Example 5.1

Compute 4-point DFT and 8-point DFT of causal three sample sequence given by,

$$x(n) = \frac{1}{3} \quad ; \quad 0 \le n \le 2$$
$$= 0 \quad ; \quad \text{else}$$

Show that DFT coefficients are samples of Fourier transform of x(n), (Refer example 4.6 of Chapter 4 for Fourier transform).

Solution

By the definition of N-point DFT, the k^{th} complex coefficient of X(k), for $0 \le k \le N-1$, is given by,

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{\frac{-j2\pi kn}{N}}$$

a) **4-point DFT ($\therefore$ N = 4)**

$$X(k) = \sum_{n=0}^{4-1} x(n)\, e^{\frac{-j2\pi kn}{4}} = \sum_{n=0}^{2} x(n)\, e^{\frac{-j\pi kn}{2}} = x(0)\,e^{0} + x(1)\,e^{\frac{-j\pi k}{2}} + x(2)\,e^{-j\pi k}$$

$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$

$$= \frac{1}{3} + \frac{1}{3}e^{\frac{-j\pi k}{2}} + \frac{1}{3}e^{-j\pi k} = \frac{1}{3}\left[1 + \cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2} + \cos\pi k - j\sin\pi k\right]$$

For 4-point DFT, X(k) has to be evaluated for k = 0, 1, 2, 3.

When k = 0 ; $X(0) = \frac{1}{3}[1 + \cos 0 - j\sin 0 + \cos 0 - j\sin 0]$

$$= \frac{1}{3}(1 + 1 - j0 + 1 - j0) = 1 = 1\angle 0$$

When k = 1 ; $X(1) = \frac{1}{3}\left[1 + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2} + \cos\pi - j\sin\pi\right]$

$$= \frac{1}{3}(1 + 0 - j - 1 - j0) = -j\frac{1}{3} = \frac{1}{3}\angle -\pi/2 = 0.333\angle -0.5\pi$$

When k = 2 ; $X(2) = \frac{1}{3}[1 + \cos\pi - j\sin\pi + \cos 2\pi - j\sin 2\pi]$

$$= \frac{1}{3}(1 - 1 - j0 + 1 - j0) = \frac{1}{3} = 0.333\angle 0$$

When k = 3 ; $X(3) = \frac{1}{3}\left[1 + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\sin 3\pi\right]$

$$= \frac{1}{3}(1 + 0 + j - 1 - j0) = j\frac{1}{3} = \frac{1}{3}\angle \pi/2 = 0.333\angle 0.5\pi$$

$\therefore$ The 4-point DFT sequence X(k) is given by,

$$X(k) = \{\, 1\angle 0,\quad 0.333\angle -0.5\pi,\quad 0.333\angle 0,\quad 0.333\angle 0.5\pi \,\}$$

Phase angles are in radians.

$\therefore$ Magnitude Function, $|X(k)| = \{\, 1,\ 0.333,\ 0.333,\ 0.333 \,\}$

Phase Function, $\angle X(k) = \{\, 0,\ -0.5\pi,\ 0,\ 0.5\pi \,\}$

b) 8-point DFT ($\therefore N = 8$)

$$X(k) = \sum_{n=0}^{8-1} x(n)\, e^{\frac{-j2\pi kn}{8}} = \sum_{n=0}^{2} x(n)\, e^{\frac{-j\pi kn}{4}} = x(0)\, e^0 + x(1)\, e^{\frac{-j\pi k}{4}} + x(2)\, e^{\frac{-j\pi k}{2}}$$

$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$

$$= \frac{1}{3} + \frac{1}{3} e^{\frac{-j\pi k}{4}} + \frac{1}{3} e^{\frac{-j\pi k}{2}} = \frac{1}{3}\left[1 + \cos\frac{\pi k}{4} - j\sin\frac{\pi k}{4} + \cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2}\right]$$

For 8-point DFT, X(k) has to be evaluated for k = 0, 1, 2, 3, 4, 5, 6, 7.

