EXAMPLE 10.90 Consider an LTI system with a system function $H(z)=\frac{1}{1-(1/2)z^{-1}}$. Find the difference equation.

Solution: Given
$$H(z)=\frac{Y(z)}{X(z)}=\frac{1}{1-(1/2)z^{-1}}$$

That is
$$Y(z)-\frac{1}{2}z^{-1}Y(z)=X(z)$$

Taking inverse Z-transform on both sides (applying the time shifting property), we get the difference equation
$$y(n)-\frac{1}{2}y(n-1)=x(n)$$

EXAMPLE 10.91 A causal system is represented by
$$H(z)=\frac{z+2}{2z^2-3z+4}$$

Find the difference equation and the frequency response of the system.

Solution: Given
$$H(z)=\frac{z+2}{2z^2-3z+4}$$

As the system is causal, $H(z)$ is expressed in negative powers of z.

Let
$$H(z)=\frac{Y(z)}{X(z)}$$
$$=\frac{z+2}{2z^2-3z+4}=\frac{z^{-1}+2z^{-2}}{2-3z^{-1}+4z^{-2}}$$

i.e.
$$2Y(z)-3z^{-1}Y(z)+4z^{-2}Y(z)=z^{-1}X(z)+2z^{-2}X(z)$$

Taking inverse Z-transform on both sides, we have
$$2y(n)-3y(n-1)+4y(n-2)=x(n-1)+2x(n-2)$$

which is the required difference equation.

Putting $z = e^{j\omega}$ in $H(z)$, we get the frequency response $H(e^{j\omega})$ of the system.

$$H(e^{j\omega}) = \left.\frac{z+2}{2z^2 - 3z + 4}\right|_{z=e^{j\omega}} = \frac{e^{j\omega} + 2}{2e^{j2\omega} - 3e^{j\omega} + 4}$$

$$= \frac{2 + \cos\omega + j\sin\omega}{4 + (2\cos 2\omega - 3\cos\omega) + j(2\sin 2\omega - 3\sin\omega)}$$

EXAMPLE 10.92 Determine the system function of a discrete-time system described by the difference equation

$$y(n) - \frac{1}{3}y(n-1) + \frac{1}{5}y(n-2) = x(n) - 2x(n-1)$$

Solution: Taking Z-transform on both sides of the given difference equation, we get

$$Y(z) - \frac{1}{3}z^{-1}Y(z) + \frac{1}{5}z^{-2}Y(z) = X(z) - 2z^{-1}X(z)$$

Hence the system function or transfer function of the given system is:

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 2z^{-1}}{1 - (1/3)z^{-1} + (1/5)z^{-2}} = \frac{z(z-2)}{z^2 - (1/3)z + (1/5)}$$

EXAMPLE 10.93 Plot the pole-zero pattern and determine which of the following systems are stable:

(a) $y(n) = y(n-1) - 0.8y(n-2) + x(n) + x(n-2)$

(b) $y(n) = 2y(n-1) - 0.8y(n-2) + x(n) + 0.8x(n-1)$

Solution:

(a) Given $\quad y(n) = y(n-1) - 0.8y(n-2) + x(n) + x(n-2)$

Taking Z-transform on both sides and neglecting the initial conditions, we have

$$Y(z) = z^{-1}Y(z) - 0.8z^{-2}Y(z) + X(z) + z^{-2}X(z)$$

i.e. $\quad Y(z)[1 - z^{-1} + 0.8z^{-2}] = X(z)(1 + z^{-2})$

The transfer function of the system is:

$$\frac{Y(z)}{X(z)} = H(z)$$

$$= \frac{1 + z^{-2}}{1 - z^{-1} + 0.8z^{-2}} = \frac{z^2 + 1}{z^2 - z + 0.8}$$

$$= \frac{(z+j)(z-j)}{(z - 0.5 - j0.74)(z - 0.50 + j0.74)}$$

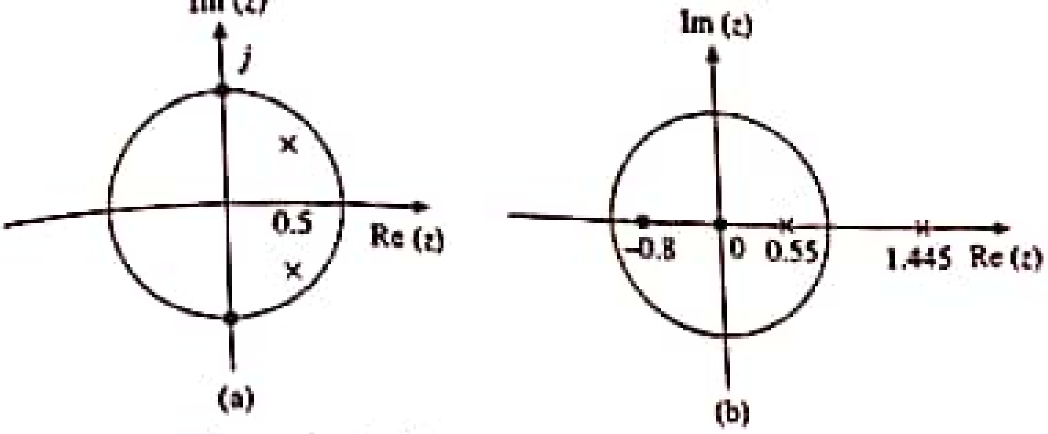

Figure 10.19 (a) Pole-zero plots for Example 10.93.

The zeros of $H(z)$ are $z = +j1$ and $z = -j1$.
The poles of $H(z)$ are $z = 0.5 - j0.74$ and $z = 0.5 + j0.74$.
The pole-zero plot is shown in Figure 10.19(a).
All the poles are inside the unit circle. Hence, the system is stable.

(b) Given $$y(n) = 2y(n-1) - 0.8y(n-2) + x(n) + 0.8x(n-1)$$

Taking Z-transform on both sides and neglecting the initial conditions, we have

$$Y(z) = 2z^{-1}Y(z) - 0.8z^{-2}Y(z) + X(z) + 0.8z^{-1}X(z)$$

i.e. $$Y(z)[1 - 2z^{-1} + 0.8z^{-2}] = X(z)[1 + 0.8z^{-1}]$$

The transfer function of the system is:

$$\therefore \quad \frac{Y(z)}{X(z)} = H(z) = \frac{1 + 0.8z^{-1}}{1 - 2z^{-1} + 0.8z^{-2}} = \frac{z(z+0.8)}{z^2 - 2z + 0.8}$$

$$= \frac{z(z+0.8)}{(z-1.445)(z-0.555)}$$

The zeros of $H(z)$ are $z = 0$ and $z = -0.8$.
The poles of $H(z)$ are $z = 1.445$ and $z = 0.555$.
The pole-zero plot is shown in Figure 10.19(b).
One pole is outside the unit circle. Therefore, the system is unstable.

EXAMPLE 10.94 A causal system has input $x(n)$ and output $y(n)$. Find the system function, frequency response and impulse response of the system if

$$x(n) = \delta(n) + \frac{1}{6}\delta(n-1) - \frac{1}{6}\delta(n-2)$$

and $$y(n) = \delta(n) - \frac{2}{3}\delta(n-1)$$

Solution: Given $$x(n) = \delta(n) + \frac{1}{6}\delta(n-1) - \frac{1}{6}\delta(n-2)$$

and $$y(n) = \delta(n) - \frac{2}{3}\delta(n-1)$$

Taking Z-transform of the above equations, we get

$$X(z) = 1 + \frac{1}{6}z^{-1} - \frac{1}{6}z^{-2}$$

and

$$Y(z) = 1 - \frac{2}{3}z^{-1}$$

The system function or the transfer function of the system is:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1-(2/3)z^{-1}}{1+(1/6)z^{-1}-(1/6)z^{-2}} = \frac{z[z-(2/3)]}{[z-(1/3)][z+(1/2)]}$$

The frequency response of the system is:

$$H(e^{j\omega}) = \left.\frac{z[z-(2/3)]}{[z-(1/3)][z+(1/2)]}\right|_{z=e^{j\omega}} = \frac{e^{j\omega}[e^{j\omega}-(2/3)]}{[e^{j\omega}-(1/3)][e^{j\omega}+(1/2)]}$$

By partial fraction expansion, we have

$$\frac{H(z)}{z} = \frac{[z-(2/3)]}{[z-(1/3)][z+(1/2)]} = \frac{A}{z-(1/3)} + \frac{B}{z+(1/2)} = \frac{-2/5}{z-(1/3)} + \frac{7/5}{z+(1/2)}$$

$$\therefore \quad H(z) = -\frac{2}{5}\left[\frac{z}{z-(1/3)}\right] + \frac{7}{5}\left[\frac{z}{z+(1/2)}\right]$$

Both the poles of $H(z)$ are inside the unit circle. So the system is stable. Taking inverse Z-transform on both sides, we get the impulse response as:

$$h(n) = -\frac{2}{5}\left(\frac{1}{3}\right)^n u(n) + \frac{7}{5}\left(-\frac{1}{2}\right)^n u(n)$$

EXAMPLE 10.95 We want to design a causal discrete-time LTI system with the property that if the input is $x(n) = \left(\frac{1}{3}\right)^n u(n) - \frac{1}{5}\left(\frac{1}{3}\right)^{n-1} u(n-1)$, then the output is $y(n) = \left(\frac{1}{2}\right)^n u(n)$. Determine the transfer function $H(z)$, the impulse response $h(n)$ and frequency response $H(e^{j\omega})$ of the system that satisfies this condition.

