* In this document *

Replace Ω by ω in all possible
-ns to make it compatible with
notations of Simon Haykin book.

Example 9.1: A signal having a spectrum ranging from dc to 10 KHz is to be sampled and converted into discrete form. What is the minimum number of samples per second that must be taken to ensure recovery?

Solution Given $f_m = 10\,\text{KHz}$

From Nyquist rate, the minimum number of samples per second that must be taken to ensure recovery is

$$\begin{aligned} f_s &= 2f_m \\ &= 20,000 \quad \text{samples/sec} \end{aligned}$$

Example 9.2:

The signal $x(t) = 10\cos(10\pi t)$ is sampled at a rate of 8 samples per second. Plot the amplitude spectrum for $|\Omega| \leq 30\pi$. Can the original signal be recovered from samples? Explain?

Solution Given $x(t) = 10\cos(10\pi t)$

$$F[x(t)] = 10\{\pi[\delta(\Omega + 10\pi) + \delta(\Omega - 10\pi)]\}$$

$$\boxed{F[\cos(\Omega_0 t] = \pi[\delta(\Omega + \Omega_0) + \delta(\Omega - \Omega_0)]}$$

The amplitude spectrum of $x(t)$ is shown in Fig. 9.9.

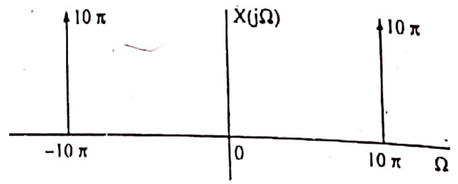

Fig. 9.9

The sampling rate $f_s = 8\,\text{Hz}$; $\Omega_s = 2\pi f_s = 2\pi(8) = 16\pi$

$$\Omega_m = 2\pi f_m = 10\pi \Rightarrow f_m = 5\,\text{Hz}$$

$$\text{Nyquist rate} = 2f_m = 10\,\text{Hz}$$

The sampling rate is less than Nyquist rate. So, the original signal cannot be recovered from the samples.

The frequency spectrum of sampled signal $x(t)$ is given by

$$X_s(j\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} 10\{\pi[\delta(\Omega+10\pi-n\Omega_s] + \pi[\delta(\Omega-10\pi-n\Omega_s)]\} \quad \left(\Omega_s = \frac{2\pi}{T}\right) \text{(use Eq. 9.12)}$$

$$= 80\pi \sum_{n=-\infty}^{\infty} [\delta(\Omega+10\pi-16n\pi)+\delta(\Omega-10\pi-16n\pi)]. \quad \left(\because \frac{1}{T} = 8\right)$$

The plot of amplitude spectrum for $|\Omega| \leq 30\pi$ is shown in Fig. 9.10.

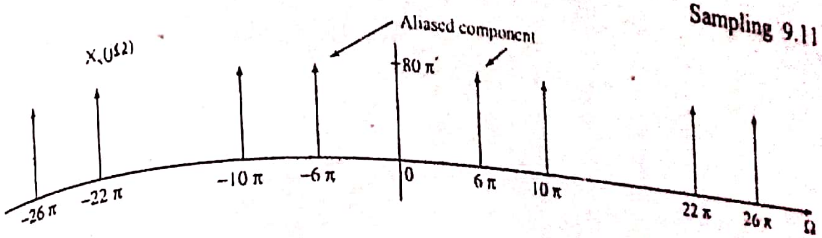

Fig. 9.10

Example 9.3: A signal $x(t) = \sin c(150\pi t)$ is sampled at a rate of (a) 100 Hz (b) 200 Hz (c) 300 Hz. For each of these three cases, can you recover the signal $x(t)$ from the sampled signal?

Solution Given $x(t) = \sin c(150\pi t)$

The spectrum of the signal $x(t)$ is a rectangular pulse with a band width (maximum frequency component) of 150 π rad/sec as shown in Fig. 9.11.

