ॐ

SIGNALS & SYSTEMS - MODULE 5

APPLICATIONS OF FOURIER REPRESENTATIONS

If a periodic signal is applied to a LTI system, the output of the system is determined by the convolution of the periodic input and aperiodic impulse response. It would be difficult to analyze the output signal in time domain. Therefore, we use the Fourier Transform to analyze the mixture of periodic and non-periodic signals.

Convolution of periodic & Non periodic Signals

W.k.t. FT of a periodic signal is given by

$$x(t) \xleftrightarrow{FT} X(j\omega)$$

$$X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\,\delta(\omega - k\omega_0) \rightarrow (1)$$

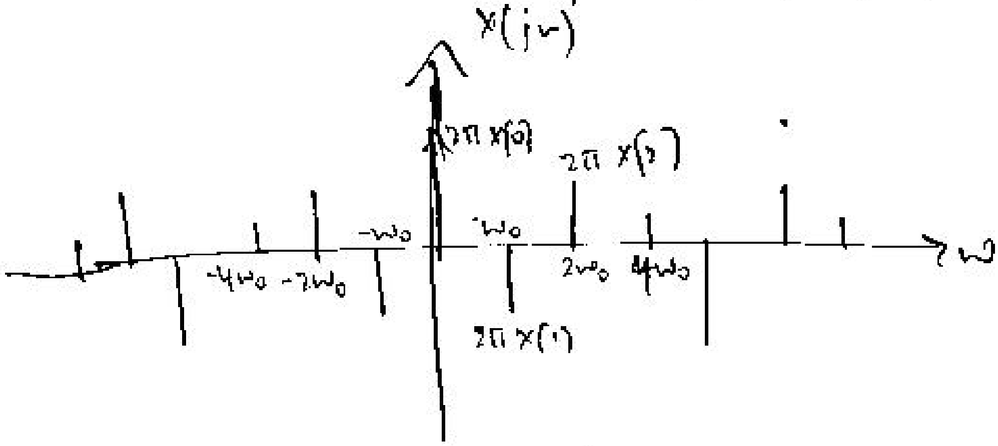
Also convolution in time domain corresponds to multiplication in frequency domain.

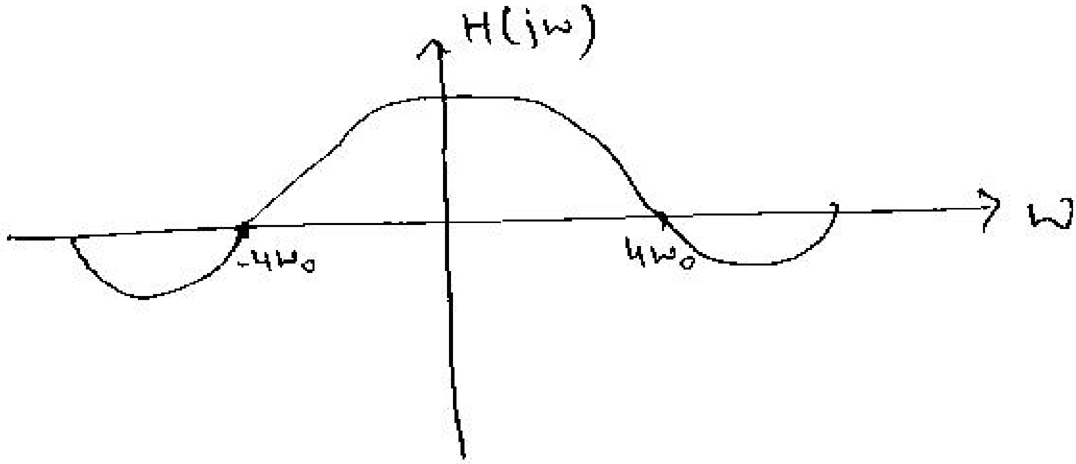
i.e. $$y(t) = x(t) * h(t) \xleftrightarrow{FT} Y(j\omega) = X(j\omega)\,H(j\omega) \rightarrow (2)$$

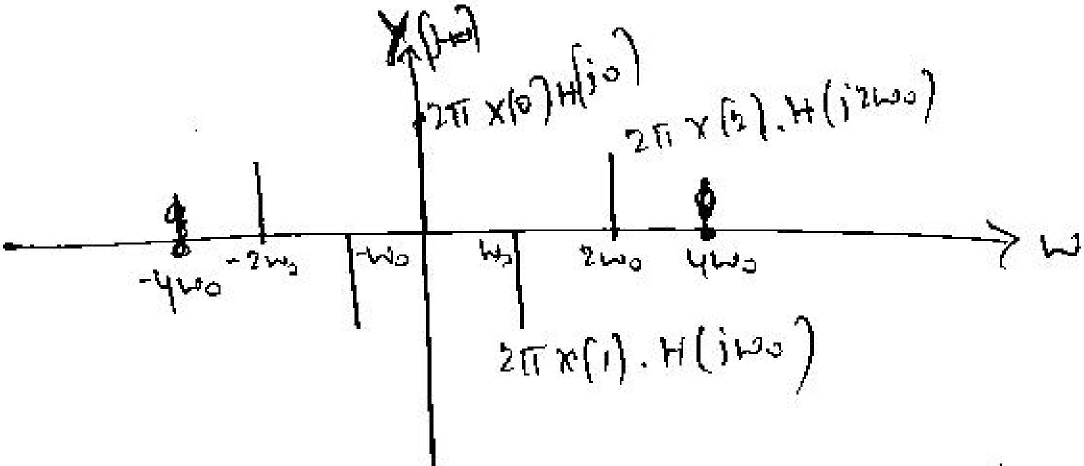
Substituting eq^n (1) in (2)

$$y(t) = x(t) * h(t) \xleftrightarrow{FT} Y(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} X[k]\,\delta(\omega - k\omega_0) . H(j\omega) \rightarrow (3)$$

$$\therefore Y(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} \underbrace{H(jk\omega_0)}_{\text{Using shifting property of Impulse}} \delta(\omega - k\omega_0)\, X(j\omega) \rightarrow (4)$$







* Strength of k^{th} impulse is adjusted by the value of $H(j\omega)$ at the frequency $k \rightarrow H(jk\omega_0)$

Ex: ~~H(jw)~~ $2\pi X[1] \cdot H(j\omega)\Big|_{\omega=\omega_0} = 2\pi X(1) \cdot H(j\omega_0)$

$2\pi X(2) \cdot H(j\omega)\Big|_{\omega=2\omega_0} = 2\pi X(2)\, H(j2\omega_0)$

* $Y(j\omega)$ corresponds to a periodic signal.

