Derivations of Time Domain Specifications

Derivation of Peak Time T_p

w.k.T

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta) \quad -(1)$$

where $\theta = \tan^{-1} \frac{\sqrt{1-\xi^2}}{\xi} \quad -(2)$

As at $t = T_p$, $c(t)$ will achieve its maxima
According to maxima theorem

$$\left.\frac{d\,c(t)}{dt}\right|_{t=T_p} = 0 \quad -(3)$$

differentiating $c(t)$ w.r.t t

$$\frac{d\,c(t)}{dt} = -\frac{e^{-\xi\omega_n t}(-\xi\omega_n)\sin(\omega_d t+\theta)}{\sqrt{1-\xi^2}} - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\omega_d(\cos\omega_d$$

$$+ \frac{e^{-\xi\omega_n t}\xi\omega_n \sin(\omega_d t+\theta)}{\sqrt{1-\xi^2}} - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\omega_d \cos(\omega_d t+\theta) = 0$$

Substitute $t = T_p$ & $\omega_d = \omega_n\sqrt{1-\xi^2}$

$$\frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\sin(\omega_d t+\theta) - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\omega_n\sqrt{1-\xi^2}\cos(\omega_d t+\theta) = 0$$

$\xi\omega_n e^{-\xi\omega_n}$

$$\frac{\xi\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\left[\xi\sin(\omega_d t+\theta) - \sqrt{1-\xi^2}\cos(\omega_d t+\theta)\right] = 0$$

$$\xi\sin(\omega_d t+\theta) - \sqrt{1-\xi^2}\cos(\omega_d t+\theta) = 0$$

$$\xi\sin(\omega_d t+\theta) = \sqrt{1-\xi^2}\cos(\omega_d t+\theta)$$

$$\tan(\omega_d t+\theta) = \frac{\sqrt{1-\xi^2}}{\xi}$$

where $\tan\theta = \frac{\sqrt{1-\xi^2}}{\xi}$.

$$\tan(\omega_d T_p+\theta) = \tan\theta \qquad -(4).$$

From trignometric formula

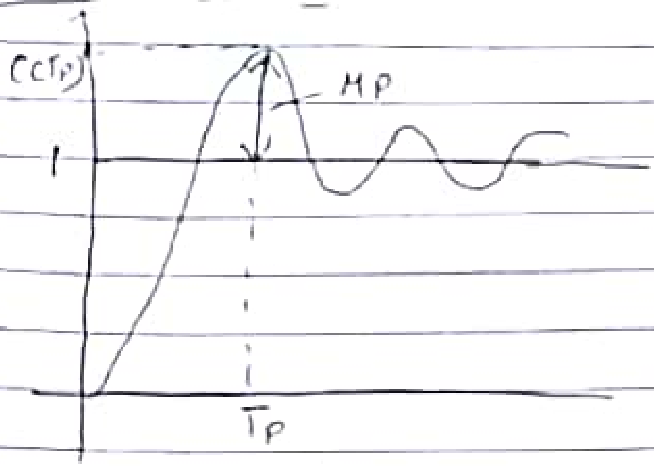
$$\tan(n\pi+\theta) = \tan\theta \qquad -(5)$$

comparing Eqn (4) & (5)

$$\omega_d T_p = n\pi$$

$$\boxed{T_p = \frac{n\pi}{\omega_d} = \frac{n\pi}{\omega_n\sqrt{1-\xi^2}}}$$

Derivation of M_p



From the figure we can write

$$M_p = c(T_p) - 1 \qquad —(1)$$

$$M_p = \left[1 - \frac{e^{-\xi \omega_n T_p}}{\sqrt{1-\xi^2}} \sin(\omega_d T_p + \theta)\right] - 1$$

$$M_p = -\frac{e^{-\xi \omega_n T_p}}{\sqrt{1-\xi^2}} \sin(\omega_d T_p + \theta) \qquad —(2)$$

w.k.T $\quad T_p = \dfrac{\pi}{\omega_d}$

Substituting T_p in Eqn (2)

$$M_p = -\frac{e^{-\xi \omega_n T_p}}{\sqrt{1-\xi^2}} \sin(\pi + \theta)$$

where $\sin(\pi+\theta) = -\sin\theta$

$$M_p = \frac{e^{-\xi\omega_n T_p}}{\sqrt{1-\xi^2}} \sin\theta \quad \text{--- (3)}$$

WKT $\tan\theta = \frac{\sqrt{1-\xi^2}}{\xi}$

where $\sin\theta = \sqrt{1-\xi^2}$

Substituting eq $\sin\theta$ in Eqn (3)

