3, $K_{m, n}$ does not contain a Hamiltonian circuit when m ≠ n.

Q.37 Show that the complete graph K_n (n ≥ 3) is a Hamiltonian graph. What is the length of that circuit ? How many circuits exist in K_n ? What is the complement of K_n ? **[SPPU : Dec.-09]**

Ans. : The complete graph K_n has n vertices, n ≥ 3 and degree of each vertex is

n − 1. As n ≥ 3.

$$d(v) = n - 1 \geq \frac{n}{2} ; \forall v \in V(K_n).$$

Therefore by theorem 2, K_n has a Hamiltonian circuit. Hence K_n is a Hamiltonian graph.

Hamiltonian circuit contains all vertices of graph and length of circuit is the number of vertices present it. Hence in K_n, the length of the Hamiltonian circuit is n and there are $\frac{(n-1)!}{2}$ Hamiltonian circuits in K_n.

The complement of K_n is the null graph on n vertices.

Q.38 Find the Hamiltonian path and circuit in $K_{4, 3}$?

[SPPU : Dec.-04]

Ans. : The complete bipartite graph $K_{4,3}$ is given by

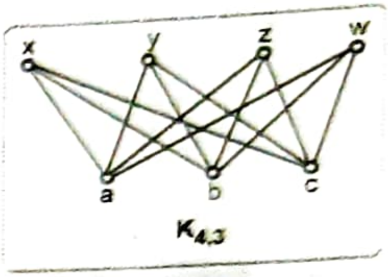


Fig. Q.38.1

In $K_{4,3}$, $4 \neq 3$ Hence it does not contain Hamiltonian circuit. Here degree of each vertex is either 3 or 4.

$\therefore$ For x, y any two vertices in $K_{4,3}$ $d(x) + d(y) = 7.1 = 6$

Hence by theorem 1, the graph $K_{4,3}$ has a Hamiltonian path. It is given by x a y b z c w.

Q.39 Give an example of the following graphs

a) Eulerian but not Hamiltonian. b) Hamiltonian but not Eulerian. c) Eulerian as well as Hamiltonian. d) Neither Eulerian nor Hamiltonian.

Ans. : a) Eulerian but not Hamiltonian graph.

Eulerian circuit : a b c d e c a

No Hamiltonian circuit.

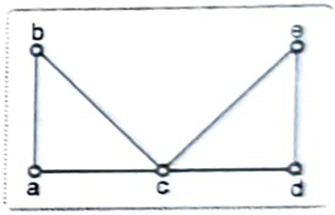


Fig. Q.39.1

b) Hamiltonian but not Eulerian

Hamiltonian circuit : abcdea

No Eulerian circuit because d (b) = 3.

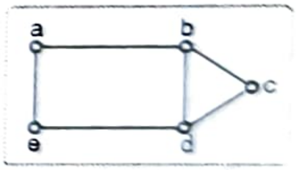


Fig. Q.39.1 (a)

c) Eulerian and Hamiltonian graph.

Hamiltonian circuit : a-b-c-a, 1-2-3-4-1

Eulerian circuit : a-b-c-a, 1-2-3-4-1

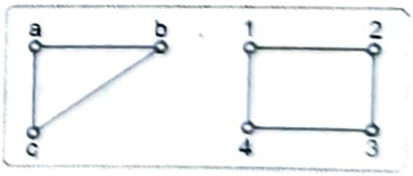


Fig.Q.39.1 (b)

d) Neither Eulerian nor Hamiltonian

Fig. Q.39.1 (c)

No Hamiltonian circuit and no Eulerian circuit.

Q.40 Determine, if the following graphs are having the Hamiltonian circuit or path. Justify your answer. [SPPU : Dec.-12]

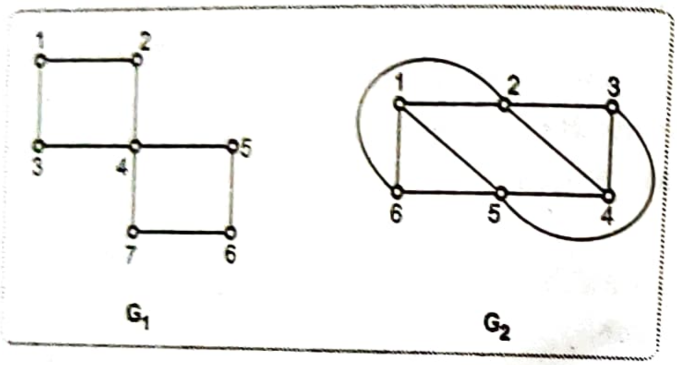


Fig. Q.40.1

Ans. : In graph G_1, there are n = 7 vertices.

d (4) = 4 and all remaining vertices is 2.

So to draw Hamiltonian circuit, we have to visit vertex 4 twice. Which is not possible in Hamiltonian path. G_1 has no Hamiltonian circuit. Hamiltonian path is 1-2-3-4-5-6-7.

In graph G_2, there are 6 vertices and degree of each vertex is 3 or 4.

If we consider two vertices of lowest degree then also their sum is 6 which is equal to the number of vertices. So there exists a Hamiltonian path in G_2. $\therefore$ Path is 1-2-3-4-5-6.

In graph G_2, $d(x) = \frac{6}{2} = 3 ; x \in V(G_2)$

$\therefore$ By theorem 2, $\exists$ a Hamiltonian circuit

$\therefore$ Hamiltonian circuit is 1-2-3-4-5-6-1.

Q.41 Which of the following have a Euler circuit or path or Hamiltonian cycle ? Write the path or circuit. [SPPU : Dec.-10]

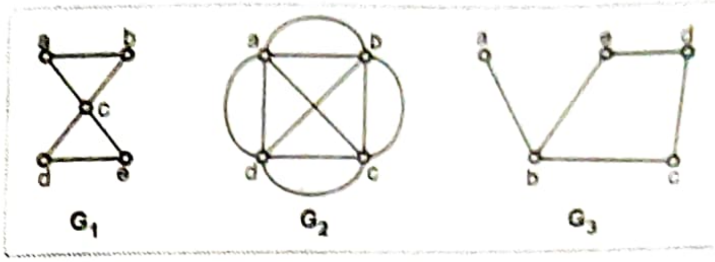


Fig. Q.41.1

Ans. : In graph G_1, degree of each vertex is an even so ∃ an Eulerian circuit which is a-b-c-d-e-c-a.

In graph G, there are 5 vertices and degree sum of every pair of vertices is 4 or greater than 4. Hence there exists a Hamiltonian path in G_1 which is given by a-b-c-d-e. But there is no any Hamiltonian circuit as vertex c is a vertex. In graph G_2, degree of each vertex is 5 which is odd integer, so there is no Eulerian path in G_2, degree of each vertex is $5 > \frac{4}{2}$. Hence there exixts a Hamiltonian circuit which is given by a-b-c-d-a.

In G_3, Eulerian path is a-b-c-d-e-b No Eulerian circuit as d (a) = 1. Hamiltonian path is a-b-c-d-e.

No Hamiltonian cyle because b is a cut vertex.

3.3 : Travelling Salesman Problem (TSP)

Q.42 Define the Travelling Salesman Problem (TSP).

Ans. : A salesman is required to travel a number of cities during a trip. Given the distance among cities, in what order should he travel so that he travels as minimum distance as possible ? This is known as Travelling Salesman Problem (TSP).

In terms of graph theory, the TSP is to a Hamiltonian circuit with the smallest weight. In the case of K_n the problem can be solved theoretically by listing all the possible Hamiltonian circuits and select one which has least weight. But this method is highly impractical for the large graphs. In fact no efficient algorithm is there to solve TSP. It is therefore desirable to obtain a reasonably good but not an optimal solution.

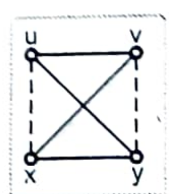


Fig. Q.42.1

One possible approach is to first find a Hamiltonian cycle and search for other Hamiltonian cycles of lesser weight. The simple method is as follows :

Let C be the Hamiltonian circuit of a graph G.

Let further uv and xy be two no-adjacent edges of C such that the vertices u, v, x and y occur in that order in C.

If ux and vy are edges such that v (u x) + w (y v) < w (uv) + w (yx) then replace the edges uv and xy in (by ux and vy). The new cycle C' would still be Hamiltonian cycle and w (C') < w (C). This process can be continued until one gets a reasonably good Hamiltonian cycle.

Q.43 Explain nearest neighbour method.

Ans. : In this method, we start with any arbitrary vertex and find the vertex which is nearest to it. Continuing this way and coming back to the starting vertex by travelling through all the vertices exactly once, we will get Hamiltonian cycle or circuit.

Consider the following steps to find Hamiltonian cycle by this method.

Step 1 : Start with any arbitrary vertex say v_1, choose the vertex closest to v_1 to form an initial path of one edge. Construct this path by selecting different vertices as described in step 2.

Step 2 : Let v_n be the latest vertex that was added to the path. Select the vertex v_{n+1} closest to v_n from all vertices that are not in the path and add this vertex to the path. Select those vertices which will not form a circuit in this stage.

Step 3 : Repeat step (2) till all the vertices of G are included in the path.

Step 4 : Lastly form a circuit by adding the edge connecting to v_1 and the last added vertex.

The circuit obtained using the nearest neighbour method will be the required Hamiltonian circuit.

Note : If we start with an arbitrary vertex in TSP then we may or may not minimum Hamiltonian circuit, But if we start with a vertex whose incident edge has the minimum weight in graph then we will get minimum Hamiltonian circuit as compared with arbitary starting vertex. For more details see Q.44.

Examples :

Q.44 Use nearest neighbour method to find the Hamiltonian circuit starting from a in the following graph. Find its weight.

[SPPU : Dec.-15]

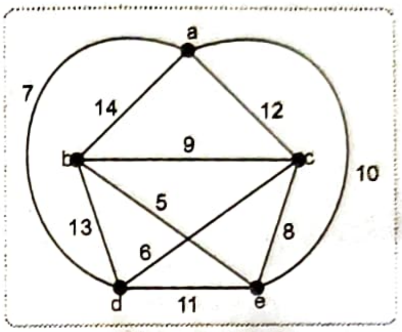


Fig. Q.44.1

Ans. : Step 1 : Let a be the starting vertex. Vertex a is adjacent to b, c, d, e. But minimum path is {a, d} which is the initial path.

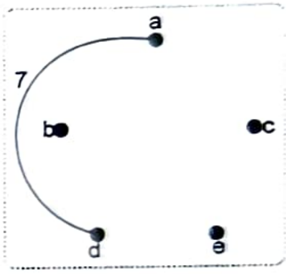


Fig. Q.44.1 (a)

Step 2 : There are three vertices adjacent to d. but closest one is C

∴ The path is {a, d, c}.

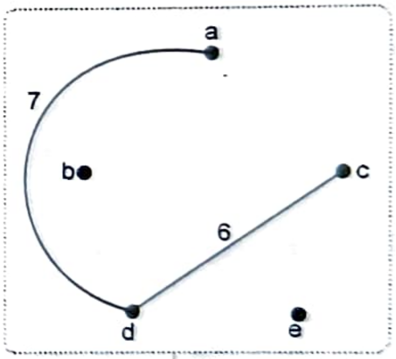


Fig. Q.44.1 (b)

Step 3 : There are 4 vertices adjacent to c. but closest is e

∴ The path is {a, d, c, e}.

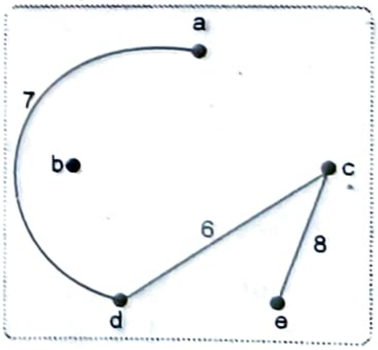


Fig. Q.44.1 (c)

Step 4 : There are 4 vertices adjacent to e. but closest is b

∴ The path is {a, d, c, e, b}.

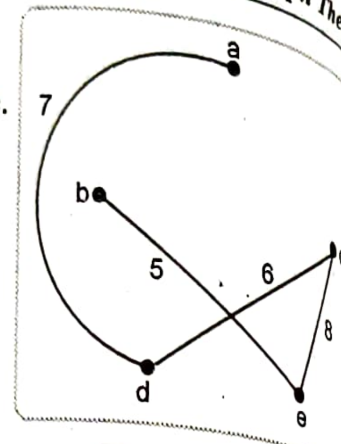


Fig. Q.44.1 (d)

Step 5 : Here are all vertices are corered so to complete Hamiltonian an circuit there should be a path from b to a.

∴ Hamiltonian circuit is {a, d, c, e, b, a}

Weight of the Hamiltonian circuit = 40.

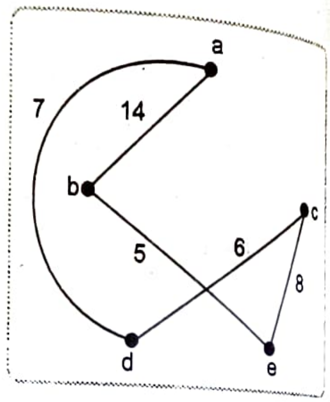


Fig. Q.44.1 (e)

3.4 : Planer Graph

Q.45 Explain planar graphs. ☞[SPPU : May-06, Dec.-08, 09, 10, 13]

Ans. : Definition : A graph is said to be planar graph if it can be drawn on a plane such that no edges intersect or cross in a point other than their end vertices.

A graph G is said to be non-planar if it is not possible to draw graph be without crossing.

1. Regions : A plane representation of a graph divides the plane into parts or regions. They are also known as faces or windows or meshes. A region or face is characterised by the set edges forming it's boundary.

A region is said to be finite if it's area is finite. A region is said to be infinite or unbounded if it's area is infinite. Every planar graph has an infinite region.

Consider the graph given below :

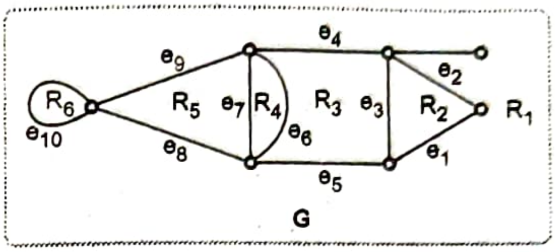


Fig. Q.45.1

The graph G has 6 regions, 7 vertices and 11 edges. Region R_1 is an infinite region known as exterior region. We have

$$R_2 = \{e_1, e_2, e_3\} = \text{Region bounded by } e_1, e_2, e_3$$

$$R_3 = \{e_3, e_4, e_5, e_6\}$$

$$R_4 = \{e_6, e_7\}, \quad R_6 = \{e_{10}\}.$$

It is observed that $n = 7$, $e = 11$, $r = 6$

$$\therefore \quad n + r - 2 = 7 + 6 - 2 = 11 = e$$

Now let us define Euler's formula.

Q.46 State and prove Euler's formula.

Ans. : Statement : For any connected planar graph G, with v number of vertices, e number of edges and r number of regions

$$\boxed{v - e + r = 2}$$

or $v + r - 2 = e$

Proof : Let G be a connected planar graph with v vertices, e edges and r regions. We shall prove the theorem by induction on e.

Step 1 : For $e = 0$, we get $v = r = 1$. Thus

$$v - e + r = 1 - 0 + 1 = 2$$

Hence result is true for $e = 0$

Step 2 : Let $e \geq 1$. Assume that the result is true for all connected planar graphs with less than e edges. Let G be a graph with v vertices, e edges and r regions.

Step 3 : Case 1 : If G has a pendent vertex say x then $G - \{x\}$ is a connected graph with $v - 1$ vertices, $e - 1$ edges and r regions.

So by induction hypothesis

$$(v - 1) - (e - 1) + r = 2$$

$$v - e + r = 2$$

Case 2 : If G has no pendent vertex the G is a connected graph with circuit. Let e_1 be the edge of a circuit in G. Then G - $\{e_1\}$ is a connected graph with v vertices, e – 1 edges and r – 1 regions {If we remove edge from a circuit, then it reduces region by 1}.

By induction hypothesis

$$v - (e - 1) + (r - 1) = 2$$

$$\Rightarrow \quad v - e + r = 2$$

Thus by the principle of mathematical induction the result is true for all e.

Q.47 If G (V, E) is a simple connected planar graph with v vertices and e edges then $e \leq 3v - 6$.

Ans. : Proof : Give that, G is a simple planar graph, so each region of G is bounded by three or more edges.

If G has r number of regions then the total number of edges in G is $e \geq 3r$.

Also each edge of G is included in exactly two regions of G. therefore $2e \geq 3r$

$$\Rightarrow \quad \frac{2e}{3} \geq r$$

Substitute these values in Euler's theorem, we get

$$v - e + r = 2$$

$$v - e + \frac{2e}{3} \geq 2$$

$$3v - e \geq 6$$

$e \leq 3v - 6$ Hence the proof.

Corollary 2 : Prove that, K_5 (the complete graph on 5 vertices) is not planar.

Proof : The complete graph on 5 vertices K_5 is given below :

K_5 has 5 vertices and 10 edges. i.e. v = 5 and e = 10.

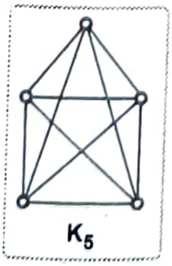


Fig. Q.47.1

Now $3v - 6 = 15 - 6 = 9$

By corollary 1, $e \leq 3v - 6$

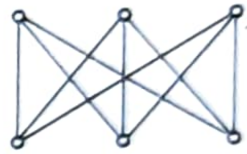
$10 \leq 9$ which is impossible.

Therefore K_5 is not planar graph.

K_5 is the smallest planar graph with respect to number of vertices.

Consider the graph $K_{3,3}$

$$\text{Here } v = 6, e = 9,$$
$$3v - 6 = 18 - 6 = 12 > 9 = e$$
$$\text{i.e.} \quad e \leq 3v - 6$$



$K_{3,3}$

Fig. Q.47.2

But $K_{3,3}$ is not a planar graph.

$\therefore$ The graph $K_{3,3}$ is the smallest non planar graph with respect to number of edges.

The graph K_5 is called the Kuratowski's first graph and $K_{3,3}$ is called the Kuratowski's second graph.

In 1930, Kuratowski gave a necessary and sufficient condition for a graph to be planar.

Kuratowski's Theorem : A graph G is a planar if G does not contain any subgraph that is isomorphic to within vertices of degree two to either K_5 or $K_{3,3}$

Two graphs are said to be isomorphic to within vertices of degree two if they are isomorphic or they can be reduced to isomorphic graphs by repeated insertion of vertices of degree 2 or by merging the edges which have exactly one common vertex of degree 2.

For example the following graph are isomorphic to within vertices of degree 2. (Homeomorphic)

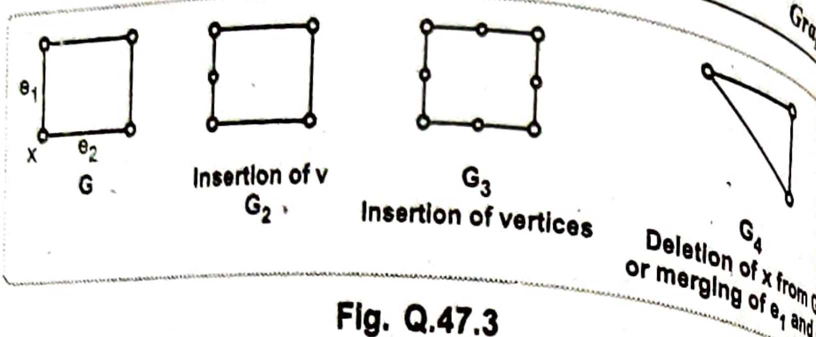


Fig. Q.47.3

Q.48 Draw a planar representation of graphs given below if poss

[SPPU : May-06, De

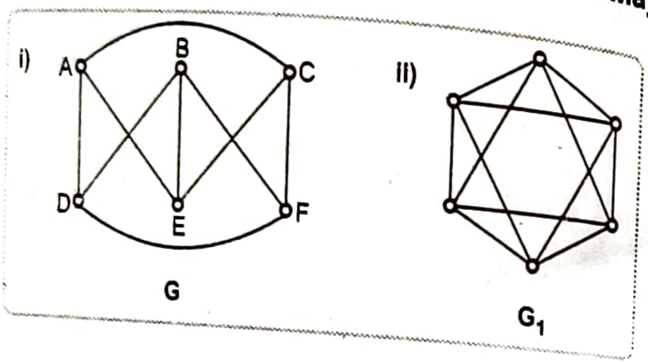


Fig. Q.48.1

Ans. : The planar representation of G_1 and G_2 is as follows :

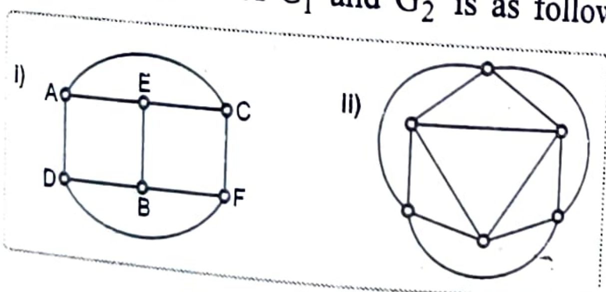


Fig. Q.48.1 (a)

Q.49 Identify whether the graphs are planar or not Justify ?

[SPPU : Dec.-0

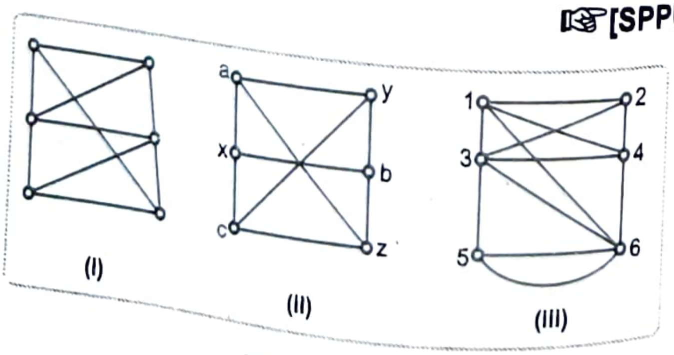


Fig. Q.49.1

Ans. : i)

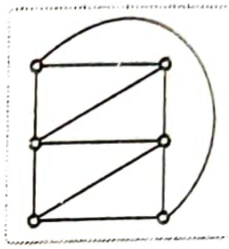


Fig. Q.49.1 (a)

Given graph is planar graph.

ii) Given graph is isomorphic to $K_{3,3}$

$\therefore$ Given graph is not planar.

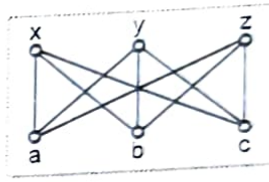


Fig. Q.49.1 (b)

iii)

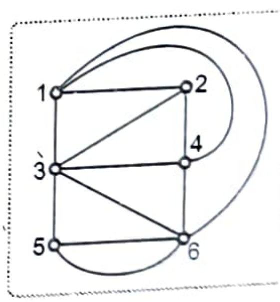


Fig. Q.49.1 (c)

$\therefore$ Given graph is planar.

Q.50 Show that in a connected planar graph with 6 vertices, 12 edges each of region is bounded by 3 edges. [SPPU : Dec.-10, 13]

Ans. : According to Eulers theorem for planar graphs.

$$v - e + r = 2$$

Here $v = 6, e = 12$

$6 - 12 + r = 2 \Rightarrow r = 8$

We know that, each edge contributed twice in a regions we have 12 edges.

So $12 \times 2 = 24$ edges are distributed among 8 regions.

$\Rightarrow \quad \frac{24}{8} = 3$ edges for each region.

So each region is bounded by 3 edges.

Q.51 Prove that $K_{3,3}$ is not planar graph.

Ans. : $K_{3,3}$ has 6 vertices and 9 edges. Suppose $K_{3,3}$ is planar, then boundary of each region has at least 4 edges because it is bipartite and contains no triangles. Each edge lies on boundary of two regions.

Therefore, $2e \geq \sum_{i=1}^{r}$ (the number of edges in the i^{th} region)

$$2e \geq 4r$$

$$2e \geq 4(2 + e - v)$$

$\Rightarrow \quad e \leq 2v - 4$ But $e = 9$ and $v = 6$

$\therefore \quad 9 \leq 12 - 4 = 8$ which is impossible.

Hence $K_{3,3}$ is not planar graph.

3.5 : Graph Colouring

Q.52 Explain coloring of graphs.

Ans. : The coloring of all vertices of a connected graph such that adjacent vertices have different colors is called a proper coloring or vertex coloring or simply a coloring of graphs.

A graph G is said to be properly colored graph if each vertex of G is colored according to a proper coloring.

e.g. 1) Consider the following graphs with proper coloring

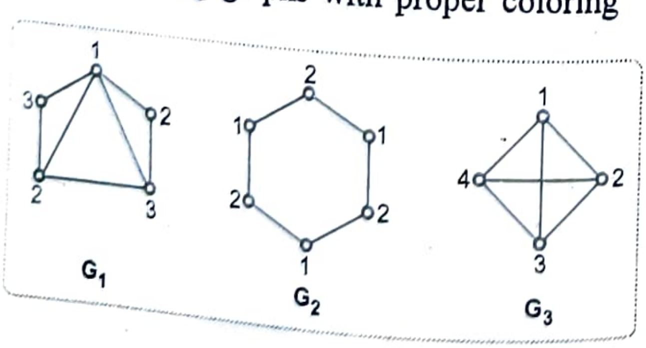


Fig. Q.52.1

1. Chromatic Number of Graph : The chromatic number of a graph G is denoted by X(G) and defined as the minimum number of colors required to color the vertices of G so that the adjacent vertices get different colors.

A graph G is said to be K-colorable if all vertices of G can be properly colored using at most K different colors. Obviously, a K-colorable graph is K+1 colorable.

If G is k-colorable then $X(G) \leq K$.

e.g. In above example (1) $X(G) = 3$, $X(G_2) = 2$, $G(G_3) = 4$. If G is any graph with $x(G) = K$ then the addition or deletion of loops or multiple edges do not change the chromatic number of that graph. Thus hereafter for a coloring of problem we consider only simple connected graphs.

2. Chromatic Polynomial : We have studied the properly coloring of graph in many different ways using a sufficiently large number of colors. The chromatic polynomial of a graph is denoted by $P_n(\lambda)$ and defined as the number of ways of properly coloring of graph using λ or fewer colors

e.g. 1) The chromatic polynomial of the complete graph K_1 is $P_1(\lambda) = \lambda$.

2) The chromatic polynomial of the complete graph K_n on n vertices is

$$P_n(\lambda) = \lambda(\lambda - 1)(\lambda - 2) \ldots (\lambda - n + 1)$$

3. Coloring of Planar Graph : A map or atlas is a plane representation of a connected planar graph. Two regions of a planar graph G are said to be adjacent if they have an edge common.

The coloring of a planar graph or map means an assignment of a color to each region of a planar graph G such that adjacent regions have different colors.

A planar graph is - n colorable if minimum n different colors are required to color graph G.

Theorem 1 : (Four Color Theorem)

Every planar graph is 4 - colorable.

Initially it was a conjecture, but in 1979 Appel and Haken proved this. That's why this conjecture became theorem.

4. Open Problem of Coloring : A lot of research is done in the coloring of planar graphs, particularly coloring of vertices, or edges or regions of a planar graph.

The following open problem is stated by Dr. H. R. Bhapkar and proved partially first time.

Open Problem :

How many minimum colors will be required to color planar graph such that

i) Adjacent vertices have different colors

ii) Incident edges have different colors.

iii) Adjacent regions have different colors.

iv) A region, boundary edges and boundary vertices of that region have different colors.

This type of coloring is known as perfect coloring of G and denoted by PC (G).

We list some observations of perfect coloring of planar graph as follows :

i) If G is a null graph then PC (G) = 2

ii) If G is a chain graph when n vertices then,

$$PC(G) = \Delta(G) + 2$$

where $\Delta(G)$ = Highest degree of a vertex in G.

Q.53 Is each of the following graphs strongly connected.

[SPPU : Dec.-16]

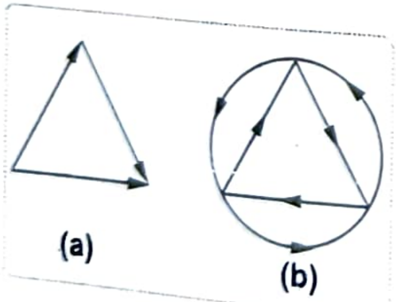


Fig. Q.53.1

Ans. : a) Graph is not strongly connected.

b) Graph is strongly connected.

3.6 : Tree

In this chapter we will study one of the simplest types of the connected graphs known as trees. This class of graphs has wide applications and has been the subject of study of many outstanding scientiests of different fields. Trees are discovered by Kircnnoff in 1847, while investigating the electrical networks. Sir Arthur Cayley used trees to study and enumerate isomers of saturated hydrocarbons.

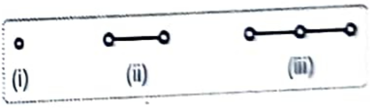
The important applications of tree include searching, sorting, syntax checking and database managements. The tree is one of the most non-linear structures used for algorithm development in computer science.

Q.54 Define tree with examples.

Ans. :

A tree is a connected graph without any circuit i.e. tree is a connected acyclic graph. The collection or set of an acyclic graphs. (not necessarily connected) is called a forest.

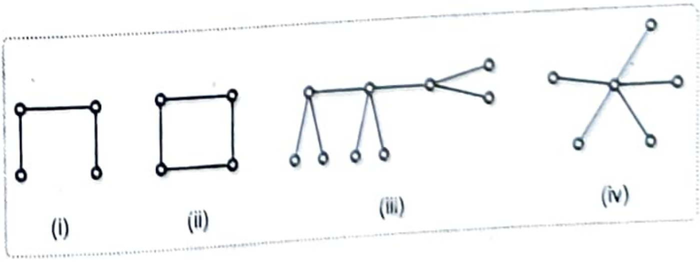
Examples :



Example 1

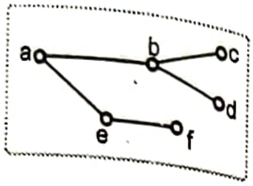
All these three graphs are trees. There are unique tree on one vertex, 2 vertices and 3 vertices.

Example 2



Graphs (i), (iii) and (iv) are trees but (ii) is not a tree as it has a cycle.

A) A vertex of degree 1 in a tree is called a **leaf** or a **terminal node**. A vertex of a degree greater than one is called a **branch node** or **internal node.**



e.g. In a tree

c, d and e are leaves or terminal nodes and a, b, c are branch nodes.

B) Some properties of tree are obvious

i) Every edge of a tree is an isthmus. Conversely if every edge in a connected graph G is an isthmus then G is a tree.

ii) Every tree is a simple graph because loops and multiple edges field cycles.

iii) Every tree is a bipartite graph with $n \geq 2$ vertices.

Theorem 1 : A graph G is a tree iff there exists a unique path between every distinct pair of vertices of G.

Proof : Suppose G is a tree. So G is a connected and without circuits. We know that a circuit forms two or more paths. But G has no circuit so it has unique path between every pair of vertices.

Conversely assume that there is a unique path between every pair of vertices in G. This implies that G is a connected and G has no cycles. i.e. G is a connected acyclic graph.

Therefore G is a tree.

Theorem 2 : A graph G on n vertices is a tree iff G is connected and has exactly $n - 1$ edges.

Proof : Suppose graph G is a tree. Therefore is a connected and acyclic graph.

It is sufficient to prove that G has $n - 1$ edges only.

We can prove this by induction principle on n.

For $n = 1$, the result is obvious

Let $n > 1$, consider $G - e'$ for any $e' \in E(G)$

As G is a tree, e' is an isthmus of G.

$\therefore G - e'$ is a disconnected graph with two components say G_1 and G_2.

Now G_1 and G_2 are connected and a cyclic graphs as G_1 and G_2 are subgraph of G.

Let G_1 has n_1 vertices and m_1 edges and G_2 has n_2 vertices and m_2 edges

$\therefore$ By induction principle

$$m_1 = n_1 - 1 \qquad \text{and} \qquad m_2 = n_2 - 1$$

Therefore the number of edges in G is given by

$$e = (e_1 + e_2) + 1 = n_1 - 1 + n_2 - 1 + 1$$

$$= n_1 + n_2 - 1$$

$$e = n - 1 \qquad (\because n = n_1 + n_2)$$

Hence G has n – 1 edges only.

Conversely assume that G is connected graph and has exactly n – 1 edges.

Claim : Prove that G is a tree.

It is suffices to prove that G is noncyclic graph.

Suppose G contains a cycle C.

Let P denote the number of vertices in C

$\therefore$ The number of edges in C = P

As G is connected graph, the remaining n – P vertices must be connected to vertices in C. To connect each vertex of G which is not in C, we required n – P edges as each edge of G can connect only one vertex to the vertices in C.

Hence the total number of edges in G is given by

$$e = (n - P) + P = n \Rightarrow e = n \text{ which is contradiction}$$

$\therefore$ G has n – 1 edges only.

Thus G is a tree.

Theorem 3 : Let G be a graph with n vertices and m edges. If any of the following is true then all are true.

i) G is a tree.

ii) G is connected and m = n – 1

iii) G is acyclic graph and m = n – 1

iv) Every edge of G is an isthmus and G is connected.

v) There is exactly one path between every pair of vertices in G.

Theorem 4 : A non trivial tree contains at least two vertices of deg 1. (i.e. pendent vertices)

Proof : Let G be a tree with n vertices, so G is connected.

$\therefore \quad d(x) \geq 1 \; ; \; \forall x \in V\,(G)$

By handshaking lemma

$$\sum_{x \in V(G)} d(x) = 2 \times \text{Number of edges in } G = 2(n-1)$$

Suppose there is no vertex of degree 1.

Then $$2n \leq \sum_{x \in V(G)} d(x) = 2n - 2$$

i.e. $2n \leq 2n - 2 \Rightarrow n \leq n - 1$ which is impossible.

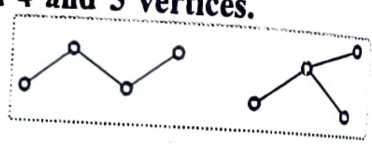
Thus there is at least one vertex of degree 1.

By a similar argument there is one more vertex of degree 1.

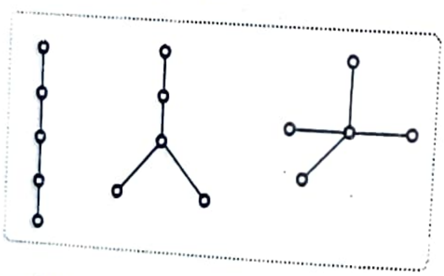
Hence every tree has at least two pendent vertices.

Q.55 Draw all non isomorphic trees on 4 and 5 vertices.

Ans. : a) Non isomorphic trees on 4 vertices



b) Non isomorphic trees on 5 vertices



Q.56 Under what conditions trees are the complete bipartite graphs.

Ans. : Suppose T is a tree which is the complete bipartite graph.

Let $$T = K_{m,\,n}$$

$\therefore$ T has m + n vertices and $(m+n-1)$ edges.

But $K_{m,\,n}$ has $m \cdot n$ number of edges.

Therefore $$mn = m + n - 1$$

$$mn - m - n + 1 = 0$$

$$(m-1)\,(n-1) = 0$$

$$m = 1 \text{ or } n = 1$$

$\Rightarrow$

i.e. $$m = 1 \text{ and any } n \quad \text{or} \quad n = 1 \text{ and any } m$$

Hence $K_{1, n}$ and $K_{m, 1}$ are the only complete bipartite graphs. These are known as star graphs.

Thus T is a star graph.

Q.57 a) Is it possible to draw a tree with 10 vertices which has vertices either of degree 1 or 3 ?

If possible draw tree. Is it possible to draw same type of tree with 13 vertices ?

b) For which values of n (number of vertices), such type of tree exist ?

Ans. : a) Given that tree T has 10 vertices so it must have 9 edges.

Let x and y be the number of vertices of degree 1 and 3 in T respectively.

$$\therefore \quad x + y = 10 \quad \text{... (Q.57.1)}$$

By handshaking lemma

$$\sum_{V \in V(T)} d(v) = 2(\text{Number of edges in T})$$

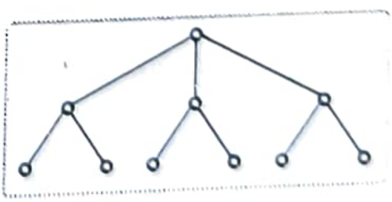
$$x + 3y = 2 \times 9 = 18$$

$$x + 3y = 18 \quad \text{... (Q.57.2)}$$

Solving equations (Q.57.1) and (Q.57.2) we get

$$y = 4 \quad \text{and} \quad x = 6$$

Therefore there are 6 vertices of degree 1 and 4 vertices of degree 3. Such type of graph is given below.



Now consider a tree with 13 vertices and 12 edges.

By using similar theory, we get

$$x + y = 13 \quad \text{... (Q.57.3)}$$

$$x + 3y = 24 \quad \text{... (Q.57.4)}$$

Solving (Q.57.3) and (Q.57.4), we get $1y = 11 \Rightarrow y = \frac{11}{2}$ and x =

which is impossible.

Therefore it is not possible to draw such type of tree.

b) Let T be a tree with n vertices.

Let x and y be the number of vertices of degree 1 and 3 in respectively. T has n – 1 edges.

Therefore $x + y = n$

By handshaking lemma $x + 3y = 2(n - 1)$ ... (Q.57.5)

$x + 3y = 2n - 2$ ... (Q.57.6)

Equation (Q.57.6) – Equation (Q.57.5) $\Rightarrow 2y = 2n - 2 - n$

$$2y = n - 2$$

$$\Rightarrow \quad y = \frac{n-2}{2} = \frac{n}{2} - 1$$

$$\Rightarrow \quad x = n - y = n - \frac{n}{2} + 1 = \frac{n}{2} + 1$$

i.e. $x = \frac{n}{2} + 1$ and $y = \frac{n}{2} - 1$. These should be non negative integers.

Case 1 : If n is even then $x = \frac{n}{2}$ is odd

and $\frac{n}{2} + 1 = x$ is an even integer

$\frac{n}{2} - 1 = y$ is an even integer

Thus the tree exists with required condition on even number of vertices. $T_2, T_4, T_6, T_8 \ldots T_{2n}$ are such type of trees.

Case 2 : If n is odd then $\frac{n}{2}$ is not an integer there x and y are not an integer.

Hence the required tree does not exist on odd number of vertices.

Q.58 Explain centre of a tree.

Ans. : We know that the distance between two vertices in a connected graph i.e. is a length of the shortest path between that vertices. Before defining centre of a tree first find the distance between vertices of a tree which is defined as follows :

Eccentricity of a Vertex

Let G be a connected graph G and $v \in V(G)$. The eccentricity of a vertex v is denoted by E(v) or e(v) and defined as the distance from v to the vertex farthest from v in G.

i.e. $E(v)$ or $e(v) = \max\{d(v, v_i)/\forall v_i \in v(G)\}$

Centre of a Graph

A vertex in a graph G with minimum eccentricity is called a centre of G and it's eccentricity is called as radius of G. It is denoted by r(G).

Q.59 Prove that every tree has either one or two centres.

Ans. :

Proof : Let T be a tree. Then e(v) for a vertex v is at a vertex farthest from v. As T is a tree, it is attained at a pendent vertex.

We know that every non trivial tree has at least two pendent vertices. Now, delete all pendent vertices from G, we get new graph G' which is also a tree. Moreover as one edge at each pendent vertex is removed, the eccentricity of any vertex will be reduced by one. Therefore centres of G will still remain centres of G'. We continue the process with G' until we arrive at K_2 or K_1. K_2 has two centres and K_1 has one centre. Hence the proof.

Q.60 Define cut vertex of a tree.

Ans. : We know that the vertex v whose removal from a connected graph G, disconnects the graph is called as cut vertex of G.

In any tree all vertices except pendent vertices are cut vertices.

3.7 : Rooted Tree

Q.61 Define rooted tree and binary tree. [SPPU : Dec.-09]

Ans. : A connected acyclic, directed graph is called a **directed tree.** In other words, a directed graph is said to be a directed tree if it will become a tree when the directions of the edges are ignored.

Q.62 Define rooted tree.

Ans. : A directed tree is called a **rooted tree** if there is exactly one vertex whose incoming degree is zero and the incoming degrees of all other vertices are one.

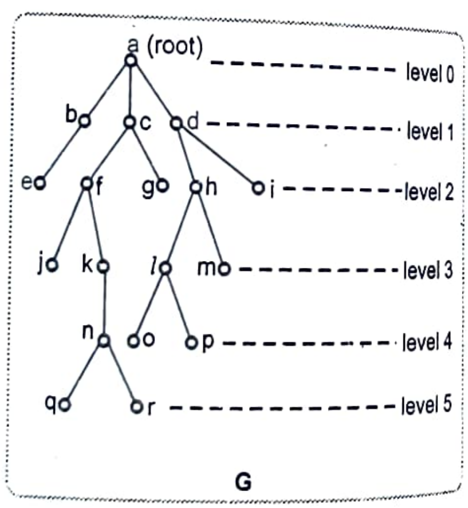
The vertex with incoming degree 0 is called **the root** of the rooted tree. The vertex whose outgoing degree is zero is called **leaf** or **terminal** node.

A vertex whose outgoing and incoming degrees are non zero is called a **branch note** or an **internal node.** Consider the following example.

Q.63 Define level and height of a tree.

Ans. : A vertex v in a rooted tree is said to be at **level n** if there is a path of length n from the root to the vertex v.

The **height** of the tree is the maximum of the levels of its vertices.



Example

In the given graph G .... a is a root

Vertices b, c, d are at level 1

Vertices e, f, g, h and i are at level 2

Vertices j, k, l and m are at level 3

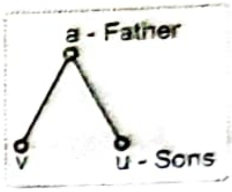
Vertices n, o and p are at level 4

Vertices q and r are at level 5

The maximum level is 5. Therefore the height of tree is 5.

Rules for rooted tree : In a rooted tree

i) If the level of a vertex u is greater than the level of vertex v then u is below v.



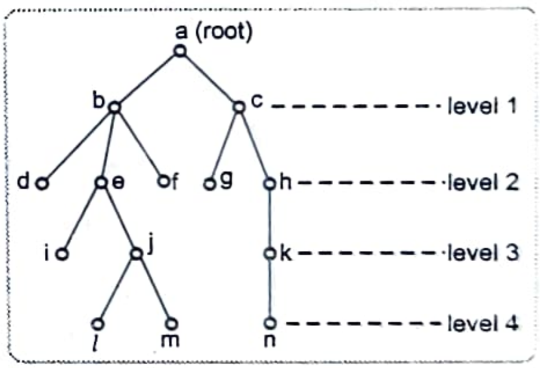
ii) If vertex u is below vertex v and there is an edge from v to u then u is said to be son of v (or child of v) and v is said to be the father of u (or parent of u) i.e.

iii) Two vertices u and v are said to be brothers if they are the sons of the same vertex.

iv) A leaf is a vertex without children.

v) If $p = \{a, v_1, v_2, v_3, \ldots v_{n-1}, b\}$ is a path from a to b then b is called as descendent of a and a is called as ancestor of b.

Consider the following example



From above figure i) a is a root of the tree.

ii) b and c lie at level 1. $\therefore$ b and c are sons of root a i.e. a is father of b and c and b and c are brothers.

iii) b has three sons d, e, f

$\therefore$ b is a father of d, e, f

c has 2 sons g and h

iv) e has two sons i and j

h has only one son k

v) i has no son j has two sons l and m. So k and l are brothers. k has one son n. n has no brother as n is a leaf.

l, m, n are known as descendent of a and a is ancestor of d, e, f, g, h, i, j, k, l, m, n and so on.

Q.64 Define Subtrees.

Ans. : Let T be a rooted tree and $x \in V(T)$.

A vertex x together will all its descendants is called the **subtree** of T rooted at x.

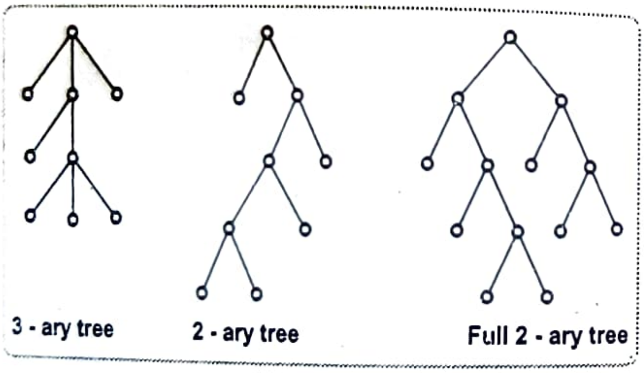
The subtree corresponding to the root node is the **entire tree.** The subtree corresponding to any other node is called a **proper subtree.**

Q.65 Define M-ary trees.

Ans. : A rooted tree in which every interior node has at most m sons is called an **m-ary tree.**

A m-ary tree is said to be **regular m-ary tree** or full m-ary tree if every branch node has exactly m sons.

Consider the following examples.



Q.66 A regular m-ary tree with p interior nodes has mp + 1 nodes at all.

[SPPU : Dec.-09]

Ans. :

Proof : Let T be a regular m-ary tree with n vertices. Out of n vertices there are p interior vertices or branch nodes.

Therefore there are $t = n - p$ number of sons or leaves in T.

But given graph is regular and p interior nodes.

So the regular m-ary tree will have mp sons.

But root is not a son.

Therefore give tree has total (mp + 1) number of vertices

Hence $\boxed{n = mp + 1}$

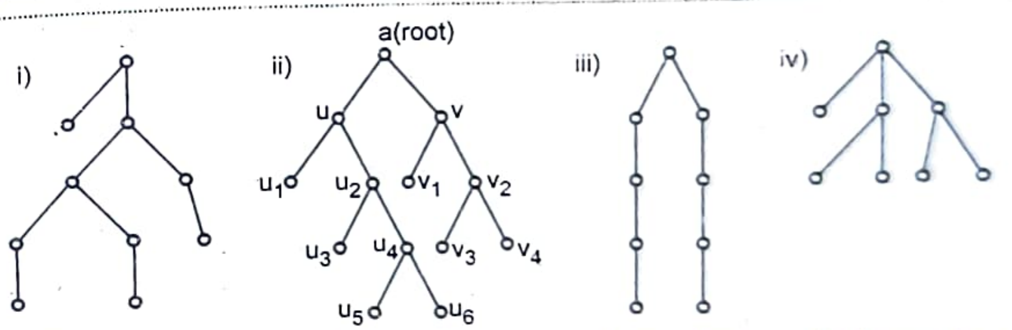
Q.67 Explain binary tree. [SPPU : Dec.-09, May-10, 12]

Ans. : An m-ary tree is known as **binary tree** if every branch node has at most 2 sons.

In other words, a tree in which there is exactly one vertex of degree 2 and each of the remaining vertices of degree or three, is called a binary tree.

A binary tree is called as **regular binary tree** or **full binary tree** if every branch node has exactly 2 sons or zero son.

Consider the following examples



Binary tree   Full binary tree   Binary tree   Not binary tree

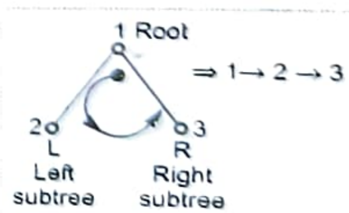
In binary trees, instead of referring the first or second subtree of a branch node, we use to the left subtree or right subtree of the node.

Q.68 Explain binary tree traversal with examples.

Ans. : Traversing means visiting or processing all the nodes of a tree. A binary tree traversal is the visiting of each node of a tree only once according to some sequence.

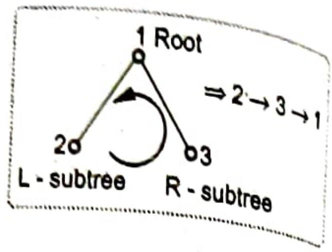
There are two types of traversing binary trees.

1) Depth-first traversal : In this method, the processing proceeds from the root or the most distant descendent of the first child.

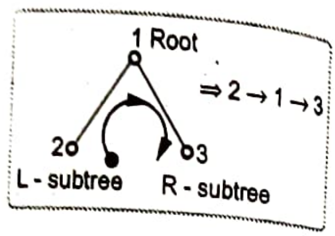


There are three types of Depth-First Traversal.

A) Pre order traversal : In this traversal, the root node is traversal first, followed by the left subtree and then the right subtree as shown below.



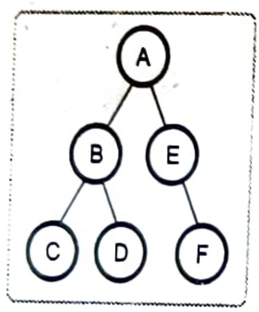
B) Post order traversal : If processes first the left subtree then the right subtree and then at the last root of the tree as shown below.



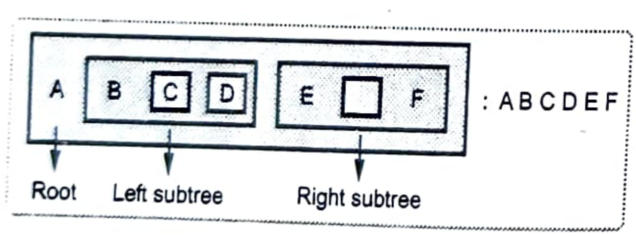
C) In order traversal : It processes the left subtree first then the root and at the last right subtree.

The prefix "in" means root is processed in between the subtrees. It is shown as below.

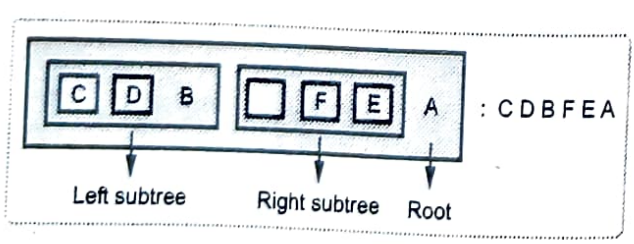
Consider the following binary tree.



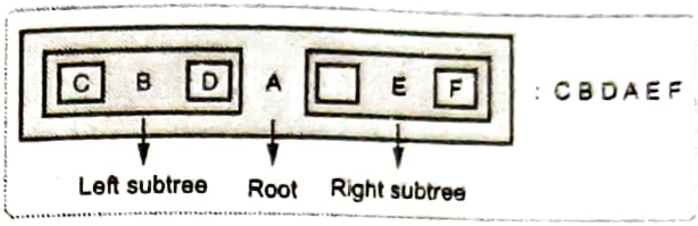
i) Pre order traversal



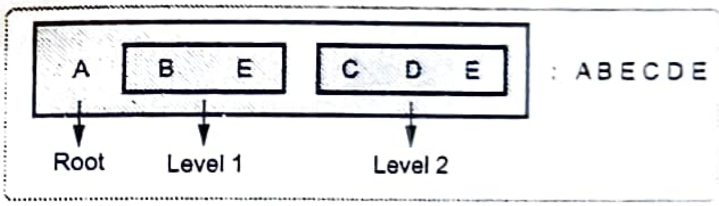
ii) Post order traversal



iii) In order traversal

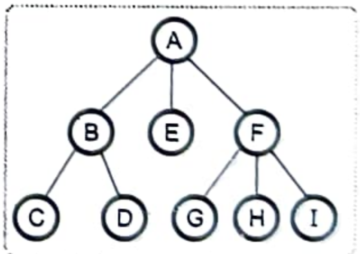


2) Breadth first traversal : In this traversal, the processing proceeds horizontally from the root of all of its children, then to its childre's children and so on until all the nodes have been processed. That is first write node of zero level, then all nodes of level 1 then level 2 and so on. The breadth first process for the above binary is shown below.