$\Rightarrow$ $y(t)$ is periodic with the same period as $x(t)$

$\rightarrow$ used to determine filter o/p with impulse response $h(t)$ and periodic i/p $x(t)$

* <u>Discrete Time analogous</u>

$$y(n) = x(n) * h(n) \xleftrightarrow{DTFT} Y(e^{j\Omega}) = X(e^{j\Omega}) \cdot H(e^{j\Omega})$$

$$X(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} X(k)\,\delta(\Omega - k\Omega_0)$$

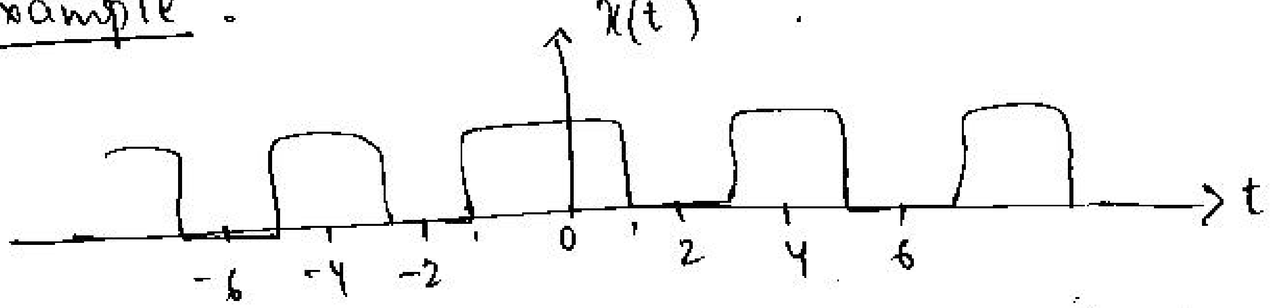
$\therefore$

$$y(n) = x(n) * h(n) \xleftrightarrow{DTFT} Y(e^{j\Omega})$$

$$Y(e^{j\Omega}) = 2\pi \sum_{k=-\infty}^{\infty} H(e^{jk\Omega_0})\, X(k)\, \delta(\Omega - k\Omega_0)$$

$\Rightarrow$ $y(n)$ is periodic with the same period as $x(n)$

<u>Example</u> :



For a s/m with input $x(t)$ and impulse response $h(t) = (1/\pi t)\sin(\pi t)$. Use the convolution property to find the output of the system.

* Find the FT of $h(t)$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt$$

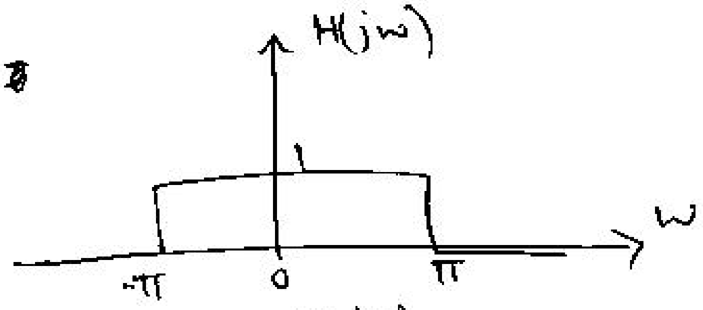
$$H(j\omega) = \int \frac{\sin \pi t}{\pi t} e^{-j\pi t} . dt$$

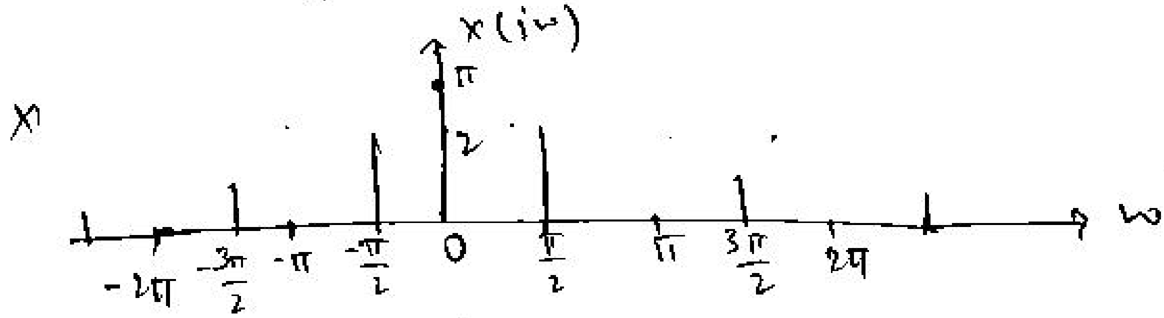
$$= \begin{cases} 1 ; & |\omega| \leq \pi \\ 0 ; & \text{otherwise} \end{cases}$$

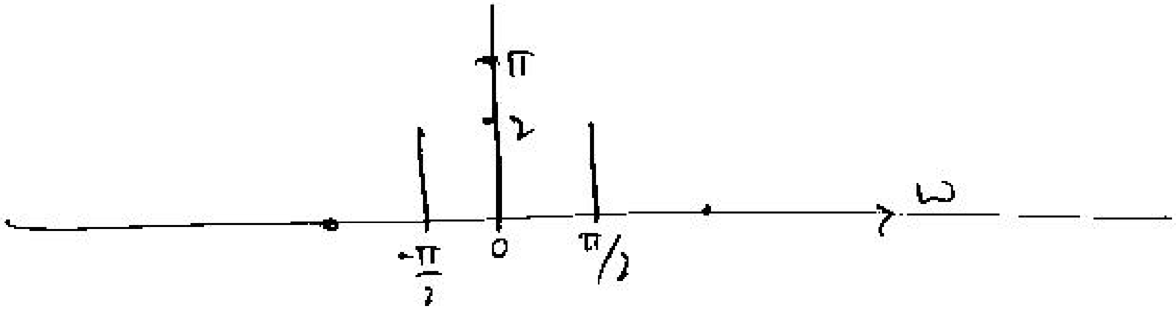
* FT of $x(t)$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} \frac{2\sin(k\pi/2)}{k} \delta\left(\omega - k\frac{\pi}{2}\right)$$

$$\therefore \quad Y(j\omega) = X(j\omega) . H(j\omega)$$







$\therefore \quad Y(j\omega) = 2\,\delta(\omega+\pi/2) + \pi\,\delta(\omega) + 2\,\delta(\omega-\pi/2)$

$\Rightarrow$ The s/m described by $H(j\omega)$ acts as a LPF i.e. It passes only the frequency components $-\pi/2$, 0 and $\pi/2$.

