Gram Schmidt Orthogonalization Procedure :-

Consider a set of M energy signals $x_1(t)$, $x_2(t), x_3(t) \ldots \ldots x_M(t)$.

ex:- DPSK modulation M=2

for bit 1 -> $x_1(t)$ is transmitted

bit 0 -> $x_2(t)$ is transmitted

In general we will have M array modulation scheme.

To find no. of basis function required to represent all M signals. ∴ We use Gram Schmidt Orthogonalization Scheme to find out no. of orthonormal basis functions

The first basis function is found as follows :-

First find energy of 1st signal as

$$E_1 = \int_0^T x_1^2(t)\,dt$$

T- symbol duration

$x_1(t)$-real valued signed

1st orthonormal basis function is given as

$$\phi_1(t) = \frac{x_1(t)}{\sqrt{E_1}}$$

orthonormal means - to have unit energy.

In order to have unit energy we normalise $x_1(t)$ by ÷ by $\sqrt{E_1}$

To find other orthonormal basis functions, we go for intermediate signal $g_i(t)$.

$$g_i(t) = x_i(t) - \sum_{j=1}^{i-1} x_{ij}\,\phi_j(t)$$

i -> index of possible transmitted wave form

i = 1, 2 . . M.

always $N \leq M$.

j -> index of basis function

j = 1, 2 . . . N -> no. of basis functions

where,

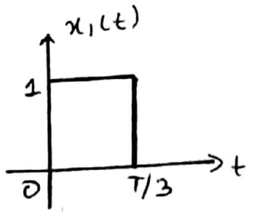
$$x_{ij} = \int_0^T x_i(t)\,\phi_j(t)\cdot dt \quad j = 1, 2 \ldots i-1$$

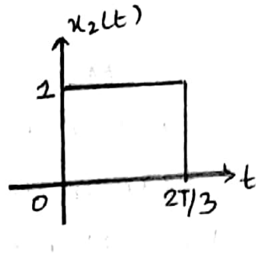
-> projection of $x_i(t)$ on $\phi_j(t)$

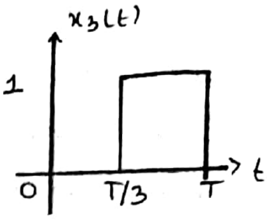
From given $g_i(t)$ {intermediate signal}, a new set of basis functions are defined as

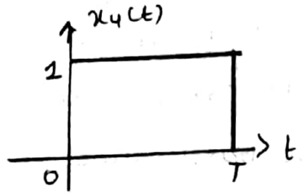
$$\phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)\,dt}} \quad i = 2, 3 \ldots N$$

Example :- Using Gram-Schmidt Orthogonalisation Procedure, find orthonormal basis functions for set of given signals







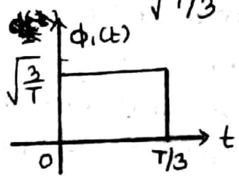


Given :- $M = 4$, $N = ?$ $\phi_1(t) = ?$ $\phi_2 = ?$ ...

Max value of $N = M$

$$\phi_1(t) = \frac{x_1(t)}{\sqrt{E_1}}, \quad E_1 = \int_0^{T/3} 1^2 \cdot dt = T/3$$

$$\phi_1(t) = \frac{x_1(t)}{\sqrt{T/3}} = \sqrt{\frac{3}{T}}\, x_1(t) \qquad 0 \le t \le T/3.$$



Intermediate signal :-

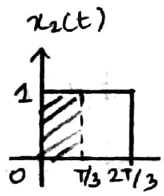
$$g_i(t) = x_i(t) - \sum_{j=1}^{i-1} x_{ij}\, \phi_j(t)$$