When k = 0 ; $X(0) = \frac{1}{3}[1 + \cos 0 - j\sin 0 + \cos 0 - j\sin 0]$

$= \frac{1}{3}(1 + 1 - j0 + 1 - j0) = 1 = 1\angle 0$

When k = 1 ; $X(1) = \frac{1}{3}\left[1 + \cos\frac{\pi}{4} - j\sin\frac{\pi}{4} + \cos\frac{\pi}{2} - j\sin\frac{\pi}{2}\right]$

$= 0.333\,(1 + 0.707 - j0.707 + 0 - j1)$

$= 0.568 - j0.568 = 0.803\angle -0.785 = 0.803\angle -0.25\pi$

$\boxed{\frac{0.785}{\pi} \times \pi = 0.25\pi}$

When k = 2 ; $X(2) = \frac{1}{3}\left[1 + \cos\frac{2\pi}{4} - j\sin\frac{2\pi}{4} + \cos\frac{2\pi}{2} - j\sin\frac{2\pi}{2}\right]$

$= 0.333\,(1 + 0 - j1 - 1 - j0)$

$= -j0.333 = 0.333\angle -\pi/2 = 0.333\angle -0.5\pi$

When k = 3 ; $X(3) = \frac{1}{3}\left[1 + \cos\frac{3\pi}{4} - j\sin\frac{3\pi}{4} + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}\right]$

$= 0.333\,(1 - 0.707 - j0.707 + 0 + j1)$

$= 0.098 + j0.098 = 0.139\angle 0.785 = 0.139\angle 0.25\pi$

When k = 4 ; $X(4) = \frac{1}{3}\left[1 + \cos\frac{4\pi}{4} - j\sin\frac{4\pi}{4} + \cos\frac{4\pi}{2} - j\sin\frac{4\pi}{2}\right]$

$= 0.333\,(1 - 1 - j0 + 1 - j0) = 0.333 = 0.333\angle 0$

When k = 5 ; $X(5) = \frac{1}{3}\left[1 + \cos\frac{5\pi}{4} - j\sin\frac{5\pi}{4} + \cos\frac{5\pi}{2} - j\sin\frac{5\pi}{2}\right]$

$= 0.333\,(1 - 0.707 + j0.707 + 0 - j1)$

$= 0.098 - j0.098 = 0.139\angle -0.785 = 0.139\angle -0.25\pi$

When k = 6 ; $X(6) = \frac{1}{3}\left[1 + \cos\frac{6\pi}{4} - j\sin\frac{6\pi}{4} + \cos\frac{6\pi}{2} - j\sin\frac{6\pi}{2}\right]$

$= 0.333\,(1 + 0 + j1 - 1 - j0)$

$= j0.333 = 0.333\angle \pi/2 = 0.333\angle 0.5\pi$

When k = 7 ; $X(7) = \frac{1}{3}\left[1 + \cos\frac{7\pi}{4} - j\sin\frac{7\pi}{4} + \cos\frac{7\pi}{2} - j\sin\frac{7\pi}{2}\right]$

$= 0.333\,(1 + 0.707 + j0.707 + 0 + j1)$

$= 0.568 + j0.568 = 0.803\angle 0.785 = 0.803\angle 0.25\pi$

Phase angles are in radians.

$\therefore$ The 8-point DFT sequence X(k) is given by,

$X(k) = \{1\angle 0,\ 0.803\angle -0.25\pi,\ 0.333\angle -0.5\pi,\ 0.139\angle 0.25\pi,\ 0.333\angle 0,\ 0.139\angle -0.25\pi,$
$0.333\angle 0.5\pi,\ 0.803\angle 0.25\pi\}$

$\therefore$ Magnitude Function, $|X(k)| = \{1,\ 0.803,\ 0.333,\ 0.139,\ 0.333,\ 0.139,\ 0.333,\ 0.803\}$

Phase Function, $\angle X(k) = \{0,\ -0.25\pi,\ -0.5\pi,\ 0.25\pi,\ 0,\ -0.25\pi,\ 0.5\pi,\ 0.25\pi\}$

The magnitude spectrum of X(k) are shown in fig 1, 2 and 3 for N = 4, N = 8, and N = 16 respectively. The curve shown in dotted line is the sketch of magnitude function of $X(e^{j\omega})$ for ω in the range 0 to 2π. Here it is observed that the magnitude of DFT coefficients are samples of magnitude function of $X(e^{j\omega})$, (Refer example 4.6 for the magnitude function of $X(e^{j\omega})$).

The phase spectrum of X(k) are shown in fig 4, 5 and 6 for N = 4, N = 8, and N = 16 respectively. The curve shown in dotted line is the sketch of phase function of $X(e^{j\omega})$ for ω in the range 0 to 2π. Here it is observed that the phase of the DFT coefficients are samples of phase function of $X(e^{j\omega})$, (Refer example 4.6 for the phase function of $X(e^{j\omega})$).