Taking the IFT of $Y(j\omega)$,

$$y(t) = \frac{1}{2} + \frac{1}{\pi}\cos(\omega\pi/2)$$

Sampling

→ Operation to generate DT signal from a CT signal.

⇒ We will show that DTFT of sampled signal is related to FT of continuous time signal.

→ Subsampling - operation on DT signals to change the ~~do~~ sampling rate

→ We will compare DTFT of sampled signal with DTFT of original signal.

Sampling of CT signals

Let $x(t)$ be a CT signal. Its sampled signal $x(n)$ is equal to samples of $x(t)$ at integer multiples of T, the sampling interval.

i.e. $x(n) = x(nT)$

* Let us relate DTFT of $x(n)$ to FT of $x(t)$ to examine effect of sampling.

Take FT of $x(n)$

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(n)\, \delta(t-nT)$$

Substitute $x(nT)$ for $x(n)$

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT)\, \delta(t-nT)$$

W.k.t. $x(t).\delta(t-nT) = x(nT).\delta(t-nT)$

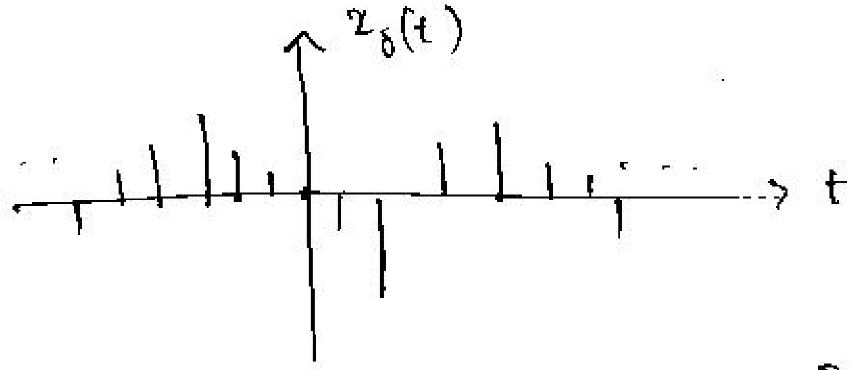
$$\therefore\ x_\delta(t) = x(t) \sum_{n=-\infty}^{\infty} \delta(t-nT)$$

$$\Rightarrow\ x_\delta(t) = x(t).p(t) \longrightarrow (1)$$

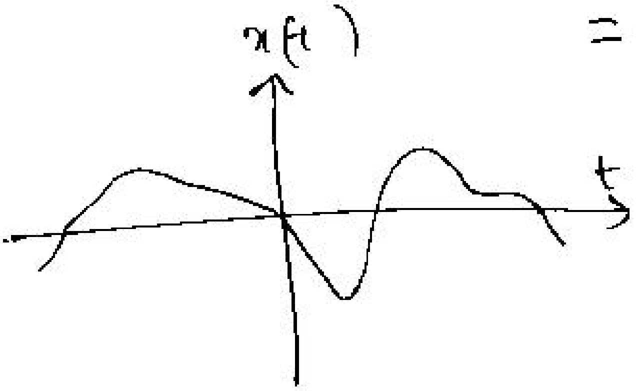
where $p(t) = \sum_{n=-\infty}^{\infty} \delta(t-nT) \rightarrow (2)$

* Eq^n (1) implies that the sampled signal $x(n)$ can be represented as the product of original CT signal and an impulse train.
* This operation is called impulse sampling.

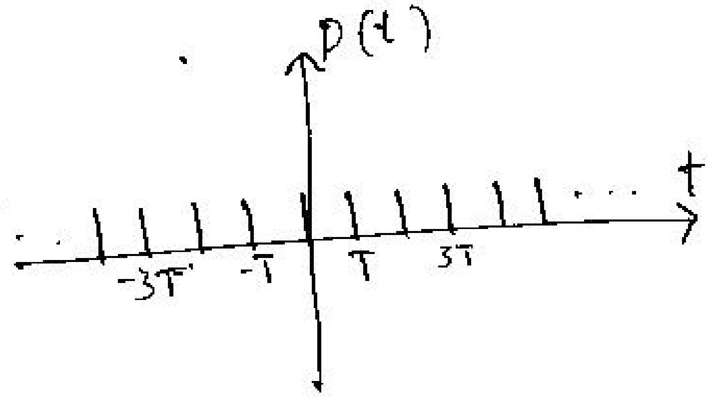
4



=



×



* Taking FT of $x_\delta(t)$ [$\Rightarrow$ Multiplication in time domain = Convolution in freq. domain]

$$X_\delta(jw) = \frac{1}{2\pi}\, X(jw) * P(jw)$$

~~next~~ Substitute FT of p(t) in the above equation.

p(t) is periodic with period T. $\Rightarrow w_0 = 2\pi/T$

FS co-efficients

$$P(k) = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t)\, e^{-jkw_0 t}\, dt$$

$$= \frac{1}{T}$$

$$\therefore P(jw) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(w - kw_0)$$

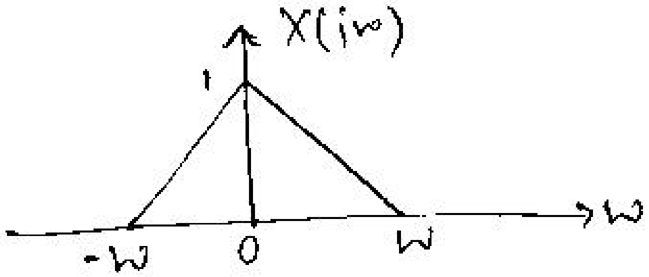
$$\therefore X_\delta(jw) = \frac{1}{2\pi}\, X(jw) * \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(w - kw_s)$$

$= \cancel{\frac{1}{T}\sum_{k=-\infty}^{\infty} X(jw)}$ $\quad w_s = \frac{2\pi}{T}$ - Sampling freq.

