## 4.11 Impulse Invariant Transformation (IIT)

If $h_a(t)$ represents the impulse response of an analog filter, then the unit sample response of a discrete-time filter $H(z)$ used in an A/D-$H(z)$-D/A structure is selected to be the sampled version of $h(t)$.

That is,

$$\begin{aligned} h(n) &= h_a(nT) \\ &= h_a(t)\big|_{t=nT} \end{aligned}$$

and the discrete (digital) transfer function is

$$H(z) = \mathcal{Z}\{h(n)\}$$

Notice that the digital transfer function $H(z)$ is the $\mathcal{Z}$-transform of the unit sample response $h(n)$, while the analog transfer function $H_a(s)$ is the Laplace transform of the unit impulse response $h_a(t)$. Don't be tempted to write $H(z) = H(s)\big|_{s=z}$, because it is incorrect.

Let us now generalize this procedure and at the same time show that $H(z)$ can be obtained directly from $H_a(s)$ without intervening steps of finding $h_a(t)$ and then $h_a(nT)$.

Consider an analog transfer function with $N$ different poles that has the $s$-domain transfer function written in partial fraction expansion form as

$$H_a(s) = \sum_{i=1}^{N} \frac{C_i}{s - s_i} \tag{4.49}$$

with the corresponding unit impulse response

$$h_a(t) = \sum_{i=1}^{N} C_i\, e^{s_i t} \tag{4.50}$$

In the above expression, $C_i$ is the constant associated with the partial fraction expansion of $H_a(s)$. If this response is sampled every $T$ seconds ($t = nT$), we have the sampled response

$$h_a(nT) = h(n) = \sum_{i=1}^{N} C_i\, e^{s_i nT} \tag{4.51}$$

Finally, we take the $\mathcal{Z}$-transform of equation (4.51) to obtain the discrete transfer function of the digital IIR filter.

$$\begin{aligned} H(z) &= \mathcal{Z}\{h(n)\} \\ &= \sum_{n=0}^{\infty} h(n) z^{-n} \end{aligned} \tag{4.52}$$

The lower limit of summation in the above equation is zero because the filter is assumed to be causal.



Substituting equation (4.51) in equation (4.52), we get

$$\begin{aligned} H(z) &= \sum_{n=0}^{\infty}\sum_{i=1}^{N} C_i\, e^{s_i nT}\, z^{-n} \\ &= \sum_{i=1}^{N} C_i \sum_{n=0}^{\infty} \left[e^{s_i T} z^{-1}\right]^n \\ &= \sum_{i=1}^{N} C_i \frac{1}{1 - e^{s_i T} z^{-1}} \end{aligned} \tag{4.53}$$

Comparing equations (4.49) and (4.53), we find that

$$\frac{1}{s - s_i} \overset{\text{IIT}}{\Longrightarrow} \frac{1}{1 - e^{s_i T} z^{-1}} = \frac{z}{z - e^{s_i T}} \tag{4.54}$$

Equation (4.54) shows that the analog pole at $s = s_i$ is mapped to a digital pole at $z_i = e^{s_i T}$. The transformed digital filter $H(z)$ has the following properties:

1. Its order is same as that of $H_a(s)$ because the common denominator on the right-hand side has degree $N$.
2. Its poles are mapped according to

$$s_i \overset{\text{IIT}}{\Longrightarrow} z_i = e^{s_i T}, \quad 1 \leqslant i \leqslant N$$

That is, the analog and digital poles are related as per the equation

$$z = e^{sT}$$

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Impulse tm

$\omega = \Omega T$

$\Omega = \omega/T$

Letting $z = re^{j\omega}$ and $s = \sigma + j\Omega$ in the above equation, we get

$$re^{j\omega} = e^{\sigma T} e^{j\Omega T}$$

Hence,

$$r = e^{\sigma T}$$

and

$$\omega = \Omega T$$

Consequently, $\sigma < 0$ implies that $0 < r < 1$ and $\sigma > 0$ implies that $r > 1$. When $\sigma = 0$, we have $r = 1$. Hence, the left-half pole is mapped inside the unit circle in the $z$-plane and the right half pole in $s$ is mapped outside the unit circle in the $z$-plane. Thus, a stable analog filter $H_a(s)$ is transformed to a stable digital filter $H(z)$. Also, the image of the $j\Omega$ axis in the $z$-plane is the unit circle as indicated above. However, the mapping of the $j\Omega$ axis is not one-to-one. The mapping, $\omega = \Omega T$ implies that the interval $-\frac{\pi}{T} \leqslant \Omega \leqslant \frac{\pi}{T}$ maps into the interval $-\pi \leqslant \omega \leqslant \pi$. In general, the interval $(2q-1)\frac{\pi}{T} \leqslant \Omega \leqslant (2q+1)\frac{\pi}{T}$ also maps into the interval $-\pi \leqslant \omega \leqslant \pi$, where $q$ is an integer. Thus, the mapping of analog frequency, $\Omega$ in to digital frequency $\omega$ is many-to-one, which simply reflects the effects of aliasing due to sampling. Fig. 4.13 shows the mapping from $s$-plane to $z$-plane using impulse invariant transformation.