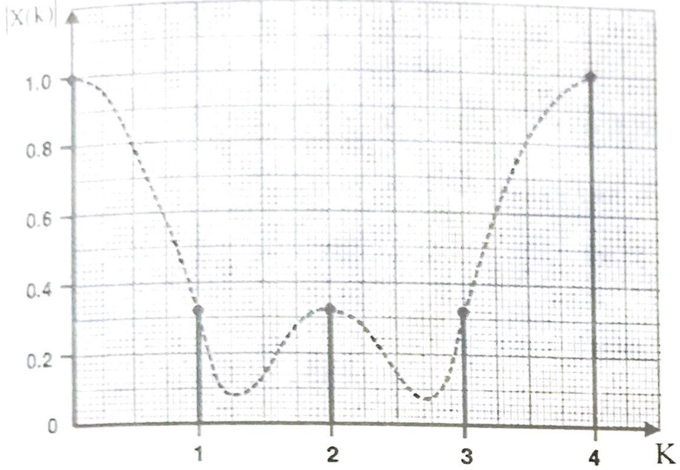

Fig 1 : Magnitude spectrum of X(k) for N=4.

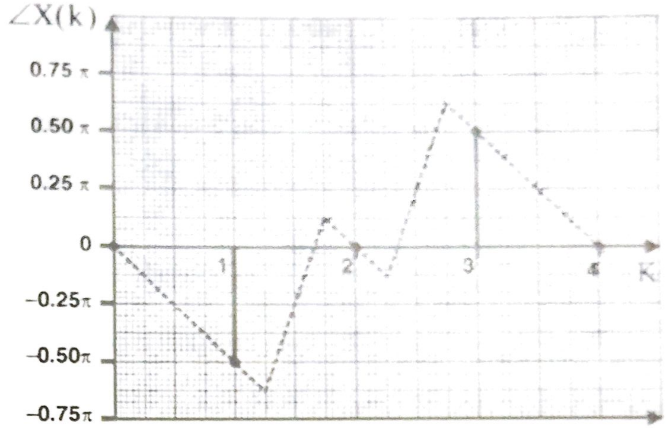

Fig 4 : Phase spectrum of X(k) for N=4

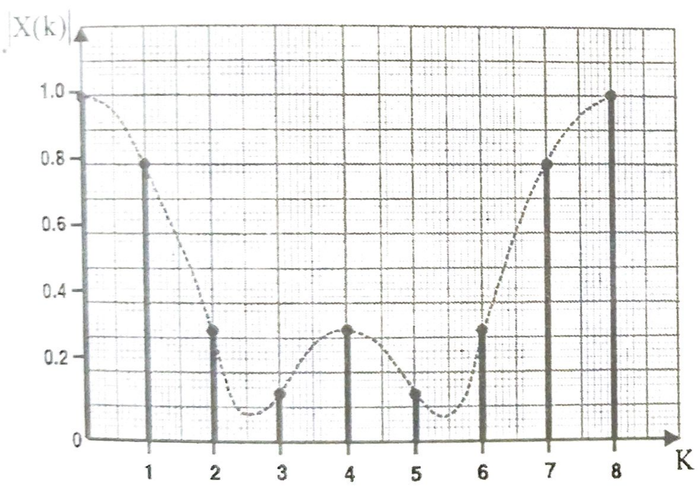

Fig 2 : Magnitude spectrum of X(k) for N=8.

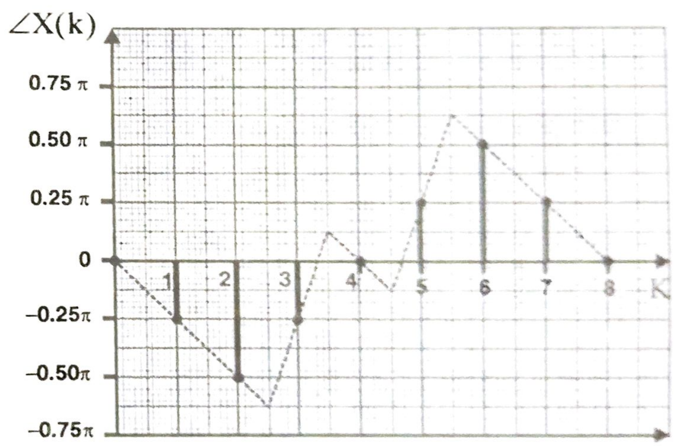

Fig 5 : Phase spectrum of X(k) for N=8.

