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Module - 4

Fourier Representation of Signals

* We know that the output of LTI sym for an i/p $x(t)$ or $x(n)$ expressed as linear combination of (weighted summation) shifted delta function lead to convolution integral (or sum)
* Similarly representing a signal as weighted sum or superposition of complex sinusoid gives useful expression for sym o/p. Along with sym o/p, representation of sig. in above manner also provides insightful characterization of signals & syms.
[complex sinusoid ⇒ $e^{j\Omega n}$ or $e^{j\omega t}$ & $e^{\pm j\theta} = \cos\theta \pm j\sin\theta$]
* The study of signals & systems using sinusoidal representations is termed as "Fourier analysis", and representation of signals (or systems) using complex sinusoids ($e^{j\Omega n}$ or $e^{j\omega t}$) is known as "Fourier representation" after Joseph Fourier (1768 to 1830) for his development of the theory

Types (or class) of Fourier Representations

There are 4 distinct Fourier representations, based on different class of signals.

1) CTP signals → FS → Fourier Series or Continuous time Fourier series → CTFS

2) CTNP signals → FT → Fourier Transform or CTFT → Continuous Time Fourier Transform.

3) DTP signals → DTFS or Discrete Time FS
4) DTNP signals → DTFT or Discrete time FT.

In summary

P	→ Series
NP	→ Transform.

CTP	→ FS
CTNP	→ FT
DT P	→ DTFS
DTNP	→ DTFT.

Complex sinusoids & system response or freq response of LTI Sims

Consider a DT sim with $h(n)$ & $x(n)$ as its impulse resp & i/p sly, the o/p of sim $y(n)$ is given by

$$y(n) = x(n) * h(n) = h(n) * x(n).$$

$$= \sum_{K=-\infty}^{\infty} h(K)\, x(n-K)$$

if $x(n) = e^{j\Omega n}$. a complex sinusoid.

then $y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) = \sum_{k=-\infty}^{\infty} h(k)\, e^{j\Omega(n-k)}$

ie $y(n) = e^{j\Omega n} \sum_{K=-\infty}^{\infty} h(K)\, e^{-j\Omega k}$.

$\therefore\ y(n) = e^{j\Omega n}\, H(e^{j\Omega})$ — (A)

where $H(e^{j\Omega})$ is purely function of Ω & is known as which is a complex number trans fer function of the sim or freq response of the sim

III^{ly} for CT sims if $x(t) = e^{j\omega t}$

$$y(t) = \int h(\tau)\, e^{j\omega(t-\tau)} d\tau = e^{j\omega t} \int h(\tau)\, e^{-j\omega\tau} d\tau$$

$$y(t) = e^{j\omega t}\, H(e^{j\omega}) \quad \text{— (B)}.$$

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Ⓐ & Ⓑ can be represented pictorially as ②

$$x(n) = e^{j\Omega n},\ x(t) = e^{j\omega t} \longrightarrow \boxed{h(n) \text{ Ⓞ } h(t)} \longrightarrow \begin{array}{l} y(n) = e^{j\Omega n} H(e^{j\Omega}) = x(n)\, H(e^{j\Omega}) \\ Y(t) = e^{j\omega t} H(e^{j\Omega}) = x(t)\, H(e^{j\Omega}) \end{array}$$

⇒ Ⓐ & Ⓑ say that when i/p to a ~~comp~~ LTI sym is a complex sinusoid of some freq, the o/p is similar sinusoid with a complex number $H(e^{j\Omega})$ Ⓞ $H(e^{j\Omega})$ multiplied with it.