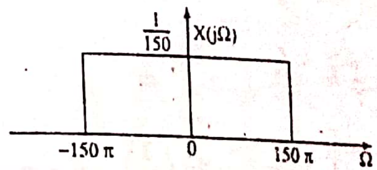

Fig. 9.11

From Fig. 9.11, we have

$$\begin{aligned} 2\pi f_m &= 150\pi \\ f_m &= 75\,\text{Hz} \\ \text{Nyquist rate} &= 2f_m = 150\,\text{Hz}. \end{aligned}$$

a) In the first case, the sampling rate is 100 Hz, which is less than Nyquist rate (under sampling). Therefore $x(t)$ cannot be recovered from its samples.

b) and (c)

In both cases the sampling rate is greater than Nyquist rate. Therefore $x(t)$ can be recovered from its samples.

Example 9.4:

Draw $|X_s(j\Omega)|$ for the following cases when $x_s(t) = x(t)\delta_T(t)$ with sampling period T, where

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

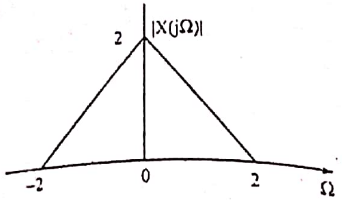

Fig. 9.12

(a) $T = \frac{\pi}{3}$ sec

(b) $T = \frac{\pi}{2}$ sec

(c) $T = 1$ sec

Solution

(a) $T = \frac{\pi}{3}$ sec $\quad f_s = \frac{1}{T} = \frac{3}{\pi}$ Hz

$$\Omega_s = 2\pi f_s = 2\pi\left(\frac{3}{\pi}\right) = 6 \text{ rad/sec}$$

The spectrum of $|X_s(j\Omega)|$ is shown in Fig. 9.13 which is periodic repitition of $X(j\Omega)$ for every 6 rad/sec.

From $X(j\Omega)$, we have

$$\Omega_m = 2 \text{ rad/sec}$$

$$f_m = \frac{1}{\pi} \text{ Hz}$$

$$\text{Nyquist rate} = 2f_m = \frac{2}{\pi} \text{ Hz}$$

$f_s > 2f_m$. Therefore the spectrum $X_s(j\Omega)$ is free from aliasing. Hence $X(j\Omega)$ can be recovered from $X_s(j\Omega)$ using an ideal low-pass filter of band width 2 rad/sec.

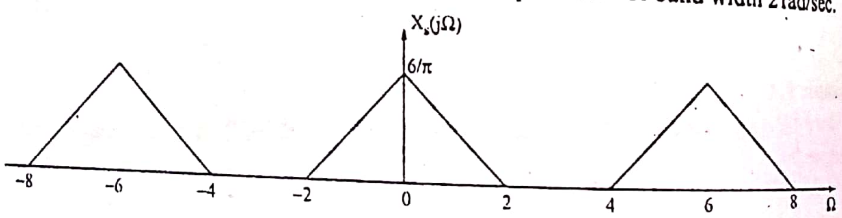

Fig. 9.13

(b) For this case the sampling period

$$T = \frac{\pi}{2} \text{ sec}$$

$$f_s = \frac{2}{\pi} \text{ Hz}$$

$$\Omega_s = 2\pi f_s = 4 \text{ rad/sec}$$

Sampling frequency is equal to Nyquist rate. Hence $X(j\Omega)$ can be recovered from $X_s(j\Omega)$.

The spectrum $|X_s(j\Omega)|$ is shown in Fig. 9.14. Here $X(j\Omega)$ repeats for every 4 rad/sec.

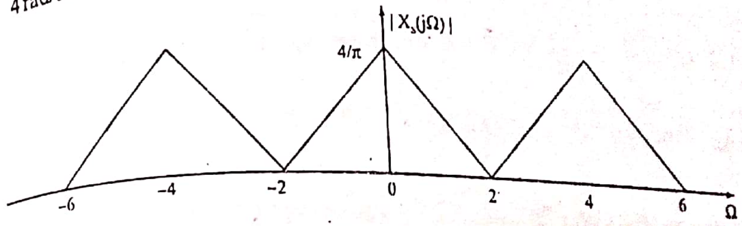

Fig. 9.14

(c) $T = \frac{2\pi}{3}$; $\quad f_s = \frac{3}{2\pi}$ sec $\quad \Omega_s = 3$ rad/sec.