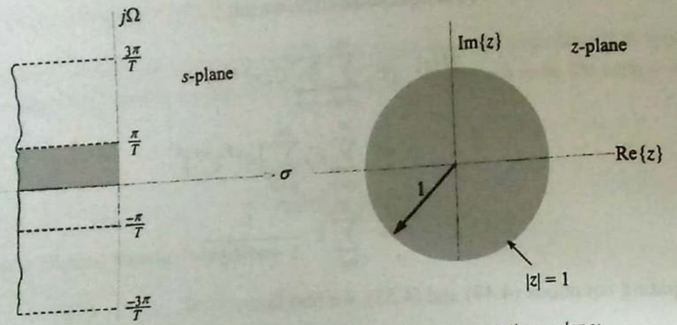


**Fig. 4.13** The impulse invariant transformation from the $s$-plane to the $z$- plane: the imaginary axis maps to the unit circle, a strip of $\frac{2\pi}{T}$ maps to the disk.

3. The frequency response $H(\omega)$ of the digital filter is related to the frequency response of the analog filter [3] $H_a(\Omega)$ by the sampling theorem,

$X_s(f) = f_s \sum X(f - nf_s)$

$$H(\omega) = \frac{1}{T}\sum_{k=-\infty}^{\infty} H\left(\frac{\omega - 2\pi k}{T}\right) \quad (4.55)$$

$\omega - n\omega_s$

Since, for a rational analog filter, $H(\Omega)$ is never band limited, the frequency response of the digital filter is always aliased. If $H_a(s)$ is lowpass or bandpass, aliasing can be made very minimal by choosing the sampling frequency $\frac{1}{T}$ high enough such that the fraction of the energy in the range $|\omega| > \frac{\pi}{T}$ will be negligible. On the otherhand, if $H_a(s)$ is highpass or bandstop, the impulse invariant method cannot be used at all. This is because the frequency response of these two filter classes does not decay to zero, so the right-hand side of equation (4.55) does not converge.

4. The zeros of $H(z)$ and $H_a(s)$ do not share a simple relationship. When the right-hand side of equation (4.53) is brought to a common denominator, the numerator will be a polynomial of degree $(N-1)$ in $z^{-1}$. This is regardless of the degree $(q)$ of the numerator polynomial of $H_a(s)$.

**Example 4.26** A third-order Butterworth lowpass filter has the transfer function

$$H(s) = \frac{1}{(s+1)(s^2+s+1)}$$

Design $H(z)$ using impulse invariant technique.

[3] $H_a(\Omega)$ was denoted earlier by $H_a(j\Omega)$.



☐ **Solution**

$$\begin{aligned} H(s) &= \frac{1}{(s+1)(s^2+s+1)} \\ &= \frac{1}{(s+1)(s+0.5-j0.866)(s+0.5+j0.866)} \\ &= \frac{C_1}{s+1} + \frac{C_2}{s+0.5-j0.866} + \frac{C_2^*}{s+0.5+j0.866} \end{aligned}$$

Using partial fraction expansion, we find

$$C_1 = 1, \quad C_2 = 0.577\, e^{-j2.62} \quad \text{and} \quad C_2^* = 0.577\, e^{j2.62}$$

Hence,

$$H(s) = \frac{1}{s+1} + \frac{0.577\, e^{-j2.62}}{s+0.5-j0.866} + \frac{0.577\, e^{j2.62}}{s+0.5+j0.866}$$

The three poles are

$$s_1 = -1, \quad s_2 = -0.5 + j0.866 \quad \text{and} \quad s_3 = -0.5 - j0.866$$

We know that

$$\begin{aligned} H(z) &= \sum_{i=1}^{3} \frac{C_i}{1 - e^{s_i T} z^{-1}} \\ &= \frac{C_1}{1 - e^{s_1 T} z^{-1}} + \frac{C_2}{1 - e^{s_2 T} z^{-1}} + \frac{C_3}{1 - e^{s_3 T} z^{-1}} \end{aligned}$$

Here,

$$C_3 = C_2^*$$

Hence,

$$\begin{aligned} H(z) &= \frac{C_1}{1 - e^{s_1 T} z^{-1}} + \frac{C_2}{1 - e^{s_2 T} z^{-1}} + \frac{C_2^*}{1 - e^{s_3 T} z^{-1}} \\ &= \frac{1}{1 - e^{-T} z^{-1}} + \frac{0.577\, e^{-j2.62}}{1 - e^{(-0.5+j0.866)T} z^{-1}} + \frac{0.577\, e^{j2.62}}{1 - e^{(-0.5-j0.866)T} z^{-1}} \\ &= \frac{1}{1 - e^{-T} z^{-1}} + \frac{0.577\, e^{-j2.62}}{1 - e^{-0.5T} e^{j0.866T} z^{-1}} + \frac{0.577\, e^{j2.62}}{1 - e^{-0.5T} e^{-j0.866T} z^{-1}} \\ &= \frac{1}{1 - e^{-T} z^{-1}} + \frac{2(0.577)\cos(-2.62) - 2(0.577)e^{-0.5T} z^{-1} \cos(-2.62 - 0.866T)}{1 - 2\, e^{-0.5T} \cos(0.866T) z^{-1} + e^{-T} z^{-2}} \end{aligned}$$

