

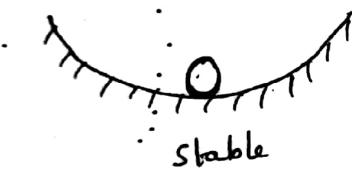
Unit 4: Stability Analysis

For a control system designer's main concern is whether designed system is stable or unstable because, stability is a very important characteristic of any control system. Almost every working system is designed to be stable. Thus, study of whether the system is stable or unstable is known as stability analysis.

"A system is said to be stable if its output is bounded for any bounded input". [BIBO]

→ Concept of stability

Consider circular object (ball) resting on different types of surfaces under the presence of gravitational force as shown in fig.



For analysis and design purpose, stability of the system can be classified as:

(1) Absolute stability

Absolute stability refers to the condition of whether the system is stable or unstable; it is a YES or NO answer.

(2) Relative stability

Relative stability is the quantitative measure of how stable is the system and this is the degree of stability.

Terms used

(1) stable system [BIBO]

A system is said to be stable if the output is bounded for any bounded input. Thus a system is said to be in equilibrium if the output eventually comes back to its equilibrium state when the system is subjected to a disturbance.

(2) Unstable system

A system is said to be unstable if the output is unbounded for a bounded input. It means with zero input the output may increase indefinitely.

(3) Absolute stable

A system is said to be absolutely stable w.r.t. a parameter of the system if it is stable for all values of these parameters.

(4) Conditionally stable

A system is said to be conditionally stable with respect to a parameter of the system if it is stable only for a certain bounded range of values of these parameters.

(5) Marginally stable (critically stable)

A system is said to be marginally stable if the system is just on the verge of becoming unstable.

(6) Relative stability

It is quantitative measure of the degree of stability, determines how stable is the system and how close it is to being unstable.

→ Time response from pole-zero plot

for all the examples shown below, the input to the system is assumed to be unit impulse function.

ex: 1

$$\text{T.F} \quad \frac{C(s)}{R(s)} = \frac{1}{s+1}$$

$$\therefore \text{o/p} \quad C(s) = \frac{1}{s+1} \quad R(s)$$

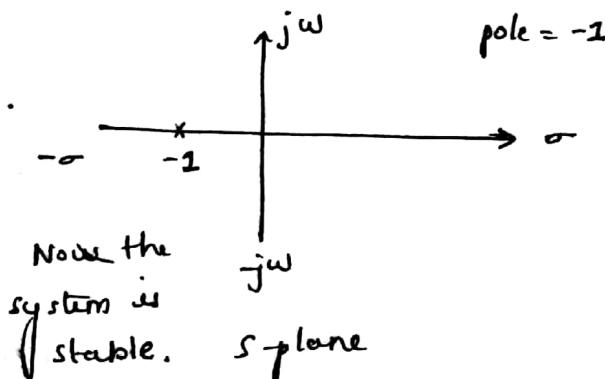
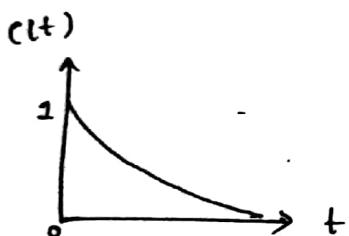
for unit impulse input. $R(s) = 1$.

$$\therefore C(s) = \frac{1}{s+1}$$

Taking Inverse L.T.

$$C(t) = \cancel{t} e^{-t}$$

$$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$$



X
o

ex: 2

$$\text{T.F} \quad \frac{C(s)}{R(s)} = \frac{s+2}{(s+1)(s+3)} \begin{array}{l} \rightarrow \text{Zero} \\ \rightarrow \text{pole} \end{array}$$

$$\text{o/p} \quad C(s) = \frac{(s+2)}{(s+1)(s+3)} \cdot R(s)$$

$s=2$

$$(s+1)(s+3)$$

$$s=-1 \quad s=-3$$

for unit ~~impulse~~ input $R(s) = 1$ $\omega(t) = f(t)$

$$R(s) = 1$$

$$\therefore C(s) = \frac{s+2}{(s+1)(s+3)}$$

using partial fractions.

$$c(s) = \frac{A}{s+1} + \frac{B}{s+3}$$

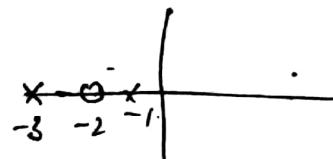
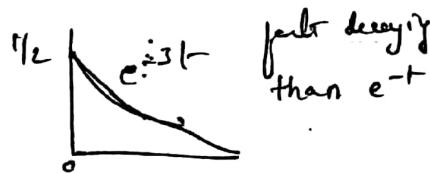
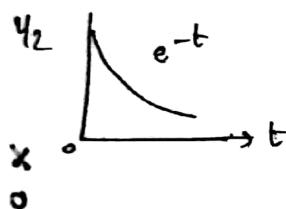
$$\begin{aligned} s+2 &= A(s+3) + B(s+1) \\ s=-3 &\quad -1 = B(-2) \\ &B = 1/2 \end{aligned} \quad \left| \begin{array}{l} s=-1 \\ 1 = 2A + 0 \\ A = 1/2 \end{array} \right.$$

$$\therefore A = 1/2, B = 1/2$$

$$c(s) = \frac{1/2}{(s+1)} + \frac{1/2}{(s+3)} \quad s = \sigma + j\omega$$

Taking inverse Laplace transform

$$c(t) = 1/2 e^{-t} + 1/2 e^{-3t}$$



This is stable system.
pole zero plot.

