Geometric Image Transformations (linear)
(2D to 2D)

Image → visual representatn of $f(x,y)$

Dynamic range → In intensity transformatn, we ↑se dynamic range for image enhancement.

applications of Intensity transformatn → enhancement denoising,

applicatn of geometric (change pixel coordinate values)

→ correct defects due to camera orientation
→ Lens distortion
→ special effects
→ Relate/combine images taken at diff time/ by diff cameras/sensors.

Geometry → lines, points & relatnship.

Algebraic geometry → apply algebra in geometric primitives.
↓
we do computatn.

Geometry $\xrightarrow[\text{inate system}]{\text{establish coord-}}$ Algebra.

① Cartesian coordinate system 2-D ⇒ (x,y)
[Euclidean space].

② Homogenous coordinate system.
[Projective space] $(x,y,1)$
↳ any n-dimensional pt. is represented by (n+1) nos./coordinate.

	Cartesian		Homogenous
Eg:	$(2,3)$	$\longrightarrow$	$(2,3;1)$
	(x,y,z)	$\longrightarrow$	$(x,y,z,1)$

Q) Why one at end?

Q) why we need homogenous when we have cartesian.

Q) why it is c/a homogenous.

$(3,2,1)$
$(6,4,2)$ → $(3,2)$ cartesian.
$(12,8,4)$

$$(kx, ky, k) \rightarrow \left(\frac{kx}{k}, \frac{ky}{y}\right)$$

To represent pt's at ∞, we use homogenous.
Pts. at ∞ are called ideal pts : $\underline{\underline{(x,y,0)}}$

$(26,5,0) \rightarrow (26/0, 5/0) \rightarrow (\infty, \infty)$
ideal pt.

$(1,0,0) \rightarrow$ in x-direcn
$(0,1,0) \rightarrow$ in y -- " --.

Algebaric geometry

✓ column vector $\begin{bmatrix} x \\ y \end{bmatrix} = \underline{X}$

X Row vector $[x \ y]^T = X^T$

Cartesian		Homogenous
Eg: $(2,3)$	$\longrightarrow$	$(2,3,1)$
(x,y,z)	$\longrightarrow$	$(x,y,z,1)$

Q) Why one at end?

Q) why we need homogenous? when we have cartesian.

Q) why it is c/a homogenous.

$(3,2,1)$
$(6,4,2)$ — $(3,2)$
$(12,8,4)$
cartesian

$$(kx, ky, k) \to \left(\frac{kx}{k}, \frac{ky}{y}\right)$$

To represent pt's at ∞, we use homogenous.
Pts. at ∞ are called ideal pts:

$\underline{(x,y,0)}$

$$(26,5,0) \to \left(\frac{26}{0}, \frac{5}{0}\right) \to (\infty, \infty)$$
ideal pt.

$(1,0,0) \to$ ideal pt in x-direcⁿ
$(0,1,0) \to$ " in y — " —

Algebraic geometry

✓ column vector $\begin{bmatrix} x \\ y \end{bmatrix} = \underline{X}$

✗ Row vector $[x \ y]^T = \underline{X}^T$

Homo & cart → columnⁿ vector is used to represent a pt.

Homogeneous representatⁿ of lines.

$$ax + by + c = 0$$

$$(a \ b \ c)^T$$

point $(x, y)^T$ lies on line $(a, b, c)^T$.

$$(x, y, 1) \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

$$\underset{\text{pt.}}{x} \cdot \underset{\text{line}}{l} = 0$$

$$l_1 \times l_2 = x \rightarrow \text{pt. of intersectⁿ}$$

line 1 → $x = 2$ → $-1, 0, 2$
line 2 → $x = 1$ → $-1, 0, 1$.

$$\begin{vmatrix} i & j & k \\ -1 & 0 & 2 \\ -1 & 0 & 1 \end{vmatrix} = i(0-0) - j(-1+2) + k(0-0)$$

