Probability error of ASK

* Derive an expression for probability of error 'P_e' of a Coherent binary ASK.

* In ASK System $S_1(t)$ & $S_2(t)$ are represented as:

$$S_1(t) = \sqrt{\frac{2E_b}{T_b}} \cos 2\pi f_c t \quad \text{and}$$

$$S_2(t) = 0.$$

Where 'E_b' is the transmitted Signal energy per bit.

* The basis function is given by

$$\phi_1(t) = \sqrt{\frac{2}{T_b}} \cos 2\pi f_c t$$

* The transmitted ASK Signals are given by

$$S_1(t) = \sqrt{E_b}\, \phi_1(t) \quad \text{for Symbol 1}$$

$$S_2(t) = 0 \quad \text{for Symbol 0}$$

Hence, we have a one dimensional Signal Space with two message points at $+\sqrt{E_b}$ and 0.

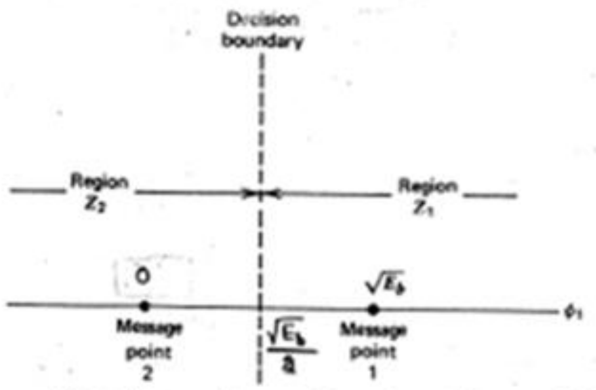
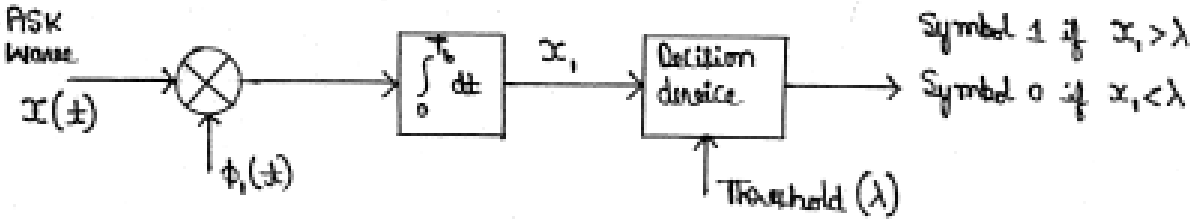


Figure Signal space diagram for coherent binary PSK system.



* Let $x(t)$ be the received Signal

$$\boxed{x(t) = s(t) + w(t)} \longrightarrow (1)$$

Where $w(t)$ is the additive white gaussian noise (AWGN) with Zero mean ($\mu = 0$) and variance (σ) of $\frac{N_0}{2}$.

$$\therefore \quad x(t) = \begin{cases} s_1(t) + w(t) & , \text{ for Symbol } 1 \\ 0 + w(t) & , \text{ for Symbol } 0 \end{cases}$$

* Let us assume that Symbol '0' is transmitted. Then o/p of the Correlator is

$$x_1 = \int_0^{T_b} x(t) \cdot \phi_1(t) \cdot dt$$

$$= \int_0^{T_b} w(t)\, \phi_1(t) \cdot dt$$

$$\boxed{x_1 = W_1}$$

$$\therefore \quad E[x_1] = E[W_1] = 0 \quad \&$$

$$\text{var}[x_1] = \frac{N_0}{2}$$

$\therefore$

$\mu = 0$
$\sigma^2 = \frac{N_0}{2}$

Computing P_e of 1st kind :-

Decision region : $\sqrt{\frac{E_b}{2}} < x_1 < \infty$

(Since x_1 has gaussian distribution, it is defined by)

$$f_{x_1}(x_1/0) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{-(x_1-\mu)^2}{2\sigma^2}} \cdot dx_1$$

Conditional PDF when Symbol '0' is transmitted is given by:

$$f_{x_1}(x_1|0) = \frac{1}{\sqrt{\cancel{2}\pi \frac{N_0}{\cancel{2}}}} \cdot e^{\frac{-(x_1-0)^2}{\cancel{2}\frac{N_0}{\cancel{2}}}}$$

$$\boxed{f_{x_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} \, e^{\left(\frac{-x_1^2}{N_0}\right)} \cdot dx_1} \longrightarrow (2)$$

