$$x(n) = \{1, 3, 2, -1, 1, \underbrace{0}_{M-1\,=\,1 \text{ zero appended}}\}$$

$$h(n) = \{2, -1, \underbrace{0, 0, 0, 0}_{L-1=4 \text{ zeros appended}}\}$$

$$\therefore\ y_L(n) = x(n) \ ⑥\ h(n)$$

$$= \begin{bmatrix} 1 & 0 & 1 & -1 & 2 & 3 \\ 3 & 1 & 0 & 1 & -1 & 2 \\ 2 & 3 & 1 & 0 & 1 & -1 \\ -1 & 2 & 3 & 1 & 0 & 1 \\ 1 & -1 & 2 & 3 & 1 & 0 \\ 0 & 1 & -1 & 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 1 \\ -4 \\ 3 \\ -1 \end{bmatrix}$$

$$\therefore\ y_L(n) = \{2, 5, 1, -4, 3, -1\}$$

2.8 LINEAR CONVOLUTION OF LONG SEQUENCES USING DFT

When we want to process or filter a long input sequence with an FIR filter in which the linear convolution is computed using the DFT, we experience some practical problem. We will have to compute the DFT of long input sequence, which is generally impractical. Furthermore, output samples are not available until all input samples are processed. This introduces an unacceptably large amount of delay. Therefore, we have to segment the long duration input sequence into small blocks, process each block using the DFT and finally combine the output sequence from the outputs of each block. This procedure is called block convolution operation. Two methods are commonly used for block convolution operation of long duration input sequence.

(i) Overlap - Save method

(ii) Overlap - Add method

These two methods are explained below.

2.8.1 Overlap – Save Method

Let the length of the input sequence $x(n)$ to a FIR filter is L_s and the length of the impulse response $h(n)$ of the filter is M.

In this method, the input sequence is divided into block of samples of size $N = L+M-1$

Each block consists of $(M-1)$ samples of previous block followed by L new sample of the input sequence to form the length $N=L+M-1$. For first block of samples the first $(M-1)$ samples are assumed to be zero. Thus the blocks of samples are,

I block : $x_1(n) = \{\underbrace{0, 0, 0, \ldots. 0}_{(M-1)\text{ zeros}}, x(0), x(1), x(2) \ldots. x(L-1)\}$

II block : $$x_2(n) = \{\underbrace{x(L-M+1), \ldots x(L-1)}_{\text{last (M-1) samples from } x_1(n)}, \underbrace{x(L), x(L+1), \ldots x(2L-1)}_{\text{L new samples}}\}$$

III block : $$x_3(n) = \{\underbrace{x(2L-M+1) \ldots x(2L-1)}_{\text{last (M-1) samples from } x_2(n)}, \underbrace{x(2L), x(2L+1) \ldots x(3L-1)}_{\text{L new samples}}\}$$

and so on.

Now the impulse response of the FIR filter is also increased to length N=L+M−1 by appending (L−1) zeros and the N point circular convolution of $x_i(n)$ with h(n) is calculated.

i.e., $y_i(n) = x_i(n) \; \circledN \; h(n)$

In the obtained $y_i(n)$, the first (M−1) samples will not agree with the linear convolution of $x_i(n)$ and h(n) because of aliasing whereas the remaining samples are identical to the linear convolution. Hence we discard the first (M−1) samples of $y_i(n)$. The remaining L samples then constitute the desired response y(n).

For example, let the length of the input sequence L_s=18 and is given by,

$$x(n) = \{x(0), x(1), x(2), \ldots\ldots\ldots\ldots x(17)\}$$

and the length of the impulse response of the FIR filter M=4.

Let us make the length of each block as N=L+M−1 = 7. Therefore, L=N−M+1 = 7−4+1 = 4.

Now the input sequence $x(n)$ can be divided into blocks of sequences as,

$$x_1(n) = \{\underbrace{0, 0, 0}_{M-1=3 \text{ zeros}}, x(0), x(1), x(2), x(3)\}$$

$$x_2(n) = \{\underbrace{x(1), x(2), x(3)}_{(M-1) = 3 \text{ samples from } x_1(n)}, \underbrace{x(4), x(5), x(6), x(7)}_{L = 4 \text{ new samples}}\}$$

$$x_3(n) = \{\underbrace{x(5), x(6), x(7)}_{(M-1)=3 \text{ samples from } x_2(n)}, \underbrace{x(8), x(9), x(10), x(11)}_{L= 4 \text{ new samples}}\}$$

Similarly, $x_4(n) = \{x(9), x(10), x(11), x(12), x(13), x(14), x(15)\}$

and $x_5(n) = \{x(13), x(14), x(15), x(16), x(17), 0, 0\}$ and so on.