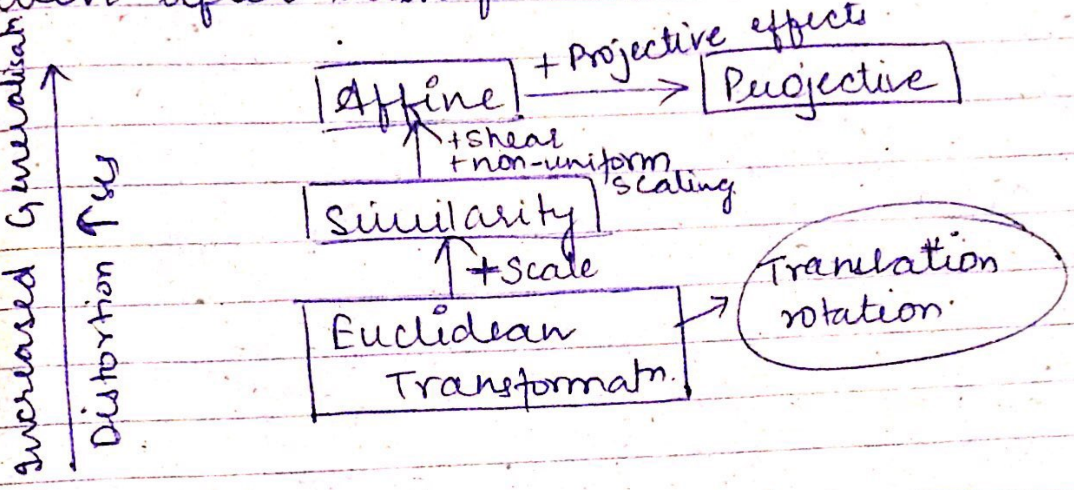
$$= i(0) + j(-1) + k(0)$$

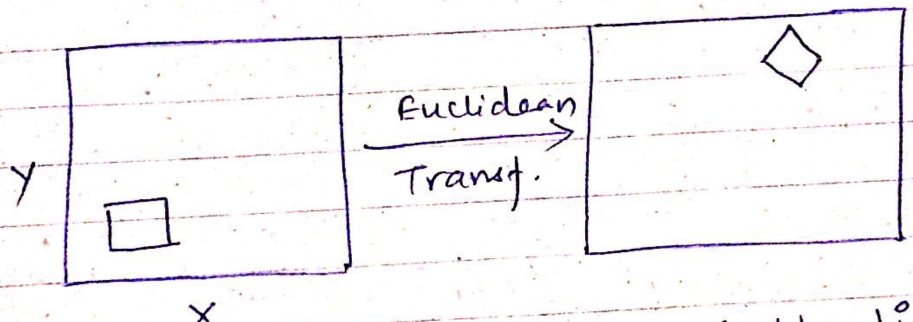
$$\begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} \longrightarrow \text{pt. at } \infty.$$

$ax + by + c = 0$
$kax + kby + kc = 0$
$k(a, b, c)^T$

Overview | hierarchy of linear transformation

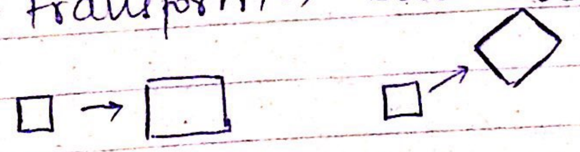
Invariants: certain quantities which remain same even after transformation





Invariants → area, angle b/w lines, size.

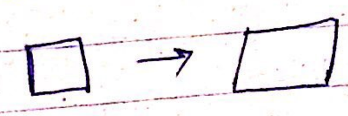
similarity transform → scale uniformly.



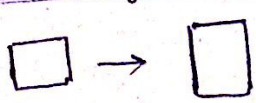
Invariants → angle b/w lines, ratio of area.

$$\frac{A'}{A} = \frac{B'}{B}$$

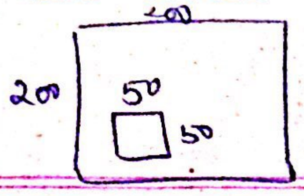
shear



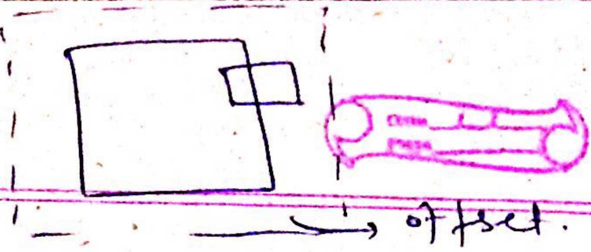
non-uniform scaling



only change size in one dire

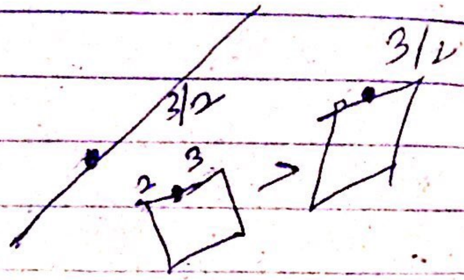


$t_x = 180$
$t_y = 180$



Invariants of affine

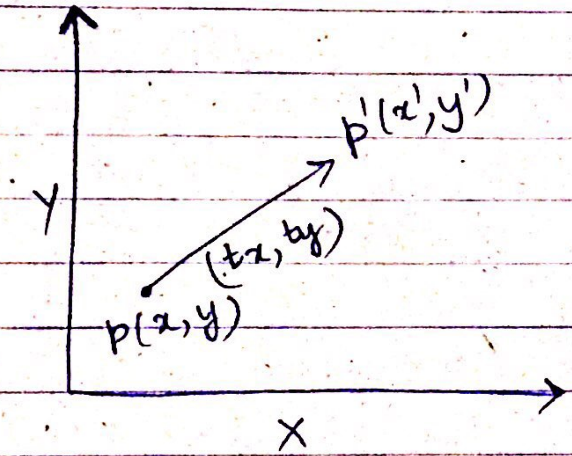
→ parallelism
→ Ratio of lengths.