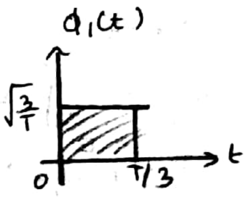
Sub. $i = 2$

$$g_2(t) = x_2(t) - \sum_{j=1}^{1} x_{2j}\, \phi_j(t)$$

Substitute $j = 1$

$$g_2(t) = x_2(t) - x_{21}\, \phi_1(t)$$



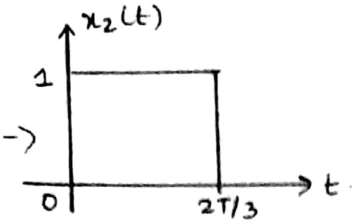


$$x_{21} = \int_0^T x_2(t)\, \phi_1(t) \cdot dt$$

$$\Rightarrow \int_0^{T/3} (1) \left(\sqrt{\frac{3}{T}}\right) \cdot dt$$

$$\Rightarrow \sqrt{\frac{3}{T}} \cdot \frac{T}{3} \Rightarrow \sqrt{\frac{T}{3}}$$

I →



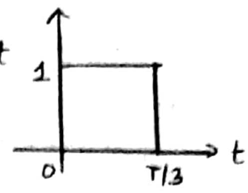
$$g_2(t) = x_2(t) - x_{21}\,\phi_1(t)$$

$$x_{21} = \sqrt{\frac{T}{3}}$$

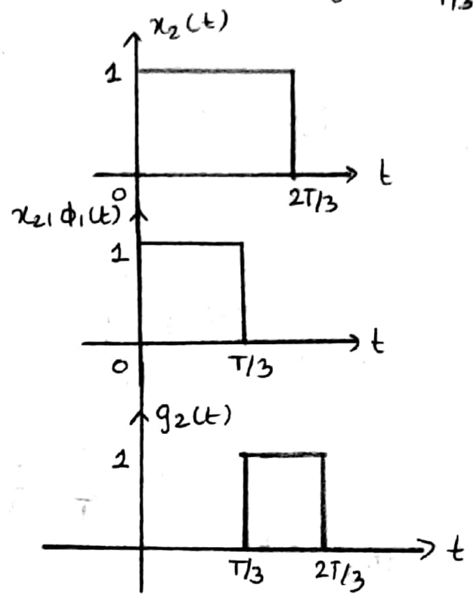
$$x_{21}\,\phi_1(t) = \sqrt{\frac{T}{3}} \times \sqrt{\frac{3}{T}} = 1$$

resultant ⟹

II →



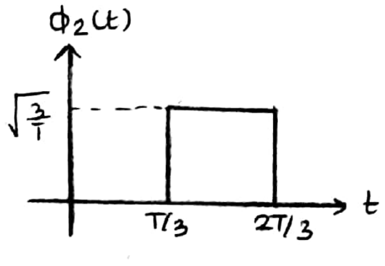
Subtract I & II



Energy of $g_2(t)$

$$= \int_0^T g_2^2(t)\cdot dt = \int_{T/3}^{2T/3} 1^2 \cdot dt = \frac{T}{3}$$

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{T/3}} = \sqrt{\frac{3}{T}}\, g_2(t), \quad T/3 \le t \le 2T/3$$



$$\boxed{\int \phi_2^2(t)\cdot dt = \frac{3}{T}\cdot\frac{T}{3} = 1}$$

↳ orthogonal.

$$\boxed{\int_0^T \phi_1(t)\,\phi_2(t)\cdot dt = 0}$$

To find 3^{rd} basis function :-

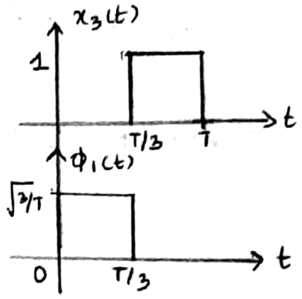
$$g_i(t) = x_i(t) - \sum_{j=1}^{i-1} x_{ij}\, \phi_j(t)$$

Sub $i = 3$.