Solution: Given

$$x(n) = \left(\frac{1}{3}\right)^n u(n) - \frac{1}{5}\left(\frac{1}{3}\right)^{n-1} u(n-1)$$

and

$$y(n) = \left(\frac{1}{2}\right)^n u(n)$$

We want to design a causal system.

$$\therefore \quad h(n) = 0 \quad \text{for } n < 0$$

Taking Z-transform on both sides of the above equations, we have

$$X(z) = \frac{z}{z-(1/3)} - \frac{1}{5}z^{-1}\frac{z}{z-(1/3)} = \frac{z}{z-(1/3)} - \frac{1}{5}\frac{1}{z-(1/3)} = \frac{z-(1/5)}{z-(1/3)}$$

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and
$$Y(z) = \frac{z}{z-(1/2)}$$

The system function $H(z)$ is:

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{z}{z-(1/2)} + \frac{z-(1/5)}{z-(1/3)} = \frac{z[z-(1/3)]}{[z-(1/2)][z-(1/5)]}$$

By partial fraction expansion of $H(z)/z$, we get

$$\therefore \quad \frac{H(z)}{z} = \frac{z-(1/3)}{[z-(1/2)][z-(1/5)]} = \frac{A}{z-(1/2)} + \frac{B}{z-(1/5)} = \frac{5/9}{z-(1/2)} + \frac{4/9}{z-(1/5)}$$

$$H(z) = \frac{5}{9}\left[\frac{z}{z-(1/2)}\right] + \frac{4}{9}\left[\frac{z}{z-(1/5)}\right]$$

Taking inverse Z-transform of $H(z)$, we get the impulse response $h(n)$ as:

$$h(n) = \frac{5}{9}\left(\frac{1}{2}\right)^n u(n) + \frac{4}{9}\left(\frac{1}{5}\right)^n u(n)$$

The frequency response is:

$$H(e^{j\omega}) = \left.\frac{z[z-(1/3)]}{[z-(1/2)][z-(1/5)]}\right|_{z=e^{j\omega}} = \frac{e^{j\omega}[e^{j\omega}-(1/3)]}{[e^{j\omega}-(1/2)][e^{j\omega}-(1/5)]}$$

EXAMPLE 10.96 A system has impulse response $h(n) = (1/3)^n u(n)$. Determine the transfer function and frequency response of the system. Also determine the input to the system if the output is given by

$$y(n) = \frac{1}{2}u(n) + \frac{1}{4}\left(-\frac{1}{3}\right)^n u(n)$$

Solution: Given
$$y(n) = \frac{1}{2}u(n) + \frac{1}{4}\left(-\frac{1}{3}\right)^n u(n)$$

and
$$h(n) = \left(\frac{1}{3}\right)^n u(n)$$

Taking Z-transform on both sides, we have the output

$$Y(z) = \frac{1}{2}\frac{z}{z-1} + \frac{1}{4}\frac{z}{z+(1/3)} = \frac{(3/4)\,z[z-(1/9)]}{(z-1)[z+(1/3)]}$$

and
$$\text{Transfer function } H(z) = \frac{z}{z-(1/3)}$$

The frequency response of the system is:

$$H(e^{j\omega}) = \left.\frac{z}{z-(1/3)}\right|_{z=e^{j\omega}} = \frac{e^{j\omega}}{e^{j\omega}-(1/3)}$$

$$\text{System function } H(z) = \frac{Y(z)}{X(z)}$$

Then

$$\text{Input } X(z) = \frac{Y(z)}{H(z)}$$

∴

$$\frac{(3/4)\,z\,[z-(1/9)]}{(z-1)\,[z+(1/3)]} \div \frac{z}{z-(1/3)} = \frac{(3/4)\,[z-(1/3)]\,[z-(1/9)]}{(z-1)\,[z+(1/3)]}$$