Ex: 3.

$$T.F. = \frac{C(s)}{R(s)} = \frac{1}{s-1}$$

$$\therefore C(s) = \frac{1}{s-1} R(s) \quad R(s) = 1$$

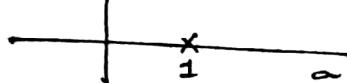
$sX \xrightarrow{s=0}$



$$\therefore C(s) = \frac{1}{s-1}$$

taking inverse L.T.

$$c(t) = e^t \cdot \text{splane}$$



$$\frac{s-1}{s-1}$$

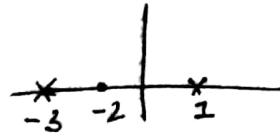
system is
unstable

Ex: 4

$$\text{T.F} \quad \frac{C(s)}{R(s)} = \frac{s+2}{(s-1)(s+3)}$$

$$R(s) = 1$$

$$C(s) = \frac{s+2}{(s-1)(s+3)}$$



$$s+2 = \frac{A}{s-1} + \frac{B}{s+3}$$

$$s+2 = A(s+3) + B(s-1)$$

$$\begin{array}{l|l} s=1 & 3 = A(4) + 0 \\ & A = 3/4 \\ \hline s=-3 & -1 = 0 + (-4B) \\ & B = 1/4 \end{array}$$

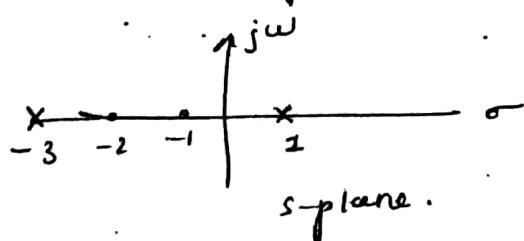
$$C(s) = \frac{3/4}{s-1} + \frac{1/4}{s+3}$$

Inverse L.T.

$$c(t) = \frac{3}{4} e^t + \frac{1}{4} e^{-3t}$$



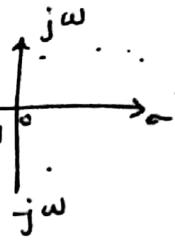
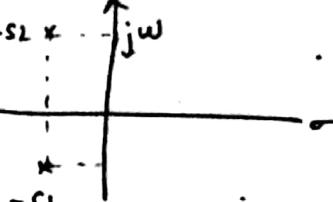
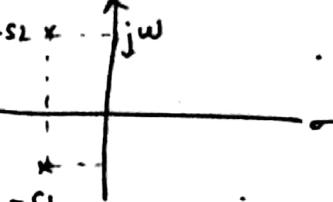
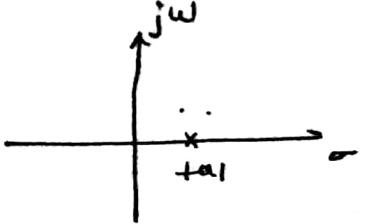
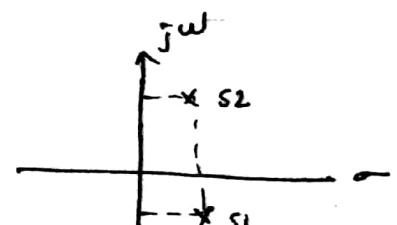
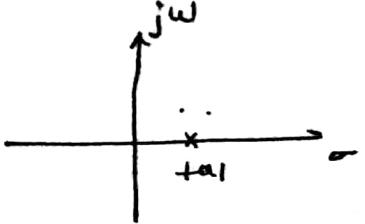
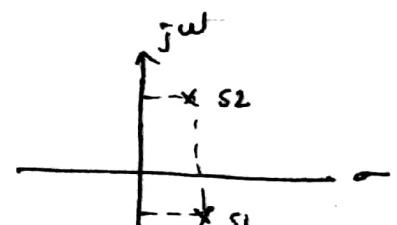
∴ The system is unstable.



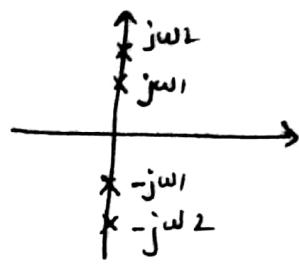
Note: (1) If all the poles of a system lie on the left of the s-plane it is a stable system

(2) If any one pole lie on the right of s plane then it is an unstable system.

- (3) Converging systems are stable
- (4) Diverging systems are unstable
- (5) The characteristics of a given system depends on the location of the poles.
- (6) The location of zero's decides the starting magnitude of the signal.
Closed loop poles = Roots of characteristic equation

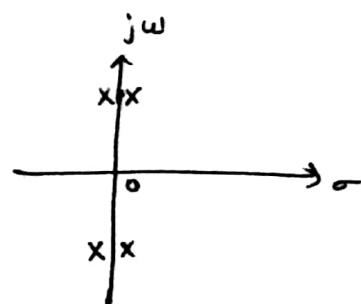
<u>Nature of Root</u>	<u>s-plane</u>	<u>z-plane</u>	<u>stability condition</u>
(1) Real; negative i.e. all the roots are in left half of s-plane			Absolutely stable
(2) Complex conjugate with negative real part i.e. L.H.S of s-plane:			Absolutely stable
(3) Real, positive i.e. R.H.S of s-plane (any one closed loop pole in right half irrespective of number of poles in left half of s-plane)			unstable
(4) Complex conjugate of positive real part			unstable

- (5) Non-repeated pair on Imaginary axis without any poles in R.H.S of s-plane



Marginally stable.

- (6) Repeated pair q on Imaginary axis without any pole in R.H.S of s-plane



unstable.

• Necessary condition for system to be stable

(1) If all the roots of the characteristic equation have a negative real part then system is stable.

(2) If any roots of the characteristic equation have a positive real part or if there is any repeated root on the Imaginary axis of s-plane then the system is said to be unstable.

(3) If some of the roots of the characteristic equation have a negative real parts and remaining roots are on the Imaginary axis then the system is said to be marginally stable

\Rightarrow Routh's Hurwitz Criterion (RH Criterion)

- It is an algebraic method of determining the stability of linear time invariant system.
- This criterion provides information about whether system is stable or not depending upon the position of the roots of the characteristic equation whether they lies on left half of s-plane or right half of s-plane.
- It deals with absolute stability.

