(c) $x_3(n) = \alpha^n \qquad 0 \le n < N$

(d) $x_4(n) = u(n) - u(n - n_0)$, where $0 < n_0 < N$

(a) The DFT of the unit sample may be easily evaluated from the definition of the DFT:

$$X_1(k) = \sum_{n=0}^{N-1} \delta(n) W_N^{nk} = 1 \qquad k = 0, 1, \dots, N-1$$

Another approach, however, is to recall that the DFT corresponds to samples of the z-transform $X_1(z)$ at N equally spaced points around the unit circle. Because $X_1(z) = 1$, it follows that $X_1(k) = 1$.

(b) For the second sequence, we may again evaluate the DFT directly from the definition of the DFT. Let us instead, however, sample the z-transform. We know that $X_2(z) = z^{-n_0}$. Therefore, sampling $X_2(z)$ at the points $z = W_N^{-k}$ for $k = 0, 1, \dots, N-1$, we find

$$X_2(k) = W_N^{n_0 k} \qquad k = 0, 1, \dots, N-1$$

(c) For $x_3(n)$, the DFT may be found directly as follows:

$$\begin{aligned} X_3(k) &= \sum_{n=0}^{N-1} x_3(n) W_N^{nk} = \sum_{n=0}^{N-1} \alpha^n W_N^{nk} \\ &= \sum_{n=0}^{N-1} \left(\alpha W_N^k\right)^n = \frac{1 - \left(\alpha W_N^k\right)^N}{1 - \alpha W_N^k} \qquad k = 0, 1, \dots, N-1 \end{aligned}$$

(d) The DFT of the pulse, $x_4(n) = u(n) - u(n - n_0)$, may be evaluated directly as follows:

$$X_4(k) = \sum_{n=0}^{n_0 - 1} W_N^{nk} = \frac{1 - W_N^{kn_0}}{1 - W_N^k}$$

Factoring out a complex exponential $W_N^{kn_0/2}$ from the numerator and a complex exponential $W_N^{k/2}$ from the denominator, the DFT may be written as

$$X_4(k) = W_N^{k(n_0-1)/2} \frac{W_N^{-kn_0/2} - W_N^{kn_0/2}}{W_N^{-k/2} - W_N^{k/2}} = e^{-j\frac{2\pi k}{N}\left(\frac{n_0-1}{2}\right)} \frac{\sin(n_0 \pi k/N)}{\sin(\pi k/N)} \qquad k = 0, 1, \dots, N-1$$

6.6 Find the 10-point inverse DFT of

$$X(k) = \begin{cases} 3 & k = 0 \\ 1 & 1 \le k \le 9 \end{cases}$$

To find the inverse DFT, note that $X(k)$ may be expressed as follows:

$$X(k) = 1 + 2\delta(k) \qquad 0 \le k \le 9$$

Written in this way, the inverse DFT may be easily determined. Specifically, note that the inverse DFT of a constant is a unit sample:

$$x_1(n) = \delta(n) \overset{DFT}{\Longleftrightarrow} X_1(k) = 1$$

Similarly, the DFT of a constant is a unit sample:

$$x_2(n) = 1 \overset{DFT}{\Longleftrightarrow} X_2(k) = N\delta(k)$$

Therefore, it follows that

$$x(n) = \tfrac{1}{5} + \delta(n)$$

6.7 Find the N-point DFT of the sequence

$$x(n) = \cos(n\omega_0) \qquad 0 \le n \le N-1$$

Compare the values of the DFT coefficients $X(k)$ when $\omega_0 = 2\pi k_0/N$ to those when $\omega_0 \neq 2\pi k_0/N$. Explain the difference.

To find the N-point DFT of this sequence, it is easier if we write the cosine in terms of complex exponentials:

$$x(n) = \tfrac{1}{2}e^{jn\omega_0} + \tfrac{1}{2}e^{-jn\omega_0}$$

Evaluating the DFT of each of these terms, we find

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}nk} = \tfrac{1}{2}\sum_{n=0}^{N-1} e^{-jn(\frac{2\pi}{N}k-\omega_0)} + \tfrac{1}{2}\sum_{n=0}^{N-1} e^{-jn(\frac{2\pi}{N}k+\omega_0)} \tag{6.18}$$