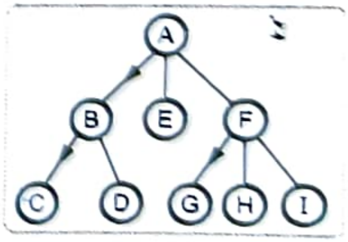
Q.69 Explain conversion of general tree to binary tree.

Ans. : Let us explain the method to convert general tree to binary tree with the help of the following example.

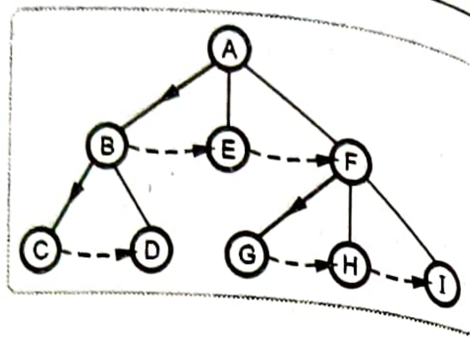


Consider the following steps

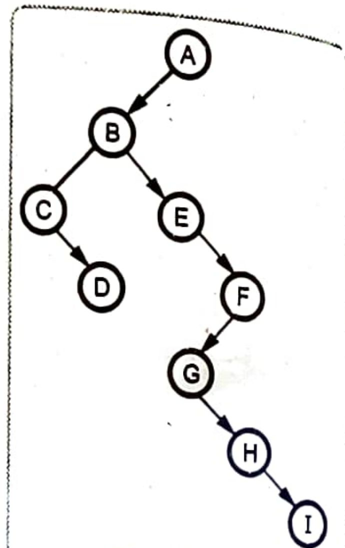
Step 1) To convert it into binary tree, we first identify the branch from the parent to its first or left most child. These branches from each parent become left pointers in the binary tree.



Step 2) Connect sibling, starting with the left most or first child, using a branch for each sibling to its right sibling. They are the right pointers in the binary tree.

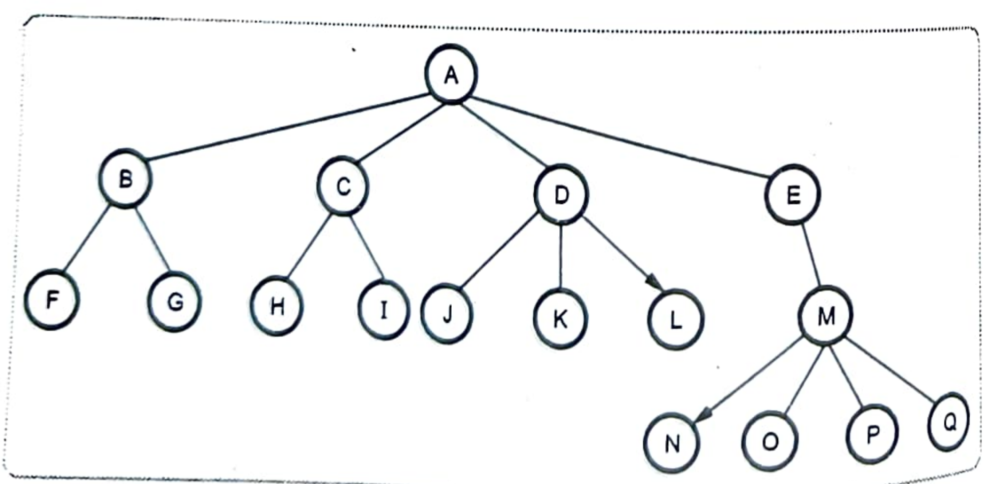


Step 3) Now remove all unneeded branches from the parent to its children. Therefore remove $A \rightarrow E$, $B \rightarrow D$, $A \rightarrow F$, $F \rightarrow H$, $F \rightarrow I$ we get the following required binary tree.

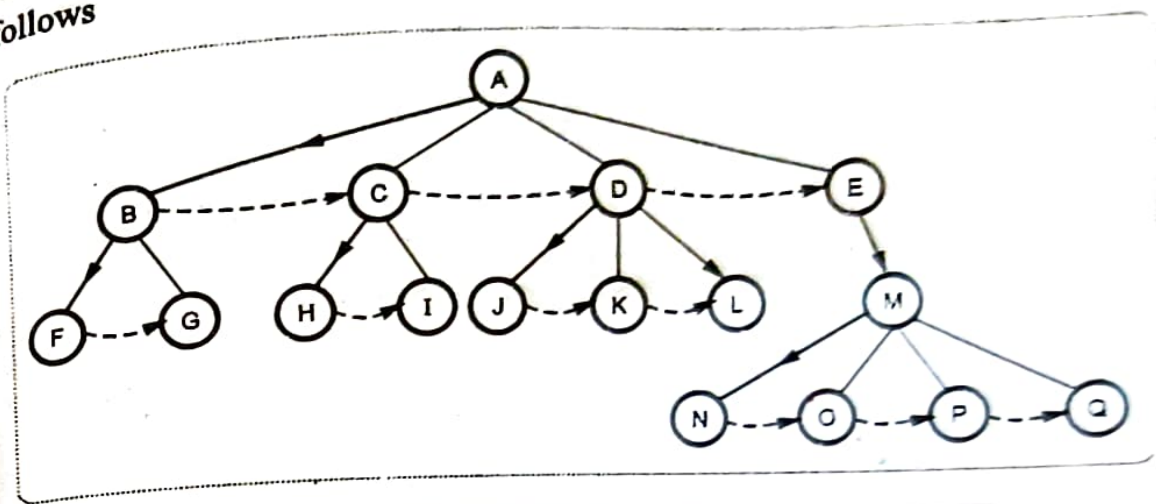


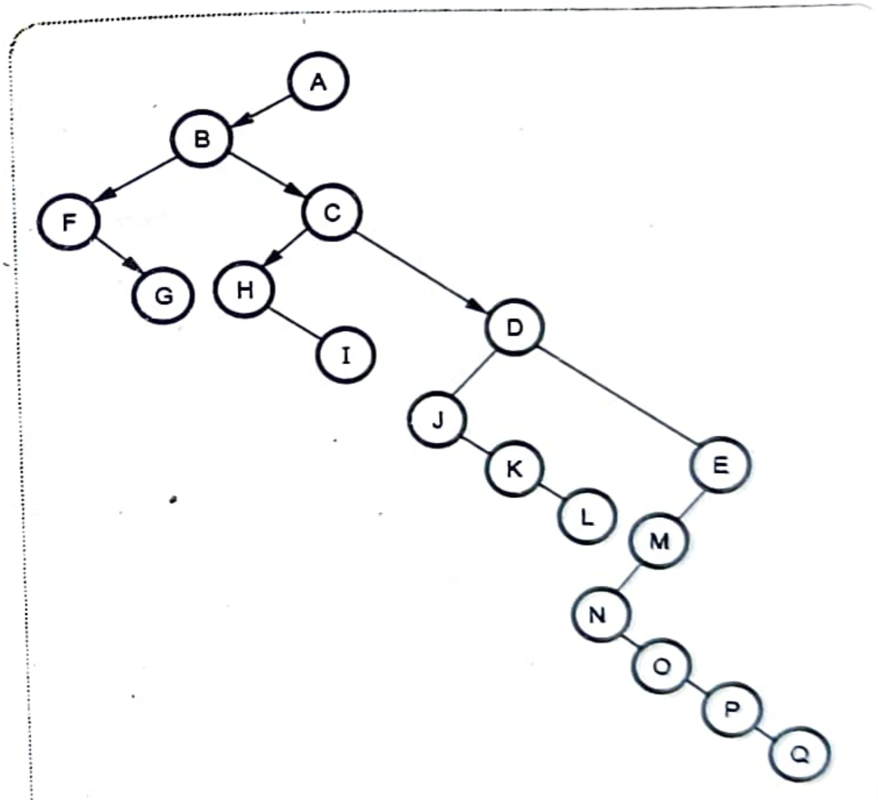
Resulting binary tree

Q.70 Convert the following tree into binary tree. [SPPU : Dec.-09]



Ans. : The steps involve to convert the given tree into binary tree are as follows

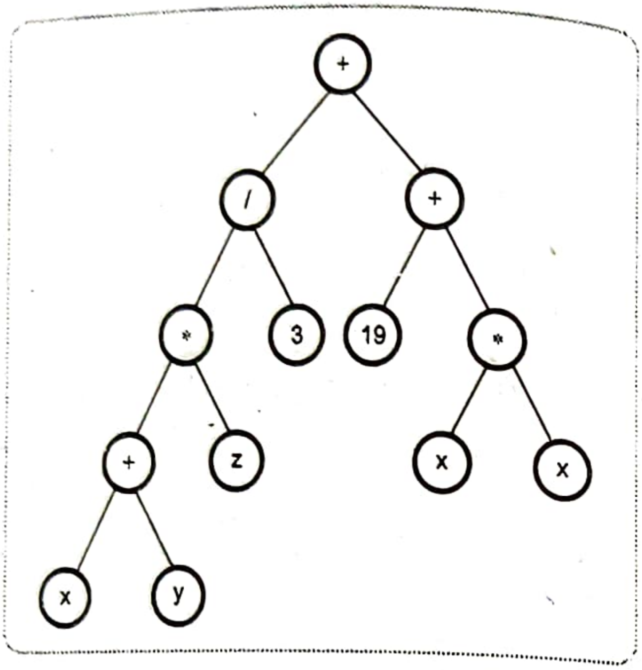




Binary tree

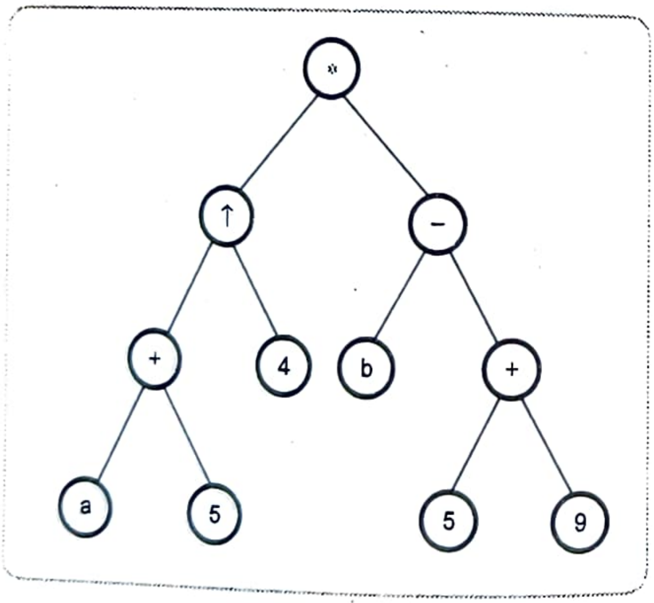
Q.71 Construct the labeled tree of the following algebraic expression $(((x+y)*z)/3)+(19+(x*x))$ **[SPPU : May-10]**

Ans. : The expression tree for the given algebraic expression is given below :

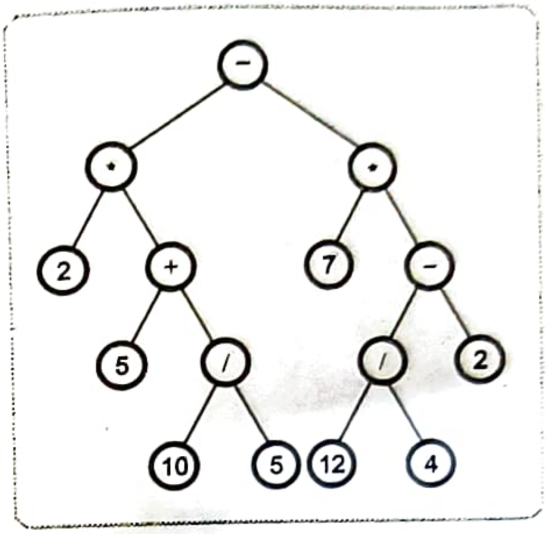


Q.72 Represent the expression $((a+5)\uparrow 4)*(b-(5+9))$ using a binary tree. [SPPU : May-12]

Ans. : The binary tree for the given expression is as follows :



Q.73 Write and evaluate the expression tree shown below. [SPPU : Dec.-09]

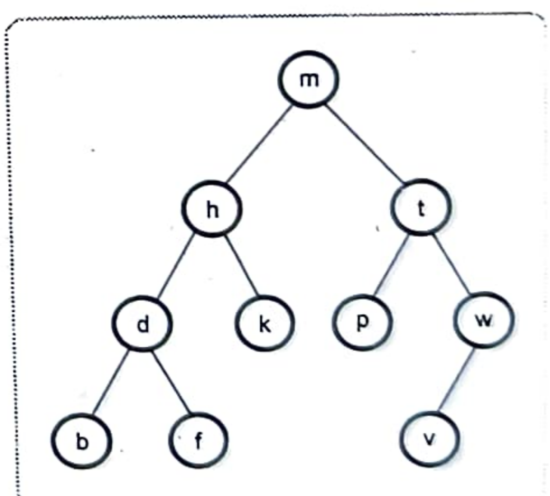


Ans. : The algebraic expression of the given binary tree is

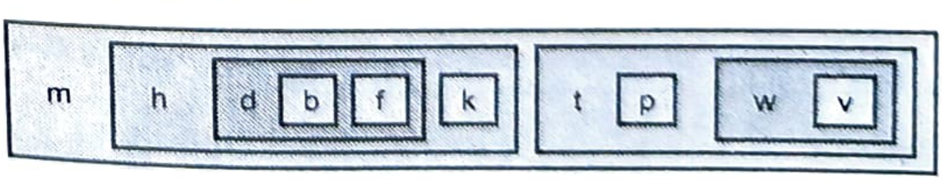
$(2*((5)+(10/5)))-((7)*(((12/4)-(2))))$

It's value is $2*(5+2)-(7*(3-1))=14-7=$ **14**

Q.74 Find the preorder, postorder and inorder traversal of the following tree. **[SPPU : May-10]**

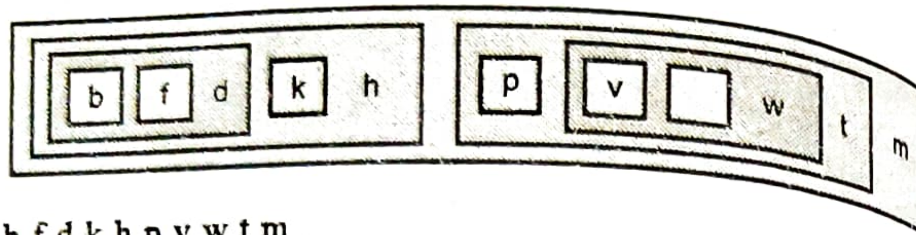


Ans. : i) Preorder traversal



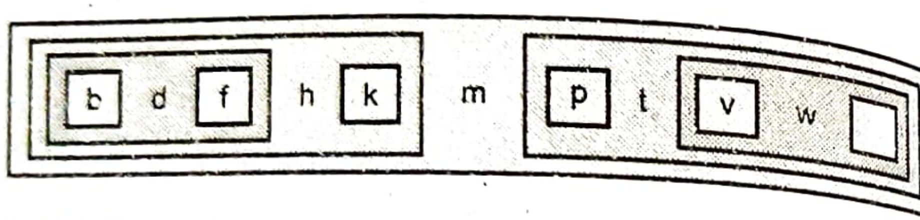
i.e. m b d b f k t p w v

ii) Postorder traversal



i.e. b f d k h p v w t m

iii) Inorder traversal



i.e. b d f h k m p t v w

Out of which 2^h are leaves.

$$\text{Hence internal vertices} = (2^{h+1} - 1) - 2^h$$
$$= 2^h(2-1) - 1$$
$$= 2^h - 1$$

Q.75 Find the maximum of possible height of a binary tree with 13 vertices and draw graph.

Ans. :

We have $n = 13$

The maximum possible height of the binary tree is

$$\frac{n-1}{2} = \frac{13-1}{2} = 6$$

The required graph as shown in Fig. Q.75.1.

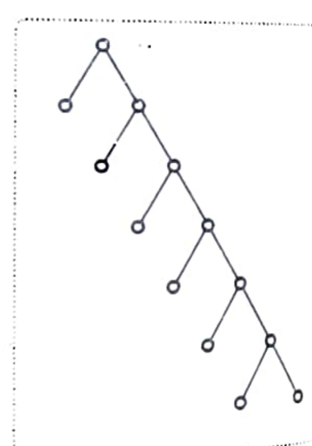


Fig. Q.75.1

Q.76 Define prefix code and binary search trees.

[SPPU : Dec.-12, 14, 15, May-08, 15]

Ans. :

1) A set of sequences is said to be a **prefix code** if no sequences in the set is a prefix of another sequence in the set.

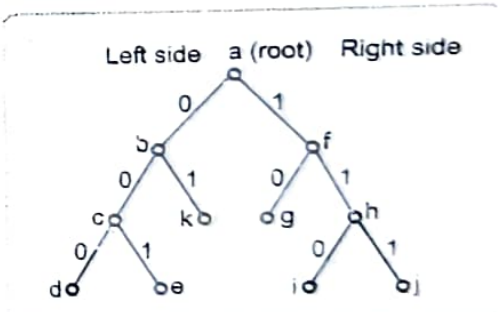
For example the set {000, 001, 01, 10, 11} is a prefix code as no sequence of symbols is present at the beginning of another sequences. All these sequence are distinct.

The set {1, 00, 000, 0001} is not a prefix code because the sequence 00 is the prefix of the sequences 000 and 0001.

A question comes in everyone's mind "How to construct prefix code to a full binary trees?"

We can solve this question by adding some flavours and binary codes to a full binary trees.

For a given full binary tree, we label the two branches incident from each internal node with 0 and 1. For the left branch we assign 0 and for the right branch we assign 1 of every rooted tree or subtree. Consider the following example of full binary tree.



In given figure root a has 2 sons. Left son is ab and right is af, so assign 0 to ab and 1 to af. Now b has two sons. Assign 0 to left son be and 1 to right son bk. Similarly assign 0 or 1 to every edge of a tree.

Now assign to each leaf, a sequence of 0's and 1's which is the sequence of labels of the edges in the path from the root to the leaf.

For example, d is a leaf and the path a to d is a – b – c – d and their respective labels are 0 – 0 – 0 so the prefix code of d is 000.

For leaf e, path is a – b – c – e and labels 0 – 0 – 1

∴ The prefix code of e is 001

Thus the prefix code of above tree is {000, 001, 01, 10, 110, 111}.

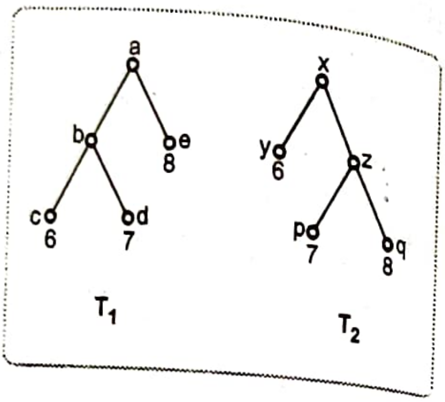
2) Optimal Tree

Let T be any full binary tree and $W_1, W_2, W_3, \ldots W_t$ be the weights of the leaves (terminal vertices) then the weight W of the full binary tree is given by

$$W(T) = \sum_{i=0}^{t} W_i l_i$$

Where $l_i = l(i)$ is the length of the path of the leaf i from the root of the tree. The full binary tree is called an optimal tree if its weight is minimum.

For example, suppose 6, 7, 8 are the weights of the leaves in a full binary tree as given below.



In T_1, $l(c) = 2$, $l(d) = 2$, $l(e) = 1$

$\therefore$ The weight of T is given by

$$W(T_1) = 6 \times l(c) + 7 \times l(d) + 8 \times l(e)$$

$$= 6 \times 2 + 7 \times 2 + 8 \times 1$$

$$= 12 + 14 + 8 = 34$$

In T_2, $l(y) = 1$, $l(p) = 2$, $l(q) = 2$

$$\therefore \quad W(T_2) = 6 \times 1 + 7 \times 2 + 8 \times 2$$

$$= 6 + 14 + 16 = 36$$

Hence $W(T_1) < W(T_2)$

Thus T_1 is the optimal tree for the weights 6, 7, 8.

Q.77 Explain huffman algorithm to find an optimal tree.

Ans. : Let $W_1, W_2, W_3, \ldots W_t$ be the weights of the leaves and it is required to construct an optimal binary tree.

The following steps of an algorithm gives the required optimal binary tree.

Step 1 : Arrange the weights in increasing order.

Step 2 : Consider two leaves with the minimum weights W_1 and W_2. Replace these two leaves and their father by a leaf. Assign weight $W_1 + W_2$ to this new leaf.

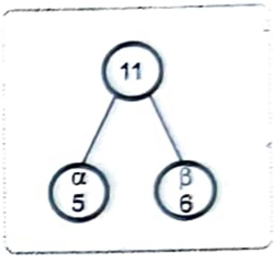
Step 3 : Repeat the step 2 for the weights $W_1, W_2, W_3, \ldots W_t$ until no weight remains.

Step 4 : The tree obtained in this way is an optimal tree for given weights and stop.

Q.78 For the following set of weights construct optimal binary prefix code. $\alpha - 5, \beta - 6, \gamma - 6, \delta - 11, \in - 20$ **[SPPU : Dec.-14]**

Ans. : Here we use Huffman's coding method.

Step 1 : Sort the weights or frequencies of the given letters in increasing order in a queue.



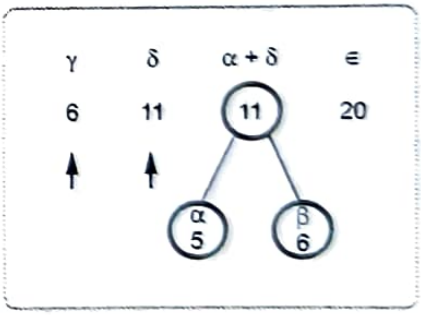
$$\begin{array}{ccccc} \alpha & \beta & \gamma & \delta & \in \\ 5 & 6 & 6 & 11 & 20 \\ \uparrow & \uparrow & & & \end{array}$$

{↑ indicates the first 2 smallest weights}

Consider the two symbols with lowest weights say α, β. The root of the first subtree has a weight
5 + 6 = 11.

As 6 < 11, it can not be placed at the beginning. It has to be placed an appropriate position means after δ. The first subtree is given below

Step 2 : Again rewrite sequence of weights in increasing order by replacing and β by new subtree weight as

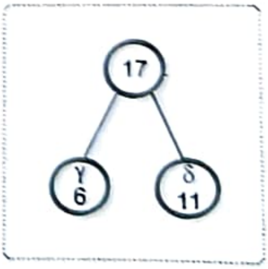


The first two smallest weights are 6 and 11.

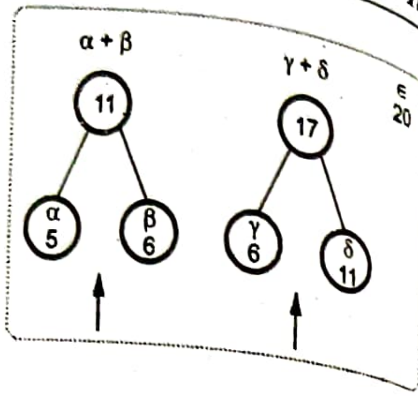
∴ The root of new subtree has a weight

$$6 + 11 = 17$$

The new subtree is as follows



Step 3 : Rewrite sequence of weight in increasing order as

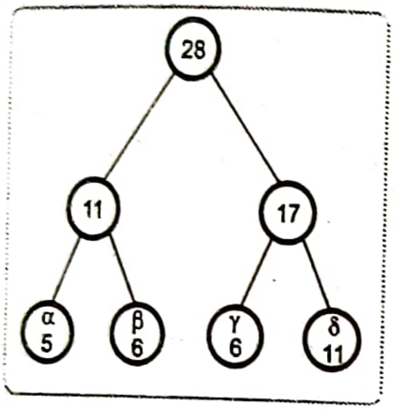


The first two smallest weights are 11 and 17.

$\therefore$ The root of new subtree has a weight

$$11 + 17 = 28$$

The new subtree is as follows



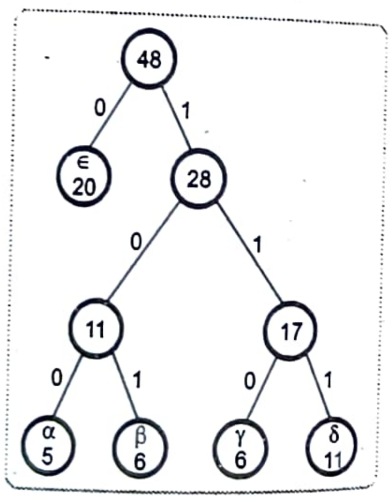
Step 4 : Rewrite the sequence of weights in increasing order

$$\begin{array}{cc} \epsilon & \\ 20 & 28 \\ \uparrow & \uparrow \end{array}$$

The root of new subtree is $20 + 28 = 48$

$\therefore$ The new subtree is as follows

This is the optimal binary tree.



Symbols leaves	α	β	γ	δ	ϵ
Binary prefix code	100	101	110	111	0

The weight of the optimal tree is

$$\begin{aligned} W &= 20 \times 1 + 5 \times 3 + 6 \times 3 + 6 \times 3 + 11 \times 3 \\ &= 20 + 5 + 18 + 18 + 33 = \mathbf{104} \end{aligned}$$

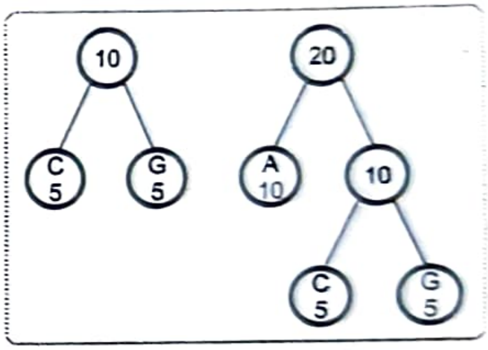
Note : All readers are requested to understand Q.78 properly and then see next examples. Hereafter we have given solutions in shortforms.

Q.79 Suppose data items A, B, C, D, E, F, G occur in the following frequencies.

Data Items	A	B	C	D	E	F	G
Weight	10	30	5	15	20	15	05

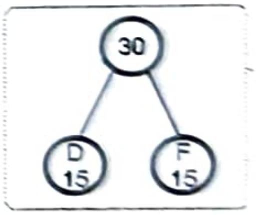
Construct a Huffman code for the data. What is the minimum weighted path length. **[SPPU : May-08, Dec.-14]**

Ans. : Consider the following steps

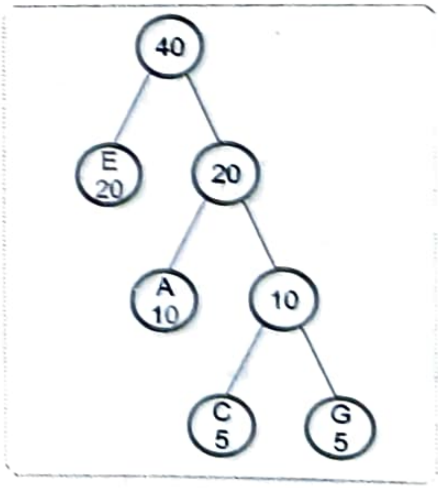


Step 1 : Sequence in increasing order is as follows

C D A D F E B
5 5 10 15 15 20 30
↑ ↑



Step 2 : Sequence : D F F
15, 15, 20, 20, 30
↑ ↑

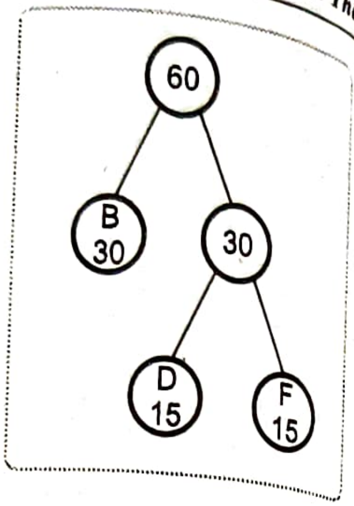


Step 3 : Sequence : E B
20, 20, 30, 30
↑ ↑

Step 4 : Sequence : B

30, 30, 40

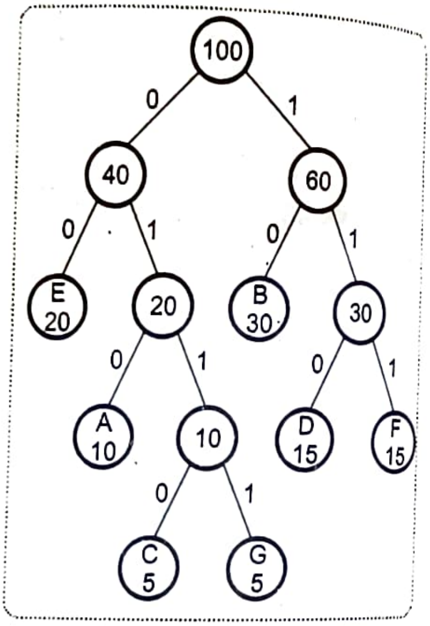
↑ ↑



Step 5 : Sequence : 40, 60

↑ ↑

Items	Binary prefix code
5	0111 or 0110
10	010
15	110 or 111
20	00
30	11



The minimum weight path length for the vertices as follows

$A \rightarrow 3$, $B \rightarrow 2$, $C \rightarrow 4$, $D \rightarrow 3$, $E \rightarrow 2$, $F \rightarrow 3$, $G \rightarrow 4$

∴ The minimum weight of tree is

$$W = 10 \times 3 + 30 \times 2 + 5 \times 4 + 15 \times 3 + 20 \times 2 + 15 \times 3 + 5 \times 4 = \mathbf{260}$$

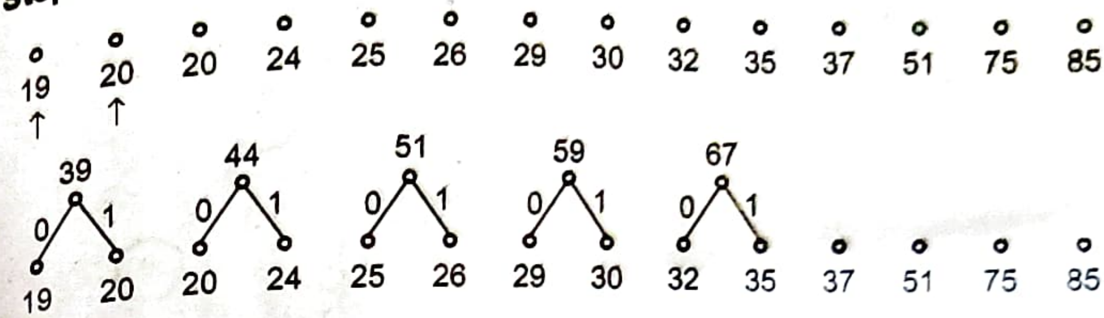
Q.80 A secondary storage media contains information in files with different formats. The frequency of different types of files is as follows.

Exe (20), bin (75), bat (20), jpeg (85), dat (51), doc (32), sys (26), c (19), cpp (25), bmp (30), avi (24), prj (29), 1st (35), zip (37).

Construct the Huffman code for this. **[SPPU : May-15, Dec.-15]**

Ans. : Consider the following steps

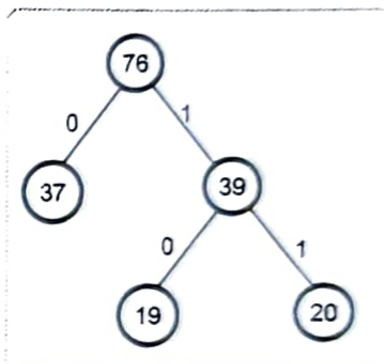
Step 1 : Sequence :



Step 2 : Sequence :

37, 39, 44, 51, 51, 59, 67, 75, 85

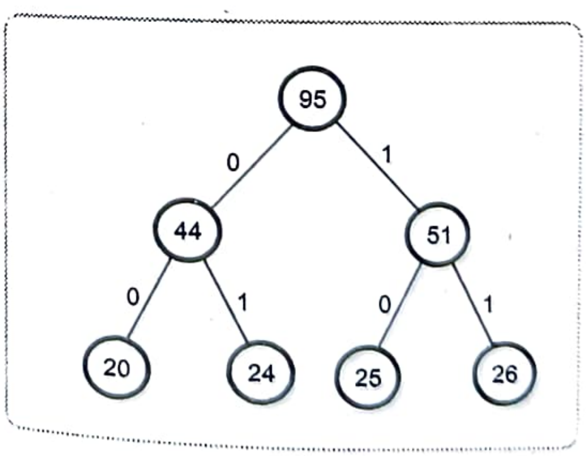
↑ ↑



Step 3 : Sequence :

44, 51, 51, 59, 67, 75, 76, 85

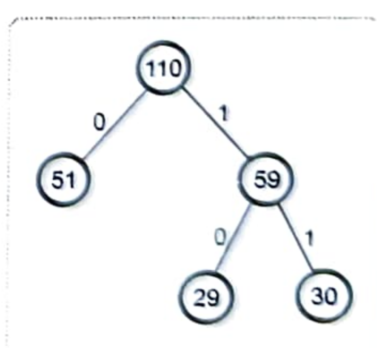
↑ ↑



Step 4 : Sequence :

51, 59, 67, 75, 76, 85, 95

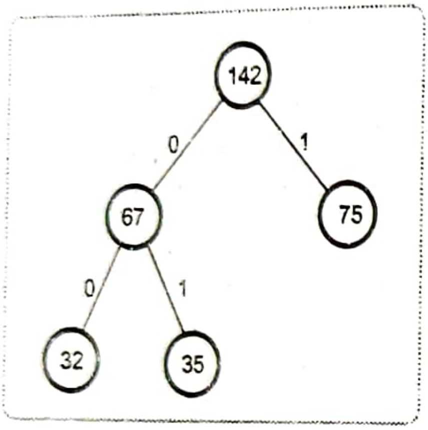
↑ ↑



Step 5 : Sequence :

67, 75, 76, 85, 95, 100

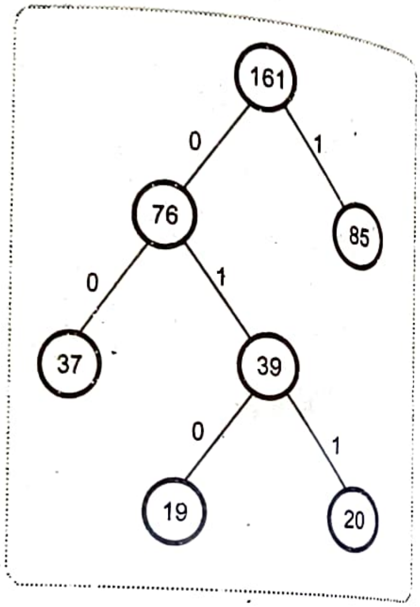
↑ ↑



Step 6 : Sequence :

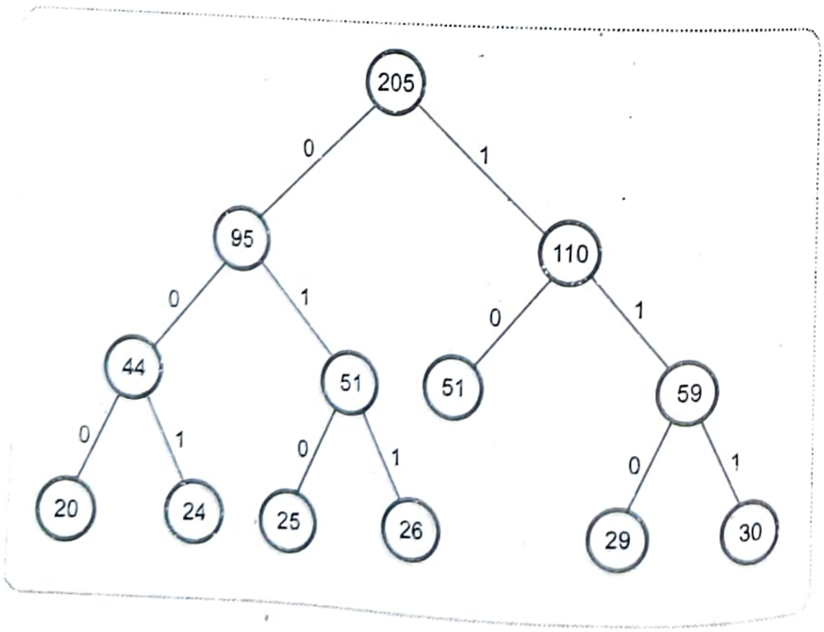
76, 85, 95, 110, 142,

↑ ↑

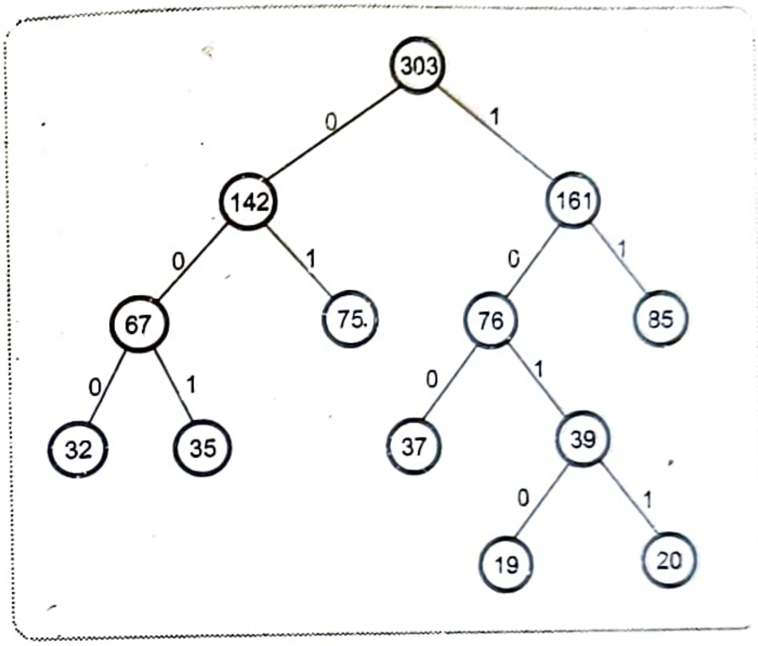


Step 7 : Sequence : 95, 110, 142, 161,

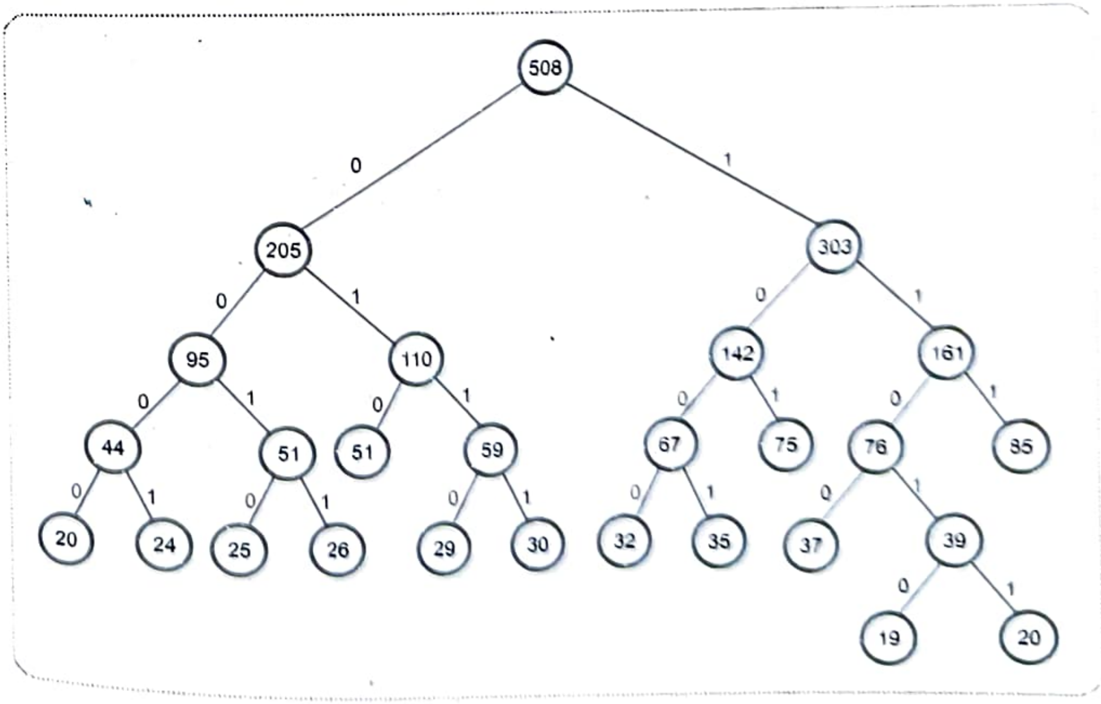
↑ ↑



Step 8 : Sequence : 142, 161, 205
↑ ↑



Step 9 : Sequence : 205, 303,
↑ ↑



Numbers	Binary prifix codes
20	0000
24	0001
25	0010
26	0011
51	010
29	0110
30	0111
32	1000
35	1001
75	101
37	1100
19	11010
20	11011
85	111

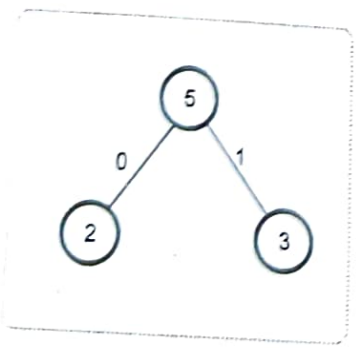
Q.81 For the following sets of weights, construct an optimal binary prefix code. For each weight in the set, give the corresponding code word i) 2, 3, 5, 7, 9, 13 ii) 8, 9, 10, 11, 13, 15, 22 **[SPPU : Dec.-17]**

Ans. : Consider the following steps.

i) **Step 1 :** Sequence :

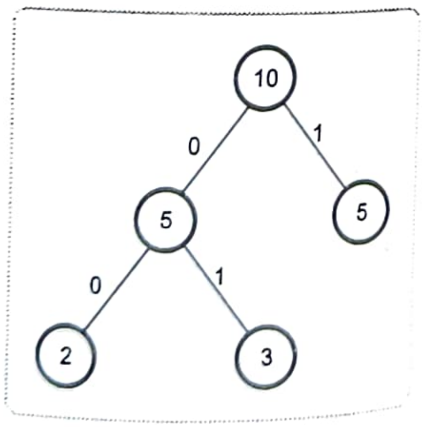
2, 3, 5, 7, 9, 13

↑ ↑



Step 2 : Sequence : 5, 5, 7, 9, 13,

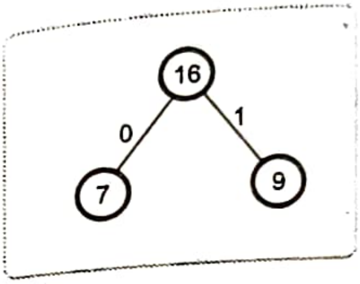
↑ ↑



Step 3 : Sequence :

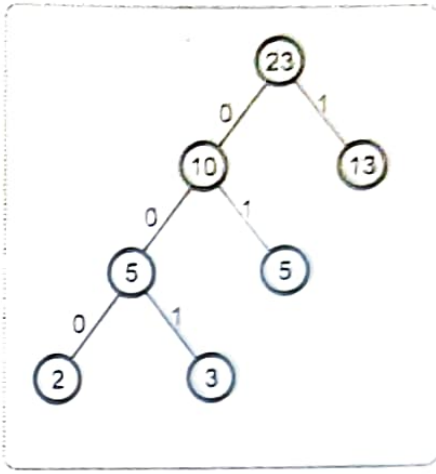
7, 9, 10, 13

↑ ↑



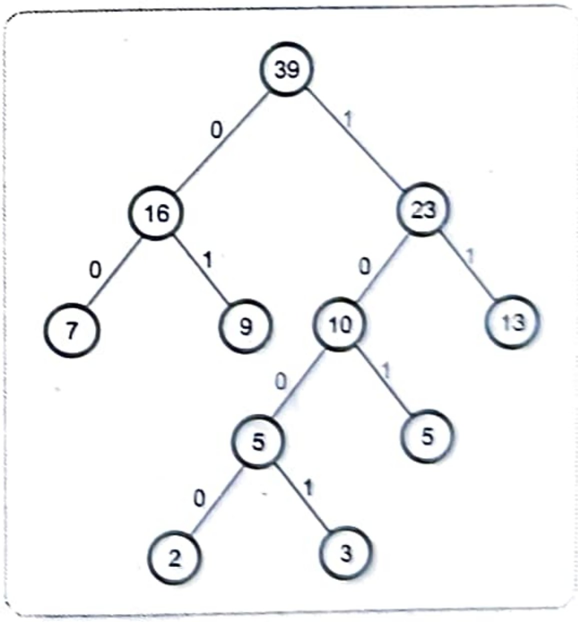
Step 4 : Sequence : 10, 13, 16,

↑ ↑



Step 5 : Sequence : 16, 23

↑ ↑



Symbols	Binary prefix code
7	00
9	01
2	1000

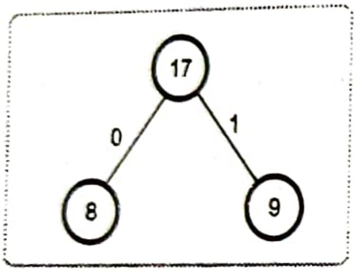
3	1001
5	101
13	11

ii) Consider the following steps

Step 1 : Sequence :

8, 9, 10, 11, 13, 15, 22

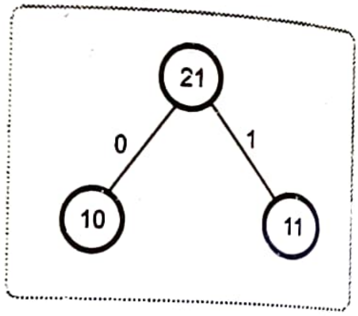
↑ ↑



Step 2 :

Sequence : 10, 11, 13, 15, 17, 22

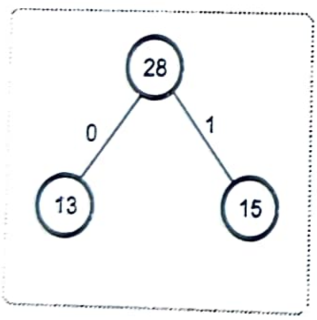
↑ ↑



Step 3 : Sequence :

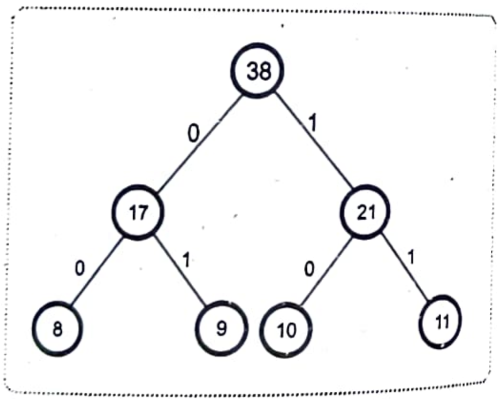
13, 15, 17, 21, 22,

↑ ↑



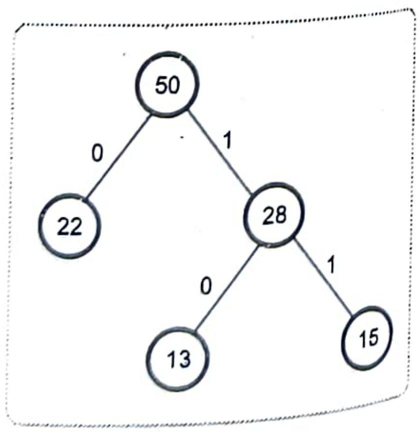
Step 4 : Sequence : 17, 21, 22, 28,

↑ ↑

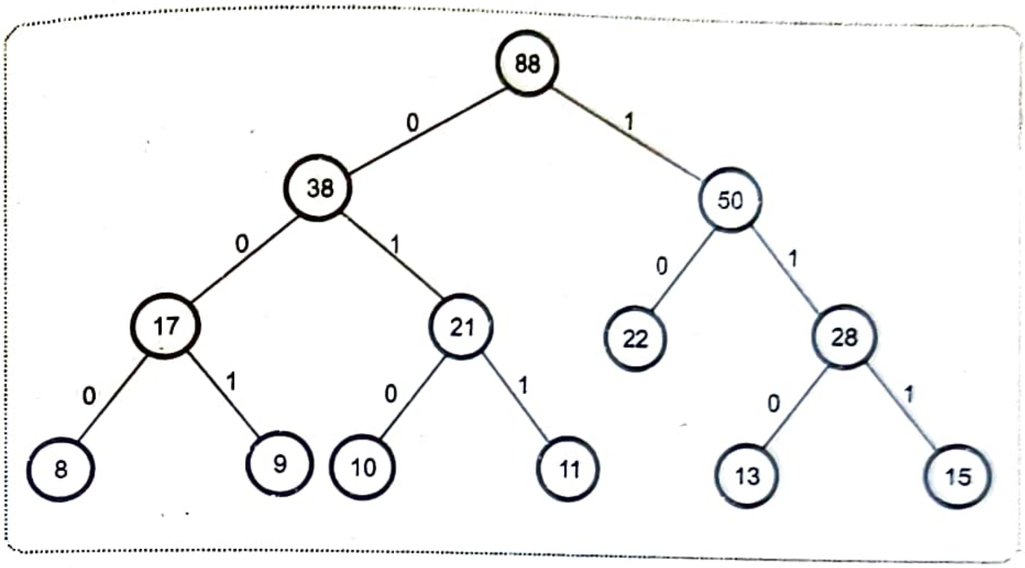


Step 5 : Sequence : 22, 28, 38

↑ ↑



Step 6 : Sequence : 38, 50
↑ ↑



Symbols	Binary prefix code
8	000
9	001
10	010
11	011
22	10
13	110
15	111

3.8 : Spanning Tree

Q.82 Define spanning trees.

[SPPU : Dec.-11, 13, 14, 15, May-05, 14, 15]

Ans. : A spanning subgraph T of a connected graph G is said to be **spanning tree** of G if T is a tree. In other words, A subgraph T of a graph G is said to be spanning tree if V(T) = V(G).

Q.83 Explain Prim's algorithm.

Ans. : Let G(V, E) be a connected weighted graph.

To construct minimum spanning tree of G consider the following steps.

Step 1 : Select any vertex v_0 in graph G.

Set $T = \{v_0, \phi\}$

Step 2 : Find edge $e_i = (v_0, v_i)$ in E such that its one end vertex is $v_0 \in T$ and its weight is minimum.

$\therefore$ New set $T = \{\{v_0, v_1\}, \{e_i\}\}$

Step 3 : Choose next edge $e_k = (v_k, v_j)$ in such a way that it's one end vertex $v_k \in T$ and other vertex $v_j \notin T$ and weight of e_k is as small as possible. Again join vertex v_j and edge e_k to T.

Step 4 : Repeat the step 3 until T contains all the vertices of G. The set T will give the minimum spanning tree of the graph G.

Q.84 Explain kruskal algorithm.

Ans. : Let G(V, E) be a weighted connect graph.

Consider the following steps.

Step 1 : Pick up an edge e_i of G such that its weight $W(e_i)$ is minimum.

