▶ **Problem 3.1** Identify the appropriate Fourier representation for each of the following signals:

(a) $x[n] = (1/2)^n u[n]$

(b) $x(t) = 1 - \cos(2\pi t) + \sin(3\pi t)$

(c) $x(t) = e^{-t}\cos(2\pi t)u(t)$

(d) $x[n] = \sum_{m=-\infty}^{\infty} \delta[n - 20m] - 2\delta[n - 2 - 20m]$

Answers:

(a) DTFT

(b) FS

(c) FT

(d) DTFS

EXAMPLE 3.2 DETERMINING DTFS COEFFICIENTS Find the frequency-domain representation of the signal depicted in Fig. 3.5

Solution: The signal has period $N = 5$, so $\Omega_o = 2\pi/5$. Also, the signal has odd symmetry, so we sum over $n = -2$ to $n = 2$ in Eq. (3.11) to obtain

$$X[k] = \frac{1}{5}\sum_{n=-2}^{2} x[n]e^{-jk2\pi n/5}$$
$$= \frac{1}{5}\{x[-2]e^{jk4\pi/5} + x[-1]e^{jk2\pi/5} + x[0]e^{j0} + x[1]e^{-jk2\pi/5} + x[2]e^{-jk4\pi/5}\}.$$

Using the values of $x[n]$, we get

$$X[k] = \frac{1}{5}\left\{1 + \frac{1}{2}e^{jk2\pi/5} - \frac{1}{2}e^{-jk2\pi/5}\right\}$$
$$= \frac{1}{5}\{1 + j\sin(k2\pi/5)\}. \tag{3.12}$$

From this equation, we identify one period of the DTFS coefficients $X[k]$, $k = -2$ to $k = 2$, in rectangular and polar coordinates as

$$X[-2] = \frac{1}{5} - j\frac{\sin(4\pi/5)}{5} = 0.232e^{-j0.531}$$

$$X[-1] = \frac{1}{5} - j\frac{\sin(2\pi/5)}{5} = 0.276e^{-j0.760}$$

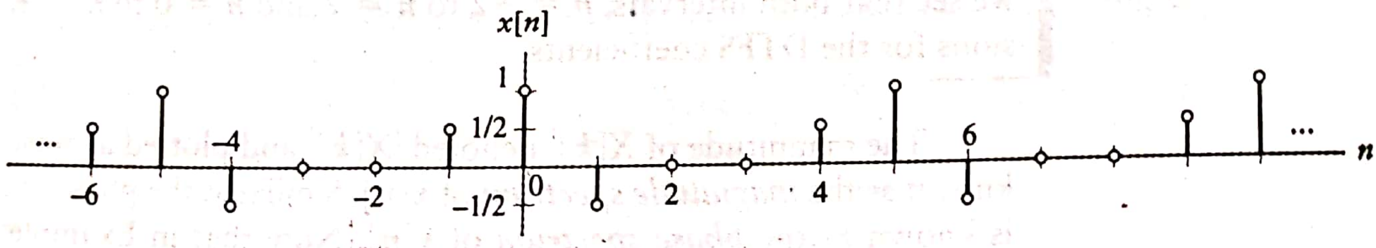

FIGURE 3.5 Time-domain signal for Example 3.2.

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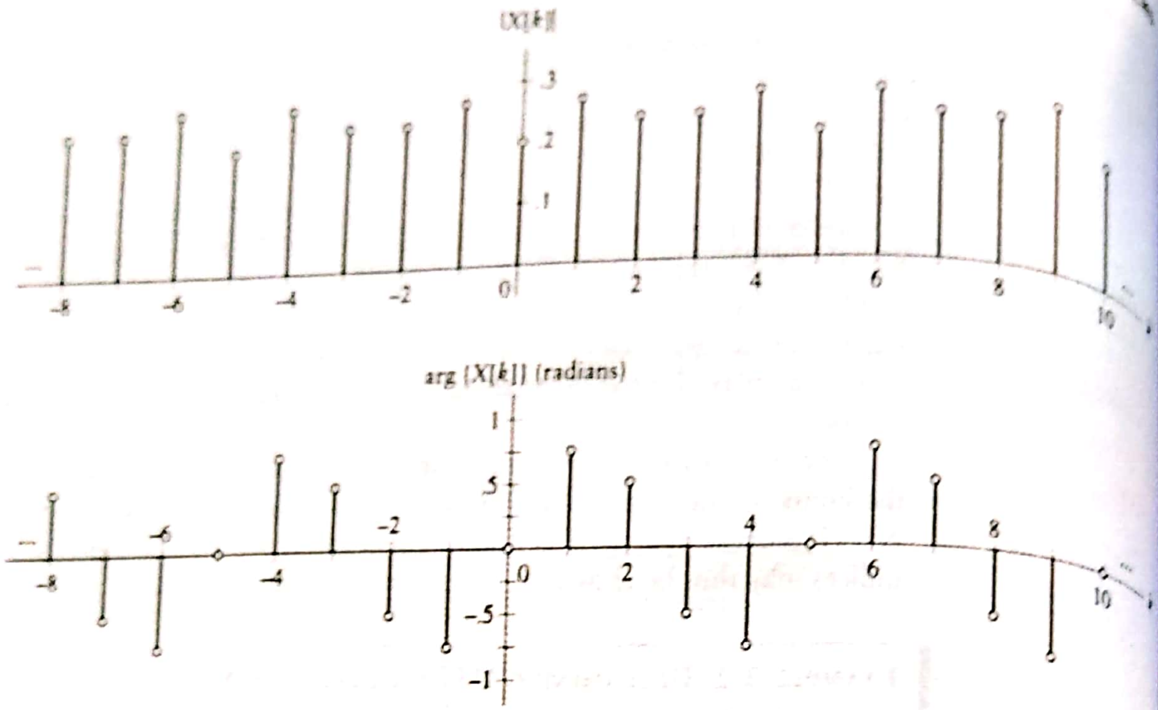


FIGURE 3.6 Magnitude and phase of the DTFS coefficients for the signal in Fig. 3.5.

$$X[0] = \frac{1}{5} = 0.2e^{j0}$$

$$X[1] = \frac{1}{5} + j\frac{\sin(2\pi/5)}{5} = 0.276e^{j0.760}$$

$$X[2] = \frac{1}{5} + j\frac{\sin(4\pi/5)}{5} = 0.232e^{j0.531}.$$

Figure 3.6 depicts the magnitude and phase of $X[k]$ as functions of the frequency index k.

