Ideal Solution / Nyquist Solution for Zero ISI

In baseband transmission of digital data, ISI is the major problem. ISI can be minimized by controlling pulse spectrum P(t) / P(f) of the overall system (including transmit filter, channel & receive filter).

P(t) → time domain, P(f) → frequency domain

In order to meet zero ISI, Nyquist criterion for distortion less baseband transmission must be fulfilled.

$P(t) = \text{sinc}(2B_0 t)$ → function which fulfills requirement of zero ISI

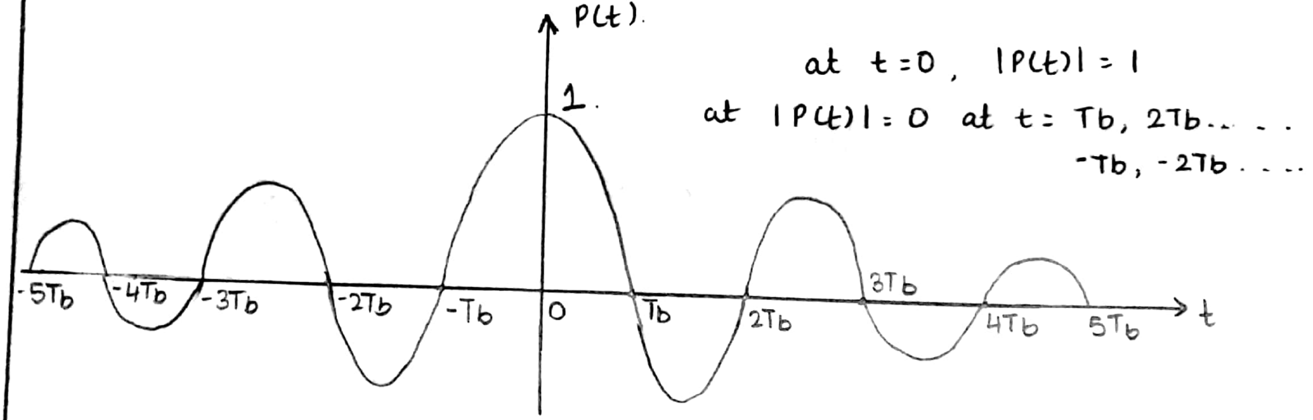
Through this func P(t), we get ideal solution.

$P(t) = \text{sinc}(2B_0 t)$ $B_0 = \frac{1}{2T_b}$ (Nyquist bandwidth)

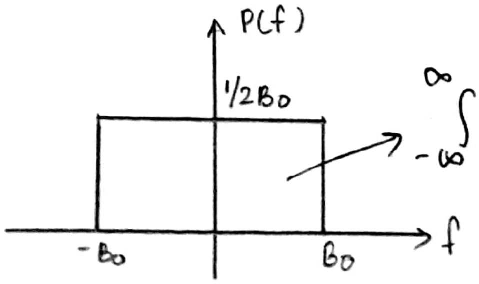
B_0 → min. B.W for zero ISI

$P(t) = \text{sinc}\left(\frac{t}{T_b}\right)$ T_b → bit duration.

P(t) is shown in fig (1). which characterise impulse responce of ideal Low Pass filter



Using Fourier Transform, $P(t) \xleftrightarrow{F.T} P(f)$ as shown in fig. rect pulse of magnitude $1/2B_0$ & width $2B_0$. Area under curve is unity



$$\int_{-\infty}^{\infty} P(f)\, df = 1.$$

P(f) mathematically given as

$$P(f) = \begin{cases} 1/2B_0, & |f| < B_0 \\ 0, & |f| > B_0 \end{cases}$$ → frequency response of ideal low pass filter.