(If there are more edges of the minimum weight then select all those edges which do not form a circuit).

Step 2 : If edges $e_1, e_2, \ldots, e_n$ have been chosen then pick an edge e_{n+1} such that

i) $e_{n+1} \neq e_i$ for $i = 1, 2, \ldots n$

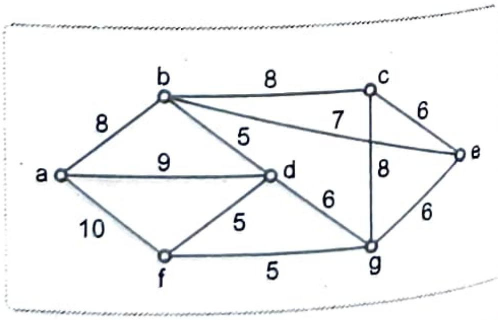
ii) The edges $e_1, e_2, \ldots, e_n, e_{n+1}$ do not form a circuit.

iii) $W(e_{n+1})$ is as small as possible subject to condition (ii).

Step 3 : Stop when step two cannot be implemented.

Q.85 Find the minimum spanning tree for the graph given in the following figure using Prim's algorithm.

[SPPU : Dec.-14]



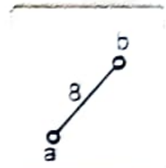
Ans. : Consider the following steps for the construction of the minimum spanning tree.

Step 1 : Select $a \in G$ as a starting vertex

$\therefore$ $T = \{\{a\}, \phi\}$

Step 2 : Vertex a is adjacent to vertices b, d and f. Among these edges minimum weight is $\{a, b\} = 8$

$\therefore$ $T = \{\{a, b\}, \{e_1\}\}$

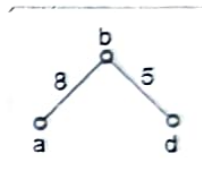


Step 3 : Vertex a is adjacent to f, d.

Vertex b is adjacent to c, d, e

The minimum weight is of an edge $\{b, d\} = 5$

$\therefore$ $T = \{\{a, b, d\}, \{e_1, e_2\}\}$



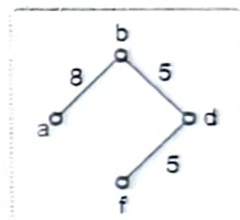
Step 4 : Vertex a is adjacent to f and af = 10

Vertex b is adjacent to c, e and bc = 8, be = 7

Vertex d is adjacent to f and g and df = 5, dg = 6

Among all these weights minimum is 5 = df

$\therefore$ $T = \{\{a, b, d, f\}\{e_1, e_2, e_3\}\}$



Step 5 : Vertex is adjacent to b, d, f but all are in T.

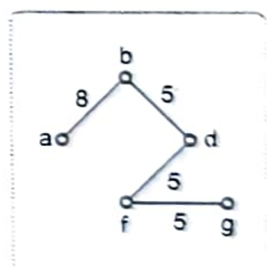
Vertex b is adjacent to c and e

Vertex d is adjacent to g.

Vertex f is adjacent to g

Among all these edges minimum weight is of fg.

$\therefore$ $T = \{\{a, b, d, f, g\}\{e_1, e_2, e_3, e_4\}\}$

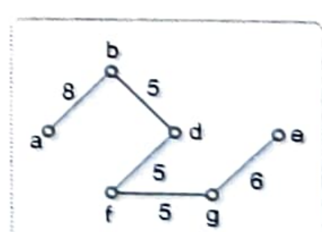


Step 6 : Vertex g is adjacent to c and e

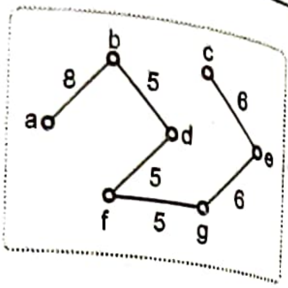
Vertex b is adjacent to c and e

Among these edges the minimum weight is ge = 6

$\therefore$ $T = \{\{a, b, d, f, g, e\}\{e_1, e_2, e_3, e_4, e_5\}\}$



Step 7 : Now only one vertex is remaining.



The vertex C is adjacent to b, e, g.

The minimum weight is ec = 6

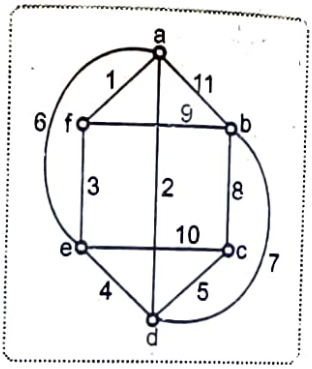
$\therefore \quad T = \{\{a, b, d, f, g, e, c\}\{e_1, e_2, e_3, e_4, e_5, e_6\}\}$

The graph obtained is the minimum spanning tree of weight

$$= 8 + 5 + 5 + 5 + 6 + 6 = 35$$

Q.86 Determine minimum spanning tree for the given graph by Prism's algorithm.

[SPPU : May-14]

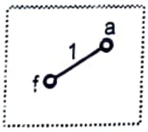


Ans. : Consider the following steps for the construction of the minimum spanning tree.

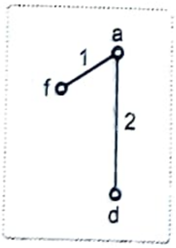
Step 1 : Starting with vertex a

$$T = \{\{a\}, \phi\}$$

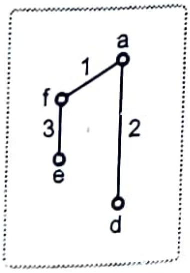
Step 2 : $T = \{\{a, f\}, e_1\}$



Step 3 : $T = \{\{a, f, d\}, \{e_1, e_2\}\}$

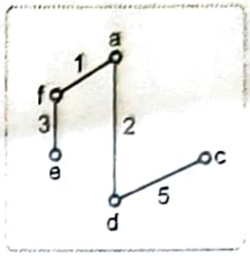


Step 4 : $T = \{\{a, f, d, e\}, \{e_1, e_2, e_3\}\}$



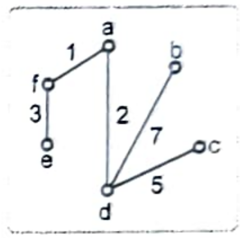
Step 5 :

$T = \{\{a, f, d, e, c\}, \{e_1, e_2, e_3, e_4\}\}$



Step 6 :

$T = \{\{a, f, d, e, c, b\}, \{e_1, e_2, e_3, e_4, e_5\}\}$

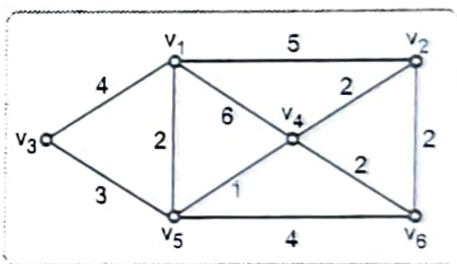


The graph obtained is the minimum spanning tree of weight

$$= 1 + 2 + 1 + 5 + 7 = 18$$

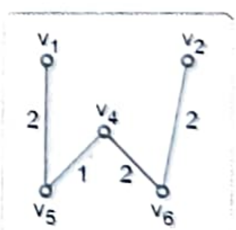
Q.87 Find the minimum spanning tree for graph given below by Kruskal's algorithm.

☞[SPPU : Dec.-13]

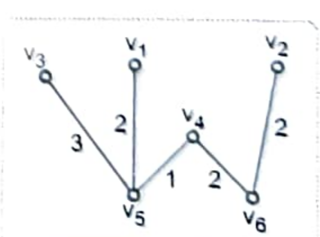


Ans. : Consider the following steps for the construction of the minimum spanning tree.

Step 1 : The minimum weight in given graph is 1, so select an edge $\{v_4, v_5\}$



Step 2 : The minimum weight is 2 in remaining graph. There are four edges of weight 2. These edges form a circuit so select edges which do not form a circuit with selected edge. We select three edges $\{v_5, v_1\}$, $\{v_4, v_6\}$, $\{v_6, v_2\}$

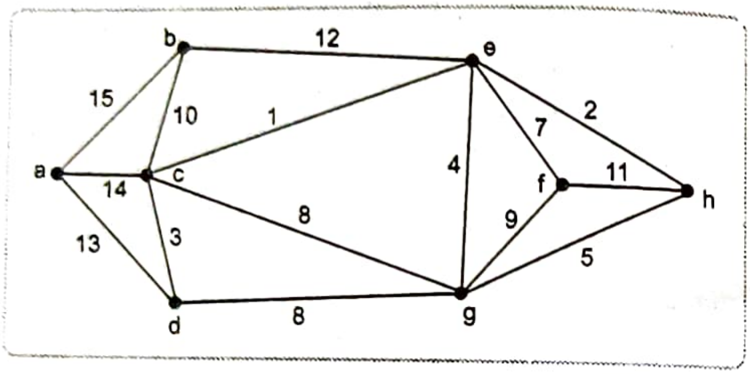


Step 3 : The minimum weight is 3 in the remaining graph so select $\{v_5, v_3\}$

The graph obtained is the minimum spanning tree of weight

$$= 3 + 2 + 1 + 2 + 2 = 10$$

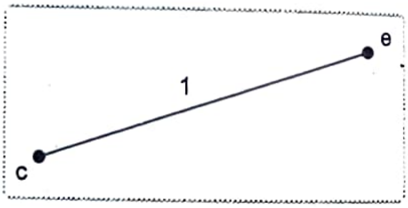
Q.88 Obtain the minimum spanning tree for the following graph. Obtain the total cost of minimum spanning tree.



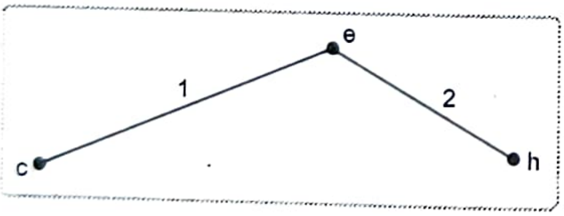
☞[SPPU : May-15, Dec.-15]

Ans. : Using Kruskal algorithm, the minimum spanning tree is obtained as follows :

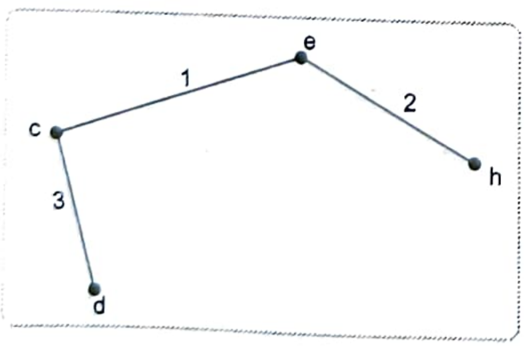
Step 1 : Minimum weight in a given graph is 1 associated with edge {a, e}



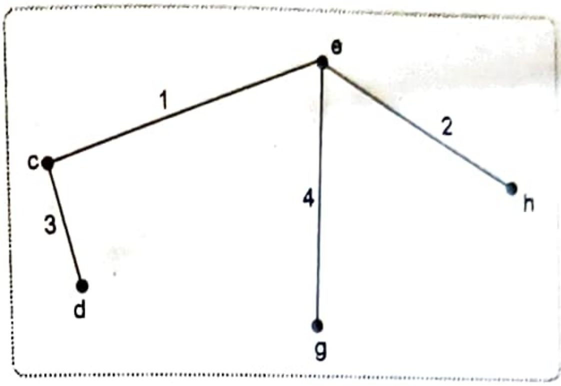
Step 2 : Minimum weight = 2



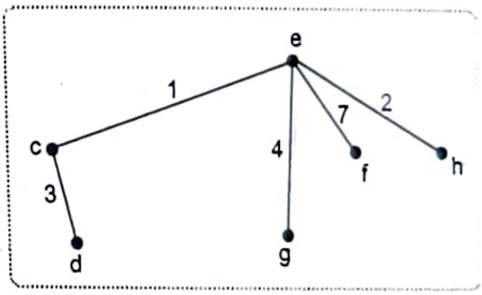
Step 3 : Minimum weight = 3



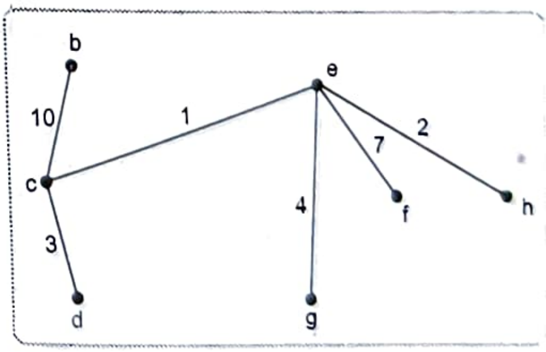
Step 4 : Minimum weight = 4



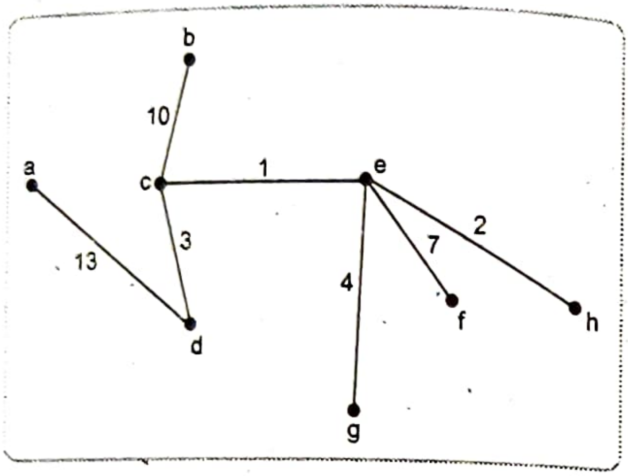
Step 5 : Minimum weight = 7



Step 6 : Minimum weight = 10



Step 7 : Minimum weight = 13



The obtained graph is the minimum spanning tree of the given graph. Its total cost is 40.

3.9 : Fundamental Cutset and Circuits

Q.89 Explain fundamental circuits and cutsets with examples.

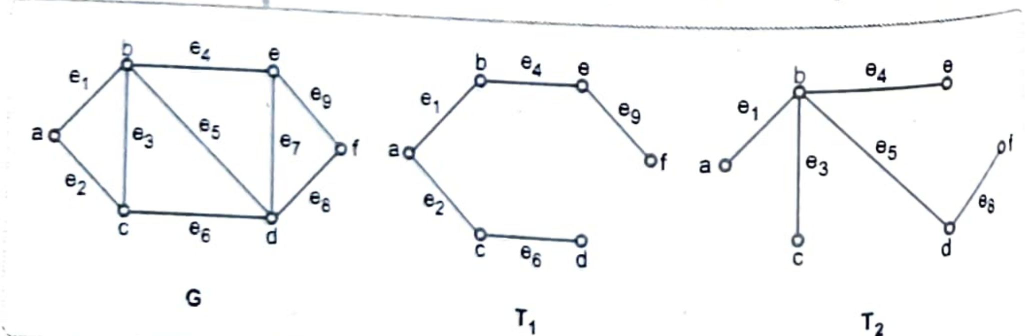
[SPPU : Dec.-07, 12, 14, 15, May-07, 15]

Ans. : Let G be a connected graph and T be a spanning tree of G. An edge of a tree is called a **branch.** An edge of G which is not in T is called chord of T. T + e contains a unique circuit called fundamental circuit of G with respect to T.

A fundamental circuit of a connected graph G is always with respect to a spanning tree of G.

Therefore the different spanning trees will have different fundamental circuits.

Consider the following graph G with two spanning trees T_1 and T_2.



In graph G, vertex set V(G) = {a, b, c, d, e, f}

Edge set E(G) = $\{e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9\}$

A) For Spanning Tree T_1 :

Branches of T_1 are e_1, e_2, e_4, e_6, e_9

Chords of T_1 are e_3, e_5, e_4, e_8

Consider the following table of chords and corresponding fundamental circuits.

Chords	Corresponding Fundamental Circuits
e_3	$\{e_1, e_2, e_3\}$
e_5	$\{e_1, e_2, e_6, e_5\}$
e_7	$\{e_1, e_2, e_4, e_6, e_7\}$
e_8	$\{e_1, e_2, e_4, e_6, e_9, e_8\}$

B) For Spanning Tree T_2

Branches of T_2 are e_1, e_3, e_5, e_4, e_8

Chords of T_2 are e_2, e_4, e_6, e_7, e_9

Consider the following table of chords and corresponding fundamental circuits.

Chords	Corresponding Fundamental Circuits
e_2	$\{e_1, e_3, e_2\}$
e_6	$\{e_3, e_5, e_6\}$
e_7	$\{e_4, e_5, e_7\}$
e_9	$\{e_4, e_5, e_8, e_9\}$

Fundamental Cutsets

Let T be k spanning tree of a connected graph G. Since every edge e of a tree is an isthmus or bridge, T-e splits into two components say T_1 and T_2. But

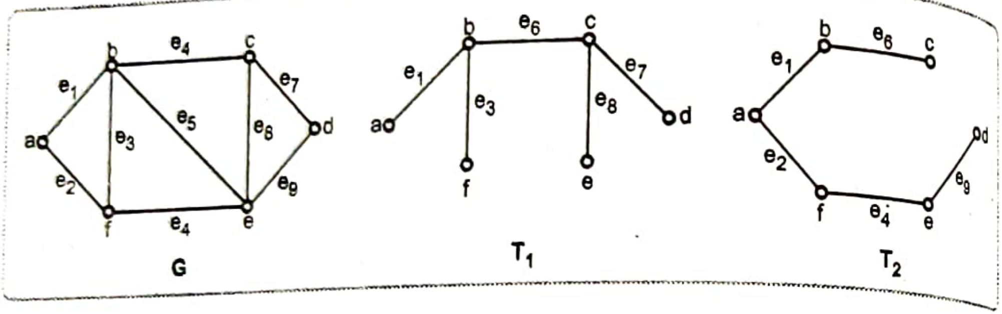
$$V(T) = V(T_1) \cup V(T_2) = V(G)$$

The set E of edges of G which join a vertex in $V(T_1)$ to a vertex in $V(T_2)$ is a cutset of G.

A cutset of G obtained in this manner is called a fundamental cutset of G with respect to T. To each edge of edge there is a fundamental cutset and every fundamental cutset is obtained in this way. Thus the number of fundamental cutsets of G w.r.t. T is the number of branches of T.

Theorem : Let G be a connected graph with n vertices then its spanning tree has $n-1$ edges and there are $n-1$ fundamental cutsets only.

e.g. Consider the following and its spanning tree.



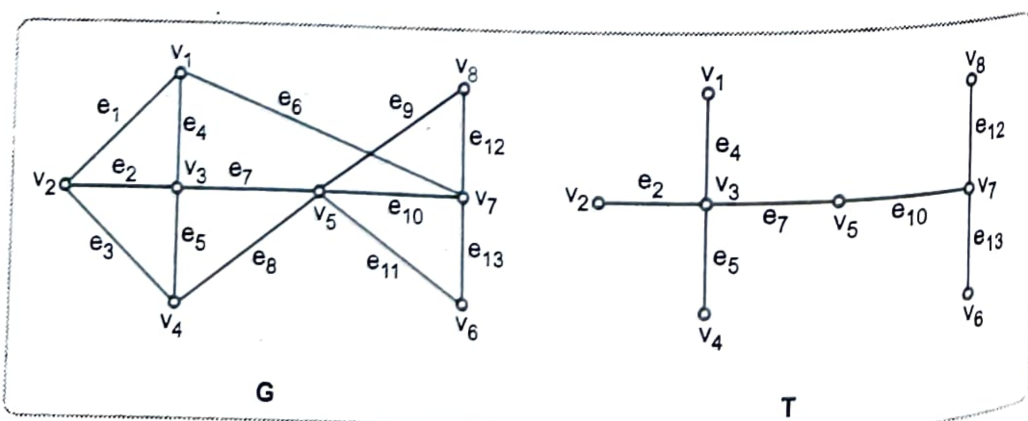
A) For Spanning Tree T_1

Branches of T_1 are e_1, e_3, e_6, e_7, e_8

Consider the following table

Branches T_1	Corresponding Fundamental Cutset
e_1	$\{e_1, e_2\}$
e_3	$\{e_3, e_2, e_4\}$
e_6	$\{e_6, e_5, e_4\}$
e_7	$\{e_7, e_9\}$
e_8	$\{e_8, e_4, e_9, e_5\}$

Q.90 Find the fundamental system of cutset for the graph G shown below w.r.t. the spanning tree T. **[SPPU : May-15, Dec.-12, 15]**

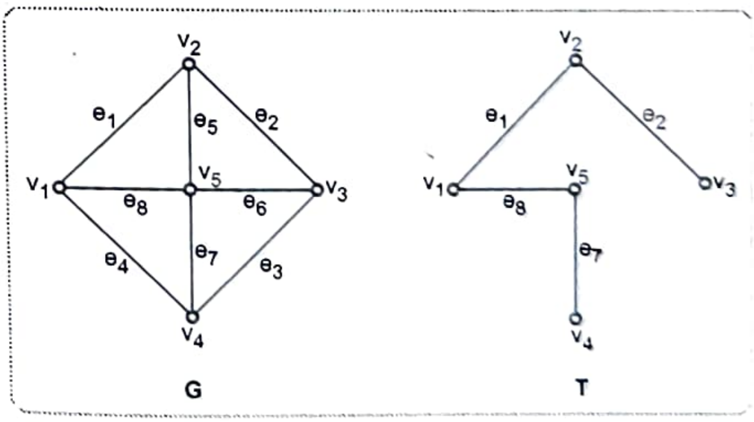


Ans. : The spanning tree T has 7 branches $\{e_2, e_4, e_5, e_7, e_{10}, e_{12}, e_{13}\}$. Therefore are seven fundamental cutsets of G w.r.t. T which are given below :

Branch	Fundamental Cutset
e_2	$\{e_2, e_1, e_3\}$
e_4	$\{e_4, e_1, e_6\}$
e_5	$\{e_5, e_3, e_8\}$
e_7	$\{e_7, e_6, e_8\}$
e_{10}	$\{e_{10}, e_6, e_9, e_{11}\}$
e_{12}	$\{e_{12}, e_9\}$
e_{13}	$\{e_{13}, e_{11}\}$

Q.91 Find the fundamental cutsets and fundamental circuits of the following graph w.r.t. given spanning tree.

[SPPU : May-07, Dec.-07, 14]



Ans. : Here the spanning tree has 4 branches e_1, e_2, e_7, e_8. Therefore there are 4 fundamental cutsets corresponding to each branch of T which are given below.

Branch	Corresponding Fundamental Cutset
e_1	$\{e_1, e_5, e_6, e_3\}$
e_2	$\{e_2, e_6, e_3\}$
e_7	$\{e_7, e_3, e_4\}$
e_8	$\{e_8, e_4, e_5, e_6, e_3\}$

The chords of T are e_3, e_4, e_5, e_6

Therefore there are 4 fundamental circuits corresponding to each chord of T which are given below :

Chord	Corresponding Fundamental Cutset
e_3	$\{e_1, e_2, e_7, e_8, e_3\}$
e_4	$\{e_4, e_7, e_8\}$
e_5	$\{e_5, e_1, e_8\}$
e_6	$\{e_6, e_1, e_2, e_8\}$

3.10 : Max Flow-Min Cut Theorem

Q.92 Explain labelling procedure for finding maximum flow in the Network.

Ans. :

1) The source a is labelled $(-, \infty)$. It means that (out from nowhere) the source can supply can infinite amount of material to the other vertices.

2) A vertex b that is adjacent from a is labelled $(a^-, \Delta b)$, where Δb is equal to $w(a, b) - \phi(a, b)$,

if $w(a, b) > \phi(a, b)$;

i.e. $$\Delta b = w(a, b) - \phi(a, b) \qquad [\text{if } w(a, b) > \phi(a, b)]$$

The vertex is not labelled if $w(a, b) = \phi(a, b)$

3) Scan and label all the remaining vertices adjacent to a. Also scan and label all the vertices adjacent to labelled vertices.

Suppose vertex q is adjacent to labelled vertex b, then q is labelled as $(b^+, \Delta q)$

where $\Delta q = \min\{\Delta b, [w(b, q) - \phi(b, q)]\}$

if $w(b, q) > \phi(b, q)$

Vertex q is not labelled if $w(b, q) = \phi(b, q)$

We can also label vertex q as $(\bar{b}, \Delta q)$ where

$$\Delta q = \min[\Delta b, \phi(q, b)] \quad \text{if} \quad \phi(q, b) > 0$$

4) Repeat step 3 till we reach to sink z.

5) If we repeat this labelling procedure, two cases shall arise while labelling the sink z.

Case I :

i) Sink z is labelled, say with a label $(y^-, \Delta z)$ [z is never labelled $(y^-, \Delta z)$]

ii) Vertex y can be labeled as $(q^-, \Delta y)$ or $(q^-, \Delta y)$ for some adjacent vertex q.

iii) If y is labelled $(q^-, \Delta y)$ (q^- means increase in flow) then we increase the flow in edge (q, y) to $\phi(q, y)$ to Δz. Similarly for the label $(q^-, \Delta y)$ we decrease the flow in edge (q, y) to $\phi(q, y) - \Delta y$.

iv) This process is continued back to source a till the value of flow is increased by amount Δz.

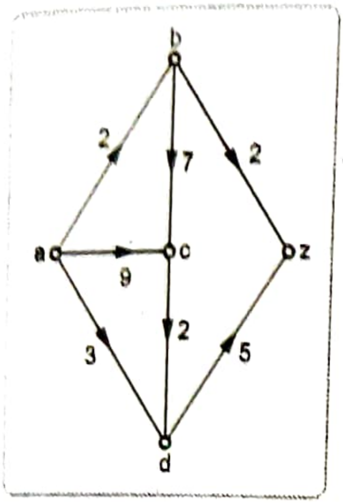
v) Again start the labelling procedure to further increase value of flow in the network.

Case II :

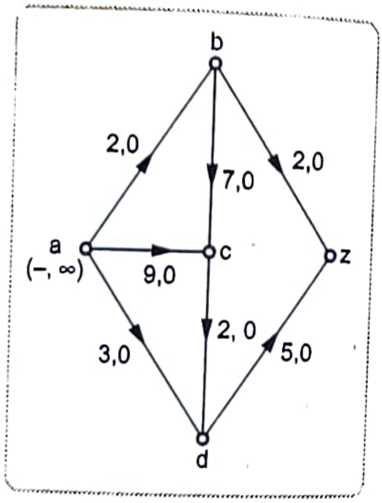
i) If the sink z is not labelled, then denote all labelled vertices as P and all unlabelled vertices as $\bar{P}$.

ii) The fact that sink z is not labeled means flow each edge directed from vertices of P to vertices of $\bar{P}$, is equal to capacity of cut $(P, \bar{P})$ is thus maximum flow.

Q.93 Determine the maximal flow in the following transport network. **[SPPU : Dec.-12, May-14]**



Ans. : Step 1 : Assign the flow zero to each edge and the label $(-, \infty)$ to the source a.



Step 2 : The vertices b, c, d are adjacent to the source a.

Therefore, we label the vertices b, c, d.

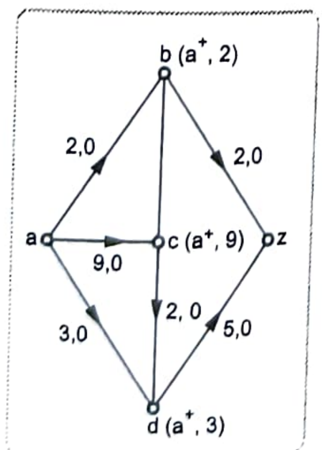
For the vertex b

$$w(a, b) - \phi(a, b) = 2$$

i.e. $\Delta b = 2$

Similarly $\Delta c = 9$ and $\Delta d = 3$

Now sink z is adjacent to both vertices b and d, so we arbitrarily choose vertex d and label.



sin z as $(d^+, 5)$ as

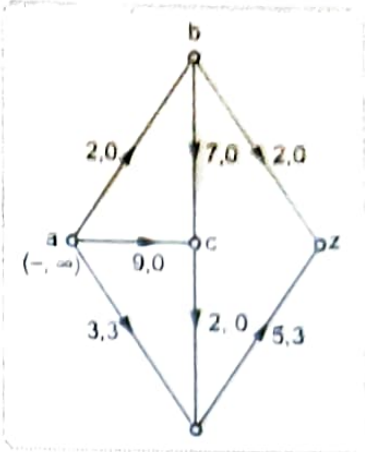
$$w(d, z) - \phi(d, z) = 5 - 0 = 5$$

and $\Delta z =$

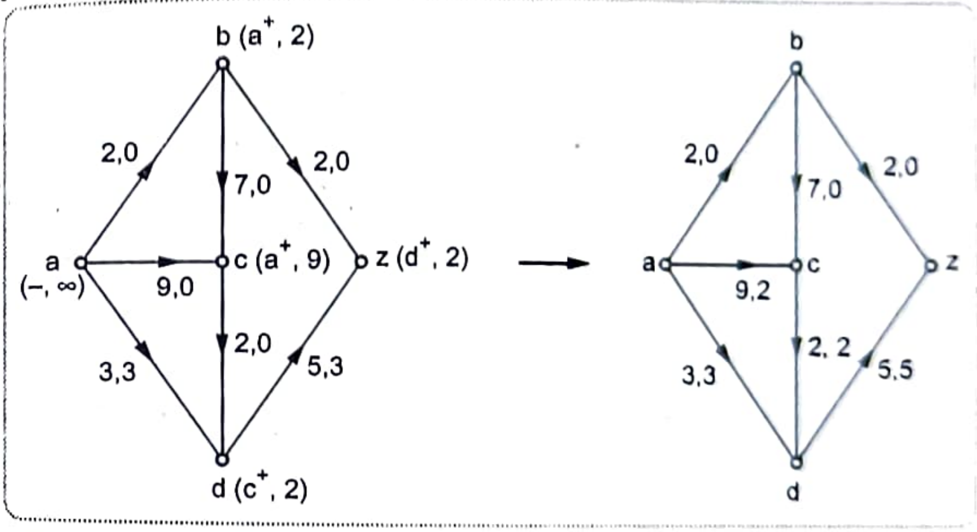
$$\min\{\Delta d, \{w(d, z) - \phi(d, z)\}$$

$$= \min\{3, 5\} = 3$$

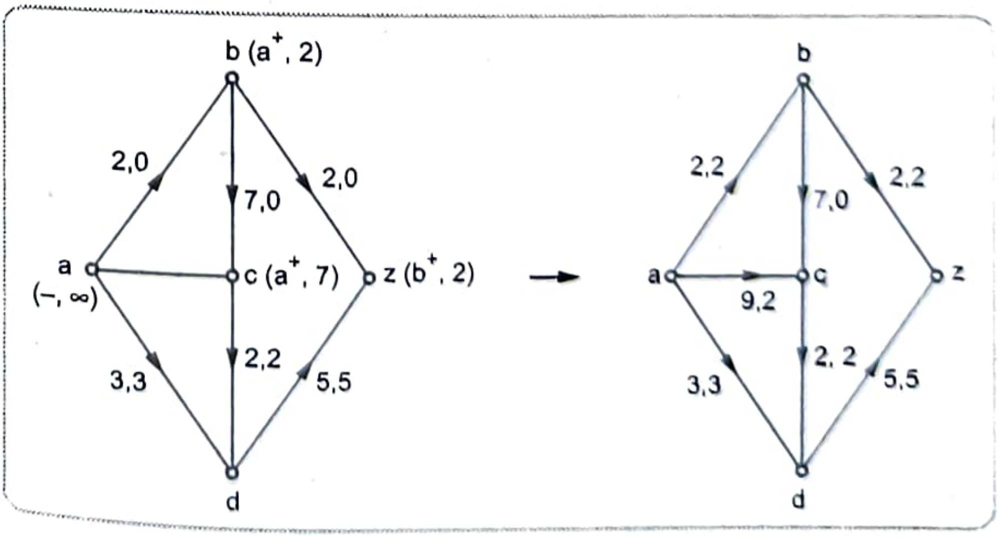
According to label of sink, we now adjust flow of edge (d, z) and edge (a, d)



Step 3 :



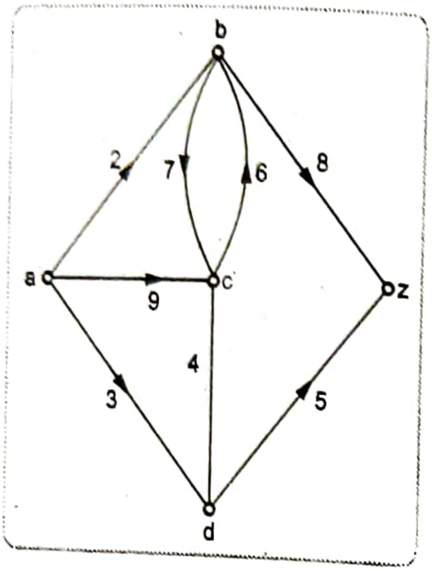
Step 4 :



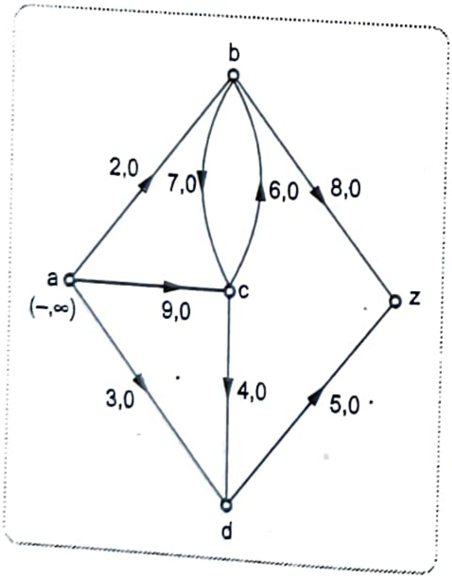
From the step 4, it is clear that the maximum flow is 7.

Q.94 Determine the maximal flow in the flowing transport network.

[SPPU : Dec.-13]



Ans. : Step 1 : Assign the flow zero to each edge and the label $(-, \infty)$ to the source a.

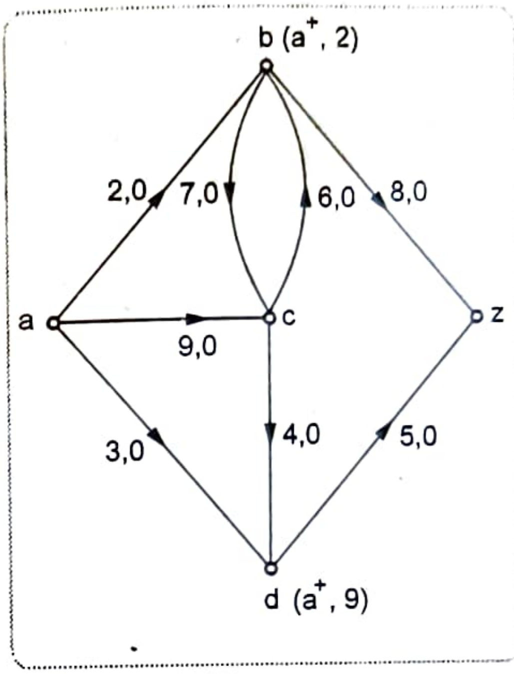


Step 2 : The vertices b, c, d are adjacent to the source a. Therefore, we label the vertices b, c, d for the vertex b.

$w(a, b) - \phi(a, b) = 2$

i.e. $\Delta b = 2,$

Similarly $\Delta c = 9, \quad \Delta d = 3$

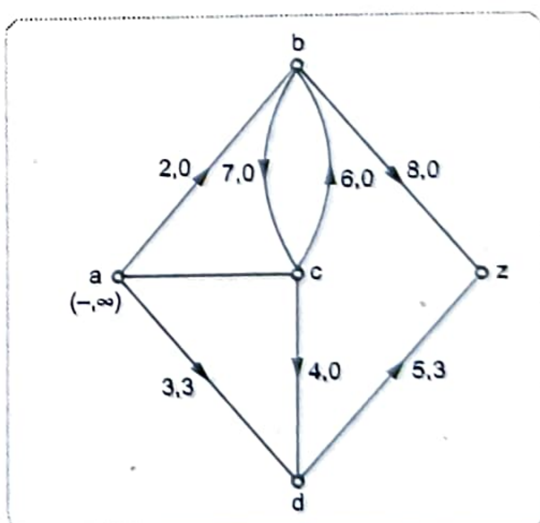


Now sink z is adjacent to both vertices b and d, so we can choose any vertex b or d for labelling of sink z. Let us choose the vertex d. The label of sink z is $(d^+, 5)$ as $w(d, z) - \phi(d, z) = 5 - 0 = 5$

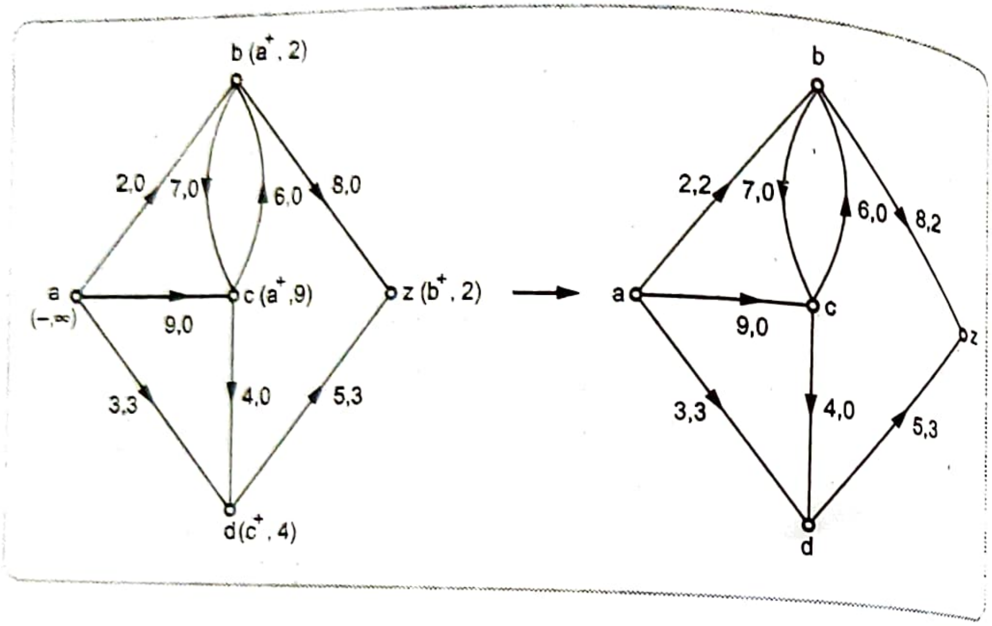
and

$$\Delta z = \min\{\Delta d, w(d, z) - \phi(d, z)\}$$
$$= \min\{3, 5\} = 3$$

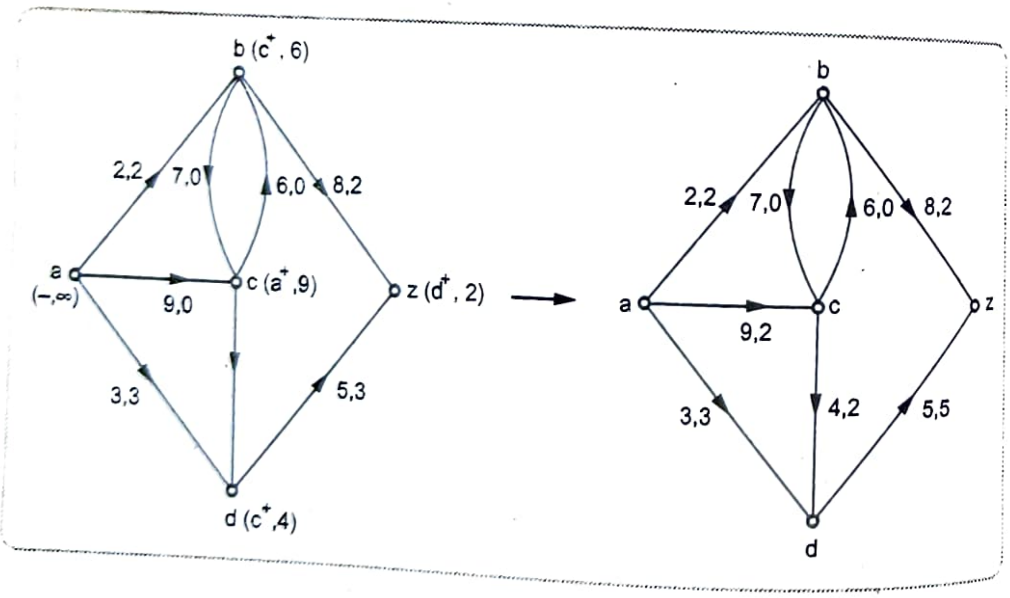
According to label of sink, we now adjust flow of edge (d, z) and edge (a, d)



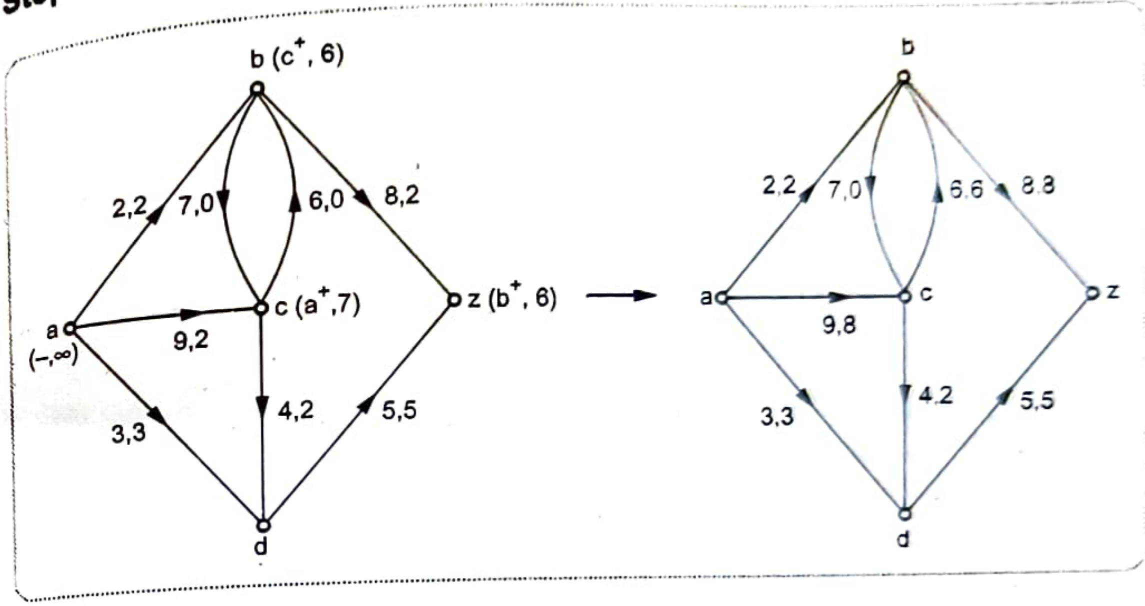
Step 3 :


b (a⁺, 2)
2,0
7,0
6,0
8,0
a
(−,∞)
c (a⁺,9)
9,0
z (b⁺, 2)
3,3
4,0
5,3
d(c⁺, 4)
b
2,2
7,0
6,0
8,2
a
c
9,0
z
3,3
4,0
5,3
d

Step 4 :


b (c⁺, 6)
2,2
7,0
6,0
8,2
a
(−,∞)
c (a⁺,9)
9,0
z (d⁺, 2)
3,3
5,3
d (c⁺,4)
b
2,2
7,0
6,0
8,2
a
c
9,2
z
3,3
4,2
5,5
d

Step 5 :



Q.95 Use labelling procedure to find a maximum flow in the transport network shown in the following Fig. Q.95.1. Determine the corresponding minimum cut.

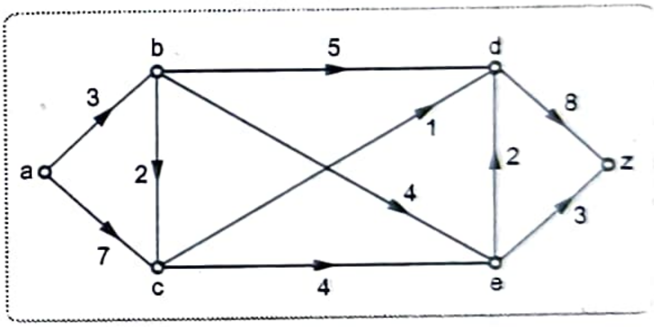
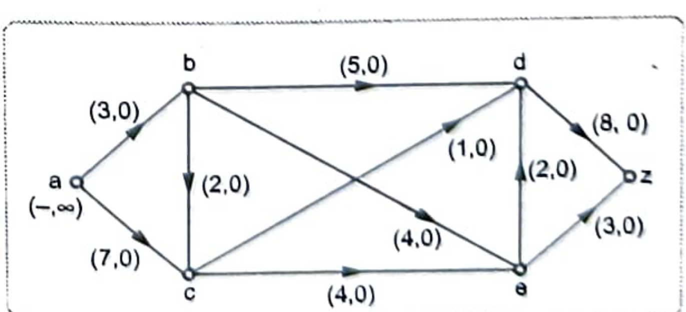


Fig. Q.95.1

[SPPU : May-07, Dec.-14]

Ans. : Step 1 : To find the maximum flow in the given transport network, assign the flow zero to each edge and label $(-, \infty)$ to the source a.



Step 2 : Since the vertices b and c are adjacent to the source a, therefore label the vertices b and c as $(a^+, \Delta b)$ and $(a^+, \Delta c)$ respectively.

Where $\Delta b = ((a, b) - f(a, b) = 3$

$\Delta c = ((a, b) - f(a, c) = 7$

$\therefore$ Label of b is $(a^+, 3)$ and label of c is $(a^+, 7)$

Now the vertices d and e are adjacent to labeled vertices b $(a^+, 3)$ and $(a^+, 7)$ therefore we label the vertices d and e as $(b^+, \cdot d)$ and $(c^+, \cdot e)$ where

$$\Delta d = \min (\Delta b, c(b, d) - f(b, d))$$
$$= \min (3, 5, -0)$$
$$= 3$$

Similarly
$$\Delta e = \min (\Delta c, c(c, e) - f(c, e))$$
$$= \min (7, 4, -0)$$
$$= 4$$

$\therefore$ Label of d is $(b^+, 3)$ and label of e is $(c^+, 4)$

Now the sink z is adjacent to both the vertices d and e, we can choose any vertex d or e to label the sink z.

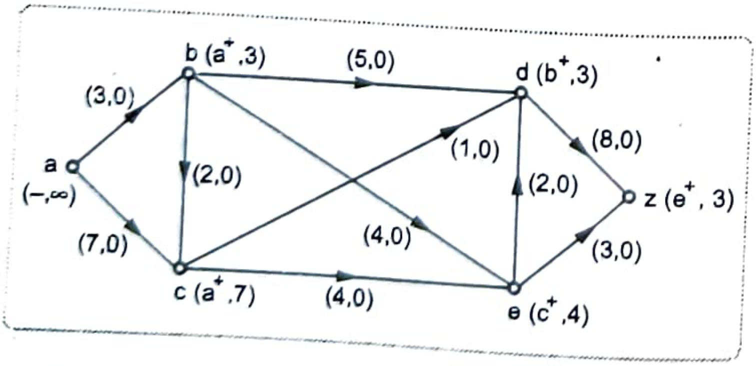
Let us choose the vertex e, which is labeled as $(c^+, 4)$

The label of z will be $(e^+, \Delta z)$ where

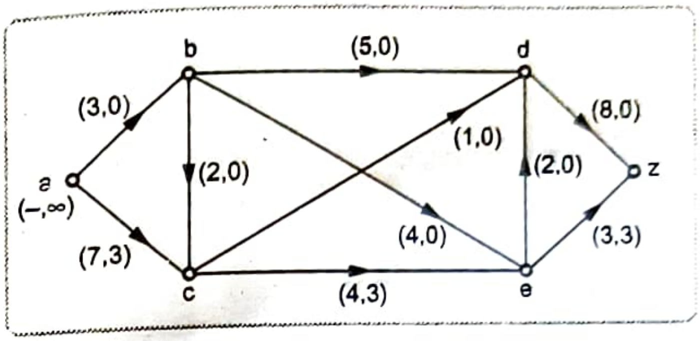
$$\Delta z = \min(4, 3, -0)$$
$$= 3$$

Hence label of z is $(e^+, 3)$

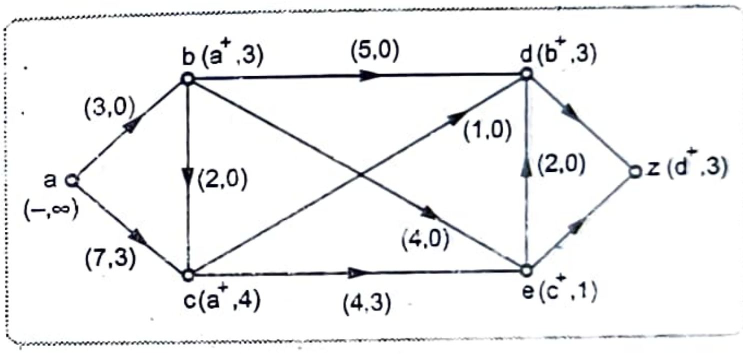
The labels of vertices can be shown as follows :



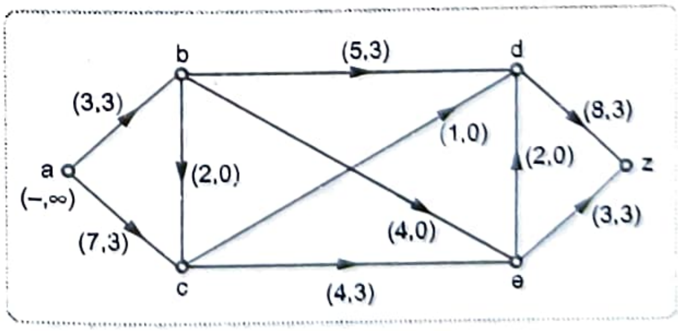
According to the label of the sink z, adjust the flow in the edges (e, z), (c, e) and (a, c)



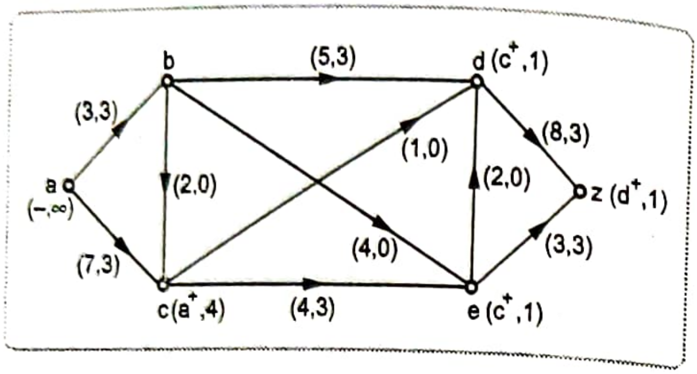
Repeat the step 2 in each pass, the new value of the flow is obtained.



