$$= \frac{1}{2\pi j} \oint X_1(z) \left\{ \sum_{n=-\infty}^{\infty} x_2(n) [(z^*)^{-1}]^{-n} \right\}^* z^{-1}\, dz$$

$$= \frac{1}{2\pi j} \oint X_1(z) \left\{ X_2[(z^*)^{-1}] \right\}^* z^{-1}\, dz$$

$$= \frac{1}{2\pi j} \oint X_1(z)\, X_2^*[(z^*)^{-1}]\, z^{-1}\, dz = \text{RHS}$$

$$\boxed{\sum_{n=-\infty}^{\infty} x_1(n)\, x_2^*(n) = \frac{1}{2\pi j} \oint X_1(z)\, X_2^*[(z^*)^{-1}]\, z^{-1}\, dz}$$

10.6.13 Initial Value Theorem

The initial value theorem of Z-transform states that, for a causal signal $x(n)$

If $$x(n) \xleftrightarrow{\text{ZT}} X(z)$$

Then $$\underset{n \to 0}{\text{Lt}}\, x(n) = x(0) = \underset{z \to \infty}{\text{Lt}}\, X(z)$$

Proof: We know that for a causal signal

$$Z[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n} = x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + \cdots$$

$$= x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \cdots$$

Taking the limit $z \to \infty$ on both sides, we have

$$\underset{z \to \infty}{\text{Lt}}\, X(z) = \underset{z \to \infty}{\text{Lt}} \left[x(0) + \frac{x(1)}{z} + \frac{x(2)}{z^2} + \frac{x(3)}{z^3} + \cdots \right] = x(0) + 0 + 0 + \cdots = x(0)$$

$\therefore$ $$\underset{n \to 0}{\text{Lt}}\, x(n) = x(0) = \underset{z \to \infty}{\text{Lt}}\, X(z)$$

$$\boxed{x(0) = \underset{z \to \infty}{\text{Lt}}\, X(z)}$$

This theorem helps us to find the initial value of $x(n)$ from $X(z)$ without taking its inverse Z-transform.

10.6.14 Final Value Theorem

The final value theorem of Z-transform states that, for a causal signal

If $$x(n) \xleftrightarrow{\text{ZT}} X(z)$$

and if $X(z)$ has no poles outside the unit circle, and it has no double or higher order poles on the unit circle centred at the origin of the z-plane, then

$$\underset{n\to\infty}{\mathrm{Lt}}\, x(n) = x(\infty) = \underset{z\to 1}{\mathrm{Lt}}\,(z-1)\,X(z)$$

Proof: We know that for a causal signal

$$Z[x(n)] = X(z) = \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

$$Z[x(n+1)] = zX(z) - zx(0) = \sum_{n=0}^{\infty} x(n+1)\, z^{-n}$$

$\therefore$ $$Z[x(n+1)] - Z[x(n)] = zX(z) - zx(0) - X(z) = \sum_{n=0}^{\infty} x(n+1)\, z^{-n} - \sum_{n=0}^{\infty} x(n)\, z^{-n}$$

i.e. $$(z-1)\,X(z) - zx(0) = \sum_{n=0}^{\infty} [x(n+1) - x(n)]\, z^{-n}$$

i.e. $$(z-1)\,X(z) - zx(0) = [x(1) - x(0)]\, z^{-0} + [x(2) - x(1)]\, z^{-1} + [x(3) - x(2)]\, z^{-2} + \cdots$$

Taking limit $z \to 1$ on both sides, we have

$$\underset{z\to 1}{\mathrm{Lt}}\,(z-1)\,X(z) - x(0) = [x(1) - x(0) + x(2) - x(1) + x(3) - x(2) + \cdots + x(\infty) - x(\infty - 1)$$

$$= x(\infty) - x(0)$$

$\therefore$ $$x(\infty) = \underset{z\to 1}{\mathrm{Lt}}\,(z-1)\,X(z)$$

or $$x(\infty) = \underset{z\to 1}{\mathrm{Lt}}\,(1-z^{-1})\,X(z)$$

$$\boxed{x(\infty) = \underset{z\to 1}{\mathrm{Lt}}\,(z-1)\,X(z)}$$

This theorem enables us to find the steady-state value of $x(n)$, i.e. $x(\infty)$ without taking the inverse Z-transform of $X(z)$.

Some common Z-transform pairs are given in Table B.7 (Appendix B). The properties of Z-transform are given in Table B.8 (Appendix B).

