Z - Transforms : (Brief review of z-transforms) properties of ROC, properties of z-transforms, Inversion of the Z-Transforms, the transfer function, system response causality & stability of Systems.

Introduction : complex sinusoidal representation of a discrete-time Signal can represent interms of complex exponential signals, its termed as z-transforms.

→ ∴ able to obtain a broader characterization of discrete-time LTI systems & their interaction with signals

→ The z-transforms of the impulse response exists for unstable LTI systems.

→ Discrete-time complex exponentials are eigen functions of LTI systems.

→ convolution of time signals corresponds to multiplication of z-transforms. (ie the o/p of LTI system is obtained by multiplying the z-transform of the i/p by the z-transform of the impulse response.

→ z-transform of the impulse response is a transfer function of a system.

→ Transfer function : Transfer function generalizes the frequency response characterization of a system's i/p-o/p behavior and offers new insights into system characteristics.

→ Appⁿ : To study the system characteristics and the derivation of computational structures for Discrete time systems on computers.

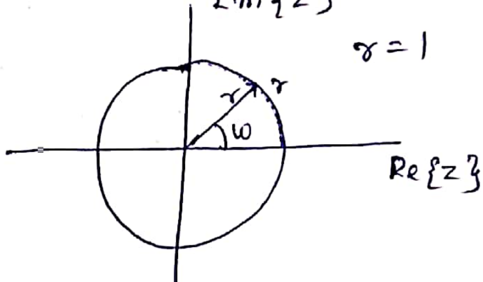
The Z-Transform :

CT (continuous time) — DT (Discrete-time)
LT (Laplace transform) — ZT (Z-Transform)

→ Representation : $x(n) \underset{IZT}{\overset{ZT}{\rightleftharpoons}} \boxed{X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}}$ bidirectional ZT

where n = integer
z = complex variable
$\boxed{r\cos\omega + jr\sin\omega = re^{j\omega}}$ (polar form)
magnitude $= |z| = r$ = damping factor
complex argument $= \omega$ = sinusoidal freq

→ $\boxed{z^{+n} = r^n \cos\omega n + j r^n \sin\omega n}$



→ $\boxed{X(z) = \sum_{n=0}^{\infty} x(n) z^{-n}}$ unidirectional ZT

→ $x(n) \rightarrow$ LTI $\rightarrow y(n) = H\{x(n)\}$

$= h(n) * x(n)$

$= \sum_{K=-\infty}^{\infty} h(K)\, x(n-K)$

we use $x(n) = z^n$

$\therefore y(n) = \sum_{K=-\infty}^{\infty} h(K)\, z^{n-K}$

$= z^n \underbrace{\left[\sum_{K=-\infty}^{\infty} h(K) z^{-K}\right]}_{\text{Transfer function } H(z)}$

$y(n) = H\{x(n)\} = H(z^n) = H(z) z^{-n}$

$\boxed{\therefore H\{z^n\} = H(z) z^{-n}}$

$\boxed{H(z) = \sum_{K=-\infty}^{\infty} h(K) z^{-K}}$

Polar form ⇒ $H(z) = |H(z)| e^{j\phi(z)}$

$\therefore y(n) = |H(z)|\, e^{j\phi(z)}\, z^n$ w.r.t $z = re^{j\omega}$

$y(n) = |H(re^{j\omega})|\, r^n \cos(\omega n + \phi(re^{j\omega})) + j\,|H(re^{j\omega})|\, r^n \sin(\omega n + \phi(re^{j\omega}))$

$r = 1 \Rightarrow y(n) = |H(e^{j\omega})| \cos(\omega n + \phi(e^{j\omega})) + j\,|H(e^{j\omega})| \sin(\omega n + \phi(e^{j\omega}))$

* Z-transform exists when the infinite sum converges. in below Sq^n

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

Condition for convergence :(i) Absolute summability of $x(n) z^{-n}$

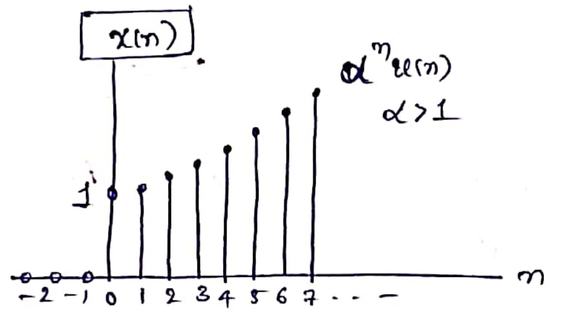
$$\sum_{n=-\infty}^{\infty} |x(n) r^{-n}| < \infty$$

where "range of r" in terms of region of convergence (ROC)

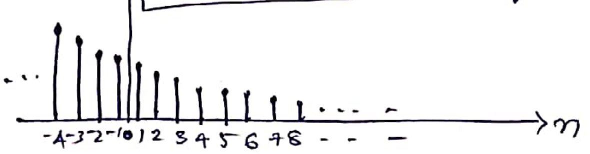
* Z-Transform exists for signals that do not have DTFT.

