**EXAMPLE 10.54** Determine the inverse Z-transform of

(a) $X(z) = \log_e\left(\frac{1}{1-a^{-1}z}\right)$; ROC; $|z|<|a|$ (b) $X(z) = \log_e\left(\frac{1}{1-az^{-1}}\right)$; ROC; $|z|>|a|$

***Solution:***

(a) Given
$$X(z) = \log_e\left(\frac{1}{1-a^{-1}z}\right);\ \text{ROC; } |z|<|a|$$

$$X(z) = \log_e\left(\frac{1}{1-a^{-1}z}\right) = -\log_e(1-a^{-1}z)$$

$$= -\left[-a^{-1}z - \frac{(a^{-1}z)^2}{2} - \frac{(a^{-1}z)^3}{3} - \frac{(a^{-1}z)^4}{4} - \cdots\right]$$

$\therefore$
$$X(z) = \cdots + \frac{1}{4a^4}z^4 + \frac{1}{3a^3}z^3 + \frac{1}{2a^2}z^2 + \frac{1}{a}z = \sum_{n=-\infty}^{-1}\left(\frac{a^n}{-n}\right)z^{-n}$$

Hence
$$x(n) = -\frac{a^n}{n} \text{ for } n<0$$

that is,
$$x(n) = \left(-\frac{a^n}{n}\right)u(-n-1)$$

(b) Given
$$X(z) = \log_e\left(\frac{1}{1-az^{-1}}\right);\ \text{ROC; } |z|>|a|$$

$$X(z) = \log_e\left(\frac{1}{1-az^{-1}}\right) = -\log_e(1-az^{-1})$$

$$= -\left[-az^{-1} - \frac{(az^{-1})^2}{2} - \frac{(az^{-1})^3}{3} - \frac{(az^{-1})^4}{4} - \cdots\right]$$

$$\therefore\ X(z) = az^{-1} + \frac{(az^{-1})^2}{2} + \frac{(az^{-1})^3}{3} + \frac{(az^{-1})^4}{4} + \cdots = \sum_{k=1}^{\infty}\frac{(az^{-1})^k}{k} = \sum_{k=1}^{\infty}\frac{a^k}{k}z^{-k}$$

Taking inverse Z-transform, we have

$$x(n) = \sum_{k=1}^{\infty}\frac{a^k}{k}\delta(n-k) = \sum_{n=1}^{\infty}\frac{a^n}{n} = \frac{a^n}{n}u(n-1)$$

**EXAMPLE 10.55** Discuss the methods by which inverse Z-transformation can be found out?

***Solution:*** The process of finding $x(n)$ from its Z-transform $X(z)$ is called the inverse Z-transform and is denoted as:

$$x(n) = Z^{-1}[X(z)]$$

There are four methods often used to find the inverse Z-transform.

1. Power series method or long division method
2. Partial fraction method
3. Complex inversion integral method or residue method
4. Convolution integral method

The long division method is simple and the advantage of this method is: it is more general and can be applied to any problem, but the disadvantage is: it is difficult to get the solution in closed form. Further it can be used only if the ROC of the given $X(z)$ is either of the form $|z| > \alpha$ or of the form $|z| < \alpha$, i.e. it is useful only if the sequence $x(n)$ is either purely right-sided or purely left-sided. It cannot be used for bidirectional sequences.

If $X(z)$ is a ratio of two polynomials, then

$$X(z) = \frac{N(z)}{D(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + a_N z^{-N}}$$

We can generate a series in $z$ by dividing the numerator by the denominator.

If ROC is $|z| > a$, it yields a causal sequence.

$$X(z) = x(0) + x(1)\, z^{-1} + x(2)\, z^{-2} + \cdots$$

So $X(z)$ is to be expressed in negative powers of $z$.

If ROC is $|z| < a$, it yields an anticausal sequence.

$$X(z) = x(0) + x(-1)\, z^{1} + x(-2)\, z^{2} + \cdots$$

So $X(z)$ is to be expressed in positive powers of $z$.

In long division method to realise a causal sequence (i.e. if ROC is $|z| > a$) both numerator and denominator are expressed either in descending powers of $z$ or in ascending powers of $z^{-1}$, and the numerator is divided by the denominator continuously. To realise an anticausal sequence (i.e. ROC is $|z| < a$) both numerator and denominator are expressed either in ascending powers of $z$ or in descending power of $z^{-1}$ and the numerator is divided by the denominator continuously.