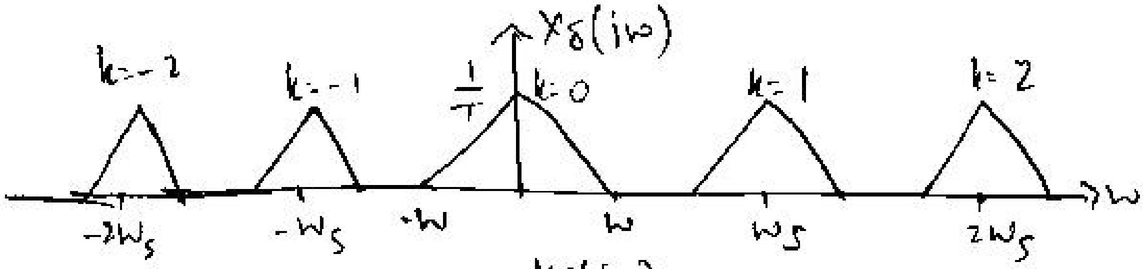
$$\therefore \quad X_\delta(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \quad \rightarrow \quad (3)$$

* Eqⁿ (3) → FT of Sampled Signal.
  ↓
  * Infinite sum of shifted versions of Original Signal's FT.
* Shifted version of $X(j\omega)$ are offset by integer multiples of ω_s.

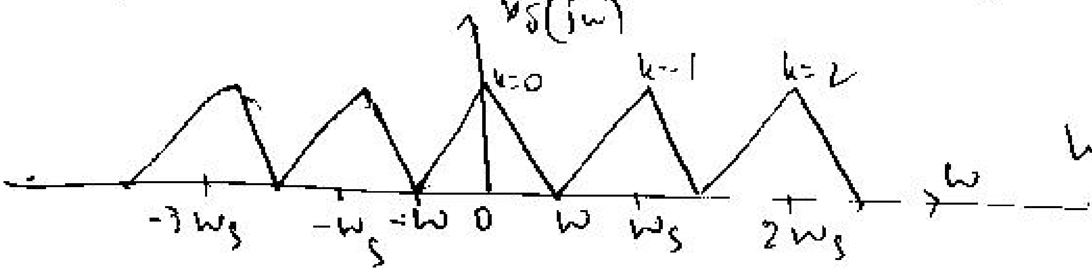
→ If we assume that $-W \le \omega \le W$, for different values of T, we illustrate eqⁿ (3) for various cases where $\omega_s = 3W$, $\omega_s = 2W$ & $\omega_s = 3/2\, W$.



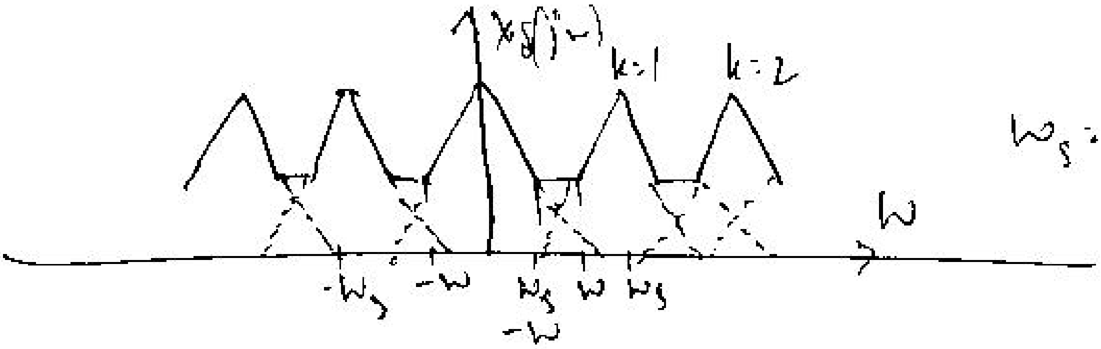
Spectrum of a CT signal



$\omega_s = 3W$



$\omega_s = 2W$



$\omega_s = \frac{3}{2}W$

From figure, we can observe that, as T increases and W_s decreases, shifted $X(jw)$ versions move closer. When $W_s < 2W$ they overlap.

↓

Aliasing

↓

High frequency component on a LF component.

→ In fig: Overlap b/w replicas of $X(jw)$ at $k=0$ & $k=1$ occurs for frequencies b/w $W_s - W$ and W. Thus it changes the shape of the spectrum.

⇒ Spectrum of sampled signal has no one to one correspondence with spectrum of CT signal & cannot be used for analysis & reconstruction of CT signal.

∴ To avoid aliasing, $W_s > 2W$

$$\Rightarrow T < \pi/W$$

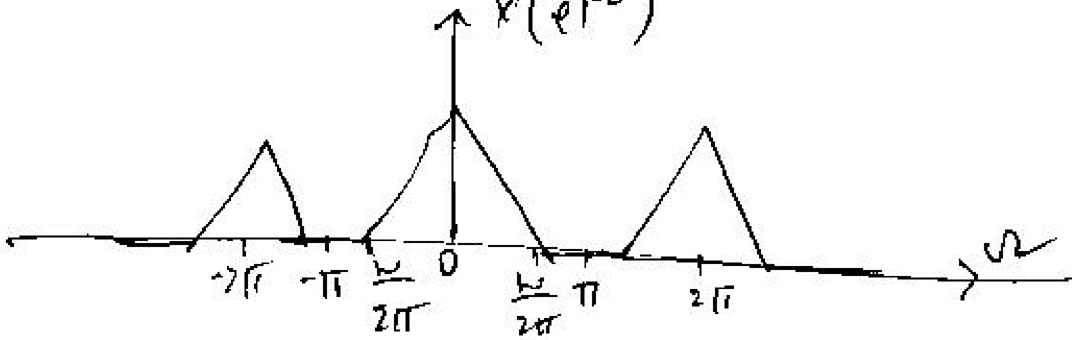
DTFT of sampled signal is obtained by using $\Omega = WT$

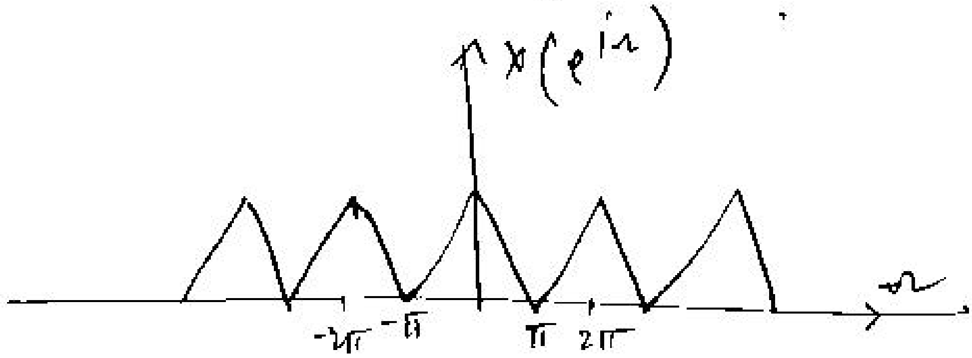
i.e. $$x[n] \xrightarrow{DTFT} X(e^{j\Omega}) = X_\delta(jw)\Big|_{w=\Omega/T}$$