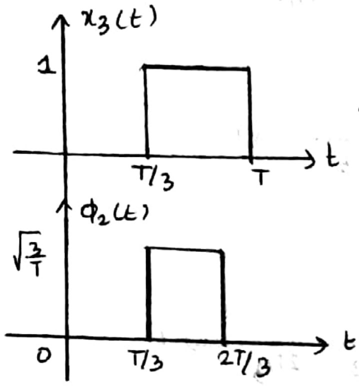
$$g_3(t) = x_3(t) - \sum_{j=1}^{2} x_{3j}\, \phi_j(t).$$

$$g_3(t) = x_3(t) - x_{31}\phi_1(t) - x_{32}\phi_2(t).$$

$$x_{31} = \int_0^T x_3(t)\,\phi_1(t) \cdot dt = 0.$$



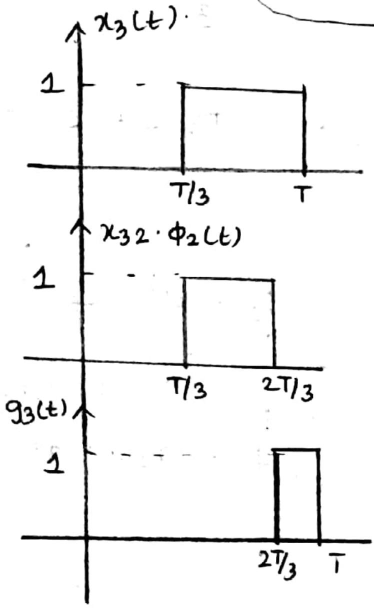
$$x_{32} = \int_0^T x_3(t)\,\phi_2(t) \cdot dt$$



$$x_{32} \Rightarrow \int_{T/3}^{2T/3} (1)\left(\sqrt{\frac{3}{T}}\right) \cdot dt$$

$$\Rightarrow \sqrt{\frac{3}{T}} \cdot \frac{T}{3} = \sqrt{\frac{T}{3}}.$$

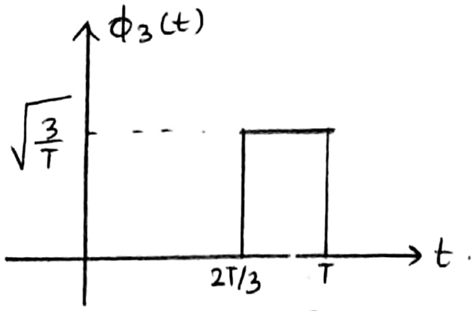
$$g_3(t) = x_3(t) - \sqrt{\frac{T}{3}}\,\phi_2(t). \rightarrow \sqrt{\frac{T}{3}} \cdot \sqrt{\frac{3}{T}} = 1 \times \phi_2(t)$$



$$\text{Energy of } g_3(t) \Rightarrow \int_{2T/3}^{T} g_3^2(t)\,dt = \int_{2T/3}^{T} 1 \cdot dt$$

$$\Rightarrow T - \frac{2T}{3} = \frac{T}{3}$$

$$\phi_3(t) = \frac{g_3(t)}{\sqrt{T/3}} \Rightarrow \sqrt{\frac{3}{T}}\, g_3(t). \qquad \frac{2T}{3} \le t \le T$$



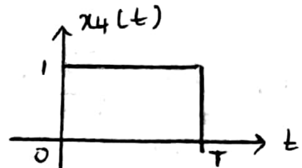
$$g_i(t) = x_i(t) - \sum_{j=1}^{i-1} x_{ij}(t)\, \phi_j(t)$$

Sub $i = 4$.