EXAMPLE 10.27 Using properties of Z-transform, find the Z-transform of the following signals:

(a) $x(n) = u(-n)$

(b) $x(n) = u(-n+1)$

(c) $x(n) = u(-n-2)$

(d) $x(n) = 2^n u(n-2)$

Solution:

(a) Given $$x(n) = u(-n)$$

We know that $$Z[u(n)] = \frac{z}{z-1} = \frac{1}{1-z^{-1}};\ \text{ROC}; |z| > 1$$

Using the time reversal property,

$$Z[u(-n)] = \left.\frac{z}{z-1}\right|_{z=(1/z)} = \frac{1/z}{(1/z)-1} = \frac{1}{1-z} = -\frac{1}{z-1}; \text{ ROC; } |z|<1$$

(b) Given $$x(n) = u(-n+1) = u[-(n-1)]$$

$\therefore$ $$Z[x(n)] = X(z) = Z[u(-n+1)] = Z\{u[-(n-1)]\}$$

$$= z^{-1} Z[u(-n)] = z^{-1}\frac{1}{1-z} = -\frac{1}{z(z-1)}$$

(c) Given $$x(n) = u(-n-2) = u[-(n+2)]$$

$\therefore$ $$Z[x(n)] = X(z) = Z[u(-n-2)] = Z\{u[-(n+2)]\}$$

$$= z^2 Z[u(-n)] = \frac{z^2}{1-z} = -\frac{z^2}{z-1}$$

(d) Given $$x(n) = 2^n u(n-2)$$

$$Z[u(n-2)] = z^{-2} Z[u(n)] = z^{-2}\frac{z}{z-1} = \frac{z^{-1}}{z-1} = \frac{1}{z(z-1)}$$

$$Z[2^n u(n-2)] = Z[u(n-2)]\Big|_{z=(z/2)} = \left.\frac{1}{z(z-1)}\right|_{z=(z/2)} = \frac{1}{(z/2)[(z/2)-1]} = \frac{4}{z(z-2)}$$

EXAMPLE 10.28 Using properties of Z-transform, find the Z-transform of the sequence

(a) $x(n) = \alpha^{n-2} u(n-2)$ (b) $x(n) = \begin{cases} 1, & \text{for } 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$

Solution:

(a) The Z-transform of the sequence $x(n) = \alpha^n u(n)$ is given by

$$X(z) = \frac{z}{z-\alpha}; \text{ ROC; } |z|>|\alpha|$$

Using the time shifting property of Z-transform, we have

$$Z[x(n-m)] = z^{-m} X(z)$$

In the same way,

$$Z[\alpha^{n-2} u(n-2)] = z^{-2} Z[\alpha^n u(n)] = z^{-2}\frac{z}{z-\alpha} = \frac{1}{z(z-\alpha)}; \text{ ROC; } |z|>|\alpha|$$

(b) Given
$$x(n) = \begin{cases} 1, & \text{for } 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases}$$

implies that $x(n) = u(n) - u(n-N)$.

We know that
$$Z[u(n)] = \frac{z}{z-1}$$

Using the time shifting property, we have
$$Z[u(n-N)] = z^{-N} Z[u(n)] = z^{-N} \frac{z}{z-1}$$

Using the linearity property, we have
$$Z[u(n) - u(n-N)] = Z[u(n)] - Z[u(n-N)] = \frac{z}{z-1} - z^{-N} \frac{z}{z-1} = \frac{z}{z-1}[1 - z^{-N}]$$

EXAMPLE 10.29 Using appropriate properties, find the Z-transform of the signal
$$x(n) = 2(3)^n u(-n)$$

Solution: Given
$$x(n) = 2(3)^n u(-n)$$

We know that
$$Z[u(n)] = \frac{1}{1 - z^{-1}}; \text{ ROC}; |z| > 1$$

Using the time reversal property, we have
$$Z[u(-n)] = Z[u(n)]\Big|_{z \to 1/z} = \frac{1}{1-z}; \text{ ROC}; |z| > |\alpha|$$

Using scaling in z-domain property, we have
$$Z[3^n u(-n)] = Z[u(-n)]\Big|_{z \to z/3} = \frac{1}{1-z}\Bigg|_{z \to z/3} = \frac{1}{1-(z/3)}; \text{ ROC}; |z| < 3$$

Using the linearity property, we have
$$Z[2(3)^n u(-n)] = 2Z[3^n u(-n)] = 2\frac{1}{1-(z/3)} = \frac{2}{1-(1/3)z}; \text{ ROC}; |z| < 3$$

EXAMPLE 10.30 Using appropriate properties, find the Z-transform of
$$x(n) = n^2 \left(\frac{1}{3}\right)^n u(n-2)$$

Solution: Given $x(n) = n^2 \left(\frac{1}{3}\right)^n u(n-2)$
$$= n^2 \left(\frac{1}{3}\right)^2 \left(\frac{1}{3}\right)^{n-2} u(n-2) = \frac{1}{9} n^2 \left(\frac{1}{3}\right)^{n-2} u(n-2)$$