* $x(n)\, r^{-n} \Rightarrow$ where $x(n) = \alpha^n u(n) \Rightarrow \alpha > 1$

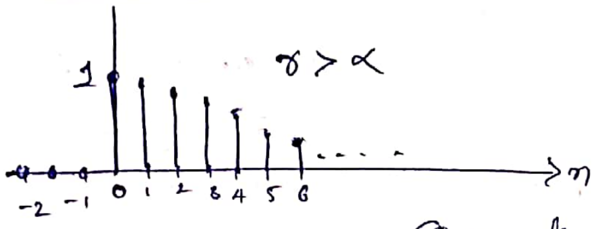
$= \alpha^n \times 1 = \alpha^n$



$\boxed{r^{-n} \Rightarrow r > \alpha} \Rightarrow$ decays faster than $x(n)$



$\boxed{x(n) \times r^{-n}}$

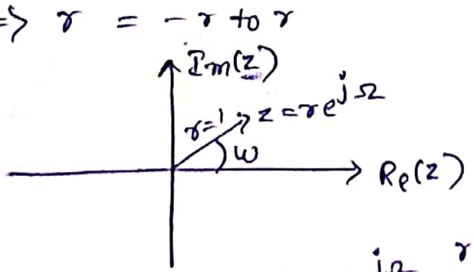


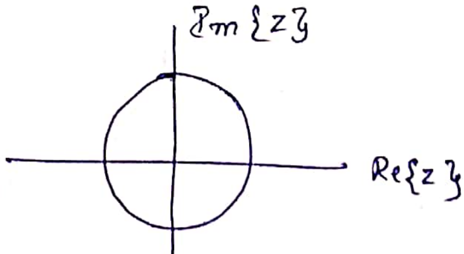
[NOTE: ω = of continuous analog
Ω = for discrete
Here ω is used for discrete for just convenience]

Ploting of z (or) z-plane : (or) complex plane $z = re^{j\omega} = r(\cos\omega + j\sin\omega)$

ω (or) $\Omega = -\pi$ to π $\quad r = 1$

$z \Rightarrow r = -r$ to r





$$z = re^{j\Omega} \overset{r=1}{=} e^{j\Omega} \Rightarrow X(z) = X(e^{j\Omega}) = X(z)\Big|_{z=e^{j\Omega}}$$

* The most commonly encountered form of the z-transform in engineering applications is a ratio of 2 polynomials in z^{-1}

$$X(z) = \frac{b_0 + b_1 z^{-1} + \dots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \dots + a_N z^{-N}}$$ rational function.

$$\left.\begin{array}{l} b_1 z^{-1} \dots b_M z^{-m} \\ a_1 z^{-1} + \dots a_N z^{-N} \end{array}\right\}$$ product terms

$$X(z) = \frac{b_0}{a_0} \frac{\prod_{K=1}^{M} (1 - C_K z^{-1})}{\prod_{K=1}^{N} (1 - d_K z^{-1})}$$

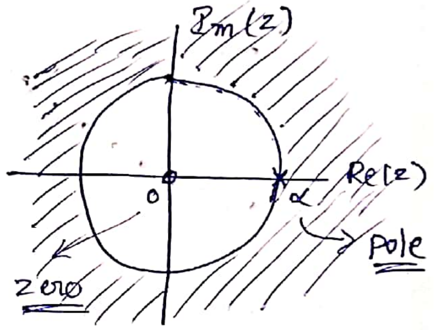
Symbol

o
x

where C_K = roots of numerator polynomial (zeros)
d_K = roots of denominator polynomial (poles)

(*) z-Transform of causal exponential signal

Determine the z-transform of the signal. $x(n) = \alpha^n u(n)$ Depict the ROC and to locations of poles and zeros of $X(z)$ in the z-plane



Example - z-plane.