$\therefore$ Probability of Symbol error, when Symbol '0' is transmitted

$$P_e(0) = \int_{\frac{\sqrt{E_b}}{2}}^{\infty} f_{x_1}(x_1|0) \cdot dx_1 \longrightarrow (3)$$

Substituting eq (2) in eq (3), we get

$$P_e(0) = \int_{\sqrt{E_b/2}}^{\infty} \frac{1}{\sqrt{\pi N_0}} \, e^{-\left[\frac{x_1}{\sqrt{N_0}}\right]^2} dx_1 \longrightarrow (4)$$

Let $Z = \frac{x_1}{\sqrt{N_0}}$

$Z\sqrt{N_0} = x_1$

$\boxed{dx_1 = dZ\sqrt{N_0}}$

limits

When $x_1 = \frac{\sqrt{E_b}}{2}$	When $x_1 = \infty$
$Z\sqrt{N_0} = \frac{\sqrt{E_b}}{2}$	$Z\sqrt{N_0} = \infty$
$Z = \frac{\sqrt{E_b}}{2} \cdot \frac{1}{\sqrt{N_0}}$	$Z = \frac{\infty}{\sqrt{N_0}}$
$Z = \sqrt{\frac{E_b}{N_0}} \times \frac{1}{2}$	$Z = \infty$

$$P_e(0) = \int_{\sqrt{\frac{E_b}{N_0}} \times \frac{1}{2}}^{\infty} \frac{1}{\sqrt{\pi \cancel{N_0}}} e^{-Z^2} \cdot dZ \cancel{\sqrt{N_0}}$$

$$P_e(0) = \int_{\frac{1}{2}\sqrt{\frac{E_b}{2}}}^{\infty} \frac{1}{\sqrt{\pi}} e^{-Z^2} dZ \longrightarrow (5)$$

WKT Complementary error function

$$\boxed{\operatorname{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_{u}^{\infty} e^{-u^2}\, du} \longrightarrow ⑥$$

From eq ⑥, we can write eq ⑤ as

$$P_e(0) = \frac{1}{\sqrt{\pi}} \, \frac{2}{2} \int_{\frac{1}{2}\sqrt{\frac{E_b}{N_0}}}^{\infty} e^{-Z^2}\, dZ$$

$$\boxed{P_e(0) = \frac{1}{2} \operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)}$$

Where $u = \frac{1}{2}\sqrt{\frac{E_b}{N_0}}$

* Similarly we can calculate probability of error for 2nd kind:

$$\boxed{P_e(1) = \frac{1}{2} \operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)}$$

Let probability of Sending '0' is $p(0) = \frac{1}{2}$

Let probability of Sending '1' is $p(1) = \frac{1}{2}$

* Then average probability of error

$$P_e = p(0)\, P_e(0) + p(1)\, P_e(1)$$

$$P_e = \frac{1}{2}\left[\frac{1}{2} \operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)\right] + \frac{1}{2}\left[\frac{1}{2} \operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)\right]$$

$$\boxed{P_e = \frac{1}{2} \operatorname{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)}$$

Probability of error of BPSK

* Let $x(t)$ be the received Signal

$$x(t) = S(t) + w(t) \longrightarrow ①$$

$$x(t) = \begin{cases} S_1(t) = \sqrt{E_b}\,\phi_1(t) + w(t) & , \text{ for Symbol } 1 \\ S_2(t) = -\sqrt{E_b}\,\phi_1(t) + w(t) & , \text{ for Symbol } 0 \end{cases}$$

* Let us assume that the Symbol 0 is transmitted. Then the o/p of the correlator is

$$x_1 = \int_0^{T_b} x(t)\,\phi_1(t)\,dt = \int_0^{T_b} [S_1(t) + w(t)]\,\phi_1(t)\,.\,dt$$

