Assignment 3

Problem1: Show that the solution of T(n)=T(n -1) + n is O(n^2) using "Elimination Method".

Solution:

T(n) = T(n-1) + n ------(0)

T(n-1)=T(n-2) + (n-1) ------(1)

T(n-2)=T(n-3) + (n-2) ------(2)

T(n-3)=T(n-4) + (n-3) ------(3)

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T(1)= 1 -------(k-1)

With loss of generality, k is assumed to be logn namely, n=2^k

So, T(n) = n + (n-1) + (n-2) + (n-3) + ….+1

$$= \sum_{i=1}^{k-1} 1$$

$$= \frac{n(n+1)}{2}$$

$$= O\left(n^2\right)$$

Problem2: Exercise 4.5-1 b (p96):Application of Master Theorem and Elimination Method

$$T(n) = 2T\left(\frac{n}{4}\right) + n^{0.5}$$

Solution:

Master method:

Given , a = 2 , b =4 and k = 0.5 and f(n) = $n^{0.5}$

Here , we check the values of a and b^k

2 = $4^{0.5}$

2=2 (here a = b^k)

Now, let's check the f(n) value,

$n^{0.5} = n^{2}$

$n^{0.5} = n^{0.5}$

Hence, a = b^{k} and f(n) = n^{a} it comes under case 2.

Elimination Method:

T(n) = $2T\left(\frac{n}{4}\right) + n^{0.5}$ ------------(1)

2 * [$T\left(\frac{n}{4}\right)$ = $2T\left(\frac{n}{4^2}\right) + \left(\frac{n}{4}\right)^{0.5}$] ------------(2)

2^2 * [$T\left(\frac{n}{4^2}\right)$ = $2T\left(\frac{n}{4^3}\right)$ + $\left(\frac{n}{4^2}\right)^{0\cdot5}$]-------------(3)

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2^k * [$T\left(\frac{n}{4^k}\right)$ = $2T\left(\frac{n}{4^{k+1}}\right)$ + $\left(\frac{n}{4^k}\right)^{0\cdot5}$]--------(k)

With loss of generality, k is assumed to be logn namely, n=4^{k} k = n and $n^{0.5} = 2^{k}$

T(n) = $n^{0.5} + 2 * \left(\frac{n}{4}\right)^{0.5} + 2^2 * \left(\frac{n}{4^2}\right)^{0\cdot5} +\ldots\ldots + 2^k * \left(\frac{n}{4^k}\right)^{0\cdot5}$

= $n^{0.5}\left(1 + 2 * \left(\frac{1}{4}\right)^{0.5} + 2^2 * \left(\frac{1}{4^2}\right)^{0\cdot5} +\ldots\ldots + 2^k * \left(\frac{1}{4^k}\right)^{0\cdot5}\right)$

= $n^{0.5}\left(1 + \left(\frac{2}{2}\right)^{1} + \left(\frac{4}{4}\right) +\ldots\ldots + \left(\frac{2^k}{2^k}\right)\right)$

= $n^{0.5}(1 + 1 + 1 + \ldots\ldots + 1)$

= $n^{0.5} \sum_{i=1}^{k} i$

= k * $n^{0.5}$

= $n^{0.5}\, n$

Problem 3: 4-1a, d,f, (p107): Master theorem and Elimination Method

Solution:

a) T(n) = $2T\left(\frac{n}{2}\right) + n^{4}$

Master method:

Here a = 2 b =2 and k =4 f(n) = n^4

First, we check the values of and b^k

2 = 2^4

2 < 2^4 (here a < b^k)

