Discrete Fourier Transform (DFT)

Recall the DTFT:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}.$$

DTFT is not suitable for DSP applications because

- In DSP, we are able to compute the spectrum only at specific discrete values of ω,
- Any signal in any DSP application can be measured only in a finite number of points.

A finite signal measured at N points:

$$x(n) = \begin{cases} 0, & n < 0, \\ y(n), & 0 \le n \le (N-1), \\ 0, & n \ge N, \end{cases}$$

where $y(n)$ are the measurements taken at N points.

Sample the spectrum $X(\omega)$ in frequency so that

$$\begin{aligned} X(k) &= X(k\Delta\omega), \quad \Delta\omega = \frac{2\pi}{N} \quad \Longrightarrow \\ X(k) &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi\frac{kn}{N}} \quad \text{DFT.} \end{aligned}$$

The **inverse DFT** is given by:

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi\frac{kn}{N}}.$$

$$\begin{aligned} x(n) &= \frac{1}{N} \sum_{k=0}^{N-1} \left\{ \sum_{m=0}^{N-1} x(m) e^{-j2\pi\frac{km}{N}} \right\} e^{j2\pi\frac{kn}{N}} \\ &= \sum_{m=0}^{N-1} x(m) \underbrace{\left\{ \frac{1}{N} \sum_{k=0}^{N-1} e^{-j2\pi\frac{k(m-n)}{N}} \right\}}_{\delta(m-n)} = x(n). \end{aligned}$$

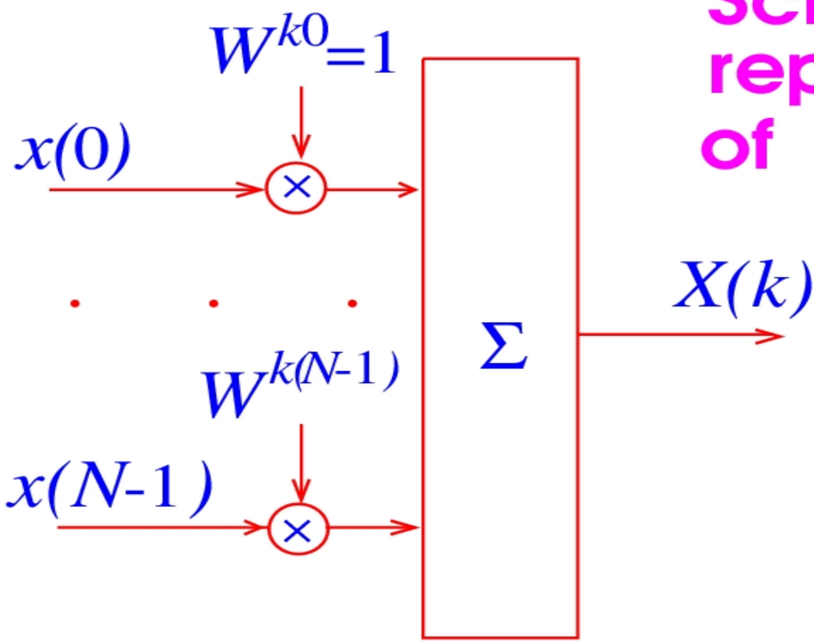
The DFT pair:

$$\boxed{\begin{aligned} X(k) &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi\frac{kn}{N}} \quad \text{analysis} \\ x(n) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi\frac{kn}{N}} \quad \text{synthesis.} \end{aligned}}$$

Alternative formulation:

$$X(k) = \sum_{n=0}^{N-1} x(n) W^{kn} \quad \longleftarrow W = e^{-j\frac{2\pi}{N}}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W^{-kn}.$$



Schematic representation of DFT

Periodicity of DFT Spectrum

$$
\begin{aligned}
X(k+N) &= \sum_{n=0}^{N-1} x(n) e^{-j2\pi\frac{(k+N)n}{N}} \\
&= \left(\sum_{n=0}^{N-1} x(n) e^{-j2\pi\frac{kn}{N}} \right) e^{-j2\pi n} \\
&= X(k) e^{-j2\pi n} = X(k) \Longrightarrow
\end{aligned}
$$

the DFT spectrum is periodic with period N (which is expected, since the DTFT spectrum is periodic as well, but with period 2π).