Soln by defn $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

$u(n) \Rightarrow 0$ to ∞

$$= \sum_{n=-\infty}^{\infty} \alpha^n u(n) z^{-n} = \sum_{n=0}^{\infty} \alpha^n \times 1 \times z^{-n}$$

$$= \sum_{n=0}^{\infty} \alpha^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{\alpha}{z}\right)^n$$

geometric series of ∞ length. Same converges provided $\frac{\alpha}{z} < 1$ (or) $|\alpha| < |z|$

$$\therefore \left|\frac{\alpha}{z}\right| < 1 \qquad |\alpha| < |z|$$

$$\therefore X(z) = \frac{1}{1 - \alpha z^{-1}}$$

$$= \frac{1}{1 - \alpha/z}$$

$$X(z) = \frac{z}{z - \alpha} \qquad |z| > |\alpha|$$

zeros $\Rightarrow z = 0$ ie at zero location

poles $\Rightarrow z - \alpha = 0$, $z = \alpha \Rightarrow$ poles at distance α

Properties of the Region of Convergence :

→ How the ROC is related to the characteristics of a signal $x(n)$

→ Can Identify the ROC from $X(z)$ & limited knowledge of the characteristics of $x(n)$.

→ The relationship b/w the ROC and the characteristics of the time-domain signal is used to find inverse z-transforms.

(1) The ROC cannot contain any poles.

∵ ROC is defined as the set of all z for which the z-transform converges.

→ $X(z)$ Must be finite for all "z" in ROC

→ "d" is pole ⇒ $|X(d)| = \infty$, z-transform does not converges at the pole. The pole cannot lie in the ROC

(2) The ROC for a finite-duration signal includes the entire z-plane, except possibility $z=0$ (or) $|z| = \infty$ (or both).

→ $x(n) \neq 0 \quad n_1 \leq n \leq n_2$

$$X(z) = \sum_{n=n_1}^{n_2} x(n) z^{-n}$$

→ Sum is converging, provided that each of its terms is finite

⇒ Signal has any nonzero causal component ie $(n_2 > 0)$, $X(z)$ involves z^{-1}, ie ROC cannot include $z=0$

$n \neq 0, -1, -2$

∴ $z^{-0} \Rightarrow 1$ ×

⇒ if $x(n)$ is non causal ie previous values $(n_1 < 0)$

$X(z)$ involves power of z ie z^{+n}

∴ ROC cannot include $|z| = \infty$

⇒ if $n_2 \leq 0$ ⇒ ROC includes $z=0$

⇒ if $n_1 \geq 0$ ⇒ ROC includes $|z| = \infty$

⇒ $x(n) = c\,\delta(n)$ ⇒ ROC is the entire z-plane.

→ Let's consider infinite-duration signals.

⇒ Condition for convergence is $|X(z)| < \infty$

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x(n)\, z^{-n} \right|$$

$$|X(e^{j\omega})| \leq \sum_{n=-\infty}^{\infty} |x(n)\, z^{-n}|$$

where $e^{j\omega} = \cos\omega + j\sin\omega$

ie magnitude of a sum of complex no's is ≤ to the sum of the individual magnitudes.

$$|X(re^{j\omega})|_{r=1} = \sum_{n=-\infty}^{\infty} |x(n)|\,|z|^{-n}$$

magnitude of a product equals the product of the magnitudes.

→ Splitting infinite sum into "−"ve & "+"ve time

$$I_-(z) = \sum_{n=-\infty}^{-1} |x(n)|\,|z|^{-n}$$

$$\& \quad I_+(z) = \sum_{n=0}^{\infty} |x(n)|\,|z|^{-n}$$

} if these two are finite

↓

ie $|X(z)| \leq I_-(z) + I_+(z)$ } This will be finite

↓

ie $|x(n)|$ should be bounded

→ If $|x(n)|$ bounded by smallest positive constants ie $A_-, A_+, r_- \& r_+$

$|x(n)| = Ar^n$

$$|x(n)| \leq A_-(r_-)^n, \quad n<0$$

$$\& \quad |x(n)| \leq A_+(r_+)^n, \quad n \geq 0$$

⇒ Signal that satisfies above two bounds grows no faster than $(r_+)^n$ for positive n and (r_-^n) for negative n.

⇒ Can construct signals that do not satisfy these bounds. (Ex: a^{n^2})

→ Let's substitute smallest positive constants in $I_-(z)$ & $I_+(z)$

⇒ WKT $I_-(z) = \sum_{n=-\infty}^{-1} |x(n)||z|^{-n}$

ROC of a left sided signal

Radius lesser than the radius of pole

$= \sum_{n=-\infty}^{-1} A_-(r_-)^n |z|^{-n}$

$= A_- \sum_{n=-\infty}^{-1} (r_-)^n |z|^{-n}$

$= A_- \sum_{n=-\infty}^{-1} \left(\frac{r_-}{|z|}\right)^n = A_- \sum_{k=1}^{\infty} \left(\frac{|z|}{r_-}\right)^k$

where $k = -n$

If $\boxed{|z| \leq r_-}$, sum converges.