Limitations

Using RH criterion even though it is possible to obtain the information regarding the stability of feedback control system, it is not possible to obtain complete information regarding the relative stability of system.

- Relative stability (graphical method)
 - (1) Root locus technique
 - (2) Bode plot
 - (3) Nyquist plot.
- Graphical methods will give better information regarding relative stability
- All the above methods are used to find the stability of closed loop system without actually solving for the roots of characteristic equation.

RH Criterion

The characteristic equation of the system is given by

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n = 0$$

Thus the roots of the characteristic equation are the closed loop poles of the system which decide the stability of the system.

a_0, a_1, a_2 are real coefficients

Necessary Condition

- (1) all the coefficients of the polynomial have the same sign
- (2) There should be not be any missing coefficients in the given characteristic equation.

Hurwitz Criterion

The sufficient condition for having all the roots of characteristic equation in left half of s-plane is given by hurwitz. It is referred as hurwitz criterion

$$H = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2n-1} \\ a_0 & a_2 & a_4 & \cdots & a_{2n-2} \\ 0 & a_1 & a_3 & \cdots & a_{2n-3} \\ 0 & a_0 & a_2 & \cdots & a_{2n-4} \\ 0 & 0 & a_1 & \cdots & a_{2n-5} \\ \vdots & & \vdots & & \vdots \\ 0 & & & \ddots & a_n \end{vmatrix}$$

$$D_1 = \begin{vmatrix} a_1 \end{vmatrix}, \quad D_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}, \quad D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}$$

for the system to be stable, all the above determinants must be positive.

ex: $s^3 + s^2 + s + 4 = 0$ determine stability of the given characteristic equation by Hurwitz method.

$$a_0 = 1, \quad a_1 = 1, \quad a_2 = 1, \quad a_3 = 4, \quad n = 3$$

$$H = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix}$$

$$D_1 = |1| = 1, \quad D_2 = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = 1 - 4 = -3$$

$$D_K = |H|$$

$$D_3 = \begin{vmatrix} 1 & -1 & 1 \\ 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix} = 4 - 16 = -12$$

$$4 - 4 \times \frac{1}{4} = -12$$

As D_2 , & D_3 are negative, given system is unstable.

ex: 2 $s^3 + s^2 + s + 4 = 0 \rightarrow$ missing term s^2 .

System is unstable.

disadvantage of Hurwitz method.

- (1) For higher order of system, the determinants of higher order is very complicated and time consuming.
- (2) Difficult to predict marginal stability of the system.
- (3) Number of roots located in right half of s plane for unstable system cannot be judged by this method.

Due to this limitation, a new method is suggested by scientist Routh called "Routh method". It is also called "Routh-Hurwitz method".

→ Routh's stability criterion

It is also called Routh's array method or RHT method. Routh suggested a method of tabulating the coefficients of characteristic equation in a particular way. Tabulation of coefficients gives an array called "Routh's array".

Array

s^n	a_0	a_2	a_4	\dots
s^{n-1}	a_1	a_3	a_5	\dots
s^{n-2}	b_1	b_2	b_3	
s^{n-3}	c_1	c_2	c_3	
s^0	\vdots			
	a_n			

Note:

The coefficients of any row are corresponding to alternate powers of s starting from the power indicated against it.

→ Coefficients for first 2 rows are written directly from characteristic equation.

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n = 0$$

→ From these 2 rows, next rows can be obtained as follows.

$$b_1 = \frac{\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}}{a_1} = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}}{a_1} = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

from 2nd & 3rd row, 4th row can be obtained by

$$c_1 = \frac{\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}}{b_1} = \frac{b_1 a_3 - a_1 b_2}{b_1}$$

$$c_2 = \frac{\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}}{b_1} = \frac{b_1 a_5 - a_1 b_3}{b_1}$$

This process is to be continued till the coefficient for s^0 is obtained which will be a_n .

\rightarrow Routh Criterion

The necessary and sufficient condition for system to be stable is "All the terms in the first column of Routh's Array must have same sign. There should not be any sign change in first column of Routh's Array."

If there is any sign changes existing then,

(a) System is unstable

(b) The number of sign changes equals the number of roots lying in right half of the S-plane.

$$\text{Ex: (1)} \quad s^3 + 6s^2 + 11s + 6 = 0$$

$$a_0 = 1, \quad a_1 = 6, \quad a_2 = 11, \quad a_3 = 6 \quad n = 3$$

s^3	1	11	
s^2	6	6	
s^1	$\frac{11 \times 6 - 6}{6} = 10$	0	
s^0	6		

$$\begin{array}{cc} 6 & 6 \\ 10 & 0 \end{array}$$

As there is no sign change in first column, system is stable

$$\begin{array}{c} 10 \times 6 - 0 \\ \hline 10 \end{array}$$

$$\frac{60}{10} = 6$$

$$\text{Ex: (2)} \quad s^3 + 4s^2 + s + 16 = 0$$

$$a_0 = 1, \quad a_1 = 4, \quad a_2 = 1, \quad a_3 = 16$$

s^3	1	1	
s^2	4	16	
s^1	$\frac{4 \times 16 - 1}{4} = -3$	0	
s^0	$\frac{-3 \times 16 - 0}{-3} = \frac{48}{-3} = -16$	0	

As there are 2 sign changes, system is unstable.

