Module-5

Discrete time Fourier Transforms (DFT)

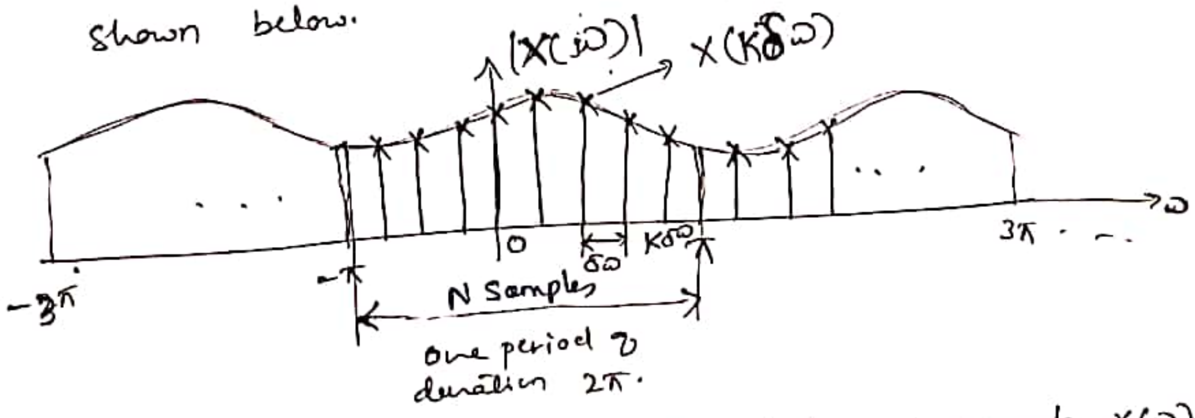
W.K.T a discrete time nonperiodic signal $x(n)$ is represented in frequency domain using Discrete Time Fourier Transform (DTFT). The DTFT is continuous in frequency and periodic with fundamental frequency 2π radians, which is not computationally convenient representation as analysis of signals is done most conviniently on digital signal processors or such as digital computers or digital hardware.

~~Therefore~~ Therefore there is a need to convert continuous frequency DTFT $X(j\omega)$ of a signal $x(n)$ in to discrete frequency, which leads to Discrete Fourier Transform. (DFT) of a signal and is represented by $X(K)$.

The DTFT of a signal $x(n)$ is given by

$$X(j\omega) = X(\omega) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} \quad \text{—(I)}$$

where ω is discrete angular freq. in rad. $X(j\omega)$ has continuous freq periodic spectrum as shown below.



To make $X(\omega)$ discrete let us take N samples $X(\omega)$ at a discrete interval $K\delta\omega$ where K varies from 0 to $(N-1)$ as ~~shown~~ shown in fig. above

$\therefore$ The spacing $\delta\omega = \frac{2\pi}{N}$.

ie $X(\omega)$ is sampled at $\omega = \omega_k = \frac{2\pi}{N}k$ to convert $X(\omega)$ in to discrete sequence of values $X(\omega_k)$

$\therefore$ $X(\omega)$ can be written as

$$X(\omega_k) = X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega_k n}$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

we can write $X\left(\frac{2\pi}{N}k\right)$ simply as $X(K)$.

$$\therefore X(K) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi}{N}kn} \quad \text{I a}$$

The infinite $\sum_{n=-\infty}^{\infty}$ can be split in to infinite sums of finite duration N and above eqn can be written as

$$X(K) = \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-lN)\, e^{-j\frac{2\pi}{N}kn}$$

(or) $$X(K) = \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x(n-lN) \cdot e^{-j\frac{2\pi}{N}kn}$$

ie $$X(K) = \sum_{n=0}^{N-1} \underbrace{x_p(n)}_{\text{periodic signal.}}\, e^{-j\frac{2\pi}{N}kn} \quad \text{— II}$$

$x_p(n)$ is periodic signal which can be expanded using fourier series as

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi}{N}kn} \quad K = 0 \text{ to } N-1 \quad \text{— III}$$

$$\& \quad x_p(n) = \sum_{k=0}^{N-1} c_k\, e^{\frac{j2\pi kn}{N}}$$

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comparing II & III

$$C_K = \frac{1}{N} X\left(\frac{2\pi}{N}k\right) = \frac{1}{N} X(K) \qquad K = 0 \text{ to } (N-1)$$

$$\therefore \; x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(K)\, e^{j\frac{2\pi}{N}kn} \qquad \text{— (IV)}$$

* It means that $x_p(n)$ can be reconstructed back by $X\left(\frac{2\pi}{N}k\right)$ (or) $X(K)$ ie samples of spectrum $X(\omega)$
* Since $x_p(n)$ is periodic extension of $x(n)$ it is clear that $x(n)$ can be obtained by $x_p(n)$ if there is no aliasing in time domain...... which is possible only if $N \geq$ length of time domain signal $x(n)$. ie L (ie $N \geq L$).