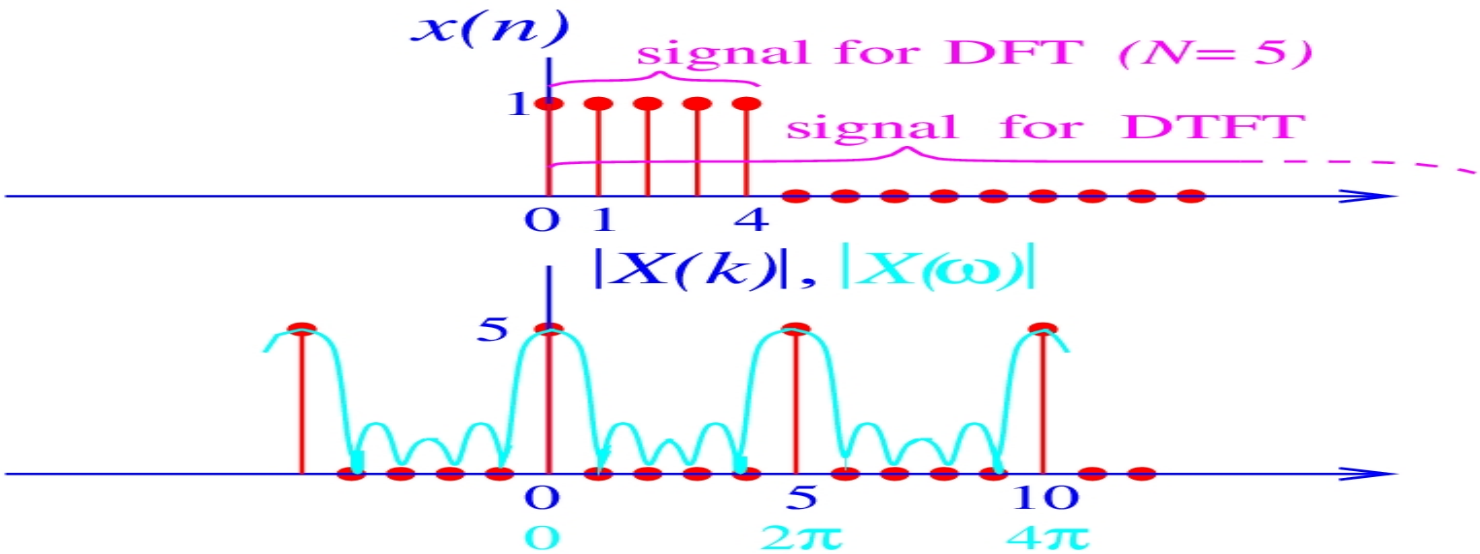
Example: DFT of a rectangular pulse:

$$
x(n) = \begin{cases} 1, & 0 \le n \le (N-1), \\ 0, & \text{otherwise.} \end{cases}
$$

$$
X(k) = \sum_{n=0}^{N-1} e^{-j2\pi\frac{kn}{N}} = N\delta(k) \Longrightarrow
$$

the rectangular pulse is "interpreted" by the DFT as a spectral line at frequency $\omega = 0$.

DFT and DTFT of a rectangular pulse (N=5)