Ideal Solution: we take $P(t) = \text{sinc}(2B_0 t)$

$B_0 = \frac{1}{2T_b}$.

as shown in fig(1). $|P(t)| = 0$ at $T_b, 2T_b \ldots$ / $-T_b, -2T_b \ldots$ } int multiple of T_b

∴ P(t) can be written as.

$$P(t - KT_b) = \text{sinc}\left[2B_0(t - KT_b)\right]$$ K - integral multiple

-ve sign → delay in time.

In baseband transmission of binary data, modified PAM signal = y(t) given as

$$y(t) = \mu \sum_{K=-\infty}^{\infty} a_K \, p(t - KT_b)$$

In order to get zero ISI, o/p y(t) is sampled at $t = 0, \pm T_b, \pm 2T_b \ldots$

To get ideal solution, put $p(t) = \text{sinc}(2B_0 t)$ in below eq.

$$y(t) = \mu \sum_{K=-\infty}^{\infty} a_K \, p(t - KT_b)$$

$p(t) = \text{sinc}(2B_0 t)$

$$y(t) = \mu \sum_{K=-\infty}^{\infty} a_K \, \text{sinc}\left\{2B_0(t - KT_b)\right\}$$

w.k.t $B_0 = \frac{1}{2T_b}$ or $2B_0 T_b = 1$

$$y(t) = \mu \sum_{K=-\infty}^{\infty} a_K \, \text{sinc}\left[2B_0 t - \underbrace{2B_0 T_b}_{=1} K\right]$$

Modified o/p is

$$y(t) = \mu \sum_{K=-\infty}^{\infty} a_K \, \text{sinc}\left[2B_0 t - K\right]$$

In ISI, there is inaccurate synchronisation of the clock in the receiver sampling circuit Here, we consider timing error(Δt) in place of (t).

O/P becomes

$$y(\Delta t) = \mu \sum_{K=-\infty}^{\infty} a_K \operatorname{sinc}[2B_0 \Delta t - K]$$

Using normalise sinc formula i.e $\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}$ we write

$$y(\Delta t) = \mu \sum_{K=-\infty}^{\infty} a_K \frac{\sin[2B_0 \pi \Delta t - \pi K]}{\pi[2B_0 \Delta t - K]}$$

We consider numerator in above expression, using $\sin(A-B)$.

we get

$$y(\Delta t) = \mu \sum_{K=-\infty}^{\infty} \frac{a_K}{\pi[2B_0 \Delta t - K]} [\sin(2B_0 \pi \Delta t) \underbrace{\cos(\pi K)}_{(-1)^K} - \cos(2B_0 \pi \Delta t) \underbrace{\sin \pi K}_{0}]$$

$\sin \pi K = 0$. $\to$ II term in num $= 0$.

replace $\cos \pi K = (-1)^K$ in I term.

$$y(\Delta t) = \mu \sum_{K=-\infty}^{\infty} \frac{a_K}{\pi[2B_0 \Delta t - K]} (-1)^K \sin(2B_0 \pi \Delta t)$$

Separate summation for $K=0$ & $K \neq 0$ we get.

$$y(\Delta t) = \underbrace{\frac{\mu a_0}{\pi[2B_0 \Delta t]} \sin(2B_0 \pi \Delta t)}_{\text{I term}} + \underbrace{\sum_{\substack{K=-\infty \\ K \neq 0}}^{\infty} \frac{\mu a_K}{\pi[2B_0 \Delta t - K]} (-1)^K \sin(2B_0 \pi \Delta t)}_{\text{II term}}$$

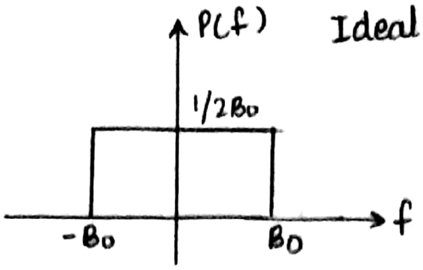
I term is desired symbol & can be written as

$\mu a_0 \operatorname{sinc}(2B_0 \Delta t)$

II term $\to$ ISI caused by timing error Δt. This term delay slowly at rate $\frac{1}{\Delta t}$ due to discontinuity of $P(f)$ at $\pm B_0$