Invariants of projective

collinearity → straight lines remain straight

Translation



$x' = x + t_x$
$y' = y + t_y$.

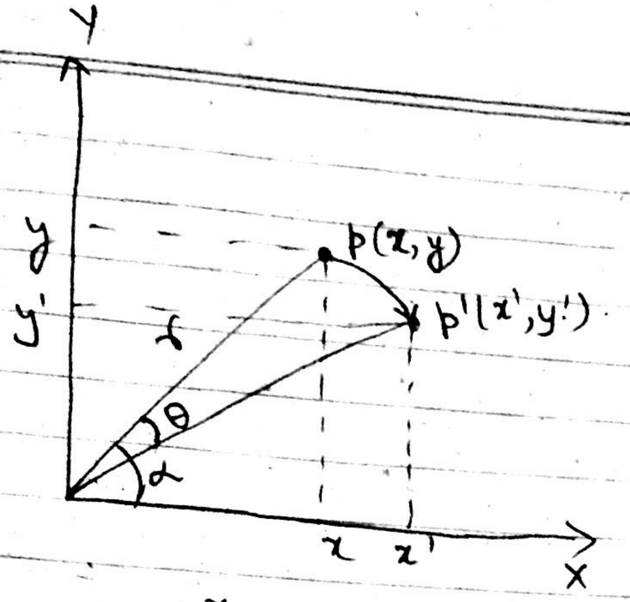
To represent them in matrix notation.

$$\underset{2\times 1}{\begin{bmatrix} x' \\ y' \end{bmatrix}} = \underset{2\times \boxed{2 \quad 2}\times 1}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}} + \underset{2\times 1}{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}$$

2×1.

Rotation

$x = r\cos\alpha$
$y = r\sin\alpha$.

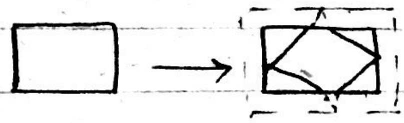


$x' = r\cos(\alpha - \theta)$
$y' = r\sin(\alpha - \theta)$

$$x' = \overbrace{r\cos\alpha}^{x}\cos\theta + \overbrace{r\sin\alpha}^{y}\sin\theta$$
$$y' = \underbrace{r\sin\alpha}_{y}\cos\theta - \underbrace{r\cos\alpha}_{x}\sin\theta$$

$$\boxed{\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= y\cos\theta - x\sin\theta \end{aligned}}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$ → rotate an image by θ degrees



→ offset

Euclidean = Rotation + Translatn.

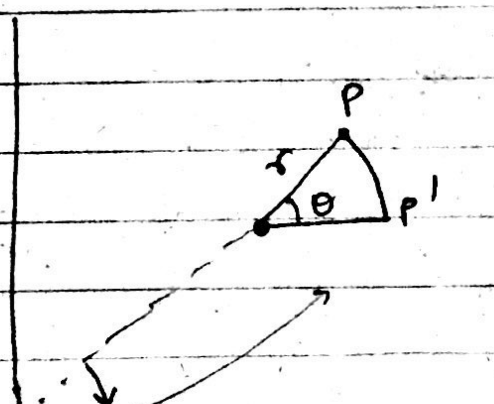
$$\underset{(2\times1)}{p'} = \underset{(2\times2)}{R}\,\underset{(2\times1)}{p} + \underset{(2\times1)}{t}$$

~~scalar~~.

$$\begin{bmatrix} \cancel{x'} \\ y' \end{bmatrix} = \begin{bmatrix} S_x & 0 \\ 1 & S_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