Now suppose we calculate $X[k]$ using $n = 0$ to $n = 4$ for the limits on the sum in Eq. (3.11), to obtain

$$\begin{aligned} X[k] &= \frac{1}{5}\{x[0]e^{j0} + x[1]e^{-jk2\pi/5} + x[2]e^{-jk4\pi/5} + x[3]e^{-jk6\pi/5} + x[4]e^{-jk8\pi/5}\} \\ &= \frac{1}{5}\left\{1 - \frac{1}{2}e^{-jk2\pi/5} + \frac{1}{2}e^{-jk8\pi/5}\right\}. \end{aligned}$$

This expression appears to differ from Eq. (3.12), which was obtained using $n = -2$ to $n = 2$. However, noting that

$$\begin{aligned} e^{-jk8\pi/5} &= e^{-jk2\pi}e^{jk2\pi/5} \\ &= e^{jk2\pi/5}, \end{aligned}$$

we see that both intervals, $n = -2$ to $n = 2$ and $n = 0$ to $n = 4$, yield equivalent expressions for the DTFS coefficients. ∎

The magnitude of $X[k]$, denoted $|X[k]|$ and plotted against the frequency index k, is known as the *magnitude spectrum* of $x[n]$. Similarly, the phase of $X[k]$, termed $\arg\{X[k]\}$, is known as the *phase spectrum* of $x[n]$. Note that in Example 3.2 $|X[k]|$ is even while $\arg\{X[k]\}$ is odd.

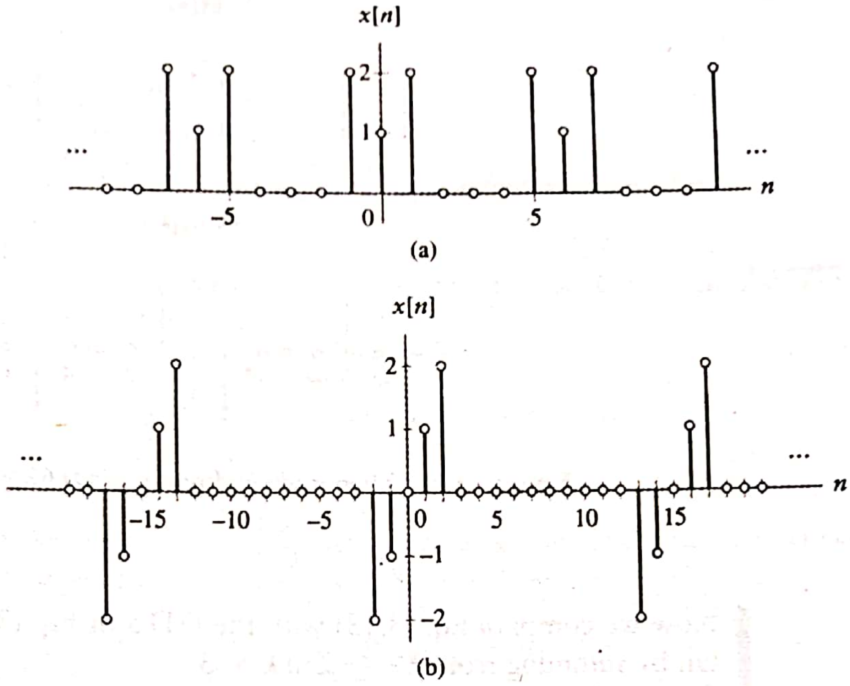


FIGURE 3.7 Signals $x[n]$ for Problem 3.2.

▶ **Problem 3.2** Determine the DTFS coefficients of the periodic signals depicted in Figs. 3.7(a) and (b).

Answers:

Fig. 3.7(a):

$$x[n] \xleftrightarrow{DTFS;\pi/3} X[k] = \frac{1}{6} + \frac{2}{3}\cos(k\pi/3)$$

Fig. 3.7(b):

$$x[n] \xleftrightarrow{DTFS;2\pi/15} X[k] = \frac{-2j}{15}(\sin(k2\pi/15) + 2\sin(k4\pi/15))$$ ◀

If $x[n]$ is composed of real or complex sinusoids, then it is often easier to determine $X[k]$ by inspection than by evaluating Eq. (3.11). The method of inspection is based on expanding all real sinusoids in terms of complex sinusoids and comparing each term in the result with each term of Eq. (3.10), as illustrated by the next example.

EXAMPLE 3.3 COMPUTATION OF DTFS COEFFICIENTS BY INSPECTION Determine the DTFS coefficients of $x[n] = \cos(\pi n/3 + \phi)$, using the method of inspection.

Solution: The period of $x[n]$ is $N = 6$. We expand the cosine by using Euler's formula and move any phase shifts in front of the complex sinusoids. The result is

$$x[n] = \frac{e^{j\left(\frac{\pi}{3}n+\phi\right)} + e^{-j\left(\frac{\pi}{3}n+\phi\right)}}{2} = \frac{1}{2}e^{-j\phi}e^{-j\frac{\pi}{3}n} + \frac{1}{2}e^{j\phi}e^{j\frac{\pi}{3}n}. \quad (3.13)$$

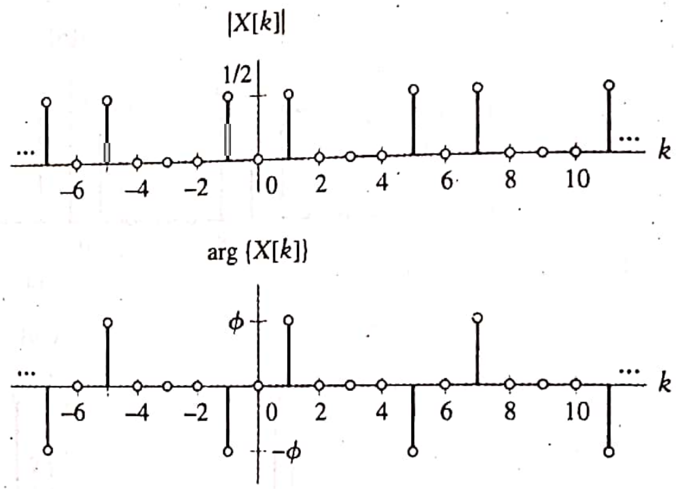


FIGURE 3.8 Magnitude and phase of DTFS coefficients for Example 3.3.

