Relating Fourier Representation of periodic & non periodic signals

(1)

FT Representation of Periodic Signals

Periodic signals ⇒ FS → FS or CTFS; → DTFS.

FT ⇒ Non periodic signals.

Bridging FT to FS.

& DTFT to DTFS.

Relating FT to FS

→ w.k.t FS of a periodic sig $x(t)$ is

$$x(t) = \sum_{k=-\infty}^{\infty} X(k) e^{jk2\pi f_0 t} = \sum_{k=-\infty}^{\infty} X(k) e^{jk\omega_0 t} \quad \text{(I)}$$

where $\omega_0 = 2\pi f_0$ is fundamental freq of signal.

→ w.k.t $1 \longleftrightarrow 2\pi\delta(\omega)$. FT of DC.

and $e^{jk\omega_0 t} \longleftrightarrow 2\pi\delta(\omega - k\omega_0)$ freq shift prop of FT. (freq shift is $k\omega_0$)

using above rel in (I) along with linearity prop ie $a x_1(t) + b x_2(t) \longleftrightarrow a X_1(k) + b X_2(k)$

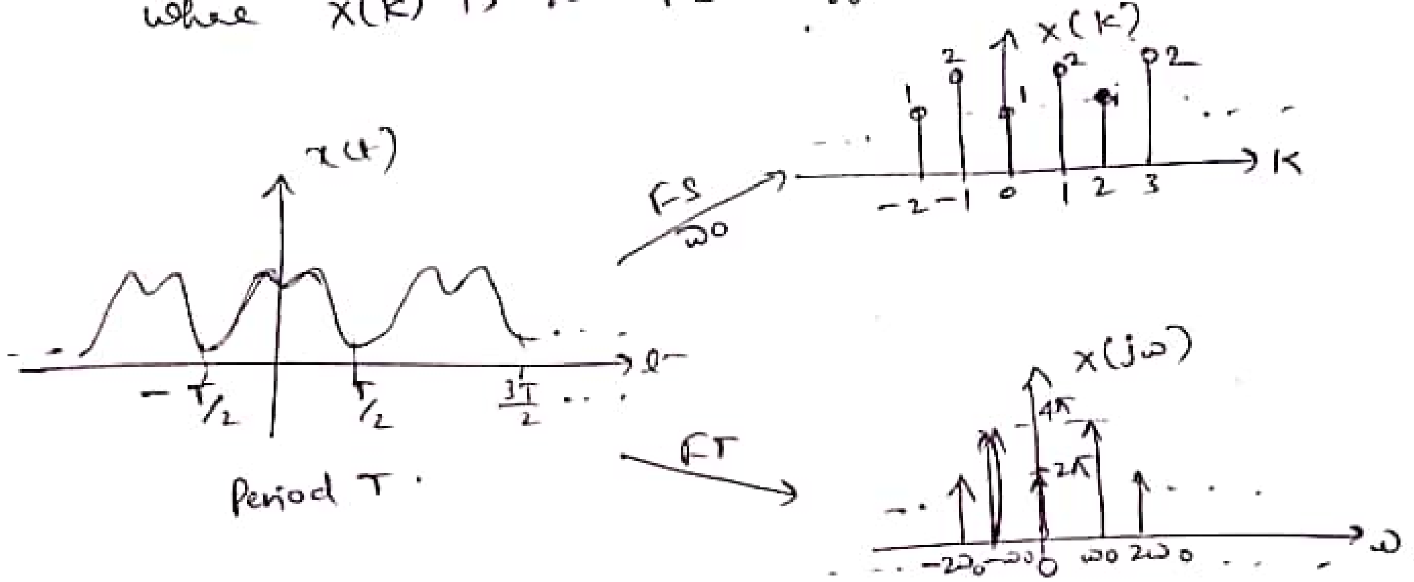
we can write

$$x(t) = \sum_{k=-\infty}^{\infty} X(k) e^{j\omega_0 k t} \xleftrightarrow{FT} 2\pi \sum_{k=-\infty}^{\infty} X(k)\delta(\omega - k\omega_0)$$

ie FT of sum of scaled complex exp with scaling factors $X(k)$ is equal to 2π sum of similar scaling factors $\delta(\omega - k\omega_0)$.