* when $N \geq L$

$$x_p(n) = x(n), \quad 0 \leq n \leq (N-1)$$

* $x(n)$ can be recovered from $X(K)$ by using (I a) when $N \geq L$. by only N values instead of infinite values

ie $X(K)$ can be written as

$$X(\omega_K) = X(K) = X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} \qquad K = 0 \text{ to } (N-1)$$

** (based on interpolation formula for reconstruction
Refer Proakis for more details) **

$$\boxed{\begin{aligned} \therefore \; X(k) &= \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}, \quad K = 0 \text{ to } (N-1) \\ \& \quad x(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X(K)\, e^{j\frac{2\pi}{N}kn}, \quad n = 0 \text{ to } (N-1) \end{aligned}}$$

are DFT pair of equations and represented as

$$x(n) \xleftrightarrow[N]{DFT} X(K).$$

<u>Example</u>

① Compute N-point DFT of $x(n) = \{1 \quad 2 \quad 3 \quad 4\}$.

Sol: $X(K) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} \qquad K = 0 \text{ to } (N-1)$

Here N = length of $x(n) = 4$.

$\therefore X(K) = \sum_{n=0}^{3} x(n)\, e^{-j\frac{2\pi}{4}Kn}, \quad K = 0 \text{ to } 3.$

$$X(K) = x(0)\, e^{-j\frac{2\pi}{4}K\cdot 0} + x(1)\, e^{-j\frac{2\pi}{4}K\cdot 1} + x(2)\, e^{-j\frac{2\pi}{4}\cdot 2K} + x(3)\, e^{-j\frac{2\pi}{4}\cdot 3K}$$

$$X(K) = x(0) + x(1)\, e^{-j\frac{\pi}{2}K} + x(2)\, e^{-j\pi K} + x(3)\, e^{-j\frac{3\pi}{2}K}.$$

$K=0,\ X(0) = 10 \qquad = 10\angle 0^c$

$1,\ X(1) = -2+2j = \cancel{2.8284\angle 2.356^c}$ (or) $2.8284\angle -0.785^c$

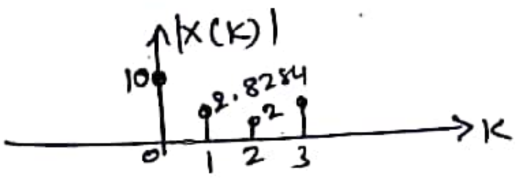
$2,\ X(2) = -2 \qquad = \cancel{2\angle \pi^c}$ (or) $0.$

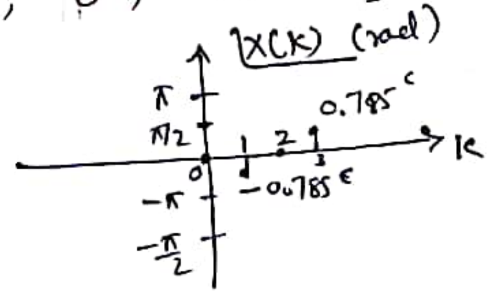
$3,\ X(3) = -2-2j = \cancel{2.8284\angle -2.356^c}$ (or) $2.8284\angle 0.785^c$

$\therefore X(K) = \{10 \quad (-2+2j) \quad -2 \quad (-2-2j)\}$

$|X(K)| = \{10,\ 2.8284,\ 2,\ 2.8284\}$ mag. Spectrum

$\angle X(K) = \{0,\ -0.785^c,\ 0,\ 0.785^c\}$ phase spectrum





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IIIly Try

1. $x(n) = \{1 \quad -1 \quad 2 \quad -3\}$

$X(K) = \{-1, (-1-2j), 7, (-1+2j)\}$

2. $x(n) = \{2 \quad -3 \quad 4 \quad -5 \quad 6\}$

$X(K) = \{4 \quad (3.736+3.2694j), (-0.736+13.849j), (-0.736-13.8496j), (3.7361-3.2694j)\}$