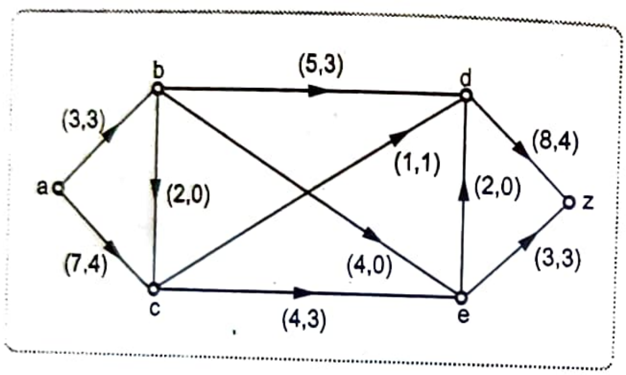
After adjusting the flow we have

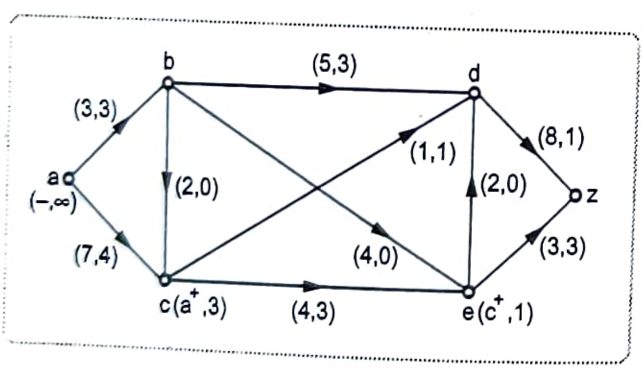


Step 4 :



Adjust the flow according to sink z label.





Since the vertex z (sink) cannot be labelled, we have to stop. The vertices b, d and z cannot be labelled because either the edges approaching to vertices are saturated or the edges in the opposite direction have flow zero. At this stage, we have minimum cut $(P, \bar{P})$.

P is the set of labelled vertices P = {a, c, e}. $\bar{P}$ is the set of unlabeled vertices, $\bar{P}$ = {b, d, z}

The edges in cut $P\bar{P}\{(a, b), (c, d), (e, z)\}$ capacity of the cut is $c(a, b) + c(c, d) + c(e, z) = 3 + 1 + 3 = 7$.

Hence the maximum flow in the given transport network 7.

Q.96 Explain game tree.

Ans. : A directed graph whose nodes are positions in a game and edges are moves, is called a game tree.

The complete game tree for a game is the game tree starting at the initial position and containing all possible moves from each position. The complete tree is the same tree as that obtained from the extensive form game representation.

We look at using trees for game planning, in particular the problems of searching game trees. We classify all games into following three types.

1) Single player path finding problems :

For examples :

a) Travelling salesman problem

b) Sliding puzzle

c) Hamiltonian paths

d) Rubik's cude

2) Two player games :

For examples :

a) Chess

b) Badminton

c) Tennis

d) Checkers

3) Constraint satisfication problems :

For examples :

a) Sudoku

b) Eight queen problem

c) Mathematical puzzles

d) Four queen problem

Each game consists of a problem space, an initial state and a set of goal states.

A problem space is a mathematical abstraction in a form of a tree :

- The root represents current state
- Nodes represent states of the game
- Edges represents moves
- Leaves represent final states (Win, loss or draw)

For example, in the 8-puzzle game

- **Nodes :** The different permentations of the tiles.
- **Edges :** Moving the black file up, down, right or left.

Minimax :

We consider game with two players in which one players gains are the result of another players losses so called zero sum games.

The minimax algorithm is a specialized search algorithm which returns the optimal sequence of moves for a player in an zero sum game.

In the game tree that result from the algorithm, each level represents a move by either of two players, say A and B player. The minimax algorithm explores the entire game tree using a depth-first search.

At each node in the tree where A-player has to move, A-player would like to play the move that miximizes the payoff. Thus, A-player will assign the maximum score amongst the children to the node where max makes a move. Similarly, B-player will minimize the pay off to A-player. The maximum and minimum scores are taken at alternating levels of the tree, Since A and B alternate turns. The minimax algorithm computes the minimx decision for the leaves of the game tree and than backs up through the tree to give the final value to the current state.

END...

Unit IV

4 Relations and Functions

4.1 : Relations

Important Points to Remember

1. **Cartesian Product :** Let A and B be two non empty sets. The Cartesian product of A and B is denoted by A B and defined as

$$A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$$

(x, y) is known as order pair.

2. **Relation :** Let A and B be two non-empty sets. A relation from A to B is any subset of $A \times B$. It is denoted by $R : A \to B$. Set A is called domain set and B is called co-domain set.

The range set of relation R, is the set of elements of B that are second elements of pairs in R.

3. **A relation R :** $A \to A$ is called the relation on set A.

4. A relation can be represented in two forms.

(a) **Matrix form :** Let $R : A \to B$ be a relation.

$A = \{a_1, a_2, a_3, \ldots\ldots, a_n\}$, $B = \{b_1, b_2, b_3, \ldots\ldots, b_m\}$.

Then R can be represented by n m matrix such that

$$m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{if } (a_i, b_j) \notin R \end{cases}$$

(b) **Digraph form :** Let R be a relation on set A. Draw small circles or dark dots for each element of A. These circles are called vertices of digraph.

Draw an edge with arrow from a_i to a_j iff $a_i \, R \, a_j$ such representation is called digraph or directed graph of R.

5. Types of Relation :

Sr. No.	Name of relation	Condition
1.	Reflexive Relation	If $(a, a) \in R$ for all $a \in A$
2.	Irreflexive	If $(a, a) \notin R$ for all $a \in A$
3.	Symmetric	Whenever $(a, b) \in R$ then $(b, a) \in R$
4.	Asymmetric	Whenever $(a, b) \in R$ then $(b, a) \notin R$
5.	Antisymmetric	Whenever $(a, b) \in R$ and $(b, a) \in R$ then $a = b$
6.	Transitive relation	If $(a, b) \in R$, $(b, c) \in R$ then $(a, c) \in R$
7.	Equivalence relation	If R is reflexive, symmetric and transitive.
8.	Partial ordering relation	If r is reflexive antisymmetric and transitive.

6. Important Properties :

i) *Complement of a relation* : Let R be a relation from A to B. The complement of relation R is denoted by $\overline{R}$ and defined as relation from A to B such that

$$\overline{R} = \{(a, b) \mid (a, b) \notin R\}$$

ii) *Inverse of relation* : Let $R : A \to B$ be a relation then the inverse relation $R^{-1} : B \to A$ is a relation such that $xR^{-1}y$ iff $y\,Rx$.

iii) *Composite relation* : Let $R_1 : A \to B$ and $R_2 : B \to C$ be two relations. Then the composition of relations R_1 and R_2 is denoted by $R_1 \cdot R_2$ or $R_1\,R_2$ and defined as

$$R_1 \cdot R_2 = \{(x, z) \mid xR_1y \text{ and } y\,R_2\,z \;;\; x \in A,\ y \in B,\ z \in C\}$$

iv) *Equivalence class* : Let $R : A \to B$ be an equivalence relation. The equivalence class of an element $a \in A$ is the set of all elements of A which are related to a. It is denoted by [a

$$[a] = \{x \mid xRa \;;\; x \in A\}$$

7. **Closure of Relations :**

i) *Reflexive closure* : Let R be a relation on a set A and R is not reflexive relation. A relation $R_1 = R \cup \Delta$ is the reflexive closure of R if $R \cup \Delta$ is the smallest reflexive relation containing R. If $A = \{x_1, x_2, x_3, x_4\}$ then take

$$\Delta = \{(x_1, x_1), (x_2, x_2), (x_3, x_3), (x_4, x_4)\}$$

ii) *Symmetric closure* : A relation R^* is the symmetric closure of relation R if $R^* = R \cup R^{-1}$.

iii) *Transitive closure* : Let $R : A \rightarrow A$ be a relation which is not transitive. The transitive closure of a relation R is the smallest transitive relation containing R.

Note : *Let A be a set with n elements and R be a relation on set A.*

$$\text{Then } R^* = R \cup R^1 \cup R^2 \cup \ldots\ldots \cup R^n$$

4.2 : Warshall's Algorithm to Find Transitive Closure

Important Points to Remember

To find the transitive closure of a relation by computing various powers of R or product of the relation matrix is quite impractical for large relations. Warshall's algorithm gives an alternate method for finding transitive closure of R. Warshall's algorithm is practical and efficient method.

Consider the following steps to find transitive closure of the relation R on a set A.

Step 1 : We have $|A| = n$

$\therefore$ We require $W_o, W_1, W_2, \ldots W_n$. Warshall sets

$$W_o = \text{Relation Matrix of } R = M_R.$$

Step 2 : To find the transitive closure of relation R on set A, with $|A| = n$

Procedure to compute W_k from W_{k-1} is as follows

i) Copy 1 to all entries in W_k from W_{k-1}, where there is a 1 in W_{k-1}.

ii) Find the row numbers $p_1, p_2, p_3 \ldots$ for which there is 1 in column k in W_{k-1} and the column numbers $q_1, q_2, q_3 \ldots$ for which there is 1 in row k of W_{k-1}.

iii) Mark entries in W_k as 1 for (p_i, q_i). If there are not already 1.

Step 3 : Stop the procedure when W_n is obtained and it is the required transitive closure of R.

Q.1 Which of the following are relations from A to B where A = {1, 2, 3, 4} ; B = {x, y, z}

a) R_1 = {(1, x), (1, y), (1, z), (4, x)}

(b) R_2 = {(x, 1), (y, 1), (z, 1), (x, 4)}

(c) R_3 = {(1, x), (2, y), (3, z), (4, w)}

(d) R_4 = {(1, 1), (2, 2)}

Ans. : Given that $R : A \to B$

$$\text{Where } A = \text{Domain set}$$

$$\text{and } B = \text{Co-domain set}$$

$$A \times B = \{(1, x), (1, y), (1, z), (2, x), (2, y), (2, z), (3, x), (3, y), (3, z), (4, x), (4, y), (4, z)\}$$

(a) R_1 is a relation from A to B because $R_1 \subset A \times B$

(b) R_2 is not a relation from A to B as $R_2 \not\subset A \times B$

(c) R_3 is not a relation from A to B as $(4, w) \in R$ but $w \not\subset B$

(d) R_4 is not a relation from A to B as $R_4 \not\subset A \times B$

Q.2 Let A = {1, 2, 3, 4, 5, 6} aRb or (a, b) ∈ R iff a is a multiple of b. Find

(i) Range set and R (6), R (3). (ii) Find relation matrix.

(iii) Draw diagraph (iv) Find in and out degree of each vertex.

Ans. : By considering given condition.

R = {(1, 1), (2, 1), (2, 2), (3, 1), (3, 3), (4, 1), (4, 2), (4, 4), (5, 1), (5, 5), (6, 1), (6, 2), (6, 3), (6, 6)}

(i) The range set of R is {1, 2, 3, 4, 5, 6}

$$R(3) = \{1, 3\} \text{ as } (3, 1) \text{ and } (3, 3) \in R$$

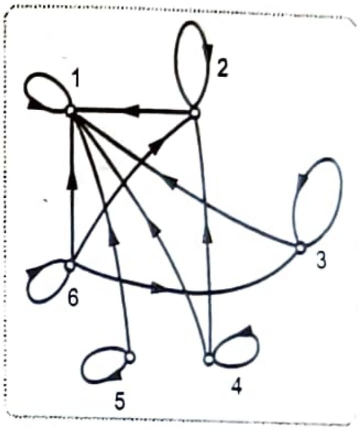
$$\text{and } R(6) = \{1, 2, 3, 6\} \text{ because } (6, 1), (6, 2), (6, 3), (6, 6) \in R$$

(ii) The relational matrix is,

$$M(R) = \begin{array}{c c} & \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} & \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

(iii) The diagraph of given relation is

vertex set = {1, 2, 3, 4, 5, 6}



(iv) The indegree of a vertex in diagraph is the number of edges coming towards that vertex. The outdegree of a vertex is defined as the number of edges going out from a vertex. Consider the following table.

Vertex	Indegee	Outdegree	Degree
1	6	1	7
2	3	2	5
3	2	2	4
4	3	1	4
5	2	1	3
6	4	1	5

Q.3 Let A = {1, 2, 3, 4, 5, 6, 7, 8, 9} and let N be the relation on A × A defined by (a, b) ~ (c, d) iff a + d = b + c.

(i) Prove that ~ is an equivalence relation.

(ii) Find equivalence class of (2, 5) **[SPPU : Dec.-11]**

Ans. : Given that $(a, b) \sim (c, d)$ iff $a + d = b + c$, $\forall$ a, b, c, d $\in$ A

(i) (a) We have $a + b = b + a \Rightarrow (a, b) \sim (a, b)$

$\Rightarrow$ '$\sim$' is reflexive relation.

(b) If $(a, b) \sim (c, d)$ then $a + d = b + c$.

$\Rightarrow b + c = a + d \Rightarrow c + b = d + a$

$\Rightarrow (c, d) \sim (a, b)$ by definition

$\Rightarrow$ '$\sim$' is symmetric relation.

(c) Suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$

then $\quad a + d = b + c$

and $\quad c + f = d + e$

$$\Rightarrow \quad a + \cancel{d} + \cancel{c} + f = b + \cancel{c} + \cancel{d} + e$$

$$a + f = b + e$$

$$\Rightarrow \quad (a, b) \sim (e, f)$$

$\therefore \sim$ is a transitive relation.

Thus '$\sim$' is reflexive, symmetric and transitive.

$\Rightarrow$ '$\sim$' is an equivalence relation.

(ii) We have $(2, 5) \in A \times A$

The equivalence class of (2, 5) is the set of elements of $A \times A$ which are equivalent to (2, 5)

$$\therefore \quad [(2, 5)] = \{(x, y) \mid (x, y) \sim (2, 5)\ ,\ x, y \in A\}$$

$$= \{(x, y) \mid x + 5 = y + 2\}$$

$$= \{(x, y) \mid x - y = -3 \text{ or } y = x + 3\ ;\ x\ ,\ y \in A\}$$

Hence, $[(2, 5)] = \{(1, 3)\ (2, 5)\ \ (3, 6)\ \ (4, 7)\ \ (5, 8)\ \ (6, 9)\}$

... $[\because (7, 10) \notin A \times A]$

Q.4 Show that $R = \{(a, b) \ / \ a \equiv b \pmod m\}$ is an equivalence relation on z. Show that $x_1 \equiv y_1$ and $x_2 \equiv y_2$. then $x_1 + x_2 \equiv y_1 + y_2$

Ans. : Given that,

$$R = \{(a, b) | a \equiv b \ (\text{mod } m)\}$$

We know that $a \equiv b \pmod m$ iff $m \mid (a - b)$

$$a - b = mk\ ,\ \ k \in z$$

(i) Let $a \in \mathbf{z}$, $a - a = 0 = m(0)$

$\therefore m|(a-a) \Rightarrow a \equiv a \pmod{m}$

$(a, a) \in R \Rightarrow R$ is reflexive relation.

(ii) Let $(a, b) \in R$

$$\Rightarrow \quad a \equiv b \pmod{m}$$

$$\Rightarrow \quad a - b = mk$$

$$\Rightarrow \quad b - a = m(-k)$$

$$\Rightarrow \quad m \mid b - a$$

$$\Rightarrow \quad b \equiv a \pmod{m}$$

$$\Rightarrow \quad (b, a) \in R$$

$\therefore$ R is symmetric relation.

(iii) Suppose (a, b) $(b, c) \in R$

then $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$

$$\Rightarrow \quad a - b = mk_1 \quad \text{and} \quad b - c = mk_2$$

$$\Rightarrow \quad a - b + b - c = mk_1 + mk_2$$

$$\Rightarrow \quad a - c = m(k_1 + k_2) = mk$$

$$\Rightarrow \quad m|(a - c) \Rightarrow a \equiv c \pmod{m}$$

$\therefore (a, c) \in R$

$\therefore$ R is transitive relation.

$\therefore$ R is reflexive, symmetric and transitive.

Thus R is an equivalence relation,

Now, $\quad x_1 \equiv y_1 \Rightarrow x_1 \equiv y_1 \pmod{m}$

$\quad x_2 \equiv y_2 \Rightarrow x_2 \equiv y_2 \pmod{m}$

$x_1 - y_1 = mk_1$ and $x_2 - y_2 = mk_2$

$$\Rightarrow \quad x_1 - y_1 + x_2 - y_2 = mk_1 + mk_2$$

$$\Rightarrow \quad (x_1 + x_2) - (y_1 + y_2) = m(k_1 + k_2)$$

$$\Rightarrow \quad m|(x_1 + x_2) - (y_1 + y_2)$$

$$\Rightarrow \quad (x_1 + x_2) \equiv (y_1 + y_2) \pmod{m}$$

$$\Rightarrow \quad x_1 + x_2 \equiv y_1 + y_2 \pmod{m}$$

$$\Rightarrow \quad x_1 + x_2 \equiv y_1 + y_2$$

Q.5 For each of these relations on set A = {1, 2, 3, 4} decide whether it is reflexive, symmetric, transitive or antisymmetric.
R_1 = {(1, 1) (2, 2) (3, 3) (4, 4)}
R_2 = {(1,1)(1, 2) (2, 2) (2,1) (3, 3) (4, 4)}, R_3 = {(1, 3) (1, 4) (2, 3) (2, 4) (3, 1) (3,4,)}.

[SPPU : Dec.-10]

Ans. : i) For R_1 :

As $\forall a \in A,\ (a,a) \in R_1$

$\therefore R_1$ is Reflexive, symmetric relation.

$\nexists$ (a, b) and (b,c)$\in R_1$ $\therefore R_1$ is transitive relation.

$\nexists$ (a, b) and (b,a)$\in R_1$ $\therefore R_1$ is antisymmetric relation.

$\therefore R_1$ reflexive, symmetric, transitive and anti-symmetric relation.

ii) For R_2 :

As $\forall a \in A\ (a, a) \in R_2$ and $aR_2b \Rightarrow bR_1 a$, for $a,b \in A$

$\therefore R_2$ reflexive and symmetric relation.

For any aR_2b and $bR_2C \Rightarrow aR_2C$

$\therefore R_2$ is transitive realtion

$\therefore R_2$ is an equivalence relation

But (1, 2) and (2, 1) $\in R_2$ and $1 \neq 2$

$\therefore R_2$ is not antisymmetric relation.

iii) For R_3 :

As $2 \in R_3$ but $(2, 2) \notin R_3$

$\therefore R_3$ is not reflexive relation.

As $(1, 4) \in R_3$ but $(4, 1) \notin R_3$

$\therefore R_3$ is not symmetric relation.

As (2, 3) and (3, 1) $\in R_3$ but $(2, 1) \notin R_3$.

$\therefore R_3$ is not transitive relation.

As (1, 3) and (3, 1) $\notin R_3$ but $1 \neq 3$.

$\therefore R_3$ is not transitive relation

As (1, 3) and (3, 1) $\in R_3$ but $1 \neq 3$

$\therefore R_3$ is not antisymmetric relation

Q.6 Consider the relation on A = { 1, 2, 3, 4, 5, 6} . R = { (i, j) | i – j| = 2}. Is R reflexive ? Is R symmetric ? Is R transitive ? **[SPPU : Dec.-14]**

Ans. : Given that A = {1, 2, 3, 4, 5, 6}

$$R = \{ (1, 3)\ (3, 1)\ (2, 4)\ (4, 2)\ (3, 5)\ (5, 3)\ (4, 6)\ (6, 4)\}$$

As $2 \in A$ but $(2, 2) \notin R \therefore R$ is not reflexive.

For any $(a, b) \in R \Rightarrow (b,a) \in R \therefore R$ is symmetric.

As (1, 3) (3, 1) $\in R$ but $(1, 1) \notin R \therefore R$ is not transitive.

Q.7 Let A = { 1, 2, 3, 4, 5, 6, 7 } and R = {(x, y) | x – y is divisible by 3}. Show that R is an equivalence relation ? Draw graph of R. **[SPPU : May-14, Dec.-12]**

Ans. : We have R = {x, y) | x – y is divisible by 3}

We know that x – x = 0 is divisible by 3

$x\,R\,x, \forall x \in A \Rightarrow R$ is reflexive relation.

As $x\,R\,y \Rightarrow x - y$ is divisible by 3

$\Rightarrow y - x$ is also divisible by 3

$\Rightarrow y - z$ is divisible by 3

$\Rightarrow y\,R\,x$ for $x, y \in A$

$\therefore$ R is a symmetric relation.

As,

$x\,R\,y$ and $y\,R \Rightarrow x - y$ and $y - z$ are divisible by 3

$\Rightarrow (x - y) + (y - z)$ is also divisible by 3

$\Rightarrow x - z$ is divisible by 3

$\Rightarrow x\,R\,z$

$\therefore$R is a transitive relation.

$\therefore$R is an equivalence relation.

and $R = \{(1, 1), (2, 2)\ (3, 3)\ (4, 4)\ (5, 5)\ (6, 6)\ (7, 7)\ (1, 4)$
$(4, 1)\ (1, 7)\ (7, 1)\ (2, 5), (5, 2)\ \ (3, 6)\ (6, 3)\}$

Its graph is as follows

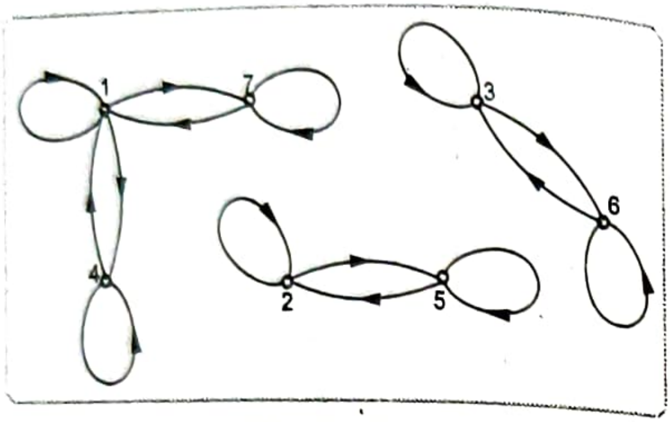


Fig. Q.7.1

Q.8 Define partition of a set. Let x = {1, 2, 3, 4, 5, 6, 7, 8, 9}. Determine whether or not each of the following is a partition of x :
A = {{2, 4, 5, 8}, {1, 9}, {3, 6, 7}}
B = {{1, 3, 6}, {2, 8}, {5, 7, 9}}

[SPPU : Dec.-11]

Ans. : Please refer section 5.8 for definition.

i) For set A :

The set A has 3 blocks,

$$A_1 = \{2, 4, 5, 8\},\quad A_2 = \{1, 9\}\ A_3 = \{3, 6, 7\}$$

- The union of all these blocks is a set X

$A_1 \cup A_2 \cup A_3 = X$

- These blocks are mutually disjoints.

$\therefore$ The set A forms a partition for the set X

ii) For set B :

$\therefore$ The set B has 3 blocks.

$$B_1 = \{1, 3, 6\},\quad B_2 = \{2, 8\} \qquad B_3 = \{5, 7, 9\}$$

As $B_1 \cup B_2 \cup B_3 \neq X$,

B is not a partition of set X.

Q.9 If S = {1, 2, 3, 4, 5, 6, 7, 8, 9}. Determine whether or not each of the following is a partition of S.

i) $A = \{\{1, 3, 5\}, \{2, 6\}, \{4, 8, 9\}\}$
ii) $B = \{\{1, 3, 5\}, \{2, 4, 6, 8\}, \{7, 9\}\}$
iii) $C = \{\{1, 3, 5\}\}, \{2, 4, 6, 8\}, \{5, 7, 9\}\}$
iv) $D = \{\{5\}\}$ **[SPPU : Dec.-12]**

Ans. : Given that

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$$

i) A is not a partition of S because S is not the union of all blocks of A.

i.e. $$S \neq A_1 \cup A_2 \cup A_3$$

ii) B is the partition of S as $B_1 \cup B_2 \cup B_3 = S$ and B_1, B_2, B_3 are mutually disjoints.

iii) As blocks of set C are not disjoints. $\therefore$ The set C is not a partition of S.

iv) $D = \{\{s\}\}$ is a partition of S, called as trivial partition.

Q.10 Find the symmetric closure of the following relations.
On $A = \{1, 2, 3\}$.
$R_1 = \{(1, 1) (2, 1)\}$ $R_2 = \{(1, 2) (2, 1) (3, 2) (2, 2)\}$
$R_3 = \{(1, 1) (2, 2) (3, 3)\}$ **[SPPU : Dec.-12]**

Ans. :

We have $A = \{1, 2, 3\}$

i) $$R_1^{-1} = \{(1, 1) (1, 2)\}$$

$\therefore$ $$R = R_1 \cup R_1^{-1} = \{(1, 1) (1, 2) (2, 1)\}$$

is the symmetric closure of R_1

ii) $$R_2^{-1} = \{(2, 1) (1, 2) (2, 3) (2, 2)\}$$

$\therefore$ $$R = R_2 \cup R_2^{-1} = \{(1, 2) (2, 1) (3, 2) (2, 3) (2, 2)\}$$

is the symmetric closure of R_2

iii) R_3 is the symmetric relation.
$\therefore$ R_3 itself is the symmetric closure.

Q.11 Find the transitive closure of R by Warshall's algorithm. Where $A = \{1, 2, 3, 4, 5, 6\}$ and $R = \{(x, y)/(x - y) = 2\}$

[SPPU : Dec.-05, 12, 13, 14, 16]

Ans. : Step 1 : We have $|A| = 6$,

$R = \{(1, 3), (3, 1), (2, 4), (4, 2), (4, 6), (6, 4), (3, 5), (5, 3)\}$

Thus we have to find Warshall's sets,

$W_0, W_1, W_2, W_3, W_4, W_5$ and W_6.

The first set W_0 is same as M_R. Which is shown below

$$W_0 = M_R = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

Step 2 : To find W_1 :

To find W_1 from W_0, we consider the first column and first row.

In a column C_1, 1 is present at R_3

In a row R_1, 1 is present at C_3

Thus add new entry in W_1, at (R_3, C_3) which is given below

$$W_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 3 : To find W_2 :

To find W_2 from W_1, we consider the second column C_2 and second row R_2.

In C_2, 1 is present at R_4

In R_2, 1 is present at C_4

Thus add new entry in W_2 at (R_4, C_4), which is given below

$$W_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 4 : To find W_3 :

To find W_3 from W_2, we consider the third column and third row.

In C_3, 1 is present at R_1, R_3, R_5

In R_3, 1 is present at C_1, C_3, C_5

Thus add new entries in W_3 at $(R_1, C_1), (R_1, C_3), (R_1, C_5)$ $(R_2, C_1), (R_2, C_3), (R_2, C_5)$ $(R_3, C_1), (R_3, C_3), (R_3, C_5)$ which is given below

$$W_3 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}$$

Step 5 : To find W_4 :

To find W_4 from W_3, we consider the fourth column and fourth row.

In C_4, 1 is present at R_2, R_4, R_6

In R_4, 1 is present at C_2, C_4, C_6

$$W_4 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}$$

Thus add new entries in W_4 at $(R_2, C_2), (R_2, C_4), (R_2, C_6), (R_4, C_2), (R_4, C_4), (R_4, C_6), (R_6, C_2), (R_6, C_4), (R_6, C_6)$ which is given below

Step 6 : To find W_5 :

To find W_5 from W_4, we consider the 5^{th} column and 5^{th} row.

In C_5, 1 is present at R_1, R_3, R_5

In R_5, 1 is present at C_1, C_3, C_5

Thus add new entries in W_5 at (R_1, C_1), (R_1, C_3), (R_1, C_5), (R_3, C_1), (R_3, C_3), (R_3, C_5), (R_5, C_1), (R_5, C_3), (R_5, C_5) which is given below

$$W_5 = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} = W_4$$

Step 7 : To find W_6 :

To find W_6 from W_5 we consider the 6^{th} column and 6^{th} row.

In C_6, 1 is present at R_2, R_4, R_6

In R_6, 1 is present at C_2, C_4, C_6

Thus add new entries in W_6 at $(R_2, C_2), (R_2, C_4), (R_2, C_6), (R_4, C_2), (R_4, C_4), (R_4, C_6), (R_6, C_2), (R_6, C_4), (R_6, C_6)$ which is given below

$$W_6 = W_5 = W_4$$

Hence W_6 is the relation matrix of R^*

$$\therefore R^* = \left\{\begin{matrix}(1, 1), (1, 3) (1, 5) (2, 2) (2, 4) (2, 6) (3, 1) (3, 3) (3, 5) \\ (4, 2) (4, 4) (4, 6) (5, 1) (5, 3) (5, 5) (6, 2) (6, 4) (6, 6)\end{matrix}\right\}$$

Q.12 Let R = {(a, d), (b, a), (b, d), (c, b), (c, d), (d, c)}
Use Warshall's algorithm to find the matrix of transitive closure where A = {a, b, c, d}

[SPPU : Dec.-15]

Ans. : Step 1 : We have

$$A = \{a, b, c, d\} \quad \therefore \quad |A| = 4$$

$$R = \{(a, d) (b, a), (b, d) (c, b), (c, d), (d, c)\}$$

Thus we have to find Warshall's sets W_0, W_1, W_2, W_3, W_4

The first set $W_0 = M_R$

$$\therefore W_0 = M_R = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & 0 & 0 & 1 \\ b & 1 & 0 & 0 & 1 \\ c & 0 & 1 & 0 & 1 \\ d & 0 & 0 & 1 & 0 \end{array}$$

Step 2 : To find W_1 :

To find W_1 from W_0, we consider the first column and first row.

In C_1, 1 is present at R_2

In R_1, 1 is present at C_4

Thus add new entry in W_1 at (R_2, C_4)

$$W_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Step 3 : To find W_2

To find W_2, from W_1, we consider the second column and second row.

In C_2, 1 is present at R_3

In R_2, 1 is present at C_1 and C_4

$$\therefore W_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Thus add new entries in W_2 at (R_3, C_1), (R_3, C_4) which is given below.

Step 4 : To find W_3 :

To find W_3 from W_2, we consider the 3rd column and 3rd row.

In C_3, 1 is present at R_4

In R_3, 1 is present at C_1, C_2, C_4

$$W_3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Thus add news entries in W_3 at (R_4, C_1), (R_4, C_2), (R_4, C_4)

Which is given below

Step 5 : To find W_4 :

To find W_4 from W_3, we consider the 4^{th} column and 4^{th} row.

In C_4, 1 is present at R_1, R_2, R_3, R_4

In R_4, 1 is present at C_1, C_2, C_3, C_4

Thus we add new entries in W_4 at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_4, C_3), (R_3, C_1), (R_3, C_2), (R_3, C_4), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4)

Which is given below

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Hence W_4 is the relation matrix of R^*.

and $$R^* = \left\{ \begin{matrix} (a, a), & (a, b) & (a, c) & (a, d) & (b, a) & (b, b) & (b, c) & (b, d) \\ (c, a), & (c, b) & (c, c) & (c, d) & (d, a) & (d, b) & (d, c) & (d, d) \end{matrix} \right\}$$

Q.13 Find the transitive closure of the relation R on
A = {1, 2, 3, 4} defined by
R = {(1, 2), (1, 3), (1, 4), (2, 1), (2, 3), (3, 4), (3, 2), (4, 2), (4, 3)}

[SPPU : Dec.-07, May-15]

Ans. : Step 1 :

We have $|A| = 4$, Thus we have to find Warshall's sets, W_0, W_1, W_2, W_3, W_4

The first set $W_0 = M_R$

$$\therefore W_0 = M_R = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Step 2 : To find W_1 :

To find W_1 from W_0, we consider the first column and first row.

In C_1, 1 is present at R_2

In R_1, 1 is present at C_2, C_3, C_4

Thus add new entries in W_1 at (R_2, C_2), (R_2, C_3), (R_2, C_4) which is given below

$$W_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

Step 3 : To find W_2 :

To find W_2 from W_1, we consider the 2^{nd} column and 2^{nd} row.

In C_2, 1 is present at R_1, R_2, R_3, R_4

In R_2, 1 is present at C_1, C_2, C_3, C_4

Thus we add new entries in W_2 at (R_1, C_1), (R_1, C_2), (R_1, C_3) (R_1, C_4), (R_2, C_1), (R_2, C_2), (R_2, C_3), (R_2, C_4), (R_3, C_1), (R_3, C_2) (R_3, C_2), (R_3, C_4), (R_4, C_1), (R_4, C_2), (R_4, C_3), (R_4, C_4)

Which is given below

$$W_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

All entries in W_2 are 1

Hence W_2 is the relation matrix of transitive closure of R.

$$\text{and } R^* = \begin{Bmatrix} (1, 1), (1, 2) & (1, 3) & (1, 4) & (2, 1) & (2, 2) & (2, 3) & (2, 4) \\ (3, 1), (3, 2) & (3, 3) & (3, 4) & (4, 1) & (4, 2) & (4, 3) & (4, 4) \end{Bmatrix}$$

Q.14 Use Warshall's Algorithm to find transitive closure of R where

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} \textbf{ and } A = \{1, 2, 3\}$$

☞[SPPU : May-06, May-08]

Solution : Step 1 : We have $|A| = 3$. Thus we have to find Warshall's sets W_0, W_1, W_2, W_3

The first set is

$$W_0 = M_R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix}$$

Step 2 : To find W_1 :

To find W_1 from W_0, we consider the first column and the first row.

In C_1, 1 is present at R_1, R_3

In R_1, 1 is present at C_1 and C_3

Thus add new entries in W_1 at (R_1, C_1), (R_1, C_3), (R_3, C_1), (R_3, C_3)

Which is given below

$$W_1 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Step 3 : To find W_2 :

To find W_2 from W_1, we consider the 2^{nd} column and 2^{nd} row.

In C_2, 1 is present at R_2, R_3

In R_2, 1 is present at C_2

Thus add new entries in W_2 at (R_2, C_2), (R_3, C_2) which is given below

$$W_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = W_1$$

Step 4 : To find W_3

To find W_3 from W_2, we consider the 3^{rd} column and 3^{rd} row.

In C_3, 1 is present at R_1, R_3

In R_3, 1 is present at C_1, C_2, C_3

Thus add new entries in W_3 at (R_1, C_1), (R_1, C_2), (R_1, C_3), (R_3, C_1), (R_3, C_2), (R_3, C_3) which is given below

$$W_3 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Hence W_3 is the relation matrix of R^*

and $R^* = \{(1, 1)\ (1, 2)\ (1, 3)\ (2, 2)\ (3, 1)\ (3, 2)\ (3, 3)\}$

4.3 : Partially Ordered Set

Important Points to Remember

A relation R on a set A is called a partially ordered relation iff R is reflexive, anti-symmetric and transitive relation.

The set A together with partially ordered relation is called a partially ordered set or POSET.

It is denoted by (A, R) or (A, ≤) where ≤ is a partially ordered relation.

Examples :

1) ($\Re$, ≤) (N, ≤) are Posets.

where '≤' is reflexive, antisymmetric and transitive relation.

2) If A = P(S) where S = (a, b, c) and for X, Y ∈ A, Define X ≤Y or XRY iff X ⊆ Y.

As $X \leq X \Rightarrow X \leq X$. ∴' ≤ ' is reflexive.

If $X \leq Y, Y \leq Z \Rightarrow X \subseteq Y$ and $Y \subseteq X \Rightarrow X = Y$

∴'≤' is antisymmetric relation.

If $X \leq Y, Y \leq Z \Rightarrow X \subseteq Y$ and $Y \subseteq Z \Rightarrow X \subseteq Z \Rightarrow X \leq Z$

⇒ '≤ is transitive relation.

∴ (P(S), ⊆) or (P(S), ≤) is a poset.

I) Comparable elements : Let (A, ≤) be a poset. Two elements a, b in A are said to be comparable elements if a ≤ b or b ≤ a. Two elements a and b of a set A are said to be non-comparable if neither a ≤ b nor b ≤ a.

In above example (2),

The comparable elements are

$$\{a\} = \{a, b\}, \{b\} = \{a, b, c\} \ \{b,c\} \subseteq \{a,b,c\}$$

Non comparable elements are

$$\{a\} \not\subseteq \{b\} \quad \{a\} \not\subseteq \{a,c\}$$

II) Totally ordered set : Let A be any nonempty set. The set A is called linearly ordered set or totally ordered set if every pair of elements in A are comparable.

i.e. for any a, b ∈ A either a ≤ b or b ≤ a.

4.4 : Hasse Diagram

Important Points to Remember

It is useful tool, which completely describes the associated partially ordered relation. It is also known as ordering diagram.

A diagram of graph which is drawn by considering comparable and non-comparable elements is called Hasse diagram of that relation.

Therefore while drawing Hasse diagram following points must be followed.

1) The elements of a relation R are called vertices and denoted by points.
2) All loops are omitted as relation is reflexive on poset.
3) If aRb or $a \leq b$ then join a to b by a straight line called an edge the vertex b appears above the level of vertex a. Therefore the arrows may be omitted from the edges in Hasse diagram.
4) If $a \nleq b$ and $b \nleq a$ i.e. a and b are non - comparable elements, then they lie on same level and there is no edge between a and b.
5) If $a \leq b$ and $b \leq c$ then $a \leq c$. So there is a path $a \rightarrow b \rightarrow c$. Therefore do not join a to c directly i.e. delete all edges that are implied by transitive relation.

Q.15 Draw Hasse diagram of a poset $(P(s), \subseteq)$ where $S = \{a, b, c\}$.

Ans. : $P(s) = \{ \phi, \{a\}, \{b\}, \{c\}, \{a, b\} \{a, c\} \{b, c\} \{a, b, c\}\}$

Now find the comparable and non comparable elements.

$\phi \subseteq \{a\}$, $\phi \subseteq \{b\}$, $\phi \subseteq \{c\}$, $\therefore$ {a}, {b}, {c} lie above the level of ϕ

$\{a\} \subseteq \{a,b\}, \{b\} \subseteq \{a,b\}, \{c\} \subseteq \{a,c\}$ $\therefore$ {a, b}, {b, c} {a,c} lies above the level of {a}, {b}, {c}.

$\{a, b\} \subseteq S$, $\{b, c\} \subseteq S$, $\{a, c\} \subseteq S$ $\therefore$ S lies above the level of {a, b}, {a, c}, {b, c}

But {a}, {b} {c} are non comparable $\therefore$ {a}, {b}, {c} lie on same level.

{a, b}, {a, c}, {b, c} are non comparable $\therefore$ lie on same level.

By considering the above observations, the Hasse diagram is as follows :

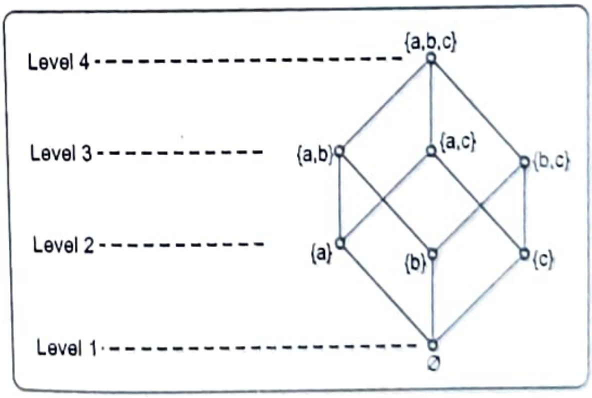


Fig. Q.15.1

4.5 : Chains, Antichains and Elements of Poset

Important Points to Remember

Let $(A, \leq)$ be a poset. A subset of A is called a chain if every pair of elements in the subset are related.

A subset of A is called antichain if no two distinct elements in a subset are related.

1) The chains are

$\{\phi, \{a\}, \{a, b\}, \{a, b, c\}\}$, $\{\{a\}, \{a, c\}, \{a, b, c\}\}$, $\{\{b, c\}, \{a, b, c\}\}$

2) Antichains are $\{\{a\}, \{b\}, \{c\}\}$

Note : 1) The number of elements in the chain is called the length of chain.

2) If the length of chain is n in a poset $(A, \leq)$ then the elements in A can be partitioned into n disjoint antichains.

3) Let $(A, \leq)$ be a poset. An element $a \in A$ is called a **maximal element** of A if there is no element $c \in A$ such that $a \leq c$.

4) An element $b \in A$ is called a **minimal element** of A if there is no element $c \in A$ such that $c \leq b$

5) Greatest element : An element $x \in A$ is called a **greatest element** of A if for all $a \in A$, $a \in x$. It is denoted by 1 and is called the unit element.

6) Least element : An element $y \in A$ is called a **least element** of A if for all $a \in A$, $y \leq a$.

It is denoted by 0 and is called as **zero element.**

7) Least upper bound (lub) : Let $(A, \leq)$ be a poset. For a, b, $c \in A$, an element C is called **upper bound** of a and b if $a \leq c$ and $b \leq c$ C is called as least upper bound of a and b in A if C is an upper bound a and b there is no upper bound d of a and b such that $d \leq c$. It is also known as **supremum.**

8) Greatest lower bound (glb) : Let $(A, \leq)$ be a poset. for a, b, $l \in A$, an element l is called the **lower bound** of a and b if $l \leq a$ and $l \leq b$.

An element l called the **greatest lower bound** of a and b if l is the lower bound of a and b and there is no lower bound f of a and b such that

• $l \leq f$.

glb is also called as **infimum.**

4.6 : Types of Lattices

Important Points to Remember

A lattice is a poset in which every pair of elements has a least upper bound (lub) and a greatest lower (glb).

Let $(A, \leq)$ be a poset and $a, b \in A$ then lub of a and b is denoted by $a \vee b$. It is called the join of a and b.

i.e. $a \vee b = \text{lub}(a, b)$

The greatest lower bound of a and b is called the meet of a and b and it is denoted by $a \wedge b$

$\therefore \quad a \wedge b = \text{glb}(a, b)$

From the above discussion, it follows that a lattic is a mathematical structure with two binary operations $\vee$ (join) and $\wedge$ (meet). It is denoted by $\{L, \vee, \wedge\}$.

Properties of a Lattice

Let $(L, \wedge, \vee)$ be a lattice and $a, b, c \notin L$. Then L satisfies the following properties.

1) Commutative property

$$a \wedge b = b \wedge a \text{ and } a \vee b = b \vee a$$

2) Associative law

$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$$

$$a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$$

3) Absorption law

$$a \wedge (a \vee b) = a \text{ and } a \vee (a \wedge b) = a$$

$$a \wedge a = a, \; a \vee a = a$$

$$a \wedge b = a \text{ iff } a \vee b = b$$

Types of Lattices

I) Bounded lattice : A lattice L is called a bounded lattice if it has a greatest element 1 and least element 0.

II) Sublattice : Let, $(L, \vee, \wedge)$ be a lattice. A non empty subset L_1 of L is called a sublattice of L if L_1 itself is a lattice w.r.t. the operations of L.

III) Distributive lattice : A lattice $(L, \vee, \wedge)$ is called a distributive lattice if for any elements $a, b, c \in L$, it satisfies the following properties,

i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

If the lattice does not satisfy the above properties then it is called a non distributive lattice.

Theorem : Let, $(L, \wedge, \vee)$ be a lattice with universal bounds 0 and 1 then for any $a \in L$, $a \vee 1 = 1, a \wedge 1 = a$, $0 \vee a = a$, $0 \wedge a = 0$.

III) Complement lattice : Let $(L, \wedge, \vee)$ be a lattice with universal bounds 0 and 1 for any $a \in L$, $b \in L$ is said to be complement of a if

$a \vee b = 1$ and

$a \wedge b = 0$.

A Lattice in which every element has a complement in that lattice, is called the complemented lattice.

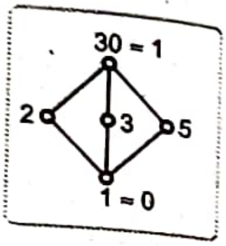


Fig. 4.1

e.g. 1) The Hasse diagram is here $0 = 1$ and $1 = 30$.

i) $2 \wedge 3 = 0$ and $2 \vee 3 = 1$, $2 \wedge 5 = 0$ and $2 \vee 5 = 1$

$\therefore$ 2 has two compliments 3 and 5

Hence the complement is not unique.

Principle of Duality

Any statement about lattice involving $\wedge, \vee \leq, \geq$ remains true if '$\wedge$' is replaced by '$\vee$', '$\vee$ by $\wedge$', '$\leq$ by $\geq$', '$\geq$ by $\leq$', '0 by 1' and '1 by 0'.

e.g. 1) $a \vee (b \wedge c) = a \wedge (b \vee c)$

2) $a \wedge (b \vee 1) = a \vee (b \wedge 0)$

Q.16 Let A be the set of positive factors of 15 and R be a relation on A s.t. R = {xRy| x divides y, x y ∈ A}. Draw Hasse diagram and give and ∧ and∨ for lattice. **[SPPU : Dec.-06]**

Ans. : We have A = {1, 3, 5, 15}

R = {(1,1) (1,3) (1,5) (1, 15) (3,15) (5,15) (15,15)}

Hasse diagram of R is :

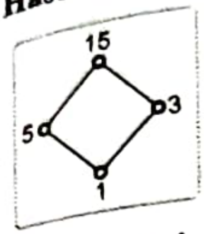


Fig. Q.16.1

Table for $\wedge$ and $\vee$

$\vee$	1	3	5	15
1	1	3	5	15
3	3	3	15	15
5	5	15	5	15
15	15	15	15	15

$\wedge$	1	3	5	15
1	1	1	1	1
3	1	3	1	3
5	1	1	5	5
15	1	3	5	15

Every pair of elements has lub and glb. $\therefore$ It is a lattice.

Q.17 Let A = {1, 2, 3, 4, 6, 9, 12} Let a relation R on a set A is R = {(a,b)/ a divides b $\forall$ a, b $\in$ A}. Give list of R. Prove that it is a partial ordering relation. Draw Hasse diagram of the same. Prove or disprove it is a lattice. **[SPPU : Dec.-11]**

Ans. : We have A = {1, 2, 3, 4, 6, 9, 12}

$$\text{and } R = \begin{Bmatrix} (1,1),(1,2),(1,3),(1,4)(1,6),(1,9),(1,12),(2,2),(2,4),(2,6), \\ (2,12)(3,3),(3,6),(3,9),(3,12),(4,4),(4,12),(6,6),(6,12),(9,9),(12,12) \end{Bmatrix}$$

We know that for any $a \in A$, $a \mid a$ $\therefore$ aRa

$\therefore$ R is a reflexive relation.

As $a \mid b$ and $b \mid a \Rightarrow a = b$ $\therefore$ R is antisymmetric relation.

As $a \mid b$ and $b \mid c \Rightarrow a \mid c \Rightarrow$ R is a transitive relation.

$\therefore$ R is reflexive antisymmetric and transitive

$\therefore$ (A, R) is a poset and R is a partial ordering relation.

Hasse diagram is as follows :

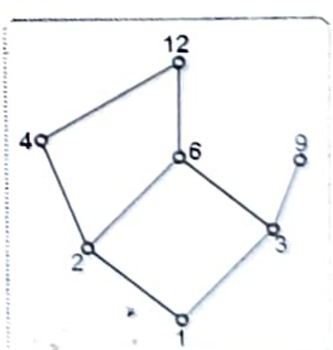


Fig. Q.17.1

In above diagram $6 \vee 9$ does not exist. $\therefore$ It is not a lattice.

Q.18 Determine whether the poset represented by each of the Hasse diagram are lattices. Justify your answer. ☞[SPPU : Dec.-10]

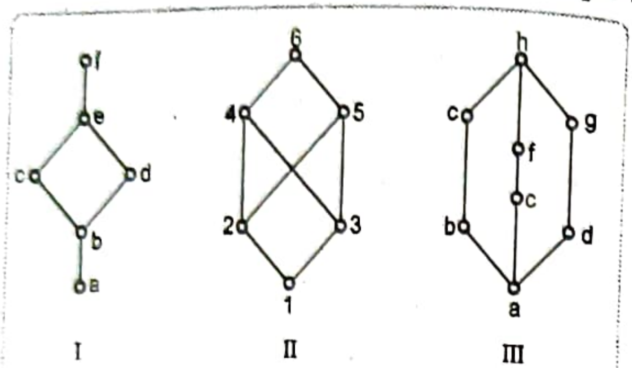


Fig. Q.18.1

Ans. : I) In Fig. Q.18.1 (I), every pair of element has glb and lub. ∴ It is a lattice.

II) In Fig. Q.18.1 (II), every pair of elements has lub and glb. ∴ It is a lattice.

III) In Fig. Q.18.1 (III), every pair of elements gas lub and glb. ∴ It is a lattice.

Q.19 Show that the set of all divisors of 36 forms a lattice. ☞[SPPU : Dec.-14]

Ans. : Let A = {1, 2, 3, 4, 6, 9,12, 18, 36 } and Let '≤' is a divisor of. It's Hasse diagram is as follows.

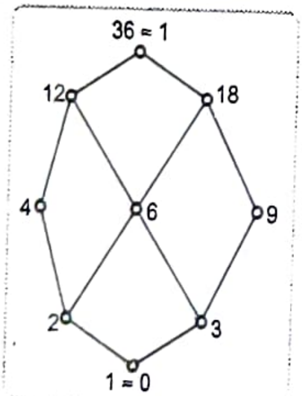


Fig. Q.19.1

The universal upper bound 1 is 36 and lower bound 0 is 1. Every pairs of elements of this poset has lub and glb.

∴ It is a lattice.

Q.20 SLet n be a positive integer, S_n be the set of all divisors of n, Let D denote the relation of divisor. Draw the diagram of lattices for n = 24, 30, 6. ☞[SPPU : May-15]

Ans. : Given that

i) We have $S_6 = \{1,2,3,6\}$, D is the relation of divisor.

ii) $S_{24} = \{1, 2, 3, 4, 6, 8, 12, 24\}$

iii) $S_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$

Diagrams of Lattices are as follows.

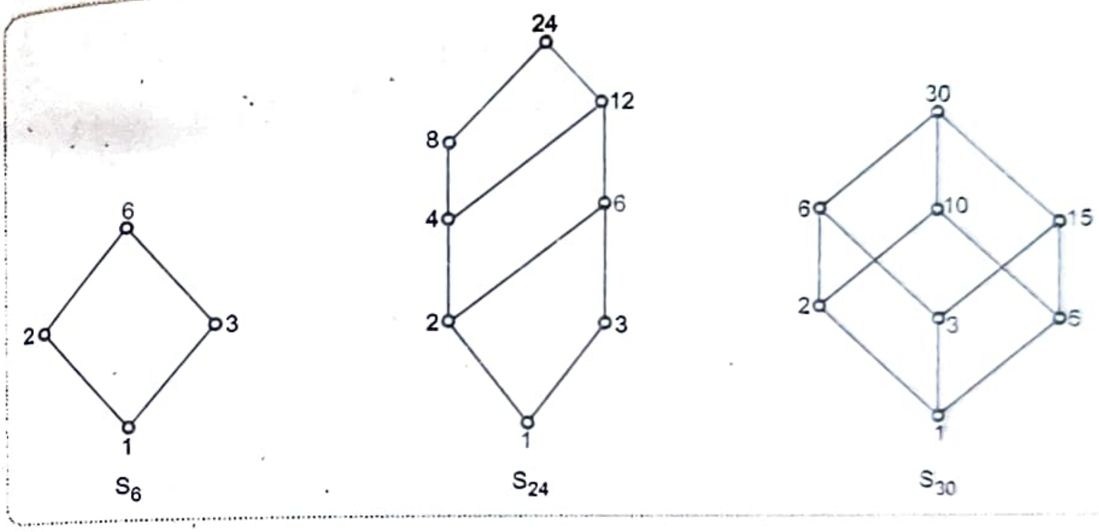


Fig. Q.20.1

Q.21 Show that the set of all divisors of 70 forms a lattice. ☞[SPPU : Dec.-13]

Ans. : Let A = {1, 2, 5, 7, 10, 14, 35, 70}

and Let '≤' is "a divisor of".