$\therefore$ FT of periodic signal is series of impulses spaced by the fundamental freq ω_0.

The k^{th} impulse has strength $2\pi X(k)$. Where $X(k)$ is k^{th} FS coefficient. as shown below.



FT of unit impulse Train

find FT of $p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT)$

Sol w.k.T $p(t)$ is periodic & its FS is given by

$$P(K) = \frac{1}{T}\int_0^T \delta(t)\, e^{-j\omega_0 K t}\, dt$$

$$= \frac{1}{T}\, e^{j0} \qquad \because \delta(t)=1 \text{ only at } t=0$$
$$=0 \quad o.w.$$

$$\therefore P(K) = \frac{1}{T}.$$

Now $P(j\omega) = \frac{2\pi}{T}\sum_{K=-\infty}^{\infty} P(K)\,\delta(\omega - K\omega_0)$

$$\boxed{P(j\omega) = \frac{2\pi}{T}\sum_{K=-\infty}^{\infty} P(K)\,\delta(\omega - K\omega_0)}$$

$||^{ly}$ find FT of

1) $x(t) = \sin \omega_0 t$.

Sol. $X(K) = \frac{1}{T_0}\int_0^{T_0} \sin\omega_0 t\, e^{-j\omega_0 t k}\, dt$

$$= \frac{1}{2jT_0}\int_0^{T_0} e^{j(1-K)\omega_0 t} - e^{-j(1+K)\omega_0 t}\, dt$$

$$= \frac{1}{j2T_0}\left[\int_0^{T_0} e^{j(1-K)\omega_0 t}\, dt - \int_0^{T_0} e^{-j(1+K)\omega_0 t}\, dt\right]$$

$$= \frac{1}{j2T_0}\left[\frac{e^{j(1-K)\omega_0 t}}{j(1-K)\omega_0}\Bigg|_0^{T_0} - \frac{e^{-j(1+K)\omega_0 t}}{-j(1+K)\omega_0}\Bigg|_0^{T_0}\right]$$

$$= \frac{1}{j^2 2T_0}\left[\frac{e^{j(1-K)\omega_0 T} - 1}{(1-K)\omega_0} + \frac{e^{-j(1+K)\omega_0 T_0} - 1}{(1+K)\omega_0}\right]$$

$$= \frac{1}{2T_0\omega_0}\left[\frac{1 - e^{j(1-K)\omega_0 T_0}}{(1-K)} + \frac{1 - e^{-j(1+K)\omega_0 T_0}}{(1+K)}\right]$$

Consider $\dfrac{1-e^{+j\omega_0(1-k)T_0}}{1-k}$.

This term $= \frac{0}{0}$ when $k=1$

& when $k \neq 1 \Rightarrow$. $\dfrac{1-e^{+j\omega_0 T_0 m}}{m}$ $m = 1-k$ are then integer.

$$= \frac{1-e^{+j\frac{2\pi}{T_0}T_0 m}}{m}$$

$$= \frac{1-e^{+j2\pi m}}{m} = \frac{1-1}{m} = 0 \text{ for any } m$$

$\therefore$ when $k=1$ using LH rule

$$\lim_{k\to 1} \frac{\frac{d}{dk}[Nr]}{\frac{d}{dk}[Dr]} = \lim_{k=1} \frac{0 - e^{+j\omega_0 T_0(1-k)} \times -j\omega_0 T_0}{0-1}$$

$$= -j\omega_0 T_0 = -j2\pi.$$

|||ly $\dfrac{1-e^{-j(1+k)\omega_0 T_0}}{1+k} = +j2\pi$ when $k=-1$

$$\therefore x(k) = \frac{1}{2\times 2\pi}\left[-j2\pi\,\delta(k-1) + j2\pi\delta(k+1)\right]$$

$$= \frac{j}{2}\left[-\delta(k-1) + \delta(k+1)\right] = \frac{1}{2j}\left[\delta(k-1) - \delta(k+1)\right]$$