Zero Padding

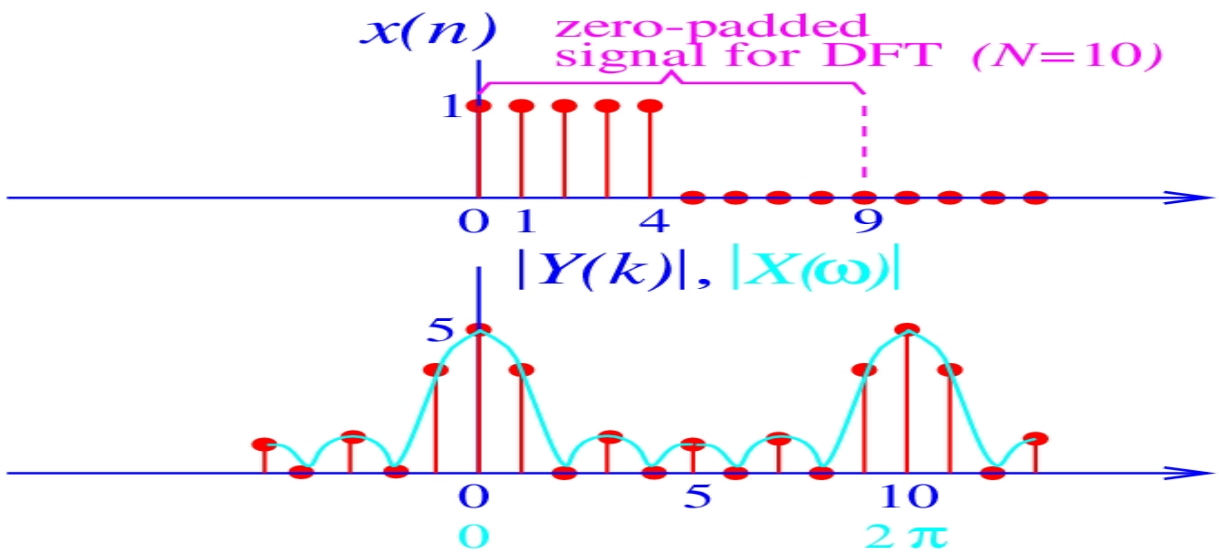
What happens with the DFT of this rectangular pulse if we increase N by *zero padding*:

$$\{y(n)\} = \{x(0), \ldots, x(M-1), \underbrace{0, 0, \ldots, 0}_{N-M \text{ positions}}\},$$

where $x(0) = \cdots = x(M-1) = 1$. Hence, DFT is

$$\begin{aligned} Y(k) &= \sum_{n=0}^{N-1} y(n) e^{-j2\pi\frac{kn}{N}} = \sum_{n=0}^{M-1} y(n) e^{-j2\pi\frac{kn}{N}} \\ &= \frac{\sin(\pi\frac{kM}{N})}{\sin(\pi\frac{k}{N})} e^{-j\pi\frac{k(M-1)}{N}}. \end{aligned}$$

DFT and DTFT of a Rectangular Pulse with Zero Padding ($N = 10,\ M = 5$)



Remarks:

- Zero padding of analyzed sequence results in "approximating" its DTFT better,
- Zero padding cannot improve the resolution of spectral components, because the resolution is "proportional" to $1/M$ rather than $1/N$,
- Zero padding is very important for fast DFT implementation (FFT).

Matrix Formulation of DFT

Introduce the $N \times 1$ vectors

$$\boldsymbol{x} = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \quad \boldsymbol{X} = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}.$$

and the $N \times N$ matrix

$$\mathcal{W} = \begin{bmatrix} W^0 & W^0 & W^0 & \cdots & W^0 \\ W^0 & W^1 & W^2 & \cdots & W^{N-1} \\ W^0 & W^2 & W^4 & \cdots & W^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ W^0 & W^{N-1} & W^{2(N-1)} & \cdots & W^{(N-1)^2} \end{bmatrix}.$$

DFT in a matrix form:

$$\boldsymbol{X} = \mathcal{W}\boldsymbol{x}.$$

Result: Inverse DFT is given by

$$\boldsymbol{x} = \frac{1}{N}\mathcal{W}^H \boldsymbol{X},$$

which follows easily by checking $\mathcal{W}^H \mathcal{W} = \mathcal{W}\mathcal{W}^H = NI$, where I denotes the identity matrix. Hermitian transpose:

$$\boldsymbol{x}^H = (x^T)^* = [x(1)^*, x(2)^*, \ldots, x(N)^*].$$

Also, "$*$" denotes complex conjugation.