Now we take N=L+M−1 = 7 point circular convolution of $x_i(n)$ and h(n) by appending L−1 = 3 zeros to the impulse response h(n) to get the output $y_i(n)$. In the outputs

$y_i(n)$, first M–1=3 samples are discarded. The remaining L samples are then constitute the desired output y(n).

i.e.,
$$y_1(n) = x_1(n) \textcircled{7} h(n) = \{\underbrace{y_1(0), y_1(1), y_1(2),}_{\text{discard}} \underbrace{y_1(3), y_1(4), y_1(5), y_1(6)}_{\text{consider}}\}$$

$$y_2(n) = x_2(n) \textcircled{7} h(n) = \{\underbrace{y_2(0), y_2(1), y_2(2),}_{\text{discard}} \underbrace{y_2(3), y_2(4), y_2(5), y_2(6)}_{\text{consider}}\}$$

$$y_3(n) = x_3(n) \textcircled{7} h(n) = \{\underbrace{y_3(0), y_3(1), y_3(2),}_{\text{discard}} \underbrace{y_3(3), y_3(4), y_3(5), y_3(6)}_{\text{consider}}\}$$

$$y_4(n) = x_4(n) \textcircled{7} h(n) = \{\underbrace{y_4(0), y_4(1), y_4(2),}_{\text{discard}} \underbrace{y_4(3), y_4(4), y_4(5), y_4(6)}_{\text{consider}}\}$$

$$y_5(n) = x_5(n) \textcircled{7} h(n) = \{\underbrace{y_5(0), y_5(1), y_5(2),}_{\text{discard}} \underbrace{y_5(3), y_5(4), y_5(5), y_5(6)}_{\text{consider}}\}$$

∴ The desired output is

$$y(n) = \{y_1(3), y_1(4), y_1(5), y_1(6), y_2(3), y_2(4), y_2(5), y_2(6), y_3(3), y_3(4), y_3(5), y_3(6), y_4(3), y_4(4), y_4(5), y_4(6), y_5(3), y_5(4), y_5(5), y_5(6)\}$$

Examples

Example2.82 Consider a FIR filter with impulse response

$$h(n) = \{3, 2, 1, 1\}$$

If the input is $x(n) = \{1, 2, 3, 3, 2, 1, -1, -2, -3, 5, 6, -1, 2, 0, 2, 1\}$, find the output. Use overlap-save method assuming the length of block is 9.

Solution:

$$x(n) = \{1, 2, 3, 3, 2, 1, -1, -2, -3, 5, 6, -1, 2, 0, 2, 1\} \qquad \therefore L_S = 16$$
$$h(n) = \{3, 2, 1, 1\} \qquad \therefore M = 4$$

The output y(n) of a FIR filter is given by performing the linear convolution of $x(n)$ and h(n).

Given that the length of the block N=L+M–1 = 9 $\quad \therefore$ L=6

The input sequence is divided into blocks as follows.

$$x_1(n) = \{0, 0, 0, 1, 2, 3, 3, 2, 1\}$$
$$x_2(n) = \{3, 2, 1, -1, -2, -3, 5, 6, -1\}$$
$$x_3(n) = \{5, 6, -1, 2, 0, 2, 1, 0, 0\}$$
$$x_4(n) = \{1, 0, 0, 0, 0, 0, 0, 0, 0\}$$

M+N−1 = L

4+N−1 = 9

Increase the length of h(n) to N=9 by appending L–1 = 5 zeros.