Multiplying the numerator and the denominator of first term on the right-hand side by $z$ and by $z^2$ for the second term on the right- hand side, the above equation becomes

$$H(z) = \frac{z}{z - e^{-T}} + \frac{-z^2 - 1.154 e^{-0.5T} \cos\left(\frac{5\pi}{6} + 0.866T\right) z}{z^2 - 2\, e^{-0.5T} \cos(0.866T) z + e^{-T}}$$

In terms of the sampling interval $T$, the filter transfer function is

$$H(z) = \frac{b_0 z^2 + b_1 z}{z^3 - a_1 z^2 - a_2 z - a_3} = \frac{b_0 z^{-1} + b_1 z^{-2}}{1 - a_1 z^{-1} - a_2 z^{-2} - a_3 z^{-3}}$$

where
$$b_0 = -2\,e^{-0.5T}\cos(0.866T) + e^{-T} + 1.154\,e^{-0.5T}\cos\left(\frac{5\pi}{6} + 0.866T\right)$$
$$b_1 = e^{-T} + 1.154\,e^{-1.5T}\cos\left(\frac{5\pi}{6} + 0.866T\right)$$
$$a_1 = e^{-T} + 2\,e^{-0.5T}\cos(0.866T)$$
$$a_2 = -e^{-T} - 2\,e^{-1.5T}\cos(0.866T)$$
$$a_3 = e^{-2T}$$

**Example 4.27** Let $H_a(s) = \frac{b}{(s+a)^2+b^2}$ be a causal second-order analog transfer function. Show that the causal second-order digital transfer function $H(z)$ obtained from $H_a(s)$ through impulse invariance method is given by

$$H(z) = \frac{e^{-aT}\sin bT\, z^{-1}}{1 - 2e^{-aT}\cos bT\, z^{-1} + e^{-2aT} z^{-2}}$$

Also, find $H(z)$ when $H_a(s) = \frac{1}{s^2+2s+2}$.

### ☐ Solution

Poles of $H_a(s)$ are obtained from

$$(s+a)^2 + b^2 = 0.$$

Solving, we get
$$s = -a \pm jb$$

Let $s_1 = -a + jb$ and $s_2 = -a - jb$

We may now write the analog transfer function in factored form as

$$H_a(s) = \frac{b}{(s+a-jb)(s+a+jb)} = \frac{C_1}{s+a-jb} + \frac{C_2}{s+a+jb}$$

where
$$C_1 = \left.\frac{b}{s+a+jb}\right|_{s=-a+jb} = \frac{1}{j2}$$

and
$$C_2 = C_1^* = -\frac{1}{j2}$$



We know that,
$$H(z) = \sum_{i=1}^{N} \frac{C_i}{1 - e^{s_i T} z^{-1}}$$

$$\Rightarrow \quad H(z) = \sum_{i=1}^{2} \frac{C_i\, z}{z - e^{s_i T}}$$

Hence,
$$H(z) = \frac{1}{j2}\left(\frac{z}{z - e^{(-a+jb)T}} - \frac{z}{z - e^{(-a-jb)T}}\right)$$
$$= \frac{1}{j2}\left(\frac{z^2 - z e^{-aT} e^{-jbT} - z^2 + z e^{-aT} e^{jbT}}{z^2 - z e^{-aT} e^{-jbT} - z e^{-aT} e^{jbT} + e^{-2aT}}\right)$$
$$= \frac{z e^{-aT}\sin bT}{z^2 - 2e^{-aT}\cos bT\, z + e^{-2aT}}$$
$$= \frac{e^{-aT}\sin bT\, z^{-1}}{1 - 2e^{-aT}\cos bT\, z^{-1} + e^{-2aT} z^{-2}}$$

Given
$$H_a(s) = \frac{1}{s^2 + 2s + 2}$$
$$\Rightarrow \quad H_a(s) = \frac{1}{(s+1)^2 + 1^2}$$

Hence,
$$H(z) = \frac{e^{-T}\sin T z^{-1}}{1 - 2e^{-T}\cos T z^{-1} + e^{-2T} z^{-2}}$$

**Example 4.28** Let $H_a(s) = \frac{s+a}{(s+a)^2+b^2}$ be a causal second-order analog transfer function. Show that the causal second-order digital transfer function $H(z)$ is obtained from $H_a(s)$ through impulse invariance method is given by

$$H(z) = \frac{1 - e^{-aT}\cos bT\, z^{-1}}{1 - 2\cos bT\, e^{-aT} z^{-1} + e^{-2aT} z^{-2}}$$