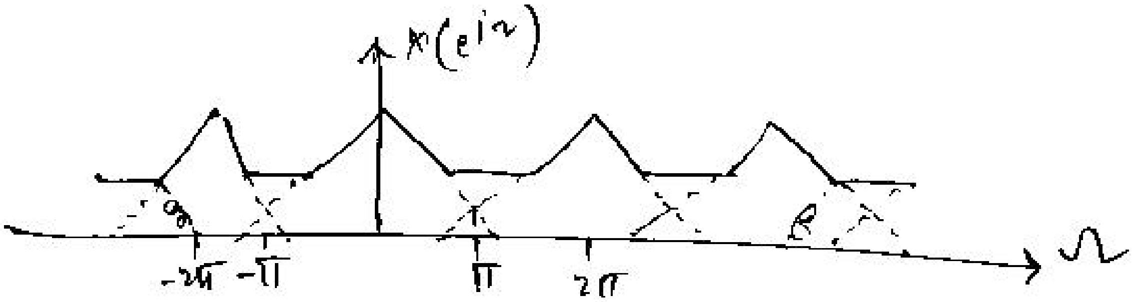
$$W = W_s \longrightarrow \Omega = 2\pi.$$

FTs have period W_s

DTFTs have period 2π


X(e^{jΩ})
-2π
-π
0
π
2π
Ω


X(e^{jΩ})
-2π
-π
π
2π
Ω


X(e^{jΩ})
-2π
-π
π
2π
Ω

Subsampling :

Sampling DT signals.

Let $y(n) = x[qn]$ be a subsampled version of $x(n)$

where $q \rightarrow$ positive integer.

→ Relate DTFT of $y(n)$ to DTFT of $x(n)$, by using FT to represent $x(n)$ as a sampled version of $x(t)$

→ Also, we express $y(n)$ as a sample of $x(t)$ by using a sampling rate 'q' times that of $x(n)$.

w.k.t. $x_\delta(t) = \sum_{n=-\infty}^{\infty} x(n)\,\delta(t - nT)$

From previous section : If $x(n) = x(nT)$

$$X_\delta(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s)) \rightarrow (1)$$

Expressing $y(n)$ as a sample of $x(t)$

$$y(n) = x[qn]$$
$$= x[nqT]$$

∴ Effective sampling rate $T' = qT$. for $y(n)$.

Now apply eqn (1) to $y(n)$.

$$y_\delta(t) = x(t)\sum_{n=-\infty}^{\infty}\delta(t-nT') \xleftrightarrow{FT} Y_\delta(jw)$$

$$Y_\delta(jw) = \frac{1}{T'}\sum_{k=-\infty}^{\infty} X(j(w-kw_s'))$$

Substitute $T' = qT$ & $w_s' = w_s/q$

$$Y_\delta(jw) = \frac{1}{qT}\sum_{k=-\infty}^{\infty} X\left(j\left(w-\frac{k}{q}w_s\right)\right) \rightarrow (2)$$

→ We have expressed $X_\delta(jw)$ & $Y_\delta(jw)$ in terms of $X(jw)$. But since $X(jw)$ is unknown, we try to express $Y_\delta(jw)$ in terms of $X_\delta(jw)$ → FT of $x(n)$.

In Eq^n (2) write k/q as a proper fraction

i.e. $\frac{k}{q} = l + \frac{m}{q}$

where l - integer part → $-\infty$ to ∞

m - remainder → 0 to $q-1$

$$\therefore Y_\delta(jw) = \frac{1}{q}\sum_{m=0}^{q-1}\underbrace{\left(\frac{1}{T}\sum_{l=-\infty}^{\infty} X\left(j\left(w-lw_s-\frac{m}{q}w_s\right)\right)\right)}_{X_\delta\left(j\left(w-\frac{m}{q}w_s\right)\right)}$$

$$\therefore Y_\delta(jw) = \frac{1}{q}\sum_{m=0}^{q-1} X_\delta\left(j\left(w-\frac{m}{q}w_s\right)\right) \rightarrow (3)$$

(3) represents ~~this~~ sum of shifted $X_\delta(j\omega)$.

* Sampling interval of $Y_\delta(j\omega)$ is T'.
* Converting back from FT to DTFT & using $\Omega = \omega T'$ in (3)

$$Y(e^{j\Omega}) = Y_\delta(j\omega)\Big|_{\omega=\Omega/T'}$$

$$= \frac{1}{q}\sum_{m=0}^{q-1} X_\delta\left(j\left(\frac{\Omega}{T'} - \frac{m}{q}\omega_s\right)\right)$$

Taking $T' = qT$

$$Y(e^{j\Omega}) = \frac{1}{q}\sum_{m=0}^{q-1} X_\delta\left(j\left(\frac{\Omega}{qT} - \frac{m}{q}\omega_s\right)\right)$$

$$= \frac{1}{q}\sum_{m=0}^{q-1} X_\delta\left(\frac{j}{T}\left(\frac{\Omega}{q} - \frac{m}{q}2\pi\right)\right) \rightarrow (4)$$

* Sampling interval of $X_\delta(j\omega)$ is T.

$$\therefore X(e^{j\Omega}) = X_\delta(j\Omega/T)$$

Substituting for X_δ term in (4)

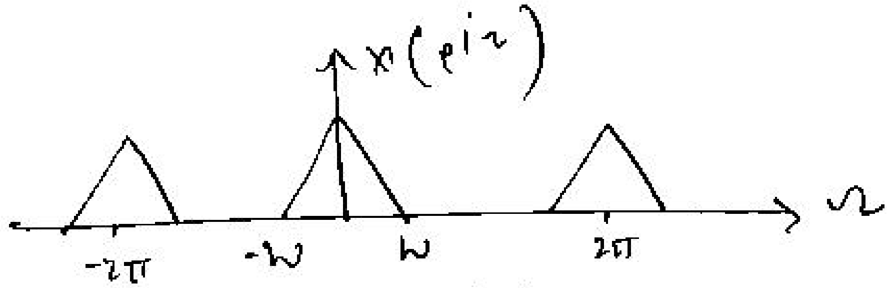
i.e. $X_\delta\left(\frac{j}{T}\left(\frac{\Omega}{q} - \frac{m}{q}2\pi\right)\right) = X\left(e^{j\left(\Omega/q - \frac{m}{q}2\pi\right)}\right)$