$k = -n$
$k = 1 \Rightarrow -n = 1$
$n = -1$
$k = \infty \Rightarrow -n = \infty$
$n = \infty$

⇒ $I_+(z) = \sum_{n=0}^{\infty} |x(n)||z|^{-n}$

ROC of a right sided signal

A radius is greater than the radius of pole

$= \sum_{n=0}^{\infty} A_+(r_+)^n |z|^{-n}$

$= A_+ \sum_{n=0}^{\infty} (r_+)^n |z|^{-n}$

$= A_+ \sum_{n=0}^{\infty} \left(\frac{r_+}{|z|}\right)^{+n}$

Sum converges only if $\boxed{|z| > r_+}$

ie if $\boxed{r_+ < |z| < r_-}$ then both $I_+(z)$ & $I_-(z)$ converges & $|X(z)|$ also converges.

ROC of a two sided signal.

if $r_+ > r_-$ ⇒ no values of z for which convergence is guaranteed.

NOTE : $|z| \leq r_- \Rightarrow |z| > |\alpha|$
$|z| > r_+ \Rightarrow |z| < |\alpha|$

→ If the signal consist of sum of exponentials

⟹ ROC is the intersection of the ROCs associated with each ~~other~~ term.

(i) have a radius greater than that of the pole of largest radius associated with right sided terms

(ii) radius less than that of the pole of smallest radius associated with left-sided terms.

Properties of the Z-Transform :-

(1) Linearity $a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{ZT} a_1 X_1(z) + a_2 X_2(z)$

ROC , $R_1 \cap R_2$

(2) Time Reversal $x(-n) \xleftrightarrow{Z} X(1/z)$ ROC: $1/R$

(3) Time Shift $x(n) \xleftrightarrow{ZT} X(z)$

$x(n-n_0) \xleftrightarrow{Z} z^{-n_0} X(z)$

(4) Multiplication by an Exponential Sequence

$a^n x(n) \xleftrightarrow{Z} X(z/a)$ ROC : aR

(5) Convolution

$$\left.\begin{array}{l} x(n) \xleftrightarrow{Z} X(z) \ \text{ROC}: R_1 \\ h(n) \xleftrightarrow{Z} H(z) \ \text{ROC}: R_2 \end{array}\right\} x(n) * h(n) \xleftrightarrow{Z} X(z)H(z) \ \text{ROC}: R_1 \cap R_2$$

(6) Differentiation in the Z-Domain / Multiplication by a Ramp

$$n\,x(n) \xleftrightarrow{Z} -z \frac{d}{dz} X(z)$$

Proof :

(1) Linearity : if $x_1(n) \xleftrightarrow{Z} X_1(z)$ ROC : R_1

$x_2(n) \xleftrightarrow{Z} X_2(z)$ ROC : R_2

then $a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow{Z} a_1 X_1(z) + a_2 X_2(z)$ ROC $R_1 \cap R_2$

Soln

$$Z\{a_1 x_1(n) + a_2 x_2(n)\} = \sum_{n=-\infty}^{\infty} (a_1 x_1(n) + a_2 x_2(n))\, z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a_1 x_1(n) z^{-n} + \sum_{n=-\infty}^{\infty} a_2 x_2(n) z^{-n}$$

$$= a_1 \underbrace{\sum_{n=-\infty}^{\infty} x_1(n) z^{-n}}_{X_1(z)} + a_2 \underbrace{\sum_{n=-\infty}^{\infty} x_2(n) z^{-n}}_{X_2(z)}$$

$$= a_1 X_1(z) + a_2 X_2(z)$$

ROC : $R_1 \cap R_2$

(2) Time Reversal property :

If $x(n) \xleftrightarrow{z} X(z)$ ROC : R

then $x(-n) \xleftrightarrow{z} X(1/z)$ ROC : 1/R

Sol^n

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n)\, z^{-n}$$

Put $n = -m \Rightarrow -n = m$

LL $\Rightarrow n = -\infty \Rightarrow m = \infty$

UL $\Rightarrow n = \infty \Rightarrow m = -\infty$

by substituting in above eq^n

$$Z\{x(-n)\} = \sum_{m=\infty}^{-\infty} x(m)\, z^{m}$$

$$Z\{x(-n)\} = \underbrace{\sum_{m=-\infty}^{\infty} x(m)\, (1/z)^{-m}}_{X(1/z)}$$

$$\boxed{Z\{x(-n)\} = X(1/z)}$$ ROC = 1/R .