We know that

$$Z[u(n)] = \frac{z}{z-1};\ \text{ROC: } |z| > 1$$

From the property of multiplication by an exponential, we have

$$Z\left[\left(\frac{1}{3}\right)^n u(n)\right] = Z[u(n)]\Big|_{z \to [z/(1/3)] = 3z} = \frac{3z}{3z-1} = \frac{z}{z-(1/3)},\ |z| > \frac{1}{3}$$

Using the time shifting property, we have

$$Z\left[\left(\frac{1}{3}\right)^{n-2} u(n-2)\right] = z^{-2} Z\left[\left(\frac{1}{3}\right)^n u(n)\right] = z^{-2}\frac{z}{z-(1/3)} = \frac{1}{z[z-(1/3)]} = \frac{1}{z^2-(1/3)z}$$

Using differentiation in z-domain property, we have

$$Z\left[n\left(\frac{1}{3}\right)^{n-2} u(n-2)\right] = -z\frac{d}{dz}\left\{Z\left[\left(\frac{1}{3}\right)^{n-2} u(n-2)\right]\right\}$$

$$= -z\frac{d}{dz}\left[\frac{1}{z^2-(1/3)z}\right] = -z\left\{\frac{-1\,[2z-(1/3)]}{[z^2-(1/3)z]^2}\right\}$$

$$= \frac{2z^2-(1/3)z}{[z^2-(1/3)z]^2}$$

Again using differentiation in z-domain property, we have

$$Z\left[n^2\left(\frac{1}{3}\right)^{n-2} u(n-2)\right]$$

$$= -z\frac{d}{dz}\left\{Z\left[n\left(\frac{1}{3}\right)^{n-2} u(n-2)\right]\right\} = -z\frac{d}{dz}\left\{\frac{2z^2-(1/3)z}{[z^2-(1/3)z]^2}\right\}$$

$$= -z\left\{\frac{[z^2-(1/3)z]^2\,[4z-(1/3)] - [2z^2-(1/3)z]\,2[z^2-(1/3)z]\,[2z-(1/3)]}{[z^2-(1/3)z]^4}\right\}$$

$$= -z\left[\frac{-4z^3+z^2-(1/9)z}{[z^2-(1/3)z]^3}\right] = \left\{\frac{z^2\,[4z^2-z+(1/9)]}{z^3\,[z-(1/3)]^3}\right\} = \left\{\frac{4z^2-z+(1/9)}{z[z-(1/3)]^3}\right\}$$

Using the linearity property, we get

$$Z\left[\frac{1}{9}n^2\left(\frac{1}{3}\right)^{n-2} u(n-2)\right] = \frac{1}{9}\left\{\frac{4z^2-z+(1/9)}{z\,[z-(1/3)]^3}\right\}$$

EXAMPLE 10.31 Using appropriate properties of Z-transform, find the Z-transform of the signal

$$x(n) = n2^n \sin\left(\frac{\pi}{2}n\right)u(n)$$

Solution: Given

$$x(n) = n2^n \sin\left(\frac{\pi}{2}n\right)u(n)$$

We know that

$$Z\left[\sin\left(\frac{\pi}{2}n\right)u(n)\right] = \frac{z \sin(\pi/2)}{z^2 - 2z\cos(\pi/2) + 1} = \frac{z}{z^2+1}$$

Using the multiplication by an exponential property, we have

$$Z\left[2^n \sin\left(\frac{\pi}{2}n\right)u(n)\right] = Z\left[\sin\left(\frac{\pi}{2}n\right)u(n)\right]\Bigg|_{z\to(z/2)}$$

$$= \frac{z}{z^2+1}\Bigg|_{z\to(z/2)} = \frac{z/2}{(z/2)^2+1}$$

$$= \frac{2z}{z^2+4}$$

Using differentiation in z-domain property, we have

$$Z\left[n2^n \sin\left(\frac{\pi}{2}n\right)u(n)\right] = -z\frac{d}{dz}\left\{Z\left[2^n \sin\left(\frac{\pi}{2}n\right)u(n)\right]\right\}$$

$$= -z\frac{d}{dz}\left(\frac{2z}{z^2+4}\right) = -z\left[\frac{(z^2+4)(2) - 2z(2z)}{(z^2+4)^2}\right]$$

$$= -z\left[\frac{-2z^2+8}{(z^2+4)^2}\right] = \frac{2z(z^2-4)}{(z^2+4)^2}$$

EXAMPLE 10.32 Determine the Z-transform of the following signals and sketch the ROC:

(a) $x_1(n) = \begin{cases} (1/2)^n, & n \ge 0 \\ (1/4)^{-n}, & n < 0 \end{cases}$

(b) Using properties of Z-transform, determine $x_2(n) = x_1(n+2)$.