⇒ Since H is function of only freq (Ω Ⓞ ω) it is known as freq. response of the system

Now $y(n) = e^{j\Omega n} H(e^{j\Omega})$

since $H(e^{j\Omega})$ is a complex number $(a+jb)$ we can express $H(e^{j\Omega})$ in polar form as

$H(e^{j\Omega}) = \sqrt{a^2+b^2}$

$|H(e^{j\Omega})| = \arg[H(e^{j\Omega})] = \tan^{-1}\left(\frac{b}{a}\right)$

$$H(e^{j\Omega}) = |H(e^{j\Omega})|\, e^{\arg(H(e^{j\Omega}))}$$

$$\therefore\ y(t) = |H(e^{j\Omega})|\, e^{j[\Omega n + \arg H(e^{j\Omega})]}$$

which means $H(e^{j\Omega})$ Ⓞ sym modifies amp. of i/p by $|H(e^{j\Omega})|$ & phase of i/p by $\arg(H(e^{j\Omega}))$.

⇒ ie The system transforms the i/p $x(n)$ by multiplying its amplitude by $|H(e^{j\Omega})|$ & phase by $\arg|H(e^{j\Omega})|$

Hence the freq function $H(e^{j\Omega})$ is also known as transfer function.

⇒ Also $H(e^{j\Omega}) = \sum_{k=0}^{\infty} h(k)\, e^{-j\Omega k}$ Ⓞ $H(e^{j\omega}) = \int h(\tau)\, e^{-j\omega\tau}\, d\tau$

$h(k)$ Ⓞ $h(\tau)$ are impulse responses of CT & DT sym

(a) in terms of $h(n)$ (or) $h(t)$

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\Omega n}$$ (or) $$h(t) = \int_{-\infty}^{\infty} h(t)\, e^{-j\omega t}\, dt$$

is definition of transfer function.

ie Transfer function is Transformation of impulse response of the system ie $H(\omega) = T[h(t)]$ (or) $H(\Omega) = T[h(n)]$

$H(\omega)$ is similar representation to $H(e^{j\omega})$

& $H(\Omega)$ is similar in representation to $H(e^{j\Omega})$.

⇒ Hence representing signal as (linear comb of weighted sum) (or) weighted superposition ( ) of complex sinusoids is very helpful for analysis of signals & system.

Fourier Series Representation — Periodic Signals

- Consider representing a periodic sig as weighted superposition of complex sinusoids.

→ Since weighted sum must have same period as signal, each sinusoid in the superposition must have the same period as signal. This implies that freq of each sinusoid must be an integer multiple of the signals fundamental freq. (Recall that sum of two (or) more periodic signals is periodic iff each individual sig. period is integer multiple of same basic freq → periodicity of sum signals) $\frac{T_1}{T_2} = \frac{m}{n} \Rightarrow nT_1 = mT_2$.

→ ∴ if $x(n)$ is a DT signal with fundamental period N, then $x(n)$ can be written as

$$\hat{x}(n) = \sum_{K} A[K]\, e^{jK\Omega_0 n} \quad \text{— (I)}$$

where $\Omega_0 = \frac{2\pi}{N}$ is fundamental freq (or) angular freq in rad.

→ the freq of k^{th} sinusoid is $k\Omega_0$, k is an integer, each of these sinusoids has fundamental period N.

→ lly for a CT signal $x(t)$ we can write weighted superposition as

$$\hat{x}(t) = \sum_{k} A[K]\, e^{j\omega_0 k t} \quad \text{— (II)}$$

where $K\omega_0$ is freq of k^{th} sinusoid &

$\omega_0 = \frac{2\pi}{T}$ is fundamental freq (or) angular freq in r/sec. of the sig

In ① & ② $\Rightarrow A[K]$ is weight applied to K^{th} harmonic

$\Rightarrow$ the hat [^] symbol denotes approximate value.

$\Rightarrow$ K is variable used to index freq $\therefore A[K]$ is a function of freq.