Now, let's check the f(n) value with n^a,

$n^4 = n^2$

$n^4 > n^1$

Hence , a < b^k and f(n) > n^a, it comes under case 3

Elimination method:

$$T(n) = 2\ T\left(\frac{n}{2}\right) + n^4$$

$$2^1[\ T\left(\frac{n}{2}\right) = 2\,T\left(\frac{n}{2^2}\right) + \left(\frac{n}{2}\right)^4]$$ --------(1)

$$2^2[\,T\left(\frac{n}{2^2}\right) = T\left(\frac{n}{2^3}\right) + \left(\frac{n}{2^2}\right)^4]$$ ---------------(2)

$$2^3[T\left(\frac{n}{2^3}\right) = T\left(\frac{n}{2^4}\right) + \left(\frac{n}{2^3}\right)^4]$$ --------------(3)

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$$2^k[T\left(\frac{n}{2^k}\right) = T\left(\frac{n}{2^{k+1}}\right) + \left(\frac{n}{2^k}\right)^4]$$ ----------(k)

Without lass of generality , k is assumed to be logn namely n=2^k then k = n

$$T(n) = n^4 + + \left(\frac{n}{2}\right)^4 ++ \left(\frac{n}{2^2}\right)^4 + + \left(\frac{n}{2^3}\right)^4 + \ldots + \left(\frac{n}{2^k}\right)^4$$

$$= n^4(1 + \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2^2}\right)^4 + \left(\frac{1}{2^3}\right)^4 + \ldots + \left(\frac{1}{2^k}\right)^4)$$

$$= n^4(1 + \frac{1}{8} + \frac{1}{8^2} + \frac{1}{8^3} + \ldots\ldots + \frac{1}{8^k})$$

$$= n^4 \sum_{k=0}^{k} \left(\frac{1}{8}\right)^i$$ (by geometric series formula we can write this as,)

$=n^4\left(\frac{1-\left(\frac{1}{8}\right)^{k+1}}{1-\left(\frac{1}{8}\right)}\right)$ (where $1-\left(\frac{1}{8}\right)=\left(\frac{7}{8}\right)$ and $\left(\frac{1}{8}\right)^{k+1}=\frac{1}{2^{3(k+1)}}=\frac{1}{2^3*2^{3k}}=\frac{1}{8*2^{3n}}$)

$=\left(\frac{8}{7}\right)n^4\left[1-\frac{1}{8*2^{3n}}\right]$

$=\left(\frac{8}{7}\right)n^4\left[1-\frac{1}{8*n^3}\right]$

$=\left(\frac{8}{7}\right)n^4-\left(\frac{1}{7}\right)$ n

Hence , T(n) = $\theta(n^4)$

d) T(n) = $7T\left(\frac{n}{3}\right)+n^2$

Solution:

Master method:

Given a =7 b =3 and k = 2 and f(n) = n^2

First, we check the values of and b^k

7 = 3^2

7 < 9 (here a < b^k)

Now, let's check the f(n) value with n^a,

$n^2 == n^7$

$n^2 > n^{1.77}$

Hence , , a < b^k and f(n) > n^a it comes under case 3

Elimination method:

T(n) = $7T\left(\frac{n}{3}\right)+n^2$ ----------(1)

7 * [$T\left(\frac{n}{3}\right)$ = 7 $T\left(\frac{n}{3^2}\right)$ + $\left(\frac{n}{3}\right)^2$] ----------(2)

7^2 * [$T\left(\frac{n}{3^2}\right)$ = 7 $T\left(\frac{n}{3^3}\right)$+$\left(\frac{n}{3^2}\right)^2$]---------(3)

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$7^k * [T\left(\frac{n}{3^k}\right) = 7\,T\left(\frac{n}{3^{k+1}}\right) + \left(\frac{n}{3^k}\right)^2]$-----(k)

With loss of generality, k is assumed to be logn namely, n=3^k k = n

T(n) = $n^2 + 7 * \left(\frac{n}{3}\right)^2 + 7^2 * \left(\frac{n}{3^2}\right)^2 + \ldots\ldots + 7^k * \left(\frac{n}{3^k}\right)^2$

$= n^2 [1 + 7 * \left(\frac{1}{3}\right)^2 + 7^2 * \left(\frac{1}{3^2}\right)^2 + \ldots\ldots + 7^k * \left(\frac{1}{3^k}\right)^2]$