Now we compare Eq. (3.13) with the DTFS of Eq. (3.10) with $\Omega_o = 2\pi/6 = \pi/3$, written by summing from $k = -2$ to $k = 3$:

$$x[n] = \sum_{k=-2}^{3} X[k]e^{jk\pi n/3}$$

$$= X[-2]e^{-j2\pi n/3} + X[-1]e^{-j\pi n/3} + X[0] + X[1]e^{j\pi n/3} + X[2]e^{j2\pi n/3} + X[3]e^{j\pi n}. \tag{3.14}$$

Equating terms in Eq. (3.13) with those in Eq. (3.14) having equal frequencies, $k\pi/3$, gives

$$x[n] \xleftrightarrow{DTFS;\frac{\pi}{3}} X[k] = \begin{cases} e^{-j\phi}/2, & k = -1 \\ e^{j\phi}/2, & k = 1 \\ 0, & \text{otherwise on } -2 \leq k \leq 3 \end{cases}$$

The magnitude spectrum, $|X[k]|$, and phase spectrum, $\arg\{X[k]\}$, are depicted in Fig. 3.8. ■

▶ **Problem 3.3** Use the method of inspection to determine the DTFS coefficients for the following signals:

(a) $x[n] = 1 + \sin(n\pi/12 + 3\pi/8)$

(b) $x[n] = \cos(n\pi/30) + 2\sin(n\pi/90)$

Answers:

(a) $$x[n] \xleftrightarrow{DTFS;2\pi/24} X[k] = \begin{cases} -e^{-j3\pi/8}/(2j), & k = -1 \\ 1, & k = 0 \\ e^{j3\pi/8}/(2j), & k = 1 \\ 0, & \text{otherwise on } -11 \leq k \leq 12 \end{cases}$$

(b) $$x[n] \xleftrightarrow{DTFS;2\pi/180} X[k] = \begin{cases} -1/j, & k = -1 \\ 1/j, & k = 1 \\ 1/2, & k = \pm 3 \\ 0, & \text{otherwise on } -89 \leq k \leq 90 \end{cases}$$ ◀

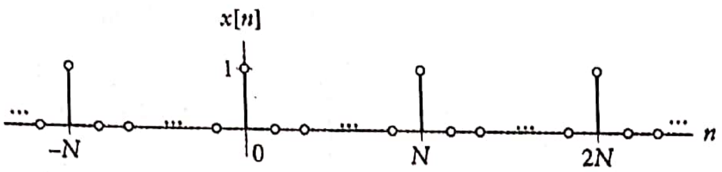


FIGURE 3.9 A discrete-time impulse train with period N.

EXAMPLE 3.4 DTFS REPRESENTATION OF AN IMPULSE TRAIN Find the DTFS coefficients of the N-periodic impulse train

$$x[n] = \sum_{l=-\infty}^{\infty} \delta[n - lN],$$

as shown in Fig. 3.9.

Solution: Since there is only one nonzero value in $x[n]$ per period, it is convenient to evaluate Eq. (3.11) over the interval $n = 0$ to $n = N - 1$ to obtain

$$\begin{aligned} X[k] &= \frac{1}{N}\sum_{n=0}^{N-1} \delta[n]e^{-jkn2\pi/N} \\ &= \frac{1}{N}. \end{aligned}$$

■

Although we have focused on evaluating the DTFS coefficients, the similarity between Eqs. (3.11) and (3.10) indicates that the same mathematical methods can be used to find the time-domain signal corresponding to a set of DTFS coefficients. Note that in cases where some of the values of $x[n]$ are zero, such as the previous example, $X[k]$ may be periodic in k with period less than N. In this case, it is not possible to determine N from $X[k]$, so N must be known in order to find the proper time signal.

EXAMPLE 3.5 THE INVERSE DTFS Use Eq. (3.10) to determine the time-domain signal $x[n]$ from the DTFS coefficients depicted in Fig. 3.10.

Solution: The DTFS coefficients have period 9, so $\Omega_o = 2\pi/9$. It is convenient to evaluate Eq. (3.10) over the interval $k = -4$ to $k = 4$ to obtain

$$\begin{aligned} x[n] &= \sum_{k=-4}^{4} X[k]e^{jk2\pi n/9} \\ &= e^{j2\pi/3}e^{-j6\pi n/9} + 2e^{j\pi/3}e^{-j4\pi n/9} - 1 + 2e^{-j\pi/3}e^{j4\pi n/9} + e^{-j2\pi/3}e^{j6\pi n/9} \\ &= 2\cos(6\pi n/9 - 2\pi/3) + 4\cos(4\pi n/9 - \pi/3) - 1. \end{aligned}$$

■

▶ **Problem 3.4** One period of the DTFS coefficients of a signal is given by

$$X[k] = (1/2)^k, \quad \text{on } 0 \le k \le 9.$$

Find the time-domain signal $x[n]$ assuming $N = 10$.

Answer:

$$x[n] = \frac{1 - (1/2)^{10}}{1 - (1/2)e^{j(\pi/5)n}}$$

◀

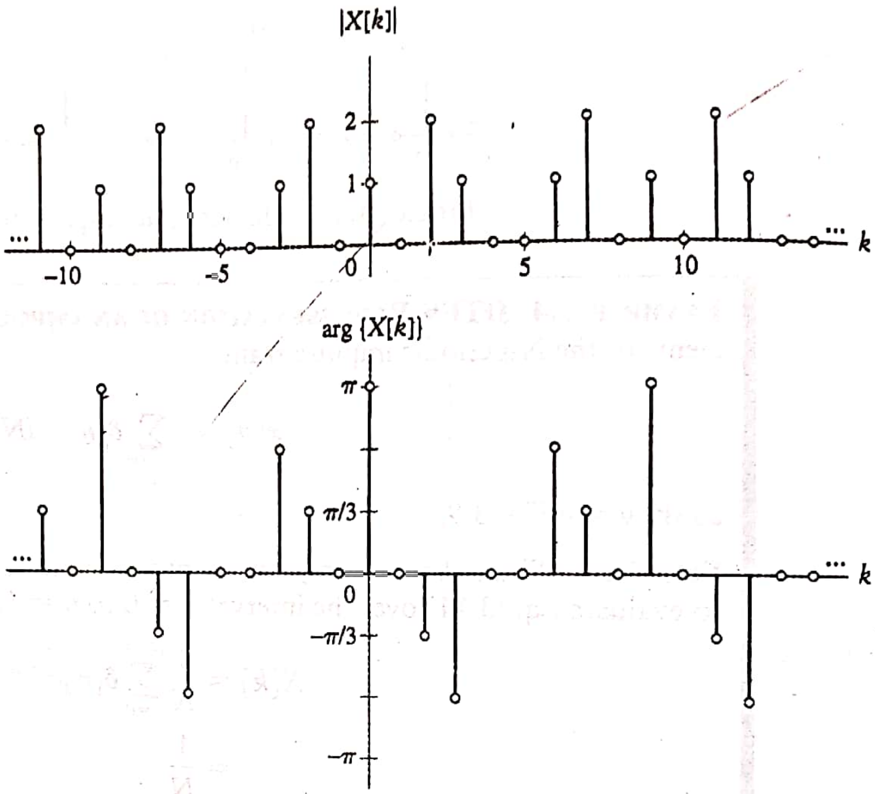


FIGURE 3.10 Magnitude and phase of DTFS coefficients for Example 3.5.

▶ **Problem 3.5** Use the method of inspection to find the time-domain signal corresponding to the DTFS coefficients

$$X[k] = \cos(k4\pi/11) + 2j\sin(k6\pi/11).$$

Answer:

$$x[n] = \begin{cases} 1/2, & n = \pm 2 \\ 1, & n = 3 \\ -1, & n = -3 \\ 0, & \text{otherwise on } -5 \le n \le 5 \end{cases}$$

◀

EXAMPLE 3.6 DTFS REPRESENTATION OF A SQUARE WAVE Find the DTFS coefficients for the N-periodic square wave given by

$$x[n] = \begin{cases} 1, & -M \le n \le M \\ 0, & M < n < N - M \end{cases}.$$

That is, each period contains $2M + 1$ consecutive ones and the remaining $N - (2M + 1)$ values are zero, as depicted in Fig. 3.11. Note that this definition requires that $N > 2M + 1$.

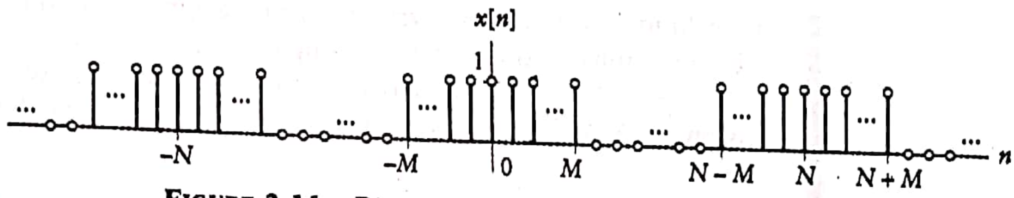


FIGURE 3.11 Discrete-time square wave for Example 3.6.

Solution: The period is N, so $\Omega_o = 2\pi/N$. It is convenient in this case to evaluate Eq. (3.11) over indices $n = -M$ to $n = N - M - 1$. We thus have

$$X[k] = \frac{1}{N}\sum_{n=-M}^{N-M-1} x[n]e^{-jk\Omega_o n}$$
$$= \frac{1}{N}\sum_{n=-M}^{M} e^{-jk\Omega_o n}.$$

We perform the change of variable on the index of summation by letting $m = n + M$ to obtain

$$X[k] = \frac{1}{N}\sum_{m=0}^{2M} e^{-jk\Omega_o(m-M)}$$
$$= \frac{1}{N}e^{jk\Omega_o M}\sum_{m=0}^{2M} e^{-jk\Omega_o m}. \tag{3.15}$$

Now, for $k = 0, \pm N, \pm 2N, \dots$ we have $e^{jk\Omega_o} = e^{-jk\Omega_o} = 1$, and Eq. (3.15) becomes

$$X[k] = \frac{1}{N}\sum_{m=0}^{2M} 1$$
$$= \frac{2M+1}{N}, \qquad k = 0, \pm N, \pm 2N, \dots.$$

For $k \neq 0, \pm N, \pm 2N, \dots$, we may sum the geometric series in Eq. (3.15) to obtain

$$X[k] = \frac{e^{jk\Omega_o M}}{N}\left(\frac{1 - e^{-jk\Omega_o(2M+1)}}{1 - e^{-jk\Omega_o}}\right), \qquad k \neq 0, \pm N, \pm 2N, \dots, \tag{3.16}$$

which may be rewritten as

$$X[k] = \frac{1}{N}\left(\frac{e^{jk\Omega_o(2M+1)/2}}{e^{jk\Omega_o/2}}\right)\left(\frac{1 - e^{-jk\Omega_o(2M+1)}}{1 - e^{-jk\Omega_o}}\right),$$
$$= \frac{1}{N}\left(\frac{e^{jk\Omega_o(2M+1)/2} - e^{-jk\Omega_o(2M+1)/2}}{e^{jk\Omega_o/2} - e^{-jk\Omega_o/2}}\right), \qquad k \neq 0, \pm N, \pm 2N, \dots.$$

At this point, we divide the numerator and denominator by $2j$ to express $X[k]$ as a ratio of two sine functions:

$$X[k] = \frac{1}{N}\frac{\sin(k\Omega_o(2M+1)/2)}{\sin(k\Omega_o/2)}, \qquad k \neq 0, \pm N, \pm 2N, \dots.$$

The technique used here to write the finite geometric-sum expression for $X[k]$ as a ratio of sine functions involves symmetrizing both the numerator, $1 - e^{-jk\Omega_o(2M+1)}$, and denominator, $1 - e^{-jk\Omega_o}$, in Eq. (3.16) with the appropriate power of $e^{jk\Omega_o}$. An alternative expression for $X[k]$ is obtained by substituting $\Omega_o = \frac{2\pi}{N}$, yielding

$$X[k] = \begin{cases} \dfrac{1}{N}\dfrac{\sin(k\pi(2M+1)/N)}{\sin(k\pi/N)}, & k \neq 0, \pm N, \pm 2N, \ldots \\ (2M+1)/N, & k = 0, \pm N, \pm 2N, \ldots \end{cases}$$