### ☐ Solution

The poles of $H_a(s)$ are obtained from

$$(s+a)^2 + b^2 = 0$$
$$\Rightarrow \quad s = -a \pm jb$$

Let $s_1 = -a + jb$ and $s_2 = -a - jb$

The analog transfer function $H_a(s)$ is written in the factored form as

$$H_a(s) = \frac{s+a}{(s+a-jb)(s+a+jb)} = \frac{C_1}{s+a-jb} + \frac{C_2}{s+a+jb}$$

where

$$C_1 = \left.\frac{s+a}{s+a+jb}\right|_{s=-a+jb} = \frac{1}{2}$$

and

$$C_2 = C_1^* = \frac{1}{2}$$

We know that

$$H(z) = \sum_{i=1}^{N} \frac{C_i\, z}{z - e^{s_i T}}$$

$$\Rightarrow \quad H(z) = \sum_{i=1}^{2} \frac{C_i\, z}{z - e^{s_i T}}$$

Hence,

$$H(z) = C_1 \frac{z}{z - e^{s_1 T}} + C_2 \frac{z}{z - e^{s_2 T}}$$

$$= \frac{1}{2}\left(\frac{z}{z - e^{(-a+jb)T}} + \frac{z}{z - e^{(-a-jb)T}}\right)$$

$$= \frac{1}{2}\left(\frac{z^2 - ze^{-aT}e^{-jbT} + z^2 - ze^{-aT}e^{jbT}}{z^2 - ze^{-aT}e^{-jbT} - ze^{-aT}e^{jbT} + e^{-2aT}}\right)$$

$$\Rightarrow \quad H(z) = \frac{z^2 - ze^{-aT}\cos bT}{z^2 - 2z\cos bT\, e^{-aT} + e^{-2aT}}$$

$$= \frac{1 - e^{-aT}\cos bT\, z^{-1}}{1 - 2\cos bT\, e^{-aT} z^{-1} + e^{-2aT} z^{-2}}$$

**Example 4.29** Transform the analog filter

$$H_a(s) = \frac{(s+1)}{s^2 + 5s + 6}$$

into $H(z)$ using impulse invariant transformation. Take $T = 0.1$ sec.

### ☐ Solution

Given,

$$H_a(s) = \frac{(s+1)}{s^2 + 5s + 6} = \frac{s+1}{(s+2)(s+3)}$$



$$= \frac{C_1}{s+2} + \frac{C_2}{s+3}$$

where

$$C_1 = \left.\frac{s+1}{s+3}\right|_{s=-2} = -1$$

and

$$C_2 = \left.\frac{s+1}{s+2}\right|_{s=-3} = 2$$

Also, the poles of $H_a(s)$ are $s_1 = -2$ and $s_2 = -3$.

We know that,

$$H(z) = \sum_{i=1}^{N} \frac{C_i}{1 - e^{s_i T} z^{-1}}$$

$$\Rightarrow \quad H(z) = \sum_{i=1}^{2} \frac{C_i}{1 - e^{s_i T} z^{-1}} = \sum_{i=1}^{2} \frac{C_i\, z}{z - e^{s_i T}}$$

Hence,

$$H(z) = \frac{C_1\, z}{z - e^{s_1 T}} + \frac{C_2\, z}{z - e^{s_2 T}}$$

$$= \frac{-z}{z - e^{-0.2}} + \frac{2z}{z - e^{-0.3}}$$

$$= \frac{-z}{z - 0.8186} + \frac{2z}{z - 0.7408}$$

$$= \frac{z^2 - 0.8964z}{z^2 - 1.559z + 0.6065}$$

$$= \frac{1 - 0.8964\, z^{-1}}{1 - 1.559\, z^{-1} + 0.6065\, z^{-2}}$$

**Example 4.30** Consider the analog filter having the transfer function

$$H_a(s) = \frac{1}{s+2}$$

a. Transform $H_a(s)$ to a digital filter $H(z)$, using impulse invariance technique. Assume that the sampling rate, $S = 2$ Hz.

b. Will the impluse response $h(n)$ match the impulse response $h(t)$ of the analog filter at the sampling instants? should it? Explain.

c. Will the step response $S(n)$ match the step response $S(t)$ of the analog filter at the sampling instants? Explain.

## ☐ Solution

a. We know that

$$H_a(s) = \frac{1}{s - s_i} \overset{\text{IIT}}{\Longrightarrow} H(z) = \frac{1}{1 - e^{s_i T} z^{-1}}$$

where $T = \frac{1}{S}$

Here, $T = \frac{1}{2}$ sec.

Hence, $H_a(s) = \frac{1}{s+2} \overset{\text{IIT}}{\Longrightarrow} H(z) = \frac{1}{1 - e^{-2\times\frac{1}{2}} z^{-1}} = \frac{1}{1 - e^{-1} z^{-1}}$

b. $$H(z) = \frac{z}{z - e^{-1}}$$

Assuming the system to be causal, we get

$$h(n) = \left(e^{-1}\right)^n u(n) = e^{-n} u(n) \quad (4.56a)$$

Given $H_a(s) = \frac{1}{s+2}$

$$\Rightarrow \quad h_a(t) = e^{-2t}, \quad t \geqslant 0$$

Letting $t = nT$, we get

$$h_a(nT) = h(n) = e^{-2nT}, \quad n \geqslant 0$$

Since $T = \frac{1}{2}$ sec, we get

$$h(n) = e^{-n} u(n) \quad (4.56b)$$

From equations (4.56a) and (4.56b), we find that the impulse responses of analog and digital filters will match at the sampling instants.