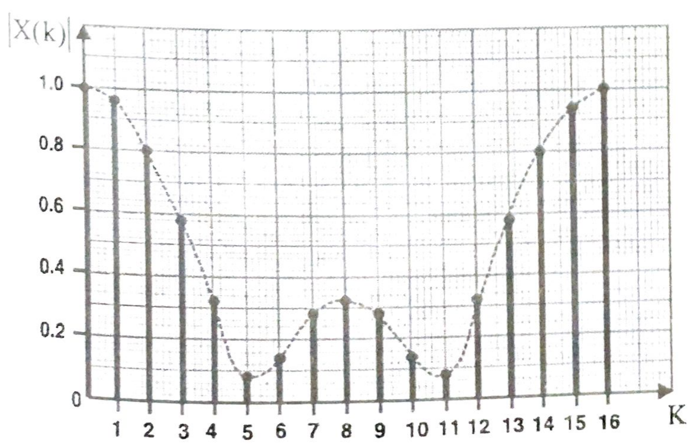

Fig 3 : Magnitude spectrum of X(k) for N=16.

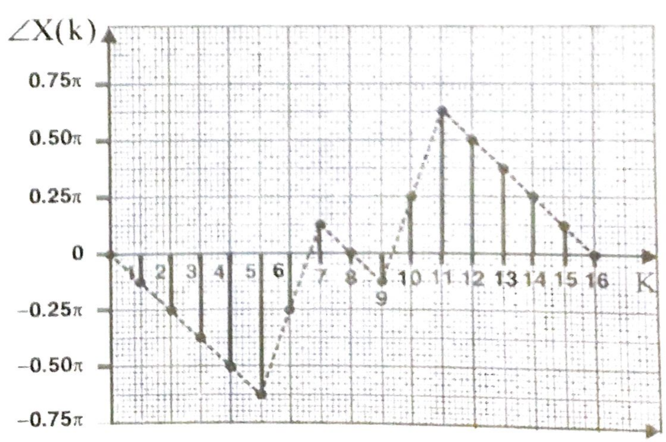

Fig 6 : Phase spectrum of X(k) for N=16.

Example 5.2

Compute the DFT of the sequence, $x(n) = \{0, 1, 2, 1\}$. Sketch the magnitude and phase spectrum.

Solution

The given signal x(n) is 4-point signal and so, let us compute 4-point DFT.

By the definition of DFT, the 4-point DFT is given by,

$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$

$$X(k) = \sum_{n=0}^{4-1} x(n)\, e^{\frac{-j2\pi kn}{4}} = \sum_{n=0}^{3} x(n)\, e^{\frac{-j\pi kn}{2}}$$

$$= x(0)\, e^{0} + x(1)\, e^{\frac{-j\pi k}{2}} + x(2)\, e^{-j\pi k} + x(3)\, e^{\frac{-j3\pi k}{2}} = 0 + e^{\frac{-j\pi k}{2}} + 2\, e^{-j\pi k} + e^{\frac{-j3\pi k}{2}}$$

$$= \cos\frac{\pi k}{2} - j\sin\frac{\pi k}{2} + 2(\cos\pi k - j\sin\pi k) + \cos\frac{3\pi k}{2} - j\sin\frac{3\pi k}{2}$$

$$= \left(\cos\frac{\pi k}{2} + 2\cos\pi k + \cos\frac{3\pi k}{2}\right) - j\left(\sin\frac{\pi k}{2} + \sin\frac{3\pi k}{2}\right)$$

$\boxed{\sin\pi k = 0 \text{ for integer } k}$

When $k = 0$; $X(0) = (\cos 0 + 2\cos 0 + \cos 0) - j(\sin 0 + \sin 0)$

$$= (1 + 2 + 1) - j(0 + 0) = 4 = 4\angle 0$$

When $k = 1$; $X(1) = \left(\cos\frac{\pi}{2} + 2\cos\pi + \cos\frac{3\pi}{2}\right) - j\left(\sin\frac{\pi}{2} + \sin\frac{3\pi}{2}\right)$

$$= (0 - 2 + 0) - j(1 - 1) = -2 = 2\angle 180^\circ = 2\angle\pi$$

When $k = 2$; $X(2) = (\cos\pi + 2\cos 2\pi + \cos 3\pi) - j(\sin\pi + \sin 3\pi)$