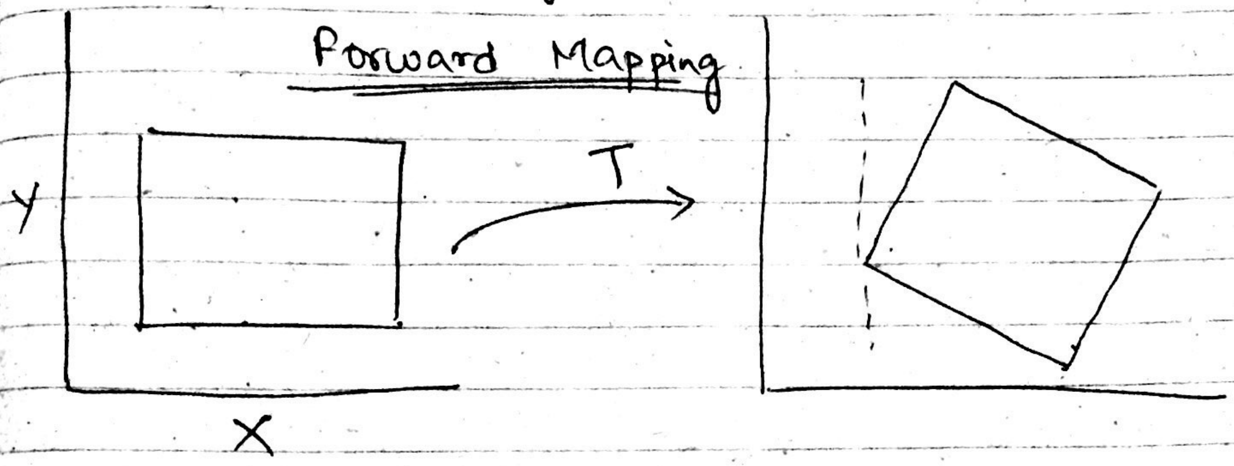
degree of freedom → t_x, t_y, θ
3



First translate to origin, then rotate & again translate

MPA

1. Forward mapping
2. Inverse mapping



$(x, y) \rightarrow$ integers

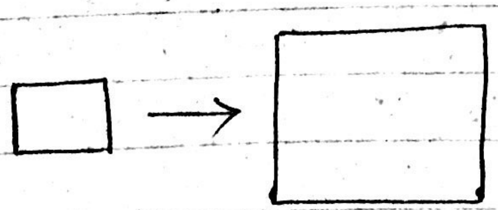
Find target image values on the basis of transformation f^n applied on source img

$$f(x,y) \xrightarrow{T} g(x', y')$$

integer — real values.

$(69.257, 135.79) \approx (69, 135)$

(68, 136) is not in target img but it was there in original $\rightarrow$ holes.



$\Rightarrow$ use interpolation

Blocking effect

Similarity Transform

uniform scaling..

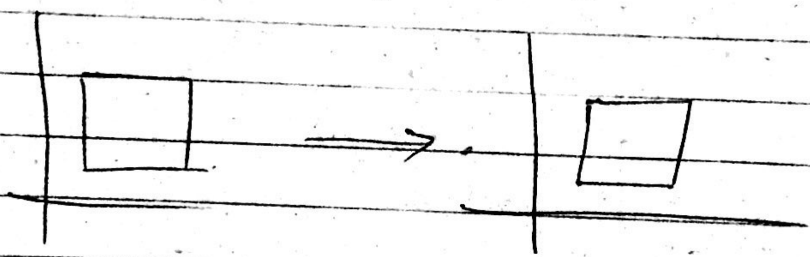
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \overset{\text{scalar}}{S_x} & 0 \\ 1 & S_y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$p' = \overset{(1\times1)}{s} R p + t$$

degree of freedom = 4

Rotation, scaling, translation
origin will always map to origin (0,0).
Invariants → angle, straight lines, parallelism.

Affine transformation



$$p' = ARp + t$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

shear   non-uniform scaling.

dof = 6

$$p' = Mp$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} c\cos\theta & c\sin\theta \\ -c\sin\theta & c\cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} c\cos\theta & c\sin\theta & t_x \\ -c\sin\theta & c\cos\theta & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

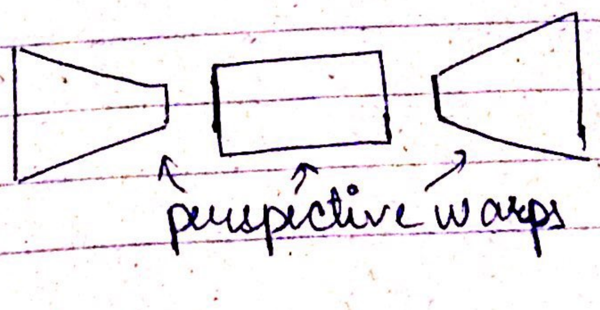
$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 0 & t_x \\ 0 & 1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{mirror / flip over y-axis}$$

$$\begin{bmatrix} -1 & 0 & t_x \\ 0 & -1 & t_y \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \text{mirror over origin } (0,0)$$

Projective transformation:

↳ always defined under homogenous quardinates because ∞ can be represented.