At this point, note that if $\omega_0 = 2\pi k_0/N$,

$$X(k) = \tfrac{1}{2}\sum_{n=0}^{N-1} e^{-jn\frac{2\pi}{N}(k-k_0)} + \tfrac{1}{2}\sum_{n=0}^{N-1} e^{-jn\frac{2\pi}{N}(k+k_0)}$$

Because the first term is a sum of a complex exponential of frequency $\omega_0 = 2\pi(k-k_0)/N$, the sum will be equal to zero unless $k = k_0$, in which case the sum is equal to N. Similarly, the second sum is equal to zero unless $k = N - k_0$, in which case the sum is equal to N. Therefore, if $\omega_0 = 2\pi k_0/N$, the DFT coefficients are

$$X(k) = \begin{cases} \dfrac{N}{2} & k = k_0 \text{ and } k = N - k_0 \\ 0 & \text{otherwise} \end{cases}$$

In the general case, when $\omega_0 \neq 2\pi k_0/N$, we must use the geometric series to evaluate Eq. (6.18):

$$\begin{aligned} X(k) &= \tfrac{1}{2}\sum_{n=0}^{N-1} e^{-jn(\frac{2\pi}{N}k-\omega_0)} + \tfrac{1}{2}\sum_{n=0}^{N-1} e^{-jn(\frac{2\pi}{N}k+\omega_0)} \\ &= \tfrac{1}{2}\frac{1-e^{-jN(\frac{2\pi}{N}k-\omega_0)}}{1-e^{-j(\frac{2\pi}{N}k-\omega_0)}} + \tfrac{1}{2}\frac{1-e^{-jN(\frac{2\pi}{N}k+\omega_0)}}{1-e^{-j(\frac{2\pi}{N}k+\omega_0)}} \end{aligned}$$

Factoring out a complex exponential from the numerator and one from the denominator, we have

$$X(k) = \tfrac{1}{2}e^{-j(\frac{N-1}{2})(\frac{2\pi}{N}k-\omega_0)}\frac{\sin\left(\pi k - \frac{N\omega_0}{2}\right)}{\sin\left(\frac{\pi k}{N} - \frac{\omega_0}{2}\right)} + \tfrac{1}{2}e^{-j(\frac{N-1}{2})(\frac{2\pi}{N}k+\omega_0)}\frac{\sin\left(\pi k + \frac{N\omega_0}{2}\right)}{\sin\left(\frac{\pi k}{N} + \frac{\omega_0}{2}\right)}$$

Note that, unless ω_0 is an integer multiple of $2\pi/N$, $X(k)$ is, in general, nonzero for each k. The reason for this difference between these two cases comes from the fact that $X(k)$ corresponds to samples of the DTFT of $x(n)$, which is

$$\begin{aligned} X(e^{j\omega}) = \sum_{n=0}^{N-1} \cos(n\omega_0)e^{-jn\omega} &= \tfrac{1}{2}e^{-j(\frac{N-1}{2})(\omega-\omega_0)}\frac{\sin N(\omega-\omega_0)/2}{\sin(\omega-\omega_0)/2} \\ &+ \tfrac{1}{2}e^{-j(\frac{N-1}{2})(\omega+\omega_0)}\frac{\sin N(\omega+\omega_0)/2}{\sin(\omega+\omega_0)/2} \end{aligned}$$

When sampled at N equally spaced points over the interval $[0, 2\pi]$, the sample values will, in general, be nonzero. However, if $\omega_0 = 2\pi k_0/N$, all of the samples except those at $k = k_0$ and $k = N - k_0$ occur at the zeros of the sine function.

6.8 Find the N-point DFT of the sequence

$$x(n) = 4 + \cos^2\left(\frac{2\pi n}{N}\right) \qquad n = 0, 1, \ldots, N-1$$

The DFT of this sequence may be evaluated by expanding the cosine as a sum of complex exponentials:

$$x(n) = 4 + \tfrac{1}{4}\left[e^{j2\pi n/N} + e^{-j2\pi n/N}\right]^2 = 4 + \tfrac{1}{2} + \tfrac{1}{4}e^{j4\pi n/N} + \tfrac{1}{4}e^{-j4\pi n/N}$$

Using the periodicity of the complex exponentials, we may write $x(n)$ as follows:

$$x(n) = \tfrac{9}{2} + \tfrac{1}{4}e^{j\frac{2\pi}{N}(2n)} + \tfrac{1}{4}e^{j\frac{2\pi}{N}(N-2)n}$$

Therefore, the DFT coefficients are

$$X(k) = \begin{cases} \tfrac{9}{2}N & k = 0 \\ \tfrac{1}{4}N & k = 2 \text{ and } k = N-2 \\ 0 & \text{else} \end{cases}$$

6.9 Suppose that we are given a program to find the DFT of a complex-valued sequence $x(n)$. How can this program be used to find the inverse DFT of $X(k)$?