c. Let $H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z - e^{-1}}$

$$\Rightarrow \quad Y(z) = \frac{z}{z - e^{-1}} X(z)$$

If $x(n) = u(n)$, we get

$$Y(z) = \frac{z}{\left(z - e^{-1}\right)} \frac{z}{z-1}; \quad \text{ROC} : |z| > 1$$



$$\Rightarrow \quad \frac{Y(z)}{z} = \frac{z}{(z-1)(z - e^{-1})} = \frac{K_1}{z-1} + \frac{K_2}{z - e^{-1}}$$

where $K_1 = \left.\frac{z}{z - e^{-1}}\right|_{z=1} = 1.582$

$$K_2 = \left.\frac{z}{z-1}\right|_{z=e^{-1}} = -0.582$$

Hence, $Y(z) = 1.582\frac{z}{z-1} - 0.582\frac{z}{z - e^{-1}}$

Taking inverse $\mathcal{Z}$-transform of $Y(z)$ we get

$$y(n) = S(n) = 1.582\, u(n) - 0.582\, e^{-n} u(n) \quad (4.56c)$$

Let $H_a(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+2}$

$$\Rightarrow \quad Y(s) = \frac{1}{s+2} X(s)$$

Since $x(t) = u(t)$, the above equation becomes

$$Y(s) = \frac{1}{s(s+2)} = \frac{K_3}{s} + \frac{K_4}{s+2}$$

where $K_3 = \left.\frac{1}{s+2}\right|_{s=0} = \frac{1}{2}$

$$K_4 = \left.\frac{1}{s}\right|_{s=-2} = -\frac{1}{2}$$

Hence, $Y(s) = \frac{1}{2s} - \frac{1}{2(s+2)}$

$$\Rightarrow \quad y(t) = \frac{1}{2} - \frac{1}{2} e^{-t}, \quad t \geqslant 0$$

Letting $t = nT$, we get

$$y(nT) = S(n) = \frac{1}{2} - \frac{1}{2} e^{-nT}, \quad n \geqslant 0$$
$$= \frac{1}{2} u(n) - \frac{1}{2} e^{-\frac{n}{2}} u(n) \quad (4.56d)$$

From equations (4.56c) and (4.56d), we find that the step responses of analog and digital filters will not match at the sampling instants.

**Example 4.31** The following causal IIR digital filters were designed using the impulse invariant transformation.

Find their respective causal analog counter parts. Take $T = 0.3$ sec.

a. $H(z) = \dfrac{2z}{z - e^{-0.9}} + \dfrac{3z}{z - e^{-1.2}}$

b. $H(z) = \dfrac{z^2 - ze^{-0.6}\cos(0.9)}{z^2 - 2ze^{-0.6}\cos(0.9) + e^{-1.2}}$

## ☐ Solution

a. Given,

$$H(z) = \frac{2z}{z - e^{-0.9}} + \frac{3z}{z - e^{-1.2}}$$

Comparing the above transfer function with

$$H(z) = \sum_{i=1}^{2} \frac{C_i\, z}{z - e^{s_i T}} = \frac{C_1\, z}{z - e^{s_1 T}} + \frac{C_2\, z}{z - e^{s_2 T}}$$

we get

$$s_1 T = -0.9 \Rightarrow s_1 = \frac{-0.9}{0.3} = -3,$$

$$s_2 T = -1.2 \Rightarrow s_2 = \frac{-1.2}{0.3} = -4,$$

$$C_1 = 2 \quad \text{and} \quad C_2 = 3$$

Hence,

$$H_a(s) = \frac{C_1}{s - s_1} + \frac{C_2}{s - s_2} = \frac{2}{s+3} + \frac{3}{s+4}$$

b.

$$H(z) = \frac{z^2 - ze^{-0.6}\cos(0.9)}{z^2 - 2ze^{-0.6}\cos(0.9) + e^{-1.2}} = \frac{1 - e^{-0.6}\cos(0.9)z^{-1}}{1 - 2e^{-0.6}\cos(0.9)z^{-1} + e^{-1.2}z^{-2}}$$

Comparing the above equation with

$$H_a(s) = \frac{s+a}{(s+a)^2 + b^2} \overset{\text{IIT}}{\Longrightarrow} H(z) = \frac{1 - e^{-aT}\cos bT\, z^{-1}}{1 - 2e^{-aT}\cos bT\, z^{-1} + e^{-2aT}\, z^{-2}}$$



we get,

$$aT = 0.6 \Rightarrow a = \frac{0.6}{0.3} = 2$$

and

$$bT = 0.9 \Rightarrow b = \frac{0.9}{0.3} = 3$$

Hence,

$$H_a(s) = \frac{s+2}{(s+2)^2 + 9}$$

**Example 4.32** Transform the causal analog filter represented by

$$H_a(s) = \frac{4s^2 + 10s + 8}{(s^2 + 2s + 3)(s+1)}$$

into a causal IIR digital filter. Assume, $T = 0.2$ secs.