$$\boxed{x(k)' = \frac{1}{2j}\delta(k-1) - \frac{1}{2j}\delta(k+1)}$$

can be taken directly also all steps of $x(k)$ are shown for under-standing

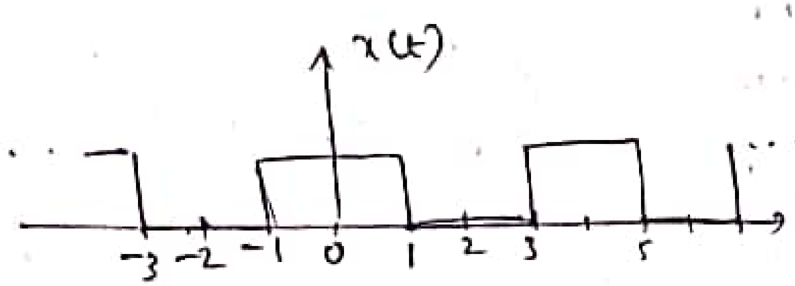
$\therefore$ FT of $x(t)$ is

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X(k)\,\delta(\omega - k\omega_0)$$

$$= 2\pi \sum_{k=-1,+1} X(k)\,\delta(\omega-\omega_0 k) = \frac{2\pi}{2j}\Big[\delta(\omega-k\omega_0)\,[\delta(k-1)-\delta(k+1)]\Big]$$

$$\boxed{X(j\omega) = \frac{\pi}{j}\left[\delta(\omega-\omega_0) - \delta(\omega+\omega_0)\right]}$$

2)



$x(t) = 1 \quad -1 \le t \le 1$
$\quad\;\; = 0 \quad -2 \le |t| \le -1$ } over one period

$$\therefore X(K) = \frac{2\sin(2\pi T_0/T)}{2K\pi} \quad (\text{or}) \quad \frac{2\sin(K\omega_0 T_0)}{KT\omega_0}$$

here $T = 4s$, $T_0 = 1\,sec$, $\therefore \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{4} = \pi/2$

$$\therefore X(K) = \frac{2\sin K\frac{\pi}{2}\cdot 1}{K\times 4\times\frac{\pi}{2}} = \frac{2\sin(K\pi/2)}{2K\pi}$$

$$X(K) = \frac{\sin(K\pi/2)}{K\pi}$$

Now $X(j\omega) = 2\pi \sum_{K=-\infty}^{\infty} X(K)\,\delta(\omega - K\omega_0)$

$$= 2\pi \sum \frac{\sin(K\pi/2)}{K\pi}\,\delta(\omega - K\omega_0)$$

$$\boxed{X(j\omega) = \sum_{K=-\infty}^{\infty} \frac{2\sin(K\pi/2)}{K}\,\delta\left(\omega - \frac{K\pi}{2}\right)}$$

iii) FT of $x(t) = \sin(\pi t)$

$$\Rightarrow X(j\omega) = \sum_{K=-\infty}^{\infty} \frac{4}{1-4K^2}\,\delta\left(\omega - K\frac{\pi}{2}\right)$$

Scanned with Cam

Reln between DTFT & DTFS

FT → CT NP signal

DTFS → DT periodic signal

ie to find DTFT of DT periodic signal.

w.k.T DTFS representation of a signal $x(n)$ is given by

$$x(n) = \sum_{K=0}^{N-1} X(K) e^{j\Omega_0 K n}$$

Also DTFT of signal multiplied by complex exp gives freq shift

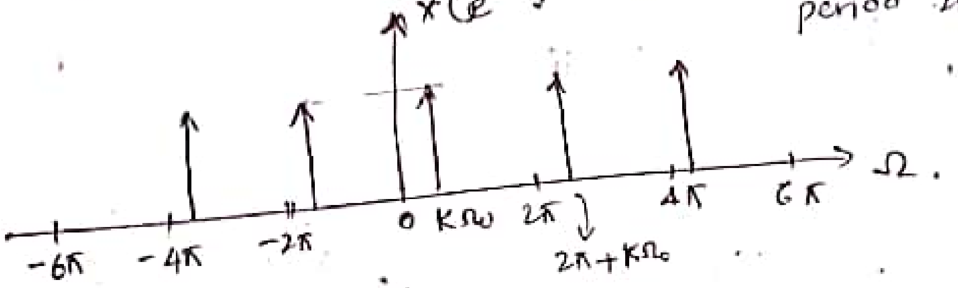
ie. $e^{j\Omega_0 K n} \xleftrightarrow{DTFT} \delta(\Omega - K\Omega_0)$. ∵ $1 \longleftrightarrow \delta(\Omega)$

$K\Omega_0$ ⇒ ∵ DTFT is periodic in freq domain with 2π

$$\therefore \quad e^{j\Omega_0 K n} \xleftrightarrow{DTFT} \sum_{m=-\infty}^{\infty} \delta(\Omega - K\Omega_0 - m2\pi) \quad \text{—(I)}$$

~~Relating DTFT to DTFS~~

This is shown in fig below. → Periodic train of pulses of period 2π shifted by $K\Omega_0$.