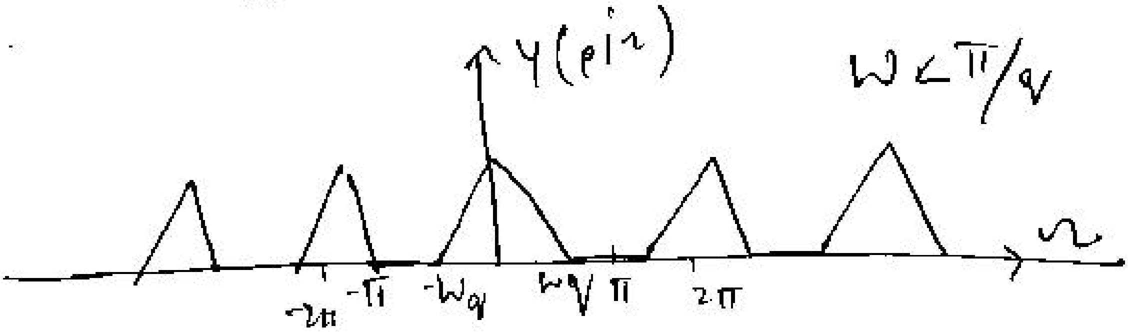
$$Y(e^{j\Omega}) = \frac{1}{q}\sum_{m=0}^{q-1} X\left(e^{j(\Omega/q - m2\pi/q)}\right)$$

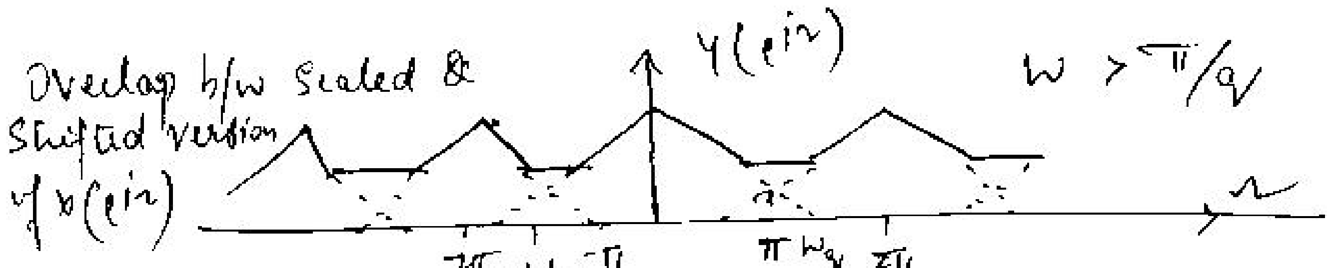
$$= \frac{1}{q}\sum_{m=0}^{q-1} X\left(e^{j(\Omega - m2\pi)/q}\right) \rightarrow (5)$$

$\Rightarrow$ $Y(e^{j\Omega})$ is obtained by sum of scaled versions of DTFT $X_q(e^{j\Omega}) = X(e^{j\Omega/q})$ shifted by multiples of 2π.

$$\Rightarrow \quad Y(e^{j\Omega}) = \frac{1}{q}\sum_{m=0}^{q-1} X_q\left(e^{j(\Omega - m2\pi)}\right)$$







$\Rightarrow$ Aliasing can be prevented if $W < \pi/q$

Highest freq. component.

Reconstruction of CT signals from samples.

→ Involves a mixture of CT & DT signals.

$$x[n] \longrightarrow \boxed{\text{Reconstruction S/m}} \longrightarrow x(t)$$

Sampling Theorem :

→ Samples are not unique identification features of a CT signal. They don't tell us anything about the behaviour of the signal in b/w the sample times.

→ Ex: $x_1(t)$ and $x_2(t)$ are 2 different signals with same set of samples ⇒ $x(n) = x_1(nT) = x_2(nT)$

∴ Set of constraints are necessary to determine how signal behaves b/w samples.

→ Signal should make smooth transitions b/w samples.

→ i.e. Rate at which time domain signal changes is directly related to maximum frequency component.

∴ Constraining smoothness in time domain corresponds to limiting bandwidth

→ One to one correspondence b/w time & frequency domain representation

⇒ To uniquely reconstruct → there must be a unique correspondence b/w FTs of CT & sampled signal.

→ FTs are uniquely related if there is no aliasing.

Sampling Theorem

Let $x(t) \xleftrightarrow{FT} X(j\omega)$ represent a band limited signal. So that $X(j\omega) = 0$ for $|\omega| > \omega_m$.

If $\omega_s > 2\omega_m$ where $\omega_s = \frac{2\pi}{T}$ is sampling freq,

then, $x(t)$ is uniquely determined by its samples

$x(nT)$, $n = 0, \pm 1, \pm 2, \ldots$

* $2\omega_m$ → minimum sampling freq → Nyquist rate
* ω_s → Nyquist frequency

Alternatively,

If $f_m = \frac{\omega_m}{2\pi}$ is the highest frequency
and f_s - sampling frequency } In Hertz

then Sampling theorem states that

$\boxed{f_s > 2f_m}$ & $f_s = 1/T$.

$\therefore$ $T < \frac{1}{2f_m}$ to satisfy sampling theorem conditions.

Example: If $x(t) = \sin(10\pi t/\pi t)$, Determine the condition on Sampling interval so that $x(t)$ can be uniquely represented by $x(n) = x(nT)$.

Sol^n: First determine the maximum frequency ω_m in $x(t)$.