$$= (-1 + 2 - 1) - j(0 + 0) = 0$$

When $k = 3$; $X(3) = \left(\cos\frac{3\pi}{2} + 2\cos 3\pi + \cos\frac{9\pi}{2}\right) - j\left(\sin\frac{3\pi}{2} + \sin\frac{9\pi}{2}\right)$

$$= (0 - 2 + 0) - j(-1 + 1) = -2 = 2\angle 180^\circ = 2\angle\pi$$

$\therefore X(k) = \{4\angle 0,\ 2\angle\pi,\ 0,\ 2\angle\pi\}$

Magnitude Spectrum, $|X(k)| = \{4,\ 2,\ 0,\ 2\}$

Phase Spectrum, $\angle X(k) = \{0,\ \pi,\ 0,\ \pi\}$

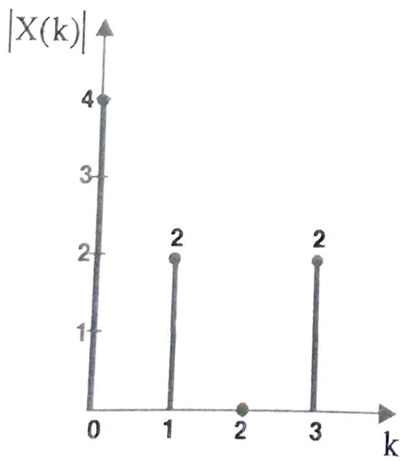

Fig 1 : Magnitude Spectrum.

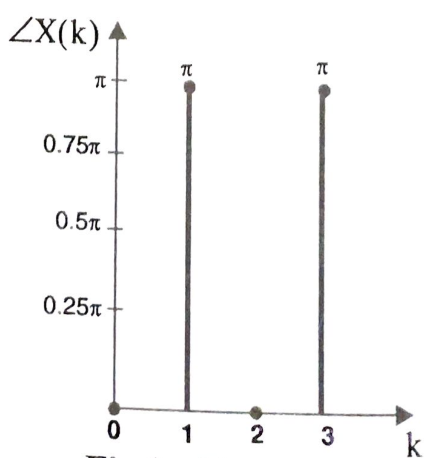

Fig 2 : Phase Spectrum.

Example 5.3

Compute circular convolution of the following two sequences using DFT.

$x_1(n) = \{\underset{\uparrow}{0}, 1, 0, 1\}$ and $x_2(n) = \{\underset{\uparrow}{1}, 2, 1, 2\}$

Solution

Given that, $x_1(n) = \{0, 1, 0, 1\}$. The 4-point DFT of $x_1(n)$ is,

$$\mathcal{DFT}\{x_1(n)\} = X_1(k) = \sum_{n=0}^{4-1} x_1(n)\, e^{\frac{-j2\pi nk}{4}} = \sum_{n=0}^{3} x_1(n)\, e^{\frac{-j\pi nk}{2}} \quad ; \quad k = 0, 1, 2, 3$$

$$= x_1(0)\, e^0 + x_1(1)\, e^{-j\frac{\pi k}{2}} + x_1(2)\, e^{-j\pi k} + x_1(3)\, e^{-j\frac{3\pi k}{2}}$$

$$= 0 + e^{-j\frac{\pi k}{2}} + 0 + e^{-j\frac{3\pi k}{2}} = e^{-j\frac{\pi k}{2}} + e^{-j\frac{3\pi k}{2}}$$

When $k = 0$; $X_1(0) = e^0 + e^0 = 1 + 1 = 2$

$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$

When $k = 1$; $X_1(1) = e^{\frac{-j\pi}{2}} + e^{\frac{-j3\pi}{2}} = -j + j = 0$

When $k = 2$; $X_1(2) = e^{-j\pi} + e^{-j3\pi} = -1 - 1 = -2$

When $k = 3$; $X_1(3) = e^{\frac{-j3\pi}{2}} + e^{\frac{-j9\pi}{2}} = j - j = 0$

Given that, $x_2(n) = \{1, 2, 1, 2\}$. The 4-point DFT of $x_2(n)$ is,

$$\mathcal{DFT}\{x_2(n)\} = X_2(k) = \sum_{n=0}^{4-1} x_2(n)\, e^{\frac{-j2\pi nk}{4}} = \sum_{n=0}^{3} x_2(n)\, e^{\frac{-j\pi nk}{2}} \quad ; \quad k = 0, 1, 2, 3$$