Number of roots located in right half of s -plane

$$= \text{Number of sign changes} = 2.$$

→ Special cases of Routh's Criterion

Special Case 1

First element of any of the rows of Routh's array is zero and the same remaining row contains at least one non-zero element.

effect: The terms in the new row become infinite and Routh test fails.

$$\text{ex: } s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$$

$$a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 6, a_4 = 2, a_5 = 1, n = 5$$

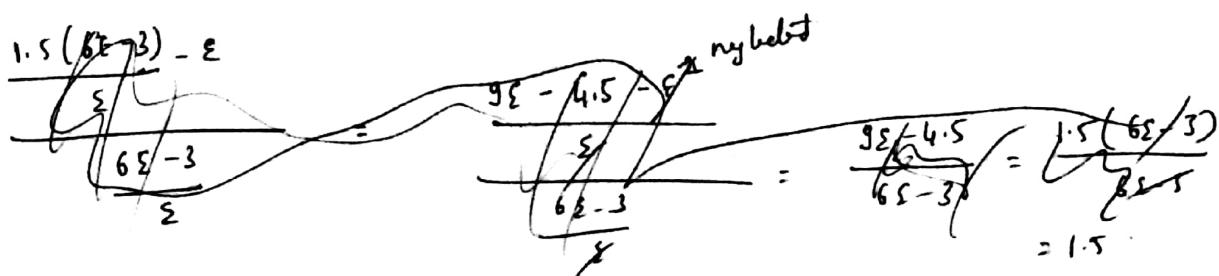
s^5	1	3	2		$\gamma_0 = \infty$
s^4	2	6	1		
s^3	0	1.5	0		
s^2	0	...		← Routh's array failed.	$b_2 = \frac{2 \times 2 - 1}{2} = \frac{4-1}{2} = 1.5$

I method

Substitute a small positive number ' ϵ ' in place of a zero occurred as a first element in a row complete the array with this number ' ϵ '. Then examine the sign change by taking $\lim_{\epsilon \rightarrow 0}$.

Consider above example,

s^5	1	3	2	
s^4	2	6	1	
s^3	ϵ	1.5	0	
s^2	$\frac{6\epsilon - 3}{\epsilon}$	1	0	
s^1	$\frac{1.5(6\epsilon - 3) - \epsilon}{\epsilon}$	0	0	
s^0	$\frac{6\epsilon - 3}{\epsilon}$			
	1	0	0	



To examine sign change,

$$\lim_{\varepsilon \rightarrow 0} \left[\frac{6\varepsilon - 3}{\varepsilon} \right] = 6 - \lim_{\varepsilon \rightarrow 0} \frac{3}{\varepsilon}$$

$$= 6 - \infty = -\infty \text{ sign is negative.}$$

(change in sign + d-ve)

$$\lim_{\varepsilon \rightarrow 0} \frac{1.5(6\varepsilon - 3) - \varepsilon^2}{6\varepsilon - 3} = \lim_{\varepsilon \rightarrow 0} \frac{9\varepsilon - 4.5 - \varepsilon^2}{6\varepsilon - 3}$$

$$= \frac{0 - 4.5 - 0}{0 - 3}$$

$$= +1.5 \text{ sign is positive}$$

s^5	1	3	2
s^4	2	6	1
s^3	ε	1.5	0
s^2	$-\infty$	1	0
s^1	+1.5	0	0
s^0	1	0	0

As there are 2 sign changes, system is unstable.

Method II

To solve the above difficulty one more method can be used. In this replace, 's' by $1/z$ in original equation. Taking LCM, rearrange characteristic equation in descending powers of 'z'. Then complete the Routh's array with this new equation in 'z' & examine the stability with this array.

(consider) $F(s) = s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$

$$s = 1/z$$

$$\frac{1}{z^5} + \frac{2}{z^4} + \frac{3}{z^3} + \frac{6}{z^2} + \frac{2}{z} + 1 = 0$$

\curvearrowleft throughout by z^5 and writing in reverse order

z^5	1	6	2	
z^4	2	3	1	
z^3	$\frac{6 \times 2 - 3}{2} = 4.5$	$\frac{2 \times 2 - 1}{2} = 1.5$	0	
z^2	$\frac{4.5 \times 3 - 2 \times 1.5}{4.5} = 2.33$	$\frac{4.5}{4.5} = 1$	0	
z^1	$\frac{2.33 \times 1.5 - 4.5}{2.33} = -0.429$	0		
z^0	1			

~~$-0.429(1) - 2.33(0)$~~
 ~~-0.429~~
 $= 1$

As there are 2 sign changes, system is
unstable. The result is same.

→ Special Case 2 :-

All the elements of a row in a Routh's array are zero.

Effect: The terms of the next row cannot be determined and Routh's test fails.

<u>ex:</u>	s^5	a	b	c	
	s^4	d	e	f	
	s^3	[0 0 0]	← Row of zero's.		

∴ This indicates Non-availability of coefficient in that row

5

procedure to eliminate this difficulty

- (1) Form an equation by using the coefficients of a row which is just above the row of zero's. Such an equation is called Auxiliary equation denoted as $A(s)$.

$$A(s) = ds^4 + es^2 + f$$

- (2) Take the derivative of an auxiliary equation w.r.t s

$$\frac{dA(s)}{ds} = 4d s^3 + 2e s$$

- (3) Replace row of zero's by the coefficients of $\frac{dA(s)}{ds}$

s^5	a	b	c
s^4	d	e	f
s^3	4d	2e	0

- (4) complete the array in terms of these new coefficients.

→ Advantages of Routh's Criterion

- (1) stability of the system can be judged without actually solving characteristic equation
- (2) No evaluation of determinants, which saves calculation time.
- (3) For unstable system it gives number of roots of characteristic equation having positive real part.
- (4) By using this criterion, critical value of system gain can be determined hence frequency of sustained oscillations can be determined.

- (5) It helps in finding out range of values of K for system stability
- (6) It helps in finding out intersection points of root locus with Imaginary axis.

→ Limitation of Routh's Criterion

(1) It is valid only for real coefficients of the characteristic equation

(2) Applicable only for linear system

(3) It does not suggest methods of stabilizing an unstable system.

