Module-5

State variable Analysis

Limitations of the Transfer function approach

1) Naturally, significant initial conditions in Obtaining precise solution of any s/m, loose their importance in conventional approach

2) The method is insufficient and troublesome to give complete time domain solution of higher order s/m

3) It is not very much convenient for the analysis of MIMO s/ms

4) It gives analysis of s/m for some specific types of i/ps like step, Ramp etc

5) It is only applicable to Linear Time Invariant s/ms

6) The classical methods like Root locus, Bode plot etc are basically trail & error procedures which fail to give the optimal solution required.

→ The concept of total internal state of the s/m considering all initial conditions. This technique which uses the concept of state is called state variable Analysis or state space Analysis.

Advantages of state variable Analysis

1) The method takes into account the effect of all initial conditions.

2) It can be applied to nonlinear as well as time varying s/ms

3) It can be conveniently applied to MIMO s/ms.

4) The slm can be designed for the optimal conditions precisely by using the modern method.

5) Any type of the i/p can be considered for designing the slm

6) As the method involves matrix algebra, can be conveniently adopted for the digital computers.

7) The state variables selected need not necessarily be the physical quantities of the slm

8) The vector matrix notation greatly simplifies the mathematical representation of the slm

→ Initial conditions i.e memory affects the slm characterisation & subsequent behaviour.

→ Initial conditions describe the status or state of the slm at $t=t_0$.

→ The state can be regarded as a compact & concise representation of the past history of the slm

→ The state of the slm in brief separates the future from the past so that the state contains all the information concerning the past history of the slm are necessarily required to determine the response of the slm for any given type of i/p.

→ The state of the slm at any time 't' is actually the combined effect of the values of the different elements of the slm which are associated with the initial conditions of the slm

→ The complete state of the slm can be considered to a vector having components which are the variable of slm which are closely associated with initial cond so state can be defined as vector x(t) called state vector x(t) ie state at any time 't' is 'n' dimensional vector ie column matrix $n \times 1$

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$x_1(t), x_2(t) \ldots\ldots x_n(t)$ constitute the state vector x(t) are called state variables of the s/m

Definitions.

(1) State :- The state of a dynamic system is defined as a minimal set of variables such that the knowledge of these variables at $t = t_0$ together with the knowledge of i/ps for $t \geq t_0$ completely determining the behaviour of the s/m for $t > t_0$.

(2) State variables :- The variables involved in determining the state of a dynamic s/m x(t), are called the state variables

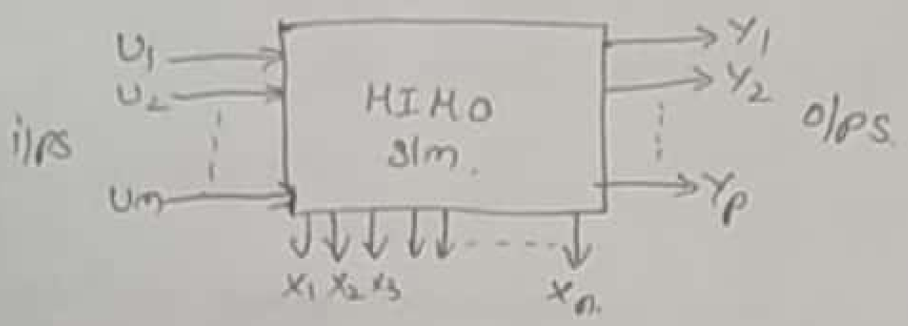
3) State vector :- The 'n' state variables necessary to describe the complete behaviour of the s/m can be considered as 'n' components of a vector x(t) called the state vector at time 't'

) State Space :- The space whose co-ordinate axes are nothing but the 'n' state variables with time as the implicit variable is called the state space.

5) **State Trajectory**:- It is the locus of the tips of the state vectors, with time as the implicit variable

State Model of Linear Systems

-> consider MIMO, nth order sym



no: of i/ps = m

no: of o/ps p

$$U(t) = \begin{bmatrix} U_1(t) \\ U_2(t) \\ \vdots \\ U_m(t) \end{bmatrix}_{m\times 1} \quad x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}_{n\times 1} \quad Y(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \\ \vdots \\ Y_p(t) \end{bmatrix}_{p\times 1}$$

The state variable representation can be arranged in the form of 'n' first order differential equations.

$$\frac{dx_1(t)}{dt} = \dot{x}_1(t) = f_1(x_1, x_2 \ldots x_n, U_1, U_2 \ldots U_m)$$

$$\frac{dx_2(t)}{dt} = \dot{x}_2(t) = f_2(x_1, x_2 \ldots x_n, U_1, U_2 \ldots U_m)$$

$$\vdots$$

$$\frac{dx_n(t)}{dt} = \dot{x}_n(t) = f_n(x_1, x_2 \ldots x_n, U_1, U_2, \ldots U_m)$$

where $f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$ is the functional operator

Integrating the above eqn

$$x_i(t) = x_i(t_0) + \int_{t_0}^{t} f_i(x_1, x_2 \dots x_n, u_1, u_2 \dots u_m)\, dt$$

$i = 1, 2 \dots n.$

Any 'n' dimensional time invariant s/m has state equations in the functional form as

$$\dot{x}(t) = f(x, u)$$

o/ps of such s/m are dependent on the state of the s/m & instantaneous i/ps.

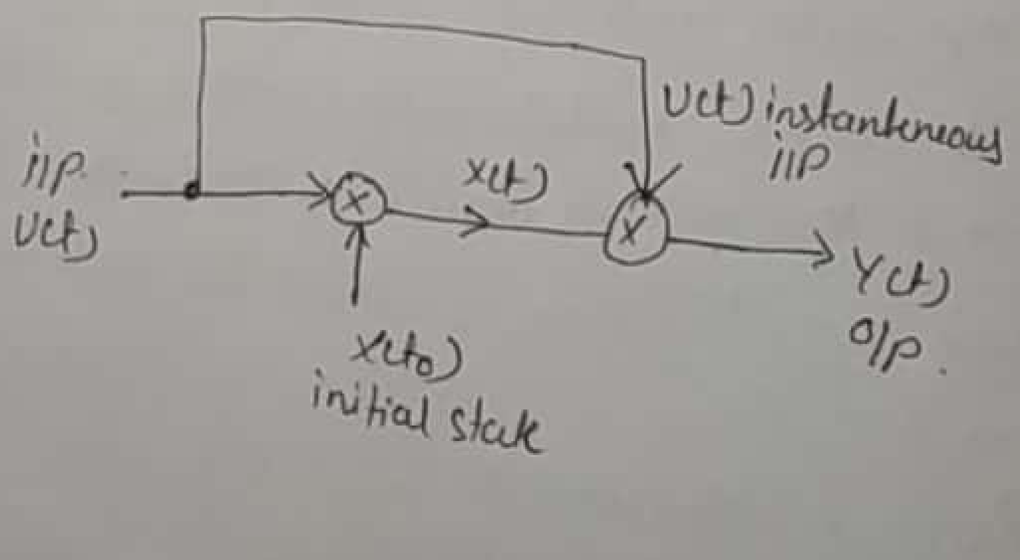
$\therefore$ functional o/p eqn can be written as

$y(t) = g(x, u)$ where 'g' is the functional operator

For time variant s/m, the same equations can be written as

$\dot{x}(t) = f(x, u, t)$ .... state eqn

$y(t) = g(x, u, t)$ .... o/p eqn



The functional eqns can be expressed interms of linear combination of s/m states & the I/p as.