Here $N=5$

** N-DFT of any real signal is conjugate symmetric **

3. Find 5 point DFT of $x(n) = \{1 \quad 2 \quad 3\}$

Here $N=5$ ∴ add 2 zeros at the end of $x(n)$

ie $x(n) = \{1 \quad 2 \quad 3 \quad 0 \quad 0\}$

$$X(K) = \sum_{n=0}^{5-1} x(n)\, e^{-j\frac{2\pi}{5}kn}$$

$$= x(0)\, e^{0} + x(1)\, e^{-j\frac{2\pi}{5}K} + x(2)\, e^{-j\frac{2\pi}{5}\times 2K} + 0 + 0$$

$X(K) = \{6, (-0.809-3.66j), (0.309+1.6776j), (0.309-1.667j), (-0.809+3.665j)\}$

4. Try Find 6 point DFT of $x(n) = \{1 \quad -2 \quad 1 \quad -2\}$

$X(K) = \{-2 \quad (1.5+0.866j), (-0.5+2.598j), 6, (-0.5-2.598j), (1.5-0.866j)\}$

To find Inverse DFT (IDFT)

① Find $x(n)$ if $X(K) = \{18, (-2+2j), -2, (-2-2j)\}$

w.k.T $x(n) = IDFT[X(K)]$, here $N = 4$

ie $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) e^{j\frac{2\pi}{N}kn}$, $n = 0$ to $N-1$

ie $x(n) = \frac{1}{4} \sum_{K=0}^{3} X(K) e^{+j\frac{2\pi}{4}kn}$

$= \frac{1}{4}\left[X(0) + X(1) e^{\frac{j2\pi}{4}n} + X(2) e^{j\frac{2\pi \times 2n}{4}} + X(3) e^{\frac{j2\pi}{4}3n}\right]$

$x(n) = \frac{1}{4}\left[X(0) + X(1) e^{j\frac{\pi}{2}n} + X(2) e^{j\pi n} + X(3) e^{j\frac{3\pi}{2}n}\right]$

$n=0$, $x(0) = 3$ $n=2$, $x(2) = 5$.

$n=1$, $x(1) = 4$ $n=3$, $x(3) = 6$

$\therefore x(n) = \{3 \quad 4 \quad 5 \quad 6\}$.

② Find $x(n)$ if $X(K) = \{1, (3+2j), (2+3j), (2-3j), (3-2j)\}$

$x(n) = \{2.2, -1.542, 0.1474, -1.1946, 1.3898\}$

③ Find $x(n)$ if ; $X(K) = \{8, -0.0902, 11.09, 11.09, -0.0902\}$

$x(n) = \{6, -2, 3, 3, -2\}$

Sampling (in Time Domain)

→ It is the operation that generates the discrete time signal from a continuous time signal, which is needed to manipulate the signal on a computer or microprocessor.

→ Sampling can be performed on discrete time signals also to change the effective data rate, which is known as subsampling.

→ When the signal is sampled the Fourier representation of the signal also changes. Let us understand how sampling effects on Fourier representation of signal by representing both original signal and sampled signal using Fourier representation.

Sampling CT signal.

Let $x(t)$ be a CT signal and let us use $x[n]$ to represent samples of signal $x(t)$, at an integer time interval T_s

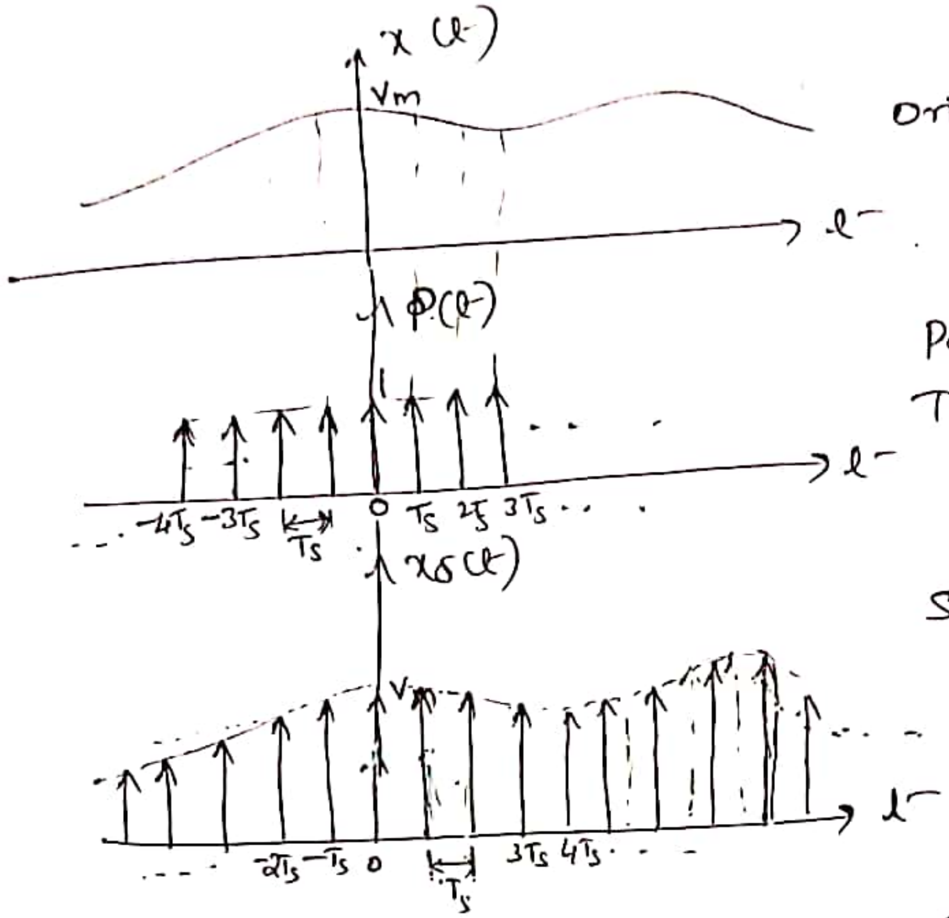
$$\therefore \quad x[n] = x(t)\big|_{t=nT_s} = x(nT_s)$$