** Now how many terms we should add or how many values K can take in ① & ② ?

$\Rightarrow$ As w.k.t for a DT complex sinusoid

$$e^{j(K+N)\Omega_0 n} = e^{jK\Omega_0 n + jK\Omega_0 N} = e^{jK\Omega_0 n} \cdot \cancel{e^{jK\Omega_0 N}}^{1}$$

$$= e^{jK\Omega_0 n} \qquad \because \Omega_0 = \frac{2\pi}{N}$$

we may have only N distinct complex sinusoids.

ie K can vary only from 0 to (N-1) or K can have only N distinct values & hence only N distinct complex sinusoids can be used in ①

$\therefore$ ① can be written as

$$\boxed{\hat{x}(n) = \sum_{K=0}^{N-1} A[K]\, e^{j\Omega_0 K n}.}$$

Σ can also be taken as $\sum_{-\left[\frac{N-1}{2}\right]}^{\left[\frac{N-1}{2}\right]}$ ~~or $\sum_{-\frac{N}{2}}^{\frac{N}{2}}$ based~~ on requirement. whether N is odd [odd or even symmetric]

~~or $x(n)$ is odd or even symmetric~~

$\Rightarrow$ We also know that any CT sinusoid with distinct freq is periodic hence in ② K can take any value $\therefore$ ② can be written as

$$\boxed{\hat{x}(t) = \sum_{K=-\infty}^{\infty} A[K]\, e^{j\omega_0 K t}} -$$

The weights $A[k]$ in above eqn's are taken in such a way that $\hat{x}(n)$ & $\hat{x}(t)$ are good approximations of $x(n)$ & $x(t)$, rly.

Hence this gives rise to FS transformation pair as follows (for DT & CT periodic signals).

DT P signals: The DTFS

→ A periodic signal $x(n)$ with fundamental period N & fundamental freq $\Omega_0 = \frac{2\pi}{N}$ is given by

$$\boxed{\begin{aligned} x(n) &= \sum_{k=0}^{N-1} X(k)\, e^{j\Omega_0 k n} \quad n = 0 \text{ to } (N-1) \\ X(k) &= \frac{1}{N}\sum_{n=0}^{N-1} x(n)\, e^{-jk\Omega_0 n}. \quad k = 0 \text{ to } (N-1) \end{aligned}}$$

where

are DTFS coefficients [$A[k]$ in previous eqn.s] of $x(n)$.

→ we say that $x(n)$ & $X(k)$ are DTFS pairs & this relationship is denoted by

$$\boxed{x(n) \xleftrightarrow[\Omega_0]{DTFS.} X(k)}$$

ie by N values of $X(k)$ we can determine $x(n)$ & by N values of $x(n)$ we can determine $X(k)$

$X(k)$ are termed as freq domain representation of $x(n)$ [a DT periodic sig with period N]

→ DTFS is the only Fourier representation that can be numerically evaluated & manipulated on a computer, as both $x(n)$ & $X(K)$ are characteri-zed by finite set of N numbers.

