Unit 5: Root locus Technique

- The performance characteristics such as relative stability and transient response of a closed loop C.S are directly related to location of roots of char^c equation in the s-plane
- The basic idea of the root locus is to study the the property of root locus graphically.
- Root locus is the path traced by the roots of char^c equation when one of the system parameters usually the gain constant K is varied from (0 to ∞)
- Complementary root locus is the path traced by roots of char^c equation when system gain K is varied from ($-\infty$ to 0)
- The path traced by roots of char^c equation when system gain K is varied from ($-\infty$ to $+\infty$) is known as complete root locus.

General rules and construction of Root locus

The plot representing the root locus is always drawn in s-plane, s is complex quantity, which is expressed as $\sigma + j\omega$, where $\sigma \rightarrow$ real axis, $j\omega$ is imaginary axis

$\Rightarrow$ To construct the root locus open loop poles and open loop zeros are necessary.
These are obtained as follows

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C.S

let $G(s)\,H(s)$ be the open loop T.F of

It can be expressed in pole-zero form.

$$G(s)\ H(s) = \frac{K(s+z_1)(s+z_2)\cdots}{s^n(s+p_1)(s+p_2)\cdots}$$

where K is loop gain or system gain

$z_1, z_2 \cdots \rightarrow$ open loop zero's represented by circle (o)

$p_1, p_2 \cdots \rightarrow$ open loop pole's represented by (x)

$n \rightarrow$ Number of pole's at origin

<u>Note</u>: (1) poles and zero's at origin cannot co-exist

(2) char^c^ equation of the system is $1+G(s)H(s)=0$

<u>Rule 1</u>: Consider the char^c^ equation as

$$1+G(s)\ H(s) = 0$$

Write the char^c^ equ in pole-zero form.

$$1+\underbrace{\frac{K}{s^n}}_{\text{poles at origin}} \frac{\overbrace{\sum_{i=1}^{1}(s+z_i)}^{\text{open loop zero's}}}{\underbrace{\sum_{j=1}^{m}(s+p_i)}_{\text{open loop poles}}} = 0.$$

<u>Rule 2</u>: locate the open loop zero's (with small circle 'o') and open loop poles (with the cross 'x') on the planes.

<u>Rule 3</u>: Determine the number of branches and its direction (i.e starting point and terminating point).

In general, number of branches (N) is equal to the number of open loop poles (P) or number of open loop zeros (Z), which ever is greater.

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$N = P$ for $P > Z$

$N = Z$ for $Z > P$

Whatever may be the condition i.e, $p > z$ or $z > p$, branch direction always starts from open loop poles and ends at open loop zero's.

No branch will start or terminate at ∞ (when $p = z$)

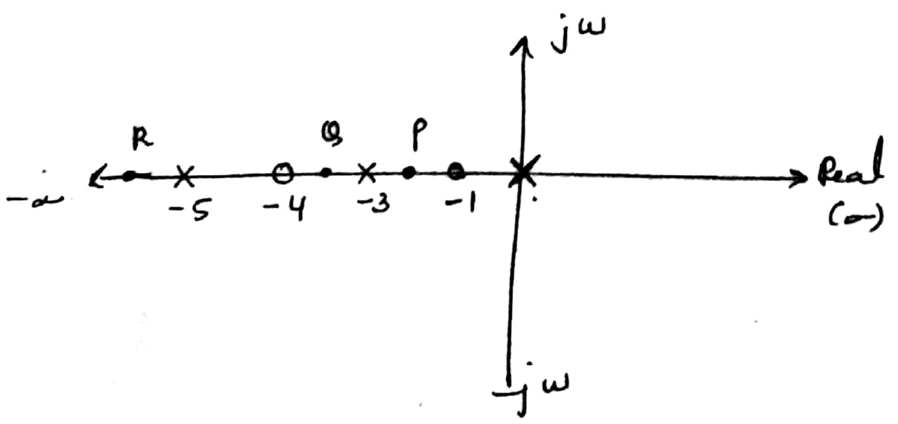
<u>Rule 4</u>: Determine the segment/section of real axis which belongs to the root locus.