The universal upper bound 1 is 70 and the lower bound 0 is 1.

It's Hasse diagram is as follows :

Every pair of elements of A has $\wedge$ and $\vee$.

∴It is a lattice [write table of $\wedge$ and $\vee$].

Q.22 Let A = {1, 2, 3, 4 , 5, 6, 7, 8 ,9, 12, 18, 24} be ordered by the relation x divides y. Show that the relation is a partial ordering and draw Hasse diagram. ☞[SPPU : Dec-15]

Ans. : We have $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 12, 18, 24\}$

$R = \{(x,y) \mid x \text{ divides } y, \text{ for } x, y \in A\}$

$R = \{(1, 2) (1, 3), (1, 4), (1, 5) (1, 6) (1, 7) (1, 8) (1, 9)$
$(1, 12) (1, 18) (1, 24)$
$(2, 2) (2, 4) (2, 6) (2, 8) (2, 12) (2, 18) (2, 24) (5, 5)$
$(6, 6) (6, 12) (6, 18)$
$(6, 24), (7, 7) (8, 8) (8, 24) (9, 9) (9, 18) (12, 12)$
$(12, 24) (18, 18) (4, 24)\}$

We have for any $x \in A$, $x|x \Rightarrow R$ is a reflexive for $x|y$ and $y|x \Rightarrow x = 0 \Rightarrow R$ is antisymmetric. If $x|y$ and $y|z \Rightarrow x|z$ $\therefore$ R is a transitive relation.

$\therefore$ R is a partial ordering relation, It's Hasse diagram is as follows.

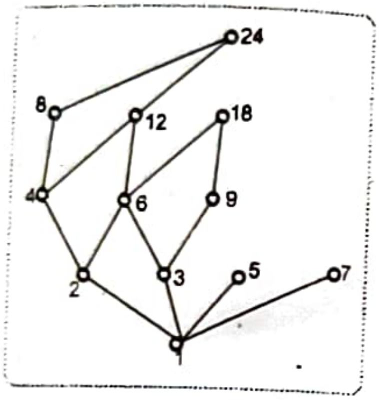


Fig. Q.22.1

Q.23 Let $x = \{2, 3, 6, 12, 24, 36\}$ and $x \le y$ iff x divides y find i) Maximal element ii) Minimal element iii) Chain iv) Antichain v) Is Poset lattice [SPPU : May-14]

Ans. : We have $x = \{2, 3, 6, 12, 24, 36\}$

The relation 'R' = '≤

$\therefore R = \{(2, 2) (2, 6) (2, 12) (2, 24) (2, 36) (3, 3) (3, 6) (3, 12) (3, 24)$
$(3, 36), (6, 6) (6, 12) (6, 24) (6, 36) (12, 12)(12, 24) (12, 36) (24, 24)$
$(36, 36)\}$.

It's Hasse diagram is as follows .

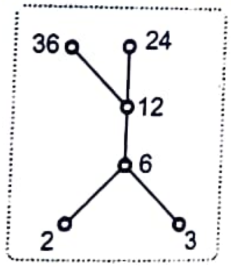


Fig. Q.23.1

i) Maximal elements are 24, 36

ii) Minimal elements are 2, 3

iii) Chain { 2, 6, 12, 24}, { 2, 6, 12, 36}, {3, 6, 12, 24}, {3, 6, 12, 36}

iv) Antichain : {2, 3} {24, 36}

v) The given poset is not a lattice as $2 \wedge 3$ does not exist.

4.7 : Functions

Important Points to Remember

1. **Function :** Let A and B be two non empty sets. A function from A to B, denoted as $f : A \rightarrow B$, is a relation from A to B such that for every element $a \in A$, there exists a unique element $b \in B$ such that (a, b) f or $f(a) = b$.

where A = Domain set,

B = Codomain set

Range set = $\{b \in B \mid \exists\, a \in A$ s.t. $f(a) = b\}$

= {Set of all image points}

2. **Equality of functions :** two functions f and g from A to B are said to be equal iff $f(a) = g(a)\ \forall\ a \in A$. we write as $f \equiv g$.

3. **Types of functions :**

   (a) *Identity function :* Let A be any non empty set. A function f from $A \rightarrow A$ is said to be the Identity function iff $f(a) = a,\ \forall\ a \in A$.

   (b) *One to one function :* (Injective function) : A function $f : A \rightarrow B$ is said to be one to one function if for all elements x, y $\in A$ such that $f(x) = f(y) \Rightarrow x = y$.

   (c) *Into function :* A function $f : A \rightarrow B$ is said to be into function if range set of f is a proper subset of B i.e. $\exists$ at least one element in B which has no pre–image in A.

   (d) *Onto function (Surjective function) :* A function $f : A \rightarrow B$ is said to be onto function if range set of f is equal to set B i.e. every element in B has pre–image in A.

   (e) *Bijective function :* A function which is one to one (injective) as well as onto (surjective) is called bijective function.

   (f) *Inverse function :* Let A, B any non–empty sets and $f : A \rightarrow B$ is a bijection function. The inverse function $f^{-1} : B \rightarrow A$ is a function defined as

$f^{-1}(b) = a$ iff $f(a) = b$ for $a \in A$ and $b \in B$

(g) *Composite function* : Let $f : A \to B$ and $g : B \to C$ be two function. The composition of f and g is denoted by gof and gof : $A \to C$ such that

$(gof)(x) = g[f(x)], \forall x \in A.$

Q.24 Explain classification of functions with example.

Ans. : Depending upon the nature of function, there are mainly two functions.

1. **Algebraic function** : A function which consists of a finite number of terms involving powers and roots of the independent variable and four fundamental operations addition, subtraction, multiplication and division, is called an algebraic function.

   There are three types of algebraic functions.

   a) *Polynomial function* : A function of the form

   $a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \quad + a_n$, where n is positive integer,

   $a_0, a_1, a_2, \quad , a_n$ are real numbers and $a_0 \; 0$, is called polynomial function of x in degree n.

   e.g. $f(x) = x^3 - 3x^2 + 2x - 5$

   b) *Rational function* : A function of the form $\frac{f(x)}{g(x)}$ where f(x) and g(x) are polynomial functions and $g(x) \neq 0$ is called rational function e.g. $\frac{x^2 - 3x + 5}{x^2 + 1}$.

   c) *Irrational function* : A function involving radicals is called irrational function. e.g. $f(x) = x^{2/3} + 5x^2 + 1, \sqrt[2]{x+1}$.

2. **Transcendental function** : A function which is not algebraic is called transcendental function.

   e.g. $f(x) = \sin x + x^3 + 5x$

   a) *Trigonometric function* : The six functions sin x, cos x, tan x, sec x, cosec x, cot x, where x is in radians are called trigonometric functions.

   e.g. $f(x) = \sin x + \tan x$.

b) *Inverse trigonometric function* : The six functions $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, $\operatorname{cosec}^{-1} x$, $\sec^{-1} x$ and $\cot^{-1} x$ are called inverse trigonometric functions.
e.g. $f(x) = \cos^{-1} x + 5 \tan^{-1} x$

c) *Exponential function* : A function of the form $f(x) = a^x$ $(a > 0)$ satisfying $a^x \cdot a^y = a^{x+y}$ and $a' = a$ is called exponential function. e.g. $f(x) = 5^x$.

d) *Logarithmic function* : The inverse function of the exponential function is called logarithmic function. e.g. $f(x) = \log x$.

Q.25 Let $f(x) = x+2$, $g(x) = x - 2$, $h(x) = 3x$, for $x \in \Re$ Where $\Re$ is the set of real numbers. Find i) gof ii) fog iii) fof iv) hog v) gog vi) foh vii) hof viii) fohog ix) gofoh. [SPPU : May-08, 15, Dec.-12, Marks 4]

Ans. : Let $x \in \Re$ be any real number.

i) $gof(x) = g[f(x)] = g[x + 2] = x + 2 - 2 = x$

ii) $fog(x) = f[g(x)] = f[x - 2] = x - 2 + 2 = x$

iii) $fof(x) = f[f(x)] = f[x + 2] = x + 2 + 2 = x + 4$

iv) $hog(x) = h[g(x)] = h[x - 2] = 3(x - 2) = 3x - 6$

v) $gog(x) = g[g(x)] = g[x - 2] = x - 2 - 2 = x - 4$

vi) $foh(x) = f[h(x)] = f[3x] = 3x + 2$

vii) $hof(x) = h[f(x)] = h[x + 2] = 3(x + 2) = 3x + 6$

viii) $$fohog(x) = f[h(g(x))] = f[h(x - 2)] = f[3(x - 2)] = f(3x - 6)$$
$$= 3x - 6 + 2 = 3x - 4$$

ix) $$gofoh(x) = g[f(h(x))] = g[f(3x)]$$
$$= g[3x + 2] = 3x + 2 - 2 = 3x$$

Q.26 Let functions f and g be defined by $f(x) = 2x + 1$, $g(x) = x^2 - 2$ Find a) gof(4) and fog (4) b) gof(a + 2) and fog(a + 2) c) fog(5) d) gof(a + 3) [SPPU : May-07, Dec.-07, Marks 4]

Ans. : a) $gof(4) = g[f(4)] = g[2(4) + 1] = g[9] = 9^2 - 2 = 79$

$fog(4) = f[g(4)] = f(4^2 - 2) = f(14) = 2(14) + 1 = 29$

b) $$gof(a + 2) = g[f(a + 2)] = g[2(a + 2) + 1] = g[2a + 5]$$
$$= (2a+5)^2 - 2 = 4a^2 + 20a + 23$$

$$fog(a+2) = f[g(a+2)] = f[(a+2)^2-2] = f[a^2+4a+2]$$
$$= 2[(a^2+4a+2]+1 = 2a^2+8a+5$$

c) $fog(5) = f[g(5)] = f[25-2] = f(23) = 2(23)+1 = 47$

d) $gof(a+3) = g[f(a+3)] = g[2(a+3)+1] = g[2a+7]$
$$= [2a+7]^2-2 = 4a^2+28a+47$$

Q.27 Determine if each is a function. If yes is it injective, surjective, bijective ?

a) Each person in the earth is assigned a number which corresponds to his age.

b) Each student is assigned a teacher.

c) Each country has assigned it's capital. **[SPPU : Dec.-11, Marks 4]**

Ans. : a) Every person has unique age

∴ It is a function. Two person's may have same age. ∴ It is not injective. There is no person whose age is 300 years. ∴ It is not surjective. ∴ Function is not bijective.

b) It is a function. It is not injective. It is not surjective. ∴ It is not bijective.

c) It is a function. It is injective as well as surjective. ∴ It is bijective.

Q.28 Let A = B be the set of real numbers.

$f : A \rightarrow B$ given by $f(x) = 2x^3 - 1$

$g : B \rightarrow A$ given by $g(y) = \sqrt[3]{\frac{1}{2}y + \frac{1}{2}}$

Show that f is a bijective function and g is also bijective function :

[SPPU : Dec.-10, Marks 4]

Ans. : 1) Suppose $f(x_1) = f(x_2)$

$$\Rightarrow \quad 2x_1^3 - 1 = 2x_2^3 - 1$$
$$\Rightarrow \quad x_1^3 = x_2^3$$
$$\Rightarrow \quad x_1 = x_2$$

∴ f is injective mapping

Let $y \in B$ and $f(x) = y \Rightarrow 2x^3 - 1 = y$

$$\therefore \quad 2x^3 = 1+y \Rightarrow x^3 = \frac{1+y}{2}$$

$$x = \sqrt[3]{\frac{1+y}{2}} = \sqrt[3]{\frac{y}{2}+\frac{1}{2}} \in A = \Re \text{ for any } y \in B$$

$\Rightarrow$

$\therefore$ f is a surjective mapping.

Thus f is injective as well as surjective function.

Hence f is a bijective function.

We have $f(x) = y \Rightarrow x = \sqrt[3]{\frac{1}{2}y+\frac{1}{2}} \Rightarrow f^{-1}(y) = x = \sqrt[3]{\frac{1}{2}y+\frac{1}{2}} = g(y)$

Thus $f^{-1} = g$. We know that if f is bijective function then f^{-1} is also bijective. Hence g is bijective function.

Q.29 If f : A $\rightarrow$ B and g : B $\rightarrow$ C are bijective functions the gof is also bijective.

Ans. : Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two bijective functions then gof $A \rightarrow C$ is a function.

(i) Let $x_1, x_2 \in A$ and suppose $gof(x_1) = gof(x_2)$

$$g[f(x_1)] = g[f(x_2)]$$

$$\Rightarrow \quad f(x_1) = f(x_2) \quad \ldots (\because g \text{ is } 1-1 \text{ function})$$

$$\Rightarrow \quad x_1 = x_2 \quad \ldots (\because f \text{ is } 1-1 \text{ function})$$

$\Rightarrow$ gof is $1-1$ function.

(ii) Let $z \in C$.

$\therefore \exists$ y in B such that g (y) = z and $\exists$ x in A such that f(x) = y.

$\therefore g(y) = z \quad \Rightarrow \quad z = g(f(x)) = gof(x)$ where $x \in A$

$\Rightarrow$ gof is onto function.

Hence gof is bijective function.

Q.30 Let f : X $\rightarrow$ Y and X and Y are set of real numbers. Find f^{-1} if (i) $f(x) = x^2$; (ii) $f(x) = \frac{2x-1}{5}$

Ans. : Let $f : X \rightarrow Y$ and $X, Y \subseteq \Re$

(i) $\quad f(x) = x^2$

Let $\quad f(x) = y$

$$x^2 = y$$

$$\Rightarrow \quad y = x^2$$

$$\Rightarrow \quad y = \pm\sqrt{x}$$

Therefore, given function is not one to one. Hence f^{-1} does not exist.

(ii) Let $f(x) = y$

$$\frac{2x-1}{5} = y$$

$$\Rightarrow \quad 2x = 5y + 1$$

$$x = \frac{5y+1}{2} \quad \forall \ y \in \Re$$

Function f is 1 – 1 and onto.

∴ The inverse function of f is given as,

$$f^{-1}(x) = \frac{1+5y}{2}$$

4.8 : Pigeon Hole Principle

Important Points to Remember

I) This principle states that if there are n + 1 pigeons and only n pigeon holes then two pigeons will share the same whole.

This principle is stated by using the analogy of the bijective mapping i.e. If A and B are any two sets such that $|A| > |B|$ then there does not exist bijective mapping from A to B.

II) The extended pigeon hole principle

If n pigeons are assigned to m pigeon holes, then one of the pigeon holes must be occupied by at least $\left[\frac{n-1}{m}\right] + 1$ pigeons. It is also known as generalized pigeon hole principle. Here $\left[\frac{n-1}{m}\right]$ is the integer division of n – 1 by m. e.g. $\left[\frac{9}{2}\right] = 4$, $\left[\frac{16}{5}\right] = 3$, $\left[\frac{8}{3}\right] = 2$.

Q.31 If 11 shoes are selected from 10 pairs of shoes then there must be a pair and matched shoes among the selection.

Ans. : In the pigeonhole principle, 11 shoes are pigeons and the 10 pairs are the pigeon holes.

Q.32 Show that if seven numbers from 1 to 12 are chosen then two of them will add upto 13.

Ans. : We have A = {1, 2, 3, 4, 5, ... 12}

We form the six different sets each containing 2 numbers that add uptot 13.

$A_1 = \{1, 12\}$, $A_2 = \{2, 11\}$, $A_3 = \{3, 10\}$, $A_4 = \{4, 9\}$, $A_5 = \{5, 8\}$, $A_6 = \{6, 7\}$

Each of the seven numbers chosen must belong to one of these sets. As there are only six sets, by pigeonhole principle two of the chosen numbers must belong to the same set and their sum is 13.

Q.33 Show that 7 colours are used to paint 50 bicycles, then at least 8 bicycles will be of same colour. **[SPPU : Dec.-09, 12, Marks 4]**

Ans. : By the extended pigeonhole principle, at least $\left\lceil \frac{n-1}{m} \right\rceil + 1$ pigeons will occupy one piegeonhole.

Here $n = 50$, $m = 7$ and $m < n$ then

$$\left\lceil \frac{50-1}{7} \right\rceil + 1 = 7 + 1 = 8$$

Thus 8 bicycles will be of the same colour.

Q.34 Write generalized pigeonhole principle. Use any form of pigeonhole principle to solve the given problem.

i) Assume that there are 3 mens and 5 womens in a party show that if these people are lined up in a row at least two women will be next to each other.

ii) Find the minimum number of students in the class to be sure that three of them are born in the same month.

[SPPU : Dec.-11, Marks 4]

Ans. :

This principle states that if there are n + 1 pigeons and only n pigeon holes then two pigeons will share the same whole.

This principle is stated by using the analogy of the bijective mapping i.e. If A and B are any two sets such that $|A| > |B|$ then there does not exist bijective mapping from A to B.

i) By using analogy of pigeon hole principle, we get

3 men = pigeonholes and 5 women = pigeon

Pigeons are more than pigeon holes.

$\therefore$ At least two pigeons share the same pigeon hole i.e. at least two women in a row will be next to each other.

ii) Let h = Number of pigeons = Number of students

n = Number of pigeon holes = Number of months = 12

Given that three students in the class are born in the same month.

$$\therefore \quad \left[\frac{n-1}{m}\right]+1 = 3$$

$$\Rightarrow \quad \frac{n-1}{12} = 3-1 = 2$$

$$n = 2\times 12+1 = 25$$

Therefore there are 25 minimum number of students in the class.

4.9 : Generating Function

Important Points to Remember

Let $a = \{a_0, a_1, a_2, \ldots a_r, \ldots\}$ be a numeric function.

An infinite series

$$A(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots + a_r z^r + \ldots$$

is called the generating function of the sequence or numeric function a.

Note : The sum and product of two generating functions is a generating function.

I) Types of generating functions

Depending upon the nature of numeric sequence, there are two types of generating functions.

1) Ordinary generating function :

An infinite sum $\sum_{r=0}^{\infty} a_r z^r = a_0 + a_1 z + a_2 z^2 + \ldots + a_r z^r + \ldots$ is called the ordinary generating function of the numeric function $\{a_r\}_{r=0}^{\infty}$.

2) Exponential generating function :

An infinite sum $\sum_{r=0}^{\infty} a_r \frac{z^r}{r!} = a_0 + a_1 z + a_2 \frac{z^2}{2!} + a_3 \frac{z^3}{3!} + \ldots + a_r \frac{z^r}{r!}$ $+\ldots$ is called an exponential generating function of $\{a_r\}_{r=0}^{\infty}$.

II) Generating functions for some numeric functions

Sr. No.	Numeric function (a_r)	Generating function
1	a^r	$\frac{1}{1-az}$
2	$k\,a^r$	$\frac{k}{1-az}$
3	r	$\frac{z}{(1-z)^2}$
4	$r+1$	$\frac{1}{(1-z)^2}$
5	$(r+1)\,a^r$	$\frac{1}{(1-az)^2}$
6	$r\,a^r$	$\frac{az}{(1-az)^2}$
7	$\frac{1}{r!}$	e^z
8	n_{C_r} ; $0 \le r \le n$ 0 ; $r > n$	$(1+z)^n$

Q.35 Find the generating function for $a_r = \begin{cases} 0 & ; \quad r = \text{odd} \\ 2^{r+1} & ; \quad r = \text{even} \end{cases}$

Ans. : The numeric function is

$$\{a_r\}_{r=0}^{\infty} = \{2, 0, 2^3, 0, 2^5, 0, 2^7, \cdots\}$$

$$\therefore \quad a_r = 2^r + (-2)^r \; ; r \ge 0$$

$\therefore$ It's generating function is

$$A(z) = \frac{1}{1-2z} + \frac{1}{1+2z} = \frac{2}{1-4z^2}$$

Q.36 Find the numeric function for $A(z) = \frac{2}{1-4z^2}$.

[SPPU : Dec.-05, 07, Marks 4]

Ans. : We have $A(z) = \dfrac{2}{1-4z^2} = \dfrac{2}{(1+2z)(1-2z)}$

$$= \frac{A}{1-2z} + \frac{B}{1+2z} \quad \text{by partial fractions}$$

$$= \frac{1}{1-2z} + \frac{2}{1+2z}$$

$$A(z) = \sum_{r=0}^{\infty} (2z)^r + \sum_{r=0}^{\infty} (-2z)^r$$

Thus numeric function for A(z) is

$$a_r = 2^r + (-2)^r \; ; r \geq 0$$

4.10 : Recurrence Relation

I) In the previous section, we have studied about the generating functions and its discrete numeric functions or sequences. In that computations, a_r is given in terms of r only. But there so many problems in which previous are used to define next terms i.e. to define a_r we use a_{r-1}, a_{r-2} and so on.

A relation in which previous terms are used to define next terms, is called the **recurrence relation.** A recurrence relation is also called as difference equation.

e.g. i) $a_r = a_{r-1} + a_{r-2} \; ; r \geq 2$

$a_r = a_{r-1} + 5$ are recurrence relations

ii) $a_r = r^2 + 5$ is not recurrence relation

II) A recurrence relation of the form

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} + \cdots + c_k a_{r-k} = f(r) \text{ for } r \geq k \quad \ldots \text{(i)}$$

Where $c_0, c_1, c_2, \cdots c_k$ are constants, is called a linear recurrence relation with constant coefficients of order k provided c_0 and c_k are non zero.

e.g. $a_r - 3a_{r-1} = 2r^2$

is the first order linear recurrence relation with coefficients.

$a_r + a_{r-1} + 5a_{r-3} = 0$

is the third order linear recurrence relation with coefficients.

III) A recurrence relation (i) is called **homogeneous recurrence relation** if $f(r) = 0$.

i.e. $c_0 a_r + c_1 a_{r-1} + \dots + c_k a_{r-k} = 0$

IV) The **characteristic equation** of the k^{th} order homogeneous relation is $c_0 \alpha^k + c_1 \alpha^{k-1} + c_2 \alpha^{k-2} + \dots + c_k = 0$ which is the k^{th} degree polynomial in α.

The roots of this equation are called the characteristic roots.

A solution of a recurrence relation which is obtained from homogeneous recurrence relation is called the homogeneous solution. It is denoted by $a_r^{(h)}$.

The homogeneous solution depends upon the nature of characteristic roots.

The following table gives the details about homogeneous solution and roots of characteristic equations.

Sr. No.	Nature of roots	Homogeneous solution $a_r^{(h)}$
1.	Distinct real roots i) α_1, α_2 ii) $\alpha_1, \alpha_2 \dots \alpha_n$	$a_r^{(h)} = A_1 \alpha_1^r + A_2 \alpha_2^r$ $a_r^{(h)} = A_1 \alpha_1^r + A_2 \alpha_2^r + \dots + A_n \alpha_n^r$
2.	Repeated real roots i) α_1, α_1 ii) $\alpha_1, \alpha_1, \alpha_1$	$a_r^{(h)} = (A_1 + A_2 r) \alpha_1^r$ $a_r^{(h)} = (A_1 + A_2 r + A_3 r^2) \alpha_1^r$
3.	Distinct complex roots $\alpha + i\beta, \alpha - i\beta$	$a_r^{(h)} = r_1^r (A \cos r\theta + B \sin r\theta)$ where $r_1 = \sqrt{\alpha^2 + \beta^2}$, $\theta = \tan^{-1}\left(\frac{\beta}{\alpha}\right)$

V) Particular solution

The solution which satisfies the linear recurrence relation with $f(r)$ on right side is called the particular solution. It is denoted by $a_r^{(p)}$. There is no general method to determine the particular solution. It depends on the nature of $f(r)$. Consider the following table of particular solutions for given right hand side function $f(r)$.

Sr. No.	f(r) (Right hand side function)	Form of particular solution
1.	$f(r) = d = \text{constant}$	$a_r^{(p)} = p$
2.	$d_0 + d_1 r$	$p_0 + p_1 r$
3.	$d_0 + d_1 r + d_2 r^2 + \ldots + d_n r^n$	$p_0 + p_1 r + p_2 r^2 + \ldots + p_n r^n$
4.	$d\, b^r$ if b is not characteristic root	$p\, b^r$
5.	$d\, b^r$ if b is the root of characteristic equation with multiplicity m	$p\, r^m\, b^r$

VI) Total solution

The total solution or general solution of a recurrence relation is the sum of homogeneous solution and particular solution.

i.e. $$a_r = a_r^{(h)} + a_r^{(p)}$$

Q.37 Solve the following recurrence relation :
$a_r - 3a_{r-1} = 2,\ r \geq 1,\ a_0 = 1$ **[SPPU : May-07, Marks 4]**

Ans. : Step 1 : The characteristic equation is

$$(\lambda - 3) = 0$$

$\Rightarrow$ $\lambda = 3$ Which is real and distinct characteristic root

Hence homogeneous solution is

$$a_r^{(h)} = A(3)^r$$

Particular solution is of the type P (constant)

Hence $a_r = P$

Therefore $a_{r-1} = P$

Substituting the value of a_r and a_{r-1} in the given recurrence relation.

$$P - 3P = 2$$

$\Rightarrow$ $-2P = 2$

$\Rightarrow$ $P = -1$

Hence $a_r = A(3)^r - 1$

Using initial condition $a_0 = 1$

$$1 = A - 1 \Rightarrow A = 2$$

Hence $$a_r = 2(3^r) - 1$$

Q.38 Find the homogeneous solution for the recurrence relation $a_n = 6\, a_{n-1} - 11\, a_{n-2} + 6 a_{n-3}$ with $a_0 = 2$, $a_1 = 5$, $a_2 = 15$

☞[SPPU : Dec.-10, 12, Marks 6]

Ans. : We have

$$a_n - 6\, a_{n-1} + 11\, a_{n-2} + 6\, a_{n-3} = 0 \quad \text{... (Q.38.1)}$$

Step 1 : The characteristic equation is

$$\alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0$$

$\alpha = 1$ is a trial root

$\therefore$ By synthetic division method

1	1	− 6	11	− 6
		1	− 5	6
	1	− 5	6	0

$$\therefore \quad (\alpha - 1)(\alpha^2 - 5\alpha + 6) = 0$$

$$(\alpha - 1)(\alpha - 2)(\alpha - 3) = 0$$

$\alpha = 1, 2, 3$ are real and distinct roots

$\therefore$ It's homogeneous solution is

$$a_n^{(h)} = A_1\, 1^n + A_2\, 2^n + A_3\, 3^n \quad \text{... (Q.38.2)}$$

Step 2 : But given that $a_0 = 2$, $a_1 = 5$, $a_2 = 15$

$$\therefore \quad a_0 = A_1 + A_2 + A_3 \Rightarrow 2 = A_1 + A_2 + A_3 \quad \text{...(Q.38.3)}$$

$$a_1 = A_1 + 2\, A_2 + 3\, A_3 \Rightarrow 5 = A_1 + 2\, A_2 + 3\, A_3 \quad \text{...(Q.38.4)}$$

$$a_2 = A_1 + 4\, A_2 + 9\, A_3 \Rightarrow 15 = A_1 + 4\, A_2 + 9\, A_3 \quad \text{...(Q.38.5)}$$

equation (Q.38.4) + equation (Q.38.3) and equation (Q.38.5) − equation (Q.38.3)

$$\Rightarrow \quad 3 = A_2 + 2\, A_3 \quad \text{... (Q.38.6)}$$

and $$13 = 3\, A_2 + 8\, A_3 \quad \text{... (Q.38.7)}$$

equation (Q.38.7) – 3 × equation (Q.38.6)

$$4 = 0 + 2\,A_3 \Rightarrow \quad \boxed{A_3 = 2}$$

equation (Q.38.6) ⇒ $A_2 = 3 - 2\,A_3 = 3 - 4 = -1$

equation (Q.38.3) ⇒ $A_1 = 2 - A_2 - A_3 = 2 - (-1) - 2 = 1$

Hence $a_n^{(h)} = 1 + (-1)\,2^n + 2\,(3)^n$

Q.39 Solve $a_r - 4a_{r-1} + 4a_{r-2} = 0$ given that $a_0 = 0$ and $a_1 = 6$.

[SPPU : Dec.-16]

Ans. : We have

$$a_r - 4a_{r-1} + 4a_{r-2} = 0 \qquad \text{... (Q.39.1)}$$

Step 1 : The characteristic equation is

$$\alpha^2 - 4\alpha + 4 = 0$$

$$(\alpha - 2)^2 = 0$$

$\alpha = 2, 2$ which are real and repeated.

∴ The homogeneous solution is

$$a_r = (A_1 + A_2 r)2^r \qquad \text{... (Q.39.2)}$$

Step 2 : Given that, $a_0 = 0$ and $a_1 = 6$

∴ Put r = 0 in equation (Q.39.2), we get

$$a_0 = (A_1 + 0)2^0 \Rightarrow A_1 = 0$$

∴ Put r = 1 in equation (Q.39.2), we get

$$a_1 = (A_1 + A_2)2^1$$

$$6 = (0 + A_2)2$$

$$\Rightarrow A_2 = 3 \text{ and } a_r^{(p)} = 0$$

∴ Equation (Q.39.2) becomes,

$$a_r = 3r\,2^r$$

Q.40 Solve the recurrence relation :

$a_n + 6a_{n-1} + 12a_{n-2} + 8a_{n-3} = 2^n$, $n \geq 3$ $a_0 = 0, a_1 = 0, a_2 = 2$

[SPPU : May-05, Marks 6]

Ans. : Step 1 : The characteristic equation is

$$\lambda^3 + 6\lambda^2 + 12\lambda + 8 = 0$$

$$\Rightarrow \quad (\lambda+2)^3 = 0$$

Therefore $\lambda = -2, -2, -2$ which are real and repeated roots

Hence the homogeneous solution is

$$a_n^{(h)} = (A_1 n^2 + A_2 n + A_3)(-2)^n$$

Step 2 : The particular solution will be of the form $P(2^n)$. Substituting the given recurrence relation,

we get,

$$P(2^n) + 6P(2^{n-1}) + 12P(2^{n-2}) + 8P(2^{n-3}) = 2^n$$

$$\Rightarrow \quad 2^n P\left[1 + \frac{6}{2} + \frac{12}{4} + \frac{8}{8}\right] = 2^n$$

Or $$P[1+3+3+1] = 1 \Rightarrow P = \frac{1}{8}$$

$$\therefore \quad a_n^{(P)} = \frac{1}{8}(2^n) = 2^{n-3}$$

Step 3 : Total solution is

$$a_n = a_n^{(h)} + a_n^{(P)} = (A_1 n^2 + A_2 n + A_3)(-2)^n + 2^{n-3}$$

Using the initial conditions

$$a_0 = 0, a_1 = 0, a_2 = 2$$

$$0 = A_3 + \frac{1}{8}$$

$$\Rightarrow \quad A_3 = -\frac{1}{8} \quad \text{... (Q.40.1)}$$

$$0 = (A_1 + A_2 + A_3)(-2)^1 + 2^{1-3}$$

$$A_1 + A_2 + A_3 = \frac{1}{8} \quad \text{... (Q.40.2)}$$

Also $$2 = (A_1 4 + A_2 \cdot 2 + A_3)(-2)^2 + 2^{2-3}$$

$$2 = (4A_1 + 2A_2 + A_3)(4) + \frac{1}{2}$$

$$\frac{3}{2} = 4(4A_1 + 2A_2 + A_3)$$

$$\Rightarrow \quad 4A_1 + 2A_2 + A_3 = \frac{3}{8} \qquad \ldots (Q.40.3)$$

From equations (Q.40.1), (Q.40.2) and (Q.40.3) $A_1 = 0$, $A_2 = \frac{1}{4}$,

$A_3 = -\frac{1}{8}$

$$a_n = \left(\frac{1}{4}n - \frac{1}{8}\right)(-2)^n + 2^{n-3}$$

END...

Unit V

5 Introduction to Number Theory

5.1 : Binary Operations

Q.1 What are the two basic binary operations on the set of integers ? Write properties of these two operations.

Ans. : There are two basic binary operations (i) Addition denoted by '+' and (ii) Multiplication denoted by '.' on the set of integers Z. If $a, b \in Z$, then $a + b$ is called the **sum** and $a \cdot b$ or more simply written as ab is called the **product** of a nd b. The basic properties of these two operations are as given below :

A_1. **Closure for addition :** $a + b \in Z \ \forall\ a, b \in Z$.

A_2. **Commutativity of addition :** $a + b = b + a \ \forall\ a, b \in Z$.

A_3. **Associativity of addition :** $a + (b + c) = (a + b) + c \ \forall\ a, b, c \in Z$.

A_4. **Existence of identity for addition :** There exists a unique integer '0' such that

$$a + 0 = a = 0 + a \ \forall\ a \in Z.$$

This integer 0 is called the **additive identity**.

A_5. **Existence of additive inverse of each integer :** If $a \in Z$, then there exists a unique integer $-a \in Z$ such that $-a + a = 0 = a + (-a)$.

The integer $-a$ is called the **negative** or the **additive inverse** of the integer a.

M_1. **Closure for multiplication :** $ab \in Z$ for all $a, b \in Z$.

M_2. **Commutativity of multiplication :** $ab = ba \ \forall\ a, b \in Z$.

M_3. **Associativity of multiplication :** $(ab)c = a(bc) \ \forall\ a, b, c \in Z$.

M_4. **Existence of identity for multiplication :** There exist a unique integer '1' such that,

$$1a = a = a1 \ \forall\ a \in Z.$$

The integer is called the **multiplicative identity**.

- **Difference of two integers :** The difference of two integers a and b denoted by 'a – b' is defined as,

$$a - b = a + (-b)$$

e.g. $a = 10, b = 3$

$$\therefore \quad 10 - 3 = 10 + (-3) = 7$$

5.2 : Order Relations

Q.2 What do you mean by order relations on the set of integers ? List properties of it.

Ans. : Order Relations on the set of integers

Definition : If $a, b \in Z$ and $a - b \in Z^+$, then we say that a is greater than b and write $a > b$. Alternatively we say that b is less than a and write $b < a$.

If $a < b$ or $a = b$, we write $a \leq b$ and if $a > b$ or $a = b$, we write $a \geq b$. Obviously, a is positive iff $a > 0$ and a is negative iff $a < 0$. Also if $a \in Z$, then one and only one of the following is true :

$$a \in Z^+, a = 0, -a \in Z^+$$

i.e. $\quad a > 0, a = 0, a < 0.$

Some important properties of the order relations on Z

1. If $a, b \in Z$, then one and only one of the following is true : $a > b$, $a = b$, $a < b$.
2. Transitivity of the order relations, if $a, b, c \in Z$, then,
   i) $a < b, b < c \Rightarrow a < c$ and ii) $a > b, b > c \Rightarrow a > c$
3. If $a, b, c \in Z$, then,
   i) $a < b \Rightarrow a + c < b + c$ and $a - c < b - c$
   ii) $a > b \Rightarrow a + c > b + c$ and $a - c > b - c$
4. If $a, b, c \in Z$, then,
   i) $a > b, c > 0 \Rightarrow ac > bc$ and ii) $a > b, c < 0 \Rightarrow ac < bc$
5. If $a \in Z, a \neq 0$, then $a^2 = a \cdot a > 0$.

Q.3 What is well ordering principle ? Explain with few examples.

Ans. : 1) Least integer in a subset of Z : Let S be a non empty subset of Z. If there exists an integer $m \in S$ such that $x \geq m$ for all $x \in S$ then m is said to be smallest or the least integer in S.

If there exists an integer $n \in S$ such that $x \leq n$ for all $x \in S$, then n is said to be the greatest integer in S.

2) **Well ordering principle :** The well ordering principle states that "every non empty subset of the set of positive integers has a least member."

Example :

i) If $Z^+ = \{1, 2, 3, \ldots\}$ then 1 is the smallest positive integer.

ii) If $S = \{10, 11, 12, \ldots, 100\}$ then 10 is the smallest positive integer.

Note : If $k \in Z^+$ then there exists no integer a such that $k < a < k + 1$.

Q.4 Write two porms of principle of mathematical induction.

Ans. : First Form : Let K be a subset of N such that i) $1 \in K$ and ii) $n \in K \Rightarrow n + 1 \in K$, then $K = N$.

Second Form : Let K be a subset of N such that i) $1 \in K$ and ii) $k \in K$ for all k satisfying $1 \leq k < n \Rightarrow n \in K$, then $K = N$.

5.3 : Division Algorithm

Q.5 Define a) Divisors b) Proper Divisor c) Improper divisors.

Ans. : Divisor : Let a, b be two integers and $a \neq 0$. If there exists an integer c such that $b = ac$, then we say that a divides **b or a is a divisor of b** or **a is a factor of b** or **b is a multiple of a.**

When a is a divisor of b, we write, "$a|b$". This is read as 'a is a divisor of b'. If a is a divisor of b, then b is a multiple of a and we also write it as $b = M(a)$. Here M(a) is read as '**integral multiple of a**'.

If a is not a divisor of b, then we write 'a b' which is read as 'a is not a divisor of b'.

For example,

i) $3|18$ as $18 = 36$

ii) $(-5)|30$ as $30 = (-5)(-6)$

iii) $a|0$, for all $a \in Z$ and $a \neq 0$, since $0 = a \cdot 0$.

Thus 0 is a multiple of every integer or every non-zero integer a is a divisor of 0.

iv) 3 5 i.e. 3 is not a divisor of 5 because there exists no integer q such that $5 = 3q$.

Thus division is not everywhere defined in Z.

Improper divisors : For every integer $a \neq 0$, ± 1 and $\pm a$ are always divisors of a. These are called improper divisors of a.

Proper divisors : If a has any divisors other than these, then they are called **proper divisors** of a. For example the only divisors of 7 are ± 1 and ± 7 and so 7 has no proper divisors. On the other hand 8 possesses proper divisors. Besides ± 1 and ± 8, ± 2 and ± 4 are also divisors of 8 and these are proper divisors of 8.

Q.6 State and prove the division algorithm.

Ans. : The theorem known as division algorithm is of great importance in the development of number theory.

Theorem 1 If a is any integer and $b \neq 0$, then there exist unique integers q, r such that,

$$a = bq + r, \qquad \text{where } 0 \leq r < |b|$$

Proof : Consider the set $S = \{a - bx : x \in Z\}$. Since $a = a - b \cdot 0$ where $0 \in Z$, therefore at least $a \in S$ and thus S is not empty.

If $b < 0$ i.e., $b \leq -1$, then $b \cdot |a| \leq -|a| \leq a$

$\Rightarrow (a-b) \cdot |a| \geq 0$ i.e., $(a-b) \cdot |a|$ is non-negative.

Now $(a-b) \cdot |a| \in S$ because $|a| \in Z$

If $b < 0$, then S contains at least one non-negative integer i.e., $(a-b) \cdot |a|$.

If $b > 0$ i.e., if $b \geq 1$, then $b(-|a|) \leq -|a| \leq a$

$\Rightarrow a - b \cdot (-|a|) \geq 0$.

Now $a - b \cdot (-|a|) \in S$ because $-|a| \in Z$.

Therefore if $b > 0$, then S contains at least one non-negative integer i.e. $a - b \cdot (-|a|)$

Thus whether $b > 0$ or $b < 0$, the set S always contains non-negative integers. Therefore by the well-ordering principle the non-empty subset of S consisting of non-negative integers has a least number. Let $r = a - bq$ where $q \in Z$, be the smallest non-negative integer belonging to S. Since r is non-negative, therefore $0 \leq r$. We claim that $r < |b|$.

To fulfil our claim first we show that $r - |b| \in S$ whether $b > 0$ or $b < 0$. If $b > 0$, then $r - |b| = r - b = a - bq - b = a - (q + 1)b \in S$ since $(q + 1) \in Z$. If $b < 0$, then

$r - |b| = r - (-b) = a - bq + b = a - (q - 1)\,b \in S$ since $(q - 1) \in Z$.

Now $b \neq 0$. Therefore $r - |b| < r$. If $r \geq |b|$, then $r - |b| \geq 0$ i.e., $r - |b|$ is non-negative. Thus if $r \geq |b|$, them $r - |b|$ is a non-negative integer belonging to S and $r-|b| < r$. This is against the choice of r as the smallest non-negative integer $\in$ S. Hence we must have $r < |b|$.

Thus there exist integers q and r such that,

$$r = a - bq$$

i.e.,

$$a = bq + r \text{ and } 0 \le r < |b|$$

Now to show that the integers q, r are unique. Suppose we should find another pair q' and r' such that

$$a = bq' + r', 0 \le r' < |b|$$

The $bq' + r' = bq + r \Rightarrow b(q'-q) = r - r'$

$\Rightarrow b|(r-r')$

Without any loss of generality we can assume that $r \ge r'$.

- Then $0 \le r < |b|$ and $0 \le r' < |b'| \Rightarrow 0 \le r - r' < |b|$. Therefore r is a divisor of $r - r'$ is positive only if $r - r' = 0$. Therefore $r' = r$.

$\therefore$ Putting $r = r'$, we get $bq' + r' = bq + r \Rightarrow bq' = bq \Rightarrow q = q'$. Thus $r = r'$ and $q = q'$. Hence q and r are unique. Hence the proof.

5.4 : Greatest Common Divisor

Q.7 Define : a) Greatest common divisor

b) Greatest common divisor of more than two integers.

Ans. : Let a and b be two integers not both zero (at least one of them is non zero). Then the Greatest Common Division (GCD) of a and b is a positive integer d such that,

i) $d|a$ and $d|b$ i.e. d is a common divisor of a and b and

ii) If an integer c, $c|a$ and $c|b$ then $c|d$ i.e. every common divisor of a and b is a divisor of d.

If d is the greatest common divisor of a and b then it is denoted by $d = (a, b)$. It is also known as the Highest Common Factor (HCF).

Example :

1, 3, 5 and 15 are common divisors of 45 and 60. Out of these each of 1, 3 and 5 is a divisor of 15.

$\therefore$ $\gcd(45, 60) = 15$

Note : $(a, b) = (-a, b) = (a, -b)$

Greatest Common Divisor of more than two Integers

Definition : Let $\{a_1, a_2, \ldots, a_n\}$ be a finite set of integers, not all zero. If there exists a positive integer d such that

i) d is a common divisor of $a_1, a_2, \ldots, a_n$ and ii) each common divisor of $a_1, a_2, \ldots, a_n$ is also a divisor of d, then d is called the greatest common divisor of $a_1, a_2, \ldots, a_n$. Symbolically, we write

$$d = (a_1, a_2, \ldots, a_n)$$

For example, $(20, 60, 45, -30) = 5$

Important Theorems to be Remembered

I) Existence and uniqueness of Greatest Common Divisor Theorem : Every pair of integers a and b, not both zero, has a unique greatest common divisor (a, b) which can be expressed in the form

$$(a, b) = xa + yb, \text{ for some integers } x \text{ and } y.$$

II) Construction of G.C.D. by Repeated use of Division Algorithm.

If $a, b \in Z$, $b \neq 0$ and $a = bq + r$, $0 \leq r \leq |b|$ then $(a, b) = (b, r)$.

III) Theorem : If $a, b \in Z$, $b \neq 0$ and $a = bq + r$, $0 \leq r < |b|$ then $(a, b) = (b, r)$.

Q.8 By using the Euclidean algorithm, find the greatest common divisor d of the numbers 1109 and 4999 and then find integers x and y to satisfy d = 1109x + 4999y.

Ans. : By repeatedly applying the process of division algorithm, we get,

$$4999 = (1109) \cdot 4 + 563, \quad \ldots(Q.8.1)$$

$$1109 = (563) \cdot 1 + 546, \quad \ldots(Q.8.2)$$

$$563 = (546) \cdot 1 + 17, \quad \ldots(Q.8.3)$$

$$546 = (17) \cdot 32 + 2, \quad \ldots(Q.8.4)$$

$$17 = (2) \cdot 8 + 1, \quad \ldots(Q.8.5)$$

$$2 = (1) \cdot 2 + 0. \quad \ldots(Q.8.6)$$

Hence (1109, 4999) = the last non-zero remainder in the above repeated divisions = 1.

Now substituting backwords, we have

$$1 = 17 - (2) \cdot 8 \qquad \text{[by equation(Q.8.5)]}$$

$$= 17 - [546 - (17) \cdot 32] \cdot 8,$$

substituting for 2 from equation(Q.8.4)

$$= (17) \cdot 257 - 546 \cdot 8$$

$$= [563 - (546) \cdot 1] \cdot 257 - 546 \cdot 8,$$

substituting for 17 from equation(Q.8.3)

$= 563 \cdot 257 - 546 \cdot 265$

$= 563 \cdot 257 - [1109 - (563) \cdot 1] \cdot 265,$

substituting for 546 from equation (Q.8.2)

$= 563 \cdot 522 - 1109 \cdot 265$

$= [4999 - (1109) \cdot 4] \cdot 522 - 1109 \cdot 265,$

substituting for 563 from equation (Q.8.1)

$= 4999 \cdot 522 - 1109 \cdot 2353$

Hence $(1109, 4999) = 1 = 1109 \cdot (-2353) + 4999 \cdot (522)$

$= 1109x + 4999y$, where $x = -2353$, $y = 522$.

Q.9 Find the G.C.D. of 275 and 200 and express it in the form $m \cdot 275 + n \cdot 200$.

Ans. : By repeatedly applying the process of division algorithm, we get,

$$275 = (200) \cdot 1 + 75, \quad \text{...(Q.9.1)}$$

$$200 = (75) \cdot 2 + 50, \quad \text{...(Q.9.2)}$$

$$75 = (50) \cdot 1 + 25, \quad \text{...(Q.9.3)}$$

$$50 = (25) \cdot 2 + 0, \quad \text{...(Q.9.4)}$$

Hence (275, 200) = the last non-zero remainder in the above repeated divisions = 25.

Now substituting backwords, we have,

$25 = 75 - (50) \cdot 1$ [by equation(Q.9.3)]

$= 75 - [200 - (75) \cdot 2] \cdot 1,$ substituting for 50 from equation(Q.9.2)

$= 75 \cdot 3 - 200 \cdot 1$

$= [275 - (200) \cdot 1] \cdot 3 - 200 \cdot 1,$

substituting for 75 from equation(Q.9.1)

$= 275 \cdot 3 - 200 \cdot 4$

$= (3) \cdot 275 + (-4) \cdot 200$

Hence $(275, 200) = 25 = (3) \cdot 275 + (-4) \cdot 200$ so that $m = 3$, $n = -4$.

Q.10 If $a = -427$, $b = 616$, find (a, b) and express it in the form $(a, b) = ax + by$.

Ans. : Since $(a, b) = (|a|, |b|)$, therefore in order to find $(-427, 616)$, we shall find (427, 616).

We construct Euclidean algorithm for 427, 616 :

$$616 = (427) \cdot 1 + 189, \quad \ldots(Q.10.1)$$

$$427 = (189) \cdot 2 + 49, \quad \ldots(Q.10.2)$$

$$189 = (49) \cdot 3 + 42, \quad \ldots(Q.10.3)$$

$$49 = (42) \cdot 1 + 7, \quad \ldots(Q.10.4)$$

$$42 = (7) \cdot 6 + 0. \quad \ldots(Q.10.5)$$

Hence (427, 616) = the last non-zero remainder in the above repeated divisions = 7.

Now substituting backwords, we have

$$\begin{aligned} 7 &= 49 - (42) \cdot 1 && \text{[by equation(Q.10.4)]} \\ &= 49 - [189 - (49) \cdot 3] \cdot 1 && \text{[by equation(Q.10.3)]} \\ &= (49) \cdot 4 - 189 \\ &= [427 - (189) \cdot 2] \cdot 4 - 189 && \text{[by equation(Q.10.2)]} \\ &= (427) \cdot 4 - (189) \cdot 9 \\ &= (427) \cdot 4 - [616 - (427) \cdot 1] \cdot 9 \\ &= (427) \cdot 13 + (616) \cdot (9) \end{aligned}$$

Hence $(-427, 616) = 7 = (-427) \cdot (-13) + (616) \cdot (-9)$

$= ax + by,$ where $x = -13$, $y = -9$.

Q.11 Write short note on Extended Euclidean Algorithm.

Ans. : Used to find GCD, Bezout's coefficients s and t and the multiplicative inverse of an integer.

- Algorithm : (Find multiplicative inverse of x with respect to m)

1. Let $r_1 = x$ $r_2 = m$, $s_1 = 1$, $s_2 = 0$ $t_1 = 0$, $t_2 = 1$ and r, s, t be integers
2. While (r_2 != 0)
3. do

   i. $r = r_1 - q \cdot r_2$;

   ii. $s = s_1 - q \cdot s_2$;

   iii. $t = t_1 - q \cdot t_2$;

   iv. $r_1 = r_2$; $r_2 = r$

   v. $s_1 = s_2$; $s_2 = s$

   vi. $t_1 = t_2$; $t_2 = t$

4. Done

5. GCD = r_1; Bezout's coefficient = s_1 and t_1, and Multiplicative inverse of $x = r_1$ with respect to $m = r_2$ is s_1 or vice versa Multiplicative inverse of $m = r_2$ with respect to $x = r_1$ is t_1.

Example - Extended Euclidean Algorithm

q	r_1	r_2	r	s_1	s_2	s	t_1	t_2	t
0	33	86	33	1	0	1	0	1	0
2	86	33	20	0	1	–2	1	0	1
1	33	20	13	1	–2	3	0	1	–1
1	20	13	7	–2	3	–5	1	–1	2
1	13	7	6	3	–5	8	–1	2	–3
1	7	6	1	–5	8	–13	2	–3	5
6	6	1	0	8	–13	86	–3	5	–33
--	1 = GCD	0	--	–13 = s	86	--	5 = t	–33	--

Multiplicative Inverse of 33 is – 13 ≡ 73 (mod 86) and Multiplicative Inverse of 86 is 5 (mod 33)

Bezout's Identity is $1 = -13 * 33 + 5 * 86$

Example - Extended Euclidean Algorithm

q	r_1	r_2	r	s_1	s_2	s
0	33	86	33	1	0	1
2	86	33	20	0	1	–2
1	33	20	13	1	–2	3
1	20	13	7	–2	3	–5
1	13	7	6	3	–5	8
1	7	6	1	–5	8	–13
6	6	1	0	8	–13	86
--	1 = GCD	0	--	–13 = s	86	--

If the intension is to find multiplicative inverse, then you need to find only variable s which gives you inverse of r_1 with respect to r_2 (modulus m), no need to do calculations for t.

Multiplicative Inverse of 33 (r_1) is –13 ≡ 73 [mod 86 (r_2)]

Q.12 Define a) Relatively prime integers, b) Least common multiple, c) Prime integers, d) Composite integers.

Ans. : a) Relatively prime integers : Two integers a and b are said to be relatively prime if their greatest common divisor is 1 i.e., if $(a, b) = 1$. If $(a, b) = 1$, we also say that a and b are co-prime or prime to each other. If a and b are relatively prime, then a and b have no common factors except 1 or – 1. For example,

i) 14, – 9 are relatively prime integers since $(14, -9) = 1$.

ii) 18, 14 are not relatively prime integers since $(18, 14) = 2$.

b) Least common multiple : Let a and b be two non-zero integers. The Least Common Multiple (L.C.M.) of a and b is the unique positive integer m such that

i) $a \mid m$, $b \mid m$ and ii) $a \mid s$, $b \mid s \Rightarrow m \mid s$

Notation : L.C.M. of a and b is denoted by [a, b]

Example : i) $[12, 18] = 36$, ii) $[10, 25] = 50$

c) Prime integers : A non-zero integer p is called a **prime** if it is neither 1 nor – 1 and if its only divisors are $1, -1, p, -p$.