For partial fraction expansion method, $X(z)/z$ must be proper and the denominator should be in factored form. If it is not proper, it is to be written as the sum of a polynomial and a proper transfer function. The proper function $X(z)/z$ is written in terms of partial fractions and inverse Z-transform of each partial fraction is found by using the table of standard Z-transform pairs and all of them are added.

In the residue method, the inverse Z-transform of $X(z)$ can be obtained using the equation:

$$x(n) = \frac{1}{2\pi j} \oint_c X(z)\, z^{n-1}\, dz$$

where $c$ is a circle in the z-plane in the ROC of $X(z)$. The above equation can also be written as

$$x(n) = \sum \text{Residues of } X(z)\, z^{n-1} \text{ at the poles inside } c$$

$$= \sum_i (z - z_i)\, X(z) z^{n-1} \Big|_{z=z_i}$$

If $X(z)$ has repeated poles of order $k$, the residue at that pole is given by

$$\frac{1}{(k-1)!}\frac{d^{k-1}}{dz^{k-1}}(z-z_i)^k X(z)\, z^{n-1}\Big|_{z=z_i}$$

The convolution integral method uses convolution property of Z-transforms to determine the inverse Z-transform and can be used when the given $X(z)$ can be written as the product of two functions. In this method, the given $X(z)$ is written as the product of two functions $X_1(z)$ and $X_2(z)$ and the inverse Z-transform of those two functions, i.e. $x_1(n)$ and $x_2(n)$ are determined separately. The inverse Z-transform of $X(z)$ is then obtained by convolving $x_1(n)$ and $x_2(n)$ in time domain.

**EXAMPLE 10.56** Determine the inverse Z-transform of

$$X(z) = \log_e(1 + az^{-1});\ \text{ROC};\ |z| > a$$

***Solution:*** Given $$X(z) = \log_e(1 + az^{-1})$$

Differentiating the given Z-transform with respect to $z$, we get

$$\frac{dX(z)}{dz} = \frac{1}{1+az^{-1}}(-az^{-2}) = \frac{-az^{-2}}{1+az^{-1}}$$

or $$-z\frac{dX(z)}{dz} = \frac{az^{-1}}{1+az^{-1}} = az^{-1}\left[\frac{1}{1-(-a)z^{-1}}\right]$$

But from the property of differentiation in $z$-domain, we have

$$nx(n) \xleftrightarrow{\text{ZT}} -z\frac{dX(z)}{dz}$$

Hence $$nx(n) = Z^{-1}\left[az^{-1}\left(\frac{1}{1-(-a)z^{-1}}\right)\right] = -Z^{-1}\left[\frac{(-a)z^{-1}}{1-(-a)z^{-1}}\right]$$

$$= -(-a)\,Z^{-1}\left[\frac{1}{z-(-a)}\right] = -(-a)[(-a)^{n-1}u(n-1)] = -(-a)^n u(n-1)$$

$$\therefore \quad x(n) = -\frac{(-a)^n}{n}u(n-1)$$

**Alternately**

**EXAMPLE 10.57** Using long division, determine the inverse Z-transform of

$$X(z) = \frac{z^2 + 2z}{z^3 - 3z^2 + 4z + 1};\ \text{ROC: } |z| > 1$$

**Solution:** Since ROC is $|z| > 1$, $x(n)$ must be a causal sequence. For getting a causal sequence, the $N(z)$ and $D(z)$ of $X(z)$ must be put either in descending powers of $z$ or in ascending powers of $z^{-1}$ before performing long division.