## ☐ Solution

Given,

$$H_a(s) = \frac{4s^2 + 10s + 8}{(s^2 + 2s + 3)\,(s+1)}$$

Making use of partial fraction expansion, we can write

$$H_a(s) = \frac{1}{s+1} + \frac{3s+5}{s^2 + 2s + 3} = \frac{1}{s+1} + \frac{3(s+1)}{(s+1)^2 + \left(\sqrt{2}\right)^2} + \frac{\sqrt{2}\left(\sqrt{2}\right)}{(s+1)^2 + \left(\sqrt{2}\right)^2}$$

We know that

1. $\dfrac{1}{s - s_i} \overset{\text{IIT}}{\Longrightarrow} \dfrac{1}{1 - e^{s_i T} z^{-1}}$

2. $\dfrac{s+a}{(s+a)^2 + b^2} \overset{\text{IIT}}{\Longrightarrow} \dfrac{1 - e^{-aT}\cos bT\, z^{-1}}{1 - 2e^{-aT}\cos bT\, z^{-1} + e^{-2aT}\, z^{-2}}$

3. $\dfrac{b}{(s+a)^2 + b^2} \overset{\text{IIT}}{\Longrightarrow} \dfrac{e^{-aT}\sin bT\, z^{-1}}{1 - 2e^{-aT}\cos bT\, z^{-1} + e^{-2aT}\, z^{-2}}$

Hence,

$$H(z) = \frac{1}{1 - e^{-T}z^{-1}} + 3\left[\frac{1 - e^{-T}\cos\left(\sqrt{2}T\right)z^{-1}}{1 - 2e^{-T}\cos\left(\sqrt{2}T\right)z^{-1} + e^{-2T}z^{-2}}\right] + \sqrt{2}\left[\frac{e^{-T}\sin\left(\sqrt{2}T\right)z^{-1}}{1 - 2e^{-T}\cos\left(\sqrt{2}T\right)z^{-1} + e^{-2T}z^{-2}}\right]$$

Letting $T = 0.2$ secs in the above equation, we get

$$H(z) = \frac{1}{1-0.81873z^{-1}} + \frac{3-2.3585z^{-1}}{1-1.5724z^{-1}+0.67032z^{-2}} + \frac{0.32314z^{-1}}{1-1.5724z^{-1}+0.67032z^{-2}}$$

$$= \frac{1}{1-0.81873z^{-1}} + \frac{3-2.03536z^{-1}}{1-1.5724z^{-1}+0.67032z^{-2}}$$

**Example 4.33** A digital lowpass filter is required to meet the following specifications:

$$20\log|H(\omega)|_{\omega=0.2\pi} \geqslant -1.9328 \text{ dB}$$
$$20\log|H(\omega)|_{\omega=0.6\pi} \leqslant -13.9794 \text{ dB}$$

The filter must have a maximally flat frequency response. Find $H(z)$ to meet the above specifications using impulse invariant transformation.

### ☐ Solution

We are given the following specifications for the digital filter:

$$K_P = -1.9328 \text{ dB}, \quad \omega_P = 0.2\,\pi$$
$$K_S = -13.9794 \text{ dB}, \quad \omega_S = 0.6\,\pi$$

**Step 1:** Convert the edge-band digital frequencies into analog frequencies using the formula $\Omega = \frac{\omega}{T}$ with $T = 1$ sec.

Hence,
$$\Omega_P = 0.2\pi \text{ rad/sec}, \quad K_P = -1.9328 \text{ dB}$$
$$\Omega_S = 0.6\pi \text{ rad/sec}, \quad K_S = -13.9794 \text{ dB}$$

The effect of $T$ gets cancelled out in the design. Hence, $T = 1$ is taken as a matter of convenience.

**Step 2:** A Butterworth analog filter is chosen as the analog prototype, to meet the maximally flat condition for the frequency response. Using the analog specifications determined in step 1, let us design an analog lowpass filter, $H_a(s)$.

$$N = \frac{\log\left[\left(10^{\frac{-K_P}{10}}-1\right)\Big/\left(10^{\frac{-K_S}{10}}-1\right)\right]}{2\log\left(\frac{\Omega_P}{\Omega_S}\right)} = 1.7$$

Rounding off to the next larger integer, we get $N = 2$.



The poles of the second-order normalized lowpass Butterworth filter are as found as follows:

$$s_k = 1\,\underline{/\theta_k}$$

where
$$\theta_k = \frac{\pi}{N}k + \frac{\pi}{2N} + \frac{\pi}{2}, \quad k = 0, 1, \cdots 2N-1$$

| $k$ | $s_k$ |
|---|---|
| 0 | $-0.707 + j0.707$ |
| 1 | $-0.707 - j0.707$ |

Hence,
$$H_2(s) = \frac{1}{\prod_{\substack{\text{LHP}\\\text{only}}}(s-s_k)} = \frac{1}{(s-s_0)(s-s_1)}$$
$$= \frac{1}{(s+0.707-j0.707)(s+0.707+j0.707)}$$
$$= \frac{1}{(s+0.707)^2+(0.707)^2}$$
$$= \frac{1}{s^2+1.414s+1}$$

Let us determine the cutoff frequency $\Omega_C$ to meet the passband requirement precisely.

$$\Omega_C = \frac{\Omega_P}{\left[10^{\frac{-K_P}{10}}-1\right]^{\frac{1}{2N}}} = 0.7255$$

The required lowpass analog protype is obtained by applying lowpass-to-lowpass analog frequency transformation to $H_2(s)$.