Using L'Hôpital's rule by treating k as a real number, it is easy to show that

$$\lim_{k \to 0, \pm N, \pm 2N, \ldots} \left(\frac{1}{N}\frac{\sin(k\pi(2M+1)/N)}{\sin(k\pi/N)} \right) = \frac{2M+1}{N}.$$

For this reason, the expression for $X[k]$ is commonly written as

$$X[k] = \frac{1}{N}\frac{\sin(k\pi(2M+1)/N)}{\sin(k\pi/N)}.$$

In this form, it is understood that the value of $X[k]$ for $k = 0, \pm N, \pm 2N, \ldots$ is obtained from the limit as $k \to 0$. A plot of two periods of $X[k]$ as a function of k is depicted in Fig. 3.12 for both $M = 4$ and $M = 12$, assuming $N = 50$. Note that in this example $X[k]$ is real; hence, the magnitude spectrum is the absolute value of $X[k]$, and the phase spectrum is 0 for $X[k]$ positive and π for $X[k]$ negative. ■

▶ **Problem 3.6** Find the DTFS coefficients of the signals depicted in Figs. 3.13(a) and (b).

Answers:

(a)

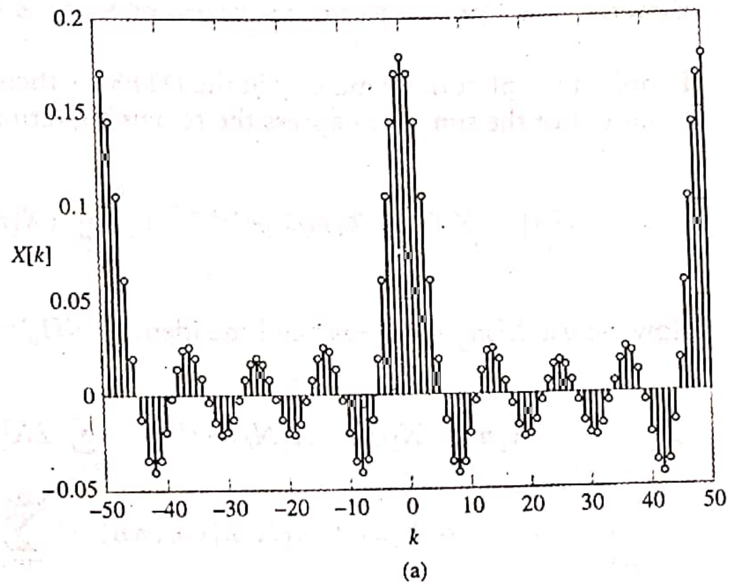
$$X[k] = \frac{8}{125} e^{jk2\pi/5} \frac{1 - \left(\frac{5}{4}e^{-jk\pi/5}\right)^7}{1 - \frac{5}{4}e^{-jk\pi/5}}$$

(b)

$$X[k] = -\frac{j}{5}\sin(k\pi/2)\frac{\sin(k2\pi/5)}{\sin(k\pi/10)}$$ ◀

It is instructive to consider the contribution of each term in the DTFS of Eq. (3.10) to the representation of the signal. We do so by examining the series representation of the square wave in Example 3.6. Evaluating the contribution of each term is particularly simple for this waveform because the DTFS coefficients have even symmetry (i.e., $X[k] = X[-k]$). Therefore, we may rewrite the DTFS of Eq. (3.10) as a series involving harmonically related cosines. Assume for convenience that N is even, so that $N/2$ is integer, and let k range from $-N/2 + 1$ to $N/2$. We thus write

$$x[n] = \sum_{k=-N/2+1}^{N/2} X[k] e^{jk\Omega_o n}.$$



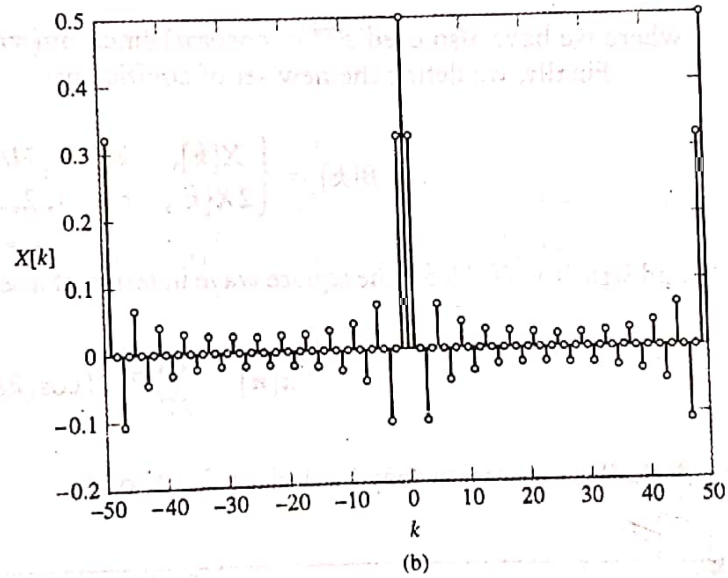


FIGURE 3.12 The DTFS coefficients for the square wave shown in Fig. 3.11, assuming a period $N = 50$: (a) $M = 4$. (b) $M = 12$.