The inverse DTFT of above eqn is given by

$$\frac{1}{2\pi} e^{jK\Omega_0 n} \longleftrightarrow \sum_{K=-\infty}^{\infty} \delta(\Omega - K\Omega_0 - m2\pi) \quad \text{(II)}$$

(using sifting prop of impulse fun.)

Hence complex sinusoid and freq shifted impulses are DTFT pair.

Now using linearity and eqn. (II) we can find DTFT of periodic signal $x(n)$ as

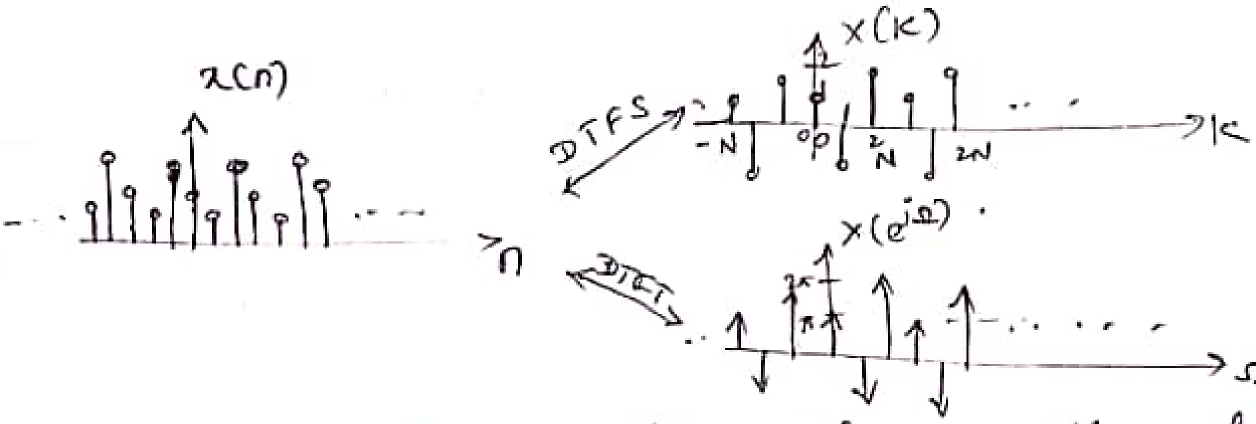
$$x(n) = \sum_{k=0}^{N-1} X(k) e^{jk\Omega_0 n} \xleftrightarrow{DTFT} X(e^{j\Omega}) = 2\pi \sum_{k=0}^{N-1} X(k) \sum \delta(\Omega - k\Omega_0 - m2\pi)$$

— (III)

Since $X(k)$ is periodic and $N\Omega_0 = 2\pi$. we can combine 2 Σ in (III) and rewrite.

$$x(n) = \sum_{k=0}^{N-1} X(k) e^{jk\Omega_0 n} \xleftrightarrow{DTFT} X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X(k)\, \delta(\Omega - k\Omega_0)$$

∴ DTFT of periodic signal is a periodic series of impulses spaced by the fundamental freq Ω_0. The k^{th} impulse has the strength $2\pi X(k)$. where $X(k)$ is k^{th} DTFS coefficient of $x(n)$.