A program to find the DFT of a sequence $x(n)$ evaluates the sum

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} \tag{6.19}$$

and produces the sequence of DFT coefficients $X(k)$. What we would like to do is to use this program to find the inverse DFT of $X(k)$, which is

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk} \tag{6.20}$$

Note that the only difference between the forward and the inverse DFT is the factor of $1/N$ in the inverse DFT and the sign of the complex exponentials. Therefore, if we conjugate both sides of Eq. (*6.20*) and multiply by N, we have

$$N x^*(n) = \sum_{k=0}^{N-1} X^*(k) W_N^{nk}$$

Comparing this to Eq. (*6.19*), we see that the sum on the right is the DFT of the sequence $X^*(k)$. Thus, if $X^*(k)$ is used as the input in the DFT program, the output will be $Nx^*(n)$. Conjugating this output and dividing by N produces the sequence $x(n)$. Therefore, the procedure is as follows:

1. Conjugate the DFT coefficients $X(k)$ to produce the sequence $X^*(k)$.
2. Use the program to find the DFT of the sequence $X^*(k)$.
3. Conjugate the result obtained in step 2, and divide by N.

DFT Properties

6.10 Consider the finite-length sequence

$$x(n) = \delta(n) + 2\delta(n-5)$$

(*a*) Find the 10-point discrete Fourier transform of $x(n)$.

(*b*) Find the sequence that has a discrete Fourier transform

$$Y(k) = e^{j2k\frac{2\pi}{10}} X(k)$$

where $X(k)$ is the 10-point DFT of $x(n)$.

(*c*) Find the 10-point sequence $y(n)$ that has a discrete Fourier transform

$$Y(k) = X(k)W(k)$$

where $X(k)$ is the 10-point DFT of $x(n)$, and $W(k)$ is the 10-point DFT of the sequence

$$w(n) = \begin{cases} 1 & 0 \le n \le 6 \\ 0 & \text{otherwise} \end{cases}$$

(*a*) The DFT of $x(n)$ is easily seen to be

$$X(k) = 1 + 2W_N^{5k} = 1 + 2e^{-j\frac{2\pi}{10}5k} = 1 + 2(-1)^k$$

(*b*) Multiplying $X(k)$ by a complex exponential of the form $W_N^{kn_0}$ corresponds to a circular shift of $x(n)$ by n_0. In this case, because $n_0 = -2$, $x(n)$ is circularly shifted to the left by 2, and we have

$$y(n) = x((n+2))_{10} = 2\delta(n-3) + \delta(n-8)$$

(*c*) Multiplying $X(k)$ by $W(k)$ corresponds to the circular convolution of $x(n)$ with $w(n)$. To perform the circular convolution, we may find the linear convolution and alias the result. The linear convolution of $x(n)$ with $w(n)$ is

$$z(n) = x(n) * w(n) = [1, 1, 1, 1, 1, 3, 3, 2, 2, 2, 2, 2]$$

and the circular convolution is

$$y(n) = \left[\sum_{k=-\infty}^{\infty} z(n-10k)\right]\mathcal{R}_{10}(n)$$

Because $z(n)$ and $z(n+10)$ are the only two sequences in the sum that have nonzero values for $0 \le n < 10$, using a table to list the values of $z(n)$ and $z(n+10)$, and summing for $n = 0, 1, 2, \ldots, 9$, we have

n	0	1	2	3	4	5	6	7	8	9	10	11
$z(n)$	1	1	1	1	1	3	3	2	2	2	2	2
$z(n+10)$	2	2	0	0	0	0	0	0	0	0	0	0
$y(n)$	3	3	1	1	1	3	3	2	2	2	–	–

Thus, the 10-point circular convolution is

$$y(n) = [3, 3, 1, 1, 1, 3, 3, 2, 2, 2]$$

6.11 Consider the sequence

$$x(n) = 4\delta(n) + 3\delta(n-1) + 2\delta(n-2) + \delta(n-3)$$

Let $X(k)$ be the six-point DFT of $x(n)$.