(3) Time Shift : if $x(n) \xleftrightarrow{z} X(z)$ ROC : R

then $x(n-n_0) \xleftrightarrow{z} z^{-n_0} X(z)$ ROC = R, except $z=0$ & $z=\infty$

Sol^n

$$Z\{x(n-n_0)\} = \sum_{n=-\infty}^{\infty} x(n-n_0)\, z^{-n}$$

let $n - n_0 = m \Rightarrow n = m + n_0$

LL $n = -\infty \Rightarrow m + n_0 = -\infty \Rightarrow m = -\infty - n_0$

$m = -\infty$

UL $n = \infty \Rightarrow m + n_0 = \infty \Rightarrow m = \infty - n_0 = \infty$

$m = \infty$

$$= \sum_{m=-\infty}^{\infty} x(m)\, z^{-(m+n_0)}$$

$$= \sum_{m=-\infty}^{\infty} x(m) \cdot z^{-m} z^{-n_0} = z^{-n_0} \underbrace{\sum_{m=-\infty}^{\infty} x(m)\, z^{-m}}_{X(z)}$$

$$Z\{x(n-n_0)\} = z^{-n_0} X(z)$$ ROC is R Except for $z=0$ & $z=\infty$

If $n_0 > 0$, poles will be introduced at origin ie $z=0$, which will overwrite the already existing zeros at origin ie $z=0$. due to $X(z)$

If $n_0 < 0$, The multiplier introduces zeros as origin which may overwrite already existing poles due to $X(z)$

(4) Multiplication by an Exponential sequence:

If $x(n) \xleftrightarrow{Z} X(z)$ ROC: R

then $a^n x(n) \xleftrightarrow{Z} X(z/a)$ ROC: Ra

Soln

$$Z\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$Z\{a^n x(n)\} = \sum_{n=-\infty}^{\infty} a^n x(n) z^{-n}$$

$$= \underbrace{\sum_{n=-\infty}^{\infty} x(n)(z/a)^{-n}}_{X(z)\ z = z/a}$$

$$Z\{a^n x(n)\} = X(z/a) \qquad \text{ROC}: aR$$

(5) Convolution:

If $x(n) \xleftrightarrow{Z} X(z)$ ROC: R_1

$h(n) \xleftrightarrow{Z} H(z)$ ROC: R_2

then $\underbrace{x(n) * h(n)}_{x_1(n)} \xleftrightarrow{Z} X(z) H(z)$ ROC $= R_1 \cap R_2$

Soln $Z\{x_1(n)\} = \sum_{n=-\infty}^{\infty} x_1(n) z^{-n}$

$$Z\{x(n) * h(n)\} = \sum_{n=-\infty}^{\infty} (x(n) * h(n)) z^{-n}$$

But wkt $x(n) * h(n) = \sum_{k=\infty}^{\infty} x(k) h(n-k)$

$$\therefore Z\{x(n) * h(n)\} = \sum_{n=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(k) h(n-k) z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x(k) \sum_{n=-\infty}^{\infty} h(n-k) z^{-n}$$

let $n-k = m \Rightarrow n = m+k$

LL $n = -\infty \Rightarrow m+k = -\infty \Rightarrow m = -\infty - k = -\infty$

UL $n = \infty \Rightarrow m+k = \infty \Rightarrow m = \infty - k = \infty$

$$\therefore = \sum_{k=-\infty}^{\infty} x(k) \underbrace{\sum_{m=-\infty}^{\infty} h(m) z^{-m}}_{H(z)} z^{-k}$$

$$= \underbrace{\sum_{k=-\infty}^{\infty} x(k) z^{-k}}_{X(z)} H(z)$$

$$\boxed{Z\{x(n) * h(n)\} = X(z) H(z)}$$

ROC : $R_1 \cap R_2$

(6) Multiplication by a Ramp / differentiation in z-domain

If $x(n) \xleftrightarrow{Z} X(z)$ ROC : R

then $n\, x(n) \xleftrightarrow{Z} -z \frac{d}{dz} X(z)$

sol^n $X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$

differentiat wrt z

$$\frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x(n) \, -n \, z^{-n-1}$$

$\times^{ly}$ by $-z$

$$-z \frac{d}{dz} X(z) = \sum_{n=-\infty}^{\infty} x(n)\, n\, z^{-n\cancel{-1}} \cancel{z}$$

$$-z \frac{d}{dz}(X(z)) = \underbrace{\sum_{n=-\infty}^{\infty} \{n x(n)\} z^{-n}}_{n\, X(z)}$$

$$\boxed{Z\{n x(n)\} = -z \frac{d}{dz}[X(z)]}$$

ROC : R