To find the line segment that is part of root locus that lies on the real axis

- Identify the point on the real axis that lies on the root locus if the sum of the number of open loop zero's and open loop poles on the real axis to right of this point is <u>odd</u>, then that line segment is part of root locus.

<u>Note</u>: Complex poles or zero's need not to be considered while counting number poles & zero's to right of pt.

- <u>ex</u>: $$G(s)\,H(s) = \frac{K(s+1)(s+4)}{s(s+3)(s+5)}$$



Consider pt p on real axis. Then consider right half of real axis at this pt p. on right side there is 1 pole & 1 zero.

$p = 1,\ z = 1$ sum $= 1 + 1 = 2$

Hence according rule p cannot be root locus.

$Q \rightarrow 2p + 1z = 3$ odd root locus

$R \rightarrow 3p + 2z = 5$ odd. root locus.

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Rule 5: Check the Symmetry of the root locus about the real axis.

Note that root locus is always symmetric about the real axis because complex open loop zero's and complex open loop poles occurs in conjugate pair.

Rule 6: Determine the asymptotes of the root locus and asymptotes angle (θ_A)

Asymptotes gives the Information about the branches approaching towards Infinity, thus Asymptotes are the guide lines to branches approaching towards ∞ along straight line.

$$\text{Angle of Asymptotes}, \ \theta_A = \frac{(2q+1)\,180^\circ}{(p-z)}$$

where $q = 0, 1, 2, \ldots (p-z-1)$

= No of Asymptotes starting with zero.

Asymptotes are always symmetrically located about real axis

Rule 7: Determine the centroid (σ_A)

- Centroid is always real, it may be located on negative or positive real axis. It may or may not be part of root locus.
- It is common pt of real axis where all the asymptotes Intersect the real axis.

$$\sigma_A = \frac{\Sigma \text{ real parts of open loop poles} - \Sigma \text{ Real part of open loop zeros}}{p-z}.$$

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ex: $G(s)\,H(s) = \dfrac{K}{(s+1)\,(s+2+j2)\,(s+2-2j)}$

Calculate angle of asymptotes and centroid.

Sol: $p=3,\quad z=0 \qquad N = p = 3.$

$p - z = 3 - 0 = 3$ branches approaching ∞

poles located at $s = -1,\ -2 \pm j2$

Angle of asymptotes are given by

$q = 0, 1, 2 \qquad \theta = \dfrac{(2q+1)\,180^\circ}{p - z}$

- Number of asymptotes = Number of branches approaching ∞

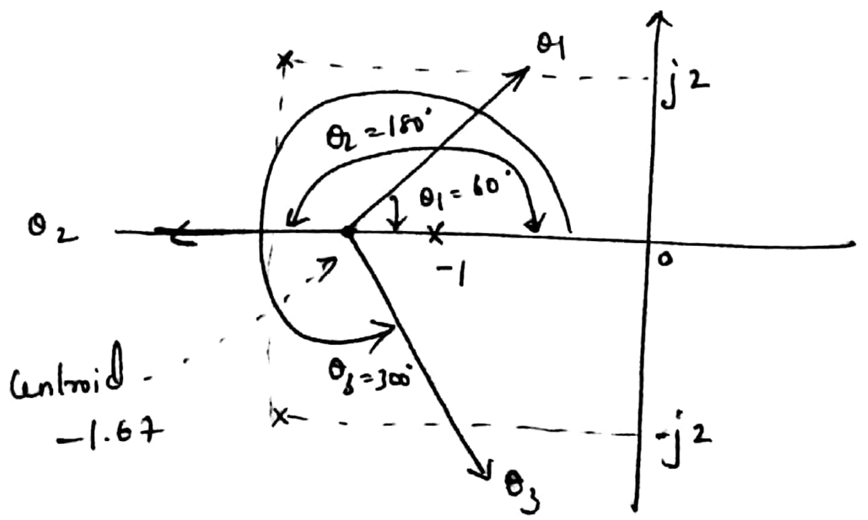
for $q = 0,\quad \theta_1 = \dfrac{180}{3} = 60^\circ$