Physical Realizability of Nyquist Channel :- (type of Low Pass channel) having ideal amplitude response P(f)

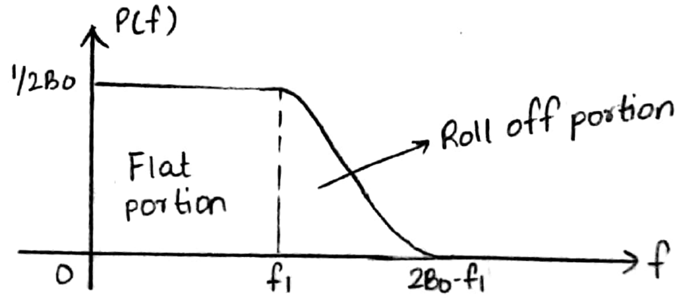
Ideal Amplitude Response



In freq response P(f), there is discontinuity at $\pm B_0$, so frequency response decreases towards zero abruptly, which is physically unrealisable.

To make it realisable with many desirable features, we make some modifications. The modified P(f) is called <u>Raised Cosine</u> - roll off Nyquist filter. which has roll off portion in addition to flat portion.

P(f) = flat portion + Roll-off portion as shown in fig.



Here the abrupt discontinuity is converted into roll-off portion from frequency f_1 to $2B_0 - f_1$. Flat portion has magnitude $1/2B_0$ from freq (0 to f_1). This characteristic is called raised cosine roll off filter. The roll-off portion introduce parameter $\underline{\alpha}$ called <u>roll-off factor</u>

Roll off factor, $\alpha = 1 - \dfrac{f_1}{B_0}$ → Relation b/w freq f_1 & Nyquist B.W B_0.

Mathematically raised cosine roll-off filter $P(f)$ is given by.

$$P(f) = \begin{cases} 1/2B_0 \dots & |f| < f_1 \\ \frac{1}{4B_0}\left[1 + \cos\left(\frac{\pi(|f| - f_1)}{2B_0 - 2f_1}\right)\right] \dots & f_1 \le |f| < 2B_0 - f_1 \\ 0 \dots & |f| \geq 2B_0 - f_1 \end{cases}$$

→ shows flat portion

shows roll off portion

In roll-off portion when $|f| = f_1$, $P(f) = 1/2B_0$

$|f| = 2B_0 - f_1$, $P(f) = 0$.

This characteristic is called raised cosine function, which is physically realisable having transmission B.W $B_T = 2B_0 - f_1$

Significance of roll-off factor

$\alpha = 1 - \frac{f_1}{B_0}$, when $\alpha = 0$.

$f_1 = B_0$ → represents ideal low pass amplitude responce with $B_T = B_0$ (minimum) Nyquist B.W.

when

$\alpha = 1$, $f_1 = 0$ ⇒ we get full cosine roll-off characteristic with $B_T = 2B_0$ (Twice of ideal).

Time Domain Analysis

Taking IFT of $P(f) = p(t) = \mathrm{sinc}(2B_0t) \frac{\cos(2\pi\alpha B_0 t)}{1 - 16\alpha^2 B_0^2 t^2}$