For example,

i) The integers 5 and – 11 are primes, while $12 = 4 \times 3$ and $-36 = 3 \cdot (-12)$ are not primes.

ii) The first 10 positive primes are,
2, 3, 5, 7, 11, 13, 17, 19, 23, 29.

Note (1) : By definition, 1 is not prime. It is obvious that $-p$ is a prime iff p is a prime.

Note (2) : 2 is the only even integer which is a prime. Every other even integer has 2 as a factor and so it cannot be a prime. Therefore if p is a prime and $p \neq 2$, then p must be an odd integer.

Note (3) : If p is a prime and a is any integer, then either $p \mid a$ or $(p, a) = 1$.

d) Composite integers : If an integer a can be written as $a = bc$, where b and c are integers such that $|b| > 1$ and $|c| > 1$, then a is called a **composite integer**.

For example $6 = 2 \cdot 3$ where $|2| > 1$ and $|3| > 1$. Therefore 6 is a **composite integer.**

Every integer a $\neq 0, \pm 1$ is either a prime or a composite.

If a is a positive integer, then a is a composite iff there exist two positive integers b and c such that,

$$a = bc, \qquad \text{where } 1 < b < a,\ 1 < c < a.$$

Important to be Remembered

- **Theorem :** Two integers a and b are relatively prime if and only if we can find integers x and y such that $ax + by = 1$
- **Euclid's Lemma :** If p is a prime number and a, b are any integers then $p \mid ab \Rightarrow p \mid a$ or $p \mid b$.
- **Corollary :** If p is a prime and $a, b \in Z$ are such that $0 < a < p$, $0 < b < p$, then p cannot be a divisor of ab.
- **Theorem :** If a prime number p divides the product $a_1 . a_2 ... . a_n$ of certain integers, then p must divide at least one of $a_1 . a_2 ... . a_n$.
- **Theorem :** If a is a possible integer greater than 1 then a has a prime factor.
- **Theorem :** The fundamental theorem of **arithmetic** or the **Unique Factorisation theorem.** Every positive integer $a > 1$ can be expressed uniquely as a product of positive primes.

Q.13 Find the G.C.D. and the L.C.M. of a = 5040, b = 14850 by writing each of the numbers a and b in prime factorization canonical form.

Ans. : We have $a = 2^4 \cdot 3^2 \cdot 5^1 \cdot 7^1$ and $b = 2^1 \cdot 3^3 \cdot 5^2 \cdot (11)^1$

$\therefore$ G.C.D. of a and b i.e., $(a, b) = 2^1 \cdot 3^2 \cdot 5^1 = 90$.

Also L.C.M. of a and b i.e. $[a, b] = 2^4 \cdot 3^3 \cdot 5^2 \cdot 7^1 \cdot (11)^1 = 831600$

Q.14 Find the number of distinct positive integral divisors and their sum for the integer 56700.

Solution : Expressing in prime factorization canonical form, we have

$$\begin{aligned} 56700 &= 2 \times 28350 = 2^2 \times 14175 \\ &= 2^2 \times 3 \times 4725 = 2^2 \times 3^2 \times 1575 \\ &= 2^2 \times 3^3 \times 525 = 2^2 \times 3^4 \times 175 \\ &= 2^2 \times 3^4 \times 5 \times 35 = 2^2 \times 3^4 \times 5^2 \times 7^1, \end{aligned}$$

where $\qquad p_1 = 2,\ p_2 = 3,\ p_3 = 5,\ p_4 = 7$ and

$$\alpha_1 = 2, \alpha_2 = 4, \alpha_3 = 2, \alpha_4 = 1.$$

$\therefore$ $\tau(56700)$ = The number of distinct positive integral divisors of 56700

$$= (\alpha_1 + 1)(\alpha_2 + 1)(\alpha_3 + 1)(\alpha_4 + 1)$$

$$= (2+1)(4+1)(2+1)(1+1) = 3 \times 5 \times 3 \times 2$$

$$= 90$$

Also $\sigma(56700)$ = The sum of all the distinct positive integral divisors of 56700

$$= \frac{2^{2+1}-1}{2-1} \cdot \frac{3^{4+1}-1}{3-1} \cdot \frac{5^{2+1}-1}{5-1} \cdot \frac{7^{1+1}-1}{7-1}$$

$$= \frac{2^3-1}{1} \cdot \frac{3^5-1}{2} \cdot \frac{5^3-1}{4} \cdot \frac{7^2-1}{6}$$

$$= 7 \times 121 \times 31 \times 8 = 210056.$$

Q.15 If $p = 2^n - 1$ is a prime, prove that n is a prime.

Solution : Let n be not a prime. Then there exist positive integers r and s such that

$$n = rs \qquad \text{where } 1 < r < n,\ 1 < s < n.$$

$$\therefore \quad p = 2^n - 1 = 2^{rs} - 1 = (2^r)^s - 1$$

$$= a^s - 1 \qquad \text{where } a = 2^r$$

$$= (a-1)(a^{s-1} + a^{s-2} + \ldots + a + 1) \qquad \ldots(Q.14.1)$$

Now $a - 1 > 1$ because $a = 2^r > 2$. Also $a - 1 = 2^r - 1 < p$ because $p = (2^r)^s - 1$. Thus $a - 1$ is a positive integer such that $1 < a - 1 < p$ and from (1), $a - 1$ is a divisor of p.

Hence p is not a prime. But this is a contradiction.

Hence n must be a prime.

Note : The converse of the above statement is not true. For example if we taken $n = 11$, then n is prime. But $2^{11} - 1 = 2047 = 23 \times 89$ and thus $2^{11} - 1$ is not prime. Hence n is prime does not necessarily imply that $2^n - 1$ is prime.

Q.16 Define Mersenne Numbers.

Ans. : The members of the form $M_n = 2^n - 1$, where n is a prime are known as **Mersenne numbers**. All the Mersenne numbers are not prime. For example $M_{11} = 2^{11} - 1 = 2047$ is composite because 23 is a divisor of 2047.

Q.17 Prove that the product of r consecutive natural numbers is divisible by r !.

Ans. : Let $x \in N$ and $x, x+1, x+2, \ldots, x+r-1$ be r consecutive natural numbers.

$$\begin{aligned} P &= \text{Product of these r consecutive natural numbers} \\ &= x(x+1)(x+2)\ldots(x+r-1) \\ &= \frac{(x+r-1)!}{(x-1)!} \end{aligned}$$

We have to prove that P is divisible by r ! i.e. we have to prove that $\frac{(x+r-1)!}{(x-1)!\, r!}$ is an integer.

For this we have to show that the highest power of every prime factor p contained in the product $(x-1)!\, r!$ is not greater than the highest power of p in $(x+r-1)!$

Now we know that if a, b are any real numbers, then

$$I(a+b) \geq I(a) + I(b),$$

Where I(a)

stands for the integral part of a.

$$\therefore I\left(\frac{x+r-1}{p}\right) \geq I\left(\frac{x-1}{p}\right) + I\left(\frac{r}{p}\right),$$

$$I\left(\frac{x+r-1}{p^2}\right) \geq I\left(\frac{x-1}{p^2}\right) + I\left(\frac{r}{p^2}\right),$$

$$I\left(\frac{x+r-1}{p^3}\right) \geq I\left(\frac{x-1}{p^3}\right) + I\left(\frac{r}{p^3}\right),$$

... ...

... ...

Adding the above inequalities, we have

$$I\left(\frac{x+r-1}{p}\right)+I\left(\frac{x+r-1}{p^2}\right)+I\left(\frac{x+r-1}{p^3}\right)+\ldots$$

$$\geq\left[I\left(\frac{x-1}{p}\right)+I\left(\frac{x-1}{p^2}\right)+\ldots\right]+\left[I\left(\frac{r}{p}\right)+I\left(\frac{r}{p^2}\right)+\ldots\right]$$

∴ The higest power of p in (x + r − 1) ! ≥ the highest power of p in the product (x−1) ! r !.

Hence (x − 1) ! r ! is a divisor of (x + r − 1) ! and consequently r ! is a divisor of
x (x + 1)... (x + 2)...((x + r − 1).

Note : From the above result we can immediately deduce that the product of any r consecutive integers is divisible by r !

Q.18 If n > 2, show that $n^5 - 5n^3 + 4n$ divisible by 120.

Solution : We have $n^5 - 5n^3 + 4n = n\,(n^4 - 5n^2 + 4)$

$$= n(n^2-1)(n^2-4)$$

$$= n\,(n-1)\,(n+1)(n-2)(n+2)$$

$$= (n-2)(n-1)\,n\,(n+1)(n+2)$$

Thus if n > 2, then $n^5 - 5n^3 + 4n$ has been expressed as a product of five consecutive natural numbers and so it is divisible by 5 ! i.e., 120.

Q.19 Write short note on congruence of integers.

Ans. : Definition : Let m be any positive integer i.e. m > 0. Then an integer 'a' is said to be congruent to another integer b modulo m if m | (a − b) i.e. if m is a division of (a − b).

Symbolically we write

$$a \equiv b \pmod{m}.$$

It will be read as "a is congruent to b modulo m".

Thus a ≡ b (mod m) iff a − b = km ffor some integer k i.e., iff a − b is a mulitple of m.

If m is not a divisor of a − b, then say that 'a is not congruent to b modulo m' and we write $a \not\equiv b \pmod{m}$.

For example :

$$89 \equiv 25 \pmod 4 \text{ since } 89 - 25 = 64 \text{ and } 4 \mid 64$$

$$25 \equiv 1 \pmod 4 \text{ since } 25 - 1 = 24 \text{ and } 4 \mid 24$$

$$153 \equiv -7 \pmod 8 \text{ since } 153 - (-7) = 160 \text{ and } 8 \mid 160$$

$$13 \equiv 3 \pmod 5 \text{ since } 13 - 3 = 10 \text{ and } 5 \mid 10.$$

But $24 \not\equiv 3 \pmod 5$ since $24 - 3 = 21$ and 5 is not a divisor of 21.

Also note that $m \mid a \Leftrightarrow a \equiv 0 \pmod m$.

Q.20 Prove that 'the relation' congruence modulo m is on equivalence relation in the set of integers.

Ans. : Theorem 1 : The relation "congruence modulo m" is an equivalence relation in the set of integers.

Proof : Let Z be the set integers. If m is any fixed positive integer, then we say that $a \equiv b \pmod m$ if $m \mid (a - b)$. We shall show that this defines an equivalence realtion on the set Z.

Reflexivity : Let a be any integer. Then $a - a = 0$ and $m \mid 0$.

Thus $a \equiv a \pmod m \ \forall \ a \in Z$. Therefore the relation is reflexive.

Symmetry : Let $a, b \in Z$ be such that $a \equiv b \pmod m$. Then we have

$$m \mid (a - b) \Rightarrow a - b = km \text{ for some } k \in Z$$

$$\Rightarrow b - a = (-k)\, m \qquad \text{where } -k \in Z$$

$$\Rightarrow m \mid (b - a) \Rightarrow b \equiv a \pmod m$$

Thus $a \equiv b \pmod m \Rightarrow b \equiv a \pmod m$ and therefore the relation is symmetric.

Transitivity : Let $a, b, c \in Z$ be such that $a \equiv b \pmod m$, $b \equiv c \pmod m$. Then we have,

$$m \mid (a - b) \text{ and } m \mid (b - c)$$

$$\Rightarrow m \mid \{(a - b) + (b - c)\} \Rightarrow m \mid (a - c)$$

$$\Rightarrow a \equiv c \pmod m$$

Thus $a \equiv b \pmod m$ and $b \equiv c \pmod m \Rightarrow a \equiv c \pmod m$. Therefore the relation is transitive.

Hence congruence modulo m is an equivalence relation on Z.

Q.21 Prove that if $a \equiv b \pmod{m}$, then, for all $x \in Z$, $a + x \equiv b + x \pmod{m}$ and $ax \equiv bx \pmod{m}$.

Ans. : We have $a \equiv b \pmod{m} \Rightarrow m \mid (a-b)$

$\Rightarrow m \mid \{(a+x)-(b+x)\}\ \forall\ x \in Z$

$\Rightarrow a + x \equiv b + x \pmod{m}\ \forall\ x \in Z$

Similarly, $a \equiv b \pmod{m} \Rightarrow m \mid (a - b)$

$\Rightarrow m \mid x(a-b)$ for all $x \in Z$

$\Rightarrow m \mid (ax-bx)\ \forall\ x \in Z$

$\Rightarrow ax \equiv bx \pmod{m}\ \forall\ x \in Z.$

Q.22 Prove that if $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then $a + c \equiv b + d \pmod{m}$,
$a - c \equiv b - d \pmod{m}$, $ac \equiv bd \pmod{m}$.

Ans. : We have ,

$$a \equiv b \pmod{m} \Rightarrow m \mid (a - b)$$

and $$c \equiv d \pmod{m} \Rightarrow m \mid (c - d)$$

Now $m \mid (a - b)$ and $m \mid (c - d)$

$\Rightarrow m \mid \{(a - b) + (c - d)\} \Rightarrow m \mid \{(a + c) - (b + d)\}$

$\Rightarrow a + c \equiv b + d \pmod{m}$

Similarly $m \mid (a - b)$ and $m \mid (c - d)$

$\Rightarrow m \mid \{(a - b) - (c - d) \Rightarrow m \mid \{(a - c) - (b - d)\}$

$\Rightarrow a - c \equiv b - d \pmod{m}.$

Finally $m \mid (a - b)$ and $m \mid (a - b)$

$\Rightarrow m \mid \{c\,(a - b) + b\,(c - d)\} \Rightarrow m \mid (ac - bd)$

Q.23 If $a \equiv b \pmod{m}$ and m_1 is a positive divisor of m, then $a \equiv b \pmod{m_1}$.

Ans. : If m_1 is a positive divisor of m, then $m = m_1 q_1$ for some $q_1 \in Z^+$.

Now $\quad a \equiv b \pmod{m}\ m \mid (a - b)$

$\Rightarrow \quad a - b = q_2 m$ for some $q_2 \in Z$

$\Rightarrow \quad a - b = q_2 (m_1 q_1) = m_1 (q_1 q_2)$ where $q_1 q_2 \in Z$

$\Rightarrow m_1 \mid (a - b) \Rightarrow a \equiv b \pmod{m_1}$

5.5 : Residue Classes

Important Points to Remember

1) **Residue classes : Definition :** We know that if m is a fixed positive integer, then 'congruence modulo m' is an equivalence relation on the set of integers Z. Consequently it partitions Z into a collection of mutually disjoint equivalence classes. These equivalence classes are called **'residue classes modulo m'**.

We shall denote the set of all residue classes of integers modulo m by Z_m or by Z/(m). It is also called the set of integers modulo m.

2) Let m be a fixed positive integer and

$$S = \{0, 1, 2, ... m - 1\}$$

Then no two integers of S are congruent modulo m to each other and every $x \in Z$ is congruent modulo m to one of the integers of S.

3) **Euler's ϕ Function : Definition :** The Euler ϕ-function is the function $\phi : Z^+ \to Z^+$ defined as follows :

i) $\phi(1) = 1$ and

ii) for $n\ (> 1) \in Z^+$, $(\phi)\ n$ = The number of positive integers less than n and relatively prime to n.

4) **Theorem 1 :** If m and n are relatively prime positive integers i.e. (m, n) = 1 then,

$\phi(m, n) = \phi(m) \cdot \phi(n)$

5) **Theorem 2 :** If n > 1 and $p_1, p_2, \ldots, p_m$ are the distinct prime factors of n, then

$$\phi(n) = n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\ldots\left(1-\frac{1}{p_m}\right)$$

6) **Theorem 3 :** Fermat's Theorem. If p is a positive prime and a is any integer such that p is not a divisor of a so that (a, p) =1, then

$$a^{p-1} \equiv (\bmod\ p)$$

7) **Corollary :** If p is a positive prime and a is any integer, then $a^P \equiv a \pmod p$ i.e., (p) $a^P - a$ is a multiple of p.

8) **Theorem 4 : Euler's theorem :** If m is a positive integer and a is any integer such that (a, m) =1, then $\alpha^{\phi(m)} \equiv 1 \pmod m$.

9) **Corollary :** Fermat's theorem as a corollary of Euler's theorem.

If in Euler's theorem, we take m = p where p is a prime, then

$$\phi(m) = \phi(p) = p-1$$

$\therefore$ The result $a^{\phi(m)} \equiv 1 \pmod m$ takes the form

$$a^{p-1} \equiv 1 \pmod p, \qquad \text{which is Fermat's theorem.}$$

10) **Theorem 5 :** Wilson's theorem. If p is a positive prime then $(p-1)!+1 \equiv 0 \pmod p$ i.e., $(p-1)!+1$ is a multiple of p.

11) **Fermat's little Thm : Fermat's little theorem** states that if p is a prime number, then for any integer x, the number $x^p - x$ is an integer multiple of p. In the notation of modular arithmetic, this is expressed as

$$x_p \equiv x \pmod p \text{ [Example } 8^{13} \equiv 8 \bmod 13]$$

- If a is not divisible by p, Fermat's little theorem is equivalent to the statement that $a^{p-1} - 1$ is an integer multiple of p, or in symbols :

$$x^{p-1} \equiv 1 \pmod p \text{ [Example } 40^{12} \equiv 1 \bmod 13]$$

Example $87^{25} \bmod 7 \equiv (87^{6*4+1} \bmod 7) \equiv (87^6 \bmod 7)^4 * (87 \bmod 7) \equiv 3$

12) **Chinese Remain der Thm : Statement :** Let $n_1, n_2, n_3, \ldots, n_r$ be positive integers such that $\gcd(n_i, n_j) = 1$ for $i \neq j$. The system of linear congruences

$$\begin{aligned} x &\equiv a_1 \pmod{n_1} \\ x &\equiv a_2 \pmod{n_2} \\ x &\equiv a_3 \pmod{n_3} \\ &\vdots \\ x &\equiv a_r \pmod{n_r} \end{aligned}$$

has a simultaneous solution, which is unique modulo the integer $n_1 n_2 n_3 \ldots n_r$.

Procedure to solve examples :

Consider $n = n_1 n_2 n_3 \ldots n_r$

For each $k = 1, 2, 3, \ldots r$, Let

$$N_k = \frac{n}{n_k} = n_1 n_2 \ldots n_{k-1} n_{k+1} \ldots n_r$$

Solve $N_k x \equiv 1 \pmod{n_k}$ and call the unique solution x_k.

Now $\bar{x} = a_1 N_1 x_1 + a_2 N_2 x_2 + \ldots + a_r N_r x_r$

Therefore $\bar{x} \equiv x' \pmod n$ where $n \mid (\bar{x} - x')$

Q.24 Find the number of positive integers < 3600 that are relatively prime to 3600.

Ans. : We have to find $\phi(3600)$.

We shall first express 3600 in canonical form

We have $3600 = 2^4 \times 3^2 \times 5^2$

$$\therefore \quad \phi(3600) = \phi(2^4 \times 3^2 \times 5^2) = \phi(2^4) \cdot \phi(3^2) \cdot \phi(5^2)$$

$$= 2^4\left(1-\frac{1}{2}\right) \cdot 3^2\left(1-\frac{1}{3}\right) \cdot 5^2\left(1-\frac{1}{5}\right)$$

$$= 2^4 \cdot 3^2 \cdot 5^2\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right)$$

$$= 3600 \cdot \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{4}{5} = 960$$

Q.25 Solve $x \equiv 2 \pmod 3$, $x \equiv 3 \pmod 5$, $x \equiv 2 \pmod 7$.

Ans. : We have $a_1 = 2$, $a_2 = 3$, $a_3 = 2$ and $n_1 = 3$, $n_2 = 5$ and $n_3 = 7$.

n_1, n_2 and n_3 are relatively primes

$$\therefore \quad n = n_1 \cdot n_2 \cdot n_3 = 3 \cdot 5 \cdot 7 = 105$$

and $$N_1 = \frac{n}{n_1} = \frac{3 \cdot 5 \cdot 7}{3} = 35, \quad N_2 = \frac{n}{n_2} = 3 \times 7 = 21$$

$$N_3 = \frac{n}{n_3} = 3.5 = 15$$

Now, Linear congruences are

$$N_1 x \equiv 1 \pmod{n_1}$$

$$35x \equiv 1 \pmod 3, \; 21x \equiv 1 \pmod 5 \text{ and } 15x \equiv 1 \pmod 7$$

These equations are satisfied by $x_1 = 2$, $x_2 = 1$, $x_3 = 1$ respectively

Therefore, the solution of the system is given by

$$\bar{x} = a_1 N_1 x_1 + a_2 N_2 x_2 + a_3 N_3 x_3$$

$$\bar{x} = 2 \cdot 35 \cdot 2 + 3 \cdot 21 \cdot 1 + 2 \cdot 15 \cdot 1 = 233$$

Hence $$\bar{x} \equiv x' \pmod n$$

$$233 \equiv x' \pmod{105}$$

$$233 \equiv 23 \pmod{105}$$

This is the required unique solution.

END...

Unit VI

6 Algebraic Structures

6.1 : Algebraic Structure

Q.1 Define binary operation. [SPPU : Dec.-05]

Ans. : Binary operation : I) Let A be any non empty set. A function $f : A \times A \rightarrow A$ is called the binary operation on the set A.

'$*$' is a binary operation on the set A iff $a * b \in A\ \forall\ a, b \in A$ and $a * b$ is unique.

II) An n-ary operation on a set $A \neq \phi$,
is a function $f : A \times A \times \ldots \times A$ (n times) $\rightarrow A$ i.e. $f : A^n \rightarrow A$

The n-ary operation is defined for each n-tuple $(a_1, a_2, \ldots a_n) \in A$ for $a_2 \in A$

If n = 1 then f is called unary operation

If n = 2 then f is called binary operation

If n = 3 then f is called ternary operation

Examples

1) A function $f : \Re \rightarrow \Re$ such that $f(x) = x$; $\forall x \in \Re$ then f is a unary operation.
2) A function $f : \Re \times \Re \rightarrow R$ such that $f(x, y) = x + y$ then f is a binary operation.
3) A function $f : \mathbf{Z} \times \mathbf{Z} \times \mathbf{Z} \rightarrow \mathbf{Z}$ such that $f(x, y, z) = x + y + z$. then f is a ternary operation.

Properties of binary operations

I) Commutative property

A binary operation '$*$' on A is said to be commutative if $a * b = b * a$, for all $a, b \in A$. e.g. $x + y = y + x$ and $x \times y = y \times x$ for all $x, y \in \Re$

$\therefore$ '+' and '×' are commutative binary operations on $\Re$

But '−' is not commutative on $\Re$

II) Associative property :

A binary operation '∗' on set A is said to be associative if

$a*(b*c) = (a*b)*c$; $\forall$ a, b, c $\in$ A

e.g. '+' and '×' are associative on the set of real numbers '−' is not associative on $\Re$

III) Idempotent

A binary operation '∗' on set A is said to be idempotent if $a*a = a$; for all $a \in A$

e.g.

1) 1 is idempotent element in $\Re$ w.r.t. binary operation 'X'.
2) 0 is idempotent element in $\Re$ w.r.t. '+'.
3) 1 is not idempotent element in $\Re$ w.r.t. '+'.

Q.2 For each of the following, determine whether ∗ is a binary operation.

i) R is the set of real numbers and $a * b$ = ab

ii) z^+ is the set of positive integers and $a * b$ = a/b.

iii) On z^+ where $a * b$ = a − b. iv) On R, where $a * b$ = min {a, b}

v) On R, where $a * b = a \times |b|$ vi) On z, where $a * b = a^b$.

Ans. : i) Yes, since f : $R^2 \to R$ defined as f(a, b) = ab is a function, with a b $\in$ R.

ii) No, since $(a, b) \in z^+ \times z^+$ does not imply that $a*b = a/b \in z$

$(1, 2) \in z^+ \times z^+$, but $1/2 \notin z^+$

iii) No, since $(1, 2) \in z^+ \times z^+$ but

iv) Yes, since, ∗ is a function with $\min\{a, b\} \in R$

v) Yes, since ∗ is a function with $a \times |b| \in R$

vi) No, since $2*(-1) = 2^{-1} = \frac{1}{2} \notin z$

Q.3 Determine whether or not following operations on the set of integers Z are associative. i) Division ii) Exponentiation

[SPPU : Dec.-05, Marks 3]

Ans. : i) Division on the set of integers is not associative as

$$(a/b)/c \neq a/(b/c)$$

ii) Exponentiation on the set of integers is not associative as

$$(a^b)^c \neq a^{(b^c)}$$

Q.4 Consider the binary operation * defined on the set A = {a, b, c, d} by the following table.

*	A	b	c	d
a	a	c	b	d
b	d	a	b	c
c	c	d	a	a
d	d	b	a	c

Find i) $c*d$ and $d*c$ ii) $b*d$ and $d*b$
iii) $a*(b*c)$ and $(a*b)*c$ iv) Is * commutative, associative ?

Ans. :

i) $c*d = a$ $\quad d*c = a$

ii) $b*d = c$ $\quad d*b = b$

iii) $b*c = b$ $\quad a*(b*c) = a*b = c$

$a*b = c$ $\quad$ Hence $(a*b)*c = c*c = a$

iv) * is not commutative, since $b*d \neq d*b$

* is also not associative, since $a*(b*c) \neq (a*b)*c$

Q.5 Define groupoid, semigroup, monoid with examples.

[SPPU : Dec.-06, 13]

Ans. : 1. Groupoid : A non empty set k with binary operation '*' is called groupoid if the binary operation '*' satisfies $\forall a, b \in G, a * b \in G$
In other words, every algebraic structure is groupoid.

e.g. $(\Re, +)$, $(\Re, -)$, $(Z, +)$, $(Z, \times)$ are groupoids.

2. Semigroup : A non empty set G with binary operation '*' is called a semigroup if it satisfies the following properties.

$$a*(b*c) = (a*b)*c \; ; \; \forall a, b, c \in G$$

i.e. '*' is associative in G

A semigroup is said to be commutative if * is commutative.

e.g. i) ($\Re$ +), (N, +), (Z, +) are commutative semigroups.

ii) ($\Re$ ×), (Z, ×) are commutative semigroups.

iii) (Z, –) is not a semigroup as '–' is not associative.

3. Monoid

Let G be a non empty set and * be a binary operation on G. (G, *) is called monoid if.

i) Associative property : $a * (b * c) = (a * b) * c$; $\forall a, b, c \in G$

ii) Existence of identity : $\exists$ an element $e \in G$ such that $e * a = a * e = a$; $\forall a \in G$

The element 'e' is called the identity element.

e.g. 1) ($\Re$ +) is a monoid as $a + b \in \Re$ $\forall a, b \in \Re$

i.e. ($\Re$ +) is closed

and $a + (b + c) = (a + b) + c$

and $0 + a = a + 0 = a$, $\forall a \in \Re$

0 is the identity element in $\Re$ w.r.t. '+'.

Example 1 : (C, +), (C, ×), (Z, +), (Z, ×), (Q, +) are monoids.

Example 2 : (N, ×) is a monoid but (N, +) is not monoid as $0 \notin N$

Q.6 Show that the algebraic system (A, +) is a monoid, where A is the set of integers and '+' is a binary operation giving addition of two integers. **[SPPU : Dec.-06, Marks 4]**

Ans. : Let A be the set of all integers and '+' defined on A.

i) Closure property : $a + b \in A$; $\forall a, b, \in A$ as $a + b$ is an integer.

ii) Associative property :

$a + (b + c) = (a + b) + c$; $\forall a, b, c \in A$

iii) Existence of identity element :

For any $a \in A$, $\exists$ $0 \in A$ such that

$a + 0 = 0 + a = a$

Therefore (A, +) is a monoid.

6.2 : Group and Abelian Group

Q.7 Define group and abelian group.

☞[SPPU : Dec.-08,09,10,12, May-14, Marks 4]

Ans. : Let G be a non empty set equipped with a binary operation '$*$'. $(G, *)$ is called a group if it satisfies the following postulates or axioms.

i) Associativity : $a*(b*c) = (a*b)*c$; $\forall$ a, b, c $\in$ G

ii) Existence of the identity : For any $a \in G$, $\exists$ $e \in G$

s.t. $\quad a*e = e*a = a$;

An element e is called the identity element in $(G, *)$

iii) Existence of the inverse :

For all $a \in G$, $\exists$ $b \in G$ such that

$$a*b = b*a = e$$

Then b is called the inverse of a in $(G, *)$

$\therefore \quad b = a^{-1}$

Abelian Group or Commutative Group :

A group $(G, *)$ is called an Abelian group if

$$a*b = b*a \quad ; \quad \forall\ a, b \in G$$

i.e. $*$ is commutative in $(G, *)$

Q.8 Explain properties of group with proof.

☞[SPPU : Dec.10, Marks 2]

Ans. : I) The identity element in a group is unique.

Proof : Suppose e_1 and e_2 are two identity elements in group G.

We have, $e_1 e_2 = e_1$; if e_2 is identity element in G.

and $\quad e_1 e_2 = e_2$; If e_1 is identity element in G.

$\Rightarrow \quad e_1 e_2 = e_1 = e_2$. Hence the identity element in group G is unique.

II) The inverse of each element in group G is unique.

☞[SPPU : Dec.-10, Marks 4]

Let a be any element of a group G and let e be identity element in group G.

Suppose b and c are two inverses of a in G.

$\therefore \quad ba = ab = e$ and $ac = ca = e$

We have, $b = be = b\,(ac)$

$b = (ba)\,c$

$b = ec$

$b = c$

Hence, the inverse of each element is unique.

III) The inverse of an inverse of the element is the original element. i.e. If the inverse of a is a^{-1} then $(a^{-1})^{-1} = a$.

Proof : Let $e \in G$ be the identity element of the group G.

Let $a \in G$.

We have, $a^{-1}a = e$

$[(a^{-1})^{-1}\,(a^{-1}]a = (a^{-1})^{-1}\,e$ ... (Multiplying by $(a^{-1})^{-1}$)

$[(a^{-1})^{-1}\,(a^{-1}]\,a = (a^{-1})^{-1}$ ... ($\because$ Associativity of e identity)

$e\,a = (a^{-1})^{-1}$

$\Rightarrow \quad a = (a^{-1})^{-1}$ Hence, the proof

IV) Prove that the inverse of the product of two elements of a group G is the product of the inverses taken in reverse order. i.e. $(ab)^{-1} = b^{-1}\,a^{-1}$, $\forall$ a ; b $\in$ G.

Proof : Let a^{-1} and b^{-1} be the inverses of a and b in a group G respectively. Let e be the identity in G. Then $a^{-1}\,a = a\,a^{-1} = e$ and $b^{-1}\,b = b^{-1}\,b = e$

Consider, $(ab)\,(b^{-1}\,a^{-1}) = a\,(bb^{-1})\,a^{-1}$ ($\because$ Associativity)

$= a\,(e)\,a^{-1}$ ($\because bb^{-1} = e$)

$= (ae)\,a^{-1} = aa^{-1} = e$ ... (1)

Similarly, $(b^{-1}a^{-1})\,(ab) = b^{-1}\,(a^{-1}\,a)\,b = b^{-1}\,(e)\,b$

$= b^{-1}\,b = e$ ... (2)

From equation (1) and equation (2), $(ab)\,(b^{-1}a^{-1}) = (b^{-1}\,a^{-1})\,(ab)$

$b^{-1}\,a^{-1}$ is the inverse of ab

$$(ab)^{-1} = b^{-1}\,a^{-1}$$

V) Prove that the cancellation laws hold in a group.i.e. If a, b, c $\in$ G then ab = ac $\Rightarrow$ b = c and ba = ca $\Rightarrow$ b = c.

Proof : Let a be any element in G and e be the identity element of a group G.

Now we have, ab = ac

Permultiplying by a^{-1} we get,

$$a^{-1}(ab) = a^{-1}(ac)$$

$$(a^{-1}a)\,b = (a^{-1}a)\,c \quad \ldots (\because \text{Associativity})$$

$$eb = ec \quad \ldots (\because aa^{-1} = e)$$

$$b = c$$

Similarly, $ba = ca$

$\Rightarrow$ $(ba)\,a^{-1} = (ca)\,a^{-1} \Rightarrow be = ce$

$\Rightarrow$ $b = c$

Hence, the proof

VI) If a, b are any elements of a group G then equation ax = b and ya = b have unique solutions in G.

Proof : Let $a \in G$

$\therefore \exists\, a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$

$a, a^{-1} \in G$ and $b \in G \Rightarrow a^{-1}b \in G$.

Now substituting $a^{-1}b$ for x in the L.H.S. given equation, we get,

$$ax = a\,(a^{-1}b) = (aa^{-1})\,b = eb = b$$

Thus, $x = a^{-1}\,b$ is the solution of $ax = b$

Let us suppose that x_1 and x_2 are two solutions of $ax = b$.

$\therefore$ $ax_1 = b$ and $ax_2 = b$

$\Rightarrow$ $b = ax_1 = ax_2 \Rightarrow x_1 = x_2$

Hence solution is unique.

Similarly prove for $ya = b$.

Q.9 If set Q_1 of all rational numbers other than 1 with $a * b = a + b - ab$. Show that (G, *) is a group. **[SPPU : Dec.-09, Marks 4]**

Ans. : We have, $a * b = a + b - ab,\ \forall\ a, b \in G$.

(i) Closure property : Let $a, b \in Q_1$, $a \neq 1$, $b \neq 1$ $\therefore ab \neq 1$.

$\therefore$ $a * b = a + b - ab \neq 1$ and $a * b \in Q_1$

Q_1 is closed w.r.t. *

(ii) Associativity : Let $a, b, c \in Q_1$.

$$(a * b) * c = (a + b - ab) * c = (a + b - ab + c - (a + b - ab)\,c$$

$$= a + b - ab + c - ac - bc - abc \quad \ldots (1)$$

$$a * (b * c) = a * (b + c - bc) = a + b + c - bc$$

$$= a\,(b + c - bc) = a + b + c - bc - ab - ac + abc \quad \ldots (2)$$

$\therefore$ From equation (1) and equation (2)

$$(a * b) * c = a * (b * c)$$

$\therefore$ * is associative

(iii) Existence of the identity : Let e be the identity element in Q_1

$$a * e = a$$

$\Rightarrow$ $a + e = ae = a$

$\Rightarrow$ $e - ae = 0$

$\Rightarrow$ $e(1 - a) = 0$ (But $a \neq 1$)

$\Rightarrow$ $e = 0$

$\therefore$ 0 is the identity element.

(iv) Existence of the inverse : Let $a \in Q_1$, $a \neq 1$.

Suppose $b \in Q_1$ is the inverse of a.

$\therefore$ $a * b = e$

$\Rightarrow$ $a + b - ab = 0$

$\Rightarrow$ $a + b(1 - a) = 0$

$b(1 - a) = -a$

$\Rightarrow$ $b = \dfrac{-a}{1-a} = \dfrac{a}{a-1} \neq 1$ and $b \in Q_1$.

$\therefore$ The inverse exist for all a in Q_1.

Thus, $(Q_1, *)$ is a group.

Q.10 If $S = \{(a, b) \mid a \neq 0, a, b, \in \Re\}$ and $(a, b) * (c, d) = (ac, bc + d)$ then show that G is a group but not abelian group w.r.t.*

Ans. : (i) Closure property : Let $(a, b)\ (c, d) \in S$; $a \neq 0$; $c \neq 0$

$\therefore$ $ac \neq 0$.

$(a, b) * (c, d) = (ac, bc + d) \in S$

(ii) Associativity : Let (a, b), (c, d) and $(e, f) \in S$

Consider

$[(a, b) * (c, d)] * (e, f) = [ac, bc + d) * (e, f) = ([ac]\ e, [bc + d]\ e + f)$

$= (ace, bce + de + f)$ ... (Q.10.1)

and$(a, b) * [(c, d) * (e, f)] = (a, b) * [ce, de + f] = [ace, b(ce) + de + f]$

$= (ace, bce + de + f)$ ... (Q.10.2)

From equation (Q.10.1) and equation (Q.10.2) $*$ is associative operation.

(iii) Existence of the identity : Let $(a, b) \in S$ and $(x, y) \in S$, $x \neq 0$

Consider, $(a, b) * (x, y) = (a, b)$

$$(ax, bx + y) = (a, b)$$

$\Rightarrow$ $ax = a$ and $bx + y = b$

$\Rightarrow$ $x = 1$ and $b + y = b$

$\Rightarrow$ $y = 0$

Similarly, $(x, y) * (a, b) = (a, b)$

(1, 0) is the identity element in S.

(iv) Existence of the inverse : Let (a, b) and $(c, d) \in S$.

$$(a, b) * (c, d) = (1, 0)$$

$$(ac, bc + d) = (1, 0)$$

$\Rightarrow$ $ac = 1$

and $bc + d = 0$

$\therefore$ $c = \frac{1}{a}$

and $d = -bc = -\frac{b}{a}$ and $c \neq 0$

Thus $\left(\frac{1}{a}, -\frac{b}{a}\right)$ is the inverse of (a, b) in S.

Thus (S, $*$) is a group.

Consider, $(a, b) * (c, d) = (ac, bc + d)$

and $(c, d) * (a, b) = (ca, da + b)$

$\Rightarrow$ $(a, b) * (c, d) \neq (c, d) * (a, b)$

Thus, (S, $*$) is not an abelian group.

Q.11 If G = $\left\{\begin{bmatrix} x & x \\ x & x \end{bmatrix} \middle/ x \text{ is non zero real number}\right\}$. **Show that G is an abelian group w.r.t. matrix multiplication.**

Ans. : (i) Closure property :

Let $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix}$ and $B = \begin{bmatrix} y & y \\ y & y \end{bmatrix} \in G$

then $$AB = \begin{bmatrix} x & x \\ x & x \end{bmatrix}\begin{bmatrix} y & y \\ y & y \end{bmatrix} = \begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix} \in G$$

(ii) **Associativity :** We know that any matrix multiplication is associative.

(iii) **Existence of the identity :** Let $E = \begin{bmatrix} e & e \\ e & e \end{bmatrix}$ such that

$AE = A \Rightarrow$
$$\begin{bmatrix} x & x \\ x & x \end{bmatrix}\begin{bmatrix} e & e \\ e & e \end{bmatrix} = \begin{bmatrix} e & e \\ e & e \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2xe & 2xe \\ 2xe & 2xe \end{bmatrix} = \begin{bmatrix} e & e \\ e & e \end{bmatrix}$$

$$\Rightarrow \quad 2xe = e$$

$$\Rightarrow \quad 2x = 1$$

$$\Rightarrow \quad x = \frac{1}{2}$$

$$\therefore \quad E = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$ is the identity element of G.

(iv) **Existence of the inverse :** Let $A = \begin{bmatrix} x & x \\ x & x \end{bmatrix} \in G$ and

$B = \begin{bmatrix} y & y \\ y & y \end{bmatrix} \in G$ such that $AB = E$

$$\Rightarrow \begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$\Rightarrow \quad 2xy = \frac{1}{2}$$

$$\Rightarrow \quad xy = \frac{1}{4}$$

$$\Rightarrow \qquad y = \frac{1}{4x}$$

$$A^{-1} = \begin{bmatrix} \frac{1}{4x} & \frac{1}{4x} \\ \frac{1}{4x} & \frac{1}{4x} \end{bmatrix}$$

(v) Commutative property :

We have, $AB = BA = \begin{bmatrix} 2xy & 2xy \\ 2xy & 2xy \end{bmatrix} \in G$

Thus G is an abelian group w.r.t. matrix multiplication.

Q.12 Let G be the set of all non-zero real numbers and let $a * b = \frac{ab}{2}$. Show that (G, *) is an abelian group.

[SPPU : Dec.-08, Marks]

Ans. : (i) Closure property : Let $a, b \in G$.

$$a * b = \frac{ab}{2} \in G \text{ as } ab \neq 0$$

(ii) Associativity : Let $a, b, c \in G$

Consider $$a * (b * c) = a * \left(\frac{bc}{2}\right) = \frac{a(bc)}{4} = \frac{abc}{4}$$

$$(a * b) * c = \left(\frac{ab}{2}\right) * c = \frac{(ab)c}{4} = \frac{abc}{4}$$

$\Rightarrow$ * is associative in G.

(iii) Existence of the identity : Let $a \in G$ and $\exists$ e such that

$$a * e = \frac{ae}{2} = a$$

$$\Rightarrow \qquad ae = 2a$$

$$\Rightarrow \qquad e = 2$$

$\therefore$ 2 is the identity element in G.

(iv) Existence of the inverse : Let $a \in G$ and $b \in G$ such that $a * b = e = 2$.

$$\Rightarrow \qquad \frac{ab}{2} = 2$$

$$\Rightarrow \qquad ab = 4$$

$$b = \frac{4}{a}$$

$\Rightarrow$

$\therefore$ The inverse of a is $\frac{4}{a}$, $\forall\, a \in G$.

(v) Commutativity : Let a, b $\in$ G

$$a * b = \frac{ab}{2}$$

and $$b * a = \frac{ba}{2} = \frac{ab}{2}$$

$\Rightarrow$ * is commutative

Thus, (G, *) is an abelian group.

Q.13 Show that the set G = {1, w, w^2} where w is the cube root of unity is a group with respect to multiplication.

☞[SPPU : Dec.-12, Marks 6]

Ans. : Consider the following composition table of G.

	1	w	w^2
1	1	w	w^2
w	w	w^2	1
w^2	w^2	1	w

(i) Closure property : From table all elements belong to G.

$\therefore$ G is closed w.r.t. multiplication.

(ii) Associativity : As all elements of G are complex numbers and multiplication of complex numbers is associative.

$\therefore$ (G, x) is associative.

(iii) Existence of the identity : From the composition table

$$1\,(1) = 1\ ,\ 1\,(w) = 1\ ,\ 1\,(w^2) = w^2$$

$\therefore$ 1 is multiplicative identity in G.

(iv) Existence of the inverse : From table, the inverses of 1, w, w^2 are 1, w^2, w respectively. Thus (G, x) is a group.

Q.14 Let (A, *) be a group. Show that (A, *) is abelian group iff $a^2 * b^2 = (a * b)^2$

☞[SPPU ; Dec.-12, Marks 4]

Ans. : Let (A, *) be an abelian group.

$$a * b = b * a$$

Consider, $$a^2 * b^2 = (a * a) * (b * b)$$
$$= a * (a * b) * b \quad \text{(* is associative)}$$
$$= a * (b * a) * b = (a * b) * (a * b)$$
$$a^2 * b^2 = (a * b)^2$$

Conversely suppose

$$a^2 * b^2 = (a * b)^2 \quad \text{(Q.14.1)}$$

$$\text{L.H.S.} = a^2 * b^2 = (a * a) * (b * b) = a * (a * b) * b$$

$$\text{R.H.S.} = (a * b)^2 = (a * b) * (a * b) = a * (b * a) * b$$

Equation (1) $\Rightarrow$

$$a * (a * b) * b = a * (b * a) * b$$

By cancellation laws,

$$a * b = b * a$$

$\Rightarrow$ (A, *) is an abelian group.

Q.15 Show that the set of all idempotents in a commutative monoid S is a submonoid of S.

☞[SPPU : Dec.-12, Marks 4]

Ans. : We know that an element $x \in S$ is called an idempotent if $x * x = x$. Let T be the set of all idempotents in S.

i) For any $x, y \in T$, Consider

$$(x*y)*(x*y) = ((x*y)*x)*y = (y*(x*x))*y = (y*x)*y$$
$$= (x*y)*y = x*(y*y) = x*y$$

$\therefore$ $x*y \in T$ for all $x, y \in T$

T is closed w.r.t. '$*$'

ii) Associativity, Let $x, y, z \in T \subseteq S$

$\therefore$ $x*(y*z) = (x*y)*z$ in T

$*$ is associative in T

iii) Existence of the identity :

Let e be the identity element in S

As $e*e = e \quad \therefore \; e \in T$

$\therefore$ e is the identity element in T.

$\therefore$ T is a monoid

Thus T is a submonoid of S.

Q.16 Prove that the set $\mathbb{Z}$ of all integers with binary operation $*$ defined by $a * b = a + b + 1$ such that $\forall\, a, b \in \mathbb{Z}$ is an abelian group : **[SPPU : May-14, Marks 4]**

Ans. : We have $a * b = a + b + 1 \;,\; \forall\, a, b \in \mathbb{Z}$

i) Closure property

For $a, b \in \mathbb{Z} \Rightarrow a+b+1 \in \mathbb{Z} \Rightarrow a * b \in \mathbb{Z}$

$\therefore$ $\mathbb{Z}$ is closed w.r.t. $*$

ii) Associative : Let a, b, c $\in \mathbb{Z}$

$$(a * b) * c = (a + b + 1) * c$$

$$= a + b + 1 + c + 1 = a + b + c + 2 \qquad \text{... (Q.16.1)}$$

and
$$a * (b * c) = a * (b + c + 1)$$

$$= a + b + c + 1 + 1 = a + b + c + 2 \qquad \text{... (Q.16.2)}$$

From equation (Q.16.1) and equation (Q.16.2) $(a * b) * c = a * (b * c)$

$\therefore$ $*$ is associative in $\mathbb{Z}$

iii) Existence of the identity :

Let e be the identity in $\mathbb{Z}$

$\therefore$ For any $a \in \mathbb{Z}$, $a * e = e * a = a$

$$\Rightarrow a + e + 1 = a$$

$$\Rightarrow e + 1 = a$$

$\Rightarrow$ $e = -1$ is the identity element in $\mathbb{Z}$

iv) Existence of the inverse :

Let $a \in \mathbb{Z}$. Suppose $b \in \mathbb{Z}$ is the inverse of a in $\mathbb{Z}$

$\therefore$
$$a * b = e$$

$$a + b + 1 = -1$$

$$a + b = -2$$

$$b = -a - 2 \in \mathbb{Z}$$

$\therefore$ The inverse exists for all $a \in \mathbb{Z}$

Thus $(\mathbb{Z}, *)$ is a group.

Now consider $a * b = a + b + 1 \quad (\because a + b = b + a)$

$$= b + a + 1 = b * a \;, \quad \forall\, a, b \in \mathbf{Z}$$

$\therefore$ $*$ is commutative in $\mathbf{Z}$

$\therefore$ $(\mathbf{Z}, *)$ is an abelian group.

Q.17 Define modulo m. [SPPU : Dec.-11, May-08]

Ans. : I) Let and b are any integers and m is a fixed positive integer then the addition modulo m denoted by $a +_m b$ and defined as $a +_m b = r$; $0 \le r \le m$

Where r is the least non negative remainder when a + b is divided by m.

e.g. $5 +_3 9 = 2$ as $5 + 9 = 14$ and $14 = 3 \times 4 + 2$

$15 +_5 25 = 0$ as $15 + 25 = 40$ and $40 = 5 \times 8 + 0$

II) Let a and b be any integers and m is a fixed positive integer. Then the multiplication modulo m is denoted by $a \times_m b$ and defined as

$$a \times_m b = r \;;\; 0 \le r \le m$$

Where r is the least non negative remainder when $a \times b$ is divided by m.

e.g. $3 \times_4 5 = 3$ as $3 \times 5 = 15$ and $15 = 4 \times 3 + 3$

$9 \times_6 4 = 0$ as $9 \times 4 = 36$ and $36 = 6 \times 6 + 0$

III) If a and be are any two integers such that a – b is divisible by a fixed positive integer m, is called "a congruent to b modulo m".

It is denoted by $a \equiv b \pmod{m}$

Q.18 Show that $(z_6, +)$ is an abelian group. [SPPU : May-08]

Ans. : Let z_6 be the set of residue classes modulo6.

$\therefore \quad z_6 = \{[0], [1], [2], [3], [4], [5]\}$

Or $\quad z_6 = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}\}$

Consider the following table

+	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{0}$	$\boxed{\bar{0}}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{1}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\boxed{\bar{0}}$
$\bar{2}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\boxed{\bar{0}}$	$\bar{1}$
$\bar{3}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\boxed{\bar{0}}$	$\bar{1}$	$\bar{2}$
$\bar{4}$	$\bar{4}$	$\bar{5}$	$\boxed{\bar{0}}$	$\bar{1}$	$\bar{2}$	$\bar{1}$
$\bar{5}$	$\bar{5}$	$\boxed{\bar{0}}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$

I) Closure Property : Every element of a table belongs to z_6

i.e. $\bar{a}+\bar{b} \in z_6 \quad \forall \bar{a}, \bar{b} \in z_6$

$\therefore$ + is closed in z_6

II) Associativity : By table, for any $\bar{a}, \bar{b}, \bar{c} \in z_6$

$$\bar{a}+(\bar{b}+\bar{c}) = (\bar{a}+\bar{b})+\bar{c}$$

$\therefore$ '+' is associative in z_6

III) Existence of the Identity : By observing the first row and the first column,

For any $\bar{a} \in z_6, \exists\ \bar{0} \in z_6$ such that

$$\bar{a}+\bar{0} = \bar{0}+\bar{a} = \bar{a}$$

$\therefore$ $\bar{0}$ is the identity element in z_6.

IV) Existence of the Inverse

From table, identify the identity elements

As $\bar{0}+\bar{0} = \bar{0}$, $\bar{0}$ is the inverse of $\bar{0}$

As $\bar{1}+\bar{5} = \bar{5}+\bar{1} = \bar{0}$, $\bar{1}$ is the inverse of $\bar{5}$ and $\bar{5}$ is the inverse of $\bar{1}$

As $\bar{2}+\bar{4} = \bar{4}+\bar{2} = \bar{0}$, $\bar{2}$ is the inverse of $\bar{4}$ and $\bar{4}$ is the inverse of $\bar{2}$

As $\bar{3}+\bar{3} = \bar{0}$, $\bar{3}$ is the inverse of $\bar{3}$

$\therefore$ Every element of z_6 has inverse in z_6.

v) Commutative Property

From table, For all $\bar{a}, \bar{b} \in z_6$

$$\bar{a}+\bar{b} = \bar{b}+\bar{a}$$

$\therefore$ '+' is commutative in z_6.

Hence $(z_6, +)$ is an abelian group.

Q.19 Let z = {0, 1, 2, 3, 4, ... (n – 1)} and '◊' be a binary operation such that $a \lozenge b$ = remainder of abelian when divided by n. Construct a table for n = 4,

Is $(z_4, \lozenge)$ is groupoid, monoid, semigroup and abelian group ?