The sampling frequency is less than the Nyquist rate. The spectra $X(j\Omega)$ repeats for every 3 rad/sec and the successive spectrum overlap as shown in Fig. 9.15. Therefore $x(t)$ cannot be recovered from its samples.

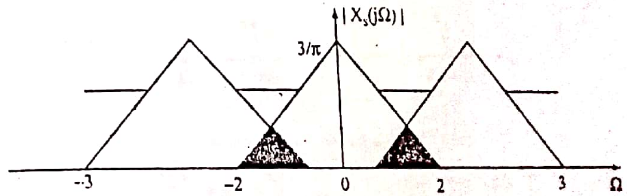

Fig. 9.15

Example 9.5: Consider the following sampling and reconstruction block

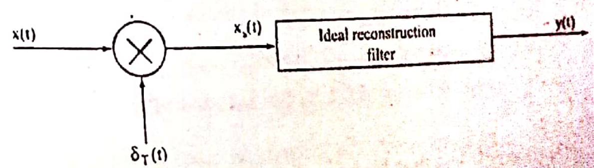

Fig. 9.16

The output of the ideal reconstruction filter can be found by sending signal $x_s(t)$ through an ideal lowpass filter having characteristics as shown in Fig. 9.17.

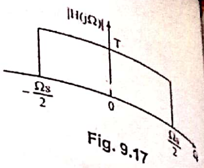

Fig. 9.17

For the following signals draw the spectrum of $|X_s(j\Omega)|$ and find expressions for $x(nT)$ and $y(t)$

(i) $x(t) = 2 + \cos(100\,\pi t)$ for $T = 0.0125$ sec.

(ii) $x(t) = 2 + \cos(100\,\pi t)$ for $T = \dfrac{1}{150}$ sec.

(iii) $x(t) = 1 + \cos(10\,\pi t) + \cos(30\pi t)$ for $T = 0.04$ sec.

(iv) If $X(j\Omega) = \dfrac{1}{2 + j\Omega}$ and $T = 2$ sec. Draw $|X_s(j\Omega)|$. Test for aliasing.

Solution Given $x(t) = 2 + \cos(100\,\pi t)$

$$= 2 + \frac{1}{2}\left[e^{j100\pi t} + e^{-j100\pi t}\right] \qquad \boxed{F[1] = 2\pi\delta(\Omega)}$$

$$X(j\Omega) = 4\pi\delta(\Omega) + \pi\left[\delta(\Omega + 100\pi) + \delta(\Omega - 100\pi)\right]$$

$$\boxed{F\left[e^{j\Omega_o t}\right] = 2\pi\delta(\Omega + \Omega_o)}$$

The amplitude spectrum $|X(j\Omega)|$ is shown in Fig 9.18.

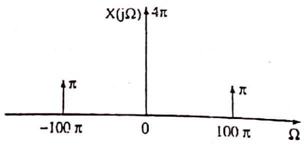

Fig. 9.18

The sampling time $T = 0.0125$

$$f_s = \frac{1}{T} = 80\text{ Hz}$$

$$\Omega_s = 2\pi f_s = 160\text{ rad/sec}$$

$$\text{Given } f_m = 50\text{ Hz}$$

$$\text{Nyquist rate} = 100\text{ Hz}$$

The sampling frequency is less than Nyquist rate. Therefore the successive spectrum $X_s(j\Omega)$ overlap and the signal $x(t)$ cannot be recovered from its samples.

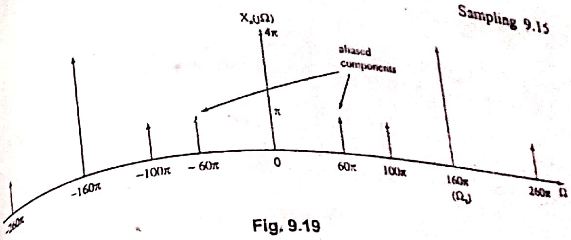

Fig. 9.19

The spectrum $X_s(j\Omega)$ of the sampled signal $x(t)$ is periodic repetition of $X(j\Omega)$ for every 160π rad/sec (80 Hz). The spectrum $X_s(j\Omega)$ is as shown in Fig. 9.19. The frequency characteristics of the reconstruction filter is shown in Fig. 9.20. When the signal with spectrum $|X_s(j\Omega)|$ as shown in Fig. 9.19 is passed through the reconstruction filter (low pass filter), the output of the filter consists of the frequency components as shown in Fig. 9.21. That is