& $h(n) = \{3, 2, 1, 1, 0, 0, 0, 0, 0\}$

Now $y_1(n) = x_1(n) \text{ (9) } h(n)$

$$= \begin{bmatrix} 0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 & 3 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 & 3 & 3 & 2 & 1 \\ 1 & 0 & 0 & 0 & 1 & 2 & 3 & 3 & 2 \\ 2 & 1 & 0 & 0 & 0 & 1 & 2 & 3 & 3 \\ 3 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 3 \\ 3 & 3 & 2 & 1 & 0 & 0 & 0 & 1 & 2 \\ 2 & 3 & 3 & 2 & 1 & 0 & 0 & 0 & 1 \\ 1 & 2 & 3 & 3 & 2 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 1 \\ 3 \\ 8 \\ 14 \\ 18 \\ 17 \\ 13 \end{bmatrix}$$

$$\therefore y_1(n) = x_1(n) \text{ (9) } h(n)$$
$$= \{\underbrace{7, 3, 1,}_{\text{discard}} \underbrace{3, 8, 14, 18, 17, 13}_{\text{consider}}\}$$

Similarly, $y_2(n) = x_2(n) \text{ (9) } h(n)$
$$= \{\underbrace{18, 17, 9,}_{\text{discard}} \underbrace{4, -5, -13, 6, 23, 11}_{\text{consider}}\}$$

$$y_3(n) = x_3(n) \text{ (9) } h(n)$$
$$= \{\underbrace{16, 28, 14,}_{\text{discard}} \underbrace{15, 9, 7, 9, 4, 3}_{\text{consider}}\}$$

$$y_4(n) = x_4(n) \text{ (9) } h(n)$$
$$= \{\underbrace{3, 2, 1,}_{\text{discard}} \underbrace{1, 0, 0, 0, 0, 0}_{\text{consider}}\}$$

$\therefore$ The output the FIR filter

$$\therefore y(n) = x(n) * h(n)$$
$$= \{3, 8, 14, 18, 17, 13, 4, -5, -13, 6, 23, 11, 15, 9, 7, 9, 4, 3, 1\}$$

Example 2.83 Find the output y(n) of a filter whose impulse response $h(n) = \{1, 1, 1\}$ and input signal $x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, 1\}$ using overlap-save method.

Solution :

$$x(n) = \{3, -1, 0, 1, 3, 2, 0, 1, 2, 1\}$$ $\therefore L_S = 10$

$$h(n) = \{1, 1, 1\}$$ $M = 3$

Assume that the length of the block $N = L + M - 1 = 5$ $\therefore L = 3$

The input sequence is divided into blocks as follows.

$$x_1(n) = \{0, 0, 3, -1, 0\}$$
$$x_2(n) = \{-1, 0, 1, 3, 2\}$$
$$x_3(n) = \{3, 2, 0, 1, 2\}$$
$$x_4(n) = \{1, 2, 1, 0, 0\}$$

Increase the length of h(n) to N=5 by appending L–1 = 2 zeros.

$$\therefore\ h(n) = \{1, 1, 1, 0, 0\}$$

Now $y_1(n) = x_1(n)$ ⑤ $h(n)$

$$= \begin{bmatrix} 0 & 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & -1 & 3 \\ 3 & 0 & 0 & 0 & -1 \\ -1 & 3 & 0 & 0 & 0 \\ 0 & -1 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 3 \\ 2 \\ 2 \end{bmatrix}$$

$$y_1(n) = \{\underbrace{-1, 0,}_{\text{discard}} \underbrace{3, 2, 2}_{\text{consider}}\}$$

Similarly, $$y_2(n) = \{\underbrace{4, 1,}_{\text{discard}} \underbrace{0, 4, 6}_{\text{consider}}\}$$

$$y_3(n) = \{\underbrace{6, 7,}_{\text{discard}} \underbrace{5, 3, 3}_{\text{consider}}\}$$

$$y_4(n) = \{\underbrace{1, 3,}_{\text{discard}} \underbrace{4, 3, 1}_{\text{consider}}\}$$

∴ The output of the filter

$$y(n) = \{3, 2, 2, 0, 4, 6, 5, 3, 3, 4, 3, 1\}$$

2.8.2 Overlap – Add Method

Let the length of the input sequence $x(n)$ to a FIR filter is L_s and the length of the impulse response h(n) is M.

In this method, the input sequence is divided into blocks of sequences having length L and M–1 zeros are added to it to make the total length N=L+M–1.

Thus the blocks generated may be written as,

$$x_1(n) = \{x(0), x(1), x(2), \ldots\ldots\ x(L-1), \underbrace{0, 0, \ldots\ldots 0}_{(M-1)\text{ zeros appended}}\}$$

$$x_2(n) = \{x(L), x(L+1), x(L+2)\ \ldots\ldots\ x(2L-1), \underbrace{0, 0, \ldots\ldots 0}_{(M-1)\text{ zeros appended}}\}$$