Taking FT

$$X(j\omega) = \begin{cases} 1 ; & |\omega| \le 10\pi \\ 0 ; & |\omega| > 10\pi \end{cases}$$

$$\therefore \omega_m = 10\pi$$

Condition for sampling $\quad \frac{2\pi}{T} > 2\omega_m$

$$\Rightarrow \frac{2\pi}{T} > 2 \times 10\pi$$

$$\therefore \boxed{T < \frac{1}{10}}$$

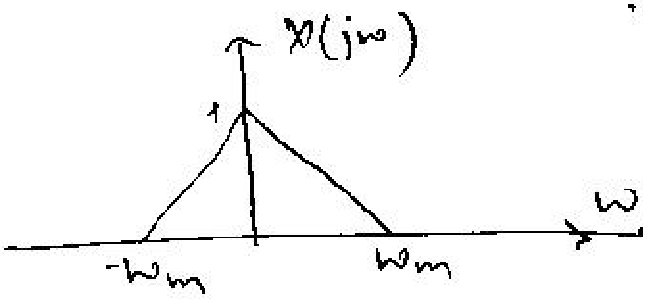
Anti-aliasing filter → used to prevent aliasing.

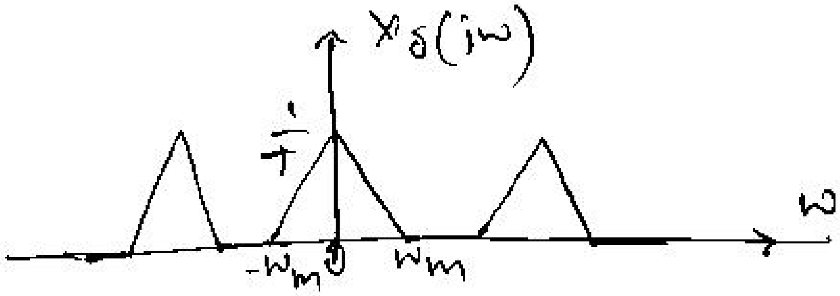
→ If we want to sample the signal at a rate, ω_s, less than twice the maximum frequency component, (Interested only in low frequency components), we can use a CT LPF prior to sampling & suppress any frequency above $\omega_s/2$. Such filter is called anti-aliasing filter.

Ideal Reconstruction

→ If $x(t) \xleftrightarrow{FT} X(j\omega)$,

$$X_\delta(j\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X(j\omega - jk\omega_s)$$

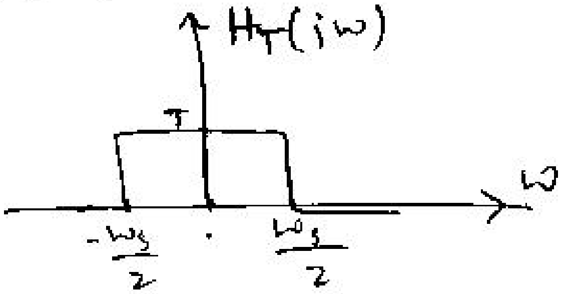




Reconstruction → Apply some operation to $X_\delta(j\omega)$ that converts it back to $X(j\omega)$.

→ Eliminate replicates of $X(j\omega)$ that appear at $k\omega_s$

* Multiply $X_\delta(j\omega)$ with $H_T(j\omega)$



$$\therefore X(j\omega) = X_\delta(j\omega) \cdot H_T(j\omega)$$

* Multiplication in freq. domain → Convolution in time domain.

$$\therefore \quad x(t) = x_\delta(t) * h_T(t)$$

Substituting for $x_\delta(t)$

$$x(t) = h_T(t) * \sum_{n=-\infty}^{\infty} x(n)\,\delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x(n)\, h_T(t-nT)$$

Now, $$h_T(t) = \frac{T \sin\left(\frac{w_s}{2} t\right)}{\pi t}$$

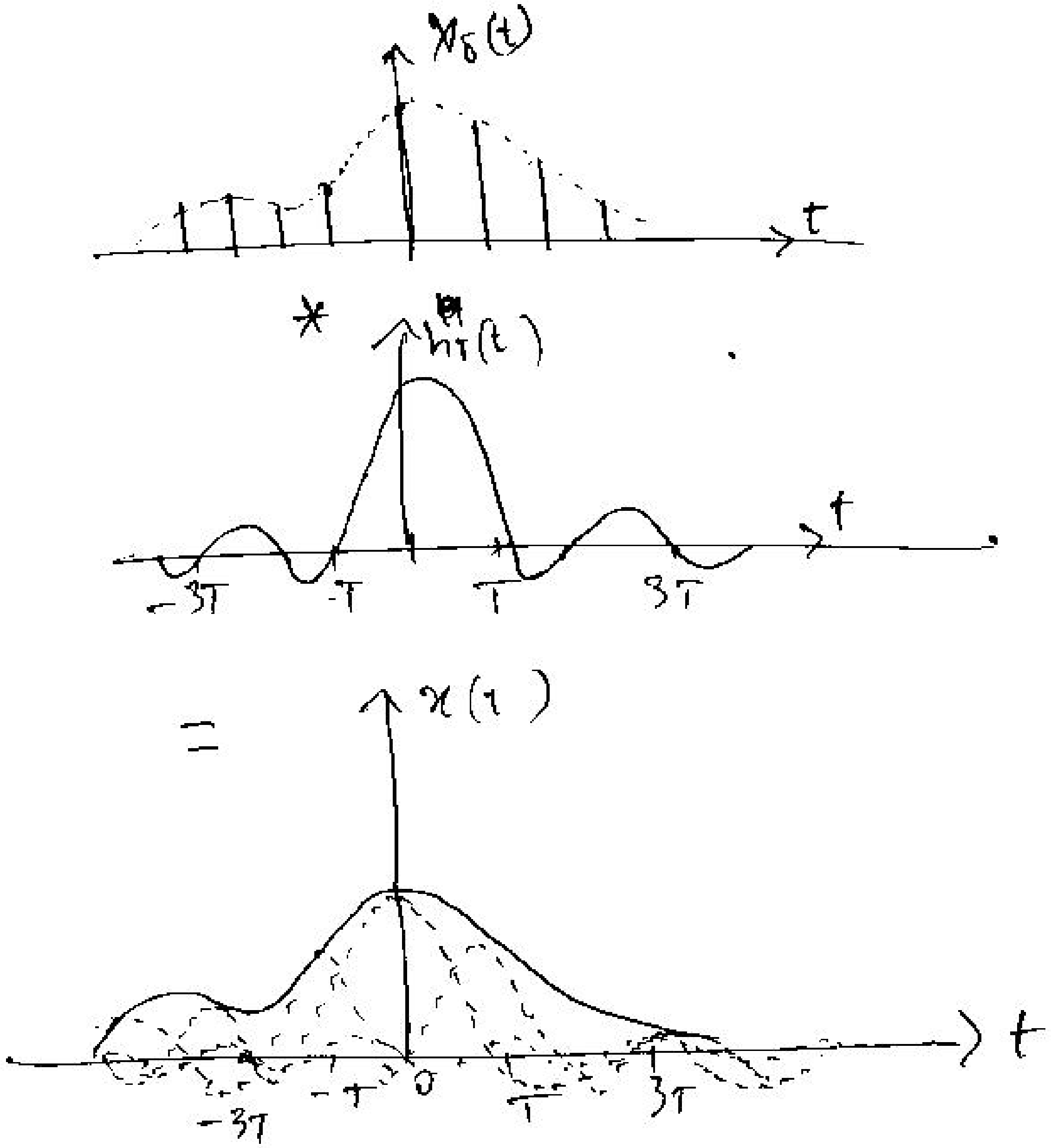
$$\therefore \quad x(t) = \sum_{n=-\infty}^{\infty} x(n)\,\text{sinc}\left(\frac{w_s}{2\pi}(t-nT)\right)$$