$$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n + b_{11}U_1 + b_{12}U_2 + \cdots b_{1m}U_m$$

$$\dot{x}_2 = a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n + b_{21}U_1 + b_{22}U_2 + \cdots b_{2m}U_m$$

$$\vdots$$

$$\dot{x}_n = a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n + b_{n1}U_1 + b_{n2}U_2 + \cdots b_{nm}U_m$$

$$\dot{x}(t) = A\,x(t) + B\,U(t)$$

$x(t)$ = state vector matrix of order $n \times 1$

$U(t)$ = I/p vector matrix of order $m \times 1$

A = s/m matrix or Evaluation matrix of order $n \times n$

B = I/p matrix or control matrix of order $n \times m$

||ly O/p Variables

$$Y_1(t) = C_{11}x_1(t) + \cdots C_{1n}x_n(t) + d_{11}U_1(t) \cdots + d_{1m}U_m(t)$$

$$\vdots$$

$$Y_p(t) = C_{p1}x_1(t) + \cdots + C_{pn}x_n(t) + d_{p1}U_1(t) \cdots + d_{pm}U_m(t)$$

$$Y(t) = C\,x(t) + D\,U(t)$$

$Y(t)$ = O/p vector matrix of order $p \times 1$

C = O/p matrix or observation matrix of order $p \times n$

D = direct transmission matrix of order $p \times m$

The two vector eqns together is called the state model of the linear s/m

$$\dot{x}(t) = A\,x(t) + B\,U(t)$$
$$Y(t) = C\,x(t) + D\,U(t)$$

This is the state model of a s/m.

For linear time variant slms, the matrices A, B, C & D are also time dependent

$$\dot{x}(t) = A(t)x(t) + B(t)U(t)$$
$$y(t) = C(t)x(t) + D(t)U(t)$$

State Model of Single Input Single output Slm

$$\dot{x}(t) = Ax(t) + BU(t)$$
$$Y(t) = Cx(t) + dU(t)$$

A = $n \times n$ matrix
B = $n \times 1$ matrix
C = $1 \times n$ matrix
d = constant
$U(t)$ = Single scalar i/p variable.

State variable Representation using Physical variables

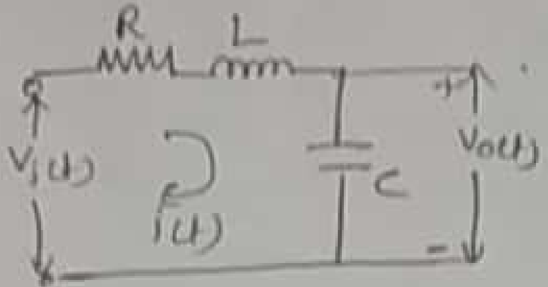
→ For the electrical slms, the c/ts through various inductors & the vtg across the various capacitors are selected to be the state variables.

→ The Equation for differentiation of one state variable should not involve the differentiation of any other state variable

→ The Mechanical slms the displacements & velocities of Energy storing elements such as spring & friction are selected as the state variable

→ The Physical variables associated with energy storing elements, which are responsible for initial conditions, are selected as the state variables of the given slm

Ex) Obtain the state model of the given Electrical s/m



There are two energy storing elements L & C, so the two state variables are c/t through inductor $i(t)$ & V/g across capacitor i.e $v_o(t)$

$$x_1(t) = i(t) \quad x_2(t) = v_o(t)$$

$$U(t) = V_i(t) = \text{i/p variable}$$

applying KVL

$$V_i(t) = i(t)R + L\frac{di(t)}{dt} + V_o(t) \quad \text{where } v_o(t) = \frac{1}{C}\int i(t)dt$$

arrange the above Eqn

$$L\frac{di(t)}{dt} = V_i(t) - i(t)R - V_o(t)$$

$$\frac{di(t)}{dt} = \frac{1}{L}V_i(t) - \frac{R}{L}i(t) - \frac{1}{L}V_o(t)$$

$$\dot{x}_1(t) = \frac{1}{L}U(t) - \frac{R}{L}x_1(t) - \frac{1}{L}x_2(t)$$

$$V_o(t) = \frac{1}{C}\int i(t)dt$$

$$\frac{dV_o(t)}{dt} = \frac{1}{C}i(t)$$

$$\dot{x}_2(t) = \frac{1}{C}x_1(t)$$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L} \\ 0 \end{bmatrix}U(t)$$

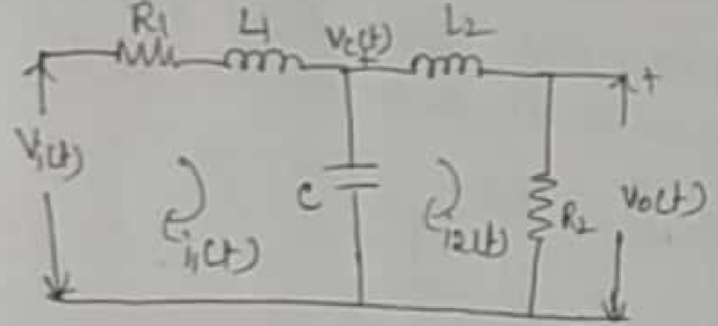
$$\dot{x}(t) = Ax(t) + Bu(t)$$

O/P Variable $y(t) = V_o(t) = x_2(t)$

$$y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\boxed{y(t) = C\,x(t)} \qquad D = 0.$$

2) Obtain the State model for the Electrical S/m given in fig. choosing the state Variables as $i_1(t)$, $i_2(t)$ & $V_c(t)$



Applying KVL to the loops:

$$V_1(t) = R_1 i_1(t) + L_1 \frac{di_1(t)}{dt} + V_c(t)$$

$$\frac{di_1(t)}{dt} = \frac{1}{L_1} V_1(t) - \frac{R_1}{L_1} i_1(t) - \frac{1}{L_1} V_c(t)$$

$$\dot{x}_1(t) = \frac{1}{L_1} u(t) - \frac{R_1}{L_1} x_1(t) - \frac{1}{L_1} x_3(t) \quad \text{—(1)}.$$

$$V_c(t) = L_2 \frac{di_2(t)}{dt} + R_2 i_2(t)$$

$$L_2 \frac{di_2(t)}{dt} = V_c(t) - R_2 i_2(t)$$

$$\frac{di_2(t)}{dt} = \frac{1}{L_2} V_c(t) - \frac{R_2}{L_2} i_2(t)$$

$$\dot{x}_2(t) = \frac{1}{L_2} x_3(t) - \frac{R_2}{L_2} x_2(t) \quad \text{—(2)}$$

Yet $V_o(t) = R_2 i_2(t)$

$Y(t) = R_2 x_2(t)$

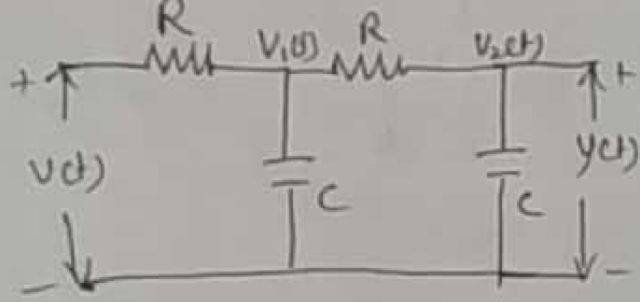
$C\frac{dv_c(t)}{dt} = i_1(t) - i_2(t)$

$x_3'(t) = \frac{1}{C}x_1(t) - \frac{1}{C}x_2(t)$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix} U(t)$$

$$Y(t) = \begin{bmatrix} 0 & R_2 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