$x(n)$ can be obtained by multiplying $x(t)$ with periodic impulse train $p(t)$ as shown in fig.

$$x_\delta(t) = x(t)\, p(t) \quad \text{— (I)}$$

Also we know that a DT signal $x(n)$ can be represented as linear combination of shifted impulses as

$$x(n) = \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t-nT_s) \quad \text{— (II)}$$



Original analog signal

Periodic impulse Train of $f_s = \frac{1}{T_s}$

Sampled signal.

also $P(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$ — (III)

$\therefore x_\delta(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$

$= \sum_{n=-\infty}^{\infty} x(t)\, \delta(t - nT_s)$

$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s)$ — (IV)

comparing (II) & (IV)

$x(n) = x_\delta(t)$

$\therefore x(n) = \sum_{n=-\infty}^{\infty} x(nT_s)\, \delta(t - nT_s)$

→ Here (I) implies that we may mathematically represent sampled signal as product of original CT signal & Impulse Train. This representation is known as impulse sampling used only for analyzing sampling.

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Representation of Sampling in freq domain

→ Let us use FT of $x(t)$ and $x_\delta(t)$ to analyze Sampling in freq domain

$$FT \text{ of } x_\delta(t) = X_\delta(j\omega)$$

w.k.T $\quad x_\delta(t) = x(t)\, p(t)$

$$\therefore \quad FT[x_\delta(t)] = FT[x(t)\, p(t)]$$

$$= \frac{1}{2\pi} FT[x(t)] * FT[p(t)] \quad \leftarrow \text{convoln prop in freq domain}$$

$$\therefore \quad X_\delta(j\omega) = \frac{X(j\omega) * P(j\omega)}{2\pi}$$

$$FT[p(t)] = FT\left[\sum_{n=-\infty}^{\infty} \delta(t - nT_s)\right]$$

$$P(j\omega) = \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

$$\therefore \quad X_\delta(j\omega) = \frac{1}{2\pi}\left[X(j\omega) * \frac{2\pi}{T_s} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)\right]$$

$$= \frac{1}{T_s}\left[X(j\omega) * \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)\right]$$

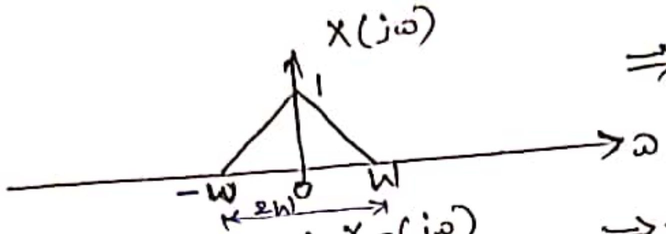
$$= \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j\omega) * \delta(\omega - k\omega_s)$$

$$\boxed{X_\delta(j\omega) = \frac{1}{T_s} \sum_{k=-\infty}^{\infty} X(j[\omega - k\omega_s])}$$

Thus the FT of sampled is given by infinite sum of shifted versions of the original signal's FT.