$= n^2 [1 + \left(\frac{7}{9}\right)^1 + \left(\frac{7}{9}\right)^2 + \ldots\ldots + \left(\frac{7}{9}\right)^k]$

$= n^2 \sum_{k=0}^{k} \left(\frac{7}{9}\right)^i$ (by geometric series formula)

$= n^2 \left[\frac{1-\left(\frac{7}{9}\right)^{k+1}}{1-\frac{7}{9}}\right]$(where, $1 - \frac{7}{9} = \frac{9}{2}$ and k = n)

$= \frac{9}{2}n^2[1 - \left(\frac{7}{9}\right)^{n+1}]$ (where log(a+b)=log a * log b)

$= \frac{9}{2}n^2\left[1 - \left(\frac{3^{1\cdot7}}{3^2}\right)^{log_3^n} * \left(\frac{3^{1\cdot7}}{3^2}\right)\right]$(where , $3^{1\cdot7log_3^n} = n^{1\cdot7}$)

$= \frac{9}{2}n^2$ [1- $\left(\frac{n^{1\cdot7}}{n^2}\right) * \frac{7}{9}$]

$= \frac{1}{2}$ [9n^2 - 7$n^{1.77}$]

$= \theta(n^2)$

f) Refer problem 2 for answer.

Problem 4: 4.3-a in CLRS textbook. (P108) : Master and Elimination Method

T(n) = $4T\left(\frac{n}{3}\right)$ +n logn

Solution:

Master method:

Here a = 4 b =3 and k =1 f(n) = n logn

First, we check the values of and b^k

$4 = 3^1$

$4 > 3^1$ (here a > b^k)

Now, let's check the f(n) value with = n^a ,

$$nlogn = n^a$$

$$nlogn < n^{1.26}$$

Hence , , a > b^k and f(n) < n^a it comes under case 1

Elimination method:

T(n) = $4T\left(\frac{n}{3}\right)$ +n logn -------(1)

4 * [$T\left(\frac{n}{3}\right)$ = 4 $T\left(\frac{n}{3^2}\right)$ +$\frac{nlogn}{3}$] --------(2)

4^2 * [$T\left(\frac{n}{3^2}\right) = 4\,T\left(\frac{n}{3^3}\right) + \left(\frac{nlogn}{3^2}\right)$-------(3)

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4^k * [$T\left(\frac{n}{3^k}\right) = 4T\left(\frac{n}{3^{k+1}}\right) + \left(\frac{nlogn}{3^k}\right)$]-----(k)

With loss of generality, k is assumed to be logn namely, n=3^k k = n

T(n) = nlogn + 4 * $\frac{nlogn}{3} + 4^2\left(\frac{nlogn}{3^2}\right) + \ldots\ldots + 4^k * \left(\frac{nlogn}{3^k}\right)$

= nlogn(1+$\frac{4}{3} + \frac{4^2}{3^2}$+ .....+$\frac{4^k}{3^k}$)

= n logn $\sum_{k=0}^{k}\left(\frac{4}{3}\right)^k$

= $n\ logn\left[\frac{1-\left(\frac{4}{3}\right)^{k+1}}{1-\frac{4}{3}}\right]$ (where $1 - \frac{4}{3} = -\frac{1}{3}$ and k =n)

= -3n logn [1 - $\left(\frac{4}{3}\right)^{n+1}$]

= -3n logn [$1 - \left(\frac{4}{3}\right)^{log_3^n} * \left(\frac{4}{3}\right)$]

$= -3n \log n[1 - \left(\frac{3^{1\cdot26}}{3}\right)^{log_3^n} * \left(\frac{4}{3}\right)]$

$= -3n \log n[1 - \left(\frac{n^{1\cdot26}}{n}\right) * \left(\frac{4}{3}\right)]$

$= -3n \log n[\frac{3n-4n^{1\cdot26}}{3n}]$(here, 3n gets cancelled)

$= \log n[\ 4n^{1\cdot26} - 3n]$