(3) obtain an appropriate state model for a sim representing by an electric circuit as shown in



sol^n) $V_1(t)$ & $V_2(t)$ are state variables

$V_1(t) = X_1(t)$ $V_2(t) = X_2(t)$

~~v(t)~~ apply KCL at node

$$\frac{U(t) - V_1(t)}{R} + C\frac{dV_1(t)}{dt} + \frac{V_1(t) - V_2(t)}{R} = 0$$

$$C\frac{dV_1(t)}{dt} = -\left(\frac{-U(t) + V_1(t)}{R}\right) - \left(\frac{V_1(t) - V_2(t)}{R}\right).$$

$$\frac{dv_1(t)}{dt} = -\frac{1}{RC}U(t) - \frac{1}{RC}V_1(t) - \frac{1}{RC}V_1(t) + \frac{V_2(t)}{RC}$$

$$\frac{dv_1(t)}{dt} = -\frac{1}{RC}U(t) + \frac{1}{RC}V_2(t) - \frac{2}{RC}V_1(t)$$

choice of state variable is generally output variable $y(t)$ itself And other state variables are derivatives of the selected state variable $y(t)$

$$x_1(t) = y(t)$$
$$x_2(t) = \dot{y}(t) = \dot{x}_1(t)$$
$$x_3(t) = \ddot{y}(t) = \ddot{x}_1(t) = \dot{x}_2(t)$$
$$\vdots$$

Thus the various state Eqns are

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = x_3(t)$$
$$\vdots$$
$$\dot{x}_{n-1}(t) = x_n(t)$$
$$\dot{x}_n(t) = ?$$

Thus we have $y(t) = x_1$, $\dot{y}(t) = x_2$ $\ddot{y}(t) = x_3, \ldots$ $y^{n-1}(t) = x_n(t)$, $y^n(t) = \dot{x}_n(t)$

$$\dot{x}_n(t) + a_{n-1}x_n(t) + a_{n-2}x_{n-1}(t) \ldots\ldots + a_1x_2(t) + a_0x_1(t) = b_0u(t)$$

$$\dot{x}_n(t) = -a_0x_1(t) - a_1x_2(t) \ldots\ldots - a_{n-2}x_{n-1}(t) - a_{n-1}x_n(t) + b_0u(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -a_0 & -a_1 & \cdots & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ b_0 \end{bmatrix} u(t)$$

$$\dot{x} = Ax(t) + Bu(t)$$

$$y(t) = x_1(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

$y(t) = Cx(t)$

$D = 0$

$$\dot{x}_1(t) = -\frac{1}{RC}v(t) + \frac{1}{RC}x_2(t)$$

KCL at node V_2

$$C\frac{dv_2(t)}{dt} + \frac{V_2(t) - V_1(t)}{R} = 0$$

$$C\frac{dv_2(t)}{dt} = -\frac{V_2(t)}{R} + \frac{V_1(t)}{R}.$$

$$\frac{dv_2(t)}{dt} = -\frac{1}{RC}V_2(t) + \frac{1}{RC}V_1(t)$$

$$\dot{x}_2(t) = -\frac{1}{RC}x_2(t) + \frac{1}{RC}x_1(t)$$

o/p ~~$y(t) = C\frac{dv_2}{dt}$~~ $Y(t) = V_2(t)$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} -2/RC & 1/RC \\ \frac{1}{RC} & -1/RC \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} -\frac{1}{RC} \\ 0 \end{bmatrix} v(t)$$

$$Y(t) = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$A = \begin{bmatrix} -2/RC & 1/RC \\ 1/RC & -1/RC \end{bmatrix}, \quad B = \begin{bmatrix} -1/RC \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad D = 0.$$

State Model from Differential Equation

Consider a linear continuous time slm represented by n^{th} order differential eqn as

$$y^n + a_{n-1}y^{n-1} + a_{n-2}y^{n-2} + \dots a_1\dot{y} + a_0 y(t) = b_0 u + b_1\dot{u} + \dots b_{m-1}u^{m-1} + b_m u^m$$

where $y^n(t) = \frac{dy^n(t)}{dt^n} = n^{th}$ derivative of $y(t)$

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54. Construct the state model using Phase variables if the slm is described by the differential eqn.

$$\frac{d^3y(t)}{dt^3} + 4\frac{d^2y(t)}{dt^2} + 7\frac{dy(t)}{dt} + 2y(t) = 5u(t)$$

Soln)

$x_1(t) = y(t)$

$x_2(t) = \dot{y}(t) = \dot{x}_1(t) = \frac{dy(t)}{dt}$

$x_3(t) = \ddot{y}(t) = \dot{x}_2(t) = \frac{d^2y(t)}{dt}$

$\therefore \quad \dot{x}_1(t) = x_2(t)$

$\dot{x}_2(t) = x_3(t)$

$\dot{x}_3(t) + 4x_3(t) + 7x_2(t) + 2x_1(t) = 5u(t)$

$\dot{x}_3(t) = -4x_3(t) - 7x_2(t) - 2x_1(t) + 5u(t)$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \dot{x}_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -7 & -4 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} u(t)$$

$y(t) = x_1(t)$

$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t) \quad D = 0$

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -7 & -4 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad D = 0$$

Transfer function from State model

Consider a standard state model derived for linear time invariant Sm as.

$$\dot{X}(t) = AX(t) + BU(t)$$
$$Y(t) = CX(t) + DU(t)$$

Taking laplace Transform on both side

$$SX(s) = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

$$\Rightarrow SX(s) - AX(s) = BU(s)$$
$$\Rightarrow SIX(s) - AX(s) = BU(s)$$
$$[SI - A]X(s) = BU(s)$$
$$X(s) = [SI-A]^{-1} BU(s)$$
$$Y(s) = CX(s) + DU(s)$$
$$= C[SI-A]^{-1} BU(s) + DU(s)$$
$$Y(s) = [C[SI-A]^{-1}B + D]U(s)$$

$$TF = \frac{Y(s)}{U(s)} = C[SI-A]^{-1}B + D$$

$$[SI-A]^{-1} = \frac{Adj[SI-A]}{|SI-A|}$$

$$T(s) = \frac{C\,Adj[SI-A]B}{|SI-A|} + D.$$

Characteristic eqn of the sim is

$|sI - A| = 0$

Ex 5) Consider a sim having state model

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 3 \\ 5 \end{bmatrix} u \;\&\; Y = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

with D=0. Obtain T.F

Soln)

$$TF = C[sI - A]^{-1} + B$$

$$[sI - A] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} s - \begin{bmatrix} -2 & -3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} s+2 & 3 \\ -4 & s-2 \end{bmatrix}$$

$$\text{adj}[sI - A] = \begin{bmatrix} s-2 & -3 \\ 4 & s+2 \end{bmatrix}$$

$$|sI - A| = (s+2)(s-2) + 12 = s^2 - 4 + 12 = s^2 + 8$$

$$[sI - A]^{-1} = \frac{\begin{bmatrix} s-2 & -3 \\ 4 & s+2 \end{bmatrix}}{s^2 + 8}$$

$$TF = \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{\begin{bmatrix} s-2 & -3 \\ 4 & s+2 \end{bmatrix}}{s^2+8} * \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \frac{\begin{bmatrix} 3s-21 \\ 5s+22 \end{bmatrix}}{s^2+8} = \frac{8s+1}{s^2+8}$$

6) Find the T.F of the S/m having state model

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \quad \& \quad Y = \begin{bmatrix} 1 & 0 \end{bmatrix} x$$

Soln) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ $D = 0$

$$TF = C[SI - A]^{-1} B + D$$

$$[SI - A] = S\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} S & -1 \\ 2 & S+3 \end{bmatrix}$$

$$\text{adj}[SI - A] = \begin{bmatrix} S+3 & +1 \\ -2 & S \end{bmatrix}$$

$$|SI - A| = S(S+3) + 2 = S^2 + 3S + 2 = (S+1)(S+2)$$

$$TF = \frac{C \, \text{adj}[SI - A] B}{|SI - A|}$$

$$= \frac{\begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} S+3 & 1 \\ -2 & S \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}}{(S+1)(S+2)}$$

$$= \frac{S+3}{(S+1)(S+2)}$$

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Solution of State Equation

Consider the state eqn of linear time invariant sm as

$$\dot{x}(t) = A x(t) + B u(t)$$

The matrices A & B are constant matrices. This state eqn can be of two types

(1) Homogeneous (2) Non homogenous.