In the given $X(z)$ both $N(z)$ and $D(z)$ are already in descending powers of $z$.

$$\begin{array}{r|l}
 & z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \\
\hline
z^3 - 3z^2 + 4z + 1 & z^2 + 2z \\
 & z^2 - 3z + 4 + z^{-1} \\
 & 5z - 4 - z^{-1} \\
 & 5z - 15 + 20z^{-1} + 5z^{-2} \\
 & 11 - 21z^{-1} - 5z^{-2} \\
 & 11 - 33z^{-1} + 44z^{-2} + 11z^{-3} \\
 & 12z^{-1} - 49z^{-2} - 11z^{-3} \\
 & 12z^{-1} - 36z^{-2} + 48z^{-3} + 12z^{-4} \\
 & -13z^{-2} - 59z^{-3} - 12z^{-4}
\end{array}$$

$$\therefore \quad X(z) = z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \ldots$$

$$\therefore \quad x(n) = \{0, 1, 5, 11, 12, -13, \ldots\}$$

Writing $N(z)$ and $D(z)$ of $X(z)$ in ascending powers of $z^{-1}$, we have

$$X(z) = \frac{N(z)}{D(z)} = \frac{z^2 + 2z}{z^3 - 3z^2 + 4z + 1} = \frac{z^{-1} + 2z^{-2}}{1 - 3z^{-1} + 4z^{-2} + z^{-3}}$$

$$\begin{array}{r|l}
 & z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \\
\hline
1 - 3z^{-1} + 4z^{-2} + z^{-3} & z^{-1} + 2z^{-2} \\
 & z^{-1} - 3z^{-2} + 4z^{-3} + z^{-4} \\
 & 5z^{-2} - 4^{-3} - z^{-4} \\
 & 5z^{-2} - 15z^{-3} + 20z^{-4} + 5z^{-5} \\
 & 11z^{-3} - 21z^{-4} - 5z^{-5} \\
 & 11z^{-3} - 33z^{-4} + 44z^{-5} + 11z^{-6} \\
 & 12z^{-4} - 49z^{-5} - 11z^{-6} \\
 & 12z^{-4} - 36z^{-5} + 48z^{-6} + 12z^{-7} \\
 & -13z^{-5} - 59z^{-6} - 12z^{-7}
\end{array}$$

$$\therefore \quad X(z) = z^{-1} + 5z^{-2} + 11z^{-3} + 12z^{-4} - 13z^{-5} \ldots$$

$$\therefore \quad x(n) = \{0, 1, 5, 11, 12, -13, \ldots\}$$

Observe that both the methods give the same sequence $x(n)$.

**EXAMPLE 10.58** Using long division, determine the inverse Z-transform of

$$X(z)=\frac{z^2+z+2}{z^3-2z^2+3z+4};\ \text{ROC: } |z|<1$$

***Solution:*** Since ROC is $|z|<1$, $x(n)$ must be a non-causal sequence. For getting a non-causal sequence, the $N(z)$ and $D(z)$ must be put either in ascending powers of $z$ or in descending powers of $z^{-1}$ before performing long division.

$$X(z)=\frac{z^2+z+2}{z^3-2z^2+3z+4}=\frac{2+z+z^2}{4+3z-2z^2+z^3}$$

$$\begin{array}{r|l}
 & \frac{1}{2}-\frac{1}{8}z+\frac{19}{32}z^2-\frac{81}{128}z^3+\frac{411}{512}z^4 \\
\hline
4+3z-2z^2+z^3 & 2+z+z^2 \\
 & 2+\frac{3}{2}z-z^2+\frac{1}{2}z^3 \\
\hline
 & -\frac{1}{2}z+2z^2-\frac{1}{2}z^3 \\
 & -\frac{1}{2}z-\frac{3}{8}z^2+\frac{1}{4}z^3-\frac{1}{8}z^4 \\
\hline
 & \frac{19}{8}z^2-\frac{3}{4}z^3+\frac{1}{8}z^4 \\
 & \frac{19}{8}z^2+\frac{57}{32}z^3-\frac{19}{16}z^4+\frac{19}{32}z^5 \\
\hline
 & -\frac{81}{32}z^3+\frac{21}{16}z^4-\frac{19}{32}z^5 \\
 & -\frac{81}{32}z^3-\frac{243}{128}z^4+\frac{81}{64}z^5-\frac{81}{128}z^6 \\
\hline
 & \frac{411}{128}z^4-129z^5+\frac{81}{128}z^6
\end{array}$$

$\therefore$
$$X(z)=\frac{1}{2}-\frac{1}{8}z+\frac{19}{32}z^2-\frac{81}{128}z^3+\frac{411}{512}z^4\ \ldots$$