$x'/h,\ y'/h,\ 1$

homography or homogeneous matrix

$$\begin{bmatrix} x' \\ y' \\ h \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} x \\ y \\ h \end{bmatrix}$$

put $i = 1$

multiplying this matrix by scalar non-zero quantity still makes it projective transform.

dof = 8

Invariant → collinearity.

affine and projective

origin (0,0) not always maps to origin.

$p' = M p$

Find M?

corresponding pts.

$x' = x + t_x$

$y' = y + t_y$

Translation → 1

Rotation → 2

Affine → 3

Projective → 4

Morphing → Gradually, converting one Image to another.

MPA

Can we fill holes using interpolation?

Feature preserving morphing.

1. Linearly interpolate coordinate values.
2. Linearly —"— colour values.

[#0] Source image

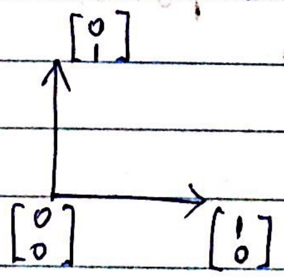
[#k] Intermediate image

[#N] Destination image

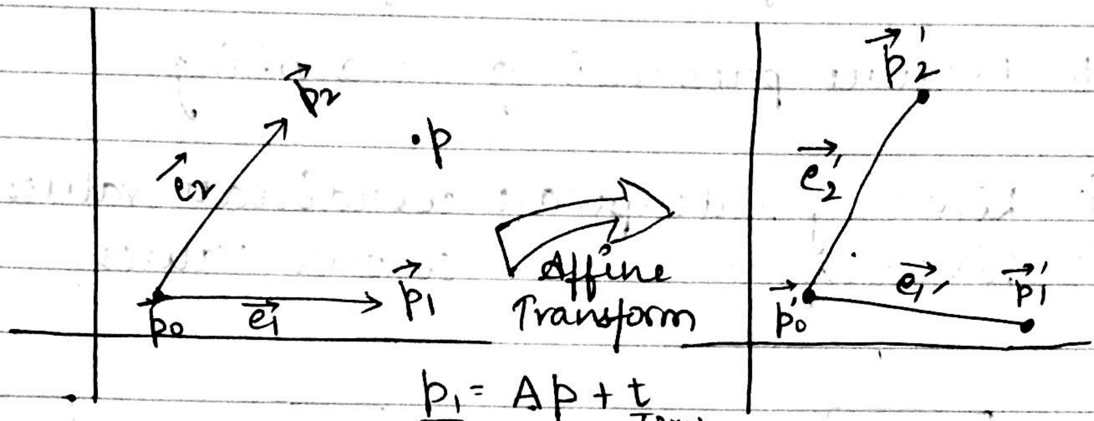
If only coordinate values change ⇒ warping

Basis → LI vectors and all other vectors in space can be represented by its linear combinatn

Unit vectors make basis vector in Euclidean space.



Affine basis → any 3 non-collinear pts.



$$\underset{2\times1}{p_1} = \underset{(2\times2)(2\times1)}{Ap} + \underset{(2\times1)}{t}$$

$$\left.\begin{aligned}\vec{e_1} &= \vec{p_1} - \vec{p_0}\\ \vec{e_2} &= \vec{p_2} - \vec{p_0}\end{aligned}\right] \rightarrow \text{affine Basis.}$$

Acc. to definition of affine coordinates:

$$\boxed{\vec{p} - \vec{p_0} = \alpha\vec{e_1} + \beta\vec{e_2}} \quad (1)$$

$\langle\alpha, \beta\rangle \rightarrow$ affine coordinates of pt. p.

After transform,

To prove:- $\boxed{\vec{p}' - \vec{p_0'} = \alpha\vec{e_1'} + \beta\vec{e_2'}}$ $\langle\alpha, \beta\rangle$ remains same.