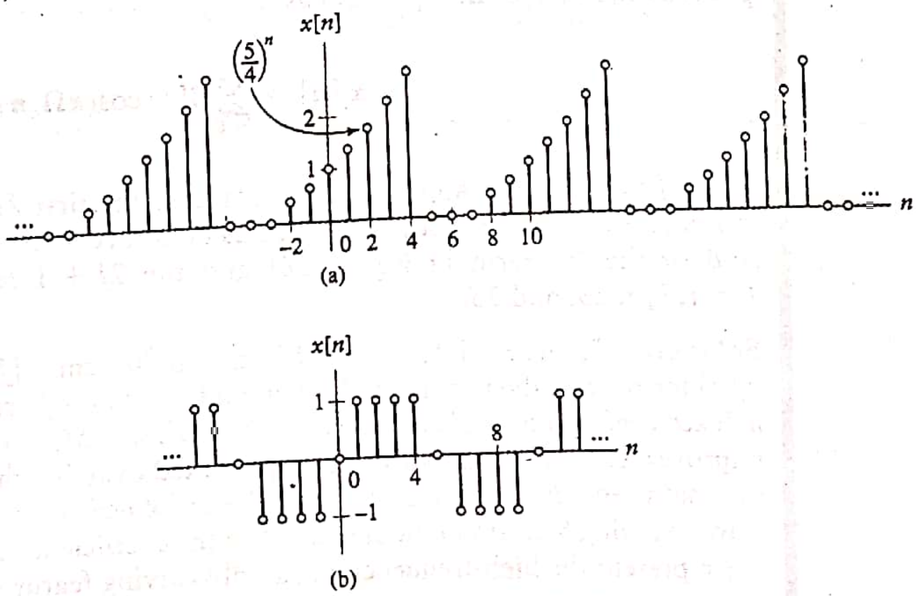


FIGURE 3.13 Signals $x[n]$ for Problem 3.6.

In order to exploit the symmetry in the DTFS coefficients, we pull the $k = 0$ and $k = N/2$ terms out of the sum and express the remaining terms using the positive index m:

$$x[n] = X[0] + X[N/2]e^{jN\Omega_o n/2} + \sum_{m=1}^{N/2-1} \left(X[m]e^{jm\Omega_o n} + X[-m]e^{-jm\Omega_o n}\right).$$

Now we use $X[m] = X[-m]$ and the identity $N\Omega_o = 2\pi$ to obtain

$$\begin{aligned} x[n] &= X[0] + X[N/2]e^{j\pi n} + \sum_{m=1}^{N/2-1} 2X[m]\left(\frac{e^{jm\Omega_o n} + e^{-jm\Omega_o n}}{2}\right) \\ &= X[0] + X[N/2]\cos(\pi n) + \sum_{m=1}^{N/2-1} 2X[m]\cos(m\Omega_o n), \end{aligned}$$

where we have also used $e^{j\pi n} = \cos(\pi n)$ since $\sin(\pi n) = 0$ for integer n.

Finally, we define the new set of coefficients

$$B[k] = \begin{cases} X[k], & k = 0, N/2 \\ 2X[k], & k = 1, 2, \ldots, N/2 - 1 \end{cases}$$

and write the DTFS for the square wave in terms of a series of harmonically related cosines as

$$x[n] = \sum_{k=0}^{N/2} B[k]\cos(k\Omega_o n). \tag{3.17}$$

A similar expression may be derived for N odd.

EXAMPLE 3.7 BUILDING A SQUARE WAVE FROM DTFS COEFFICIENTS The contribution of each term to the square wave may be illustrated by defining the partial-sum approximation to $x[n]$ in Eq. (3.17) as

$$\hat{x}_J[n] = \sum_{k=0}^{J} B[k]\cos(k\Omega_o n), \tag{3.18}$$

where $J \le N/2$. This approximation contains the first $2J + 1$ terms centered on $k = 0$ in Eq. (3.10). Assume a square wave has period $N = 50$ and $M = 12$. Evaluate one period of the Jth term in Eq. (3.18) and the $2J + 1$ term approximation $\hat{x}_J[n]$ for $J = 1, 3, 5, 23$, and 25.

Solution: Figure 3.14 depicts the Jth term in the sum, $B[J]\cos(J\Omega_o n)$, and one period of $\hat{x}_J[n]$ for the specified values of J. Only odd values for J are considered, because the even-indexed coefficients $B[k]$ are zero when $N = 25$ and $M = 12$. Note that the approximation improves as J increases, with $x[n]$ represented exactly when $J = N/2 = 25$. In general, the coefficients $B[k]$ associated with values of k near zero represent the low-frequency or slowly varying features in the signal, while the coefficients associated with values of k near $\pm\frac{N}{2}$ represent the high-frequency or rapidly varying features in the signal. ■