$\therefore$
$$x(n)=\left\{\ldots,\frac{411}{512},-\frac{81}{128},\frac{19}{32},-\frac{1}{8},\underset{\uparrow}{\frac{1}{2}}\right\}$$

Also
$$X(z)=\frac{2+z+z^2}{4+3z-2z^2+z^3}=\frac{2z^{-3}+z^{-2}+z^{-1}}{4z^{-3}+3z^{-2}-2z^{-1}+1}$$

$$\therefore \quad X(z) = \frac{z}{z^2 - 1.5z + 0.5} = 2z^2 + 6z^3 + 14z^4 + 30z^5 + \cdots$$

$$\therefore \quad x(n) = 2\delta(n+2) + 6\delta(n+3) + 14\delta(n+4) + 30\delta(n+5) + \cdots$$
$$= \{\ldots, 30, 14, 6, 2, 0, 0\}$$

### 10.7.2 Partial Fraction Expansion Method

To find the inverse Z-transform of $X(z)$ using partial fraction expansion method, its denominator must be in factored form. It is similar to the partial fraction expansion method used earlier for the inversion of Laplace transforms. However, in this case, we try to obtain the partial fraction expansion of $X(z)/z$ instead of $X(z)$. This is because, the Z-transform of time domain signals have $z$ in their numerators. This method can be applied only if $X(z)/z$ is a proper rational function (i.e. the order of its denominator is greater than the order of its numerator). If $X(z)/z$ is not proper, then it should be written as the sum of a polynomial and a proper function before applying this method. The disadvantage of this method is that, the denominator must be factored. Using known Z-transform pairs and the properties of Z-transform, the inverse Z-transform of each partial fraction can be found.

Consider a rational function $X(z)/z$ given by

$$\frac{X(z)}{z} = \frac{b_0 z^M + b_1 z^{M-1} + b_2 z^{M-2} + \cdots + b_M}{z^N + a_1 z^{N-1} + a_2 z^{N-2} + \cdots + a_N}$$

When $M < N$, it is a proper function.
When $M \geq N$, it is not a proper function, so write it as:

$$\frac{X(z)}{z} = \underbrace{c_0 z^{N-M} + c_1 z^{N-M-1} + \cdots + c_{N-M}}_{\text{polynomial}} + \underbrace{\frac{N_1(z)}{D(z)}}_{\text{Proper rational function}}$$

There are two cases for the proper rational function $X(z)/z$.

**Case 1** $X(z)/z$ has all distinct poles.

When all the poles of $X(z)/z$ are distinct, then $X(z)/z$ can be expanded in the form

$$\frac{X(z)}{z} = \frac{C_1}{z - P_1} + \frac{C_2}{z - P_2} + \cdots + \frac{C_N}{z - P_N}$$

The coefficients $C_1, C_2, \ldots, C_N$ can be determined using the formula

$$C_k = (z - P_k)\left.\frac{X(z)}{z}\right|_{z=P_k}, \quad k = 1, 2, \ldots, N$$

CASE 2 $X(z)/z$ has $l$-repeated poles and the remaining $N$-$l$ poles are simple. Let us say the $k$th pole is repeated $l$ times. Then, $X(z)/z$ can be written as:

$$\frac{X(z)}{z} = \underbrace{\frac{C_1}{z-P_1} + \frac{C_2}{z-P_2} + \cdots +}_{(N\text{-}l)\text{ terms}} \frac{C_{k1}}{z-P_k} + \frac{C_{k2}}{(z-P_k)^2} + \cdots + \frac{C_{kl}}{(z-P_k)^l}$$

where
$$C_{kl} = (z-P_k)^l \left.\frac{X(z)}{z}\right|_{z=P_k}$$

In general,
$$C_{ki} = \frac{1}{(l-i)!} \frac{d^{l-i}}{dz^{l-i}} \left[ (z-P_k)^l \frac{X(z)}{z} \right]\Bigg|_{z=P_k}$$

If $X(z)$ has a complex pole, then the partial fraction can be expressed as:

$$\frac{X(z)}{z} = \frac{C_1}{z-P_1} + \frac{C_1^*}{z-P_1^*}$$

where $C_1^*$ is complex conjugate of $C_1$ and $P_1^*$ is complex conjugate of $P_1$.

In other words, complex conjugate poles result in complex conjugate coefficients in the partial fraction expansion.