Solution:

(a) Given

$$x_1(n) = \begin{cases} (1/2)^n, & n \ge 0 \\ (1/4)^{-n}, & n < 0 \end{cases}$$

$$\therefore \qquad X_1(z) = \sum_{n=-\infty}^{\infty} x_1(n)\, z^{-n}$$

$$= \sum_{n=-\infty}^{-1} \left(\frac{1}{4}\right)^{-n} z^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n} z^{-n}$$

$$= \sum_{n=-1}^{-\infty} \left(\frac{1}{4}z\right)^{-n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^{n}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{4}z\right)^{n} + \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^{n} = \sum_{n=0}^{\infty} \left(\frac{1}{4}z\right)^{n} - 1 + \sum_{n=0}^{\infty} \left(\frac{1}{2}z^{-1}\right)^{n}$$

The first sequence converges if $|(1/4)z| < 1$, i.e. $|z| < 4$ and the second sequence converges if $|(1/2)z^{-1}| < 1$, i.e. if $|z| > 1/2$.
So $X_1(z)$ converges if $1/2 < |z| < 4$.

$$\therefore \qquad X_1(z) = \frac{(1/4)z}{1-(1/4)z} + \frac{1}{1-(1/2)z^{-1}};\ \text{ROC; } \frac{1}{2} < |z| < 4$$

$$= -\frac{z}{z-4} + \frac{z}{z-(1/2)} = \frac{-(7/2)z}{(z-4)[z-(1/2)]}$$

(b) $$x_2(n) = x_1(n+2)$$

Using the time shifting property

$$X_2(z) = Z[x_1(n+2)] = z^2 Z[x_1(n)]$$

$$= z^2 X_1(z) = \frac{-(7/2)\, z^3}{(z-4)[z-(1/2)]};\ \text{ROC; } \frac{1}{2} < |z| < 4$$

The pole-zero location and ROC are shown in Figure 10.13.

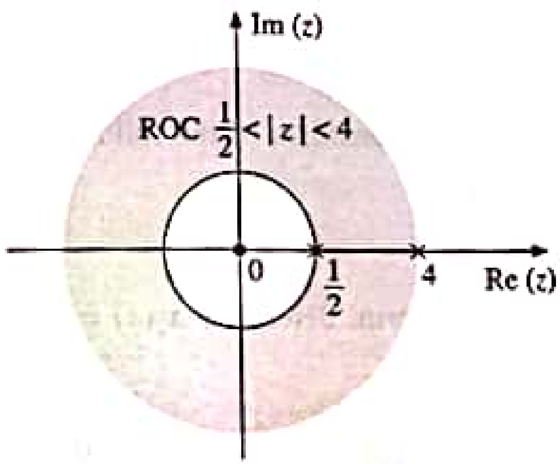

Figure 10.13 Pole-zero location and ROC for Example 10.32.

EXAMPLE 10.33 Determine the Z-transform of the following signal:

$$x(n) = \frac{1}{3}(n^2 + n)\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

Solution: Given

$$x(n) = \frac{1}{3}(n^2 + n)\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

$$= \frac{1}{3}n^2\left(\frac{1}{2}\right)^{n-1} u(n-1) + \frac{1}{3}n\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

We know that

$$Z\left[\left(\frac{1}{2}\right)^n u(n)\right] = \frac{z}{z-(1/2)}; \text{ ROC; } |z| > \frac{1}{2}$$

Using the time shifting property, we have

$$Z\left[\left(\frac{1}{2}\right)^{n-1} u(n-1)\right] = z^{-1}Z\left[\left(\frac{1}{2}\right)^n u(n)\right] = z^{-1}\frac{z}{z-(1/2)} = \frac{1}{z-(1/2)}$$

Using the multiplication by n property, we have

$$Z\left[n\left(\frac{1}{2}\right)^{n-1} u(n-1)\right] = -z\frac{d}{dz}\left\{Z\left[\left(\frac{1}{2}\right)^{n-1} u(n-1)\right]\right\} = -z\frac{d}{dz}\left[\frac{1}{z-(1/2)}\right]$$

$$= -z\left[\frac{-1}{[z-(1/2)]^2}\right] = \frac{z}{[z-(1/2)]^2}$$

Using the multiplication by n property, we have

$$Z\left[n^2\left(\frac{1}{2}\right)^{n-1} u(n-1)\right] = -z\frac{d}{dz}\left\{Z\left[n\left(\frac{1}{2}\right)^{n-1} u(n-1)\right]\right\} = -z\frac{d}{dz}\left\{\frac{z}{[z-(1/2)]^2}\right\}$$