(*a*) Find the finite-length sequence $y(n)$ that has a six-point DFT

$$Y(k) = W_6^{4k}X(k)$$

(*b*) Find the finite-length sequence $w(n)$ that has a six-point DFT that is equal to the real part of $X(k)$,

$$W(k) = \text{Re}\{X(k)\}$$

(*c*) Find the finite-length sequence $q(n)$ that has a three-point DFT

$$Q(k) = X(2k) \qquad k = 0, 1, 2$$

(*a*) The sequence $y(n)$ is formed by multiplying the DFT of $x(n)$ by the complex exponential W_6^{4k}. Because this corresponds to a circular shift of $x(n)$ by 4,

$$y(n) = x((n-4))_6$$

it follows that

$$y(n) = 4\delta(n-4) + 3\delta(n-5) + 2\delta(n) + \delta(n-1)$$

(*b*) The real part of $X(k)$ is

$$\mathrm{Re}\{X(k)\} = \tfrac{1}{2}[X(k) + X^*(k)]$$

To find the inverse DFT of $\mathrm{Re}\{X(k)\}$, we need to evaluate the inverse DFT of $X^*(k)$. Because

$$X^*(k) = \left[\sum_{n=0}^{N-1} x(n)W_N^{nk}\right]^* = \sum_{n=0}^{N-1} x^*(n)W_N^{-nk}$$

$$= \sum_{n=0}^{N-1} x^*(n)W_N^{(N-n)k} = \sum_{n=0}^{N-1} x^*((N-n))_N W_N^{nk}$$

$X^*(k)$ is the DFT of $x^*((N-n))_N$. Therefore, the inverse DFT of $\mathrm{Re}\{X(k)\}$ is

$$w(n) = \tfrac{1}{2}[x(n) + x^*((N-n))_N]$$

With $N = 6$, this becomes

$$w(n) = \left[4,\ \tfrac{3}{2},\ 1,\ 1,\ 1,\ \tfrac{3}{2}\right]$$

(*c*) The sequence $q(n)$ is of length three with a DFT $Q(k) = X(2k)$ for $k = 0, 1, 2$ where $X(k)$ is the six-point DFT of $x(n)$. Because the coefficients $X(k)$ are samples of $X(z)$ at six equally spaced points around the unit circle, $X(2k)$ for $k = 0, 1, 2$ corresponds to three equally spaced samples of $X(z)$ around the unit circle. Therefore,

$$q(n) = \left[\sum_{r=-\infty}^{\infty} x(n-3r)\right]\mathcal{R}_3(n)$$

With $x(n) = 0$ outside the interval $0 \le n \le 3$, it follows that

$$q(0) = x(0) + x(3) = 5$$
$$q(1) = x(1) = 3$$
$$q(2) = x(2) = 2$$

and we have

$$q(n) = 5\delta(n) + 3\delta(n-1) + 2\delta(n-2)$$

6.12 Consider the sequence

$$x(n) = \delta(n) + 2\delta(n-2) + \delta(n-3)$$

(*a*) Find the four-point DFT of $x(n)$.

(*b*) If $y(n)$ is the four-point circular convolution of $x(n)$ with itself, find $y(n)$ and the four-point DFT $Y(k)$.

(*c*) With $h(n) = \delta(n) + \delta(n-1) + 2\delta(n-3)$, find the four-point circular convolution of $x(n)$ with $h(n)$.

(*a*) The four-point DFT of $x(n)$ is

$$X(k) = \sum_{n=0}^{3} x(n)W_4^{nk} = 1 + 2W_4^{2k} + W_4^{3k}$$

(*b*) With $y(n) = x(n) \circledast x(n)$, it follows that $Y(k) = X^2(k)$:

$$Y(k) = \left(1 + 2W_4^{2k} + W_4^{3k}\right)\left(1 + 2W_4^{2k} + W_4^{3k}\right)$$
$$= 1 + 4W_4^{2k} + 2W_4^{3k} + 4W_4^{4k} + 4W_4^{5k} + W_4^{6k}$$