Frequency Interval/Resolution: DFT's frequency resolution

$$F_{\text{res}} \sim \frac{1}{NT} \quad [\text{Hz}]$$

and covered frequency interval

$$\Delta F = N \Delta F_{\text{res}} = \frac{1}{T} = F_{\text{s}} \quad [\text{Hz}].$$

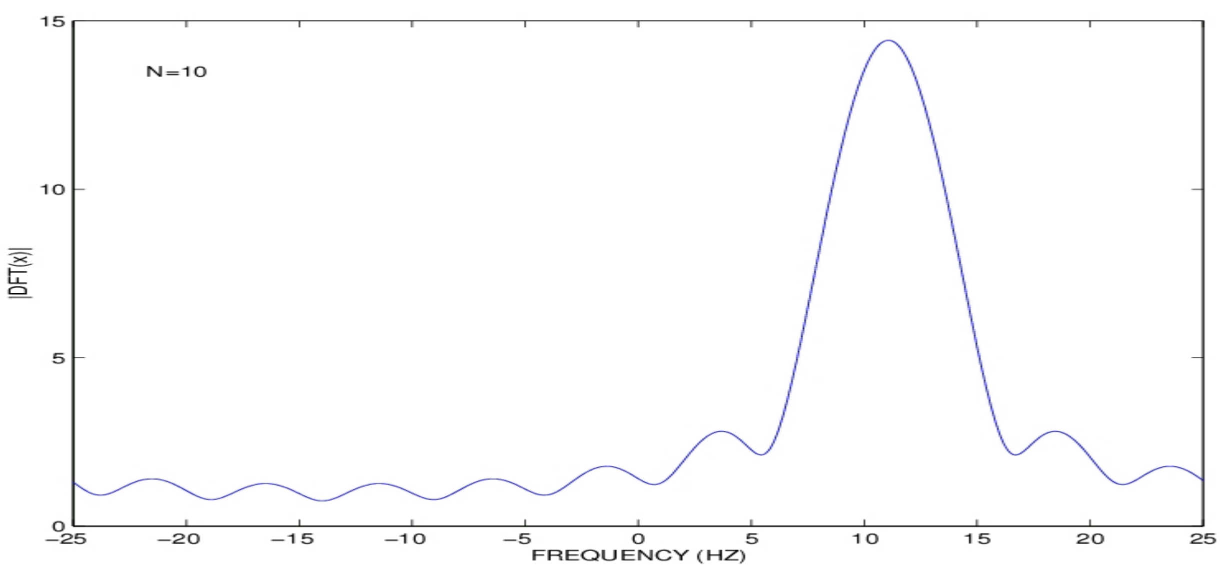
Frequency resolution is determined only by the length of the observation interval, whereas the frequency interval is determined by the length of sampling interval. Thus

- Increase sampling rate $\Longrightarrow$ expand frequency interval,
- Increase observation time $\Longrightarrow$ improve frequency resolution.