Homogeneous Equation

→ If A is a constant matrix & input control forces are zero then the eqn takes

$$\dot{x}(t) = A x(t)$$

Non Homogeneous Equation

→ If A is a constant matrix & matrix u(t) is non zero vector

$$\dot{x}(t) = A x(t) + B u(t)$$

Classical Method of Solution

Consider a scalar differential eqn as

$$\frac{dx}{dt} = ax \quad \text{where } x(0) = x_0$$

homogeneous eqn without the i/p vector

assume the solution of this eqn as

$$x(t) = b_0 + b_1 t + b_2 t^2 + \cdots b_k t^k$$

$t=0 \quad x(0) = x_0 = b_0.$

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The solution has to satisfy the original differential Eqn

$$\frac{d}{dt}\left[b_0 + b_1 t + \dots + b_k t^k\right] = a\left[b_0 + b_1 t + \dots + b_k t^k\right]$$

$$b_1 + 2b_2 t + \dots k b_k t^{k-1} = ab_0 + ab_1 t + \dots ab_k t^k$$

$b_1 = ab_0 \quad 2b_2 = ab_1 \quad kb_k = ab_{k-1}$

$$b_2 = \frac{ab_1}{2} = \frac{1}{2} a \times ab_0 = \frac{a^2 b_0}{2} = \frac{1}{2!} a^2 b_0$$

$$b_3 = \frac{ab_2}{3} = \frac{1}{3 \times 2} a^3 b_0 = \frac{1}{3!} a^3 b_0$$

$\vdots$

$$b_k = \frac{1}{k!} a^k b_0 \quad \& \quad x(0) = b_0$$

$$\therefore x(t) = b_0 + ab_0 t + \frac{1}{2!} a^2 b_0 t^2 + \frac{1}{3!} a^3 b_0 t^3 \dots + \frac{1}{k!} a^k b_0 t^k$$

$$= \left[1 + at + \frac{1}{2!} a^2 t^2 + \frac{1}{3!} a^3 t^3 + \dots + \frac{1}{k!} a^k t^k\right] x(0)$$

$$1 + at + \frac{1}{2!} a^2 t^2 + \frac{1}{3!} a^3 t^3 + \dots + \frac{1}{k!} a^k t^k = e^{at}$$

$$\dot{x}(t) = e^{at} x(0)$$

$$\dot{x}(t) = A x(t)$$

$$\dot{x}(t) = e^{At} x(0)$$

$$\therefore \phi(t) = e^{At} = \text{state transition matrix}$$

$$e^{At} = 1 + At + \frac{1}{2!} A^2 t^2 + \dots \frac{1}{k!} A^k t^k + \dots$$

If instead of initial time $t=0$, it is selected as $t=t_0$ then the state transition matrix is

$$\phi(t) = e^{A(t-t_0)}$$

Zero Input Response

→ The solution of the homogeneous state eqn is under the condition of zero i/p. Such response is called zero Input response (ZIR)

Solution of Non homogeneous eqn

Consider a non homogeneous state eqn as

$$\dot{x}(t) = Ax(t) + Bu(t)$$

$$\dot{x}(t) - Ax(t) = Bu(t)$$

Premultiplying both sides by e^{-At}

$$e^{-At}[\dot{x}(t) - Ax(t)] = e^{-At}Bu(t)$$

$$e^{-At}\dot{x}(t) - e^{-At}Ax(t) = \frac{d}{dt}[e^{-At}x(t)]$$

where $\frac{d}{dt}[e^{-At}x(t)] = e^{-At}Bu(t)$

Assuming initial time as $t=0$ & integrating both sides from $t=0$ to t

$$e^{-At}x(t)\Big|_0^t = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

$$e^{-At}x(t) - x(0) = \int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

Premultiplying both sides by e^{At}

$$\therefore \; e^{At}e^{-At}x(t) - e^{At}x(0) = e^{At}\int_0^t e^{-A\tau}Bu(\tau)\,d\tau$$

$$x(t) = \underbrace{e^{At}x(0)}_{ZIR} + \underbrace{\int_0^t e^{A(t-\tau)}Bu(\tau)\,d\tau}_{ZSR}$$

Solution of non homogeneous Eqn

Properties of State Transition Matrix

1) $\phi(0) = e^{A \times 0} = I$

2) $\phi(t) = e^{At} = (e^{-At})^{-1} = [\phi(-t)]^{-1}$

$\phi^{-1}(t) = \phi(-t)$

3) $\phi(t_1 + t_2) = e^{A(t_1+t_2)} = e^{At_1} e^{At_2} = \phi(t_1)\phi(t_2) = \phi(t_2)\phi(t_1)$

4) $e^{A(t+s)} = e^{At} e^{As}$

5) $e^{(A+B)t} = e^{At} e^{Bt}$ only if $AB = BA$

6) $[\phi(t)]^n = [e^{At}]^n = e^{Ant} = \phi(nt)$

7) $\phi(t_2 - t_1) \cdot \phi(t_1 - t_0) = \phi(t_2 - t_0)$

Solution of State Eqn by Laplace Transform method

Consider the nonhomogeneous State Eqn as

$$\dot{x}(t) = A x(t) + B u(t) \quad \text{--- (1)}$$

Taking laplace Transform on both sides

$$\cancel{S x(s) = A x(s) + B u(s)}$$

$$\cancel{S x(s) - A x(s) = B U(s)}$$

$$S x(s) - x(0) = A x(s) + B U(s)$$

$$S x(S) - A x(S) = x(0) + B U(S)$$

$$[SI - A] x(S) = x(0) + B U(S)$$

$$x(S) = [SI - A]^{-1} x(0) + [SI - A]^{-1} B U(S) \quad \text{--- (2)}$$

$$= ZIR + ZSR$$

Comparing zero input response obtained earlier

$$[SI - A]^{-1} = \phi(S) \quad \text{--- (3)}$$

$$L^{-1}[sI-A]^{-1} = \phi(t) = e^{At} \quad —(3)$$

$\phi(s) = [sI-A]^{-1}$ is called resolvant matrix of A

Taking inverse laplace Transform of Eqn (2)

$$x(t) = L^{-1}[x(s)] = L^{-1}\{[sI-A]^{-1}x(0) + [sI-A]^{-1}BU(s)\}$$

$$[sI-A]^{-1} = \phi(s)$$

$$x(t) = \underbrace{L^{-1}[\phi(s)x(0)]}_{ZIR} + \underbrace{L^{-1}[\phi(s)BU(s)]}_{ZSR}$$

$$\boxed{\phi(s) = [sI-A]^{-1} = \frac{Adj[sI-A]}{|sI-A|}}$$

$$L^{-1}[\phi(s)] = \phi(t) = e^{At}$$

State transition matrix e^{At} & hence

$$\therefore e^{At} = L^{-1}[\phi(s)] = L^{-1}[sI-A]^{-1} = L^{-1}\left\{\frac{Adj[sI-A]}{|sI-A|}\right\}$$