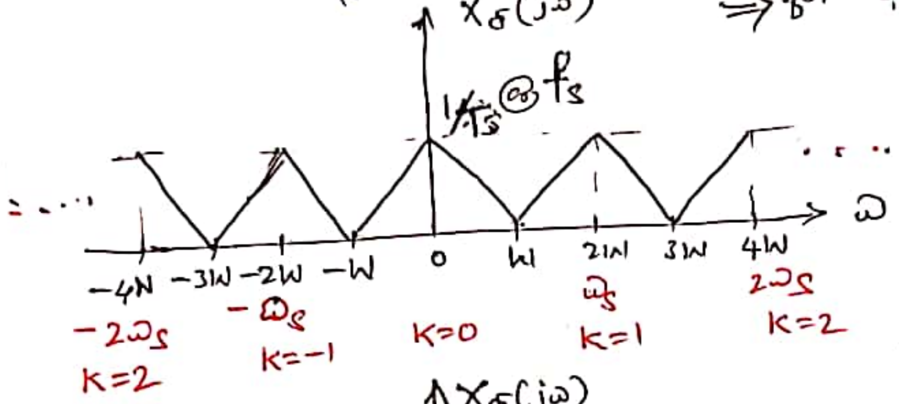
The same is shown in fig below.

fig (a)



⇒ FT of x(t)

fig (b)



FT of $X_\delta(j\omega)$

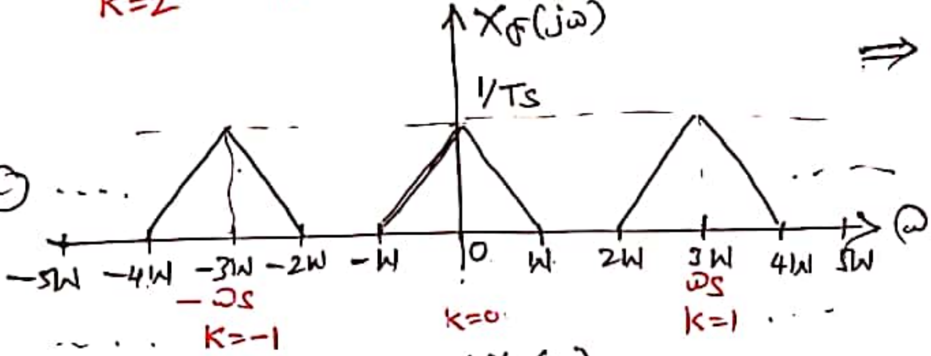
⇒ for $\omega_s = 2W$

$X_\delta(j\omega)$ for

⇒ $\omega_s > 2W$

assuming $\omega_s = 3W$

fig (c)

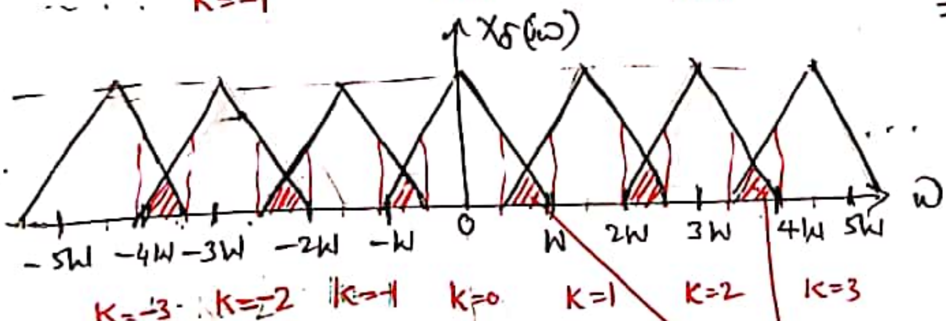


$X_\delta(j\omega)$ for

⇒ $\omega_s < 2W$

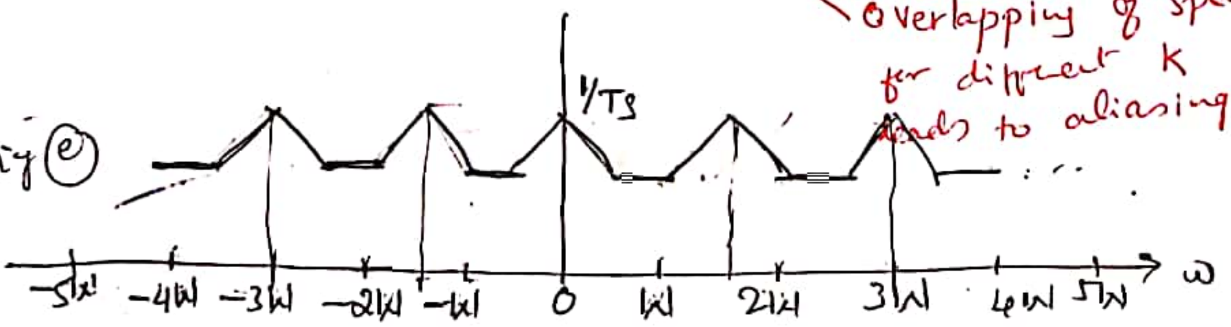
$\omega_s = 1.5W$

fig (d)