α - roll-off factor

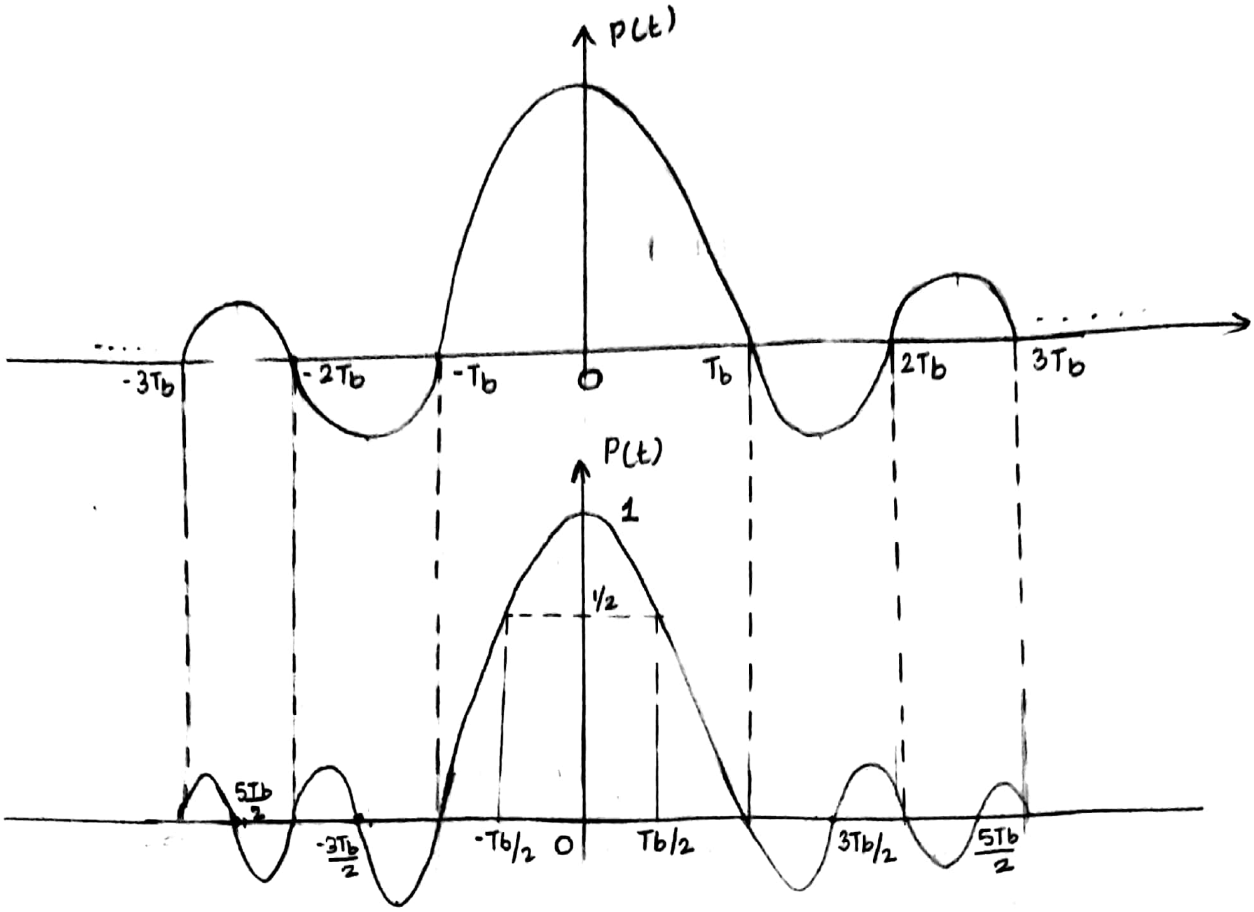
B_0 - Nyquist B.W

Time domain func $p(t)$ consists of product of 2 terms

1) → $\mathrm{sinc}(2B_0t)$ - associated with ideal solution which shows zero crossing of $p(t)$ at $t = \pm T_b, \pm 2T_b \dots$ for $(\alpha = 0)$.

at $t = \pm 1.5T_b, \pm 2.5T_b \dots$ for $\alpha = 1$

It is shown in fig. i.e Time Response of raised cosine roll-off Nyquist filter.



at $\alpha = 0$, $P(t)$ has zero crossing at $t = \pm T_b, \pm 2T_b \ldots$

at $t = 0$, $P(t) = 1$

at $\alpha = 1$, $t = \pm\frac{3T_b}{2}, \pm\frac{5T_b}{2}$ including $t = \pm T_b, \pm 2T_b$ etc. → zero crossings

$t = 0$, $p(t) = 1$.