111y CT P signals can be represented by FS-pair given by

$$x(t) = \sum_{K=-\infty}^{\infty} X(K)\, e^{j\omega_0 K t}$$

where $X(K) = \frac{1}{T}\int_0^T x(t)\, e^{-jk\omega_0 t}\, dt.$

& $x(t) \xleftrightarrow[\omega_0]{FS} X(K)$

DTFS and its nature

W.K.T DTFS is given by

$$X(K) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-j\Omega_0 k n}$$

&

$$x(n) = \sum_{n=0}^{N-1} X(K)\, e^{+j\Omega_0 k n}.$$

here $x(n)$ is a periodic signal in n with fundamental period N

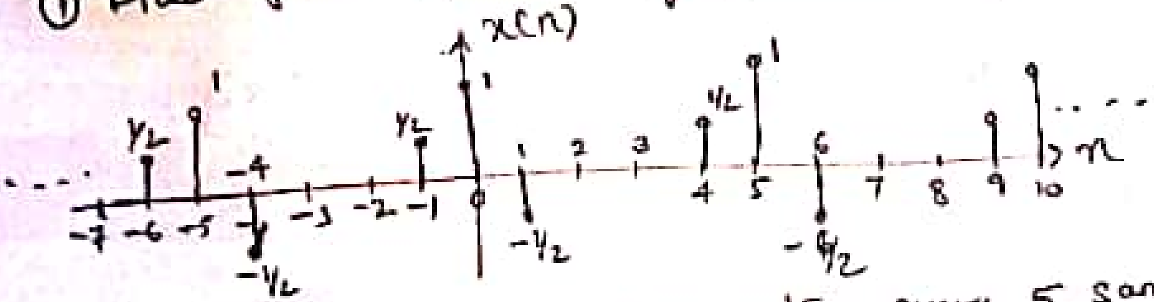
Also $X(K)$ is a periodic signal in k with fundamental period N

$\therefore$ $\boxed{\text{DTFS nature can be summarized as } \quad \underline{x(n)}: \text{ periodic in } n \text{ with period } N \quad \underline{X(K)}: \text{ periodic in } k \text{ with period } N}$

$\therefore$ range of summation can be chosen to simplify the problem in hand.

Examples on DTFS

① Find freq domain representation of $x(n)$.



→ Here $x(n)$ repeats itself after every 5 samples

$\therefore N = 5$ $\therefore$ $\Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{5}$

→ Here range of period can be considered in any convenient manner ie -1 to 3 or 0 to 4 or any thing as it is periodic.

i) considering one period → ranging from 0 to 4

$$X(K) = \frac{1}{N}\sum_{n=0}^{4} x(n)\, e^{-j\Omega_0 nk} \qquad K = 0 \text{ to } 4.$$

we get $X(0) = \frac{1}{5}\sum_{n=0}^{4} x(n) = x(0) + x(1) + x(2) + x(3) + x(4)$

$= 1 + \left(\frac{-1}{2}\right) + 0 + 0 + 0 + \left(\frac{1}{2}\right)$

$= 1/5 = 0.2$ //

IIIly $X(1) = \frac{1}{5}\sum_{n=0}^{4} x(n)\, e^{-j\Omega_0 n} = x(0) + x(1)\, e^{-j\frac{2\pi}{5}} + x(2) e^{-j\frac{4\pi}{5}}$

$+ x(3)\, e^{-j\frac{6\pi}{5}} + x(4) e^{-j\frac{8\pi}{5}}$

ie $X(1) = 1 + \left(\frac{-1}{2}\right)(\qquad) + 0 + 0 + \frac{1}{2}(\qquad)$

$X(1) = (1 - 0.95j)/5 = 0.2 - 0.190j$

IIIly $X(2) = (1 - 0.587j)/5 = 0.2 - 0.117j$

$X(3) = (1 + 0.587j)/5 = 0.2 + 0.117j$

$X(4) = (1 + 0.95j)/5 = 0.2 + 0.19j.$

(a)

$$X(K) = \frac{1}{N}\sum_{n=0}^{4} x(n)\, e^{-j\Omega_0 nk} = x(0) + x(1)\, e^{-j\Omega_0 k} + x(4) e^{-j\Omega_0 4K}$$

$X(K) = x(0) + \left(-\frac{1}{2}\right) e^{-j\frac{2\pi}{5}K} + \frac{1}{2} e^{-j\frac{8\pi}{5}K}.$

$= \frac{1}{5}\left[1 - \frac{1}{2} e^{j\frac{2\pi}{5}K} + \frac{1}{2} e^{j\frac{8\pi}{5}K}\right]$

K = 0 1 2 3 4

$\therefore X(K) = \frac{1}{5}\{1, (1 - 0.951j), (1 - 0.587j), (1 + 0.587j), (1 + 0.951j)\}$

$X(K) = \{0.2, 0.276\angle{-0.76}^c, 0.2319\angle{-0.53}^c, 0.2319\angle{0.53}^c, 0.276\angle{0.76}\}$