$$= -z\left\{\frac{[z-(1/2)]^2 - z\,2\,[z-(1/2)]}{[z-(1/2)]^4}\right\} = -z\frac{[z-(1/2)-2z]}{[z-(1/2)]^3}$$

$$= \frac{-z[-z-(1/2)]}{[z-(1/2)]^3} = \frac{z\,[z+(1/2)]}{[z-(1/2)]^3}$$

Using the linearity property, we have

$$X(z) = \frac{1}{3}\left\{Z\left[n^2\left(\frac{1}{2}\right)^{n-1} u(n-1)\right] + Z\left[n\left(\frac{1}{2}\right)^{n-1} u(n-1)\right]\right\}$$

$$= \frac{1}{3}\left\{\frac{z\,[z+(1/2)]}{[z-(1/2)]^3} + \frac{z}{[z-(1/2)]^2}\right\} = \frac{(2/3)z^2}{[z-(1/2)]^3}$$

Solution: The given sequence is:

$$x(n) = n\left(\frac{1}{2}\right)^n u(n) * \left[\delta(n) - \frac{1}{2}\delta(n-1)\right]$$

Let

$$x(n) = x_1(n) * x_2(n)$$

where

$$x_1(n) = n\left(\frac{1}{2}\right)^n u(n) \quad \text{and} \quad x_2(n) = \delta(n) - \frac{1}{2}\delta(n-1)$$

$\therefore$

$$X_1(z) = Z\left[n\left(\frac{1}{2}\right)^n u(n)\right] = -z\frac{d}{dz}\left[\frac{z}{z-(1/2)}\right]$$

$$= -\frac{z\{[z-(1/2)]-z\}}{[z-(1/2)]^2} = \frac{(1/2)\,z}{[z-(1/2)]^2}$$

$$X_2(z) = Z\left[\delta(n) - \frac{1}{2}\delta(n-1)\right] = 1 - \frac{1}{2}z^{-1} = \frac{z-(1/2)}{z}$$

Using the convolution property of Z-transforms, we get

$$X(z) = X_1(z)\,X_2(z) = \frac{(1/2)\,z}{[z-(1/2)]^2}\,\frac{z-(1/2)}{z} = \frac{(1/2)}{z-(1/2)};\ \text{ROC};\ |z| > \frac{1}{2}$$

Alternate method

$$x(n) = x_1(n) * x_2(n) = \sum_{k=-\infty}^{\infty} x_1(k)\,x_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty}\left[k\left(\frac{1}{2}\right)^k u(k)\right]\left[\delta(n-k) - \frac{1}{2}\delta(n-k-1)\right]$$

$$= \sum_{k=-\infty}^{\infty} k\left(\frac{1}{2}\right)^k u(k)\,\delta(n-k) - \frac{1}{2}\sum_{k=-\infty}^{\infty} k\left(\frac{1}{2}\right)^k u(k)\,\delta(n-k-1)$$

$$= n\left(\frac{1}{2}\right)^n u(n) - \frac{1}{2}(n-1)\left(\frac{1}{2}\right)^{n-1} u(n-1)$$

Taking Z-transform on both sides, we have

$$X(z) = \frac{(1/2)\,z}{[z-(1/2)]^2} - \frac{1}{2}z^{-1}\left[\frac{(1/2)\,z}{[z-(1/2)]^2}\right] = \frac{(1/2)}{z-(1/2)};\ \text{ROC};\ |z| > \frac{1}{2}$$

EXAMPLE 10.38 Find the Z-transform of the following signal using convolution property of Z-transforms.

$$x(n) = \left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{4}\right)^n u(n)$$

Solution: Let

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

$\therefore$

$$X_1(z) = \frac{z}{z-(1/2)}; \text{ ROC}; |z| > \frac{1}{2}$$

and

$$x_2(n) = \left(\frac{1}{4}\right)^n u(n)$$

$\therefore$

$$X_2(z) = \frac{z}{z-(1/4)}; \text{ ROC}; |z| > \frac{1}{4}$$

Now,

$$x(n) = x_1(n) * x_2(n)$$

$\therefore$

$$Z[x(n)] = X(z) = Z[x_1(n) * x_2(n)] = X_1(z)\, X_2(z); \text{ ROC}; |z| > \frac{1}{2}$$

$$= \frac{z}{z-(1/2)} \frac{z}{z-(1/4)}; \text{ ROC}; |z| > \frac{1}{2}$$

EXAMPLE 10.39 Find the Z-transform of the signal

$$x(n) = n\left[\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)\right]$$

Solution: Let

$$x_1(n) = \left(\frac{1}{2}\right)^n u(n)$$

$\therefore$

$$X_1(z) = \frac{1}{1-(1/2)z^{-1}} = \frac{z}{z-(1/2)}; \text{ ROC}; |z| > \frac{1}{2}$$

and

$$x_2(n) = \left(\frac{1}{3}\right)^n u(n)$$

$\therefore$

$$X_2(z) = \frac{1}{1-(1/3)z^{-1}} = \frac{z}{z-(1/3)}; \text{ ROC}; |z| > \frac{1}{3}$$