Overlapping of spectrum for different K leads to aliasing.

fig (e)



⇒ Reconstructed signal Spectrum for $\omega_s < 2W$

Aliasing

→ The overlapping of shifted replicas of the original spectrum is termed as <u>aliasing</u> which refers to the phenomenon of a high frequency continuous time component taking on the identity of the low frequency discrete time component.

→ Aliasing distorts the spectrum of the sampled signal as shown in fig (d) above. These replicas add & hence the resultant spectrum changes from triangle to constant as shown in fig (d) & (e).

→ Hence the spectrum of sampled signal does not have one-to-one correspondence with that of the original CT signal $x(t)$ ie we cannot use this spectrum for analysing $x(t)$ & hence can not uniquely reconstruct the original signal.

This means that the sampling interval must satisfy the condition $T_s < \frac{\pi}{W}$ for reconstruction of original signal to be feasible. W in radians.

The DTFT of the sampled signal $x_\delta(t)$ can be obtained by replacing $\Omega = \omega T_s$ in $X_\delta(j\omega)$.

ie $X(e^{j\Omega}) = X(j\omega)\Big|_{\omega=\Omega T_s}$. where Ω is Digital angular freq in rad.

ie $X(e^{j\Omega}) = \frac{1}{T_s} \sum_{K=\infty}^{\infty} X\left(e^{\frac{j}{T_s}(\Omega - K\Omega_s)}\right)$

Example

Let $x(t) = \cos(\pi t)$. Determine FT of sampled signal

for i) $T_s = 1/4$, $T_s = 1$, $T_s = 3/2$.

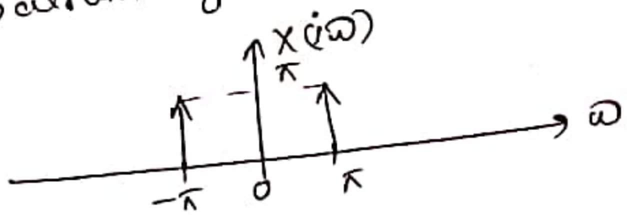
Sol here $\omega_0 = 2\pi f_0 = \pi \Rightarrow T_s = \frac{\pi}{\omega} = \cancel{\frac{\pi}{2\pi f_0}} = \frac{T_0}{2}$.

also $f_0 = \frac{1}{2}$. & $T_0 = 2$ sec.

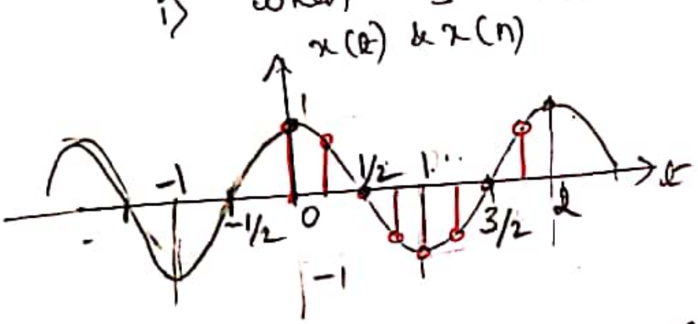
Also $X(j\omega) = \pi[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)]$

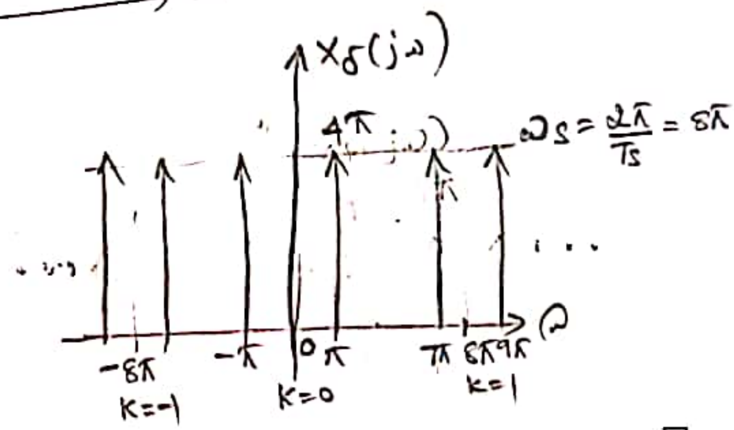
$= \pi[\delta(\omega - \pi) + \delta(\omega + \pi)]$

∴ spectrum of $x(t)$ is



i) when $T_s = 1/4$.