$$= \int_0^{T_b} S_2(t)\,\phi_1(t)\,dt + \int_0^{T_b} w(t)\,\phi_1(t)\,.\,dt$$

$$\boxed{x_1 = -\sqrt{E_b} + W_1}$$

$$s_{11} = \int_0^{T_b} s_1(t)\phi_1(t)\,dt = \sqrt{E_b}$$

$$s_{21} = \int_0^{T_b} s_2(t)\phi_1(t)\,dt = -\sqrt{E_b}$$

* Mean of the random variable X_1 is

$$E[X_1] = E[\sqrt{E_b} + W_1] = E[\sqrt{E_b}] + E[W_1] = \sqrt{E_b} + 0$$

$$\boxed{E[X_1] = -\sqrt{E_b}}$$

Since the expected value of a constant is the constant itself,

* variance of X_1 is

$$var[X_1] = var[\sqrt{E_b} + W_1] = var[\sqrt{E_b}] + var[W_1] = 0 + N_0/2$$

$$\boxed{var[X_1] = N_0/2}$$

Since, the variance of a constant is zero.

* Conditional pdf when Symbol '0' is transmitted is given by:

$$f_{X_1}(x_1/0) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\left(\frac{x-\mu}{2\sigma^2}\right)} \longrightarrow ⓐ$$

A Gaussian random variable is completely specified by its mean value and variance. Hence, the conditional probability density function of random variable X_1 given that symbol 0 is transmitted is given by

$$f_{X_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} \exp\left[\frac{-(x_1 + \sqrt{E_b})^2}{N_0}\right]$$

$$f_{X_1}(x_1|0) = \frac{1}{\sqrt{\cancel{2}\pi \frac{N_0}{\cancel{2}}}} \cdot e^{\frac{-(x_1+\sqrt{E_b})^2}{\cancel{2}\cdot\frac{N_0}{\cancel{2}}}} \qquad (e^x)$$

$$\boxed{f_{X_1}(x_1|0) = \frac{1}{\sqrt{\pi N_0}} \cdot e^{-\left[\frac{x_1+\sqrt{E_b}}{\sqrt{N_0}}\right]^2}} \longrightarrow ③$$

* Let $P_e(0)$ denotes the Conditional probability of deciding in favour of Symbol '1' when '0' is transmitted.

Region Z_1 : $0 \leq x_1 \leq +\infty$

$$\therefore \quad P_e(0) = \int_0^\infty f_{X_1}(x_1/0)\, dx_1$$

$$P_e(0) = \frac{1}{\sqrt{\pi N_0}} \int_0^\infty e^{-\left[\frac{x_1+\sqrt{E_b}}{\sqrt{N_0}}\right]^2} \cdot dx_1$$

Let $Z = \frac{x_1+\sqrt{E_b}}{\sqrt{N_0}}$

$$dZ = \frac{dx_1}{\sqrt{N_0}} + 0$$

$$dx_1 = \sqrt{N_0}\, dZ$$

Limits :-

When $x_1 = 0$	When $x_1 = \infty$
$Z = \frac{x_1+\sqrt{E_b}}{\sqrt{N_0}}$ $Z = \frac{0+\sqrt{E_b}}{\sqrt{N_0}}$ $\boxed{Z = \sqrt{\frac{E_b}{N_0}}}$	$\boxed{Z = \infty}$

$$\therefore \quad P_e(0) = \int_{\sqrt{E_b/N_0}}^{\infty} \frac{1}{\sqrt{\pi \cancel{N_0}}}\, e^{-Z^2}\, dZ \sqrt{\cancel{N_0}} \longrightarrow ④$$

$$= \frac{1}{\sqrt{\pi}} \int_{\sqrt{E_b/N_0}}^{\infty} e^{-Z^2} \cdot dZ$$

$$P_e(0) = \frac{1}{\sqrt{\pi}}\, \frac{2}{2} \int_{\sqrt{E_b/N_0}}^{\infty} e^{-Z^2}\, dZ \longrightarrow ⑤$$

WKT Complementary error function

$$\boxed{erfc(u) = \frac{2}{\sqrt{\pi}} \int_{u}^{\infty} e^{-u^2}\, du} \longrightarrow (6)$$