Ex 7) Find the state transition matrix for

$$A = \begin{bmatrix} 0 & -1 \\ 2 & -3 \end{bmatrix}$$

Soln

$$[sI-A] = \begin{bmatrix} s & +1 \\ -2 & s+3 \end{bmatrix}$$

$$adj[sI-A] = \begin{bmatrix} s+3 & -1 \\ +2 & s \end{bmatrix}$$

$$|sI-A| = (s+3)s + 2 = s^2+3s+2 = (s+1)(s+2)$$

$$[sI-A]^{-1} = \frac{adj[sI-A]}{|sI-A|} = \frac{\begin{bmatrix} s+3 & -1 \\ 2 & s \end{bmatrix}}{(s+1)(s+2)}$$

$$\phi(s) = [sI-A]^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{-1}{(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$e^{AT} = L^{-1}[sI-A]^{-1} = L^{-1}\begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{-1}{(s+1)(s+2)} \\ \frac{2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

using partial fration expression for all the s/m elements

$$e^{At} = \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{-1}{s+1} + \frac{1}{s+2} \\ \frac{2}{s+1} - \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & -e^{-t} + e^{-2t} \\ 2e^{-t} - 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} = \phi(t)$$

8) for a certain s/m, when

$$x(0) = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \text{ then } x(t) = \begin{bmatrix} e^{-3t} \\ -3e^{-3t} \end{bmatrix}$$

while $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ then $x(t) = \begin{bmatrix} e^{t} \\ e^{t} \end{bmatrix}$

Determine the s/m Matrix. Also find the state transition matrix

Sol^{n}) The solution of the Eq^{n} is

$$x(t) = e^{At} x(0)$$

$$A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}$$

Eq^{n} is $\dot{x}(t) = A\,x(t)$

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$x(t) = \begin{bmatrix} e^{-3t} \\ -3e^{-3t} \end{bmatrix}$ $\dot{x}(t) = \begin{bmatrix} -3e^{-3t} \\ 9e^{-3t} \end{bmatrix}$

$x(0) = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ $\dot{x}(0) = \begin{bmatrix} -3 \\ 9 \end{bmatrix}$

$\dot{x}(0) = A\,x(0)$

$\begin{bmatrix} -3 \\ 9 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$

$A_1 - 3A_2 = -3$

$A_3 - 3A_4 = 9$

$|||^{ly}$

$x(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$ $\dot{x}(t) = \begin{bmatrix} e^t \\ e^t \end{bmatrix}$

$x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\dot{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\dot{x}(0) = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} x(0)$

$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$A_1 + A_2 = 1$

$A_3 + A_4 = 1$

$\cancel{A_1} - 3A_2 = -3$

$\cancel{A_1} + A_2 = 1$

$(-) \quad (-) \qquad (-)$

$-4A_2 = -4$

$A_2 = 1.$

$A_1 = 0$

$A_{\cancel{3}} - 3A_4 = 9$

$\cancel{A_3} + A_4 = 1$

$(-) \quad (-) \qquad (-)$

$-4A_4 = +8$

$A_4 = -2$

$A_3 + 6 = 9$

$A_3 = 9 - 6 = 3$

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$$A = \begin{bmatrix} 0 & 1 \\ 3 & -2 \end{bmatrix}$$

$$[SI - A] = \begin{bmatrix} s & -1 \\ -3 & s+2 \end{bmatrix}$$

$$\text{adj}[SI - A] = \begin{bmatrix} s+2 & 1 \\ 3 & s \end{bmatrix}$$

$$|SI - A| = (s+2)s - 3 = s^2 + 2s - 3 = (s+3)(s-1)$$

$$[SI-A]^{-1} = \frac{\text{adj}[SI-A]}{|SI-A|} = \begin{bmatrix} \frac{s+2}{(s+3)(s-1)} & \frac{1}{(s+3)(s-1)} \\ \frac{3}{(s+3)(s-1)} & \frac{s}{(s+3)(s-1)} \end{bmatrix}$$

$$e^{At} = L^{-1}[SI - A]^{-1}.$$

$$= L^{-1} \begin{bmatrix} \frac{0.25}{s+3} + \frac{0.75}{s-1} & \frac{-0.25}{s+3} + \frac{0.25}{s-1} \\ \frac{-0.75}{s+3} + \frac{0.75}{s-1} & \frac{0.75}{s+3} + \frac{0.25}{s-1} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 0.25e^{-3t} + 0.75e^{t} & -0.25e^{-3t} + 0.25e^{t} \\ -0.75e^{-3t} + 0.75e^{t} & 0.75e^{-3t} + 0.25e^{t} \end{bmatrix}$$

13) obtain the solution of the homogeneous state Eqn $\dot{x} = Ax$ where

$$A = \begin{bmatrix} 1 & -2 \\ 1 & -4 \end{bmatrix} \quad \& \quad x(0) = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

Soln)
$$e^{At} = L^{-1}[SI - A]^{-1}$$

$$[SI - A] = \begin{bmatrix} s-1 & 2 \\ -1 & s+4 \end{bmatrix}$$

$$[sI-A]^{-1} = \frac{adj[sI-A]}{|sI-A|}$$

$$adj[sI-A] = \begin{bmatrix} s+4 & -2 \\ +1 & s-1 \end{bmatrix}$$

$$|sI-A| = (s+4)(s-1)+2 = s^2+3s-2 =$$
$$= (s-0.561)(s+3.561)$$

$$[sI-A]^{-1} = \begin{bmatrix} \frac{s+4}{(s-0.561)(s+3.561)} & \frac{-2}{(s-0.561)(s+3.561)} \\ \frac{1}{(s-0.561)(s+3.561)} & \frac{s-1}{(s-0.561)(s+3.561)} \end{bmatrix}$$

$$e^{At} = L^{-1}[sI-A]^{-1} = \begin{bmatrix} 1.106e^{0.561t} - 0.106e^{-3.561t} & -0.485e^{0.561t} + 0.485e^{-3.561t} \\ 0.242e^{+0.561} + 0.242e^{-3.561t} & -0.106e^{0.561t} + 1.106e^{-3.561t} \end{bmatrix}$$

$$x(t) = e^{At}x(0)$$

$$= e^{At}\begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.068e^{0.561t} + 0.432e^{-3.561t} \\ 0.015e^{0.561t} + 0.985e^{-3.561t} \end{bmatrix}$$

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14) A linear time invariant slm is characterised by the homogeneous state eqn

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Compute the solution of homogeneous eqn, assume the initial state vector

$$x_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Sol^n) $A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

$$[SI - A] = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$$

$$\text{adj}[SI - A] = \begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}$$

$$|SI - A| = (s-1)(s-1) - 0 = (s-1)^2$$

$$[SI-A]^{-1} = \frac{\text{adj}[SI-A]}{|SI-A|} = \frac{\begin{bmatrix} s-1 & 0 \\ 1 & s-1 \end{bmatrix}}{(s-1)^2}$$

$$= \begin{bmatrix} \frac{1}{s-1} & 0 \\ \frac{1}{(s-1)^2} & \frac{1}{s-1} \end{bmatrix}$$

$$e^{At} = L^{-1}[SI-A]^{-1} = L^{-1}\begin{bmatrix} 1/s-1 & 0 \\ \frac{1}{(s-1)^2} & 1/s-1 \end{bmatrix}$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix}$$

$$x(t) = e^{At} x(0)$$

$$= \begin{bmatrix} e^t & 0 \\ te^t & e^t \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} e^t \\ te^t \end{bmatrix}$$

15) obtain the State-transition matrix $\phi(t)$ of the following s/m $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ Also, obtain the inverse of the state transition matrix $\phi^{-1}(t)$.