i.e. $x(t)$ - weighted sum of sinc functions shifted by the sampling interval.

* weights - correspond to values of DT sequence
* Value of $x(t)$ at $t=nT$ is $x(n)$ because all shifted sinc functions go through zero at nT except the n^{th} one whose value is $x(n)$.

The last eq^n is also termed as Band limited interpolation. It cannot be implemented practically because

* It represents a non-causal s/m. o/p depends on past & future values of $x(n)$
* Influence of each sample extends over infinite time because $h_T(t)$ has infinite duration



Modulation of periodic & Non periodic Signals

Modulation property of FT

$$y(t) = g(t)\,x(t) \xleftrightarrow{FT} Y(j\omega) = \frac{1}{2\pi} G(j\omega) * X(j\omega)$$

w.k.t. $X(j\omega) = 2\pi \sum_{k=-\infty}^{\infty} x(k)\,\delta(\omega - k\omega_0)$ for periodic $x(t)$

$$\therefore Y(j\omega) = G(j\omega) * \sum_{k=-\infty}^{\infty} X(k)\,\delta(\omega - k\omega_0)$$

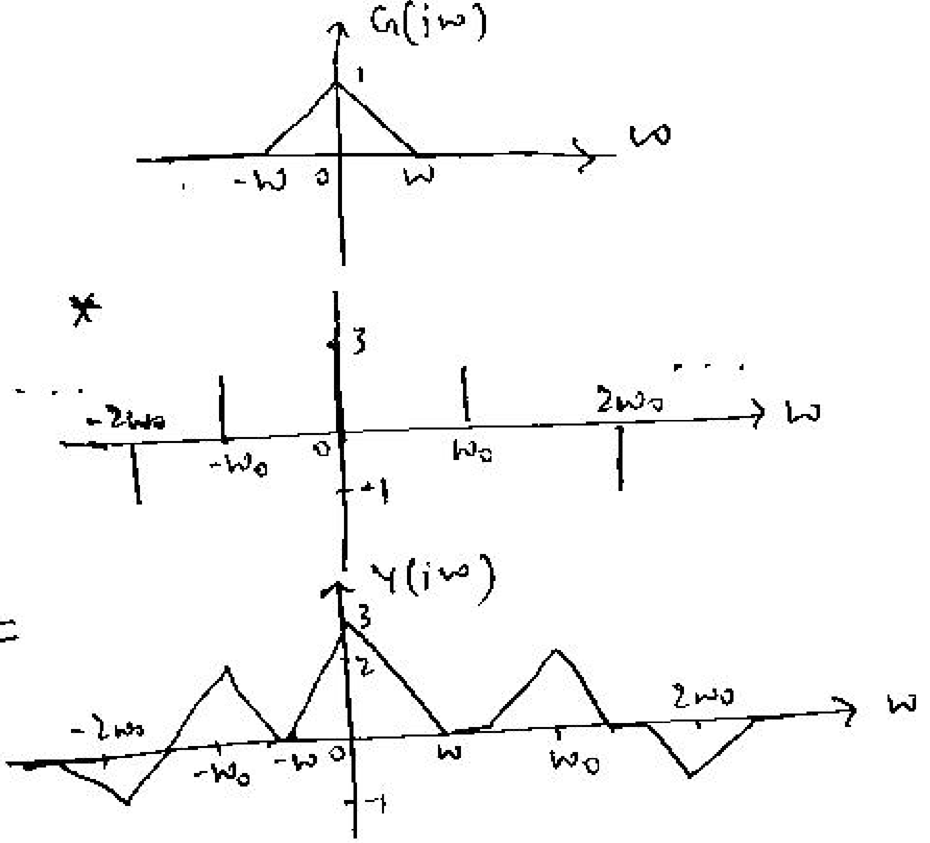
→ Convolution of any function with a shifted impulse results in shifted version of original function.

$$\therefore \quad Y(j\omega) = \sum_{k=-\infty}^{\infty} x[k]\, G(j(\omega - k\omega_0))$$

$\Rightarrow$ Modulation of $g(t)$ with periodic $x(t)$ gives a FT containing weighted sum of shifted version of $G(j\omega)$

$\rightarrow$ $Y(j\omega)$ $\rightarrow$ FT of a non periodic signal.

[~~Product~~ Periodic $\times$ Non periodic = Non periodic]



DT Modulation

$$y[n] = x[n]\,.\,g[n] \xrightarrow{DTFT} Y(e^{j\Omega}) = \frac{1}{2\pi} X(e^{j\Omega}) \circledast G(e^{j\Omega})$$

Substitute for $X(e^{j\Omega})$ & Simplify.

$$Y(e^{j\Omega}) = \int_{2\pi} \sum_{k=-\infty}^{\infty} x[k]\, \delta(\theta - k\Omega_0)\, G(e^{j(\Omega-\theta)})\, d\theta$$

Since $\Omega_0 = 2\pi/N$, in any 2π interval there are only N impulses of $\delta(\theta - k\Omega_0)$. $\therefore$ Infinite sum reduces to N values of k.

$$Y(e^{j\Omega}) = \sum_{k=\langle N\rangle} X[k] \int_{2\pi} \delta(\theta - k\Omega_0)\, G\left(e^{j(\Omega-\theta)}\right) d\theta$$

$$\therefore \quad Y(e^{j\Omega}) = \sum_{k=\langle N\rangle} X[k]\, G\left(e^{j(\Omega - k\Omega_0)}\right).$$