Because

$$W_4^{4k} = 1 \qquad W_4^{5k} = W_4^{k} \qquad W_4^{6k} = W_4^{2k}$$

the expression for $Y(k)$ may be simplified to

$$Y(k) = 5 + 4W_4^{k} + 5W_4^{2k} + 2W_4^{3k}$$

Therefore,

$$y(n) = 5\delta(n) + 4\delta(n-1) + 5\delta(n-2) + 2\delta(n-3)$$

(*c*) With $h(n) = \delta(n) + \delta(n-1) + 2\delta(n-3)$, the four-point circular convolution of $x(n)$ with $h(n)$ may be found using the tabular method. Because, the linear convolution of $x(n)$ with $h(n)$ is

$$y(n) = x(n) * h(n) = [1,\ 1,\ 2,\ 5,\ 1,\ 4,\ 2]$$

then

n	0	1	2	3	4	5	6	7	8
$y(n)$	1	1	2	5	1	4	2	0	0
$y(n+4)$	1	4	2	0	0	0	0	0	0
$z(n)$	2	5	4	5	–	–	–	–	–

or

$$z(n) = 2\delta(n) + 5\delta(n-1) + 4\delta(n-2) + 5\delta(n-3)$$

6.13 Let $x(n)$ be the sequence

$$x(n) = 2\delta(n) + \delta(n-1) + \delta(n-3)$$

The five-point DFT of $x(n)$ is computed and the resulting sequence is squared:

$$Y(k) = X^2(k)$$

A five-point inverse DFT is then computed to produce the sequence $y(n)$. Find the sequence $y(n)$.

The sequence $y(n)$ has a five-point DFT that is equal to the product $Y(k) = X(k)X(k)$. Therefore, $y(n)$ is the five-point circular convolution of $x(n)$ with itself:

$$y(n) = \left[\sum_{k=0}^{4} x(k)x((n-k))_5\right]\mathcal{R}_5(n)$$

A simple way to evaluate this circular convolution is to perform the linear convolution $y'(n) = x(n) * x(n)$ and alias the result:

$$y(n) = \left[\sum_{k=-\infty}^{\infty} y'(n-5k)\right]\mathcal{R}_5(n)$$

The linear convolution of $x(n)$ with itself is easily seen to be

$$y'(n) = [4, 4, 1, 4, 2, 0, 1]$$

Using the tabular method for computing the circular convolution, we have

n	0	1	2	3	4	5	6	7
$y'(n)$	4	4	1	4	2	0	1	0
$y'(n+5)$	0	1	0	0	0	0	0	0
$y(n)$	4	5	1	4	2	–	–	–

Therefore,

$$y(n) = 4\delta(n) + 5\delta(n-1) + \delta(n-2) + 4\delta(n-3) + 2\delta(n-4)$$

6.14 Consider the two sequences

$$x(n) = \delta(n) + 3\delta(n-1) + 3\delta(n-2) + 2\delta(n-3)$$
$$h(n) = \delta(n) + \delta(n-1) + \delta(n-2) + \delta(n-3)$$

If we form the product

$$Y(k) = X(k)H(k)$$

where $X(k)$ and $H(k)$ are the five-point DFTs of $x(n)$ and $h(n)$, respectively, and take the inverse DFT to form the sequence $y(n)$, find the sequence $y(n)$.

Because $Y(k)$ is the product of two 5-point DFTs, $H(k)$ and $X(k)$, $y(n)$ is the five-point circular convolution of $h(n)$ with $x(n)$. We may find $y(n)$ by performing the circular convolution analytically (or graphically) or by finding the linear convolution and aliasing the result or by multiplying DFTs and finding the inverse DFT. In this problem, because $h(n)$ is a simple sequence, we will use the analytic approach.

The five-point circular convolution of $x(n)$ with $h(n)$ is

$$y(n) = x(n) ⑤ h(n) = \sum_{k=0}^{4} h(k)x((n-k))_5 \qquad n = 0, 1, 2, 3, 4$$

Because $h(n) = 1$ for $n = 0, 1, 2, 3$, and $h(4) = 0$, the five-point convolution is

$$y(n) = x(n) ⑤ h(n) = \sum_{k=0}^{3} x((n-k))_5 \qquad n = 0, 1, 2, 3, 4$$

Therefore, the circular convolution is equal to the sum of the values of the circularly shifted sequence $x((n-k))_5$ from $k = 0$ to $k = 3$. Because $x(n)$ is

$$x(n) = [1,\ 3,\ 3,\ 2,\ 0]$$

(recall that $x(n)$ is considered to be a sequence of length five), $x((-n))_5$ is formed by reading the sequence values *backward*, beginning with $n = 0$:

$$x((-n))_5 = [1,\ 0,\ 2,\ 3,\ 3]$$

Thus, $y(0)$ is the sum of the first four values of $x((-n))_5$, which gives $y(0) = 6$. Circularly shifting this sequence to the right by 1, we have

$$x((1-n))_5 = [3,\ 1,\ 0,\ 2,\ 3]$$

and summing the first four values gives $y(1) = 6$. Continuing with this process, we find $y(2) = 7$, $y(3) = 9$, and $y(4) = 8$.