Sol^n)

$$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$$

$$\phi(t) = e^{At} = L^{-1}[SI - A]^{-1}$$

$$SI - A = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix}$$

$$\text{adj}[SI - A] = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$|SI - A| = s(s+3) + 2 = (s+1)(s+2)$$

$$[SI - A]^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$e^{At} = L^{-1}[SI - A]^{-1} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$\phi^{-1}(t) = \phi(-t)$$

$$= \begin{bmatrix} 2e^{t} - e^{2t} & e^{t} - e^{2t} \\ -2e^{t} + 2e^{2t} & -e^{t} + 2e^{+2t} \end{bmatrix}$$

16) obtain the time response of the following s/m

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

where $u(t)$ is the unit step occuring at $t=0$ &

$x^T(0) = [1 \quad 0]$

Soln)

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

ZIR

zero i/p response = $x(t) = \begin{bmatrix} e^t \\ te^t \end{bmatrix}$

ZSR

zero state response $= x(t) = L^{-1}\{\phi(s) B U(s)\}$

$$\phi(s) = L\{\phi(t)\} = L\{e^{At}\} = [SI-A]^{-1}$$

$$[SI-A]^{-1} = \begin{bmatrix} \frac{1}{S-1} & 0 \\ \frac{1}{(S-1)^2} & \frac{1}{S-1} \end{bmatrix} \qquad U(s) = \frac{1}{S}$$

$$\phi(S) B U(S) = \begin{bmatrix} \frac{1}{S-1} & 0 \\ \frac{1}{(S-1)^2} & \frac{1}{S-1} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \left[\frac{1}{S}\right]$$

$$= \begin{bmatrix} \frac{1}{S-1} \\ \frac{S}{(S-1)^2} \end{bmatrix} \cdot \frac{1}{S} = \begin{bmatrix} \frac{1}{S(S-1)} \\ \frac{S}{S(S-1)^2} \end{bmatrix}$$

using partial fraction

$$ZSR = L^{-1}[\phi(s)\, B\, U(s)] = L^{-1}\begin{bmatrix} \frac{1}{s-1} - \frac{1}{s} \\ \frac{1}{(s-1)^2} \end{bmatrix}$$

$$= \begin{bmatrix} e^{t} - 1 \\ te^{t} \end{bmatrix}$$

Total response $= x(t) = ZIR + ZSR$

$$= \begin{bmatrix} e^{t} \\ te^{t} \end{bmatrix} + \begin{bmatrix} e^{t} - 1 \\ te^{t} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{t} - 1 \\ 2te^{t} \end{bmatrix}$$

17) Find the response of the system,

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x + \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} u(t), \quad x(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

& $y(t) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x$ to the following i/p

$$u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix} = \begin{bmatrix} u(t) \\ e^{-3t} u(t) \end{bmatrix}$$ where $u(t)$ = unit step function

Soln) F e^{At} for $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$

$$[SI - A] = \begin{bmatrix} s & -1 \\ 2 & s+3 \end{bmatrix} \quad adj[SI-A] = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$|SI - A| = s(s+3) + 2 = s^2 + 3s + 2 = (s+2)(s+1)$$

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$$[sI-A]^{-1} = \frac{\operatorname{adj}[sI-A]}{|sI-A|} = \frac{\begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}}{(s+1)(s+2)}$$

$$= \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$ZIR = e^{At} x(0) = e^{At} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$ZSR = L^{-1}[\phi(s) B\, U(s)]$$

$$U(s) = \begin{bmatrix} 1/s \\ \frac{1}{s+3} \end{bmatrix}$$

$$= L^{-1}\left\{ \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/s \\ \frac{1}{s+3} \end{bmatrix} \right\}$$

$$= L^{-1}\left\{ \begin{bmatrix} 2\frac{s+3}{(s+1)(s+2)} & \frac{s+3}{(s+1)(s+2)} + \frac{1}{(s+1)(s+2)} \\ \frac{-4}{(s+1)(s+2)} & \frac{-2}{(s+1)(s+2)} + \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 1/s \\ 1/s+3 \end{bmatrix} \right\}$$

$$ZSR = L^{-1}\left\{ \begin{bmatrix} \dfrac{2(s+3)}{(s+1)(s+2)} & \dfrac{s+4}{(s+1)(s+2)} \\ \dfrac{-4}{(s+1)(s+2)} & \dfrac{s-2}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 1/s \\ \dfrac{1}{s+3} \end{bmatrix} \right\}$$

$$= \begin{bmatrix} \dfrac{2(s+3)}{s(s+1)(s+2)} + \dfrac{(s+4)}{(s+1)(s+2)(s+3)} \\ -\dfrac{4}{s(s+1)(s+2)} + \dfrac{(s-2)}{(s+1)(s+2)(s+3)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{2(s+3)(s+3) + s(s+4)}{s(s+1)(s+2)(s+3)} \\ \dfrac{-4(s+3) + s(s-2)}{s(s+1)(s+2)(s+3)} \end{bmatrix}$$

s^2+6s+9

$$= \begin{bmatrix} \dfrac{2s^2+12s+18+s^2+4s}{s(s+1)(s+2)(s+3)} \\ \dfrac{-4s-12+s^2-2s}{s(s+1)(s+2)(s+3)} \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{3s^2+16s+18}{s(s+1)(s+2)(s+3)} \\ \dfrac{s^2-6s-12}{s(s+1)(s+2)(s+3)} \end{bmatrix}$$

$$3s^2+16s+18 = A(s+1)(s+2)(s+3) + B\,s(s+2)(s+3) + C\,s(s+1)(s+3) + D\,s(s+1)(s+2)$$

$s = 0$

$18 = 6A$

$$A = \frac{\cancel{18}\ 9\ 3}{\cancel{6}\ \cancel{2}\ 1} = 3$$

$s = -1$

$$3 - 16 + 18 = -B(-1+2)(-1+3)$$

$$5 = -2B \quad B = -\frac{5}{2} = -2.5$$

$s = -2$

$$12 - 32 + 18 = -2C(-2+1)(-2+3)$$

$$-20 + 18 = -2C(-1)(1)$$

$$-2 = 2C$$

$$C = -1$$

$s = -3$.

$$27 - 48 + 18 = -3D(-3+1)(-3+2)$$

$$= -3D(-2)(-1)$$

$$-3 = -6D$$

$$D = \frac{-3}{-6} = \frac{1}{2} = 0.5$$

$$s^2 - 6s - 12 = A(s+1)(s+2)(s+3) + Bs(s+2)(s+3) + Cs(s+1)(s+3) + Ds(s+1)(s+2)$$

$s = 0$

$$-12 = 6A \quad A = -2$$

$s = -1$.