Using convolution in the time domain property, we have

$$Z[x_1(n) * x_2(n)] = X_1(z)\, X_2(z)$$

$$\therefore \qquad Z\left[\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)\right] = \frac{z}{z-(1/2)}\frac{z}{z-(1/3)}$$

Using differentiation in z-domain property, we have

$$Z\left\{n\left[\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)\right]\right\} = -z\frac{d}{dz}\left\{Z\left[\left(\frac{1}{2}\right)^n u(n) * \left(\frac{1}{3}\right)^n u(n)\right]\right\}$$

$$= -z\frac{d}{dz}\left\{\frac{z^2}{[z-(1/2)][z-(1/3)]}\right\}$$

$$= -z\left[\frac{[z^2-(5/6)z+(1/6)]2z - z^2[2z-(5/6)]}{[z^2-(5/6)z+(1/6)]^2}\right]$$

$$= -z\left[\frac{2z^3-(5/3)z^2+(1/3)z-2z^3+(5/6)z^2}{[z^2-(5/6)z+(1/6)]^2}\right]$$

$$= \frac{(5/6)z^2[z-(2/5)]}{[z-(1/2)]^2[z-(1/3)]^2}$$

EXAMPLE 10.40 Using Z-transform, find the convolution of the sequences

$$x_1(n) = \{2, 1, 0, -1, 3\}; \; x_2(n) = \{1, -3, 2\}$$

Solution: From the convolution property of Z-transforms, we have

$$Z\{x_1(n) * x_2(n)\} = X_1(z)X_2(z) \text{ which implies that}$$

$$x_1(n) * x_2(n) = Z^{-1}[X_1(z)X_2(z)]$$

Given $\qquad x_1(n) = \{2, 1, 0, -1, 3\}$

$\therefore \qquad X_1(z) = 2 + z^{-1} - z^{-3} + 3z^{-4}$

and $\qquad x_2(n) = \{1, -3, 2\}$

$\therefore \qquad X_2(z) = 1 - 3z^{-1} + 2z^{-2}$

$$\therefore \qquad X_1(z)X_2(z) = (2 + z^{-1} - z^{-3} + 3z^{-4})(1 - 3z^{-1} + 2z^{-2})$$

$$= 2 - 5z^{-1} + z^{-2} + z^{-3} + 6z^{-4} - 11z^{-5} + 6z^{-6}$$

Taking inverse Z-transform on both sides,

$$x(n) = \{2, -5, 1, 1, 6, -11, 6\}$$

MPLE 10.41 Find the convolution of the sequences

$$x_1(n)=\left(\frac{1}{2}\right)^n u(n) \text{ and } x_2(n)=\left(\frac{1}{3}\right)^{n-2} u(n-2)$$

: (a) Convolution property of Z-transforms and (b) Time domain method

tion:

(a) Convolution property of Z-transforms

Given $$x_1(n)=\left(\frac{1}{2}\right)^n u(n) \text{ and } x_2(n)=\left(\frac{1}{3}\right)^{n-2} u(n-2)$$

$$\therefore \quad X_1(z)=Z\left[\left(\frac{1}{2}\right)^n u(n)\right]=\frac{1}{1-(1/2)z^{-1}}=\frac{z}{z-(1/2)};\ \text{ROC};\ |z|>\frac{1}{2}$$

and $$X_2(z)=Z\left[\left(\frac{1}{3}\right)^{n-2} u(n-2)\right]=z^{-2}Z\left[\left(\frac{1}{3}\right)^n u(n)\right]$$

$$=z^{-2}\frac{1}{1-(1/3)z^{-1}}=\frac{z^{-1}}{z-(1/3)};\ \text{ROC};\ |z|>\frac{1}{3}$$

We know that

$$x(n)=x_1(n)*x_2(n)$$

$$\therefore \quad Z[x(n)]=X(z)=Z[x_1(n)*x_2(n)]=X_1(z)\,X_2(z)$$

$$\therefore \quad Z[x_1(n)*x_2(n)]=\frac{z}{z-(1/2)}\,\frac{z^{-1}}{z-(1/3)}=\frac{1}{[z-(1/2)][z-(1/3)]}$$

$$\therefore \quad x(n)=Z^{-1}\left\{\frac{1}{[z-(1/2)][z-(1/3)]}\right\}=Z^{-1}\left[\frac{1}{z-(1/2)}-\frac{1}{z-(1/3)}\right]6$$

$$=6\left[\left(\frac{1}{2}\right)^{n-1}u(n-1)-\left(\frac{1}{3}\right)^{n-1}u(n-1)\right]$$