The same problem can be solved using index of one period from -2 to $+2$. $N=5$

ie K varies from -2 to $+2$

$$\therefore X(K) = \frac{1}{5}\sum_{K=-2}^{2} x(n)\, e^{-j\frac{2\pi}{5}Kn}$$

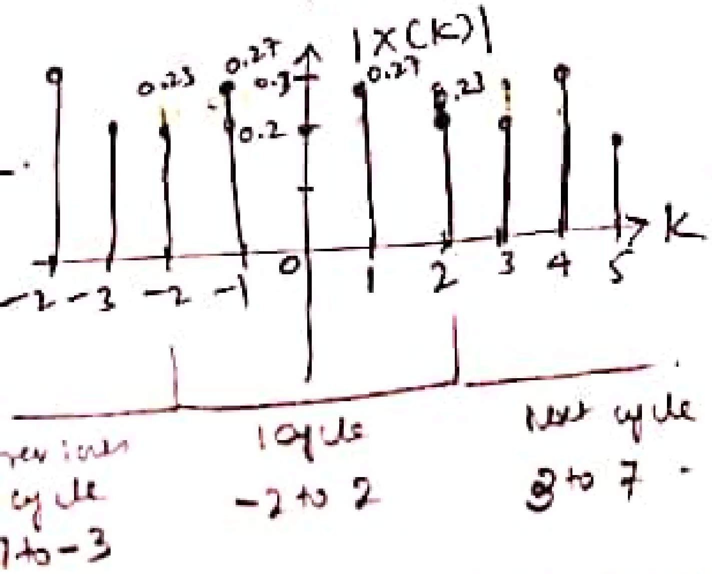
$$= \frac{1}{5}\left[x(-1)e^{j\frac{2\pi}{5}K} + x(0)e^{j0} + x(1)e^{-j\frac{2\pi}{5}K}\right]$$

$$= \frac{1}{5}\left[\frac{1}{2}e^{j\frac{2\pi}{5}K} + 1 + \left(-\frac{1}{2}\right)e^{-j\frac{2\pi}{5}K}\right] \quad \because x(-2) = x(2) = 0.$$

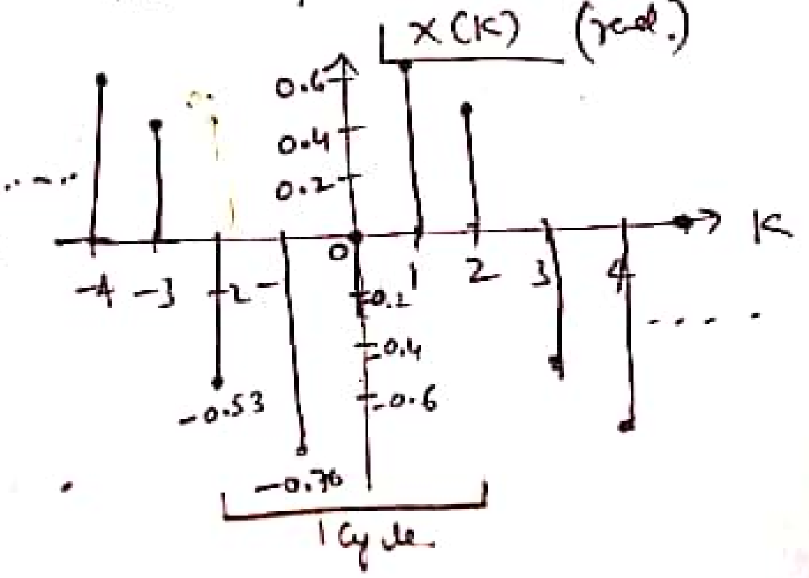
$$= \frac{1}{5}\left[1 + j\sin\left(\frac{2\pi}{5}K\right)\right]$$

$$\therefore X(K) = \Big[\underset{K=-2}{0.232\, e^{-j0.531}},\ \underset{K=-1}{0.276\, e^{-j0.76}},\ \underset{K=0}{0.2},\ \underset{K=1}{0.276\, e^{j0.76}},\ \underset{K=2}{0.232\, e^{j0.531}}\Big]$$