$$1 + 6 - 12 = -B(1)(2)$$

$$-5 = -2B \quad B = \frac{5}{2} = 2.5$$

$s = -2$

$$4 + 12 - 12 = -2C(-1)(1)$$

$$4 = 2C \quad C = 2$$

$s = -3$.

$$9 + 18 - 12 = -3D(-2)(-1)$$

$$15 = -6D$$

$$D = -\frac{\cancel{15}^{5}}{\cancel{6}_{2}} = -\frac{5}{2} = -2.5$$

$$ZSR = L^{-1}\begin{bmatrix} \frac{3}{s} - \frac{2.5}{s+1} - \frac{1}{s+2} + \frac{0.5}{s+3} \\ \frac{-2}{s} + \frac{2.5}{s+1} + \frac{1}{s+2} - \frac{2.5}{s+3} \end{bmatrix}$$

$$= L^{-1}\begin{bmatrix} 3 - 2.5e^{-t} - e^{-2t} + 0.5e^{-3t} \\ -2 + 2.5e^{-t} + 2e^{-2t} - 2.5e^{-3t} \end{bmatrix}$$

o/p response = ZIR + ZSR

= 0 + ZSR

$$\begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$\boxed{Y_1(t) = 3 - 2.5e^{-t} - e^{-2t} + 0.5e^{-3t}}$$

$$Y_2(t) = 3 - \cancel{2.5e^{-t}} - \cancel{e^{-2t}} + 0.5e^{-3t} - 2 + \cancel{2.5e^{-t}} + \cancel{2e^{-2t}} - 2.5e^{-3t}$$

$$\cancel{Y(t) = 1 - 2e^{-3t}}$$

$$Y_2(t) = 3 - 2.5e^{-t} - e^{-2t} + 0.5e^{-3t} - 2 + 2.5e^{-t} + 2e^{-2t} - 2.5e^{-3t}$$

$$\boxed{Y_2(t) = 1 + e^{-2t} - 2e^{-3t}}$$

18) Given $\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$

Find the unit step response when, $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Sol) $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $x(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$[SI - A] = \begin{bmatrix} s & -1 \\ +2 & s+3 \end{bmatrix}$$

$$\text{adj}[SI - A] = \begin{bmatrix} s+3 & 1 \\ -2 & s \end{bmatrix}$$

$$|SI - A| = s(s+3)+2 = (s+2)(s+1)$$

$$[SI - A]^{-1} = \begin{bmatrix} \frac{s+3}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix}$$

using partial fraction

$$= \begin{bmatrix} \frac{2}{s+1} - \frac{1}{s+2} & \frac{1}{s+1} - \frac{1}{s+2} \\ \frac{-2}{s+1} + \frac{2}{s+2} & \frac{-1}{s+1} + \frac{2}{s+2} \end{bmatrix}$$

$$\phi(t) = e^{At} = \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$ZIR = A\, e^{At} x(0)$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} & e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} & -e^{-t} + 2e^{-2t} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2e^{-t} - e^{-2t} + e^{-t} - e^{-2t} \\ -2e^{-t} + 2e^{-2t} - e^{-t} + 2e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix}$$

$$ZSR = L^{-1}\{\phi(s) B U(s)\}$$

$$= L^{-1}\left\{ \begin{bmatrix} \frac{(s+3)}{(s+1)(s+2)} & \frac{1}{(s+1)(s+2)} \\ \frac{-2}{(s+1)(s+2)} & \frac{s}{(s+1)(s+2)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \left[\tfrac{1}{s}\right] \right\}$$

$$= L^{-1}\left\{ \begin{bmatrix} \frac{1}{(s+1)(s+2)} \\ \frac{s}{(s+1)(s+2)} \end{bmatrix} \left[\tfrac{1}{s}\right] \right\}$$

$$= L^{-1} \begin{bmatrix} \frac{1}{s(s+1)(s+2)} \\ \frac{1}{(s+1)(s+2)} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{0.5}{s} - \frac{1}{s+1} + \frac{0.5}{s+2} \\ \frac{1}{s+1} - \frac{1}{s+2} \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$Y(t) = ZIR + ZSR$

$$= \begin{bmatrix} 3e^{-t} - 2e^{-2t} \\ -3e^{-t} + 4e^{-2t} \end{bmatrix} + \begin{bmatrix} 0.5 - e^{-t} + 0.5e^{-2t} \\ e^{-t} - e^{-2t} \end{bmatrix}$$

$$= \begin{bmatrix} 0.5 + 2e^{-t} - 1.5e^{-2t} \\ -2e^{-t} + 3e^{-2t} \end{bmatrix}$$

19) The state model of the s/m is given by.

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \quad \begin{bmatrix} x_1(0) \\ x_1(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

where $u(t) = 0$ for $t < 0$ Find the o/p response
$= e^{-t}$ for $t \geq 0$.

Sol) $A = \begin{bmatrix} 0 & 1 \\ -3 & -4 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $x(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. $u(t) = e^{-t}$ for $t \geq 0$

$$[SI - A] = \begin{bmatrix} S & -1 \\ 3 & S+4 \end{bmatrix}$$

$$\text{adj}\,[SI - A] = \begin{bmatrix} S+4 & 1 \\ -3 & S \end{bmatrix}$$

$$|SI - A| = S(S+4) + 3 = (S+3)(S+1)$$

$$[SI - A]^{-1} = \frac{\begin{bmatrix} S+4 & 1 \\ -3 & S \end{bmatrix}}{(S+3)(S+1)}$$

$$= \begin{bmatrix} \frac{S+4}{(S+1)(S+3)} & \frac{1}{(S+1)(S+3)} \\ \frac{-3}{(S+1)(S+3)} & \frac{S}{(S+1)(S+3)} \end{bmatrix}$$

$$e^{At} = \begin{bmatrix} 1.5e^{-t} - 0.5e^{-3t} & 0.5e^{-t} - 0.5e^{-3t} \\ -1.5e^{-t} + 1.5e^{-3t} & -0.5e^{-t} + 1.5e^{-3t} \end{bmatrix}$$

$$ZIR = e^{At} x(0) = e^{At}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.5e^{-t} - 0.5e^{-3t} \\ -0.5e^{-t} + 1.5e^{-3t} \end{bmatrix}$$

$$ZSR = L^{-1}\{\Phi(s) B U(s)\}$$

$$= L^{-1}\left\{\begin{bmatrix} \frac{s+4}{(s+1)(s+3)} & \frac{1}{(s+1)(s+3)} \\ \frac{-3}{(s+1)(s+3)} & \frac{s}{(s+1)(s+3)} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\left[\frac{1}{s+1}\right]\right\}$$

$$= L^{-1}\left\{\begin{bmatrix} \frac{1}{(s+1)(s+3)} \\ \frac{s}{(s+1)(s+3)} \end{bmatrix}\left[\frac{1}{s+1}\right]\right\}$$

$$= L^{-1}\begin{bmatrix} \frac{1}{(s+1)^2(s+3)} \\ \frac{s}{(s+1)^2(s+3)} \end{bmatrix} = \begin{bmatrix} \frac{0.5}{(s+1)^2} - \frac{0.25}{(s+1)} + \frac{0.25}{(s+3)} \\ \frac{-0.5}{(s+1)^2} + \frac{0.75}{(s+1)} - \frac{0.75}{s+3} \end{bmatrix}$$

$$= \begin{bmatrix} 0.5te^{-t} - 0.25e^{-t} + 0.25e^{-3t} \\ -0.5te^{-t} + 0.75e^{-t} - 0.75e^{-3t} \end{bmatrix}$$

o/p response

$$y(t) = ZIR + ZSR$$

$$y(t) = \begin{bmatrix} 0.5e^{-t} - 0.5e^{-3t} \\ -0.5e^{-t} + 1.5e^{-3t} \end{bmatrix} + \begin{bmatrix} 0.5te^{-t} - 0.25e^{-t} + 0.25e^{-3t} \\ -0.5te^{-t} + 0.75e^{-t} - 0.75e^{-3t} \end{bmatrix}$$

$$= \begin{bmatrix} 0.5te^{-t} - 0.25e^{-t} - 0.25e^{-3t} \\ -0.5te^{-t} + 0.75e^{-t} + 0.75e^{-3t} \end{bmatrix}$$

20) write state space eqn for the following eqn

$$L_a \frac{di_a}{dt} + R_a i_a + k_b \theta = V_a(t) \quad \& \quad J\frac{d^2\theta}{dt^2} + F\frac{d\theta}{dt} = T_a(t)$$

where $T_a(t) = K_t i_a$.