[SPPU : Dec.-11]

Ans. : We have $z_4 = \{0, 1, 2, 3\}$

Table of z_4 is

$\Diamond$	0	1	2	3
0	$\boxed{0}$	0	0	0
1	0	1	2	3
2	0	2	$\boxed{0}$	2
3	0	3	2	1

i) From table, for any a, b $\in z_4$, $a \Diamond b \in z_4$

$\therefore$ $(z_4, \Diamond)$ is a groupoid and $\Diamond$ is a binary operation in z_4 i.e. $\Diamond$ is closed in z_4.

ii) Semigroup : By table for any a, b, c, $\in z_4$

$$a \Diamond (b \Diamond c) = (a \Diamond b) \Diamond c$$

$\therefore$ $\Diamond$ is associative in z_4

OR Let $a \Diamond b = r$ and $b \Diamond c = t$

$\therefore$ $ab = 4p + r$ and $bc = 4l + t$

$\therefore$ $(a \Diamond b) \Diamond c = r \Diamond c = s$ where $rc = 4q + s$

$a \Diamond (b \Diamond c) = a \Diamond t = k$ where $at = 4m + k$

Prove that $s = k$

$$a(bc) = 4al + at = 4al + 4m + k \quad \text{... (Q.19.1)}$$

$$(ab)c = (4p + r)c = 4pc + rc = 4pc + 4q + s \quad \text{... (Q.19.2)}$$

Equations (Q.19.1) and equation (Q.19.2) are equal

$$\Rightarrow 4al + 4m + k = 4pc + 4q + s$$

$$\Rightarrow k = s$$

Hence $(a \Diamond b) \Diamond c = a \Diamond (b \Diamond c)$

Thus $(z_4, \Diamond)$ is a semigroup

iii) Monoid

By observing the first row and the first column of table

For any $a \in z_4$, $\exists$ $0 \in z_4$ such that

$$a + 0 = 0 + a = a$$

$\therefore$ 0 is the identity element in z_4

Thus from (i), (ii) and (iii), $(z_4, +)$ is a monoid.

iv) Existence of the inverse

From table, identify the identity elements.

x is the inverse of 3 in z_4

$\therefore \quad x \Diamond 3 = 0$ and $x \Diamond 3 = 4$ if $3x = 4m + r$

$\therefore \quad 3x - 4m = 0$

$\Rightarrow \quad 3x = 4m$

$\Rightarrow \quad x = \dfrac{4m}{3}$ which is an integer

For $m = 1, 2,$ x is not an integer

For $m = 3,$ $x = 4$ which is an integer

But $x = 4 \notin z_4$

Thus 3 does not have inverse in z_4

$\therefore (z_4, \Diamond)$ is not a group

Hence $(z_4, \Diamond)$ is not an abelian group.

Q.20 Define permutation group.

Ans. : I) Let $S = \{1, 2, 3, ..., n\}$ be a finite set with n distinct elements. If $f : S \rightarrow S$ is a bijective function then f is called a permutation of degree n.

Let $f(a_1) = b_1, \quad f(a_2) = b_2, \quad f(a_3) = b_3, \quad .... f(a_n) = b_4$

Then the permutation is denoted by

$$f = \begin{pmatrix} a_1 & a_2 & a_3 & .... & a_n \\ b_1 & b_2 & b_3 & .... & b_n \end{pmatrix} \begin{matrix} \rightarrow \text{Elements in Domain} \\ \rightarrow \text{Elements in Co-domain} \end{matrix}$$

II) The permutation corresponding to the identity function is called the identity permutation

$$I = \begin{pmatrix} a_1 & a_2 & a_3 & .... & a_n \\ b_1 & b_2 & b_3 & .... & b_n \end{pmatrix}$$

III) If $f_1 = \begin{pmatrix} a_1 & a_2 & a_3 & .... & a_n \\ b_1 & b_2 & b_3 & .... & b_n \end{pmatrix}$ and $f_2 = \begin{pmatrix} b_1 & b_2 & b_3 & .... & b_n \\ c_1 & c_2 & c_3 & .... & c_n \end{pmatrix}$

Then the product of two permutations is given by

$$f_1 f_2 = \begin{pmatrix} a_1 & a_2 & a_3 & .... & a_n \\ c_1 & c_2 & c_3 & .... & c_n \end{pmatrix}$$

(To write product use $a_1 \to b_1 \to c_1 \Rightarrow a_1 \to c_1$ and so on)

6.3 : Subgroup

Q.21 Define complexes and subgroups.

Ans. : 1) Complex of a group : Let $(G, *)$ be a group. Any non empty subset of a group G is called a complex of the group.

e.g. $H_1 = \{1, 2, 3, 4, 5\}$, $H_2 = \{1, 2, 3, ...\}$, $H_3 = \mathbb{Z}$

are complexes of a group $(\Re, +)$

2. Subgroup

Let $(G, *)$ be a group. A non empty subset H of a group G, is said to be subgroup of G if $(H, *)$ itself is a group.

Examples

1) $(\mathbb{Z}, +)$ is a subgroup of $(\Re\ +)$
2) $(\Re\ +)$ is a subgroup of $(\mathbb{C}, +)$
3) $(N, +)$ is not a subgroup of $(\mathbb{Z}, +)$
4) $(\Re_0, \cdot)$ is a subgroup of $(\mathbb{C}_0, \cdot)$
5) $(\{1\}, \cdot)$ is a subgroup of $(Q_0, \cdot)$, $(\Re_0, \cdot)$, $(\mathbb{C}, \cdot)$
6) $(\Re\ +)$ is a subgroup of $(\Re, +)$

Note :

1) $\{e\}$ and $\{G\}$ are subgroups of group G : These subgroups are called improper subgroups.
2) Subgroups which are not improper are called proper subgroups.

Q.22 Explain properties of subgroup with proof.

Ans. : Theorem 1) : Prove that the identity of a subgroup is the same as that of the group.

Let H be a subgroup of the group G.

Let e and e' be the identities of G and H respectively.

Now, $a \in H \Rightarrow e'\ a = a$ ... [$\because$ e' is identity of H]

also $a \in H \Rightarrow a \in G \Rightarrow ea = a$ [$\because$ e is identity of G]

$$e'a = ea$$

$\therefore$ in G we have,

$$e' = e \quad \text{... [by right cancellation law in G]}$$

$\Rightarrow$.

Theorem II) : Prove that the inverse of any element of a subgroup is the same as the inverse of the same regarded as an element of the group.

Let H be a subgroup of the group G.

Let e be the identity of G as well as of H.

Let $a \in H$, suppose b is the inverse of a in H and c is the inverse of a in G. Then we have,

$$ba = e$$

and

$$ca = e$$

$\therefore$ in G we have, $ba = ca \Rightarrow b = c$

Theorem III) : Prove that the order of any element of a subgroup is the same as the order of the element regarded as a member of the group.

Let H be a subgroup of the group G.

Let $a \in H$, But $a \in G$ and $a^n = e$ in G.

$a^n = e$ in H also. Hence the proof.

Theorem IV) : A necessary and sufficient condition for a non-empty subset H of a group G to be a subgroup is that $a \in H$, b H $\Rightarrow ab^{-1} \in H$ where b^{-1} is the inverse of b in G.

Suppose H is a subgroup of G.

Let $a \in H$, $b \in H$. Now each element of H must possess inverse because H itself is a group.

$$b \in H \Rightarrow b^{-1} \in H$$

Further H must be closed with respect to multiplication i.e. the composition in G.

$\therefore$ $a \in H, b^{-1} \in H \Rightarrow ab^{-1} \in H$

(i) The condition is sufficient. Now it is given that $a \in H$, $b \in H \Rightarrow ab^{-1} \in H$. We have to prove that H is a subgroup of G.

(ii) Existence of identity :

We have $\quad a \in H, a \in H \Rightarrow aa^{-1} \in H$ ... By given condition

$$\Rightarrow e \in H$$

Thus the identity e is an element of H.

(iii) Existence of inverse :

Let a be any element of H. Then by the given condition, we have,

$e \in H, a \in H \Rightarrow ea^{-1} \in H \Rightarrow a^{-1} \in H.$

Thus each element of H possesses inverse.

(iv) Closure property : Let a, b $\in$ H. Then as shown above

$b \in H \Rightarrow b^{-1} \in H$. Therefore applying the given condition we have,

$a \in H, b^{-1} \in H \Rightarrow a\,(b^{-1})^{-1} \in H \Rightarrow ab \in H.$

(v) Associativity : The elements of H are also the elements of G. The composition in G is associative. Therefore, it must also be associative in H.

Hence H itself is a group for the composition in G. Therefore, H is a subgroup of G.

Theorem V) : If H_1 and H_2 are two subgroups of a group G, then $H_1 \cap H_2$ is also a subgroup of G.

Let H_1 and H_2 be any two subgroups of G. Then $H_1 \cap H_2 \neq \phi$ Since at least the identity element e is common to both H_1 and H_2.

In order to prove that $H_1 \cap H_2$ is a subgroup it is sufficient to prove that

$a \in H_1 \cap H_2 , b \in H_1 \cap H_2 \Rightarrow ab^{-1}\ H_1 \cap H_2$

Now, $a \in H_1 \cap H_2 \Rightarrow a \in H_1$ and $a \in H_2$

$a \in H_1 \cap H_2 \Rightarrow b \in H_1$ and $b \in H_2$

But H_1, H_2 are subgroups.

$\therefore$ $a \in H_1, b \in H_1, \Rightarrow ab^{-1} \in H_1$

$a \in H_2, b \in H_2 \Rightarrow ab^{-1} \in H_2$

Finally $ab^{-1} \in H_1 , ab^{-1} \in H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$

Thus, we have shown that, $a \in H_1 \cap H_2 , b \in H_1 \cap H_2 \Rightarrow ab^{-1} \in H_1 \cap H_2$

Hence $H_1 \cap H_2$ is a subgroup of G.

Theorem VI) : Show that the union of two subgroups is a subgroup if and only if one is contained in the other.

Suppose H_1 and H_2 are two subgroups of a group G.

Let $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$

Then $H_1 \cup H_2 = H_2$ or H_1.

But H_1, H_2 are subgroups and therefore, $H_1 \cup H_2$ is also a subgroup. Conversly suppose $H_1 \cup H2$ is a subgroup. To prove that $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$.

Let us assume that H_1 is not a subset of H_2 and H_2 is also not a subset of H_1.

Now H_1 is not a subset of $H_2 \Rightarrow \exists\ a \in H_1$ and $a \notin H_2$ ... (Q.22.1)

and H_2 is not a subset of $H_1 \Rightarrow \exists\ b \in H_2$ and $B \notin H_1$ ... (Q.22.2)

From equation (Q.22.1) and (Q.22.2).

We have, $a \in H_1 \cup H_2$ and $b \in H_1 \cup H_2$

Since $H_1 \cup H_2$ is a subgroup, therefore $ab = c$ (say) is also an element of $H_1 \cup H_2$.

But $\quad ab = c \in H_1 \cup H_2 \Rightarrow ab = c \in H_1$ or H_2

Suppose $\quad ab = c \in H_1$

Then $\quad b = a^{-1} c \in H_1$,

... [$\because H_1$ is a subgroup. $\therefore a \in H_1 \Rightarrow a^{-1} \in H_1$]

But from (2), we have $b \in H_1$. Thus we get the contradiction.

Again suppose $ab = c \in H_2$

Then, $\quad a = cb^{-1} \in H_2$

... [$\because H_2$ is a subgroup, therefore $b \in H_2, \Rightarrow b^{-1} \in H_2$]

But from (1), we have $a \notin H_2$. Thus here also we get a contradiction.

Hence either $H_1 \subseteq H_2$ or $H_2 \subseteq H_1$

Q.23 Is union of two subgroups is a subgroup? If not, give example.

Ans. : The union of two subgroups is not necessarily a subgroup.

Let $\quad H_1 = \{...., -4, -2, 0, 2, 4, 6, ...\}$

$\quad H_2 = \{...., -6, -3, 0, 3, 6, ...\}$ are subgroup of $(Z, +)$

Now $\quad H_1 \cup H_2 = \{-6, -4, -3, -2, 0, 2, 3, 4, 6, ...\}$

$2, 3 \in H_1 \cup H_2$ but $2 + 3 = 5 \notin H_1 \cup H_2$

$\therefore$ + is not binary operation on $H_1 \cup H_2$

$\Rightarrow$ $H_1 \cup H_2$ is not a subgroup of G.

Q.24 Define cosets with example.

Ans. : Let $(G, *)$ be a group and H be any subgroup of G.

Let $a \in G$ be any element, then the set

$$H * a = \{h * a\} / \forall h \in H\}$$ is called a right coset of H in G.

and $$a * H = \{a * h / \forall h \in H\}$$ is called a left coset of H in G.

Note :

1) $H*a$ and $a*H$ are subsets of G.

2) If $(G, *)$ is an abelian group then $H*a = a*H$ in G.

e.g. 1) Let $(\mathbf{Z}, +)$ is a group and

$$H = \{...., -10, -5, 0, 5, 10, ...\}$$ is a subgroup of $G = \mathbf{Z}$

$\therefore$ For $1 \in \mathbf{Z}$, $H+1 = \{...., -9, -4, 1, 6, 11, ...\}$

$3 \in \mathbf{Z}$, $H+3 = \{...., -7, -2, 3, 8, 13, ...\}$

$5 \in \mathbf{Z}$, $H+5 = \{...., -5, 0, 5, 10, ...\} = H$

are right cosets of H in G.

Q.25 Order of an element of a group.

Ans. : Let $(G, \cdot)$ be a group. The smallest positive integer is called the order of an element $a \in G$ if

$$a^n = e(\text{identity element in G})$$

If is denoted by $o(a) = n$.

If no such positive number exists, then we say that a is of infinite order or zero order.

Note :

1) For the order of the group is the number of distinct elements in G.

2) The order of the identity element is 1 i.e. $o(e) = 1$

3) In a group G, $o(a) = o(a^{-1})$; $\forall a \in G$

4) In a group G, $o(a) \leq o(G)$

Q.26 Define cyclic group.

Ans. : A group G is called a cyclic group if $\exists$ at least one element $a \in G$ such that every element $x \in G$ can be written as $x = a^m$ where m is some integer.

The element a is called the generator of G and denoted by $G = \langle a \rangle$

Example 1 : Four fourth roots of unity form a cyclic group with respect to multiplication.

$$G = \{1, -1, i, -i\}$$

We have $(i)^1 = i, \quad (i)^2 = -1, \quad (i)^3 = -i, \quad (i)^4 = 1$

$\therefore$ i is the generator of G $\Rightarrow$ G is a cyclic group.

Moreover, $(-i)^1 = -i, \quad (-i)^2 = -1, (-i)^3 = i, \ (-i)^4 = 1$

$\therefore$ –i is also generator of G

$\therefore$ $$G = \langle i \rangle = \langle -i \rangle$$

Q.27 Define normal subgroups.

Ans. : A subgroup H of a group $(G, *)$ is said to be a normal subgroup of G if for all $g \in G$ and for all $h \in H$

$g * h * g^{-1} \in H$ (We may write $g\,h\,g^{-1} \in H$)

Every group G possesses at least two normal subgroups namely $\{e\}$ and G. These groups are called improper normal subgroups.

Simple Group : A group G is said to be simple group if it has only two normal subgroups, $\{e\}$ and G.

Notes :

1) Every subgroup of an abelian group is normal.
2) The intersection of normal subgroups is a normal subgroup.

Q.28 Define quotient groups. [SPPU : Dec.-14, 15, May-15]

Ans. : Let $(G, *)$ be a group and N be a normal subgroup of G. Let G/N be the collection of all cosets of N in G.

$$G/N = \{N * a / a \in G\}$$

$(G/N, *)$ is called the quotient group or factor group.

Q.29 Prove that $(G/N, *)$ is a group. [SPPU : Dec.-15, May-15]

Ans. : Let $(G, *)$ be a group and N is the normal subgroup at G

$\therefore$ $$G/N = \{N * a / \forall a \in G\}$$

Theorem :

1) Closure Property : Let $a, b \in G, \therefore N * a, N * b \in G/N$

$$(N * a) * (N * b) = N * (a * N) * b \qquad (\because N \text{ is normal})$$

$$= N * (N * a) * b = (N * N) * (a * b)$$

$$= N * c \qquad (\because N * N = N \text{ and } a * b = c \in G)$$

$\therefore$ $(N*a)*(N*b) \in G/N$

$\therefore$ G/N is closed w.r.t. $*$

2) Associativity : Let a, b, c $\in$ G

and $N*a$, $N*b$ and $N*c \in G/N$

$$\therefore (N*a)*[(N*b)*(N*c)] = N*a*[N*(b*c)]$$

$$= N*a*(b*c) \qquad \text{by (1)}$$

$$= N*(a*b)*c \qquad \text{by (1)} \quad (* \text{ is associative in } G)$$

$$= (N*(a*b))*(N*c) = [(N*a)*(N*b)]*(N*c)$$

$$= [(N*a)*(N*b)]*(N*c)$$

Thus $*$ is associative in G/N

3) Existence of the identity :

We have $N = N*e \in G/N$ and for any $N*a \in G/N$

$$(N*a)*(N*e) = N*(a*e) \qquad \text{(by (1))}$$

$$= N*a$$

$\therefore$ $N*e = N$ is the identity element in G/N.

4) Existence of the inverse : Let $a \in G$, $N*a \in G/N$

$\therefore \exists\ a^{-1} \in G$ and $N*a^{-1} \in G/N$ such that

$$(N*a)*(N*a^{-1}) = N*(a*a^{-1}) = N*e = N$$

Hence $N*a^{-1}$ is the inverse of $N*a$ in G/N

Thus $(G/N, *)$ is a group, known as quotient group.

Q.30 Prove lagranges theorem : The order of each subgroup of a finite group is a divisor of the order of the group.

Ans. : Let G be a group of finite order n. Let H be a subgroup of G and let O (H) = m. Suppose $h_1, h_2 \ldots h_m$ are the m members of H.

Let $a \in G$. Then Ha is a right coset of H in G and we have

$$Ha = \{h_1a, h_2a, \ldots\ldots, h_m a\}$$

Ha has m distinct members, since $h_i a = h_j a \Rightarrow h_i = h_j$.

Therefore each right coset of H in G has m distinct members. Any two distinct right cosets of H in G are disjoint i.e. they have no element in common. Since G is a finite group, the number of distinct right cosets of H in G will be finite, say equal to K. The union of these K distinct right cosets of H in G is equal to G.

Thus, if $Ha_1, Ha_2, \ldots\ldots Ha_k$ are the K distinct right cosets of H in G, then

$$G = Ha_1 \cup Ha_2 \cup \ldots\ldots\ldots\ldots \cup Ha_k$$

$\Rightarrow$ The number of elements in G = The number of elements in Ha_1 + The number of elements in Ha_2 + ... + The number of elements in Ha_k

... [$\because$ two distinct right cosets are mutually disjoint]

$$O(G) = Km \Rightarrow n = Km$$

$$\Rightarrow \quad K = \frac{n}{m} \Rightarrow m \text{ is a divisor of } n$$

$\Rightarrow$

$\Rightarrow$ O (H) is a divisor of O (G).

Q.31 Prove the order of every element of a finite group is a divisor of the order of the group.

Ans. : Suppose G is a finite group of order n.

Let $a \in G$ and $O(a) = m$. Prove that m is a divisor of n.

Let $H = \{ \ldots\ldots, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, \ldots\ldots\}$ be the subset of G consisting of all integral powers of a.

We know that H is a subgroup of G.

We have to show that H contains only m distinct elements and that they are a, a, $a^2, a^3, \ldots\ldots, a^m = e\ a^o$

Let $1 \leq r \leq m$, $1 \leq s \leq m$, and $r > s$.

then $a^r = a^s \Rightarrow a^r a^{-s} = a^s a^{-s} \Rightarrow a^{r-s} \Rightarrow a^0 \Rightarrow a^{r-s} = e$

Thus there exists a positive integer r – s less than m such that $a^{r-s} = e$. m is the least positive integer such that $a^m = e$.

$$\therefore \quad a^r \neq a^s$$

$\therefore$ a, $a^2, a^3, \ldots\ldots, a^m = a^0 = e$ are all distinct elements of H.

Now suppose at is any element of H where t is any integer.

By division algorithm,

We have, $\quad t = mp + q$ ... [p and q are some integers, $0 \leq q < m$]

We have, $\quad a^t = a^{mp+q} = a^{mp} a^q = \left(a^m\right)^p a^q = a^q \quad \ldots (0 \leq q < m)$

$\therefore$ a^q is one of the m element a, $a^2, \ldots\ldots, a^m = a^0$

Hence, H has only m distinct elements. Thus order of H is m.

By Lagrange's theorem m is a divisor of n.

Q.32 Prove that if G is a finite group of order n and $a \in G$, then $a^n = e$.

Ans. : In a finite group, the order of each element is finite.
Let O (a) = m. The subset H of G consisting of all integral powers of a is a subgroup of G and the order of H is m.
By Lagrange's theorem, m is a divisor of n.

Let $k = \frac{n}{m}$ then $n = mk$

Now, $a^n = a^{mk} = (a^m)^k = e^k$ ... $O(a) = m \Rightarrow am = e$

$= e$

6.4 : Cyclic Group

Q.33 Prove that every cyclic group is an abelian group.

[SPPU : Dec.-14, Marks 3]

Ans. : Let G = {a} be a cyclic group generated by a.
Let x, y be any two elements of G. Then there exist integers r and s such that $x = a^r$, $y = a^s$.

Now, $xy = a^r a^s = a^{r+s} = a^{s+r} = a^s a^r = yx$

Thus, we have, $xy = yx \ \forall \ x, y \in G$.

Therefore G is abelian.

Q.34 Prove that If a is a generator of a cyclic group G, then a^{-1} is also a generator of G.

Let G = {a} be a cyclic group generated by a.
Let a^r be any element of G, where r is some integer.

as $a^r = (a^{-1})^{-r}$

∴ − r is also some integer.

∴ Each element of G, is generated by a^{-1}

Thus, a^{-1} is also a generator of G.

Q.35 Prove that every group of order 3 is cyclic.

OR Prove that every group of prime order is cyclic.

Ans. : Suppose G is a finite group whose order is a prime number P, then to prove that G is a cyclic group. As an integer P is said to be a prime number if P ≠ 0, P ≠ ± 1, and if the only divisors of P are ± 1, ± P.

$\because$ G is a group of prime order, therefore G must contain at least 2 elements.

As 2 is the least positive prime integer.

There must exist an element $a \in G$ such that $a \neq t$, the identity element e.

Since a is not the identity element, therefore O (a) is definitely ≥ 2.

Let O (a) = m, If H is the cyclic subgroup of G generated by a then

$$O(H) = O(a) = m.$$

By Lagrange's theorem m must be divisor of P.

But P is prime and $m \geq 2$. Hence m = P

$\therefore$ H = G. Since H is cyclic. Therefore G is cyclic and a is a generator of G.

Q.36 Prove that every subgroup of a cyclic group is cyclic.

Ans. : Suppose G = {a} is a cyclic group generated by a. If H = G or {e}, then obviously H is cyclic. So let H be a proper subgroup of G. The elements of H are integral powers of a. If $a^s \in H$, then the inverse of a^s.

i.e. $a^{-s} \in H$.

$\therefore$ H contains elements which are positive as well as negative integral powers of a.

Let m be the least positive integer such that $a^m \in H$.

Then we shall prove that $H = \{a^m\}$

i.e. H is cyclic and is generated by a^m.

Let a^1 be any arbitrary element H.

By division algorithm,

there exist integers q and r such that $t = mq + r$, $0 \leq r < m$.

Now, $\quad a^m \in H \Rightarrow (a^m)^q \in H$ ... by closure property

$$\Rightarrow a^{mq} \in H \Rightarrow (a^{mq})^{-1} \in H \Rightarrow a^{-mq} \in H$$

Also, $a^t \in H \quad a^{-mq} \in H \Rightarrow a^t q^{-mq} \in H \Rightarrow a^{t-mq} \in H \Rightarrow a^r \in H$.

Now m is the least positive integer such that $a^m \in H$ and $0 \leq r < m$.

Therefore r must be equal to 0. Hence t = mq.

$$\therefore \quad a^t = a^{mq} = (a^m)^q$$

Thus every element $a^t \in H$ is of the form (a^m) q.

Therefore H is cyclic and a^m is a generator of H.

Q.37 Define the subgroup of a group. Let (G, 0) be a group. Let H = {a | a ∈ G and aob = boa for all b in G}. Show that H is normal subgroup of G.

Ans. : Let (G, x) be a group. A non empty subset H of a group G is said to be a subgroup of G if (H, *) itself is a group.

Given that, $H = \{a \mid a \in G \text{ and } a \circ b = b \circ a \,;\, \forall\, b \in G\}$

Let $a, b \in H \Rightarrow a \circ x = x \circ a$ and $b \circ x = x \circ b,\ \forall\, x \in G.$

$$\Rightarrow (b \circ x)^{-1} = (x \circ b)^{-1}$$

$$\Rightarrow x^{-1} \circ b^{-1} = b^{-1} \circ x^{1}$$

$$\Rightarrow b^{-1} \in H. \qquad \text{... (Q.37.1)}$$

Now, $(a \circ b^{-1}) \circ x = a \circ (b^{-1} \circ x)$ [∵ o is associative]

$= a \circ (x \circ b^{-1})$ [∵ use (1) or $b^{-1} \in H$]

$= (a \circ x) \circ b^{-1}$ [∵ $a \in H$]

$= (x \circ a) \circ b^{-1} = x \circ (a \circ b^{-1})$

$\Rightarrow \quad a \circ b^{-1} \in H$

Therefore H is a subgroup of group G.

Let $h \in H$ and $g \in G$ and any x in G.

Consider,

$(g \circ h \circ g^{-1}) \circ x = (g \circ g^{-1} \circ h) \circ x$ [∵ $h \in H$]

$= (e \circ h) \circ x = h \circ x = x \circ h$ (∵ $h \in H$)

$= x \circ (h \circ g \circ g^{-1}) = x \circ (g \circ h \circ g^{-1})$ (∵ $h \in H$)

$\Rightarrow g \circ h \circ g^{-1} \in H$ for any $g \in G$

∴ H is a normal subgroup of G.

Q.38 Show that the four permutations I, (a b), (c d), (a b) (c d) on four symbols a, b, c, d form a finite abelian group with respect to the multiplication.

Ans. : Let

$$f_1 = I = \begin{pmatrix} a & b & c & d \\ a & b & c & d \end{pmatrix}$$

$$f_2 = (a \ \ b) = \begin{pmatrix} a & b & c & d \\ b & a & c & d \end{pmatrix}$$

$$f_3 = (c \ \ d) = \begin{pmatrix} a & b & c & d \\ a & b & d & c \end{pmatrix}$$

$$f_4 = (a\ b)\ (c\ \ d) = \begin{pmatrix} a & b & c & d \\ b & a & d & c \end{pmatrix}$$

Let

$$G = \{f_1, f_2, f_3, f_4\}$$

Consider the multiplication table

X	f_1	f_2	f_3	f_4
f_1	f_1	f_2	f_3	f_4
f_2	f_2	f_1	f_4	f_3
f_3	f_3	f_4	f_1	f_2
f_4	f_4	f_3	f_2	f_1

(i) **Closure property :** All elements of table belong to G. (G, x) is closed w.r.t. x.

(ii) **Multiplication** of permutations is associative.

(iii) $f_1 = I$ is the identity element in G.

(iv) In G, $f_1^{-1} = f_1$, $f_2^{-1} = f_2$, $f_3^{-1} = f_3$ and $f_4^{-1} = f_4$

(v) Multiplication table is symmetric about main diagonal $\therefore$ (G, x) is commutative.

Hence (G, x) is an ablein group of order 4.

6.5 : Homomorphism and Isomorphism

Q.39 Define homomorphism of groups with properties.

[SPPU : Dec.-15, 14, 12, 11, May-15]

Ans. : Let $(G_1, *)$ and (G_2, o) be two groups. A function

$f : (G_1, *) \rightarrow (G_2, o)$ is said to be homomorphism.

If $f(a * b) = f(a)\ o\ f(b)$ for all $a, b \in G_1$.

i.e. $a * b$ in $G_1 \rightarrow f(a)\ o\ f(b)$ in G_2

A homomorphism from G to itself is called as endomorphism.

Properties of group homomorphism.

Let $f : G_1 \rightarrow G_2$ be group homomorphism and $((G_1, *)$ and $(G_2 * o)$ are groups then

i) $f(e_1) = (e_2)$

ii) $f(a^{-1}) = [f(a)]^{-1}$

Proof : 1) Let $a \in G_1$ and $f(a) \in G_1$ and e_2 is the identity element in G_2.

$\therefore \quad f(a) o e_2 = f(a) = f(a * e_1)$

$f(a) o e_2 = f(a) o f(e_1)$

$\Rightarrow f(e_1) = e_2$

2) Let $a \in G_1$ then $a^{-1} \in G$

and $\quad e_2 = f(e_1) = f(a * a^{-1})$ (f is homomorphism)

$e_2 = f(a) o f(a^{-1})$

$\Rightarrow \quad f(a^{-1}) = [f(a)]^{-1}$

Q.40 Define isomorphism of groups.

☞[SPPU : Dec.-13, 12, 11, 10, May-08, 10]

Ans. : Let $(G_1, *)$ and (G_2, o) be two groups. A function $f : (G_1, *) \rightarrow (G_2, o)$ is said to be isomorphism.

If i) f is a homomorphism from $G_1 \rightarrow G_2$, ii) f is bijective function.

If $f : G_1 \rightarrow G_2$ is an isomorphism of groups then G_1 and G_2 are called as isomorphic groups and denoted by $G_1 \equiv G_2$.

An isomorphism from G to itself is called as automorphism of group G.

Q.41 Let G be a group with identity e show that a function $f : G \rightarrow G$ defined by $f(a) \; \forall = e, a \in G$ is a homomorphism (Endomorphism).

☞[SPPU : May-08, 10, Marks 3]

Ans. : We have $f : G \rightarrow G$ and $f(a) = e, \; \forall \; a \in G$.

Let $a, b \in G \Rightarrow f(a), f(b) \in G$

$\therefore \quad f(a * b) = e$ (as $a * b \in G$)

$= e * e = f(a) * f(b)$

$\therefore$ f is a homomorphism.

Q.42 Explain homomorphism and antomorphism of groups with examples.

☞[SPPU : Dec.-11, 12, 13, Marks 3]

Ans. : Refer Q.39 and Q.40 for definition.

e.g. 1) The homomorphism $f : (\mathbf{Z}, +), (\mathbf{Z}, +)$ such that

$f(n) = -n$ is an antomorphism of group.

2) The homomorphism $f : (R_0, o) \to (R_0, o)$ such that

$f(x) = x; \ \forall x \in \Re_0$ is an antomorphism of groups.

Q.43 Let $\Re$ be the additive group of real numbers and $\Re+$ be the multiplicative group of positive real numbers. Prove that the mapping
$f : (R, +) \to (\Re_+, X)$ defined by $f(x) = e^x$, $\forall x \in \Re$ is an isomorphism of $\Re$ onto $\Re_+$

[SPPU : Dec.-10, Marks 4]

Ans. : If x is any real number, then e^x is always positive real number and e^x is unique. Therefore, $f : \Re \to R_+$ is a function such that $f(x) = e^x$.

Let $x_1, x_2 \in \Re$ then $f(x_1) = f(x_2)$

$\Rightarrow \quad e^{x1} = e^{x2} \Rightarrow x_1 = x_2$

$\therefore$ f is one to one mapping.

For any $y \in \Re_+$ then $\log y \in \Re$ such that

$f(\log y) = e^{\log y} = y$

$\therefore$ f is onto.

Now for any $x_1, x_2 \in \Re$

Consider, $\quad f(x_1 + x_2) = e^{x1+x2} = e^{x1 \times x2} = f(x_1) \times f(x_2)$

$\therefore$ f preserves compositions in $\Re$ and $\Re+$

$\therefore$ f is an isomorphism of $\Re$ onto $\Re+$.

Hence $\quad \Re \cong \Re_+$

6.6 : Rings

Q.44 Define rings, integral domains and fields.

[SPPU : Dec.-08, 10, 11, 12, 13, 14, May-14]

Ans. : 1. Rings

SPPU : Dec.-13, May-14

Let R be a non empty set equipped with two binary operations called addition and multiplication and denoted by '+' and '$\cdot$' respectively.

An algebraic structure (R,+, $\cdot$) is called a ring if it satisfies following axioms.

1) (R, +) is an abelian group i.e.

i) **Closure property :** for a, b $\in$ R, a + b $\in$ R

ii) **Associativity :** for a, b, c $\in$ R, $a + (b + c) = (a + b) + c$

iii) **Existence of the identity :** For any $a \in R$, $\exists$ $0 \in R$ s.t.

$a + 0 = 0 + a = a$.

$\therefore$ 0 is called as the additive identity element of ring.

iv) **Existence of the inverse :** for each $a \in R$, $\exists$ $-a \in R$

Such that $a + (-a) = -a + a = 0$

$-a$ is called the additive inverse of a

v) **Commutative property :** For a, b $\in$ R

$a + b = b + a$

2) (R, ·) is semigroup i.e.

i) **Closure property :** $\forall a, b \in R$, $a \cdot b \in R$

ii) **Associativity :** for a, b, c $\in$ R,

$a \cdot (b \cdot c) = (a \cdot b) \cdot c$

3) Multiplication distributes over addition $\forall a, b, c \in R$

i) $a \cdot (b + c) = a \cdot b + a \cdot c$ (Right distributive law)

ii) $(a + b) \cdot c = a \cdot c + b \cdot c$ (Left distributive law)

2. Commutative Ring : A ring (R, +, ·) is said to be commutative ring if $\forall$ $b \in R$, $a \cdot b = b \cdot a$

3. Ring with Unity : A ring (R, +, ·) is said to be ring with unity if $\forall$ $a \in R$, $\exists$, $1 \in R$ such that $a \cdot 1 = 1 \cdot a = a$.

Examples :

1) (Z, +, ·) is a commutative ring with unity.

2) (2Z, +, ·) is a commutative ring without unity where 2Z = set of even integers.

3) The set of $n \times n$ matrices over real numbers with respect to usual matrix addition and multiplication is a non commutative ring with unity.

4. Properties of a Ring : If (R, +, ·) is a ring with identity 0 and unit element 1 then following are true for all a, b, c $\in$ R.

i) $a \cdot 0 = 0 \cdot a = 0$

ii) $a \cdot (-b) = (-a) \cdot b = -(a \cdot b)$

iii) $(-a)\cdot(-b) = a\cdot b$

iv) Unit element is unique

5. Subring : Let (R, +, ·) be a ring. A non empty subset S of R is said to be subring of R if (S,+, ·) is a ring. e.g. ($\mathbb{Z}$,+, ·) is a subring of ($\Re$,+, ·).

6. Zero Divisors : Let (R, +, ·) be a commutative ring. An element $a \neq 0$ in R is said to be zero divisor if $\exists\ b \neq 0$ in R such that $a \cdot b = 0$

A ring (R, +, ·) is said to be without zero divisors

if $a \cdot b = 0 \Rightarrow a = 0$ or $b = 0$, $\forall a, b \in R$.

e.g. 1) $\bar{2}$ is a zero divisor in $(\mathbf{Z}_4, +, \cdot)$ as $\bar{2}\cdot\bar{2} = \bar{4} = 0$

2) $(M_{2\times2}(\Re), +, \cdot)$ is a ring with zero divisors.

as $$A\cdot B = \begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix} = 0 \text{ but } A \neq 0 \text{ and } B \neq 0$$

3) $(\mathbf{Z}, +, \cdot)$ is a ring without zero divisors i.e. $a\cdot b = 0 \Rightarrow a = 0$ or $b = 0$.

7. Integral Domain **[SPPU : Dec.-13, 11, 10, May-14]**

A commutative ring with zero divisors is called an integral domain.

e.g. 1) $(\Re, +, \cdot)$, $(\mathbf{Z}, +, \cdot)$, $(\mathbb{C}, +, \cdot)$ are integral domains.

2) $(\mathbf{Z}_4, +, \cdot)$ is a ring with zero divisors

$\therefore$ It is not integral domain.

8. Field **[SPPU : Dec.-13, 12, 11, 10, May-14]**

A commutative ring with unity in which every non zero element possesses their mulplicative inverse, is called as field.

A field is an integral domain.

e.g. 1) $(\Re, +, \cdot)$, $(\mathbb{C}, +, \cdot)$, $(Q, +, \cdot)$ are fields.

2) $(\mathbf{Z}, +, \cdot)$ is integral domain but not field.

9. Ring Homomorphism

Let $(R, +, *)$ and $(S, +', \xi *')$ be two rings.

A function $\phi : R \rightarrow S$ is called a ring homomorphism

If for any a, b $\in$ R

i) $$\phi(a+b) = \phi(a) + \phi(b)$$

ii) $\phi(a*b) = \phi(a)*' \phi(b)$

If ϕ is bijective then it is called as a ring isomorphism.

The kernal of ring homomorphism is defined as the set $\{a \in R | \phi(x) = 0'\}$.

It is denoted by ker (ϕ) or ker ϕ

Q.45 Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a, b \in z \right\}$ **f is the mapping that takes** $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ **to a – b.**

i) Show that f is a homomorphism. ii) Find Kernel of f.

[SPPU : Dec.-08]

Ans. : i) $f = \left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} + \begin{bmatrix} c & d \\ d & c \end{bmatrix} \right) = f\left(\begin{bmatrix} a+c & b+d \\ b+d & a+c \end{bmatrix} \right) = (a+c)-(b+d)$

$$= (a-b)+(c-d) = f\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} \right) + f\left(\begin{bmatrix} c & d \\ d & c \end{bmatrix} \right)$$

Also $f\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} \right) \cdot \left(\begin{bmatrix} c & d \\ d & c \end{bmatrix} \right) = f \begin{bmatrix} ac+bd & ad+bc \\ bc+ad & bd+ac \end{bmatrix} = (ac+bd)-(ad+bc)$

$$= (ac-bc)+(bd-ad) = (a-b)(c-d)$$

$$= f\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} \right) \cdot f\left(\begin{bmatrix} c & d \\ d & c \end{bmatrix} \right)$$

ii) $\text{Ker } f = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} : a-b=0 \right\}$ i.e. $\left\{ \begin{bmatrix} a & a \\ a & a \end{bmatrix} ; a \in z \right\}$

Q.46 Show that $S = \{a + b\sqrt{2}; a, b \in z\}$ **for the operations +, × is an integral domain but not a field.**

[SPPU : Dec.-14]

Ans. : We have,

$$(a+b\sqrt{2})+(c+d\sqrt{2}) = (a+c)+(b+d)\sqrt{2}$$

$$(a+b\sqrt{2})+(c+d\sqrt{2} = (ac+2bd)+(bc+ad)\sqrt{2}$$

Clearly S is commutative ring with unit element 1.

We have to prove S is an integral domain.

Let $\left(a+b\sqrt{2}\right)\left(c+d\sqrt{2}\right) = 0$

$$ac + 2\,bd = 0 \qquad \text{... (Q.46.1)}$$

$\therefore$

$$bc + ad = 0 \qquad \text{...(Q.46.2)}$$

and

Suppose $a = 0$; then $bd = bc = 0$

either $b = 0$ or both $d = c = 0$

Hence, if $a = 0$, $a+b\sqrt{2} = 0$

or $c+d\sqrt{2} = 0$

Assume $a \neq 0$. Multiplying equation (Q.46.1) by d

we have, $$acd + 2bd^2 = 0 \qquad \text{... (Q.46.3)}$$

From equation (Q.46.2)

$$ad = -bc$$

Hence substituting this value in equation (Q.46.3)

We have, $-bc^2 + 2bd^2 = 0$

$\Rightarrow b(2d^2 - c^2) = 0$

$\therefore b = 0$ or $c^2 = 2d^2$, i.e. $c = \sqrt{2}\,d$

Since c is an integer, $c^2 = 2d^2$ is true only if $c = d = 0$

Hence if $c^2 \neq 2d^2$, $b = 0$. But $b = 0$ implies $a = 0$

Hence, in any case either $a + b\sqrt{2} = 0$ or $c + d\sqrt{2} = 0$

Hence, S is an integral domain.

To show that S is not a field consider the element $2+\sqrt{2}$. It's multiplicative inverse does not exit in S, for $\left(2+\sqrt{2}\right)\left(c+d\sqrt{2}\right) = 1$

$$\Rightarrow \quad 2c + 2d = 1 \Rightarrow c + d = \frac{1}{2}$$

Absurd, since c, d $\in$ z.

Q.47 Let z_8 = {0, 1 ,2, 3, 4, 5, 6, 7 }. Let R is a relation under the operations addition modulo 7 and multiplication modulo 7. Does this system form a ring ? It is a commutative ring ? **[SPPU : Dec.-11]**

Ans. : Consider the following tables,

Table 1 :

$\times_7$	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	0
2	0	2	4	6	1	3	5	0
3	0	3	6	2	5	1	4	0
4	0	4	1	5	2	6	3	0
5	0	5	3	1	6	4	2	0
6	0	6	5	4	3	2	1	0
7	0	0	0	0	0	0	0	0

Table 2 :

$+_7$	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	0
1	1	2	3	4	5	6	0	1
2	2	3	4	5	6	0	1	2
3	3	4	5	6	0	1	2	3
4	4	5	6	0	1	2	3	4
5	5	6	0	1	2	3	4	5
6	6	0	1	2	3	4	5	6
7	0	1	2	3	4	5	6	0

I) S.T. $(z_8, +_7)$ is an abelian group.

i) From table 2, all element are in z_8 $\therefore$ z_8 is closed w.r.t. $+_7$

ii) Associativity : for all a, b $c \in z_8$.

$$a +_7 (b +_7 c) = (a +_7 b) +_7 c$$

iii) By observing the first row of table 2. 0 is the additive identity in z_8.

6.7 : Group Codes

Q.48 Explain group code.

Ans. : Let S_n be the set of all binary words of length n. Let $\oplus$ be a binary operation on S_n such that for all x, y $\in S_n$ where $x = (x_1, x_2 \ldots x_n)$, $y = (y_1, y_2, \ldots y_n)$,

$$x \oplus y = (x_1 \oplus y_1, x_2 \oplus y_2, \ldots x_n \oplus y_n)$$

Where the operation $\oplus$ denotes the addition modulo 2 on {0, 1} and is given by the following table.

$\oplus$	0	1
0	0	1
1	1	0

The algebraic structure $(S_n, \oplus)$ forms a group in which n tuple of 0's (0, 0, 0, ..., 0) is the identity and each element is its own inverse. In general, any code which is a group under the operation $\oplus$ is called a group code. Group code was first introduced by Hamming and it is very useful in binary encoding techniques.

e.g. if $x = (1\ 0\ 1\ 0\ 1)$ and $y = (0\ 1\ 1\ 0\ 0)$

then $x \oplus y = (1\ 1\ 0\ 0\ 1)$

Q.49 Define hamming distance. **[SPPU : Dec.-11, Dec.-15]**

Ans. : Let x be a word is S_n. The weight of x is denoted by w(x) and defined as

$$w(x) = \text{Number of one's in } x$$

e.g. $w(0\ 0\ 1\ 0\ 1) = 2$, $w(0\ 0\ 0) = 0$, $w = (1\ 1\ 1\ 1) = 4$

Let $x = (x_1, x_2, \ldots x_n)$, $y = (y_1, y_2, \ldots y_n)$ be any two elements in $(S_n, \oplus)$. The Hamming distance between x and y is denoted by d(x,y) and defined as

d (x, y) = The number of co-ordinates at which x_i and y_i are different.

e.g. If

$$x = (1\ 0\ 1\ 1\ 0\ 1)$$

$$y = (0\ 1\ 1\ 1\ 1\ 0)$$

$$\downarrow\ \downarrow \qquad \downarrow\ \downarrow$$

different

$\therefore$ d (x, y) = 4, as x and y have first second, fifth, sixth positions different)

Now $\quad x \oplus y = (1\ 1\ \ 0\ 0\ 1\ \ 1)$

$$w(x \oplus y) = 4 = d(x, y)$$

Thus for any $x, y \in S_n$, $d(x, y) = w(x \oplus y)$

Properties of Hamming Distance :

Let $x, y, z \in (S_n, \oplus)$ then

i) $d(x, y) \geq 0$

ii) $d(x, y) = 0$ iff $x = y$

iii) $d(x, y) = d(y, x)$

iv) $d(x, z) \leq d(x, y) + d(y, z)$

The minimum distance of a code is the minimum of all the distances between district pair of code words.

e.g. Let x (1 1 0 1 1 0) , y = (0 0 1 1 1 1) , z = (1 0 1 0 1 0)

$\therefore \quad d(x, y) = 4,\ d(y, z) = 3,\ d(x, z) = 3$

Out of 4, 3, 3, minimum is 3.

Therefore the minimum distance between the words x, y, z is 3

By using the weight and minimum distance, a combination of errors can be detected and corrected.

Q.50 Explain generation of codes by using parity checks.

Ans. : In 1950, Hamming developed the first complete error detecting and error correcting encoding procedure. This procedure has been frequently used in computer systems.

Hamming constructed the codes, called Hamming codes by introducing redundant digit called parity digits. In a message, that is n digits long, m digits ($m < n$) are used to represent the information part of the message and the remaining $k = n - m$ digits are used for the detection and correction of errors. These K digits are called parity checks.

Hamming single error detecting codes can be described as follows.

i) The actual message is contained in the first $(n-1)$ digits of a code word of length n and the last digit position is set to 0 or 1 so as to make the entire message contain an even numbers of 1's. Such an encoding procedure is called a even parity check.

) Odd parity check can be used by making the entire message containing an odd no. of 1's. e.g. In even parity check {00, 10, 11} becomes {000, 101, 110}. In odd parity check { 00, 01, 10, 11}

$\uparrow \quad \uparrow \quad \uparrow$

becomes {001, 010, 100, 111}

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$

The code words of length n in which information is contained m digits ($m < n$) and remaining $k = n - m$ digits are parity checks, can be generated by using $k \times n$ matrix H. This matrix is called parity check matrix, where elements are from set {0, 1} A single error correcting code of length n generated by H with k parity check is given by

$$2^k \geq n + 1 = (k + m) + 1$$

$$m \leq 2^k - k - 1$$

The number of code words generated by H is $2^m = 2^{n-k}$ and the code generated in this way called the Hamming code.

Theorem 1 : Let H be a parity check matrix which consists of k rows and n columns. Then the set of words $x = (x_1, x_2, x_3, \ldots x_n)$ which belongs to the following set.

$$C = \{ x \mid xH^t = 0 \pmod 2\}$$ is a group code under the

operation $\oplus$ (addition modulo 2) where H^t = Transpose of matrix H.

Theorem 2 : The parity check matrix H generates a code word of weight q iff $\exists$ a set of columns of H such that their k - tuple sum is zero.

For the parity check matrix H of order $k \times n$.

Where $\quad n$ = Length of code word

$\quad\quad\quad k$ = Number of parity check bits

The information digits will be $m = n - k$. The number of code words generated is $2^m = 2^{n-k}$. Consider the following steps for the generation of code words.

Step 1 : Find the system of equations from $x \cdot H^t = 0$.

Step 2 : Find the values of parity check bits interms of information digits from the equations obtained in step 1.

Step 3 : Give values 0 or 1 to information digits and calculate the values of parity check bits according to step 2. The binary words obtained in this way will be the required code words generated by H.

To find the number of errors defected and corrected first find the columns of H whose sum (addition modulo 2) is zero. The number of theses columns is equal to the minimum weight of the code and which is equal to the minimum distance of the code.

Q.51 Find the minimum distance of an encoding function $e : B^2 \to B^5$ given as .
e (0 0) = 00000, e (0 1) = 1 0 0 1 1, e (1 0) = 0 1 1 1 0,
e (1 1) = 1 1 1 1 1.

☞[SPPU : Dec.-06]

Ans. :

$$d\,[e\,(0\ 0),\ e\,(0\ 1)] = 3 \qquad d\,[e\,(0\ 0),\ e\,(1\ 1)] = 5$$
$$d\,[e\,(0\ 0),\ e\,(0\ 1)] = 3 \qquad d\,[e\,(0\ 0),\ e\,(1\ 0)] = 3$$
$$d\,[e\,(0\ 1),\ e\,(1\ 0)] = 4 \qquad d\,[e\,(0\ 1),\ e\,(1\ 1)] = 2$$
$$d\,[e\,(1\ 0),\ e\,(1\ 1)] = 2$$

The minimum along all distances is 2.

Thus the minimum distance of an encoding is 2.

Q.52 What is Hamming function (distance) ? Find the distance between the code words of C = { (0 0 0 0), (0 1 0 1), (1 0 1 1), (0 1 1 1)} Rewrite the message by adding even parity check bit and parity check bit.

☞[SPPU : Dec.-11, Marks 5]

Solution : Please refer Q.50 for the definition. We have

$$d\,[e\,(0\ 0\ 0\ 0),\ e\,(0\ 1\ 0\ 1)] = 2 \qquad d\,[e\,(0\ 0\ 0\ 0),\ e\,(1\ 0\ 1\ 1)] = 3$$
$$d\,[e\,(0\ 0\ 0\ 0),\ e\,(0\ 1\ 1\ 1)] = 3 \qquad d\,[e\,(0\ 1\ 0\ 1),\ e\,(1\ 0\ 1\ 1)] = 3$$
$$d\,[e\,(0\ 1\ 0\ 1),\ e\,(0\ 1\ 1\ 1)] = 1 \qquad d\,[e\,(1\ 0\ 1\ 1),\ e\,(0\ 1\ 1\ 1)] = 2$$

Even parity check bit of c is

{(0 0 0 0 0), (0 1 0 1 0), (1 0 1 1 1), (0 1 1 1 1)}

Odd parity check bit of c is

{(0 0 0 0 1), (0 1 0 1 1), (1 0 1 1 0), (0 1 1 1 0)}

Q.53 Given the parity check matrix $H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$ **find the minimum distance of code generated by H. How many errors can it detect and correct.**

☞[SPPU : Dec.-15, Marks 5]

Ans. : From the given matrix H, select minimum number of column whose sum is zero.

$\therefore$ Consider columns $h_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $h_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $h_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

The sum of these columns

$$h_1 \oplus h_2 \oplus h_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Hence the minimum weight of the code is 3 which is equal to its minimum distance.

We know that the code can detect all combinations of k or fever errors iff the minimum distance between any two code words is at least k +1.

Here minimum distance is $3 = k+1 \Rightarrow k = 2$.

$\therefore$ This code can detect 2 or less errors.

Also it can correct k errors if the minimum distance is 2k +1.

$$\therefore \quad 2k + 1 = 3 \Rightarrow k = 1$$

This code can correct only one error.

Therefore this is a single error correcting code.

Q.54 Find the number of code words generated by the parity check matrix H given by H = $\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$ **find all the code words generated.** **[SPPU : Dec.-12, Marks 5]**

Ans. : The parity check matrix is

$$H = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}_{3\times 6}$$

H is of order 3×6. Hence the length of the code word is 6 in which last three digits are parity check bits. The information digits are

$$M = n - k = 6 - 3 = 3$$

$\therefore$ The matrix H will generate. $2^m = 8$ code words.

They are the solution of $x \cdot H^t = 0$.

$$[x_1\ x_2\ x_3\ x_4\ x_5, x_6]\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = [0\ \ 0\ \ 0]$$

$$\Rightarrow \begin{aligned} x_1 + x_2 + \quad\quad + x_4 + \quad\quad\quad &= 0 \\ x_2 + x_3 + \quad\quad + x_5 + &= 0 \\ x_1 + \quad + x_3 + \quad\quad + x_6 &= 0 \end{aligned}$$

$$x_4 = x_1 + x_2$$

$$\Rightarrow x_5 = \quad x_2 + x_3$$

$$x_6 = x_1 + \quad + x_3$$

By giving different combinations of 0 and 1 to x_1, x_2, x_3, we get the following code words.

x_1	x_2	x_3	x_4	x_5	x_6
0	0	0	0	0	0
0	0	1	0	1	1
0	1	0	1	1	0
0	1	1	1	0	1
1	0	0	1	0	1
1	0	1	1	1	0
1	1	0	0	1	1
1	1	1	0	0	0

Hence the code is

C { (0 0 0 0 0) (0 0 1 0 1 1) (0 1 0 1 1 0)
(0 1 1 1 0 1) (1 0 0 1 0 1)
(1 0 1 1 1 0) (1 1 0 0 1 1) (1 1 1 0 0 0)}

END...