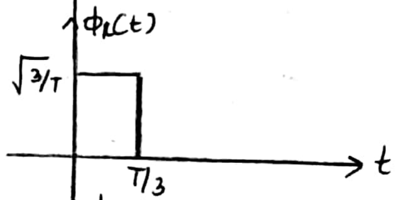
$$g_4(t) = x_4(t) - \sum_{j=1}^{4} x_{4j}(t)\, \phi_j\, t$$

$$\Rightarrow x_4(t) - x_{41}\,\phi_1(t) - x_{42}\,\phi_2(t) - x_{43}\,\phi_3(t).$$

$$x_{41} = \int_0^T x_4(t)\,\phi_1(t)\cdot dt$$

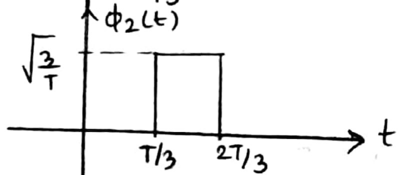


$$\Rightarrow \int_0^{T/3} (1)\left(\sqrt{\frac{3}{T}}\right)\cdot dt$$

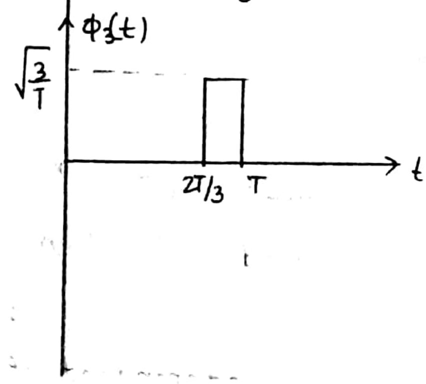


$$x_{41} \Rightarrow \sqrt{\frac{3}{T}}\cdot\frac{T}{3} = \sqrt{\frac{T}{3}}$$

$$x_{42} = \int_0^T x_4(t)\,\phi_2(t)\cdot dt$$



$$\Rightarrow \int_{T/3}^{2T/3} (1)\sqrt{\frac{3}{T}}\cdot dt$$



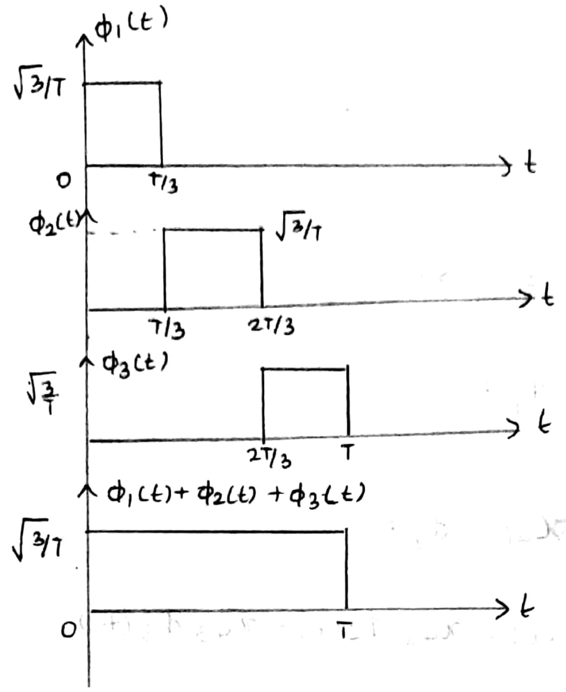
$$= \sqrt{\frac{3}{T}}\left(\frac{2T}{3} - \frac{T}{3}\right)$$

$$x_{42} \Rightarrow \sqrt{T/3}$$

$$x_{43} = \int_0^T x_4(t)\,\phi_3(t)\,dt$$

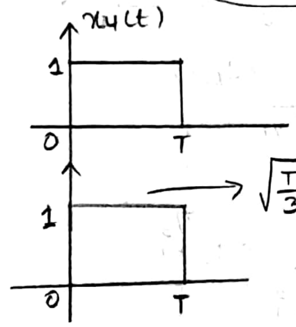
$$\Rightarrow \int_{2T/3}^{T} (1)\cdot\sqrt{\frac{3}{T}}\cdot dt = \sqrt{\frac{3}{T}}\left(\frac{2T}{3} - \frac{T}{3}\right) = \sqrt{\frac{T}{3}}$$

$$g_4(t) = x_4(t) - \sqrt{\frac{T}{3}}\left\{\phi_1(t) + \phi_2(t) + \phi_3(t)\right\}$$



$$\Rightarrow x_4(t) - \sqrt{\frac{T}{3}} \times \sqrt{\frac{3}{T}} \Rightarrow 1$$