Mag plot



Phase plot



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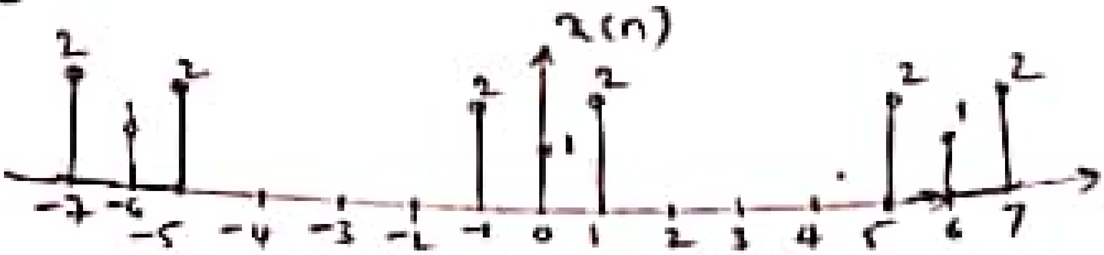
①

N = 6

K = -2 to +3

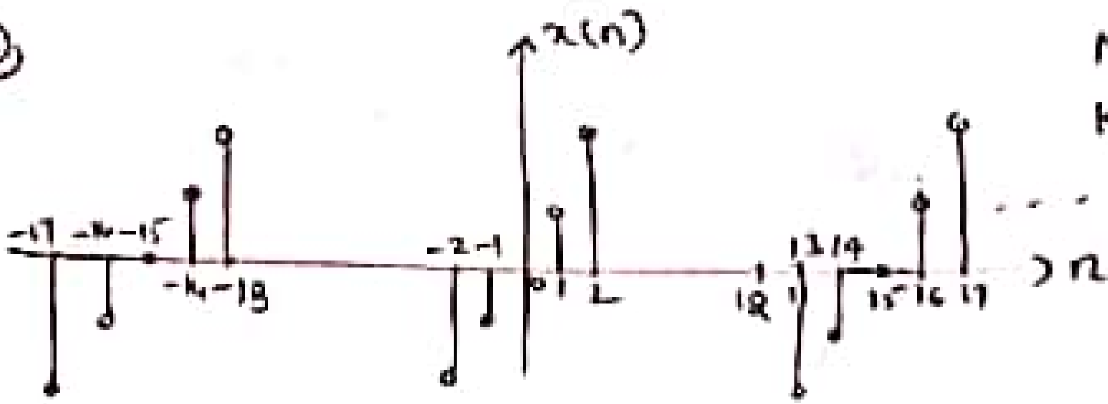


$$X(K) = \frac{1}{6} + \frac{2}{3}\cos\left(\frac{\pi K}{3}\right)$$

②

N = 15

K = -7 to 7



$$X(K) = -\frac{2j}{15}\left[\sin\left(\frac{2\pi K}{15}\right) + 2\sin\left(\frac{4\pi K}{15}\right)\right]$$

Computation of DTFS by inspection

Find DTFS of $x(n) = \cos\left(\frac{2\pi}{3} n\right)$

Here $N = 3$ $\because x(n)$ is a periodic cosine signal.

w.k.T $x(n) = \sum_{K=0}^{N-1} X(K) e^{j\Omega_0 kn}$

for $N = 3$

$\therefore x(n) = \sum_{K=0}^{2} x(n) e^{j\Omega_0 kn}$ (or) $\sum_{K=-1}^{1} X(K) e^{j\Omega_0 kn}$

ie $x(n) = \sum_{K=-1}^{1} X(K) e^{j\Omega_0 kn}$, $\Omega_0 = \frac{2\pi}{3}$.