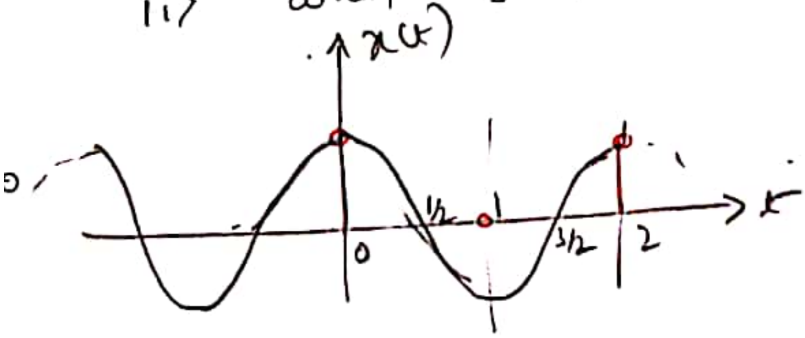


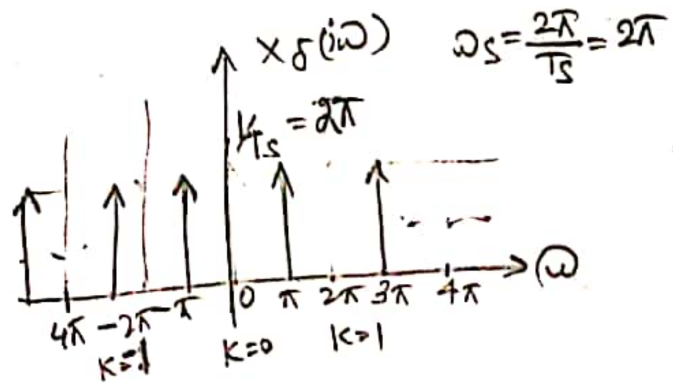


$$X(j\omega) = \frac{1}{T_s}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) = 4\sum_{k=-\infty}^{\infty} X[j(\omega - 8k)]$$

$$= \cdots \underset{k=-1}{X(j\omega+8)} + \underset{k=0}{X(j\omega)} + \underset{k=1}{X(j(\omega-8))} + \cdots$$

ii) when $T_s = 1$





(7)

$$X_\delta(j\omega) = T_s \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

$$\omega_s = 2\pi \qquad T_s = 1$$

$$X_\delta(j\omega) = \sum_{k=-\infty}^{\infty} X(j(\omega - 2\pi k))$$

$$= \dots \; X(j(\omega+2\pi)) + X(j\omega) + X(j(\omega-2\pi)) \dots$$

$$|X_\delta(j\omega)| = \dots \pi[\delta(\omega+\pi+2\pi) + \delta(\omega-\pi+2\pi)] \Rightarrow k=-1$$

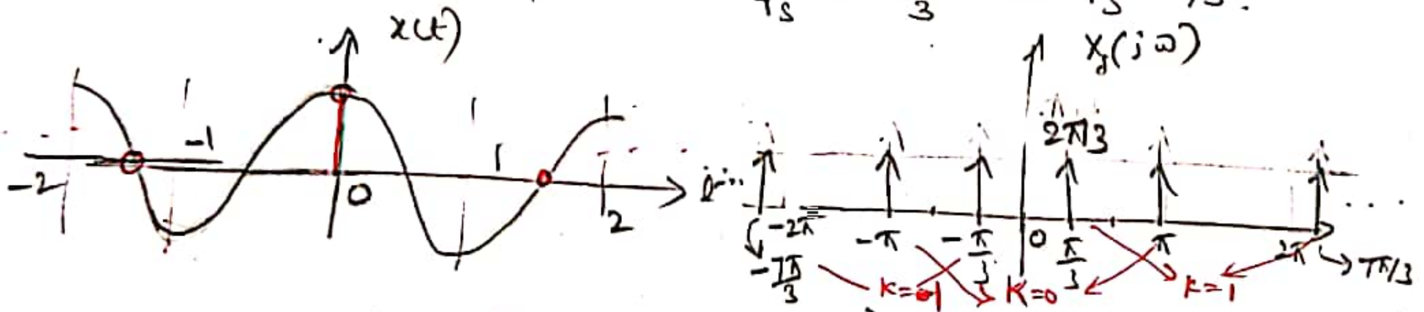
$$+ \pi[\delta(\omega+\pi) + \delta(\omega-\pi)] \Rightarrow k=0$$