$\Rightarrow$



$\rightarrow \sqrt{\frac{T}{3}}\{\phi_1 + \phi_2 + \phi_3(t)\}$

$g_4(t) = 0.$

$\phi_4(t) = 0.$

NOTE :-

* If signals $x_1(t)$, $x_2(t)$ .... $x_M(t)$ form linearly independent set, then $N = M$.
* If signals $x_1(t)$, $x_2(t)$ ... $x_M(t)$ are not linearly independent set then $N < M$.

Here in the example, $x_1(t) + x_3(t) = x_4(t)$

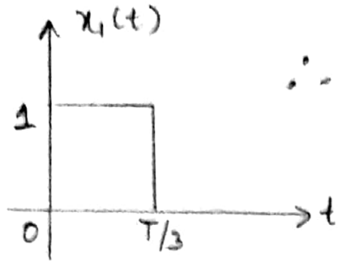
$\therefore$ No of orthonormal func = 3 (No of independent signals).

3 $\rightarrow$ linearly independent signals

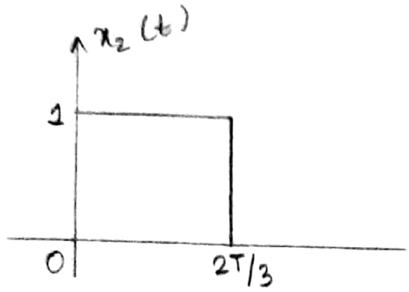
4th signal is linearly dependent (i.e $x_4(t) = x_1(t) + x_3(t)$)

To represent $x_1(t)$, $x_2(t)$, $x_3(t)$, $x_4(t)$ in terms of orthonormal func

w.r.t $\phi_1(t) = x_1(t)/\sqrt{T/3}$



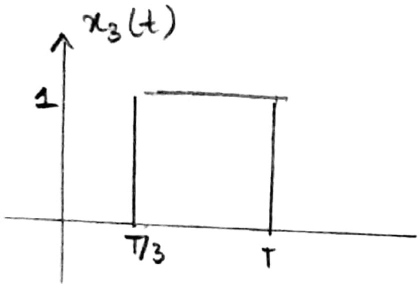
$$\therefore x_1(t) = \sqrt{\frac{T}{3}} \cdot \phi_1(t).$$



$$x_2(t) = \sqrt{\frac{T}{3}}\, \phi_1(t) + \sqrt{\frac{T}{3}}\, \phi_2(t)$$

$\phi_1(t)$ varies from $0 \le t \le T/3$

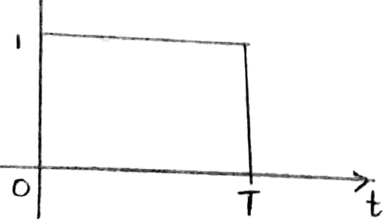
$\phi_2(t)$ varies from $T/3 \le t \le \frac{2T}{3}$



$$x_3(t) = \underbrace{\sqrt{\frac{T}{3}}}_{\text{scaling factor}}\, \phi_2(t) + \sqrt{\frac{T}{3}}\, \phi_3(t)$$

$\phi_2(t)$ varies from $T/3 \le t \le 2T/3$

$\phi_3(t)$ varies from $\frac{2T}{3} \le t \le T$



$$x_4(t) = x_1(t) + x_3(t)$$

$$= \sqrt{\frac{T}{3}} \left\{ \phi_1(t) + \phi_2(t) + \phi_3(t) \right\}$$