(b) Time domain method

$$x_1(n)*x_2(n)=\sum_{k=0}^{n} x_1(k)\,x_2(n-k)$$

$$=\sum_{k=0}^{n-2}\left(\frac{1}{2}\right)^k u(k)\left(\frac{1}{3}\right)^{n-2-k} u(n-2-k)$$

$$=\sum_{k=0}^{n-2}\left(\frac{1}{2}\right)^k\left(\frac{1}{3}\right)^{n-2-k}=\sum_{k=0}^{n-2}\left(\frac{1}{2}\right)^k\left(\frac{1}{3}\right)^n\left(\frac{1}{3}\right)^{-2}\left(\frac{1}{3}\right)^{-k}$$

$$= 9\left(\frac{1}{3}\right)^n \sum_{k=0}^{n-2}\left(\frac{3}{2}\right)^k = 9\left(\frac{1}{3}\right)^n\left[\frac{1-(3/2)^{n-1}}{1-(3/2)}\right] = -18\left(\frac{1}{3}\right)^n\left[1-\left(\frac{3}{2}\right)^{n-1}\right]$$

$$= -6\left[\left(\frac{1}{3}\right)^{n-1} u(n-1) - \left(\frac{1}{3}\right)^{n-1}\left(\frac{3}{2}\right)^{n-1} u(n-1)\right]$$

$$= -6\left[\left(\frac{1}{3}\right)^{n-1} u(n-1) - \left(\frac{1}{2}\right)^{n-1} u(n-1)\right]$$

EXAMPLE 10.42 Find the convolution of the sequences $x_1(n) = (1/3)^n\ u(n)$ and $x_2(n) = (1/5)^n\ u(n)$ using (a) Convolution property of Z-transforms and (b) Time domain method.

Solution:

(a) **Convolution property of Z-transforms**

Given $$x_1(n) = \left(\frac{1}{3}\right)^n u(n) \text{ and } x_2(n) = \left(\frac{1}{5}\right)^n u(n)$$

$$\therefore \quad X_1(z) = Z\left[\left(\frac{1}{3}\right)^n u(n)\right] = \frac{1}{1-(1/3)\,z^{-1}} = \frac{z}{z-(1/3)};\ \text{ROC; } |z| > \frac{1}{3}$$

and $$X_2(z) = Z\left[\left(\frac{1}{5}\right)^n u(n)\right] = \frac{1}{1-(1/5)\,z^{-1}} = \frac{z}{z-(1/5)};\ \text{ROC; } |z| > \frac{1}{5}$$

We know that

$$x(n) = x_1(n) * x_2(n)$$

$$\therefore \quad Z[x(n)] = X(z) = Z[x_1(n) * x_2(n)] = X_1(z)\,X_2(z)$$

$$\therefore \quad Z[x_1(n) * x_2(n)] = \frac{z}{z-(1/3)}\,\frac{z}{z-(1/5)}$$

$$\therefore \quad x(n) = Z^{-1}\left[\frac{z}{z-(1/3)}\,\frac{z}{z-(1/5)}\right] = Z^{-1}\left[\frac{5}{2}\,\frac{z}{z-(1/3)} - \frac{3}{2}\,\frac{z}{z-(1/5)}\right]$$

$$= \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n)$$

(b) **Time domain method**

Given $$x_1(n) = \left(\frac{1}{3}\right)^n u(n) \text{ and } x_2(n) = \left(\frac{1}{5}\right)^n u(n)$$

$$x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{n} x_1(k)\, x_2(n-k)$$

$$= \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k u(k) \left(\frac{1}{5}\right)^{n-k} u(n-k)$$

$$= \sum_{k=0}^{n} \left(\frac{1}{3}\right)^k \left(\frac{1}{5}\right)^n \left(\frac{1}{5}\right)^{-k} = \left(\frac{1}{5}\right)^n \sum_{k=0}^{n} \left(\frac{1}{3} \times \frac{5}{1}\right)^k = \left(\frac{1}{5}\right)^n \sum_{k=0}^{n} \left(\frac{5}{3}\right)^k$$

$$= \left(\frac{1}{5}\right)^n \left[\frac{1-(5/3)^{n+1}}{1-(5/3)}\right] = -\frac{3}{2}\left\{\left(\frac{1}{5}\right)^n \left[1 - \left(\frac{5}{3}\right)^n \frac{5}{3}\right]\right\}$$

$$= -\frac{3}{2}\left(\frac{1}{5}\right)^n + \frac{3}{2}\left(\frac{1}{5}\right)^n \left(\frac{5}{3}\right)^n \frac{5}{3} = \frac{5}{2}\left(\frac{1}{3}\right)^n u(n) - \frac{3}{2}\left(\frac{1}{5}\right)^n u(n)$$