$$+ \pi[\delta(\omega+\pi-2\pi) + \delta(\omega-\pi-2\pi)] + \dots \Rightarrow k=1$$

$$|X_\delta(j\omega)| = \pi[\dots + \delta(\omega+3\pi) + \underbrace{\delta(\omega+\pi) + \delta(\omega+\pi)}_{2\delta(\omega+\pi)} + \delta(\omega-\pi)$$

$$+ \underbrace{\delta(\omega-\pi)}_{2\delta(\omega-\pi)} + \underbrace{\delta(\omega-3\pi)}_{2 \text{ times}} + \dots]$$

iii) $T_s = 3/2 \Rightarrow \omega_s = \frac{2\pi}{T_s} = \frac{4\pi}{3}$ & $f_s = 2/3$.



$$X_\delta(j\omega) = \frac{2}{3} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) = \frac{2}{3} \sum X(j(\omega - \tfrac{4\pi}{3}k))$$

$$= \dots + \frac{2}{3} X(j(\omega + \tfrac{4\pi}{3})) + \frac{2}{3} X(j\omega) + \frac{2}{3} X(j(\omega - \tfrac{4\pi}{3}))$$

$$+ \dots$$

$$|X_\delta(j\omega)| = \frac{2\pi}{3}\Big[\dots \delta(\omega - \pi + \tfrac{4\pi}{3}) + \delta(\omega + \pi + \tfrac{4\pi}{3}) \Rightarrow k=-1$$

$$+ \delta(\omega-\pi) + \delta(\omega+\pi) \Rightarrow k=0$$

$$+ \delta(\omega - \pi - \tfrac{4\pi}{3}) + \delta(\omega + \pi - \tfrac{4\pi}{3}) + \dots\Big]$$

$$= \frac{2\pi}{3}\Big[\dots \delta(\omega + \tfrac{\pi}{3}) + \delta(\omega + \tfrac{7\pi}{3}) + \delta(\omega-\pi) + \delta(\omega+\pi)$$

$$+ \delta(\omega - \tfrac{7\pi}{3}) + \delta(\omega - \tfrac{\pi}{3}) + \dots\Big]$$

Reconstruction of Signal (CT) from Samples — The Sampling Theorem.

→ w.k.t sampling is a process of converting CT signal into DT signal. as shown above

∴ Reconstruction is a reverse process of sampling. i.e it is a process of getting CT from its samples.

→ As sampling a signal results in to one-to-one (relation) correspondence b/w time domain and frequency domain representation of signal, we may consider the problem of reconstruction in frequency domain

→ As seen before when signal is sampled with a frequency $\omega_s \geq 2W_m$ there is a one to one correspondence b/w FT of x(t) and ~~$X_s(j\omega)$~~ $x_s(t)$ and there is a aliasing effect which distorts one-to-one correspondence when $\omega_s < 2W_m$ where W_m is highest frequency of x(t)

→ ∴ Sampling theorem states that

If $x(t) \xleftrightarrow{FT} X(j\omega)$ represents a band-limited signal, such that $|X(j\omega)| = 0$ for $|\omega| > W_m$; then x(t) is uniquely determined by its samples $x(nT_s)$ iff $\omega_s \geq 2W_m$ where W_m is highest freq component of x(t) and ω_s is sampling freq & $\omega_s = \frac{2\pi}{T_s}$

T_s is known as sampling time.

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Ex

→ The minimum sampling frequency $2W$ is known as <u>Nyquist sampling rate</u> (or) <u>Nyquist Rate</u>

→ The actual sampling rate ω_s is known as Nyquist rate in rad/sec.

→ In many problems it is convenient to evaluate Sampling theorem with freq expressed in Hz.

ie If $f_m = \omega_m / 2\pi$ is the highest frequency present in $x(t)$ and f_s denotes sampling frequency then

Sampling theorem states that " A band limited signal $x(t)$ with highest frequency f_m Hz in it can be completely reconstructed from its samples $x(nT_s)$ (or) $x(n)$ if it is sampled with a frequency $f_s > 2f_m$.

or equivalently $T_s = \frac{1}{f_s} < \frac{T}{2}$ where T is fundamental time period $T = \frac{1}{f_m}$