From eq (6), we can write eq (5) as

$$\boxed{P_e(0) = \frac{1}{2}\, erfc\left(\sqrt{\frac{E_b}{N_0}}\right)}$$

Where $u = \sqrt{\frac{E_b}{N_0}}$

* Similarly we can calculate probability of error for 2nd kind :

$$\boxed{P_e(1) = \frac{1}{2}\, erfc\left(\sqrt{\frac{E_b}{N_0}}\right)}$$

Let probability of transmitting Symbol '0' is $P(0) = \frac{1}{2}$

Let probability of transmitting Symbol '1' is $P(1) = \frac{1}{2}$

* Then average probability of error

$$P_e = P(0)\, P_e(0) + P(1)\, P_e(1)$$

$$P_e = \frac{1}{2}\left[\frac{1}{2}\, erfc\left(\sqrt{\frac{E_b}{N_0}}\right)\right] + \frac{1}{2}\left[\frac{1}{2}\, erfc\left(\sqrt{\frac{E_b}{N_0}}\right)\right]$$

$$\boxed{P_e = \frac{1}{2}\, erfc\left(\sqrt{\frac{E_b}{N_0}}\right)}$$

Probability of error of BFSK

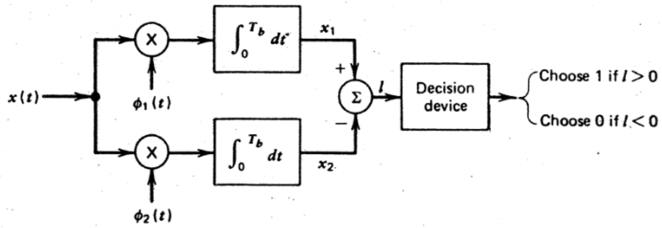
* Let $x(t)$ be the received BPSK Signal given by:

$$\boxed{x(t) = S(t) + w(t)}$$

$$x = \begin{cases} S_1(t) + w(t) & \text{for Symbol '1'} \\ S_2(t) + w(t) & \text{for Symbol '0'} \end{cases}$$

Where $w(t)$ is AWGN noise having mean $(\mu)=0$ & variance $(\sigma) = \frac{N_0}{2}$.

* Let us consider the transmission of Symbol '0' then the received Signal is $\boxed{x(t) = S_2(t) + w(t)}$



* The o/p of the top Correlator is

$$x_1 = \int_0^{T_b} x(t)\, \phi_1(t) \cdot dt$$

$$x_1 = \int_0^{T_b} [S_2(t) + w(t)]\, \phi_1(t)\, dt$$

$$= \int_0^{T_b} S_2(t)\, \phi_1(t)\, dt + \int_0^{T_b} w(t)\, \phi_1(t)\, dt$$

$$x_1 = S_{21} + W_1$$

$$\boxed{x_1 = 0 + W_1} \longrightarrow (1)$$

We can write $S(t)$ as

$$S(t) = \begin{cases} S_1(t) = \sqrt{E_b}\, \phi_1(t) & \text{for Symbol '1'} \\ S_2(t) = \sqrt{E_b}\, \phi_2(t) & \text{for Symbol '0'} \end{cases}$$

* Mean is given by

$$E[X_1] = E[0 + W_1] = 0 + 0$$

$$\boxed{E[X_1] = 0}$$

* variance of X_1 is

$$\text{var}[X_1] = \text{var}[0 + W_1] = 0 + \frac{N_0}{2}$$

$$\boxed{\text{var}[X_1] = \frac{N_0}{2}}$$

* The o/p of the bottom Correlator is

$$x_2 = \int_0^{T_b} x(t)\, \phi_2(t) \cdot dt$$

$$x_2 = \int_0^{T_b} [S_2(t) + w(t)]\, \phi_2(t)\, dt$$

$$x_2 = \int_0^{T_b} S_2(t)\, \phi_2(t)\, dt + \int_0^{T_b} w(t)\, \phi_2(t)\, dt$$

$$x_2 = S_{22} + W_2$$

$$\boxed{x_2 = \sqrt{E_b} + W_2} \longrightarrow (2)$$

* The mean of X_2 is

$$\boxed{E[X_2] = \sqrt{E_b}}$$

* The variance of X_2 is

$$\boxed{\text{var}[X_2] = \frac{N_0}{2}}$$

* Let us find the mean & variance of random variable $l = x_1 - x_2$ is also gaussian.

Mean $E[L] = E[X_1] - E[X_2]$

$$E[L] = 0 - \sqrt{E_b}$$

$$\boxed{E[L] = -\sqrt{E_b}}$$

(* The variance of the random variable 'L' is independent of which binary Symbol was transmitted. Since the random variables X_1 & X_2 are Statistically independent each with a variance equal to $N_0/2$.)