That is,
$$H_a(s) = H_2(s)\big|_{s\longrightarrow\frac{s}{\Omega_C}}$$

$$\Rightarrow \quad H_a(s) = \left.\frac{1}{s^2+1.414s+1}\right|_{s\longrightarrow\frac{s}{0.7255}}$$
$$= \frac{1}{\left(\frac{s}{0.7255}\right)^2+1.414\left(\frac{s}{0.7255}\right)+1}$$
$$= \frac{0.52635}{s^2+1.02586s+0.52635}$$
$$= \frac{0.52635}{s^2+1.02586s+0.263097-0.263097+0.52635}$$
$$= \frac{1.0259\times0.513082}{(s+0.5129298)^2+(0.513082)^2}$$

**Step 3:** Let us design $H(z)$ using IIT with $T = 1$ sec.

We know that,

$$\frac{b}{(s+a)^2+b^2} \overset{\text{IIT}}{\Longrightarrow} \frac{e^{-aT}\sin bTz^{-1}}{1-2e^{-aT}\cos bTz^{-1}+e^{-2aT}z^{-2}}$$

Hence,
$$H(z) = 1.0259\left[\frac{e^{-0.5129298}\sin(0.513082)z^{-1}}{1-2e^{-0.5129298}\cos(0.513082)z^{-1}+e^{-2\times 0.5129298}z^{-2}}\right]$$
$$= \frac{0.301512z^{-1}}{1-1.0433z^{-1}+0.3585z^{-2}}$$

**Verification**

$$H\left(e^{j\omega}\right) = H(\omega) = \frac{0.301512e^{-j\omega}}{1-1.0433e^{-j\omega}+0.3585e^{-j2\omega}}$$

$$\Rightarrow \quad |H(\omega)| = \frac{0.301512\sqrt{\cos^2\omega+\sin^2\omega}}{\sqrt{(1-1.0433\cos\omega+0.3585\cos 2\omega)^2+(1.0433\sin\omega-0.3585\sin 2\omega)^2}}$$

Therefore $\quad 20\log|H(\omega)|_{\omega=0.2\pi} = -2$ dB

and $\quad 20\log|H(\omega)|_{\omega=0.6\pi} = -14.4$ dB

It may be noted that the passband specification is slightly exceeded and this is due to aliasing. This will not be the case when $H(z)$ is designed using bilinear transformation. If the resulting $H(z)$ designed using IIT fails to meet the given specifications because of aliasing, there is no alternative with impulse invariance but to try again with a higher-order filter or with a different adjustment of the filter parameter, holding $N$ fixed.

**Example 4.34** Apply impulse invariant technique to the analog transfer function given by

$$H_a(s) = \frac{s^2+4.525}{s^2+0.692s+0.504}$$

with $T = 1$ sec.

☐ **Solution**

We can write the given analog transfer function as

$$H_a(s) = \frac{s^2+4.525+0.692s-0.692s+0.504-0.504}{s^2+0.692s+0.504}$$
$$= \frac{(s^2+0.692s+0.504)-0.692s+4.021}{s^2+0.692s+0.504}$$



$$= 1+\left[\frac{-0.692(s+0.346)+4.021+0.692\times 0.346}{(s+0.346)^2+0.(6199)^2}\right]$$
$$= 1-\frac{0.692(s+0.346)}{(s+0.346)^2+(0.6199)^2}+\frac{4.26}{(s+0.346)^2+(0.6199)^2}$$
$$= 1-\frac{0.692(s+0.346)}{(s+0.346)^2+(0.6199)^2}+6.872\frac{0.6199}{(s+0.346)^2+(0.6199)^2}$$

We know that

1. $\dfrac{s+a}{(s+a)^2+b^2} \overset{\text{IIT}}{\Longrightarrow} \dfrac{1-e^{-aT}\cos bT\,z^{-1}}{1-2e^{-aT}\cos bT\,z^{-1}}$

2. $\dfrac{b}{(s+a)^2+b^2} \overset{\text{IIT}}{\Longrightarrow} \dfrac{e^{-aT}\sin bT\,z^{-1}}{1-2e^{-aT}\cos bTz^{-1}+e^{-2aT}z^{-2}}$

3. $1 \overset{\text{IIT}}{\Longrightarrow} 1$

(please note that $\mathcal{L}\{\delta(t)\} = 1$ and $\mathcal{Z}\{\delta(n)\} = 1$)

Hence,
$$H(z) = 1-\frac{0.692\left[1-e^{-0.346}\cos(0.6199)z^{-1}\right]}{1-2\,e^{-0.346}\cos(0.6199)z^{-1}+e^{-0.692}z^{-2}}$$
$$+\frac{6.872\,e^{-0.346}\sin(0.6199)z^{-1}}{1-2\,e^{-0.346}\cos(0.6199)z^{-1}+e^{-0.692}z^{-2}}$$
$$= 1-\left[\frac{0.692-3.2234z^{-1}}{1-1.1517z^{-1}+0.5006z^{-2}}\right]$$

**Example 4.35** Design a digital Chebyshev I filter that satisfies the following constraints.

$$0.8 \leqslant |H(\omega)| \leqslant 1, \quad 0 \leqslant \omega \leqslant 0.2\pi$$
$$|H(\omega)| \leqslant 0.2, \quad 0.6\pi \leqslant \omega \leqslant \pi$$

Use impluse invariant transformation.