(4) It does not provide exact locations of the closed loop poles in left or right half of s -plane

problems

$$(1) s^5 + s^4 + 2s^3 + 2s^2 + 3s + 15 = 0$$

s^5	1	2	3
s^4	1	2	15
s^3	0	$\frac{3-15}{1} = -12$	0

Replace 0 by small positive number ϵ

s^5	1	2	3
s^4	1	2	15
s^3	$\frac{3-15}{\epsilon}$	ϵ	-12
s^2	$\frac{2\epsilon+12}{\epsilon}$	$\frac{15\epsilon-0}{\epsilon} = 15$	0

$$\begin{array}{c|ccc} s^1 & \frac{\left(\frac{2\varepsilon+12}{\varepsilon}\right)(-12) - [15\varepsilon]}{\left(\frac{2\varepsilon+12}{\varepsilon}\right)} & 0 & 0 \\ \hline s^0 & 15 & & \end{array}$$

$$V_0 = \infty$$

$$\lim_{\varepsilon \rightarrow 0} \frac{2\varepsilon+12}{\varepsilon} = 2 + \frac{12}{\varepsilon} = +\infty$$

$$\lim_{\varepsilon \rightarrow 0} \frac{\left(\frac{2\varepsilon+12}{\varepsilon}\right)(-12) - 15\varepsilon}{\frac{2\varepsilon+12}{\varepsilon}} = \lim_{\varepsilon \rightarrow 0} \frac{-24\varepsilon - 144 - 15\varepsilon^2}{2\varepsilon+12} = \frac{0 - 144 - 0}{12} = -12$$

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 3 \\ s^4 & 1 & 2 & 15 \\ s^3 & \varepsilon & -12 & 0 \\ s^2 & +\infty & 15 & 0 \\ s^1 & -12 & 0 \\ s^0 & 15 & \end{array}$$

There are 2 sign changes, so system is unstable.

$$(2) \quad s^6 + 4s^5 + 3s^4 + -16s^2 - 64s - 48 = 0$$

find the number of roots of this equation with positive real part, zero real part, & negative real part.

$$a_0 = 1 \quad a_2 = 3 \quad a_4 = -16 \quad a_6 = -48$$

$$a_1 = 4 \quad a_3 = 0 \quad a_5 = -64$$

$$\begin{array}{c|cccc}
s^6 & 1 & 3 & -16 & -48 \\
s^5 & 4 & 0 & -64 & 0 \\
s^4 & 12/4 = 3 & 0 & \frac{-16 + 64}{4} = & \\
s^3 & 0 & 0 & \frac{48 - 0}{4} = -48 & \leftarrow \text{special case 2}
\end{array}$$

$$A(s) = 3s^4 - 48 = 0$$

$$\frac{dA}{ds} = 12s^3$$

$$\begin{array}{c|ccccc}
s^6 & 1 & 3 & -16 & -48 & \\
s^5 & 4 & 0 & -64 & 0 & \\
s^4 & 3 & 0 & -48 & 0 & \\
s^3 & 12 & 0 & 0 & 0 & \leftarrow \text{Auxiliary eqn.} \\
s^2 & 0[s] & \frac{12(-48)}{12} = -48 & 0 & 0 & \\
s^1 & 0 & \frac{0 - (-48)(12)}{\varepsilon} = \frac{576}{\varepsilon} & 0 & 0 & \\
s^0 & -48 \left(\frac{576}{\varepsilon} \right) = -48 & & & &
\end{array}$$

$$\lim_{\epsilon \rightarrow 0} \frac{s+6}{\epsilon} = +\infty$$

one sign change and system is unstable
Thus there is one root in RHS of s-plane i.e. with positive real part.

Now solve $A(s) = 0$ for dominant roots

$$A(s) = 3s^4 - 48 = 0$$

$$\text{put } s^2 = y.$$

$$3y^2 = 48$$

$$\sqrt{16} = 4$$

$$y^2 = 16$$

$$y = \sqrt{16} = 4$$

$$y = \pm \sqrt{16} = \pm 4$$

$$s^2 = \pm 4$$

$$s = \pm 2$$

$$y^2 = 16$$

$$y^2 = -4$$

$$s = \pm 2j$$

So $s = \pm 2j$ are the 2 roots on Imaginary axis i.e. with zero real part. Root in RHS indicated by a sign change $s = +2$ as obtained by solving $A(s) = 0$.

\therefore Total there are 6 roots as $n = 6$

Roots with positive real part = 1 $(6-2)=0 \Rightarrow s = +2$

Roots with zero real part = 2 $(0+2j)$ $(0-2j)$

Roots with negative real part = 3 $(6-2-1=3)$
 $(s+2)=0 \Rightarrow s = -2$
 $(0+ij)=0 \Rightarrow s = -2j$

→ Marginal K and frequency of sustained oscillations

Marginal value of 'K' is that value of 'K' for which system becomes marginally stable. For a marginally stable system there must be a row of zeros occurring in Routh's array.

(3) For unity feedback system $G(s) = \frac{K}{s(1+0.4s)(1+0.25s)}$
 find range of values of K , marginal value of K and
 frequency of sustained oscillations.

\rightarrow Sol⁴

$$\text{char eqn: } 1 + G(s) H(s) = 0 \quad \& \quad H(s) = 1$$

$$1 + \frac{K}{s(1+0.4s)(1+0.25s)} \quad (1) = 0 \quad \& \quad K > 0$$

$$s[1+0.65s + 0.1s^2] + K = 0$$

$$0.1s^3 + 0.65s^2 + s + K = 0$$

$$\begin{array}{c|cc} s^3 & 0.1 & 1 \\ s^2 & 0.65 & K \\ s^1 & \frac{0.65 - 0.1K}{0.65} & 0 \\ s^0 & K & \end{array}$$

for closed loop CS to be stable, the following conditions must be true.

$$\text{from } s^0 \quad (1) \quad K > 0$$

$$s^1 \quad (2) \quad 0.65K - 0.1K > 0$$

$$0.65K > 0.1K$$

$$\therefore 6.5 > K$$

\therefore Range of values of K , $0 < K < 6.5$

Marginal value of K , $K = K_{\text{marg}}$

$$0.65 - 0.1K_{\text{marg}} = 0$$

$$K_{\text{marg}} = 6.5$$

To find freq^c, find out roots of auxiliary eqn at marginal value of K

$$A(s) = 0.65s^2 + K = s^2 = -10 \quad i.e. s = \pm j3.162$$

$$s = \pm j\omega \quad \omega = 3.162 \text{ rad/sec}$$

So value of K which makes any row of Routh array as row of zeros is called Marginal value of K .