* $\text{var}[L] = \text{var}[X_1] + \text{var}[X_2]$

$$= \frac{N_0}{2} + \frac{N_0}{2}$$

$$\boxed{\text{var}[L] = N_0}$$

* Conditional PDF when Symbol '0' transmitted is given by

$$f_L(l/0) = \frac{1}{\sqrt{2\pi\sigma^2}}\, e^{\frac{-(l-\mu)^2}{2\sigma^2}}$$

$$\boxed{\begin{array}{l}\mu = -\sqrt{E_b}\ \& \\ \sigma^2 = N_0\end{array}}$$

$$f_L(l/0) = \frac{1}{\sqrt{2\pi N_0}}\, e^{\frac{-(l+\sqrt{E_b})^2}{2N_0}}$$

$$\boxed{f_L(l/0) = \frac{1}{\sqrt{2\pi N_0}}\, e^{-\left[\frac{l+\sqrt{E_b}}{\sqrt{2N_0}}\right]^2}} \longrightarrow (3)$$

* Let $P_e(0)$ denotes the Conditional probability of deciding in favour of Symbol '1', when Symbol '0' is transmitted.

$\therefore$ Region Z_1 : $0 \leq x \leq \infty$

$$P_e(0) = \int_0^{\infty} f_L(l/0)\, dl = \int_0^{\infty} \frac{1}{\sqrt{2\pi N_0}} e^{-\left[\frac{l+\sqrt{E_b}}{\sqrt{2N_0}}\right]^2} dl$$

Let $Z = \dfrac{l+\sqrt{E_b}}{\sqrt{2N_0}}$

$dZ = \dfrac{dl}{\sqrt{2N_0}} + 0$

$\boxed{dl = dZ\sqrt{2N_0}}$

Limits

When $l = \infty$ then $Z = \infty$	When $l = 0$, $Z = \dfrac{l+\sqrt{E_b}}{\sqrt{2N_0}}$ $Z = 0 + \dfrac{\sqrt{E_b}}{\sqrt{2N_0}}$ $Z = \sqrt{\dfrac{E_b}{2N_0}}$

$$\therefore P_e(0) = \int_{\sqrt{\frac{E_b}{2N_0}}}^{\infty} \frac{1}{\sqrt{2\pi N_0}} e^{-Z^2} dZ \sqrt{2N_0} = \frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{E_b}{2N_0}}}^{\infty} e^{-Z^2} . dZ$$

$$P_e(0) = \frac{1}{\sqrt{\pi}} \frac{2}{2} \int_{\sqrt{\frac{E_b}{2N_0}}}^{\infty} e^{-Z^2} dZ \longrightarrow (4)$$

WKT Complementary error function

$$\boxed{\text{erfc}(u) = \frac{2}{\sqrt{\pi}} \int_u^{\infty} e^{-u} du} \longrightarrow (5)$$

From eq (5), we can write eq (4) as

$$\boxed{P_e(0) = \frac{1}{2} \text{erfc}\left[\sqrt{\frac{E_b}{2N_0}}\right]}$$

* Similarly we can calculate probability of error for 2nd kind :

$$\boxed{P_e(1) = \frac{1}{2} \text{erfc}\left[\sqrt{\frac{E_b}{2N_0}}\right]}$$

* If Symbol 0's & 1's are equiprobable then $p(0) = p(1) = \frac{1}{2}$

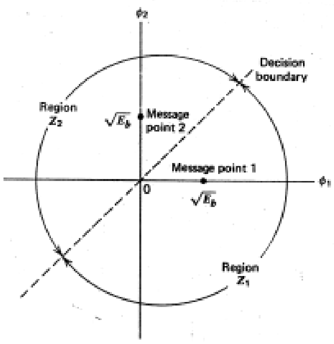
WKT $P_e(0) = P_e(1) = \frac{1}{2}\, \text{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right)$

* Then average probability of error

$$P_e = P(0)\,P_e(0) + P(1)\,P_e(1)$$

$$P_e = \frac{1}{2}\left[\frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right)\right] + \frac{1}{2}\left[\frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right)\right]$$

$$\boxed{P_e = \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right)}$$



Signal space diagram for coherent binary FSK system.

BASK	$\boxed{P_e = \frac{1}{2}\,\text{erfc}\left(\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right)}$
BPSK	$\boxed{P_e = \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)}$
BFSK	$\boxed{P_e = \frac{1}{2}\,\text{erfc}\left(\sqrt{\frac{E_b}{2N_0}}\right)}$