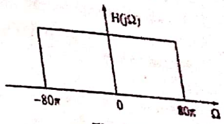

Fig. 9.20

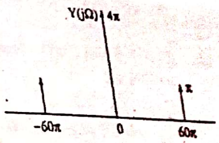

Fig. 9.21

$$Y(j\Omega) = 4\pi\delta(\Omega) + \pi\left[\delta(\Omega + 60\pi) + \delta(\Omega - 60\pi)\right]$$

$$\Rightarrow\ y(t) = 2 + \cos(60\pi t)$$

$$x(nT) = x(t)\big|_{t=nT} = 2 + \cos(100\pi t)\big|_{t=0.0125\,n}$$

$$= 2 + \cos(1.25\,n\pi)$$

$$= 2 + \cos(0.75\,n\pi)$$

ii) $x(t) = 2 + \cos(100\pi t)$ $\quad T = \dfrac{1}{150}$ sec;

$$x(nT) = x(t)\big|_{t=nT}$$

$$x(nT) = 2 + \cos(100\pi t)\big|_{t=1/150} = 2 + \cos(0.666\,n\pi)$$

$$f_m = 50\text{ Hz} \qquad f_s = 150\text{ Hz}; \qquad \Omega_s = 2\pi f_s = 300\pi$$

The amplitude spectrum is shown in Fig. 9.22.

The spectrum of sampled signal of $x(t)$, is periodic extention of $X(j\Omega)$ for every 300π (rad/sec) (150 Hz) as shown in Fig. 9.22.

The spectrum $X_s(j\Omega)$ is free from aliasing, as the sampling rate $f_s > 2f_m$ (150 Hz > 100 Hz).

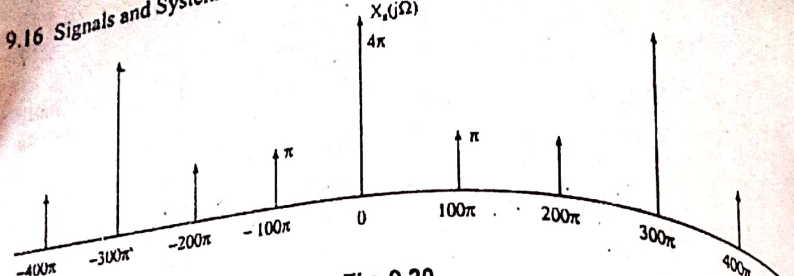

Fig. 9.20

The frequency characteristics of a reconstruction filter is shown in Fig. 9.23. If the signal $x(nT)$ with spectrum $X_s(j\Omega)$ is passed through the reconstruction filter then the resultant output spectrum $Y(j\Omega)$ is as shown in Fig. 9.24.

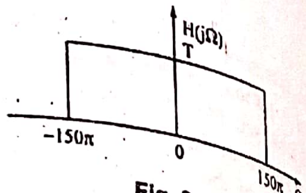

Fig. 9.23

$$Y(j\Omega) = 4\pi\delta(\Omega) + \pi\left[\delta(\Omega - 100\pi) + \delta(\Omega + 100\pi)\right]$$

$$\Rightarrow \quad y(t) = 2 + \cos(100\pi t)$$

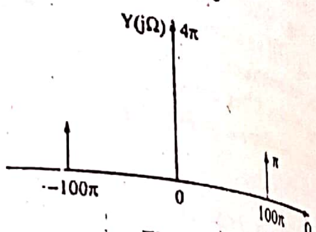

Fig. 9.24

iii) $x(t) = 1 + \cos(10\pi t) + \cos(30\pi t)$
$T = 0.04$ sec

$$x(nT) = 1 + \cos(0.4\,n\pi) + \cos(1.2\,n\pi)$$

$$f_s = \frac{1}{T} = 25\,\text{Hz}, \qquad \Omega_s = 50\,\pi \text{ rad/sec}$$

$$\Omega_m = 2\pi f_m = 30\pi \Rightarrow f_m = 15\,\text{Hz}$$

Nyquist rate = 30 Hz

The sampling frequency $f_s < 2f_m$. Therefore the successive spectrum of $X(j\Omega)$ in $X_s(j\Omega)$ overlaps and $x(t)$ cannot be recovered from $x(nT)$.