$$= x_2(0)\, e^0 + x_2(1)\, e^{\frac{-j\pi k}{2}} + x_2(2)\, e^{-j\pi k} + x_2(3)\, e^{\frac{-j3\pi k}{2}}$$

$$= 1 + 2\, e^{\frac{-j\pi k}{2}} + e^{-j\pi k} + 2\, e^{\frac{-j3\pi k}{2}}$$

When $k = 0$; $X_2(0) = 1 + 2e^0 + e^0 + 2e^0 = 1 + 2 + 1 + 2 = 6$

When $k = 1$; $X_2(1) = 1 + 2\, e^{\frac{-j\pi}{2}} + e^{-j\pi} + 2\, e^{\frac{-j3\pi}{2}} = 1 - 2j - 1 + 2j = 0$

When $k = 2$; $X_2(2) = 1 + 2\, e^{-j\pi} + e^{-j2\pi} + 2\, e^{-j3\pi} = 1 - 2 + 1 - 2 = -2$

When $k = 3$; $X_2(3) = 1 + 2\, e^{\frac{-j3\pi}{2}} + e^{-j3\pi} + 2\, e^{\frac{-j9\pi}{2}} = 1 + 2j - 1 - 2j = 0$

$$X_1(k) = \begin{cases} 2 \ ; & k = 0 \\ 0 \ ; & k = 1 \\ -2 \ ; & k = 2 \\ 0 \ ; & k = 3 \end{cases} \qquad X_2(k) = \begin{cases} 6 \ ; & k = 0 \\ 0 \ ; & k = 1 \\ -2 \ ; & k = 2 \\ 0 \ ; & k = 3 \end{cases}$$

Let, $X_3(k)$ be the product of $X_1(k)$ and $X_2(k)$.

$\therefore X_3(k) = X_1(k)\, X_2(k)$

When $k = 0$; $X_3(0) = X_1(0) \times X_2(0) = 2 \times 6 = 12$

When $k = 1$; $X_3(1) = X_1(1) \times X_2(1) = 0 \times 0 = 0$

When $k = 2$; $X_3(2) = X_1(2) \times X_2(2) = -2 \times -2 = 4$

When $k = 3$; $X_3(3) = X_1(3) \times X_2(3) = 0 \times 0 = 0$

$\therefore X_3(k) = \{12, 0, 4, 0\}$

By circular convolution theorem of DFT, we get,

$$\mathcal{DFT}\{x_1(n) \circledast x_2(n)\} = X_1(k)\,X_2(k) \quad \Rightarrow \quad x_1(n) \circledast x_2(n) = \mathcal{DFT}^{-1}\{X_1(k)\,X_2(k)\} = \mathcal{DFT}^{-1}\{X_3(k)\}$$

Let $x_3(n)$ be the 4-point sequence obtained by taking inverse DFT of $X_3(k)$.

$$\mathcal{DFT}^{-1}\{X_3(k)\} = x_3(n) = \frac{1}{4}\sum_{k=0}^{4-1} X_3(k)\, e^{\frac{j2\pi nk}{4}} = \frac{1}{4}\sum_{k=0}^{3} X_3(k)\, e^{\frac{j\pi nk}{2}} \quad ; \quad n = 0, 1, 2, 3$$

$$= \frac{1}{4}\left[X_3(0)\,e^0 + X_3(1)\,e^{\frac{j\pi n}{2}} + X_3(2)\,e^{j\pi n} + X_3(3)\,e^{\frac{j3\pi n}{2}}\right]$$

$\boxed{\sin \pi n = 0 \text{ for integer } n}$

$$= \frac{1}{4}\left[12 + 0 + 4e^{j\pi n} + 0\right] = 3 + e^{j\pi n} = 3 + \cos \pi n + j\sin \pi n = 3 + \cos \pi n$$

When $n = 0$; $x_3(0) = 3 + \cos 0 = 3 + 1 = 4$

When $n = 1$; $x_3(1) = 3 + \cos \pi = 3 - 1 = 2$

When $n = 2$; $x_3(2) = 3 + \cos 2\pi = 3 + 1 = 4$

When $n = 3$; $x_3(3) = 3 + \cos 3\pi = 3 - 1 = 2$

$$\therefore x_1(n) \circledast x_2(n) = x_3(n) = \{\underset{\uparrow}{4},\ 2,\ 4,\ 2\}$$