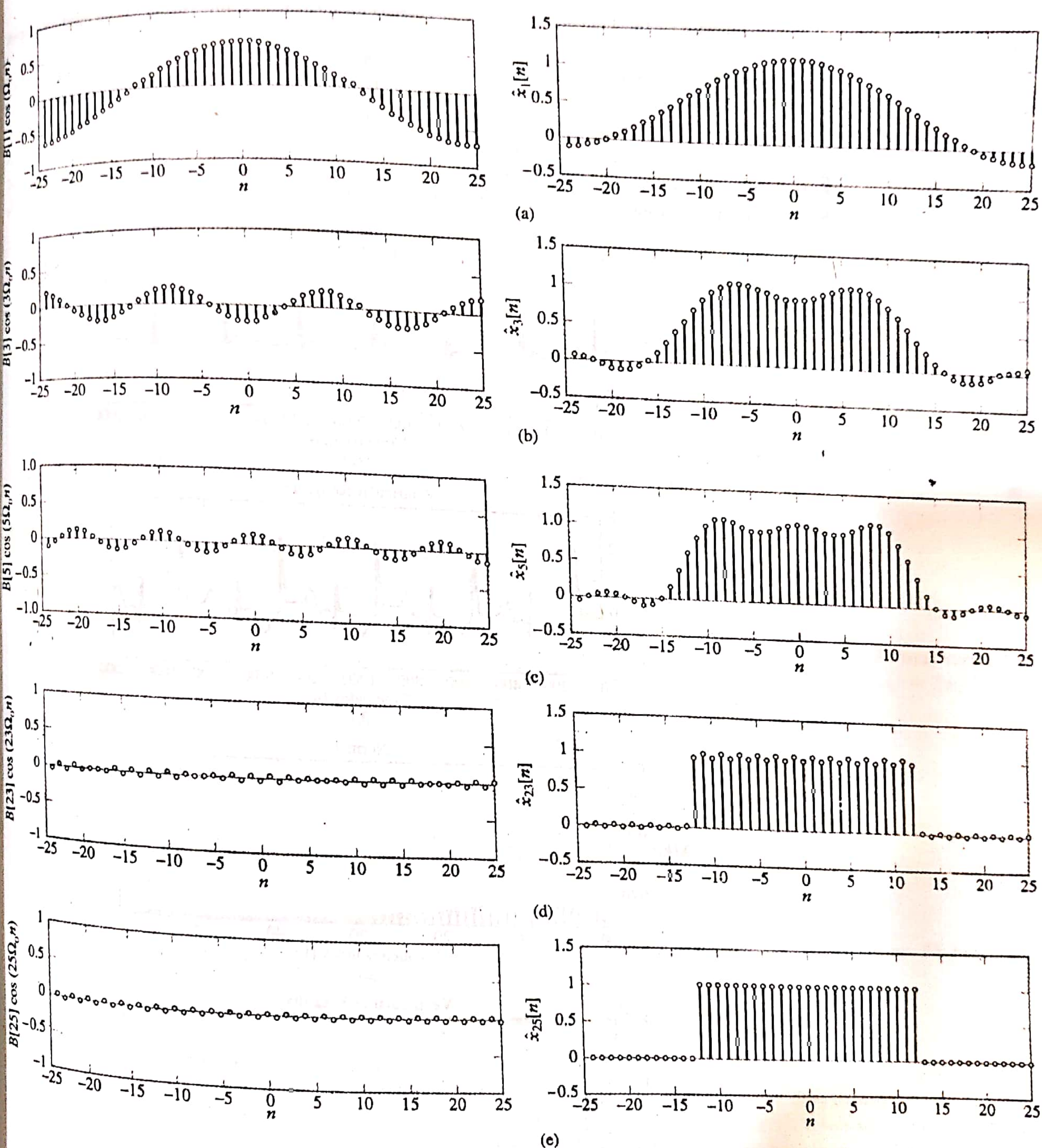


FIGURE 3.14 Individual terms in the DTFS expansion of a square wave (left panel) and the corresponding partial-sum approximations $\hat{x}_J[n]$ (right panel). The $J = 0$ term is $\hat{x}_0[n] = 1/2$ and is not shown. (a) $J = 1$. (b) $J = 3$. (c) $J = 5$. (d) $J = 23$. (e) $J = 25$.

The use of the DTFS as a numerical signal analysis tool is illustrated in the next example.

EXAMPLE 3.8 NUMERICAL ANALYSIS OF THE ECG Evaluate the DTFS representations of the two electrocardiogram (ECG) waveforms depicted in Figs. 3.15(a) and (b). Figure 3.15(a) depicts a normal ECG, while Fig. 3.15(b) depicts the ECG of a heart experiencing ventricular tachycardia. The discrete-time signals are drawn as continuous functions, due to the difficulty of depicting all 2000 values in each case. Ventricular tachycardia is a serious cardiac rhythm disturbance (i.e., an arrhythmia) that can result in death. It is characterized by a rapid,

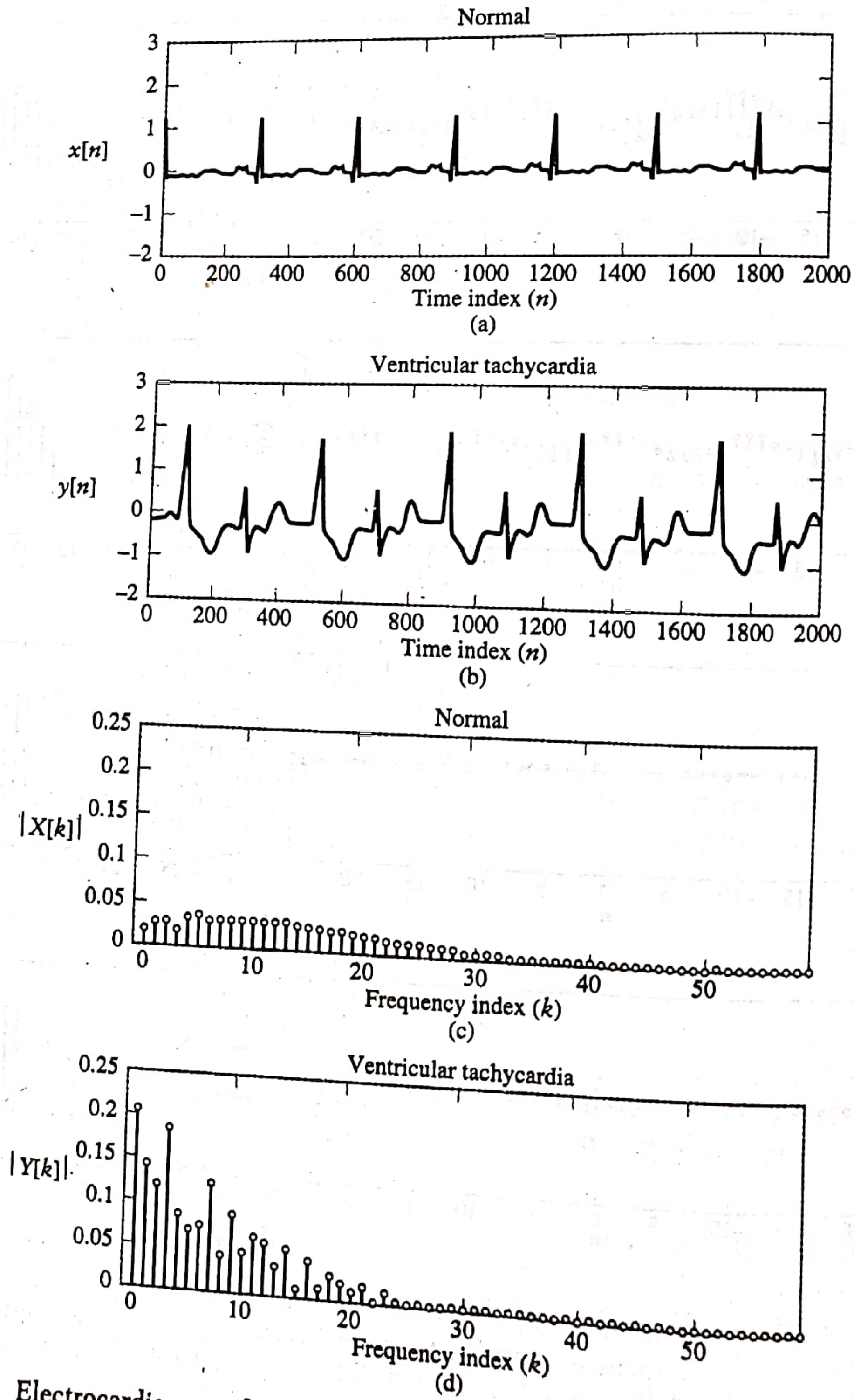


FIGURE 3.15 Electrocardiograms for two different heartbeats and the first 60 coefficients of their magnitude spectra. (a) Normal heartbeat. (b) Ventricular tachycardia. (c) Magnitude spectrum for the normal heartbeat. (d) Magnitude spectrum for ventricular tachycardia.