- Now $K=0$, makes row s^0 as row of zeros but $K=0$ cannot be marginal value, because for $K=0$, constant term in the charac equation becomes zero, i.e 1 coefficient for s^0 vanishes, which makes system unstable instead of marginally stable.

"Hence marginal value of K is a value which makes any row other than s^0 as row of zeros."

- To obtain the frequency of sustained oscillations, solve the auxiliary equation $A(s)=0$ for $K = K_{mar}$.
The magnitude of Imaginary root of $A(s)=0$, obtained for marginal value of $K(K_{mar})$ indicates the frequency of sustained oscillations, which system is going to produce.

(4) for system, $s^4 + 22s^3 + 10s^2 + s + K = 0$ find K_{mar} , ω_{mar}

s^4	1	10	K
s^3	22	1	0
s^2	9.95	$\frac{2K}{22}$	0
s^1	$\frac{9.95 - 22K}{9.95}$	0	$\frac{2K}{22}$
s^0	K		

$$\frac{22 \times 10^{-1}}{22} = \frac{22}{22} = 9.95$$

Marginal value of K which makes row s^1 as row of zeros

$$9.95 - 22K_{mar} = 0$$

$$K_{mar} = 0.4524$$

$$A(s) = 9.95s^2 + K = 0$$

$$9.95s^2 + 0.4524 = 0$$

$$s^2 = -0.04546$$

$$s = \pm j 0.2132$$

Hence the freq of osc $\omega = 0.2132 \text{ rad/sec}$.

(5) for a system with charc equation, examine stability

$$f(s) = s^6 + 3s^5 + 4s^4 + 6s^3 + 5s^2 + 3s + 2 = 0$$

$$a_0 = 1 \quad a_2 = 4 \quad a_4 = 5 \quad a_6 = 2 \quad n = 6$$

$$a_1 = 3 \quad a_3 = 6 \quad a_5 = 3 \quad a_7 = 0$$

s^6	1	4	5	2	
s^5	3	6	3	0	
s^4	2	4	2	0	
s^3	0	0	0	0	← special case 2 Row of zero's

$$A(s) = 2s^4 + 4s^2 + 2 = 0$$

$$\div 2 \quad = s^4 + 2s^2 + 1 = 0$$

$$\frac{dA(s)}{ds} = 4s^3 + 4s$$

s^6	1	4	5	2	
s^5	3	6	3	0	
s^4	2	4	2	0	
s^3	4	4	0	0	
s^2	2	2	0	0	
s^1	0	0	0	0	← special case 2
s^0					

$$A'(s) = 2s^2 + 2$$

$$\frac{dA'(s)}{s} = 4s$$

s^6	1	4	5	2
s^5	3	6	3	0
s^4	2	4	2	0
s^3	4	4	0	0
s^2	2	2	0	0
s^1	4	0	0	0
s^0	2	0	0	0

$$s^4 + 2s^2 + 1 = 0$$

$$(s^2 + 1)^2 = 0$$

No sign change, hence no root in RHS of s-plane
 As now 2 zero's occur system may be marginally stable or unstable

→ To examine this find the roots of auxiliary equation

$$A(s) = s^4 + 2s^2 + 1 = 0$$

$$(s^2 + 1)^2 = 0$$

n=4

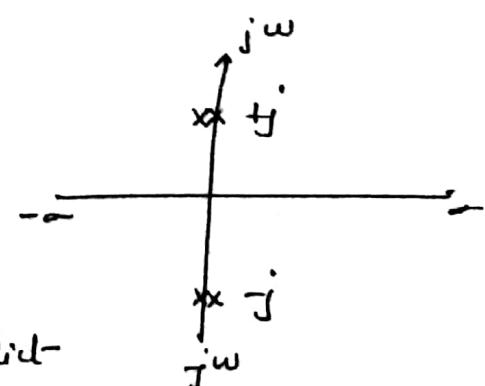
$$s^2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-2 \pm \sqrt{4 - 4}}{2} = -1$$

$$s^2 = -1 \quad | \quad s^2 = -1$$

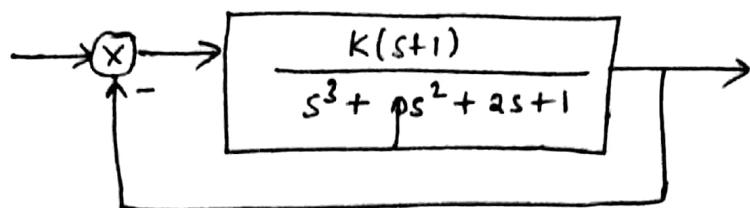
$$s_{1,2} = \pm j \quad | \quad s_{3,4} = \pm j$$

The roots of $A'(s) = 0$ are the roots of $A(s) = 0$, so don't solve II auxiliary eqn. predict the stability from the nature of roots of I auxiliary equation

As there are repeated roots on Imaginary axis, system is unstable



(6) st given system oscillates with freq c 2 rad/sec
find values of 'K_{max}' and 'p'. No poles are in RHS



Solⁿ: As system oscillates, it is marginally stable and value of 'K' at this situation is marginal value of 'K'. As system is marginally stable there must be row of zero's occurring in Routh's array.