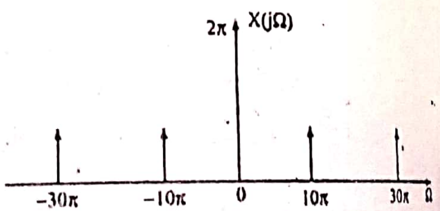

Fig. 9.25

$$X(j\Omega) = 2\pi\delta(\Omega) + \pi\left[\delta(\Omega + 10\pi) + \delta(\Omega - 10\pi)\right] + \pi\left[\delta(\Omega + 30\pi) + \delta(\Omega - 30\pi)\right]$$

$\Omega_s = 50\,\pi$ rad/sec. Therefore the spectrum of $X_s(j\Omega)$ can be obtained by periodically repeating $X(j\Omega)$ for every 50π rad/sec (25 Hz). The spectrum $X_s(j\Omega)$ is shown in Fig. 9.26.

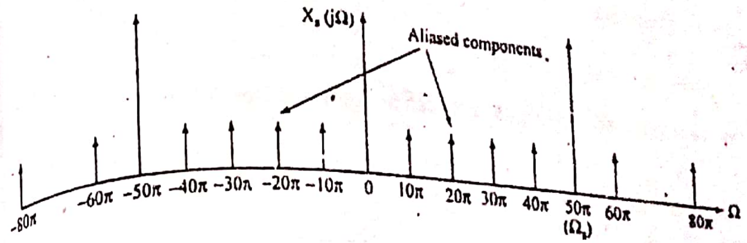

Fig. 9.26

The frequency spectrum of the lowpass filter is shown in Fig. 9.27. The output spectrum of lowpass filter is shown in Fig. 9.28.

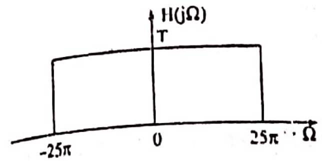

Fig. 9.27

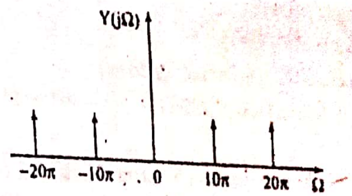

Fig. 9.28

$$\Rightarrow y(t) = 1 + \cos(10\pi t) + \cos(20\pi t)$$

iv) Given $T = 2$ sec $f_s = 0.5$ sec

$$\Omega_s = \pi \text{ rad/sec}$$

$$X(j\Omega) = \frac{1}{2 + j\Omega}$$

$$|X(j\Omega)| = \frac{1}{\sqrt{4 + \Omega^2}}$$

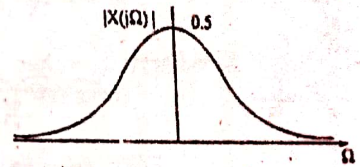

Fig. 9.29

The spectrum $|X(j\Omega)|$ is plotted as shown in Fig. 9.29.

The spectrum $|X(j\Omega)|$ is not band limited. Therefore aliasing of successive frequency components occurs even Ω_s is large. The spectra $|X_s(j\Omega)|$ is shown in Fig. 9.30.

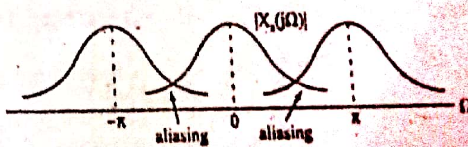

Fig. 9.30

Exercise 9.1:
Consider a continuous time signal $x(t)$ with frequency spectrum shown in Fig. 9.31. Find the frequency spectrum of its sampled sequences if the sampling frequency

(i) $\Omega_s = 20$ rad/sec
(ii) $\Omega_s = 40$ rad/sec
(iii) $\Omega_s = 60$ rad/sec

Also find in which cases the signal $x(t)$ can be recovered from it's samples.