$q = 1,\quad \theta_2 = \dfrac{(2+1)\,180}{3} = 180^\circ$

$q = 2\quad \theta_3 = \dfrac{(4+1)\,180}{3} = 300^\circ$

All these asymptotes are going to Intersect at a common point on real axis called Centroid.

- $\sigma = \dfrac{\Sigma p - \Sigma z}{p - z} = \dfrac{-1-2-2-0}{3} = -5/3 = -1.67$



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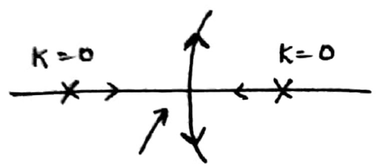
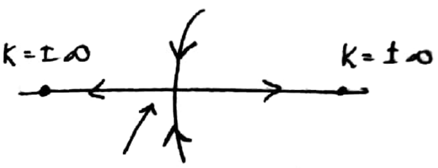
Rule 8: Determine breakaway point and break in point on the real axis (if any)

- A point on the real axis, which represents a multiple root of characteristic equation where the root locus leaves the real axis and enters the Imaginary (complex) plane is known as breakaway point, where as root locus returns to the real axis is known as breakin point. There can be complex conjugate in s-plane.

s-plane



Break away pt



Break in point.

- It can be determined from the equation $\frac{dK}{ds} = 0$

ex: for $G(s)\,H(s) = \frac{K}{s(s+1)(s+4)}$, determine the co-ordinates of valid breakaway points.

Sol^n: Char^c equ: $1 + G(s)\,H(s) = 0$

$$1 + \frac{K}{s(s+1)(s+4)} = 0$$

$$s^3 + 5s^2 + 4s + K = 0.$$

Write the value of K in terms of s

$$K = -s^3 - 5s^2 - 4s$$

$$\frac{dK}{ds} = -3s^2 - 10s - 4 = 0$$

$$\therefore \quad 3s^2 + 10s + 4 = 0.$$

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$\therefore$ Breakaway points. $\dfrac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$= \frac{-10 \pm \sqrt{100 - 4\times4\times3}}{2\times3} = -0.46, \ -2.86.$$

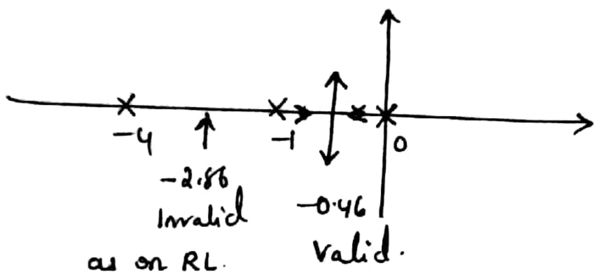
Subs in expression for K.

for $s = -0.46$, $K = +0.8793$.

$s = -2.86$, $K = -6.064$

$\therefore$ for $s = -0.46$, K is positive

$\therefore$ its valid breakaway pt for the root locus.



Note: If the value of K is positive that breakaway point is valid for the root locus. The breakaway points for which values of K is negative, are Invalid for direct RL but are valid for Inverse RL.

Rule 9. Intersection of Root locus with Imaginary axis.

Determine the point at which the root locus crosses the Imaginary axis (if it does so) using RH criterion and find corresponding value of K.

ex: $G(s)\,H(s) = \dfrac{K}{s(s+1)(s+4)}$.

Soln: char^c equ $\quad 1 + G(s)\,H(s) = 0$.

$$1 + \frac{K}{s(s+1)(s+4)} = 0$$

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i.e $s^3 + 5s^2 + 4s + K = 0$.