LHS $\quad \vec{p}' - \vec{p_0'} = (A\vec{p} + t) - (A\vec{p_0} + t)$

$$= A(\vec{p} - \vec{p_0})$$

From eqn (1),

$$= A(\alpha\vec{e_1} + \beta\vec{e_2})$$

$$= \alpha A(\vec{p_1} - \vec{p_0}) + \beta A(\vec{p_2} - \vec{p_0})$$

$$= \alpha(A\vec{p_1} - A\vec{p_0}) + \beta(A\vec{p_2} - A\vec{p_0})$$

adding & subtracting t,

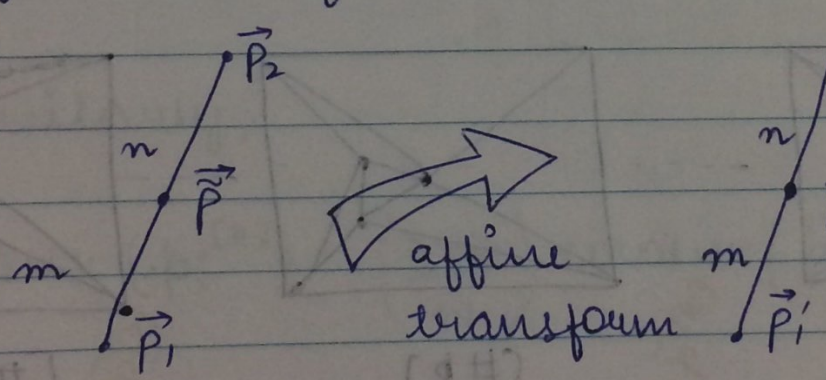
$$= \alpha\left[(A\vec{p_1} + t) - (A\vec{p_0} + t)\right] + \beta\left[(A\vec{p_2} + t) - (A\vec{p_0} + t)\right]$$

$$= \alpha [\vec{p}_1' - \vec{p}_0'] + \beta [\vec{p}_2' - \vec{p}_0']$$

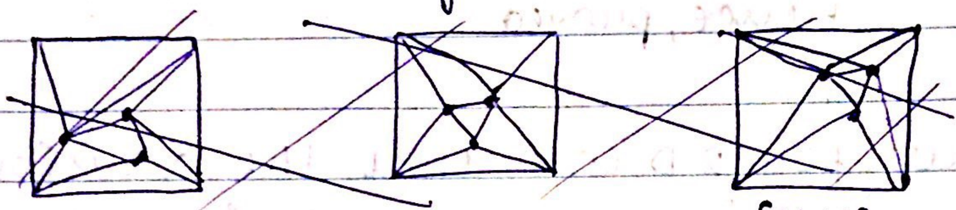
$$= \alpha \vec{e}_1' + \beta \vec{e}_2' = \text{RHS}.$$

Hence, proved.

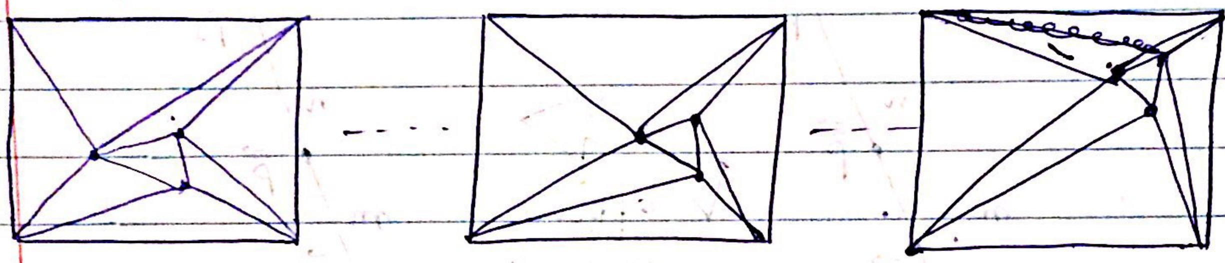
Q. Show (for 2-D case) that the division of a line in the ratio $m:n$ is invariant to an affine transformation.



Feature preserving morphing (same size images).

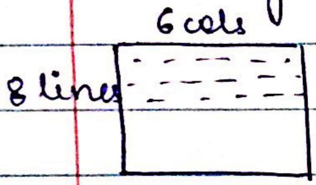


[#0] [#N]



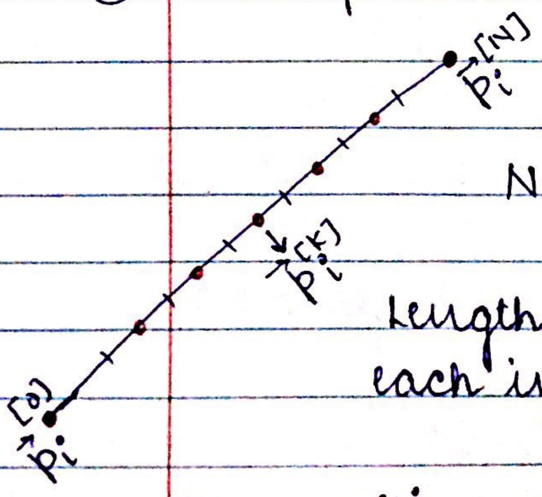
[#1] [#k] [#N]

① Triangulate these images.



source_coord.txt.

② Interpolate coordinate colour values.



N-intervals

N-1 intermediate pts.