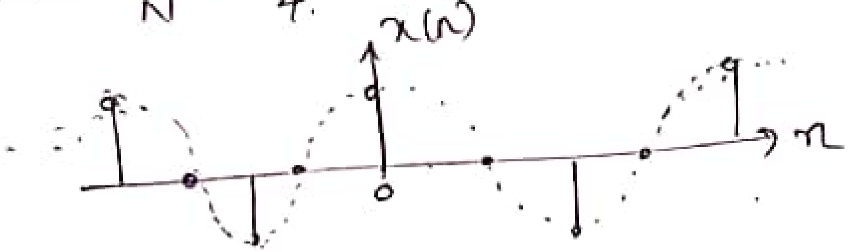
⇒ ie DTFT of periodic signal $x(n)$ with fundamental freq. $\Omega_0 = \frac{2\pi}{N}$, period N, and DTFS coefficients $X(K)$ is given by placing impulses at $K\Omega_0$ scaled by 2π times DTFS coefficients $X(K)$

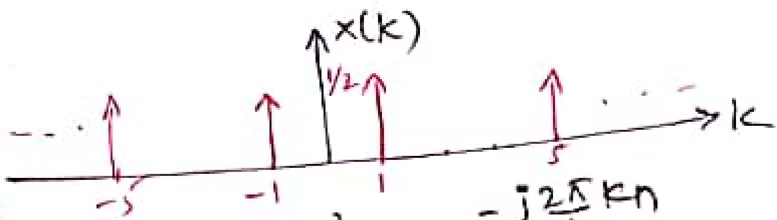
⇒ i.e. similarly if DTFT contains train of impulses then DTFS coefficients are obtained by dividing amplitudes of impulses by 2π. The spacing b/w impulses is fundamental freq of $x(n)$.

Ex Find DTFT representation of $x(n) = \cos(\Omega_0 n)$

$$\Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{4}.$$

Soln.





$$X(K) = \frac{1}{4}\sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi}{4}Kn} = \frac{1}{8}\sum_{n=0}^{3} e^{j\Omega_0(1-K)n} + \frac{1}{8}\sum_{n=0}^{3} e^{-j\Omega_0(1+K)n}$$

$$= \frac{1}{2}\left[\delta(K-1) + \delta(K+1)\right]$$

$$\therefore\ X(e^{j\Omega}) = \text{DTFT}\,[x(n)] = 2\pi \sum_{k=-\infty}^{\infty} X(k)\, \delta(\Omega - k\Omega_0)$$

$$= 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{2}\,[\delta(k-1) + \delta(k-1)]\, \delta(\Omega - k\Omega_0).$$

exists only for $k=1$ & $k=-1$.

$$= \frac{2\pi}{2}\left[\delta(k-1)\,\delta(\Omega - k\Omega_0)\Big|_{k=1} + \delta(k+1)\,\delta(\Omega - k\Omega_0)\Big|_{k=-1}\right]$$

$$= \pi\,[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)].$$

$\therefore$ DTFT of periodic signal $x(n) = \cos(\Omega_0 n)$

is $\pi\,[\delta(\Omega - \Omega_0) + \delta(\Omega + \Omega_0)]$.

∴ Reln B/w FT & FS i.e Fourier Transform & representation of periodic signals is

FT → FS

$$x(t) = \sum_{k=-\infty}^{\infty} X(k)\, e^{j\omega_0 k t} \xleftrightarrow{FT} 2\pi \sum_{k=-\infty}^{\infty} X(k)\,\delta(\omega - k\omega_0)$$

DT FT → DT FS

$$x(n) = \sum_{k=0}^{N-1} X(k)\, e^{+j\Omega_0 k n} \xleftrightarrow{FT} 2\pi \sum_{k=-\infty}^{\infty} X(k)\,\delta(\Omega - k\Omega_0)$$

Convolution and multiplication of mixture of periodic & nonperiodic signals

If a sym has impulse resp $h(n)$ & i/p $x(n)$ as ($h(t)$ & $x(t)$) shown below.

Ex: To find resp of sym having $h(n)$ as rectangular pulse of &To for i/p as sinusoidal signal.

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

(or)

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

The response of sym can be obtained by

$y = x * h$.

if x is periodic & h is nonperiodic (or)

if x is nonperiodic & h is periodic

and if we wish to use FT tool to find response of system then we need to

→ find FT of Non Periodic signal.

→ find FT equivalent of periodic signal

→ use convolution prop of FT

ie $x * h \longleftrightarrow X(j\omega)\, H(j\omega)$

to find $Y(j\omega)$ and $y(n)$ is obtained by taking inverse transform of $Y(j\omega)$.