~~Mp =~~

$$M_p = \frac{e^{-\xi\omega_n T_p}}{\cancel{\sqrt{1-\xi^2}}} \cancel{\sqrt{1-\xi^2}}$$

$$M_p = e^{-\xi\omega_n T_p}$$

where $T_p = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$

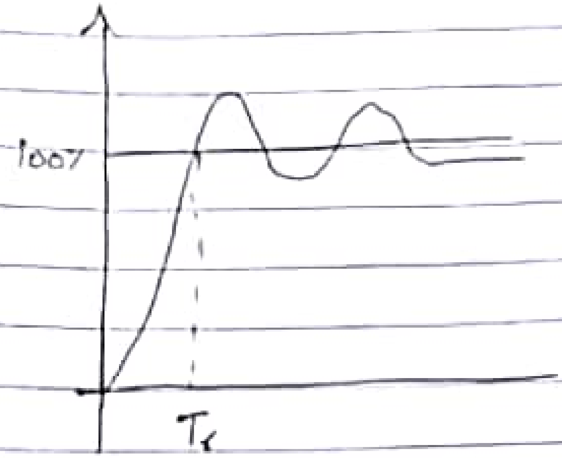
$$M_p = e^{-\xi\omega_n \frac{\pi}{\omega_n\sqrt{1-\xi^2}}}$$

$$M_p = e^{-\xi\pi/\sqrt{1-\xi^2}}$$

$$\boxed{\% M_p = e^{-\xi\pi/\sqrt{1-\xi^2}} \times 100}$$

Derivation of T_r

Time required by output to achieve 100% of its final value, starting from zero during the first attempt is the rise time



$$c(t)|_{t=T_r} = 1 \quad \text{for unit step input} \quad -(1)$$

$$1 - \frac{e^{-\xi\omega_n T_r}}{\sqrt{1-\xi^2}} \sin(\omega_d T_r + \theta) = 1$$

$$-\frac{e^{-\xi\omega_n T_r}}{\sqrt{1-\xi^2}} \sin(\omega_d T_r + \theta) = 0$$

$$\sin(\omega_d T_r + \theta) = 0. \quad -(2)$$

Trignometrically this is true only if

$$\sin n\pi = 0$$

$$\sin(\omega_d T_r + \theta) = \sin n\pi.$$

$$\omega_d T_r + \theta = n\pi \quad \text{where } n = 1, 2, 3 \ldots$$

$$\omega_d T_r = n\pi - \theta$$

$$\boxed{T_r = \frac{n\pi - \theta}{\omega_d} \text{ sec}}$$

Derivation of T_s

For 2% tolerance band.

T_s is the time when output becomes 98% of its final value and remains within the range of ±2%

$$\therefore \quad c(t)\Big|_{t=T_s} = 0.98$$

At $t = T_s$, the transient oscillatory term completely vanishes.

Thus $\frac{1}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$ vanishes, which determine the oscillatory behaviour.

$$c(t)_{t=T_s} = 0.98$$

$$1 - e^{-\xi \omega_n T_s} = 0.98$$

$$e^{-\xi \omega_n T_s} = 1 - 0.98$$

$$e^{-\xi \omega_n T_s} = 0.02$$

Applying ln on both sides

$$\ln(e^{-\xi \omega_n T_s}) = \ln(0.02)$$

$$-\xi \omega_n T_s = -3.912$$

$$T_s = \frac{3.912}{\xi \omega_n}$$

$$\boxed{T_s \cong \frac{4}{\xi \omega_n}}$$

Similarly for ±5% of tolerance

$$c(t)|_{t=T_s} = 0.95$$

$$1 - e^{-\xi \omega_n T_s} = 0.95$$

$$e^{-\xi \omega_n T_s} = 0.05$$

$$\ln(e^{-\xi \omega_n T_s}) = \ln(0.05)$$

$$-\xi \omega_n T_s = -2.995$$

$$T_s = \frac{2.995}{\xi \omega_n}$$

$$\boxed{T_s \cong \frac{3}{\xi \omega_n}}$$

Derivation of Delay time T_d.

$$c(t)\big|_{t=T_d} = 0.5$$

$$1 - \frac{e^{-\xi\omega_n T_d}}{\sqrt{1-\xi^2}} \sin(\omega_d T_d + \theta) = 0.5$$

$$\frac{e^{-\xi\omega_n T_d}}{\sqrt{1-\xi^2}} \sin(\omega_d T_d + \theta) = 0.5$$

By using linear approximation we get

$$T_d = \frac{1 + 0.7\xi}{\omega_n}$$