Length of each interval $= \Delta l = \dfrac{\vec{p}_i^{[N]} - \vec{p}_i^{[0]}}{N}$

The position of i^{th} pt. in #k image →

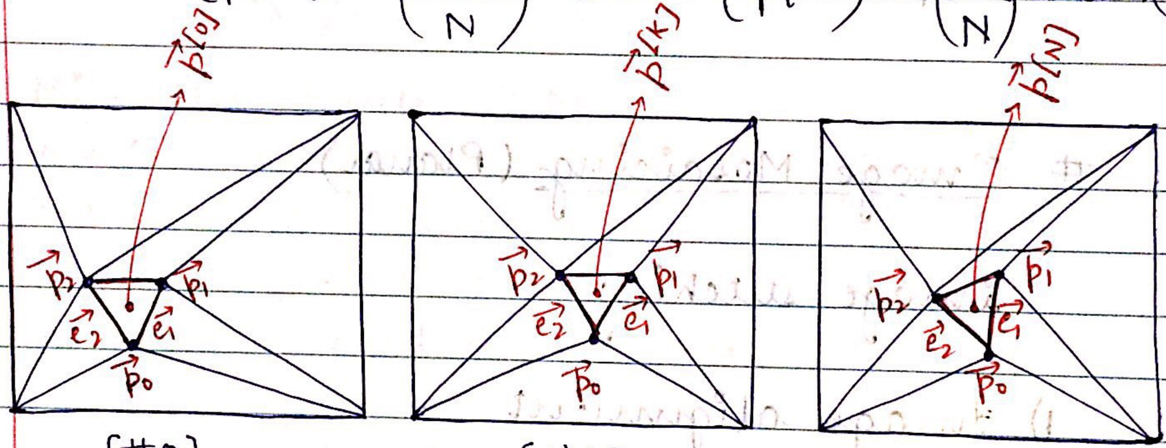
$$p_i^{[k]} = p_i^{[0]} + k\Delta l$$

$$= p_i^{[0]} + k\left[\frac{\vec{p}_i^{[N]} - \vec{p}_i^{[0]}}{N}\right]$$

$$p_i^{[k]} = \left(\frac{N-k}{N}\right)\vec{p}_i^{[0]} + \left(\frac{k}{N}\right)\vec{p}_i^{[N]}$$

Similarly,

$$\text{colour}\left(p_i^{[k]}\right) = \left(\frac{N-k}{N}\right)\text{colour}\left(\vec{p}_i^{[0]}\right) + \left(\frac{k}{N}\right)\text{colour}\left(\vec{p}_i^{[N]}\right)$$



[#0] [#k] [#N]

$N = 10$

$$\underbrace{\vec{p}^{[k]} - \vec{p}_0^{[k]}}_{} = \alpha e_1^{[k]} + \beta e_2^{[k]}$$

unknowns $\rightarrow \alpha, \beta$

$$\vec{p}^{[0]} - \vec{p}_0^{[0]} = \alpha\vec{e}_1^{[0]} + \beta\vec{e}_2^{[0]}$$

$$\vec{p}^{[N]} - \vec{p}_0^{[N]} = \alpha\vec{e}_1^{[N]} + \beta\vec{e}_2^{[N]}$$

For α & β,

$$p - p_0 = \alpha e_1 + \beta e_2$$

$$\begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \alpha \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \beta \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$$\begin{bmatrix} e \\ f \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix}^{-1} \cdot \begin{bmatrix} e \\ f \end{bmatrix}$$

Image Mosaicing (Planar)

image stitching

1) Image alignment
2) cut & Paste
3) Blending

Image Mosaics

1) To increase field of view without increasing cost.
2) Two images of a planar object are related by Homography.

$$\underset{3\times1}{\bar{p}'} = \underset{3\times3}{H}\,\underset{3\times1}{\bar{p}}$$

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

Projective camera eqⁿ

$$\boxed{\underset{1\times1}{\lambda} \underset{3\times1}{\vec{p}} = \underset{3\times4}{M} \underset{4\times1}{\vec{P}_w}} \quad \text{--- (1)}$$

~~Multiply pt both sides by~~

$$\underset{4\times1}{\bar{P}_w} = \lambda \underset{4\times3}{M^*} \underset{3\times1}{\bar{p}} + \underset{\substack{1\times1 \\ \text{scalar}}}{s} \underset{4\times1}{\bar{n}}$$

$M^* \rightarrow$ left inverse of M.

$p \rightarrow$ 2-D image point. $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \because$ projective always homogenous.

$M \rightarrow$ camera matrix.

$\bar{n} \rightarrow$ null space of M.