EXAMPLE 10.43 Using final value theorem, find $x(\infty)$, if $X(z)$ is given by

(a) $\dfrac{z+1}{(z-0.6)^2}$ (b) $\dfrac{z+2}{4(z-1)(z+0.7)}$ (c) $\dfrac{2z+3}{(z+1)(z+3)(z-1)}$

Solution:

(a) Given

$$X(z) = \frac{z+1}{(z-0.6)^2}$$

Looking at $X(z)$, we notice that the ROC of $X(z)$ is $|z| > 0.6$ and $(z-1)\,X(z)$ has no poles on or outside the unit circle. Therefore,

$$x(\infty) = \underset{z\to 1}{\text{Lt}}\ (z-1)\,X(z) = \underset{z\to 1}{\text{Lt}}\ (z-1)\frac{z+1}{(z-0.6)^2} = 0$$

(b) Given

$$X(z) = \frac{z+2}{4(z-1)(z+0.7)}$$

$$(z-1)\,X(z) = \frac{z+2}{4(z+0.7)}$$

$(z-1)\,X(z)$ has no poles on or outside the unit circle.

$$\therefore \quad x(\infty) = \underset{z\to 1}{\text{Lt}}\ (z-1)\,X(z) = \underset{z\to 1}{\text{Lt}}\left[\frac{z+2}{4(z+0.7)}\right] = \frac{3}{6.8} = 0.44$$

(c) Given

$$X(z) = \frac{2z+3}{(z+1)(z+3)(z-1)}$$

$$(z-1)\,X(z) = \frac{(z-1)(2z+3)}{(z+1)(z+3)(z-1)} = \frac{2z+3}{(z+1)(z+3)}$$

$(z-1)\,X(z)$ has one pole on the unit circle and one pole outside the unit circle. So $x(\infty)$ tends to infinity as $n \to \infty$.

EXAMPLE 10.44 Find $x(0)$ if $X(z)$ is given by

(a) $\dfrac{z^2+2z+2}{(z+1)(z+0.5)}$ (b) $\dfrac{z+3}{(z+1)(z+2)}$

Solution:

(a) Given $$X(z)=\frac{z^2+2z+2}{(z+1)(z+0.5)}=\frac{1+(2/z)+(2/z^2)}{[1+(1/z)][1+(0.5/z)]}$$

$$x(0)=\underset{z\to\infty}{\text{Lt}}\,X(z)=\underset{z\to\infty}{\text{Lt}}\,\frac{[1+(2/z)+(2/z^2)]}{[1+(1/z)][1+(0.5/z)]}=1$$

(b) Given $$X(z)=\frac{z+3}{(z+1)(z+2)}=\frac{z[1+(3/z)]}{z^2[1+(1/z)][1+(2/z)]}=\frac{1}{z}\,\frac{1+(3/z)}{[1+(1/z)][1+(2/z)]}$$

$$x(0)=\underset{z\to\infty}{\text{Lt}}\,X(z)=\underset{z\to\infty}{\text{Lt}}\,\frac{1}{z}\,\frac{1+(3/z)}{[1+(1/z)][1+(2/z)]}=0$$

EXAMPLE 10.45 Prove that the final value of $x(n)$ for $X(z)=z^2/(z-1)(z-0.2)$ is 1.25 and its initial value is unity.

Solution: Given $$X(z)=\frac{z^2}{(z-1)(z-0.2)}$$

The final value theorem states that

$$\underset{n\to\infty}{\text{Lt}}\,x(n)=x(\infty)=\underset{z\to 1}{\text{Lt}}\,(z-1)\,X(z)$$

$\therefore$ $$x(\infty)=\underset{z\to 1}{\text{Lt}}\,(z-1)\frac{z^2}{(z-1)(z-0.2)}=\underset{z\to 1}{\text{Lt}}\,\frac{z^2}{z-0.2}=\frac{1}{1-0.2}=1.25$$

The initial value theorem states that

$$\underset{n\to 0}{\text{Lt}}\,x(n)=x(0)=\underset{z\to\infty}{\text{Lt}}\,X(z)$$

$\therefore$ $$x(0)=\underset{z\to\infty}{\text{Lt}}\,\frac{1}{[1-(1/z)][1-(0.2/z)]}=1$$