6.15 Let $x(n)$ and $h(n)$ be finite-length sequences that are six points long, and let $X(k)$ and $H(k)$ be the eight-point DFTs of $x(n)$ and $h(n)$, respectively. If we form the product

$$Y(k) = X(k)H(k)$$

and take the inverse DFT to form the sequence $y(n)$, find the values of n for which $y(n)$ is equal to the linear convolution

$$z(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

If the linear convolution of two sequences is M points long, for an N-point circular convolution with $N < M$, the first $M - N$ points will be aliased. With $x(n)$ and $h(n)$ both of length six, $z(n) = x(n) * h(n)$ will be 11 points long. Therefore, with an eight-point circular convolution, the first three points will be aliased, and the last five will be equal to the linear convolution.

6.16 If $Y(k) = H(k)X(k)$ where $H(k)$ and $X(k)$ are the N-point DFTs of the finite-length sequences $h(n)$ and $x(n)$, respectively, show that

$$y(n) = \left[\sum_{k=0}^{N-1} \tilde{h}(k)\tilde{x}(n-k)\right]\mathcal{R}_N(n)$$

The sequence that has an N-point DFT equal to $Y(k) = H(k)X(k)$ is

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)X(k)W_N^{-nk} \qquad n = 0, 1, \ldots, N-1$$

Because we would like to express $y(n)$ in terms of $x(n)$ and $h(n)$, let us substitute

$$H(k) = \sum_{l=0}^{N-1} h(l) W_N^{lk}$$

into the expression for $y(n)$ as follows:

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sum_{l=0}^{N-1} h(l) W_N^{lk} W_N^{-nk} \qquad n = 0, 1, \ldots, N-1$$

Interchanging the order of the summations gives

$$y(n) = \sum_{l=0}^{N-1} h(l) \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-k(n-l)} \right] \qquad n = 0, 1, \ldots, N-1$$

However, note that the term in brackets is equal to $x((n-l))_N$. Therefore, it follows that

$$y(n) = \sum_{l=0}^{N-1} h(l) x((n-l))_N \qquad n = 0, 1, \ldots, N-1$$

which is equivalent to

$$y(n) = \left[\sum_{k=0}^{N-1} \tilde{h}(k) \tilde{x}(n-k) \right] \mathcal{R}_N(n)$$

as was to be shown.

6.17 Let $y(n)$ be the linear convolution of the two finite-length sequences, $h(n)$ and $x(n)$, of length N,

$$y(n) = h(n) * x(n)$$

and let $y_N(n)$ be the N-point circular convolution

$$y_N(n) = h(n) \circledN x(n) = \left[\sum_{k=0}^{N-1} h(k) \tilde{x}(n-k) \right] \mathcal{R}_N(n)$$

Derive the following relationship between $y(n)$ and $y_N(n)$:

$$y_N(n) = \left[\sum_{k=-\infty}^{\infty} y(n+kN) \right] \mathcal{R}_N(n)$$

There are several ways to derive this relationship. One is to examine what happens when the DTFT of $y(n)$ is sampled. Alternatively, this result may be derived from a systems point of view as follows. First, note that $y_N(n)$ is equal to one period of the *linear* convolution of the finite-length sequence $h(n)$ with the periodic sequence $\tilde{x}(n)$:

$$y_N(n) = [h(n) * \tilde{x}(n)] \mathcal{R}_N(n)$$

If we let

$$p_N(n) = \sum_{k=-\infty}^{\infty} \delta(n-kN)$$

then the periodic sequence $\tilde{x}(n)$ is formed by linearly convolving $x(n)$ with $p_N(n)$:

$$\tilde{x}(n) = x(n) * p_N(n)$$

Therefore, the N-point circular convolution may be written as

$$y_N(n) = \{h(n) * [x(n) * p_N(n)]\} \mathcal{R}_N(n)$$