Routh's array,

s^3	1	4
s^2	5	K
s^1	$\frac{20-K}{5}$	0
s^0	K	

from s^0, $K > 0$

s^1, $20 - K > 0$ $K < 20$

$\therefore$ $K_{max} = 20$ that makes row corresponding to s^1 as row of zero's.

$$A(s) = 5s^2 + K = 0.$$

$$K = K_{max} = 20.$$

$$5s^2 + 20 = 0.$$

$$s^2 = -4$$

$$\therefore s = \pm j2.$$

So, $s = \pm j2$, are the points of Intersection of root locus with Imaginary axis. If K_{max} is positive there is valid Intersection of RL with Imaginary axis.

<u>Note:</u> If K_{max} is positive, RL Intersects with Imaginary axis. but if K_{max} is negative, RL doesnot Intersect with Imaginary axis and lies totally in left half of s-plane.

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Rule 10 : Determine the angle of departure from complex conjugate poles and angle of arrival at the complex conjugate zero's.

→ Angle of departure from complex conjugate pole

The angle at which branch departs from complex open loop pole (complex conjugate pole) is called angle of departure. Represented by ϕ_d.

$$\phi_d = 180 - \phi.$$

where $\phi = \Sigma \phi_p - \Sigma \phi_z$.

$\Sigma \phi_p$ = Contributions by the angle made by remaining open loop poles at the pole at which ϕ_d is to be calculated.

$\Sigma \phi_z$ = Contributions by the angle made by open loop zero's at the pole at which ϕ_d to be calculated

→ Angle of arrival at a complex zero

It can be calculated by using.

$$\phi_a = 180 + \phi$$

$$\phi = \Sigma \phi_p - \Sigma \phi_z.$$

Note: To calculate $\Sigma \phi_p$, join all the remaining poles to complex pole under consideration. Add all the angles subtended by joining pole to pole under consideration. Similarly join all zero's to pole under consideration and adding all angles determine $\Sigma \phi_z$.

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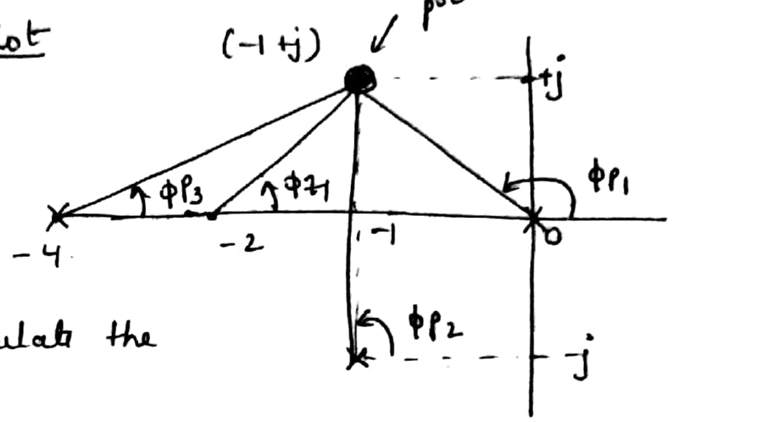
ex: for $G(s)\,H(s) = \dfrac{K(s+2)}{s(s+4)(s^2+2s+2)}$, calculate angles of departure at complex conjugate poles

Sol^n: $p = 4$, $z = 1$

poles are at $s=0$, $s=-4$, $s=-1\pm j$

zeros are at $s=-2$

→ Draw pole-zero plot



- let us calculate ϕd at the pole $s=-1+j$
- join all other poles to this pole and calculate the angles ϕp_1, ϕp_2, ϕp_3.
- join all zero's to this pole and calculate ϕz_1