$x(n) = \cancel{\Sigma}\ X(-1) e^{-j\Omega_0 n} + X(0) + X(1) e^{j\Omega_0 n}$. — (I)

Also given $x(n)$ can be written as

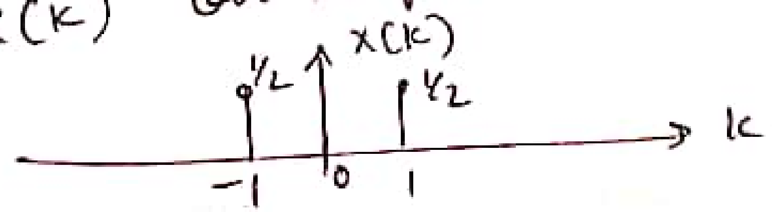
$$x(n) = \frac{e^{j\frac{2\pi}{3}n} + e^{-j\frac{2\pi}{3}n}}{2}$$

$$= \frac{1}{2} e^{j\frac{2\pi}{3}n} + e^{-j\frac{2\pi}{3}n}. \quad \text{— (II)}$$

comparing (I) & (II)

$X(-1) = \frac{1}{2}$ & $X(1) = \frac{1}{2}$.

$\therefore X(K)$ can be plotted as



Try

1) $x(n) = 1 + \sin\left[\frac{n\pi}{12} + \frac{3\pi}{8}\right]$

2) $x(n) = \cos\left(\frac{n\pi}{30}\right) + 2\sin\left(\frac{n\pi}{90}\right)$

1) $$X(K) = \begin{cases} \frac{e^{-j\frac{3\pi}{8}}}{2j} & K=-1 \\ 1 & K=0 \\ \frac{e^{j\frac{3\pi}{8}}}{2j} & K=1 \\ 0 & \text{o.w.} \end{cases}$$

2) $$X(K) = \begin{matrix} -1/j & K=-1 \\ 1/j & K=1 \\ 1/2 & K=\pm 3 \\ 0 & \text{o.w.} \end{matrix}$$

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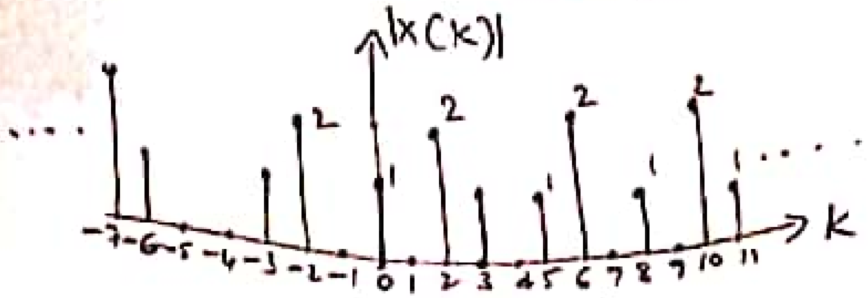
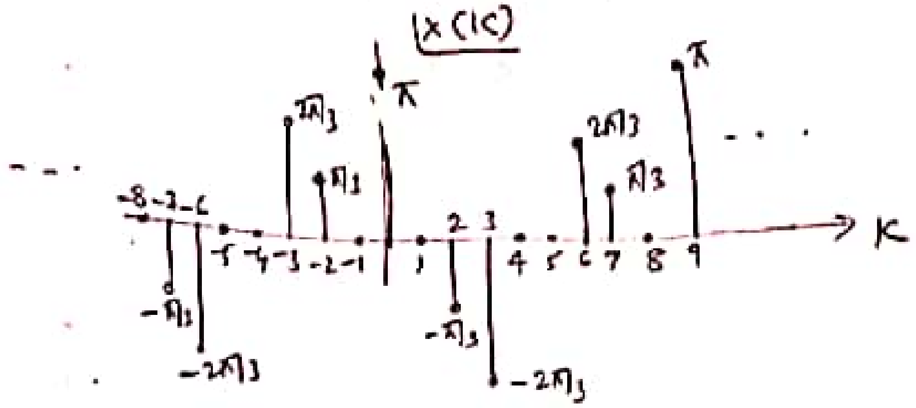
Inverse DTFS