In general if we wish to find convolution of two signals among which one is periodic & other is non periodic then represent periodic sig. using FT & ~~multiply~~ FT of other signal to get

FT equivalent of convolution. & then find inverse to find convoln in time domain

$\|^{ly}$ if It is required to multiply two time domain signals among which one is periodic & other is non periodic use same procedure as above to get FT equivalent of multiplication & then take inv FT to find time domain multn

ie $x(t)h(t) \xleftrightarrow{FT} \frac{1}{2\pi} X(j\omega) * h(j\omega)$

$x(n)h(n) \xleftrightarrow{DTFT} \frac{1}{2\pi} X(e^{j\Omega}) * H(e^{j\Omega})$

To summarize convolution and modulation of mixed signals

CT=> $y(t) = x_p(t) * h(t) \xleftrightarrow{FT} 2\pi \sum_{k=-\infty}^{\infty} X(k)\delta(\omega - k\omega_0) \times H(j\omega)$

conv

ie $Y(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X(k) H(jk\omega) \delta(\omega - k\omega_0)$

$\|^{ly}$ DT=> $y(n) = x_p(n) * h(n) \xleftrightarrow{DTFT} Y(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X(k) H(e^{jk\Omega_0}) \delta(\Omega - k\Omega_0)$

conv

lly **For multiplication**

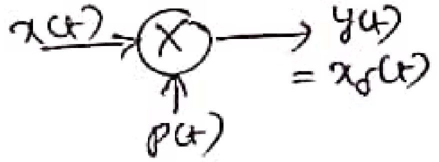
CT mult $\Rightarrow$ $y(t) = x_p(t)\, g(t) \xleftrightarrow{FT} \sum_{k=-\infty}^{\infty} X(k)\, G(j(\omega - k\omega_0))$

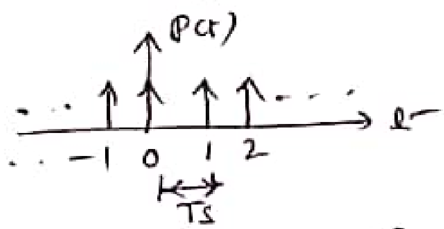
and DT mult $\Rightarrow$ $y(n) = x_p(n)\, g(n) \xleftrightarrow{DTFT} \sum_{k=0}^{N-1} X(k)\, G(e^{j(\Omega - k\Omega_0)})$.

Using above prop/concept

In sampling process where $x_\delta(t) = x(t)\, p(t)$

where $p(t)$ is periodic train of impulses & $x(t)$ is a non periodic signal





$y(t) = x_\delta(t) = x(t)\, p(t)$

$X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * P(j\omega)$

$P(k) = \frac{1}{T_s} \int_0^{T_s} \delta(t)\, e^{-j\omega_0 k t}\, dt$

$= \frac{1}{T_s} e^{j0} = \frac{1}{T_s}.$

freq $= \omega = 2\pi f$

$\omega_s = 2\pi f_s$

$= \frac{2\pi}{T_s}.$

$\therefore P(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} P(k)\, \delta(\omega - k\omega_s)$

$= 2\pi \sum_{k=-\infty}^{\infty} \frac{1}{T_s}\, \delta(\omega - k\omega_s)$

$\therefore X_\delta(j\omega) = \frac{1}{2\pi} X(j\omega) * \sum_{k=-\infty}^{\infty} \frac{2\pi}{T_s}\, \delta(\omega - k\omega_s)$

ie $X_\delta(j\omega) = \frac{2\pi}{2\pi T_s} \sum_{k=-\infty}^{\infty} X(j\omega) * \delta(\omega - k\omega_s)$

$$\boxed{X_\delta(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))}$$

(or) Directly using

$$x_\delta(t) = p(t)\, x(t) \xleftrightarrow{FT} \sum_{k=-\infty}^{\infty} P(k)\, X(j(\omega - k\omega_0)) = X_\delta(j\omega)$$

$$X_\delta(j\omega) = \sum_{k=-\infty}^{\infty} P(k)\, X(j(\omega - k\omega_0))$$

$$= \sum_{k=-\infty}^{\infty} \frac{1}{T_s}\, X(j(\omega - k\omega_0))$$

ie $$\boxed{X_\delta(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_0))}$$

Scanned with Cam