SOLVED MODEL QUESTION PAPER (In Sem)

Discrete Mathematics

S.E. (IT) Semester - III [As Per 2019 Pattern]

Time : 1 Hour] **[Maximum Marks : 30**

N.B. : ***i) Attempt Q.1 or Q.2 and Q.3 or Q.4.***

ii) Neat diagrams must be drawn wherever necessary.

iii) Figures to the right side indicate full marks.

iv) Assume suitable data, if necessary.

Q.1 a) *Draw Venn diagram and prove the expression. Also write the dual of each of the given statements.*

i) $(A \cup B \cup C)^C = (A \cup C)^C \cap (A \cup B)^C$

ii) $(U \cap A) \cup (B \cap A) = A$ **(Refer Q.6 of Chapter - 1)** **[5]**

b) *Prove by mathematical induction for* $n \geq 1$.

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + n\,(n+1) = \frac{n\,(n+1)\,(n+2)}{3}$$

(Refer Q.19 of Chapter - 1) **[5]**

c) *Let p denote the statement, "The material is interesting'. q denote the statement, "The exercises are challenging", and r denote the statement, "The course is enjoyable".*

Write the following statements in symbolic form :

i) The material is interesting and exercises are challenging.

ii) The material is interesting means the exercises are challenging and conversely.

iii) Either the material is interesting or the exercises are not challenging but not both.

iv) If the material is not interesting and exercises are not challenging, then the course is not enjoyable.

v) The material is uninteresting, the exercises are not challenging and the course is not enjoyable. **(Refer Q.33 of Chapter - 1)** **[5]**

OR

Q.2 a) *State and prove the principle of inclusion and exclusion sets.* **(Refer Q.11 of Chapter - 1)**

b) *Show that $n^4 - 4n^2$ is divisible by 3 for all $n \geq 2$.*
(Refer Q.25 of Chapter - 1)

c) *Prove by truth table $P \rightarrow (Q \vee R) \equiv (P \rightarrow Q) \vee (P \rightarrow R)$.*
(Refer Q.40 of Chapter - 1)

Q.3 a) *How many 4 digits numbers can be formed by using the digits 2, 4, 6, 8 when repetition of digits are allowed ?*
(Refer Q.12 of Chapter - 2) [5]

b) *A committee of 5 people is to be formed from a group of 4 men and 7 women. How many possible committees can be formed if at least 3 women are on the committee ?* **(Refer Q.25 of Chapter - 2)** [5]

c) *Three students A, B and C are swimming in the race A and B have same probability of winning and each each is twice as likely to win as C. Find the probability that :*
i) B wins ii) C wins iii) B or C wins **(Refer Q.60 of Chapter - 2)** [5]

OR

Q.4 a) *Determine the number of ways in which 5 software engineers and 6 electronics engineers can be sitted at a round table so that no two software engineers can sit together.* **(Refer Q.19 of Chapter - 2)** [5]

b) *A bag contains 3 red and 5 black balls and a 2^{nd} bag contains 6 red and 4 black balls. A ball is drawn from each bag. Find the probability that 1) one is red and other is black. 2) both are red 3) both are black.* **(Refer Q.48 of Chapter - 2)** [5]

c) *The contents of urns I, II, III are as follows respectively.*
I $\rightarrow$ 1 white, 2 black, 3 red balls
II $\rightarrow$ 2 white, 1 black, 1 red balls
III $\rightarrow$ 4 white, 5 black, 3 red balls
One urn is chosen at random and two balls are drawn. They happen to be white and red.
What is the probability that they come from urn I, II or III ?
(Refer Q.55 of Chapter - 2) [5]

END...

JUNE - 2022 [END SEM] [5869]-281 **Solved Paper**

Course 2019

Time : $2\frac{1}{2}$ Hours] [Max. Marks : 70

Instructions to the candidates :

1) *Answer Q.1 or Q.2, Q.3 or Q.4, Q.5 or Q.6, Q.7 or Q.8.*

2) *Figures to the right indicate full marks.*

3) *Draw neat diagrams wherever necessary.*

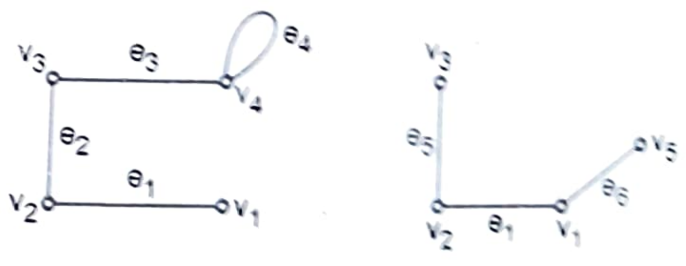
4) *Use of scientific calculators is allowed.*

5) *Assume suitable data if necessary.*

Q.1 a) *What are various operations on graph ? Explain it in detail ?* [4]

Ans. : **A) Intersection of two graphs :** The intersection of two graphs G_1 (V_1, E_1) and G_2 (V_2, E_2) is a graph G (V, E) whose vertex set is $V = V_1 \cap V_2$ and edge set is $E = E_1 \cap E_2$. The intersection of G_1 and G_2 is denoted by $G_1 \cap G_2$.

e.g.



G_1 G_2

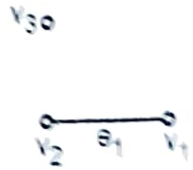
Fig. 1

$V_1 = \{v_1, v_2, v_3, v_4\}$ $V_2 = \{v_1, v_2, v_3, v_5\}$

$E_1 = \{e_1, e_2, e_3, e_4\}$ $E_2 = \{e_1, e_5, e_6\}$

Therefore $G = G_1 \cap G_2$ (v, E) where

$V = V_1 \cap V_2 = \{v_1, v_2, v_3\}$, $E = E_1 \cap E_2 = \{e_1\}$



$G_1 \cap G_2$

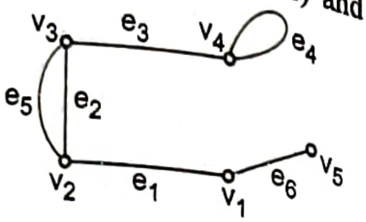
Fig. 2

B) Union of two graphs : Let $G_1(V_1, E_1)$, $G_2(V_2, E_2)$ be two graphs. The union of G_1 and G_2 is denoted by $G_1 \cup G_2 = G$ (v, E) and it is a graph whose vertex set is

$V = V_1 \cup V_2$ and Edge set is $E = E_1 \cup E_2$

Consider the graphs G_1 and G_2 as shown in above example :

The union of G_1 and G_2 is given by G (v, E)



$G_1 \cup G_2$

Fig. 3

where

$$V = V_1 \cup V_2 = \{v_1, v_2, v_3, v_4, v_5\}$$

$$V = V_1 \cup V_2 = \{e_1, e_2, e_3, e_4, e_5, e_6\}$$

Note : Both graphs G_1 and G_2 are subgraphs of $G_1 \cup G_2$.

C) The ring sum of two graphs : The ring sum of two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is denoted by $G = G_1 \oplus G_2$ (V, E) whose vertex set is $V = V_1 \cup V_2$ and the edge set consists of those edges which are either in E_1 or in E_2 but not in both i.e.

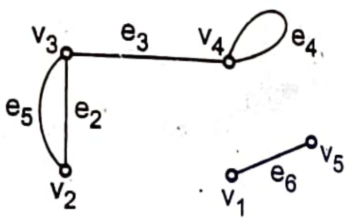


Fig. 4

$E = (E_1 \cup E_2) - (E_1 \cap E_2)$.

The ring sum of above graphs G_1 and G_2 is given by $G(V, E) = G_1 \oplus G_2$

$$V = \{v_1, v_2, v_3, v_4, v_5\} = V_1 \cup V_2$$

$$E = (E_1 \cup E_2) - (E_1 \cap E_2) = \{e_2, e_3, e_4, e_5, e_6\}$$

D) Sum of two graphs : The sum of two vertex disjoint graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ is denoted by $G_1 + G_2 = G(V, E)$ is defined as the graph whose vertex set is $V(G_1 \cup G_2)$ and consisting of edges which are in G_1 or G_2 together with the edges obtained by joining each vertex of G_1 to each vertex of G_2. Thus $G_1 + G_2$ is nothing but the graph $G_1 \cup G_2$ in which each vertex of G_1 is joined to each vertex of G_2 by an edge.

e.g. If

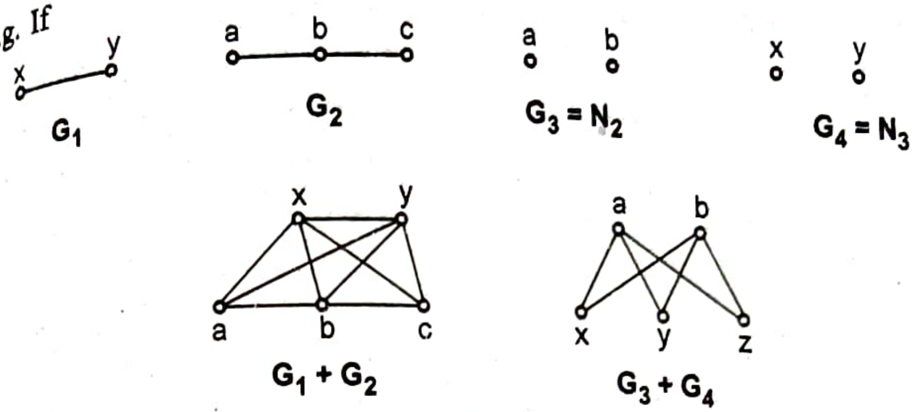


Fig. 5

Note : The sum $N_m + N_n$ of null graphs is nothing but the complete bipartite graph $K_{m, n}$.

E) Product of two graphs : Let $G_1 (V_1, E_1)$ and $G_2 (V_2, E_2)$ be two vertex disjoint graphs then the product of G_1 and G_2 is denoted by $G_1 \times G_2 = G (V, E)$ is a graph whose vertex set is $V = V_1 \times V_2$ and two edges (x_1, x_2) and (y_1, y_2) are adjacent if

$x_1 = y_1$ and x_2 is adjacent to y_2 in G_2 or $x_2 = y_2$ and x_1 is adjacent to y_1 in G_1.

e.g. If

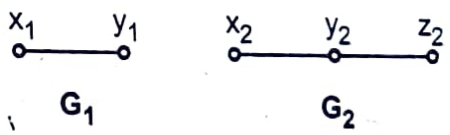


Fig. 6

Then $G_1 \times G_2$ is given below :

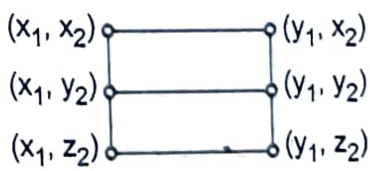


Fig. 7

F) Decomposition : A graph G is said to have been decomposed into two subgraphs H and K if $H \cup K = G$ and $H \cap K$ = Null graph i.e. each edge of G occurs either in H or in K but not in both. But vertices may occur in both. In this context isolated vertices are not considered.

e.g. The decomposition of G into H and K is given below :

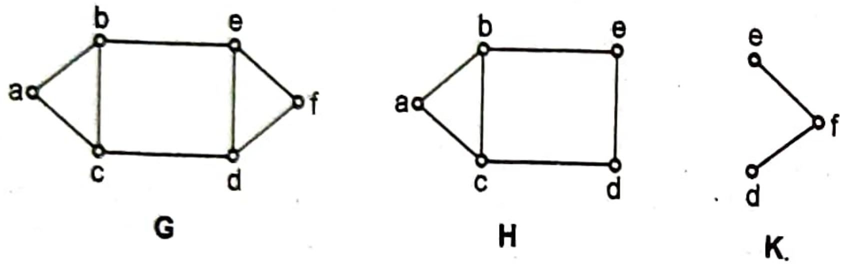
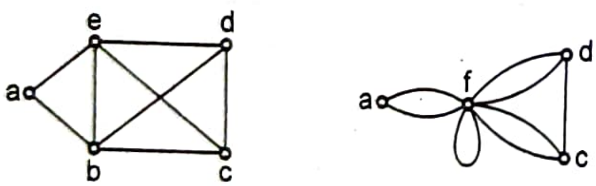


Fig. 8

G) Fusion of vertices : A pair of vertices a and b in a graph G are said to be fused if a and b are replaced by a single new vertex say c such that every edge that was incident on either a or b or both is incident on the new vertex c. The fusion of two vertices do not change the number of edges but reduced number of vertices by 1.

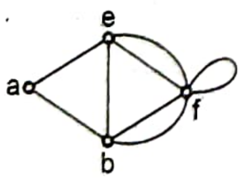
e.g.



G

Graph after fusion of b and e

Fig. 9

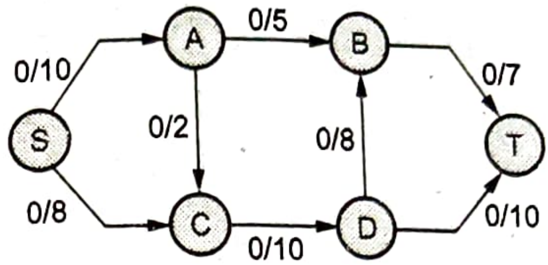


Graph after fusion of c and d

Fig. 10

b) *Find the maximum flow in the given network.* **[8]**

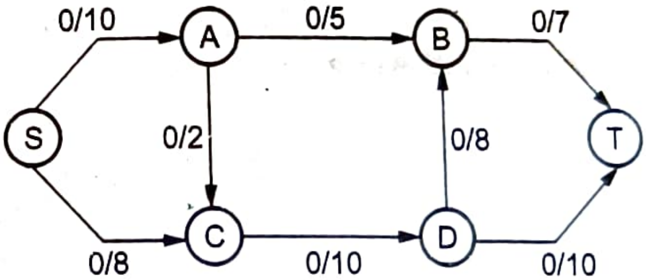
Network (G)



Flow = 0

Fig. 11

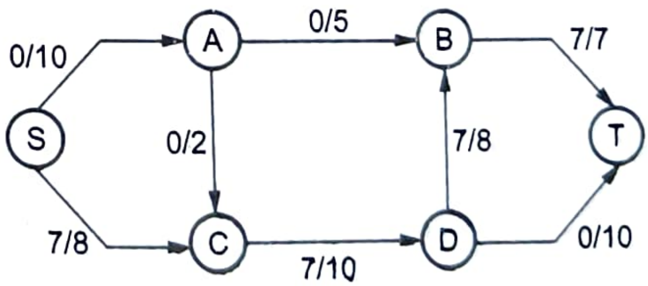
Ans. :



Flow = 0
Network (G)

Fig. 12

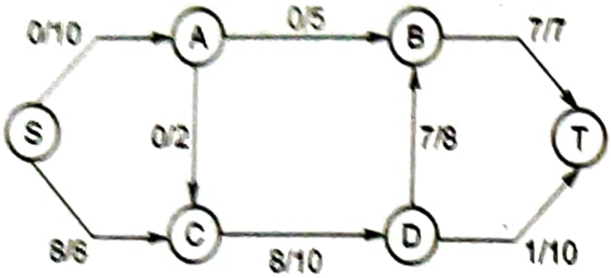
Step 1 : Select an arbitrary path S-C-D-B-T from the source vertex S to sink vertex T. This path can carry a flow of 7 units as the arc BT can carry a maximum of 7 units.



Flow = 0 + 7

Fig. 13

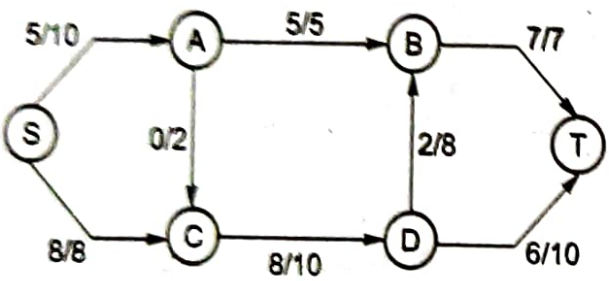
Step 2 : Now, select the path S-C-D-T. This path can carry a flow of 01 unit as the arc D.T will get saturated in a flow of 1 unit.



Flow = 0 + 1

Fig. 14

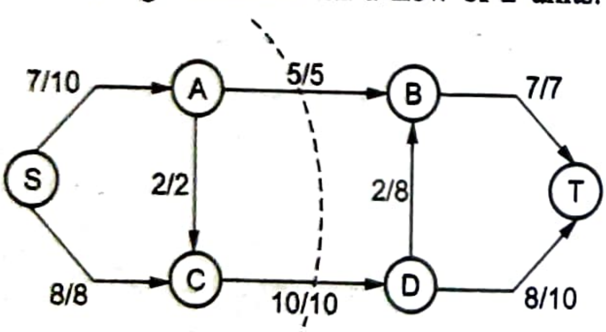
Step 3 : Now, select the path S-A-B-T. This path can carry an additional flow of unit 5 unit as the arc AB will get saturated on a flow of 5 unit.



Flow = 0 + 1

Fig. 15

Step 4 : Now, select the path S-A-C-D-T. This path can carry a flow of 2 units as the arc AC will get saturated on a flow of 2 units.



Flow = 0 + 5

Fig. 16

No more paths left, ∴ Maximum flow = 15

c) *Find the shortest path using Dijikstra's algorithm.* **[6]**

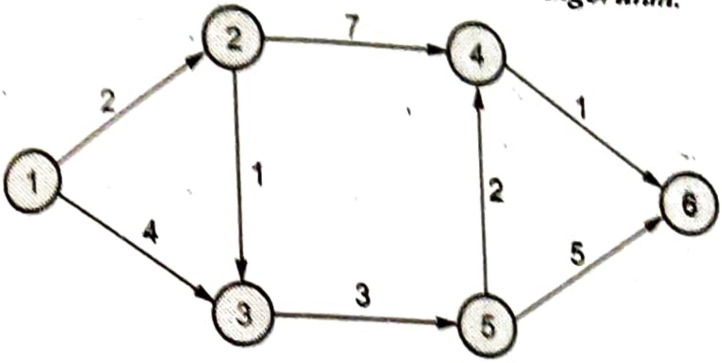


Fig. 17

Ans. : Step 1 : Select any arbitrary path of given network. Let 1 3 5 6 path selected 1 vertex is source and 6 vertex is sink. 35 arc can carry maximum of 3 units.

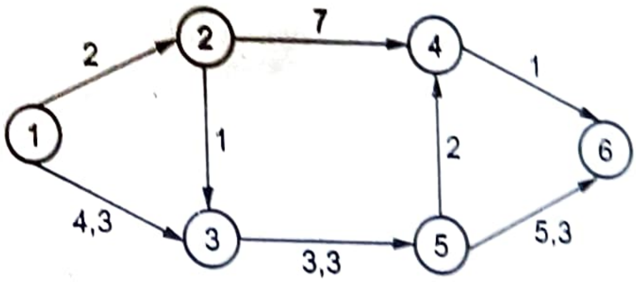


Fig. 18

Step 2 : Now select 1 2 4 6 path. This path can carry a flow of 2 units as the arc 12 will get saturated on a flow of 2 units.

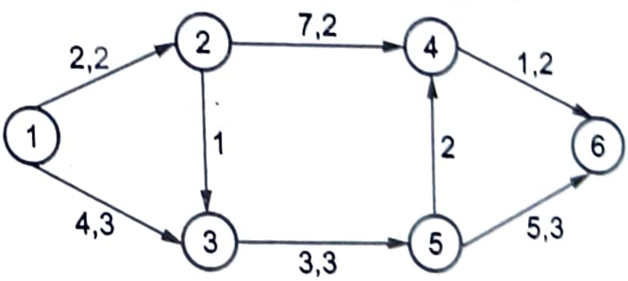


Fig. 19

OR

Q.2 a) *Let 'G' be a connected planar graph with 20 vertices and the degree of each vertex is 3. Find the number of edges and regions in the graph.* **[6]**

Ans. : Let G be a connected planar graph with n = 20 vertices.

i.e. D(v) = 3.

Using handshaking Lemma

$$\sum_{i=1}^{20} d(V_i) = 2e$$

$$\Rightarrow \quad 20 \times 3 = 2e$$

$$\Rightarrow \quad e = 30 = \text{No. of edges}$$

Using Euler's formula of connected plannar graph with n vertices, e edges and f faces or regions

$$n - e + f = 2$$

$$f = 2 - n + e$$

$$2 - 20 + 30 = 12$$

b) *Explain the following types of graphs with the help of examples : i) Bipartite graph ii) Complete graph iii) Regular graph iv) Spanning subgraph* **[6]**

Ans. : Bipartite graph : A graph G (v, E) is said to be bipartite graph if its vertex set can be partitioned into two disjoint subsets say v_1 and v_2 such that $v_1 \cup v_2 = v$ and $v_1 \cap v_2 = \phi$ and every edge of G joins a vertex of v_1 to a vertex of v_2.

In Bipartite graph, vertices of v_1 should not be adjacent. It is free from loops.

Following graphs are bipartite graphs

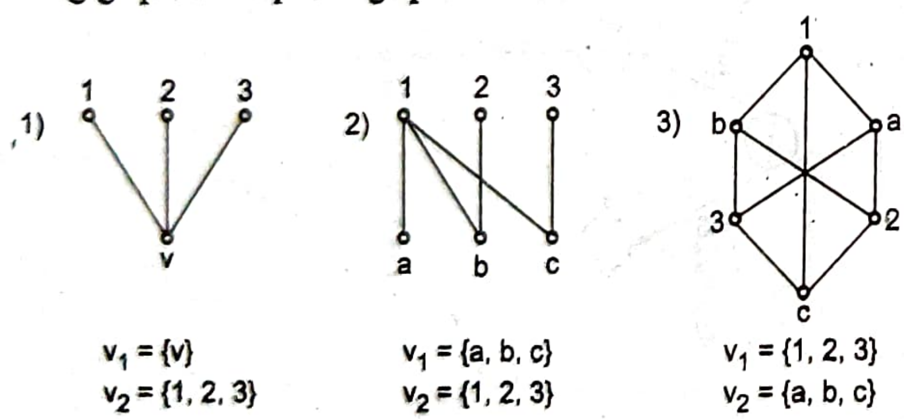


Fig. 20

Complete graph : A simple graph G in which every pair of distinct vertices are adjacent is called a complete graph. If G is a complete graph on n vertices then it is denoted by K_n.

In a complete graph, there is an edge between every pair of distinct vertices.

In graph K_n, every vertex is adjacent to remaining n–1 vertices so degree of each vertex is n – 1.

Thus K_n is a (n – 1) - regular graph.

K_n has exactly $\frac{n(n-1)}{2}$ edges.

Consider the following examples :

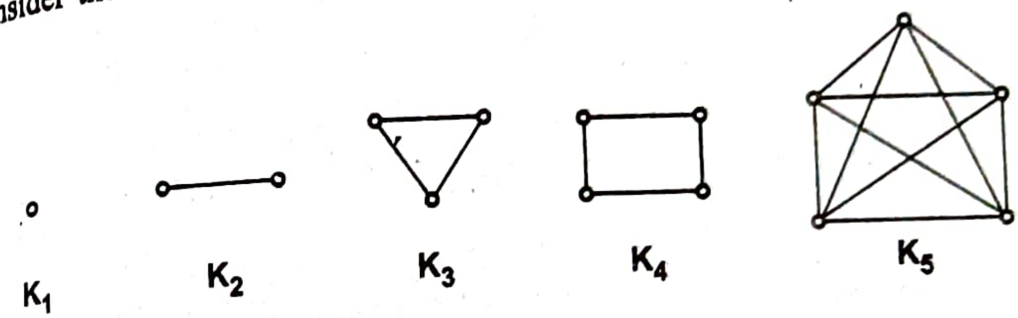


Fig. 21

Regular graph : A graph G is said to be r-regular graph if every vertex of G has degree r.

i) Regular graph of degree zero is called null graph.

ii) A regular graph of degree 3 is called cubic graph.

e.g.

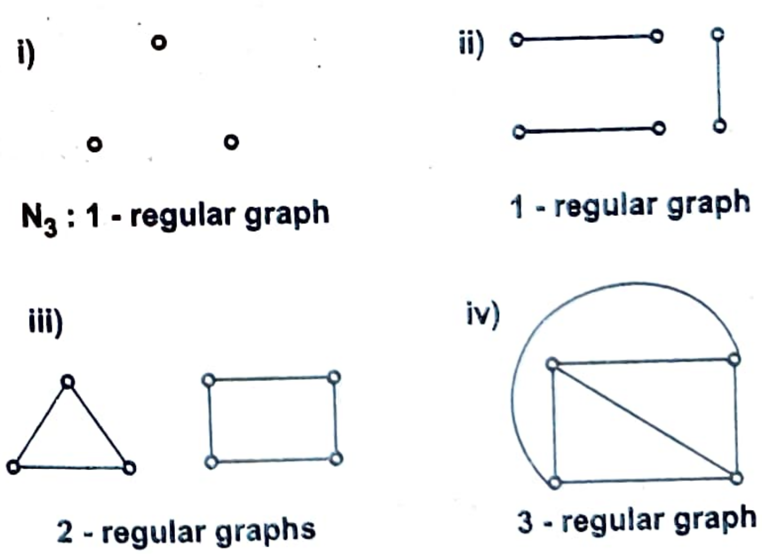


Fig. 22

Spanning subgp : Let G (V, E) be any graph. A subgraph H of a graph G is said to be spanning subgraph if V (G) = V (H).

Example : Let G be the following graph :

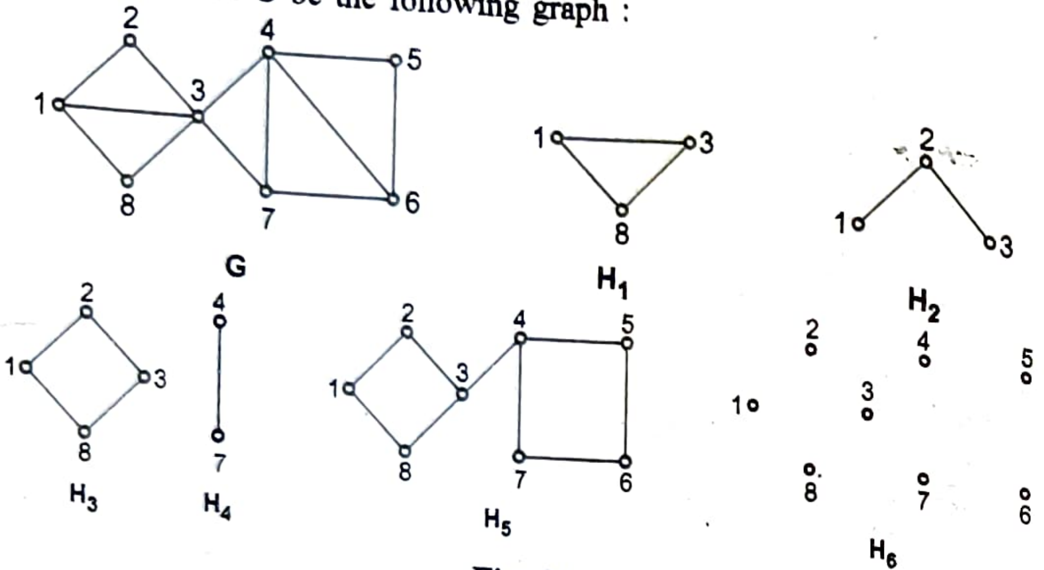


Fig. 23

Graphs H_1, H_2, ... H_6 are subgraphs of G.

H_1 and H_2 are edge disjoint subgraphs but not vertex disjoint subgraphs.

H_3 and H_4 are vertex disjoint subgraphs as well as edge disjoint subgraphs.

Subgraphs H_5 and H_6 are spanning subgraphs of G as V (H_5) = V (H_6) = V (G).

c) *Find under what conditions $K_{m,n}$ the complete bipartite graph will have an Eulerian circuit.* **[6]**

Ans. : In $K_{m,\,n}$ consider the following cases.

Case 1 : m = n and both m and n are even :

In this case, degree of each vertex is even, Hence by theorem 1, $K_{m,\,n}$ will have an Eulerian circuit. For example $K_{1,\,2}$ and $K_{4,\,4}$.

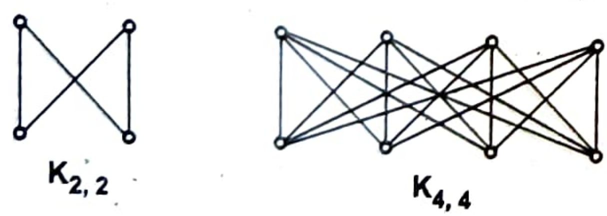
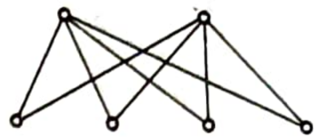


Fig. 24

Case 2 : If m = n and m, n are odd :

In this case degree of each vertex is odd. Hence Eulerian circuit will not exist.



$K_{2,4}$

Fig. 25

Case 3 : If m ≠ n but m and n are even :

In this case, degree of each vertex is even. So there exists an Eulerian circuit.

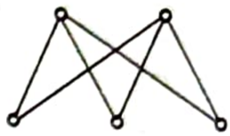


Fig. 26

Case 4 :

If m ≠ n and either m is odd or n is odd or both are odd : then graph will have vertices of odd degree. Hence Eulerian circuit does not exist. e.g. K_2, 3.

Q.3 a) *Suppose that the relation R on a set is represented by the matrix M_R. Is reflexive, symmetric and/or anti-symmetric ?* **[6]**

$$\begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

Ans. : Let, A = {1,2,3} R be a relation defined on A. M_R is relation matrix.

$$M_R = \begin{array}{c|ccc} & 1 & 2 & 3 \\ \hline 1 & 1 & 1 & 0 \\ 2 & 1 & 1 & 1 \\ 3 & 0 & 1 & 1 \end{array}$$

$$R = \{(1,1), (1,2), (2,1), (2,2), (2,3), (3,2), (3,3)\}$$

i) ∵ (1,1), (2,2), (3,3) ∈ R

So that R is reflexive.

ii) Here, R is symmetric because for all (a,b) ∈ R there is aRb and bRa.

iii) Here, R is anti-symmetric because (a,b) ∈ R (b,a) ∈ R and a = b.

b) *Find the homogeneous solution for the recurrence relation* **[6]**

$A_n - 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$ *with* $a_0 = 2,\ a_1 = 5,\ a_2 = 15$

Ans. : We have

$$a_n - 6\,a_{n-1} + 11\,a_{n-2} + 6\,a_{n-3} = 0 \qquad \dots (1)$$

Step 1 : The characteristic equation is

$$\alpha^3 - 6\alpha^2 + 11\alpha - 6 = 0$$

$\alpha = 1$ is a trial root

$\therefore$ By synthetic division method

1	1	− 6	11	− 6
		1	− 5	6
	1	− 5	6	0

$$\therefore \quad (\alpha - 1)(\alpha^2 - 5\alpha + 6) = 0$$

$$(\alpha - 1)(\alpha - 2)(\alpha - 3) = 0$$

$\alpha = 1, 2, 3$ are real and distinct roots

$\therefore$ It's homogeneous solution is

$$a_n^{(h)} = A_1\, 1^n + A_2\, 2^n + A_3\, 3^n \qquad \dots (2)$$

Step 2 : But given that $a_0 = 2$, $a_1 = 5$, $a_2 = 15$

$$\therefore \quad a_0 = A_1 + A_2 + A_3 \Rightarrow 2 = A_1 + A_2 + A_3 \qquad \dots (3)$$

$$a_1 = A_1 + 2\,A_2 + 3\,A_3 \Rightarrow 5 = A_1 + 2\,A_2 + 3\,A_3 \qquad \dots (4)$$

$$a_2 = A_1 + 4\,A_2 + 9\,A_3 \Rightarrow 15 = A_1 + 4\,A_2 + 9\,A_3 \qquad \dots (5)$$

Equation (4) + Equation (3) and equation (5) – Equation (3)

$$\Rightarrow \quad 3 = A_2 + 2\,A_3 \qquad \dots (6)$$

and

$$13 = 3\,A_2 + 8\,A_3 \qquad \dots (7)$$

Equation (7) – 3 × Equation (6)

$$4 = 0 + 2\,A_3 \Rightarrow \boxed{A_3 = 2}$$

Equation (6) $\Rightarrow \quad A_2 = 3 - 2\,A_3 = 3 - 4 = -1$

Equation (3) ⇒ $A_1 = 2 - A_2 - A_3 = 2 - (-1) - 2 = 1$

Hence $a_n^{(h)} = 1 + (-1)\, 2^n + 2\, (3)^n$

c) *Let f(x) = x + 2, g(x) = x – 2, h(x) = 3x, for x ε R where R is the set of real numbers Find i) gof ii) fog iii) fof iv) hog v) gog.* **[5]**

Ans. :

i) $gof(x) = g[f(x)] = g[x + 2] = x + 2 - 2 = x$

ii) $fog(x) = f[g(x)] = f[x - 2] = x - 2 + 2 = x$

iii) $fof(x) = f[f(x)] = f[x + 2] = x + 2 + 2 = x + 4$

iv) $hog(x) = h[g(x)] = h[x - 2] = 3(x - 2) = 3x - 6$

v) $gog(x) = g[g(x)] = g[x - 2] = x - 2 - 2 = x - 4$

OR

Q.4 a) *Find relation matrix,* **[6]**

i) If A = {1,2, 3, 4, 5, 6} and a R b if a divides b for a, b ε A.

ii) R = {(a, b) /a < b} for a, b ε A.

Ans. : i) R = {(1, 2} (1, 3), (1, 4), (1, 5), (1, 6), (2, 4) (2, 6), (3, 6), (1, 1) (2, 2), (3, 3), (4, 4), (5, 5) (6, 6)}

$$\therefore \quad \text{Relation Matrix} = M_R = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}_{6\times 6} \end{array}$$

ii) We have

$A = \{1, 2, 3, 4, 5, 6\}$

$R = \{(a, b)/a(b, \forall, a, b \in A)\}$

$= \{(1, 2)\ (1, 3)\ (1, 4)\ (1, 5)\ (1, 6)\ (2, 3)$
$(2, 4)\ (2, 5)\ (2, 6)\ (3, 4)\ (3, 5)\ (3, 6)$
$(4, 5)\ (4, 6)\ (5, 6)\}$

$$\text{Relation matrix is } M = \begin{array}{c} \\ 1 \\ 2 \\ \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}_{6\times 6} \end{array}$$

b) *Let $A = \{1, 2, 3, 4\}$, $B = \{a, b\}$. and $R = \{(1, a), (2, a), (3, a), (4, a)\}$, $S = \{(4, a), (4, b), (3, a), (3, b)\}$*

Find : i) $A \times B$ ii) $\sim R$ iii) $\sim S$ iv) $\sim R \cup \sim S$ **[6]**

Ans. : $A = \{1, 2, 3, 4\}$ and $B = \{a, b\}$

$R = \{(1, a), (2, a), (3, a), (4, a)\}$

$S = \{(4, a), (4, b), (3, a), (3, b)\}$

i) $A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b), (4, a), (4, b)\}$

ii) $\sim R = (A \times B) - R = \{(1, b), (2, b), (3, b), (4, b)\}$

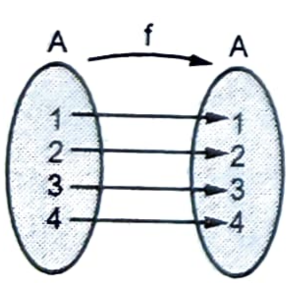
iii) $\sim S = (A \times B) - S = \{(1, a), (1, b), (2, a), (2, b)\}$

iv) $\sim R \cup \sim S = \{(1, a), (1, b), (2, a), (2, b), (3, b), (4, b)\}$.

c) *Describe : i) Identity function ii) Composite function iii) Inverse function* **[5]**

Ans. : i) Identify function : Let A be any non empty set and function $f : A \rightarrow A$ is said to be the identity function if $f(x) = x$, $\forall\ x \in A$.

e.g.



$f : A \rightarrow A$ is the identity function

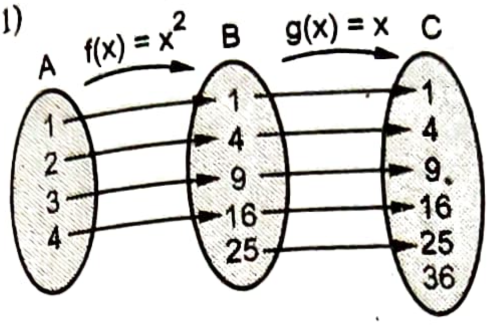
Fig. 27

ii) Composite function : Let $f : A \rightarrow B$ and $g : B \rightarrow C$ be two functions.

The composite function of f and g is denoted by gof and defined as $gof : A \rightarrow C$ is a function such that $(gof)(a) = g[f(a)]\ \forall a \in A$.

Note : gof is defined only when the range of f is a subset of the domain of g.

e.g. 1)



Therefore gof : A $\rightarrow$ C

gof (1) = g[f(1)] = g(1) = 1

gof (2) = g[f(2)] = g(4) = 4

gof (3) = g[f(3)] = g(9) = 9

gof (4) = g[f(4)] = g(16) = 16

Fig. 28

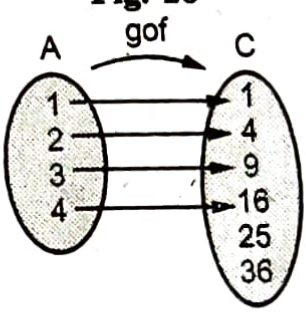


Fig. 29

iii) **Inverse function :** Let a function f : $A \rightarrow B$ be a bijective function then f^{-1} : $B \rightarrow A$ is called the inverse mapping of f and defined as $f(b)^{-1} = a$ iff f(a) = b

It is also known as **invertible mapping.**

Q.5 a) *Find the prime factorization of each of the following integer.*

i) 6647 *ii) 45500* *iii) 10 !* **[6]**

Ans. : i) The prime factors are : 17 × 17 × 23 written in exponential form : $17^2 \times 23^1$

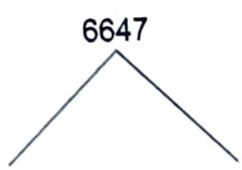


Fig. 30

ii) Prime factors of 45500 = 2, 2, 5, 5, 5, 7, 13 written in experimential form = $2^2 \times 5^3 \times 7 \times 13$

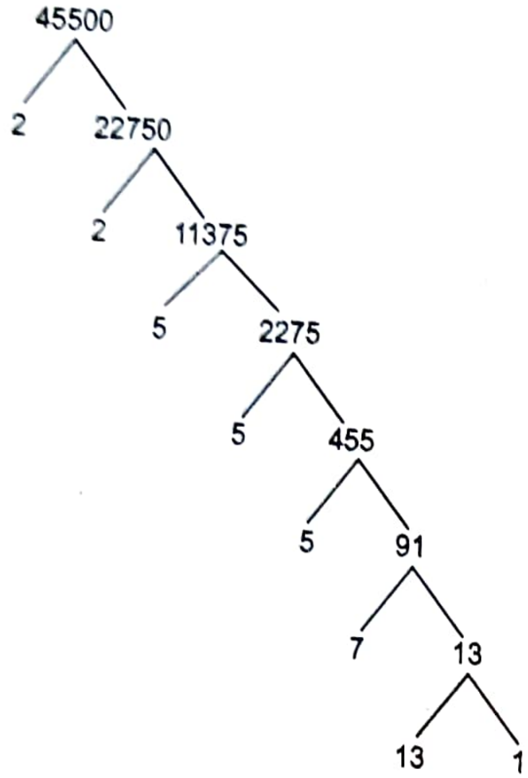


Fig. 31

iii) $$10! = 10 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$$
$$= 2^8 \times 3^4 \times 5^2 \times 7^1$$

We can count the no. of factors of 10! which will be $(8+1) \times (4+1) \times (2+1) \times (1+1) = 270$

For each of the factor d of any number x, we can get x by multiplying d by $\frac{x}{d}$ where $\frac{x}{d}$ is another factor of x - 50, number of ways we can choose (a,b) such that a $\times$ b = 10! will be $\frac{270}{2} = 135$. In multiplication there is no need to consider a $\times$ b different from b $\times$ a, otherwise answer would be 270.

b) *Find integers p and q such that 51p + 36q = 3 using Extended Euclidian algorithm. Also find GCD.* **[6]**

Ans. : By Euclids Lemma, for integers a and b s.t $a > b$ ... b and r s.t.

$a = bq + r$ where $0 \leq r \leq b$

Take $a = 51$ and $b = 36$

$$51 = 36 \times 1 + 15 \quad ...(1)$$

$\therefore$ for 36 and 15 we have $36 = 15 \times 2 + 6$ ...(2)

$$15 = 6 \times 2 + 3 \quad ...(3)$$

$$6 = 3 \times 2 + 0$$

The remainder has zero $\therefore$ Our procedure stops here and by back substitution.

$$3 = 15 - 6 \times 2 \quad \text{by equation (3)}$$

$$= 15 - (36 - 15 \times 2) \times 2 \quad \text{by equation (2)}$$

$$= 15 - 36(2) + 15(4)$$

$$= 15(5) - 36(2) \quad \text{by equation (1)}$$

$$= [51 - 36(1)]\,(5) - 36(2)$$

$$= 51(5) - 36(5) - 36(2)$$

$$3 = 51(5) - 36(7)$$

$\therefore$ $$3 = 51p + 36(q)$$

$\therefore$ $\boxed{p = 5}$ and $\boxed{q = -7}$

and gcd of 51 and 36 = 3

c) *Find the values of the following using modular arithmetic.* **[6]**

i) 77 mod 9 *ii) 3110 mod 13*

Ans. : i) We first divide the dividend (77) by the divisor (9)

2^{nd} multiply the whole part of the quotient in the previous step by the divisor (9).

Then, finally subtract answer in the 2^{nd} step from the dividend (77) to get the answer.

$$\frac{77}{9} = 8.555556$$

$$8 \times 9 = 72$$

$$77 - 72 = 5$$

Thus the answer is 5.

ii) We first divide the dividend (3110) by the divisor (13)

2^{nd} we multiply the whole part of the quotient in the previous step by the divisor (13).

Then, finally subtract the answer in 2^{nd} step from the dividend (3110) to get the answer.

$$\frac{3110}{13} = 239.230769$$

$$239 \times 13 = 3107$$

$$3110 - 3107 = 3$$

Thus, the answer is 3.

OR

Q.6 a) *Solve the following using Fermat's Little theorem.* **[6]**

i) 7^{69} *mod 23* *ii)* 3^{101} *mod 13*

Ans. : i) $7^{69} = y$ mod 23, here n = 7 and p = 23

By Fermat's Little theorem is,

$$n^{p-1} = 1 \text{ mod } p$$

By substituting the values of a and p and rewrite the equation :

$$7^{(23-1)} = 1 \text{ mod } 23$$

$$7^{(22)} = 1 \text{ mod } 23$$

We can write 7^{69} as $(7^{22})^3 * 7^3$

Therefore,

$$7^{69} = y \bmod (23)$$

And can be written as

$$7^{69} = 7^{66} * 7^3$$

$$7^{69} = (7^{22})* 7^3 \bmod 23$$

$$7^{69} = (1)^3 * 7^3 \bmod 23$$

$$7^{69} = 343 \bmod 23$$

Therefore, the smallest positive residue y = 21.

ii) $3^{101} = y \bmod 13$

Here n = 3, P = 13 again by Fermat's Little theorem

$$n^{P-1} = 1 \bmod P$$

$$(3)^{(13-1)} = 1 \bmod 13$$

$$3^{(12)} = 1 \bmod 13$$

We can write 3^{101} as $(3^{12})^8 * 3^5$

Therefore, $3^{101} = y \bmod 13$

And can be written as : $3^{101} = 3^6 * 3^5$

$$3^{101} = (3^{12})^8 * 3^5 \bmod 13$$

$$3^{101} = (1)^8 * 3^5 \bmod 13$$

$$3^{101} = 243 \bmod 13$$

$$3^{101} = 9 \bmod 13$$

Therefore, the smallest positive residue y = 9.

b) *Find Euler Totient Function of the following numbers.* **[6]**

i) 75 *ii) 5488* *iii) 77*

Ans. : i) $\phi(75)$, $n = 75 = 15 \times 5$

Here, n = 75, n is prime number

$$\phi(75) = \phi(15) * \phi(5)$$
$$= 14 * 4$$
$$= 56$$

ii) $\phi(5488)$

Here,

$$n = 5488 = 2^4 \times 7^3$$

$$\phi(5488) = n \times \left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right)$$
$$= 5488 \times \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{7}\right)$$
$$= 5488 \times \frac{1}{2} \times \frac{6}{7}$$
$$= 392$$

2	5488
2	2744
2	1372
2	686
7	343
7	49
7	7
	1

iii) $\phi(77)$

Here,

$$n = 77 = 11 \times 7$$

$$\phi(77) = \phi(11) * \phi(7)$$
$$= 10 \times 6$$
$$= 60$$

c) *Compute GCD of the following using Euclidean algorithm.* **[6]**

i) GCD (831, 366)

ii) GCD (2222, 1234)

Ans. : i) GCD (831, 866)

$$831 = 366 * 2 + 99$$

$$366 = 99 * 3 + 69$$

$$99 = 69 * 1 + 30$$

$$69 = 30 * 2 + 9$$

$$30 = 9 * 3 + 3$$

$$9 = 3 * 3 + 0$$

Hence, gcd(831, 366) = 3

ii) GCD (2222, 1234)

$$2222 = 1234 * 1 + 988$$

$$1234 = 989 * 1 + 246$$

$$988 = 246 * 4 + 4$$

$$246 = 4 * 61 + 2$$

$$61 = 2 * 3 + 1$$

$$2 = 1 * 2 + 0$$

Hence, GCD (2222, 1234) = 0

Q.7 a) *Consider the (2, 6) encoding function e.e(00) = 100000,* **[7]**

e(10) = 101010

e(01) = 001110, e(11) = 101001

Find minimum distance of e.

How many errors will e detect ?

Ans. :

d (100000, 101010) = 2

d (100000, 001110) = 4

d (100000, 101001) = 2

d (101010, 001110) = 2

d (101010, 001110) = 2

d (001110, 101001) = 4

∴ Minimum distance of the code = Minimum of all the distance = 2

∴ Code can detect (2 – 1) or fewer errors.

b) *Let R = {0°, 60°, 120°, 180°, 240°, 300°} and * = binary operation, so that a * b is overall angular rotation corresponding to successive rotations by a and then by b. Show that (R, *) is a Group.* **[6]**

Ans. : Given that R = {0° = 360°, 60°, 120°, 180°, 240°, 300°}

Consider the following table.

*	0	60	120	180	240	300
0	0	60	120	180	240	300
60	60	120	180	240	300	0
120	120	180	240	300	0	60
180	180	240	300	0	60	120
240	240	300	0	60	120	240
300	300	0	60	120	180	240

From this table we get,

a) * is closed

i.e. For any a, b ∈ R, a * b ∈ R

b) * is associative

i.e. a * (b * c) = (a * b) * c

c) Existence of the identity

0° is the identity element in R.

d) Existence of the inverse

Element	0	60	120	180	240	300
Inverse	0	300	240	180	120	60

$\therefore$ (R, *) is a group.

c) *Prove that the following table on relation of elements of set $G = \{0, 1, 2, 3, 4, 5\}$ multiplication mod 6 is not a group.* **[4]**

	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	0
5	5	0	1	2	3	4

Ans. : $G = \{0, 1, 2, 3, 4, 5\} = Z_6$ (mod 6)

To check whether $G = Z_6$ is a group under multiplication, we have to check if it satisfies four conditions.

1) Closure 2) Associativity 3) Identity and 4) Inverse.

Closure : Multiplication is closed i.e. for any $a, b \in G$ we have $a \cdot b \in G$

Associativity - Modulo multiplication is associative, for any $a, b, c \in G$ we have $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ (mod 6)

Identity : Take $e = 1 \in G$

$\therefore$ We have for any $a \in G$

$$a \cdot e = e \cdot a = a$$

Multiplication Inverse : For this condition we want a and b such that

$$a \cdot b = e = 1$$

But for $a = 0 \in G$ we can not find any $b \in G$ such that

$$0 \cdot b = 1$$

$\therefore$ Inverse property does not satisfies.

$\therefore$ G is not a group under multiplication.

OR

Q.8 a) *Determine whether description of * is a valid definition of a binary operation on the set.* **[6]**

*i) On R, a * b = ab (ordinary multiplication)*

ii) On z^+*, a * b = a / b*

*iii) On z, a * b = ab*

*iv) On z+, a * b = a – b*

*v) On z, a * b = 2a + b*

*vi) On R, a * b = ab/3*

Ans. :

i) Yes, since $f : R^2 \rightarrow R$ defined as $f(a, b) = ab$ is a function, with $a\,b \in R$

ii) No, since $(a, b) \in z^+ \times z^+$ does not imply that $a * b = {}^a\!/\!_b \in z$

$(1, 2) \in z^+ \times z^+$, but $1/2 \notin z^+$

iii) No, since $(1, 2) \in z^+ \times z^+$ but

iv) Yes, since, $*$ is a function with $\min\{a, b\} \in R$

v) Yes, since $*$ is a function with $a \times |b| \in R$

vi) No, since $2 * (-1) = 2^{-1} = \dfrac{1}{2} \notin z$

b) *S = {1, 2, 3, 6, 12} where a * b is defined as LCM (a, b). Determine whether it is an Abelian Group or not.* **[7]**

Ans. : $S = \{1, 2, 3, 6, 12\}$

$a * b = \text{LCM}(a, b)$

i) Take $a = 2, b = 3$

$a * b = 6$ since 6 is in set S.

Also 3 * 1, 2 * 1, 6 * 2, 3 * 2 all possibility in sets.

Hence closure axiom is true.

ii) $(a * b) * c = $ (LCM of $a * b) * c$

Let's Take $a = 2$, $b = 3$, $c = 6$

$$(a * b) * c = 6 * 6$$
$$= 6$$
$$a * (b * c) = a * (6)$$
$$= 2 * 6$$
$$= 6$$

Hence, associative axiom it true.

iii) $a * e = $ (LCM of a and e)
$$= 2 \times 1 = 2$$

2×1, 3×1, 6×1, 12×1 satisfying $a * e = e$.

so $e = 1$. Hence identity axiom is true.

iv) $a * a^{-1} = a^{-1} * a = e$ for inverse axiom.

For each element it is observed that no inverse element exist so it is not Abelian group.

c) *Define ring.* **[4]**

Ans. : Let R be a non empty set equipped with two binary operations called addition and multiplication and denoted by '+' and '·' respectively.

An algebraic structure $(R,+, \cdot)$ is called a ring if it satisfies following axioms.

1) $(R, +)$ is an abelian group i.e.

i) **Closure property :** for $a, b \in R$, $a + b \in R$

ii) **Associativity :** for $a, b, c \in R$, $a + (b + c) = (a + b) + c$

iii) **Existence of the identity :** For any $a \in R$, $\exists\ o \in R$ s.t., $a + 0 = 0 + a = a$.

$\therefore$ 0 is called as the additive identity element of ring.

iv) Existence of the inverse : For each $a \in R$, $\exists -a \in R$

Such that $a + (-a) = -a + a = 0$

$-a$ is called the additive inverse of a

v) Commutative property : For $a, b \in R$

$$a + b = b + a$$

2) $(R, \cdot)$ is semigroup i.e.

i) **Closure property :** $\forall a, b \in R, a \cdot b \in R$

ii) **Associativity :** for $a, b, c \in R$,

$$a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

3) Multiplication distributes over addition $\forall a, b, c \in R$

i) $a \cdot (b+c) = a \cdot b + a \cdot c$ (Right distributive law)

ii) $(a+b) \cdot c = a \cdot c + b \cdot c$ (Left distributive law)

END...