Find inv. DTFS of $X(K)$ given





From given fig we see that $N=9$.

let $k=-4$ to 4

$$\therefore x(n)=\sum_{K=-4}^{4} X(K)\, e^{j\Omega_0 Kn} \qquad n=0 \text{ to } 8.$$

Also from fig

$$X(K)=\{\underset{-4}{0\angle 0},\ \underset{-3}{1\angle 2\pi/3},\ \underset{-2}{2\angle \pi/3},\ \underset{-1}{0\angle 0},\ \underset{0}{1\angle \pi},\ \underset{1}{0\angle 0},\ \underset{2}{2\angle -\pi/3},\ \underset{3}{1\angle},\ \underset{4}{0}\}$$

$$\therefore x(n)=\sum_{K=-4}^{4} X(K)\, e^{j\Omega_0 Kn} = \sum_{K=-4}^{4} |X(K)|\, e^{j\angle X(K)}\, e^{j\Omega_0 Kn}$$

$$= \underbrace{1e^{j\frac{2\pi}{3}} e^{j\Omega_0(-3)n}}_{K=-3} + \underbrace{2e^{j\frac{\pi}{3}} e^{j\Omega_0(-2)n}}_{K=-2} + \underbrace{1\cancelto{-1}{e^{j\pi}}\ \cancelto{1}{e^{j\Omega_0(0)n}}}_{k=0}$$

$$+ \underbrace{2e^{-j\frac{\pi}{3}} e^{j\Omega_0(2)n}}_{K=2} + 1e^{-j\frac{2\pi}{3}} e^{j\Omega_0 n}, \text{ here } \Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{9}$$

$$\text{ie } x(n) = e^{j\frac{2\pi}{3}} e^{-j\frac{6\pi}{9}n} + 2e^{j\frac{\pi}{3}} e^{-j\frac{4\pi}{9}n} - 1 + 2e^{-j\frac{\pi}{3}} e^{j\frac{4\pi n}{9}} + e^{-j\frac{2\pi}{3}} e^{j\frac{6\pi}{9}n}.$$

$$\therefore x(n) = 2\cos\left(\frac{6\pi n}{9} - \frac{2\pi}{3}\right) + 4\cos\left(\frac{4\pi n}{9} - \frac{\pi}{3}\right) - 1$$

Q) Find time domain signal $x[n]$ for given DTFS coeffici-cients for one fundamental period.

$$X(K) = \left(\frac{1}{2}\right)^{K} \qquad 0 \leq K \leq 9.$$

Sol. here $N = 10$ as $X(K)$ has one period extending from 0 to 9.

$$\therefore \quad x(n) = \sum_{n=0}^{9} X(K)\, e^{j\Omega_0 n K} \quad , \Omega_0 = \frac{2\pi}{N} = \frac{2\pi}{10}$$

$$= \sum_{n=0}^{9} \left(\frac{1}{2}\right)^{K} e^{j\frac{2\pi}{10}Kn}.$$

$$x(n) = \sum_{K=0}^{9} \left[\frac{1}{2} e^{j\frac{2\pi}{10}n}\right]^{K}.$$

It is in the form $\sum_{K=0}^{M} a^K = \frac{1-a^{M+1}}{1-a}$

with $a = \frac{1}{2} e^{j\frac{2\pi}{10}n}$.

$$\therefore \quad x(n) = \frac{1 - \left[\frac{1}{2} e^{j\frac{2\pi}{10}n}\right]^{10}}{1 - \frac{1}{2} e^{j\frac{2\pi}{10}n}}$$

$$= \frac{1 - \left(\frac{1}{2}\right)^{n}}{1 - \frac{1}{2} e^{j\frac{\pi}{5}n}} \qquad \because e^{j\frac{2\pi}{10}n \times 10} = e^{j2n\pi} = 1.$$