**EXAMPLE 10.63** Find the inverse Z-transform of

$$X(z) = \frac{z^{-1}}{3 - 4z^{-1} + z^{-2}}; \text{ ROC; } |z| > 1$$

***Solution:*** Given
$$X(z) = \frac{z^{-1}}{3-4z^{-1}+z^{-2}} = \frac{z}{3z^2 - 4z + 1}$$

$$= \frac{z}{3[z^2 - (4z/3) + (1/3)]} = \frac{1}{3} \frac{z}{(z-1)[z-(1/3)]}$$

$$\therefore \quad \frac{X(z)}{z} = \frac{1}{3} \frac{1}{(z-1)[z-(1/3)]} = \frac{A}{z-1} + \frac{B}{z-(1/3)}$$

where $A$ and $B$ can be evaluated as follows:

$$A = (z-1)\left.\frac{X(z)}{z}\right|_{z=1} = (z-1)\frac{1}{3}\left.\frac{1}{(z-1)[z-(1/3)]}\right|_{z=1} = \frac{1}{3}\frac{1}{1-(1/3)} = \frac{1}{2}$$

$$B = \left(z - \frac{1}{3}\right)\left.\frac{X(z)}{z}\right|_{z=1/3} = \left(z-\frac{1}{3}\right)\frac{1}{3}\left.\frac{1}{(z-1)[z-(1/3)]}\right|_{z=1/3} = \frac{1}{3}\frac{1}{(1/3)-1} = -\frac{1}{2}$$

$$\therefore \quad \frac{X(z)}{z} = \frac{1}{2}\frac{1}{z-1} - \frac{1}{2}\frac{1}{z-(1/3)}$$

or
$$X(z) = \frac{1}{2}\left[\frac{z}{z-1} - \frac{z}{z-(1/3)}\right]; \text{ ROC; } |z| > 1$$

Since ROC is $|z| > 1$, both the sequences must be causal. Therefore, taking inverse Z-transform, we have

$$x(n) = \frac{1}{2}\left[u(n) - \left(\frac{1}{3}\right)^n u(n)\right]; \text{ ROC; } |z| > 1$$

**EXAMPLE 10.64** Find the inverse Z-transform of the following:

$$X(z) = \frac{(1/6)\,z^{-1}}{[1-(1/2)\,z^{-1}][1-(1/3)\,z^{-1}]}; \text{ ROC; } |z| > \frac{1}{2}$$

***Solution:*** Given
$$X(z) = \frac{(1/6)\,z^{-1}}{[1-(1/2)\,z^{-1}][1-(1/3)\,z^{-1}]}; \text{ ROC; } |z| > \frac{1}{2}$$

Multiplying the numerator and denominator of $X(z)$ by $z^2$, we have

$$X(z) = \frac{(1/6)\,z}{[z-(1/2)][z-(1/3)]}$$

The above equation can be expressed in partial fraction form as:

$$\frac{X(z)}{z} = \frac{1/6}{[z-(1/2)][z-(1/3)]} = \frac{C_1}{z-(1/2)} + \frac{C_2}{z-(1/3)}$$

where $C_1$ and $C_2$ can be evaluated as follows:

$$C_1 = \left(z - \frac{1}{2}\right)\frac{X(z)}{z}\bigg|_{z=1/2} = \frac{1}{6}\,\frac{1}{z-(1/3)}\bigg|_{z=1/2} = \frac{1}{6}\,\frac{1}{(1/2)-(1/3)} = 1$$

$$C_2 = \left(z - \frac{1}{3}\right)\frac{X(z)}{z}\bigg|_{z=1/3} = \frac{1}{6}\,\frac{1}{z-(1/2)}\bigg|_{z=1/3} = \frac{1}{6}\,\frac{1/6}{(1/3)-(1/2)} = -1$$

$$\therefore \quad \frac{X(z)}{z} = \frac{1}{z-(1/2)} - \frac{1}{z-(1/3)}$$

or
$$X(z) = \frac{z}{z-(1/2)} - \frac{z}{z-(1/3)}; \text{ ROC; } |z| > \frac{1}{2}$$

Since ROC is $|z| > 1/2$, both the sequences must be causal. Taking inverse Z-transform, we have

$$x(n) = \left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{3}\right)^n u(n)$$