regular heart rate of approximately 150 beats per minute. Ventricular complexes are wide (about 160 ms in duration) compared with normal complexes (less than 110 ms) and have an abnormal shape. Both signals appear nearly periodic, with only slight variations in the amplitude and length of each period. The DTFS of one period of each ECG may be computed numerically. The period of the normal ECG is $N = 305$, while the period of the ECG showing ventricular tachycardia is $N = 421$. One period of each waveform is available. Evaluate the DTFS coefficients of each, and plot their magnitude spectrum.

Solution: The magnitude spectrum of the first 60 DTFS coefficients is depicted in Figs. 3.15(c) and (d). The higher indexed coefficients are very small and thus are not shown.

The time waveforms differ, as do the DTFS coefficients. The normal ECG is dominated by a sharp spike or impulsive feature. Recall that the DTFS coefficients of an impulse train have constant magnitude, as shown in Example 3.4. The DTFS coefficients of the normal ECG are approximately constant, exhibiting a gradual decrease in amplitude as the frequency increases. They also have a fairly small magnitude, since there is relatively little power in the impulsive signal. In contrast, the ventricular tachycardia ECG contains smoother features in addition to sharp spikes, and thus the DTFS coefficients have a greater dynamic range, with the low-frequency coefficients containing a large proportion of the total power. Also, because the ventricular tachycardia ECG has greater power than the normal ECG, the DTFS coefficients have a larger amplitude. ■

3.5 Continuous-Time Periodic Signals: The Fourier Series

Continuous-time periodic signals are represented by the Fourier series (FS). We may write the FS of a signal $x(t)$ with fundamental period T and fundamental frequency $\omega_o = 2\pi/T$ as

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t},} \tag{3.19}$$

where

$$\boxed{X[k] = \frac{1}{T}\int_0^T x(t)e^{-jk\omega_o t}\,dt} \tag{3.20}$$

are the FS coefficients of the signal $x(t)$. We say that $x(t)$ and $X[k]$ are an *FS pair* and denote this relationship as

$$x(t) \xleftrightarrow{FS;\,\omega_o} X[k].$$

From the FS coefficients $X[k]$, we may determine $x(t)$ by using Eq. (3.19), and from $x(t)$, we may determine $X[k]$ by using Eq. (3.20). We shall see later that in some problems it is advantageous to represent the signal in the time domain as $x(t)$, while in others the FS coefficients $X[k]$ offer a more convenient description. The FS coefficients are known as a *frequency-domain representation* of $x(t)$ because each FS coefficient is associated with a complex sinusoid of a different frequency. As in the DTFS, the variable k determines the frequency of the complex sinusoid associated with $X[k]$ in Eq. (3.19).

The FS representation is most often used in electrical engineering to analyze the effect of systems on periodic signals.

The infinite series in Eq. (3.19) is not guaranteed to converge for all possible signals. In this regard, suppose we define

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_o t}$$

and choose the coefficients $X[k]$ according to Eq. (3.20). Under what conditions does $\hat{x}(t)$ actually converge to $x(t)$? A detailed analysis of this question is beyond the scope of this text; however, we can state several results. First, if $x(t)$ is square integrable—that is, if

$$\frac{1}{T}\int_0^T |x(t)|^2\,dt < \infty,$$

then the MSE between $x(t)$ and $\hat{x}(t)$ is zero, or, mathematically,

$$\frac{1}{T}\int_0^T |x(t) - \hat{x}(t)|^2\,dt = 0.$$

This is a useful result that applies to a broad class of signals encountered in engineering practice. Note that, in contrast to the discrete-time case, an MSE of zero does not imply that $x(t)$ and $\hat{x}(t)$ are equal pointwise, or $x(t) = \hat{x}(t)$ at all values of t; it simply implies that there is zero power in their difference.

Pointwise convergence of $\hat{x}(t)$ to $x(t)$ is guaranteed at all values of t except those corresponding to discontinuities if the Dirichlet conditions are satisfied:

- $x(t)$ is bounded.
- $x(t)$ has a finite number of maxima and minima in one period.
- $x(t)$ has a finite number of discontinuities in one period.

If a signal $x(t)$ satisfies the Dirichlet conditions and is not continuous, then $\hat{x}(t)$ converges to the midpoint of the left and right limits of $x(t)$ at each discontinuity.

The next three examples illustrate how the FS representation is determined.

EXAMPLE 3.9 DIRECT CALCULATION OF FS COEFFICIENTS Determine the FS coefficients for the signal $x(t)$ depicted in Fig. 3.16.

Solution: The period of $x(t)$ is $T = 2$, so $\omega_o = 2\pi/2 = \pi$. On the interval $0 \le t \le 2$, one period of $x(t)$ is expressed as $x(t) = e^{-2t}$, so Eq. (3.20) yields

$$X[k] = \frac{1}{2}\int_0^2 e^{-2t}e^{-jk\pi t}\,dt$$
$$= \frac{1}{2}\int_0^2 e^{-(2+jk\pi)t}\,dt.$$

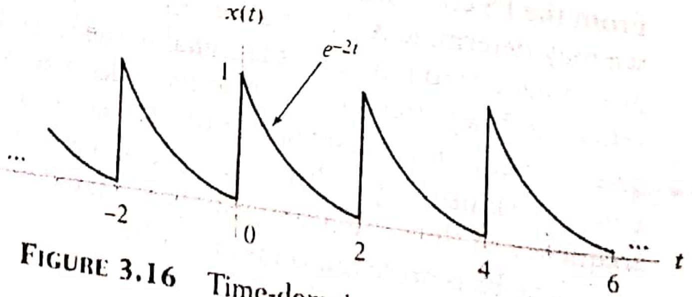


FIGURE 3.16 Time-domain signal for Example 3.9.