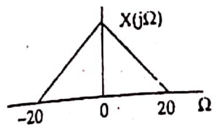

Fig. 9.31

Example 9.6: Determine the Nyquist sampling rate and Nyquist sampling intervals for the following signals

(i) $\sin c^2(200\pi t)$
(ii) $0.5 \sin c^2(200\pi t)$
(iii) $\sin c(200\pi t) + 3 \sin c^2(120\pi t)$
(iv) $\sin c(100\pi t) \sin c(200\pi t)$

Solution We know that the Fourier transform of triangular function is square of (sinc) function. It can be graphically represented as follows

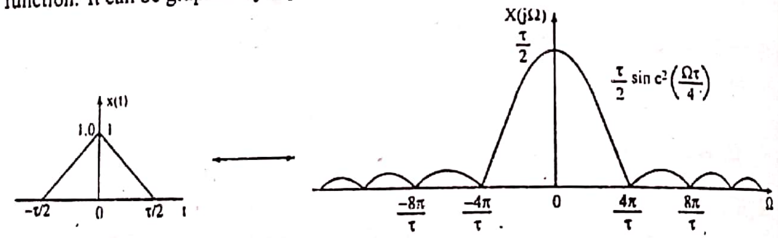

Fig. 9.32

By using Dual property, we have

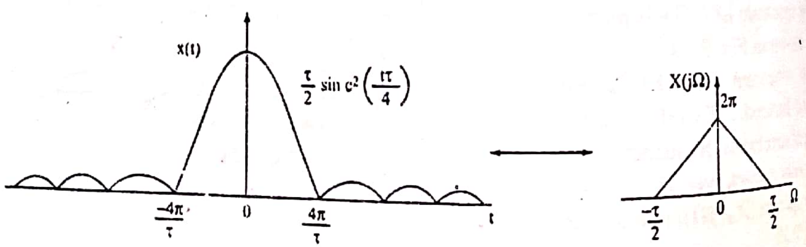

Fig. 9.33

Similarly Fourier transform of a rectangular pulse is a sinc function.

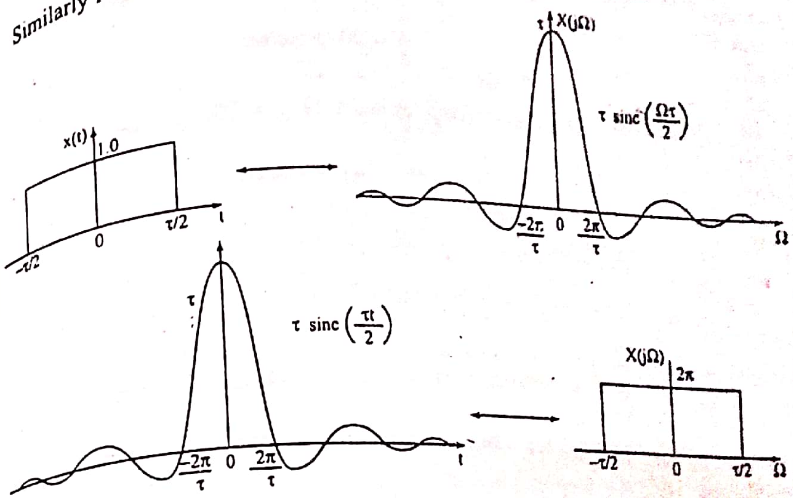

Fig. 9.34

(i) Given

$$x(t) = \sin c^2(200\pi t)$$

Comparing the signal $x(t)$ with the signal in Fig. 9.33 we can find

$$\frac{t\tau}{4} = 200\pi t$$
$$\Rightarrow \tau = 800\pi$$
$$\tau/2 = 400\pi$$

That is the maximum frequency component present in the signal is $400\,\pi$ rad/sec (or) the signal is band limited to $400\,\pi$ rad/sec

$$\Omega_m = 400\pi$$
$$f_m = 200\,\text{Hz}$$
$$\text{Nyquist rate } f_s = 2f_m = 400\,\text{Hz}$$
$$\text{Sampling interval } T = \frac{1}{400} = 2.5\,\text{m sec}$$