Example 5.4

Compute linear and circular convolution of the following two sequences using DFT.

$$x(n) = \{\underset{\uparrow}{1},\ 2\} \text{ and } h(n) = \{\underset{\uparrow}{2},\ 1\}$$

Solution

Linear Convolution by DFT

The linear convolution of x(n) and h(n) will produce a 3 sample sequence. To avoid time aliasing let us convert the 2 sample input sequences into 3-sample sequences by padding with zeros.

$$\therefore x(n) = \{\underset{\uparrow}{1},\ 2,\ 0\} \text{ and } h(n) = \{\underset{\uparrow}{2},\ 1,\ 0\}$$

By the definition of N-point DFT, the three point DFT of x(n) is,

$$X(k) = \sum_{n=0}^{3-1} x(n)\, e^{\frac{-j2\pi kn}{3}} = x(0)\,e^0 + x(1)\,e^{\frac{-j2\pi k}{3}} + x(2)\,e^{\frac{-j4\pi k}{3}} = 1 + 2\,e^{\frac{-j2\pi k}{3}}$$

When $k = 0$; $X(0) = 1 + 2e^0 = 1 + 2 = 3$

When $k = 1$; $X(1) = 1 + 2\,e^{\frac{-j2\pi}{3}} = 1 + 2(-0.5 - j0.866) = -\,j1.732$

$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$

When $k = 2$; $X(2) = 1 + 2\,e^{\frac{-j4\pi}{3}} = 1 + 2(-0.5 + j0.866) = j1.732$

By the definition of N-point DFT, the three point DFT of h(n) is,

$$H(k) = \sum_{n=0}^{3-1} h(n)\, e^{\frac{-j2\pi kn}{3}} = h(0)\,e^0 + h(1)\,e^{\frac{-j2\pi k}{3}} + h(2)\,e^{\frac{-j4\pi k}{3}} = 2 + e^{\frac{-j2\pi k}{3}}$$

When $k = 0$; $H(0) = 2 + e^0 = 2 + 1 = 3$

When $k = 1$; $H(1) = 2 + e^{\frac{-j2\pi}{3}} = 2 - 0.5 - j0.866 = 1.5 - j0.866$

When $k = 2$; $H(2) = 2 + e^{\frac{-j4\pi}{3}} = 2 - 0.5 + j0.866 = 1.5 + j0.866$

Let, $Y(k) = X(k)\,H(k)$; for $k = 0, 1, 2$

When $k=0$; $Y(0) = X(0)\,H(0) = 3 \times 3 = 9$

When $k=1$; $Y(1) = X(1)\,H(1) = (-j1.732) \times (1.5-j0.866) = -1.5 - j2.598$

When $k=2$; $Y(2) = X(2)\,H(2) = (j1.732) \times (1.5+j0.866) = -1.5 + j2.598$

$$\therefore Y(k) = \{\underset{\uparrow}{9},\ -1.5 - j2.598,\ -1.5 + j2.598\}$$

The sequence y(n) is obtained from inverse DFT of Y(k). By definition of inverse DFT,

$$y(n) = \mathcal{DFT}^{-1}\{Y(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\, e^{\frac{j2\pi kn}{N}} \quad ; \quad \text{for } n = 0, 1, 2, \ldots, N-1$$

$$\therefore y(n) = \frac{1}{3}\sum_{k=0}^{2} Y(k)\, e^{\frac{j2\pi kn}{3}}$$

$$= \frac{1}{3}\left[Y(0)\,e^0 + Y(1)\,e^{\frac{j2\pi n}{3}} + Y(2)\,e^{\frac{j4\pi n}{3}}\right] \quad ; \quad \text{for } n = 0, 1, 2$$

$$= \frac{1}{3}\left[9 + (-1.5 - j2.598)\,e^{\frac{j2\pi n}{3}} + (-1.5 + j2.598)\,e^{\frac{j4\pi n}{3}}\right]$$

$$= 3 + (-0.5 - j0.866)\,e^{\frac{j2\pi n}{3}} + (-0.5 + j0.866)\,e^{\frac{j4\pi n}{3}}$$