* at $t = \pm\frac{T_b}{2}$ or $\pm\frac{1}{4B_0}$, $P(t) = 0.5$.

On comparison, for $\alpha = 1$, side lobe height is decreased as compared to $\alpha = 0$, which makes physical realisability of Nyquist channel for zero ISI

2) $1/t^2$ → factor reduces for large value at 't';

This factor reduce tail of sinc function (side lobe func) which reduces sampling errors. compared to ideal soln

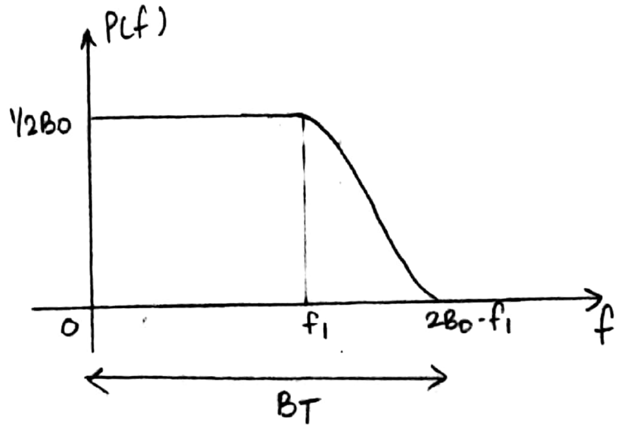
When $\alpha = 1$ i.e $1 = 1 - \frac{f_1}{B_0}$ or $f_1 = 0$ & $P(f)$ given as

$$P(f) = \begin{cases} \frac{1}{4B_0}\left[1+\cos\left(\frac{\pi|f|}{2B_0}\right)\right], & 0 < |f| < 2B_0 \\ 0, & |f| > 2B_0 \end{cases}$$

After taking IFT of $P(f)$ we get $p(t)$

$$p(t) = \frac{\text{sinc}(4B_0 t)}{1 - 16B_0^2 t^2}$$ → which is called raised cosine roll-off filter.

Transmission B.W Requirement for Raised Cosine Filter.



Here transmission B.W

$B_T = 2B_0 - f_1$

$B_0 = \frac{1}{2T_b}$ ··· Nyquist B.W.

Roll of portion from freq (f_1 to $2B_0 - f_1$) depend on roll-off factor (α) $\alpha = 1 - \frac{f_1}{B_0}$

we can write

$f_1 = B_0(1-\alpha)$

put f_1 in B_T eq.

$B_T = 2B_0 - B_0(1-\alpha)$

$B_T = B_0(1+\alpha)$ → in terms of Nyquist B.W & roll off factor

when $\alpha = 0$, $B_T = B_0$ (Nyquist B.W)

$\alpha = 1$, $B_T = 2B_0$ (Twice of ideal soln)

We consider $B_T = B_0 (1+\alpha)$

$$B_T = B_0 + \alpha B_0$$

$B_0 \rightarrow$ Nyquist B.W

$\alpha B_0 \rightarrow$ * Excess BW that the transmission BW requirement of raised cosine spectrum

On basis of physical realisability, there is need of excess BW (αB_0) which make raised cosine spectrum.

Ratio of $\frac{\text{excess BW}}{\text{Nyquist BW}} = \frac{\alpha B_0}{B_0} = \alpha$ (Roll off factor)

$\alpha \rightarrow$ also called excess BW factor

Summary :-

IDEAL CASE

When Roll-off factor $\alpha = 0$

Excess BW $(\alpha B_0) = 0$

$$B_T = B_0 = \frac{1}{2T_b}$$

- minimum possible value

PRACTICAL CASE

When $\alpha = 1$

$\alpha B_0 = B_0$ (excess BW)

$B_T = 2B_0 = \frac{1}{T_b}$ (double the ideal).

Note :- $\alpha = 1$ provides basis for synchronizing the receiver & transmitter