Making partial fraction expansion of $X(z)/z$, we get

$$\therefore \quad \frac{X(z)}{z} = \frac{3}{4}\,\frac{[z-(1/3)]\,[z-(1/9)]}{z(z-1)\,[z+(1/3)]} = \frac{A}{z} + \frac{B}{z-1} + \frac{C}{z+(1/3)} = \frac{1/12}{z} + \frac{5/12}{z-1} + \frac{1/4}{z+(1/3)}$$

$$X(z) = \frac{1}{12} + \frac{5}{12}\,\frac{z}{z-1} + \frac{1}{4}\,\frac{z}{z+(1/3)}$$

Taking inverse Z-transform, we get

$$x(n) = \frac{1}{12}\delta(n) + \frac{5}{12}u(n) + \frac{1}{4}\left(-\frac{1}{3}\right)^n u(n)$$

EXAMPLE 10.97 Find the transfer function and impulse response of the system described by the difference equation

$$y(n) - \frac{1}{3}y(n-1) = 4x(n-1)$$

Solution: The given difference equation is:

$$y(n) - \frac{1}{3}y(n-1) = 4x(n-1)$$

Taking Z-transform on both sides, we get

$$Y(z) - \frac{1}{3}z^{-1}Y(z) = 4z^{-1}X(z)$$

i.e.

$$Y(z)\left(1 - \frac{1}{3}z^{-1}\right) = 4z^{-1}X(z)$$

The transfer function of the system is:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{4z^{-1}}{1-(1/3)z^{-1}} = \frac{4}{z-(1/3)}$$

Taking inverse Z-transform, the impulse response is:

$$h(n) = 4\left(\frac{1}{3}\right)^{n-1} u(n-1)$$

EXAMPLE 10.98 A causal LTI system is described by the difference equation

$$y(n) = y(n-1) + y(n-2) + x(n) + 2x(n-1)$$

Find the system function and frequency response of the system. Plot the poles and zeros and indicate the ROC. Also determine the stability and impulse response of the system.

Solution: The given difference equation is:

$$y(n) = y(n-1) + y(n-2) + x(n) + 2x(n-1)$$

Taking Z-transform on both sides, we have

$$Y(z) = z^{-1}Y(z) + z^{-2}Y(z) + X(z) + 2z^{-1}X(z)$$

i.e.

$$Y(z)(1 - z^{-1} - z^{-2}) = X(z)(1 + 2z^{-1})$$

The system function is:

$$\frac{Y(z)}{X(z)} = H(z) = \frac{1+2z^{-1}}{1-z^{-1}-z^{-2}} = \frac{z(z+2)}{z^2-z-1} = \frac{z(z+2)}{(z-1.62)(z+0.62)}$$

The frequency response of the system is:

$$H(e^{j\omega}) = \left.\frac{z(z+2)}{(z-1.62)(z+0.62)}\right|_{z=e^{j\omega}} = \frac{e^{j\omega}(e^{j\omega}+2)}{(e^{j\omega}-1.62)(e^{j\omega}+0.62)}$$

$H(z)$ has the zeros at $z = 0$ and $z = -2$.

$H(z)$ has the poles at $z = 1.62$ and $z = -0.62$. One of the pole is outside the unit circle. So the system is unstable. The poles and zeros and the ROC are shown in Figure 10.20.

To find the impulse response $h(n)$, partial fraction expansion of $H(z)/z$ gives

$$\frac{H(z)}{z} = \frac{z+2}{(z-1.62)(z+0.62)} = \frac{A}{z-1.62} + \frac{B}{z+0.62} = \frac{1.62}{z-1.62} - \frac{0.62}{z+0.62}$$

$\therefore$

$$H(z) = 1.62\left(\frac{z}{z-1.62}\right) - 0.62\left(\frac{z}{z+0.62}\right)$$

Taking inverse Z-transform, the impulse response is:

$$h(n) = 1.62(1.62)^n u(n) - 0.62(-0.62)^n u(n)$$

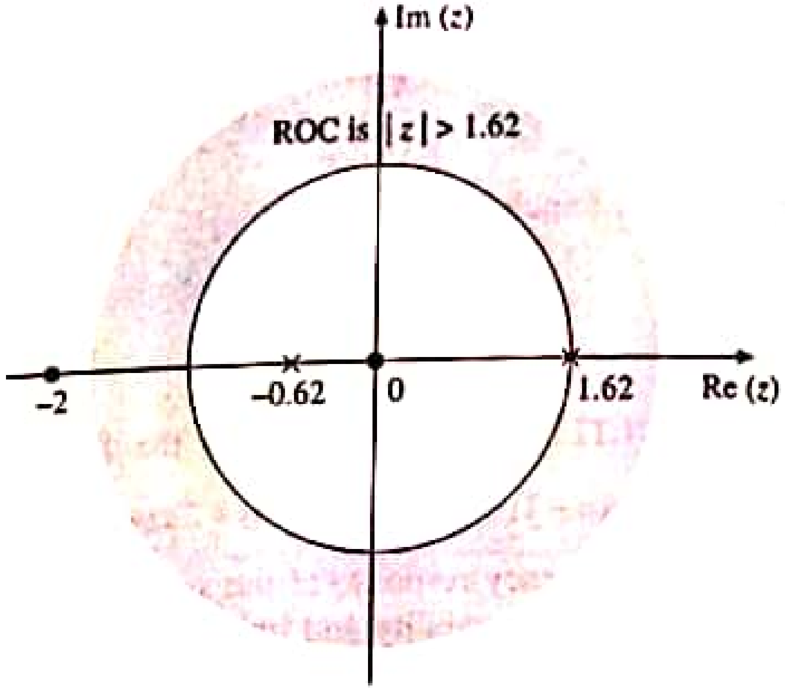

Figure 10.20 Pole-zero plot and ROC for Example 10.98.

EXAMPLE 10.99 Determine whether the following systems are both causal and stable.

(a) $H(z)=\dfrac{3+z^{-1}}{1+z^{-1}-(4/9)z^{-2}}$ (b) $H(z)=\dfrac{1+2z^{-1}}{1+(6/5)z^{-1}+(9/25)z^{-2}}$

Solution:

(a) Given $H(z)=\dfrac{3+z^{-1}}{1+z^{-1}-(4/9)z^{-2}}=\dfrac{z(z+3)}{z^2+z-(4/9)}=\dfrac{z(z+3)}{[z-(1/3)][z+(4/3)]}$

The poles of $H(z)$ are: $z = 1/3$ and $z = -4/3$

For a causal system to be stable, the ROC must include the unit circle. Now, for a causal $H(z)$, the ROC is $|z| > 4/3$. Since one pole is lying outside the unit circle, the given system is not both causal and stable.

(b) Given $H(z)=\dfrac{1+2z^{-1}}{1+(6/5)z^{-1}+(9/25)z^{-2}}=\dfrac{z(z+2)}{z^2+(6/5)z+(9/25)}=\dfrac{z(z+2)}{[z+(3/5)]^2}$

The location of the poles is at $z = -3/5$.

Since all the poles are lying inside the unit circle in the z-plane, the system is both causal and stable.

EXAMPLE 10.100 An LTI system is described by the difference equation.

$$y(n)-\frac{9}{4}y(n-1)+\frac{1}{2}y(n-2)=x(n)-3x(n-1)$$

Specify the ROC of $H(z)$, and determine $h(n)$ for the following conditions:

(a) The system is stable.

(b) The system is causal.

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EXAMPLE 10.102 Test the condition for stability of the first order infinite impulse response (IIR) filter governed by the equation

$$y(n) = x(n) + bx(n-1)$$

Solution: Given difference equation is:

$$y(n) = x(n) + bx(n-1)$$

Taking Z-transform on both sides, we have

$$Y(z) = X(z) + bz^{-1}X(z)$$

$$= X(z)[1 + bz^{-1}]$$

The transfer function of the system $H(z)$ is:

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= 1 + bz^{-1} = \frac{z+b}{z}$$

The system has a zero at $z = -b$ and a pole at $z = 0$. So the system is stable for all values of b.

EXAMPLE 10.103 Determine the impulse response of the system described by the difference equation

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

using Z-transform.

Solution: Given difference equation is:

$$y(n) - 3y(n-1) - 4y(n-2) = x(n) + 2x(n-1)$$

Taking Z-transform on both sides, we have

$$Y(z) - 3z^{-1}Y(z) - 4z^{-2}Y(z) = X(z) + 2z^{-1}X(z)$$

i.e.

$$Y(z)(1 - 3z^{-1} - 4z^{-2}) = X(z)(1 + 2z^{-1})$$

The transfer function of the system is:

$$H(z) = \frac{Y(z)}{X(z)}$$

$$= \frac{1 + 2z^{-1}}{1 - 3z^{-1} - 4z^{-2}} = \frac{z(z+2)}{z^2 - 3z - 4}$$

Taking partial fractions of $H(z)/z$, we have

$$\therefore \quad \frac{H(z)}{z} = \frac{z+2}{z^2 - 3z - 4} = \frac{z+2}{(z-4)(z+1)} = \frac{A}{z-4} + \frac{B}{z+1} = \frac{1.2}{z-4} - \frac{0.2}{z+1}$$