When $n = 0$;
$$y(0) = 3 + (-0.5 - j0.866)e^0 + (-0.5 + j0.866)e^0$$
$$= 3 - 0.5 - j0.866 - 0.5 + j0.866 = 2$$

$$\boxed{e^{\pm j\theta} = \cos\theta \pm j\sin\theta}$$

When $n = 1$;
$$y(1) = 3 + (-0.5 - j0.866)e^{\frac{j2\pi}{3}} + (-0.5 + j0.866)e^{\frac{j4\pi}{3}}$$
$$= 3 + (-0.5 - j0.866)(-0.5 + j0.866) + (-0.5 + j0.866)(-0.5 - j0.866)$$
$$= 3 + (0.5^2 + 0.866^2) + (0.5^2 + 0.866^2) = 3 + 1 + 1 = 5$$

When $n = 2$;
$$y(2) = 3 + (-0.5 - j0.866)e^{\frac{j4\pi}{3}} + (-0.5 + j0.866)e^{\frac{j8\pi}{3}}$$
$$= 3 + (-0.5 - j0.866)(-0.5 - j0.866) + (-0.5 + j0.866)(-0.5 + j0.866)$$
$$= 3 + (-0.5 - j0.866)^2 + (-0.5 + j0.866)^2$$
$$= 3 - 0.5 + j0.866 - 0.5 - j0.866 = 2$$

$$\therefore x(n) * h(n) = y(n) = \{\underset{\uparrow}{2},\ 5,\ 2\}$$

Circular Convolution by DFT

The given sequences are 2-point sequences. Hence 2-point DFT of the sequences are obtained as follows.

The 2-point DFT of x(n) is given by,

$$X(k) = \sum_{n=0}^{2-1} x(n)\, e^{\frac{-j2\pi kn}{2}} = x(0)\,e^0 + x(1)\,e^{-j\pi k} = 1 + 2\,e^{-j\pi k} \quad ; \text{ for } k = 0, 1$$

When $k = 0$; $X(0) = 1 + 2\,e^0 = 1 + 2 = 3$

When $k = 1$; $X(1) = 1 + 2\,e^{-j\pi} = 1 - 2 = -1$

$$\therefore X(k) = \{\underset{\uparrow}{3},\ -1\}$$

The 2-point DFT of h(n) is given by,

$$H(k) = \sum_{n=0}^{2-1} h(n)\, e^{\frac{-j2\pi kn}{2}} = h(0)\,e^0 + h(1)\,e^{-j\pi k} = 2 + e^{-j\pi k} \quad ; \quad \text{for } k = 0, 1$$

When $k = 0$; $\quad H(0) = 2 + e^0 = 2+1=3$

When $k = 1$; $\quad H(1) = 2 + e^{-j\pi} = 2 - 1 = 1$

$$\therefore \quad H(k) = \{\underset{\uparrow}{3},\ 1\}$$

Let the product of X(k) and H(k) be equal to Y(k).

$$\therefore \quad Y(k) = X(k)\,H(k) \quad ; \quad \text{for } k = 0,\ 1$$

When $k = 0$; $\quad Y(0) = X(0)\,H(0) = 3 \times 3 = 9$

When $k = 1$; $\quad Y(1) = X(1)\,H(1) = -1 \times 1 = -1$

$$\therefore \quad Y(k) = \{\underset{\uparrow}{9},\ -1\}$$

The sequence y(n) is obtained from inverse DFT of Y(k). By the definition of inverse DFT,

$$y(n) = \mathcal{DFT}^{-1}\{Y(k)\} = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)\,e^{\frac{j2\pi kn}{N}} \quad ; \text{ for } n = 0,\ 1,\ 2,\ \ldots\ldots,\ N-1$$

Here, $N = 2$

$$\therefore \quad y(n) = \frac{1}{2}\sum_{k=0}^{1} Y(k)\,e^{\frac{j2\pi kn}{2}} = \frac{1}{2}\left[Y(0) + Y(1)\,e^{j\pi n}\right] = \frac{1}{2}\left[9 - e^{j\pi n}\right] = 4.5 - 0.5\,e^{j\pi n}$$

When $n = 0$; $\quad y(0) = 4.5 - 0.5\,e^0 = 4.5 - 0.5 = 4$

When $n = 1$; $\quad y(1) = 4.5 - 0.5\,e^{j\pi} = 4.5 + 0.5 = 5$ $\quad \boxed{e^{j\pi} = -1}$

$$\therefore \quad x(n) \circledast h(n) = y(n) = \{\underset{\uparrow}{4},\ 5\}$$