$$\Sigma \phi_p = \phi p_1 + \phi p_2 + \phi p_3$$

$$\Sigma \phi_z = \phi z_1$$

from geometry, $\phi p_1 = 135°$, $\phi p_2 = 90°$, $\phi p_3 = 18.43°$

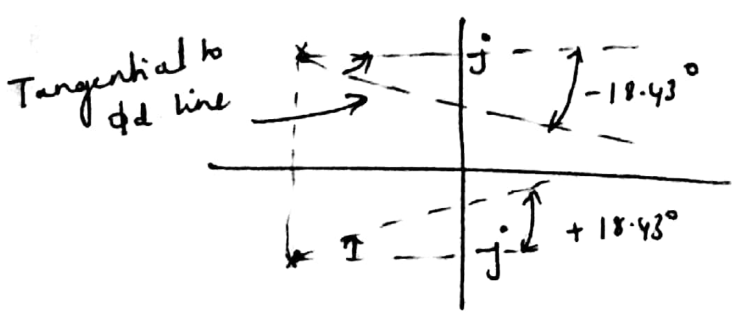
$$\therefore \Sigma \phi_p = 135° + 90° + 18.43° = 243.43°$$

$$\Sigma \phi_z = 45°$$

$\tan\theta = \frac{\text{opp}}{\text{Adj}}$

$$\phi = \Sigma\phi_p - \Sigma\phi_z = 243.43° - 45° = 198.43°$$

$$\phi d = 180 - \phi = 180 - 198.43° = -18.43°$$



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∴ Root locus branch leaving this pole will depart tangentially to the line whose angle is given by $\phi d = -18.43°$ as shown.

$s = -1-j$, will depart tangentially to the line whose angle is $\phi d = +18.43°$.

Rule 11 : Graphical determination of 'K' for specified Damping ratio 'ξ'

- determine Value of K at any pt s using magnitude condition and for given Value of damping ratio.

→ Value of K using Magnitude Cond^n.

$$|G(s)\ H(s)|_{s=s_1} = |1|$$

$$= \frac{\text{product of lengths from pt } s \text{ to open loop poles}}{\text{product of lengths from pt } s \text{ to open loop zero's}}.$$

Value of K using damping ratio, ξ

$$\xi = 0.707$$

find $\theta = \cos^{-1}\xi = \cos^{-1}(0.707) = 45°$

steps.

(1) Draw a RL to scale on graph paper. preferably choose same scale for x and y axis.

(2) Get value of $\theta = \cos^{-1}\xi$

(3) Draw a line at angle θ' from origin such that θ' is measured from negative real axis in clockwise direction.

(4) determine the Intersection point of this line with RL sketched to the scale.

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(5) For this point, apply the magnitude condition to decide the corresponding system gain 'K'

→ General steps to solve the problem on Root locus.

step 1 : Get the general Information about number of open loop poles, zeros, number of branches from $G(s) \cdot H(s)$.

step 2 : Draw the pole-zero plot. Identify sections of real axis for the existence of the root locus. And predict minimum number of breakaway points by using general predictions

step 3 : Calculate angles of Asymptotes
step 4 : Determine centroid, sketch a separate sketch for step 3 & step 4.
step 5 : calculate the breakaway and breakin points.
If breakaway points are complex conjugates, then use angle condition to check them for there validity as breakaway points.

step 6 : calculate the Intersection points of root locus with the Imaginary axis.

step 7 : Calculate the angles of departures or arrivals if applicable.

step 8 : Combine step 1 to 7 and draw final sketch of the root locus.

step 9 : predict the stability and performance of the given system by using root locus.

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→ Advantages of Root locus Method

(1) RL analysis also helps in deciding the stability of the controll systems with time delay.

(2) Gain margin and phase margin can be determined from RL.

(3) Relative stability about a particular value of $s = -\sigma$ can be determined.

(4) Information about settling time of the system also can be determined from RL.

(5) Using RL, value of the system gain 'K' for any pt on the RL can be determined, by using magnitude condition.

(6) Absolute stability of the system can be predicted from the locations of the roots in the s-plane.

→ Effect of Addition of open loop poles and Zero's

(1) Addition of pole

$$s^4 + 6s^3 + 11s^2 + 6s + K = 0$$

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