(ii) Same values as above

(iii) Given $x(t) = \sin c(200\pi t) + 3 \sin c^2(120\pi t)$

Let $x_1(t) = \sin c(200\pi t)$ and $x_2(t) = 3 \sin c^2(120\,\pi t)$

Comparing the signal $x_1(t)$ with the signal in Fig. 9.34, we get

$$\frac{\tau}{2} = 200\pi \text{ rad/sec}$$

That is the signal $x_1(t)$ is band limited to $\Omega_{m_1} = 200\,\pi\text{rad/sec}$

$$\Rightarrow \quad f_{m1} = 100 \text{ Hz}$$

For the signal $x_2(t)$.

$$\tau/4 = 120\pi$$
$$\Rightarrow \tau/2 = 240\pi$$

The signal $x_2(t)$ is band limited to $\Omega_{m2} = 240\,\pi\text{rad/sec}$.

when both signals are added, the maximum frequency component is

$$\Omega_m = \Omega_{m2} = 240\,\pi\text{rad/sec}$$
$$f_m = 120 \text{ Hz}$$
$$\text{Nyquist rate } f_s = 2f_m = 240 \text{ Hz}$$
$$\text{Nyquist interval } T_s = \frac{1}{240} = 4.167 \text{ m sec}$$

(iv) $x(t) = \text{sin c}(100\pi t)\,\text{sin c}(200\pi t)$

Let $x_1(t) = \text{sin c}(100\pi t)$; $x_2(t) = \text{sin c}(200\pi t)$

The signal $x_1(t)$ is band limited to frequency $100\,\pi$rad/sec and the signal $x_2(t)$ is band limited to frequency $200\,\pi$rad/sec

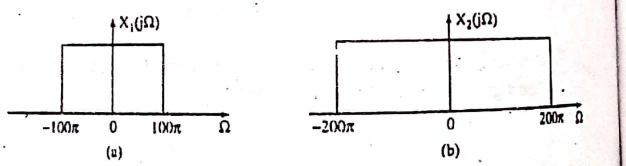

Fig. 9.35

The frequency spectrum of $x(t)$ can be obtained by convolving the individual spectrum of $x_1(t)$ and $x_2(t)$. That is

$$F[\text{sin c}(100\pi t)\,\text{sin c}(200\pi t)] = X_1(j\Omega) * X_2(j\Omega)$$

The convolution of $X_1(j\Omega)$ and $X_2(j\Omega)$ results a triangular shape frequency spectrum with $\Omega_m = 300\,\pi\text{rad/sec}$ as shown in Fig. 9.36.

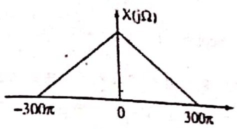

Fig. 9.36

Therefore

$$f_m = 150 \text{ Hz}$$
$$\text{Nyquist rate} = 2f_m = 300 \text{ Hz}$$
$$\text{Nyquist interval} = \frac{1}{300} = 3.33 \text{ m sec}$$

Exercise 9.2:

1. The amplitude spectrum of a continuous time signal is shown in Fig. 9.37. When the signal shown in figure is sampled with a sampling frequency of $\Omega_s = 200\,\pi\text{rad/sec}$, sketch the amplitude spectrum of the sampled signal

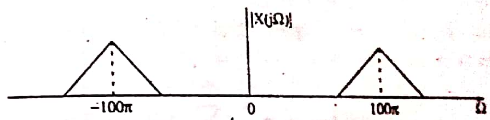

Fig. 9.37

2. The amplitude spectrum of $x(t)$ is shown in Fig. 9.38. This signal is amplitude modulated by $\cos(75\pi t)$. When the modulated signal is sampled with sampling period $T = 0.01$ sec, sketch the amplitude spectrum of the sampled signal.

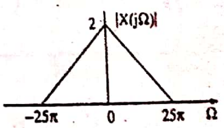

Fig. 9.38

9.5 Sampling of Bandpass singals

Let us consider a continuous time signal band limited to a range $\Omega_L \le |\Omega| \le \Omega_H$, where $\Omega_L > 0$. This type of signal is known as a bandpass signal. We can sample such type of signals with a sampling rate greater than twice the highest frequency

$$\Omega_{sf} \ge 2\Omega_H \qquad \text{(9.28a)}$$

to prevent aliasing.