☐ **Solution**

We are given the following digital specifications:

Passband ripple: $\delta_P = 1-0.8 = 0.2$.

Passband-edge frequency: $\omega_P = 0.2\pi$.

Stopband tolerance: $\delta_S = 0.2$.

Stopband-edge frequency: $\omega_S = 0.6\pi$.

**Step 1:** Convert the above edge-band digital frequencies into analog frquencies using the formu… $\Omega = \frac{\omega}{T}$ with $T = 1$ sec.

Hence,

$$\Omega_P = 0.2\pi, \quad \delta_P = 1 - 0.8 = 0.2$$
$$\Rightarrow \quad K_P = 20\log(1-\delta_P) = -1.94 \text{ dB}$$
$$\Omega_S = 0.6\,\pi, \quad \delta_S = 0.2$$
$$\Rightarrow \quad K_S = 20\log\delta_S = -14 \text{ dB}$$

**Step 2:** Design a chebyshev I lowpass analog prototype filter to meet the specifications listed … step 1.

$$d = \sqrt{\frac{(1-\delta_P)^{-2}-1}{\delta_S^{-2}-1}} = 0.153$$

$$K = \frac{\Omega_P}{\Omega_S} = 0.33$$

Filter order:

$$N \geqslant \frac{\cosh^{-1}\left(\frac{1}{d}\right)}{\cosh^{-1}\left(\frac{1}{K}\right)}$$
$$\Rightarrow \quad N \geqslant 1.446$$

Hence, the minimum filter order is $N = 2$.

We know that

$$1 - \delta_P = \frac{1}{\sqrt{1+\epsilon^2}}$$
$$\Rightarrow \quad \epsilon = 0.75$$

$$a = \frac{1}{2}\left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{1}{N}} - \frac{1}{2}\left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{-1}{N}}$$
$$= 0.57735$$

$$b = \frac{1}{2}\left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{1}{N}} + \frac{1}{2}\left[\frac{1+\sqrt{1+\epsilon^2}}{\epsilon}\right]^{\frac{-1}{N}}$$
$$= 1.1547$$

$$\sigma_k = -a\sin\left[(2k-1)\frac{\pi}{2N}\right], \quad k = 1, 2, 3, 4$$

$$\Omega_k = b\cos\left[(2k-1)\frac{\pi}{2N}\right], \quad k = 1, 2, 3, 4$$

| $k$ | $\sigma_k$ | $\Omega_k$ | $s_k$ |
|---|---|---|---|
| 1 | −0.4082481 | 0.8164962 | $-0.4082481 + j0.8164962$ |
| 2 | −0.4082481 | −0.8164962 | $-0.4082481 - j0.8164962$ |
| 3 | 0.4082481 | −0.8164962 | $0.4082481 - j0.8164962$ |
| 4 | 0.4082481 | 0.8164962 | $0.4082481 + j0.8164962$ |



Hence,

$$H_2(s) = \frac{K_N}{\prod\limits_{\substack{\text{LHP}\\\text{only}}}(s-s_k)} = \frac{K_N}{(s-s_1)(s-s_2)}$$
$$= \frac{K_N}{(s+0.4082481-j0.8164962)(s+0.4082481+j0.8164962)}$$
$$= \frac{K_N}{(s+0.4082481)^2+(0.8164962)^2}$$
$$= \frac{K_N}{s^2+0.8164962s+0.833333}$$

where

$$K_N = \frac{b_0}{\sqrt{1+\epsilon^2}} = \frac{0.833333}{\sqrt{1+(0.75)^2}} = 0.667$$

Thus,

$$H_2(s) = \frac{0.667}{s^2+0.8164962s+0.833333}$$

Since, we want the cutoff at $\Omega_P = 0.2\pi$, let us apply lowpass-to-lowpass transformation on $H_2(s)$ and get $H_a(s)$.

That is,

$$H_a(s) = H_2(s)\big|_{s\longrightarrow\frac{s}{0.2\pi}}$$
$$= \frac{0.667}{\left(\frac{s}{0.2\pi}\right)^2+0.8164962\left(\frac{s}{0.2\pi}\right)+0.833333}$$
$$= \frac{0.263321}{s^2+0.51302s+0.32899}$$
$$= \frac{0.263321}{(s+0.25651)^2+(0.51302)^2}$$
$$= \frac{0.263321}{0.51302}\times\frac{0.51302}{(s+0.25651)^2+(0.51302)^2}$$
$$= 0.513276\times\frac{0.51302}{(s+0.25651)^2+(0.51302)^2}$$

**Step 3:** Design $H(z)$ using IIT with $T = 1$ sec.

Recall :

$$\frac{b}{(s+a)^2+b^3} \overset{\text{IIT}}{\Longrightarrow} \frac{e^{-aT}\sin bT z^{-1}}{1-2e^{-aT}\cos bT z^{-1}+e^{-2aT}z^{-2}}$$

Hence,

$$H(z) = \frac{0.513276\times e^{-0.25651}\sin(0.51302)z^{-1}}{1-2e^{-0.25651}\cos(0.51302)z^{-1}+e^{-2\times0.25651}z^{-2}}$$
$$= \frac{0.19492z^{-1}}{1-1.34828z^{-1}+0.598685z^{-2}}$$