$P_w \rightarrow$ pt. in world coordinate.

$$\lambda \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{bmatrix} x_w \\ y_w \\ z_w \\ 1 \end{bmatrix}$$

The eqⁿ of a plane in world coordinate can be written as—

$$ax + by + cz + dH = 0$$

$\bar{c} \rightarrow$ vector of coeffⁿˢ. $\boxed{\bar{C} \cdot \bar{P}_w = 0}$ — (2)

Scanned with CamScanner

By putting eqⁿ 2 in eqⁿ 1

$$\bar{P}_W = M^*\bar{p} + s\bar{n}.$$
$$= \bar{C}^T M^*\bar{p} + s\bar{C}^T\bar{n} = 0.$$

$$\boxed{s = -\frac{\bar{C}^T M^*\bar{p}}{\bar{C}^T\bar{n}}} \quad (3)$$

Putting 3 back in eqⁿ 1,

$$\bar{P}_W = \lambda M^*\bar{p} + s\bar{n}$$
$$P_W = \lambda M^*\bar{p} - \bar{n}\left(\frac{\bar{C}^T_{1\times4}\, M^*_{4\times3}\, \bar{p}_{3\times1}}{\bar{C}^T_{1\times4}\, \bar{n}_{4\times1}}\right)$$

Taking $M^*\bar{p}$ common,

$$\bar{P}_{4\times1} = \underbrace{\left(I_{4\times4} - \frac{\bar{n}_{4\times1}\cdot\bar{C}^T_{1\times4}}{\underbrace{\bar{C}^T_{1\times4}\, n_{4\times1}}_{\text{scalar}}}\right) \underbrace{M^*_{4\times3}\,\bar{p}_{3\times}}_{4\times1}}$$

$$\boxed{\bar{P}_{4\times1} = \tilde{M}_{4\times3}\,\bar{p}_{3\times1}} \quad (4) \rightarrow \text{for one view.}$$

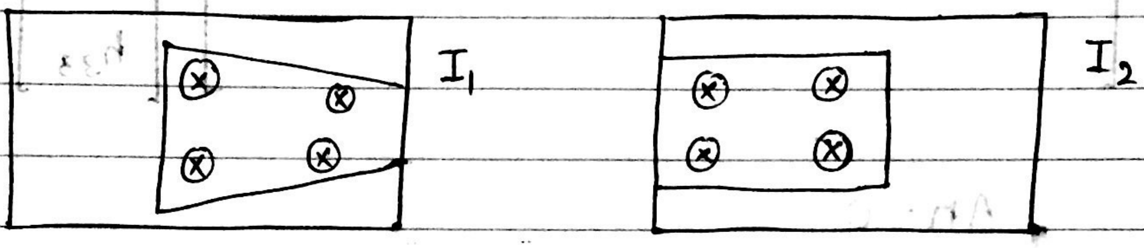
For another view,

$$p'_{3\times1} = M'_{3\times4}\, P_{W_{4\times1}}$$

using eqⁿ 4,

$$\boxed{\bar{p}' = \underbrace{M_{3\times4}\,\tilde{M}_{4\times3}}_{H}\,\bar{p}_{3\times1}}$$ → only for planar objects

Blackboard image mosaics



$$\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{32} \\ h_{31} & h_{32} & h_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$\frac{x'}{1}, \frac{y'}{1}$

$$\frac{x'}{1} = \frac{x h_{11} + y h_{12} + h_{13}}{x h_{31} + y h_{32} + 1}$$

$$y' = \frac{x h_{21} + y h_{22} + h_{23}}{x h_{31} + y h_{32} + 1}$$

$$\begin{bmatrix} 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} h_{11} & h_{12} & h_{12} \\ & & \\ & & \end{bmatrix} \begin{bmatrix} 5 \\ 9 \\ 1 \end{bmatrix}$$

$10h_{31} + 18h_{32} + 2 = 5h_{11} + 9h_{12} + h_{13}$

$20h_{31} + 36h_{32} + 4 = 5h_{21} + 9h_{22} + h_{23}$

we will get 8 eqns using 4 coordinates.

$$\begin{array}{cc} & \text{9 cols} \rightarrow \\ \text{8 rows} \downarrow & \begin{bmatrix} 5 & 9 & 1 & 0 & 0 & 0 & -10 & -18 & -2 \\ 0 & 0 & 0 & 5 & 9 & 1 & -20 & -36 & -4 \\ \vdots & & & & & & & & \vdots \end{bmatrix} \begin{bmatrix} h_{11} \\ \vdots \\ h_{33} \end{bmatrix} = 0 \end{array}$$

$Ah = 0$

$\boxed{\text{SVD}}$ → single valued decompositn.

$$\boxed{\begin{array}{cccc} A = & U & S & V^T \\ M\times N & M\times M & M\times N & N\times N \\ 8\times 9 & 8\times 8 & 8\times 9 & 9\times 9 \end{array}} \rightarrow \text{use inbuilt fn.}$$

↳ last column of V gives us h (homography).