Soln) $x_1 = \theta \quad x_2 = \dot{\theta} = \dot{x}_1 \quad \dot{\theta} = \frac{d^2\theta}{dt^2} = \dot{x}_2$

$x_3 = i_a$.

$$L_a \dot{x}_3 + R_a x_3 + K_b x_1 = u(t)$$

$$J\dot{x}_2 + f x_2 = K_t x_3$$

$$\dot{x}_3 = \frac{u(t)}{L_a} - \frac{R_a}{L_a} x_3 - \frac{K_b}{L_a} x_1$$

$$\dot{x}_2 = \frac{K_t}{J} x_3 - \frac{f}{J} x_2$$

$$\dot{x}_1 = x_2$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -f/J & kt/J \\ -\frac{K_b}{L_a} & 0 & -\frac{R_a}{L_a} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1/L_a \end{bmatrix} u(t)$$

Q) The following Eqn defines a seperately excited DC motor in the form of differential Eqn

$$\omega + \left(\frac{B}{J}\right)\frac{d\omega}{dt} + \left(\frac{k^2}{LJ}\right)\omega = \left(\frac{k}{LJ}\right)V$$

The above Eqn in state space form as follows

$$\begin{bmatrix}\omega \\ \dot{\omega}\end{bmatrix} = P\begin{bmatrix}\theta \\ \omega\end{bmatrix} + QV$$

Find P matrix, if V is vtg i/p & ω is angular velocity

Soln)

$$\omega = \frac{d\theta}{dt} \quad \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$$

$$\frac{d\theta}{dt} + \left(\frac{B}{J}\right)\frac{d^2\theta}{dt^2} + \left(\frac{k^2}{LJ}\right)\frac{d\theta}{dt} = \left(\frac{k}{LJ}\right)V$$

$$x_1 = \theta, \quad \dot{x}_1 = \dot{\theta} \quad \dot{x}_2 = \frac{d^2\theta}{dt^2}$$

$$\dot{x}_1 = x_2 = \frac{d\theta}{dt}$$

$$x_2 + \left(\frac{B}{J}\right)\dot{x}_2 + \left(\frac{k^2}{LJ}\right)x_2 = \left[\frac{k}{LJ}\right]V$$

$$\left(\frac{B}{J}\right)\dot{x}_2 = \left[\frac{k}{LJ}\right]V - \left[1 + \frac{k^2}{LJ}\right]x_2$$

$$\dot{x}_2 = \frac{k}{BL}V - \left[\frac{k^2+LJ}{BL}\right]x_2$$

$$\begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1 \\ 0 & -\left[\frac{k^2+LJ}{BL}\right]\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ \frac{k}{BL}\end{bmatrix}V$$

W.K.T $x_1 = \theta \quad \dot{x}_1 = \frac{d\theta}{dt} = \omega$

$$\begin{bmatrix} \omega \\ \dot{\omega} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{(k^2+LJ)}{BL} \end{bmatrix} \begin{bmatrix} 0 \\ \omega \end{bmatrix} = QV$$

$$P = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{(k^2+LJ)}{BL} \end{bmatrix}$$

22) consider a slm given by

$\dddot{y} + 9\ddot{y} + 26\dot{y} + 24y = 6U$ obtain state model

Soln)

$y = x_1$

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

$$\dot{x}_3 + 9x_3 + 26x_2 + 24x_1 = 6U$$

$$\dot{x}_3 = -24x_1 - 26x_2 - 9x_3 + 6U$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix} U(t)$$

$$Y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

23) obtain the state model for the given TF

$$\frac{Y(s)}{U(s)} = \frac{24}{s^3+9s^2+26s+24}$$

Soln

$$s^3Y(s) + 9s^2Y(s) + 26sY(s) + 24Y(s) = 24U(s)$$

$\cancel{Y(s) = x_1(t)}$

$\cancel{x_2(t) = \dot{x}_1(t)}$

$x_3(t) = \dot{x}_2(t)$

$$\dddot{y} + 9\ddot{y} + 26\dot{y} + 24y = 24u.$$

$y = x_1$

$\dot{x}_1 = x_2$

$\dot{x}_2 = x_3$

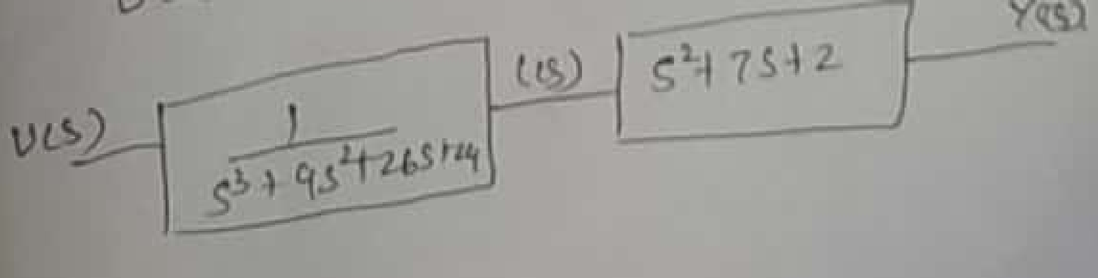
$$\dot{x}_3 + 9x_3 + 26x_2 + 24x_1 = 24u.$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -24 & -26 & -9\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 24\end{bmatrix}u.$$

$$y(t) = \begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

24) obtain the state model for the given slm

$$\frac{Y(s)}{U(s)} = \frac{s^2+7s+2}{s^3+9s^2+26s+24}$$



$$\frac{C(s)}{U(s)} = \frac{1}{s^3+9s^2+26s+24}$$

$$s^3C(s) + 9s^2C(s) + 26s\,C(s) + 24\,C(s) = U(s)$$

$$\dddot{c} + 9\ddot{c} + 26\dot{c} + 24c = u$$

$$c = x_1$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 + 9x_3 + 26x_2 + 24x_1 = u.$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -24 & -26 & -9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u.$$

$$\frac{Y(s)}{C(s)} = s^2+7s+2$$

$$Y(s) = s^2C(s) + 7s\,C(s) + 2C(s)$$

$$y(t) = \ddot{c}(t) + 7\dot{c}(t) + 2c(t)$$

$$c(t) = x_1$$

$$\dot{x}_1 = x_2$$

$$y(t) = 2x_1 + 7x_2 + \dot{x}_3$$

$$y(t) = \begin{bmatrix} 2 & 7 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$