char^c eq u : $1 + G(s) H(s) = 0 \quad H(s) = 1$

$$1 + \frac{K(s+1)}{s^3 + ps^2 + 2s + 1} (1) = 0$$

$$s^3 + ps^2 + 2s + 1 + ks + K = 0$$

$$s^3 + ps^2 + (2+k)s + (1+K) = 0$$

$$\begin{array}{c|cc} s^3 & 1 & 2+K \\ s^2 & p & 1+K \\ s^1 & \frac{p(2+K) - (1+K)}{p} & 0 \\ s^0 & 1+K & \end{array}$$

At marginal value of 'K'

$$(2+K)p - (1+K) = 0$$

$$(2+K)p = 1+K$$

$$p = \frac{K+1}{K+2}$$

Now at this value,

$$A(s) = ps^2 + K + 1 = 0$$

$$s^2 = -\frac{(K+1)}{p}$$

$$s = \pm j \sqrt{\frac{K+1}{p}}$$

Compare with $s = \pm j\omega$, $\omega = 2$.

$$\sqrt{\frac{K+1}{p}} = 2$$

Take square,

$$\frac{K+1}{p} = 4$$

$$p = \frac{K+1}{4}$$

$$\frac{K+1}{K+2} = \frac{K+1}{4}$$

$$K+2 = 4$$

$$\boxed{K = 2}$$

$$K = K_{max} = 2$$

$$p = \frac{K_{max} + 1}{2+q} = \frac{3}{4} = 0.75$$

(7) A unity feedback cs has $G(s) = \frac{K(s+13)}{s(s+3)(s+7)}$
using Routh's criterion calculate the range of K for which
the system is (i) stable, (ii) has its closed loop
poles more negative than -1

Soln: charc equ. $I + G(s) \cdot H(s) = 0$ $H(s) = I$

$$I + \frac{K(s+13)}{s(s+3)(s+7)} (I) = 0$$

$$s^3 + 10s^2 + 21s + Ks + 13K = 0$$

$$s^3 + 10s^2 + s(K+21) + 13K = 0$$

Routh's array

s^3	1	$K+21$
s^2	10	$13K$
s^1	$\frac{210 - 3K}{10}$	0
s^0	13K	

(i) for stability

$$\text{from } s^0, \quad 13K > 0, \quad \therefore K > 0.$$

$$\text{from } s^1, \quad 210 - 3K = 0 \quad 210 > 3K \\ \therefore K < 70.$$

$$\therefore \text{range of } K \quad 0 < K < 70.$$

(ii) for all closed loop poles more negative than -1

$$s = s' - 1$$

$$(s' - 1)^3 + 10(s' - 1)^2 + \underbrace{(s' - 1)(K + 21)}_{-21 + 31K} + 13K = 0$$

$$(s')^3 - 3(s')^2 + 3s' - 1 + 10(s')^2 - 20s' + 10 + Ks' - K + 21s' \\ - 21 + 31K = 0.$$

$$\therefore (s')^3 + 7(s')^2 + s'[4 + K] + [12K - 12] = 0.$$

Routh's array

$(s')^3$	1	$4 + K$
$(s')^2$	7	$12K - 12$
$(s')^1$	$\frac{28 + 7K - 12K + 12}{7}$	0
$(s')^0$	12K - 12	

$$(a-b)^3 = a^3 - b^3 - 3ab(a-b)$$

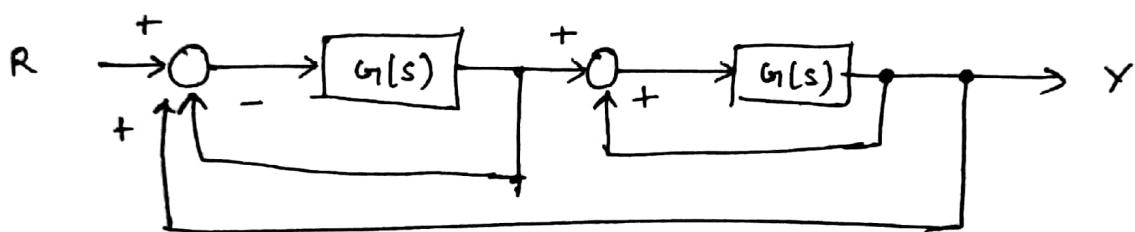
$$\text{from } (s^0) \quad 12K - 12 > 0, \quad K > 1$$

$$\text{from } (s^1) \quad -5K + 40 > 0 \quad 40 > 5K \\ K < 8$$

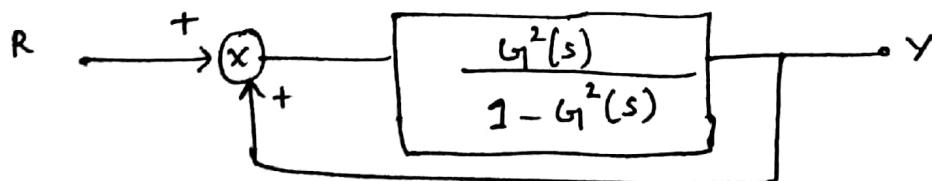
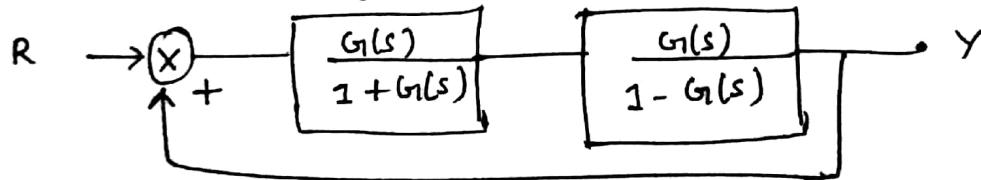
$$\text{range of } K, \quad 1 < K < 8$$

This is range of K to get closed loop poles more negative than -1 .

- (8) The block diagram of a feedback c.s is shown in fig. Apply RH Criterion to determine the range of K for stability if $G_1(s) = \frac{K}{(s+\alpha)(s+\beta)}$



Soln: Eliminating minor feedback loops



$$\frac{Y(s)}{R(s)} = \frac{\frac{G_1^2(s)}{1 - G_1^2(s)}}{1 - \frac{G_1^2(s)}{1 - G_1^2(s)} (z)} = \frac{G_1^2(s)}{1 - 2G_1^2(s)}$$

Substituting the value of $G_1(s) = \frac{K}{(s+4)(s+5)}$

$$\frac{Y(s)}{R(s)} = \frac{\frac{K^2}{(s+4)^2(s+5)^2}}{1 - \frac{2K^2}{(s+4)^2(s+5)^2}}$$

$$\frac{Y(s)}{R(s)} = \frac{K^2}{(s+4)^2(s+5)^2 - 2K^2}$$

$$\frac{Y(s)}{R(s)} = \frac{K^2}{\underbrace{s^4 + 18s^3 + 121s^2 + 360s + 400 - 2K^2}_{} = 0.$$

∴ characteristic equation $1 + G(s) H(s) = 0$.

$$s^4 + 18s^3 + 121s^2 + 360s + \underbrace{400 - 2K^2}_{} = 0$$

Routh's array is,

s^4	1	121	$400 - 2K^2$
s^3	18	360	0
s^2	101	$\frac{18(400 - 2K^2)}{18}$	0
s^1	$360 - 18(400 - 2K^2)$	0	
s^0	101		
	$400 - 2K^2$		

$$\text{for } s^0; \quad 400 - 2K^2 > 0$$

$$400 > 2K^2$$

$$200 > K^2$$

$$K < \sqrt{200}$$

$$K < 14.1421$$

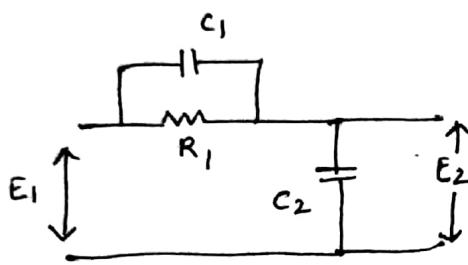
$$0 < K < 14.14$$

$$\frac{18 \times 121 - (1)(360)}{18} = 101$$

for any value of K between $0 < K < \infty$, the term s' is always positive. Thus range of K for stability

$$0 < K < 14.1421.$$

- (g) The open loop T.F of a unity feedback c.s is given by $G(s) = \frac{50}{s(1+0.05s)(1+0.2s)}$. Apply Routh criterion show that the system is unstable. Confirm that the introduction of the 2 terminal pair now connected in cascade with $G(s)$ make the system stable.



$$C_1 = 0.5 \mu F$$

$$C_2 = 10 \mu F$$

$$R_1 = 1 M\Omega$$

Sol^u:

$$G_1(s) = \frac{50}{s(1+0.05s)(1+0.2s)} \quad H(s) = 1$$

$$F(s) = 1 + G_1(s) H(s) = 0$$

$$= 1 + \frac{50}{s(1+0.05s)(1+0.2s)} =$$

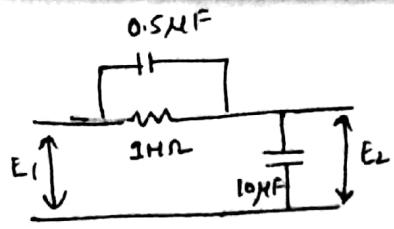
$$\begin{aligned} s(1+0.05s)(1+0.2s) + 50 &= 0 \\ s[1+0.2s + 0.05s + 0.01s^2] + 50 &= 0 \\ s + 0.2s^2 + 0.05s^2 + 0.01s^3 + 50 &= 0 \\ 0.01s^3 + 0.05s^2 + s + 50 &= 0 \end{aligned}$$

$$\text{i.e. } 0.01s^3 + 0.25s^2 + s + 50 = 0$$

Routh's array

s^3	0.01	1
s^2	0.25	50
s	-1	0
s^0	50	

There are 2 sign changes in 1 column of Routh's array hence the system is unstable



$$Z_1(s) = R_1 + jC_1$$

$$= \frac{R_1 + j/C_1}{R_1 + j/C_1} = \frac{R_1}{1 + sR_1C_1}$$

$$Z_2(s) = \frac{1}{10 \times 10^{-6}s}$$

$$= \frac{1 \times 10^6}{1 + s \times 1 \times 10^6 \times 0.5 \times 10^{-6}}$$

$$Z_2(s) = \frac{10^5}{s}$$

$$Z_2(s) = \frac{10^6}{1 + s0.5}$$

By voltage division rule,

$$\frac{E_2(s)}{E_1(s)} = \frac{Z_2(s)}{Z_1(s) + Z_2(s)} = \frac{\frac{10^5}{s}}{\frac{10^6}{1+0.5s} + \frac{10^6}{s}} = \frac{10^5(1+0.5s)}{10^6s + 10^5(1+0.5s)}$$

$$\frac{E_2(s)}{E_1(s)} = \frac{1+0.5s}{1+10.5s}$$

$$\therefore G'(s) = \frac{50(1+0.5s)}{s(1+0.05s)(1+0.2s)(1+10.5s)}, H(s) = 1.$$

$$F(s) = G'(s)H(s) + 1 = 0$$

$$(0.105s^4 + 2.635s^3 + 10.75s^2 + 265 + 50 = 0) \text{ check ..}$$

→ New Routh's array

s^4	0.105	10.75	50
s^3	2.635	265	0
s^2	0.190	50	0
s^1	12.4369	0	
s^0	50		

$$a(s) = \frac{50}{s(1+0.05s)(1+0.2s)}$$

$$U'(s) = U(s) \cdot \frac{E_1(s)}{E_1(s)}$$

As there are no sign changes in I column of Routh's array, the system is stable. This shows the n/w connected in cascade with $G(s)$, make the system stable.