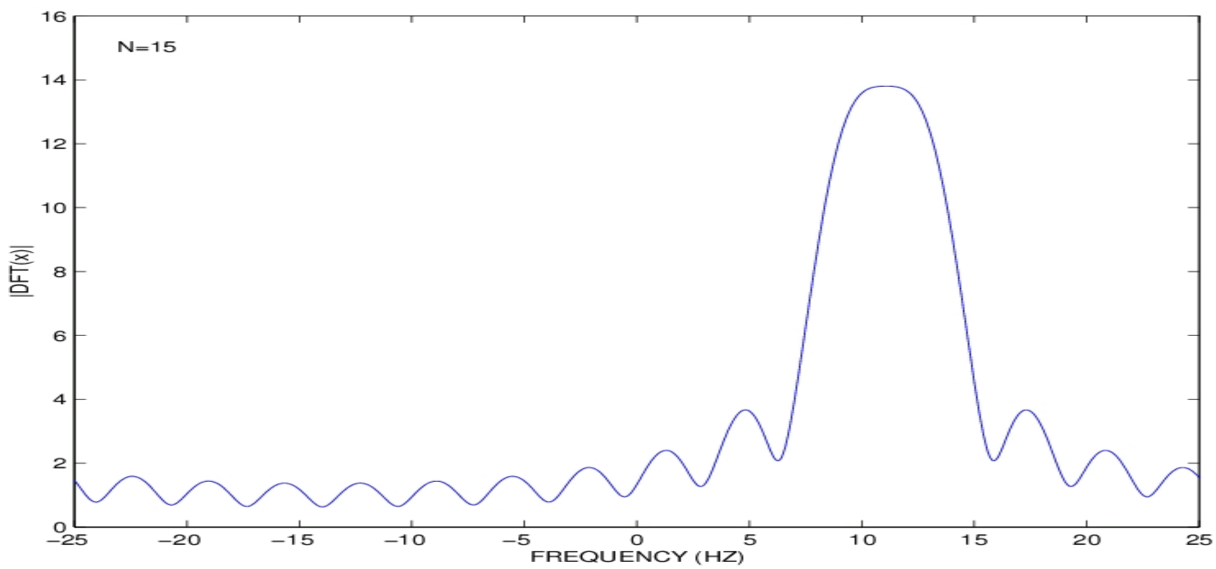
Question: Does zero padding alter the frequency resolution?

Answer: No, because resolution is determined by the length of observation interval, and zero padding does not increase this length.

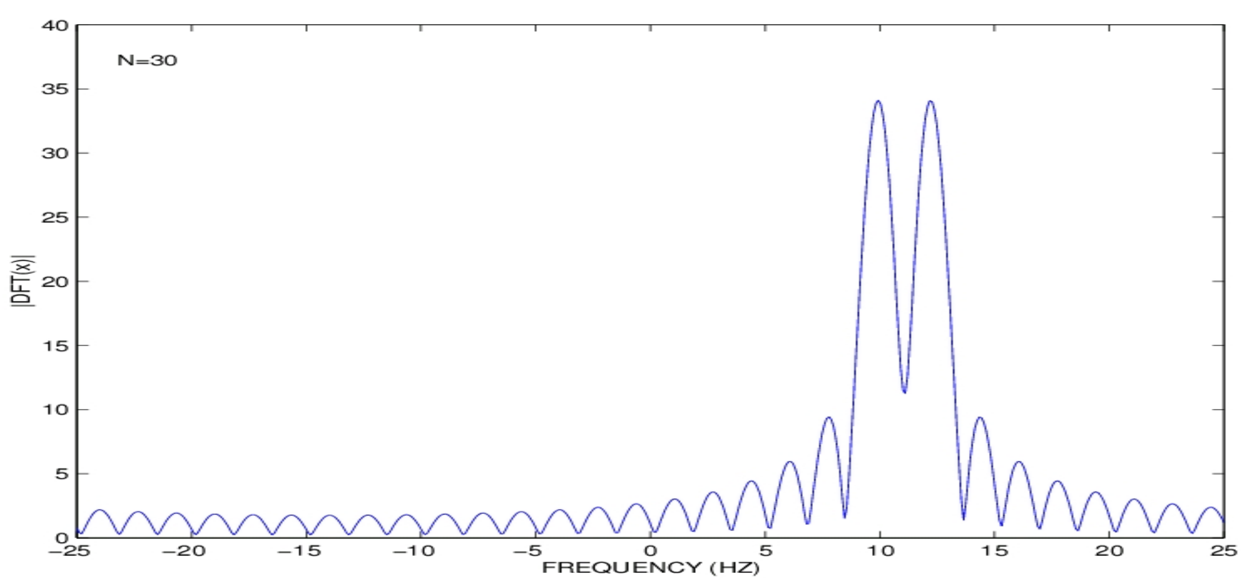
Example (DFT Resolution): Two complex exponentials with two close frequencies $F_1 = 10$ Hz and $F_2 = 12$ Hz sampled with the sampling interval $T = 0.02$ seconds. Consider various data lengths $N = 10, 15, 30, 100$ with zero padding to 512 points.



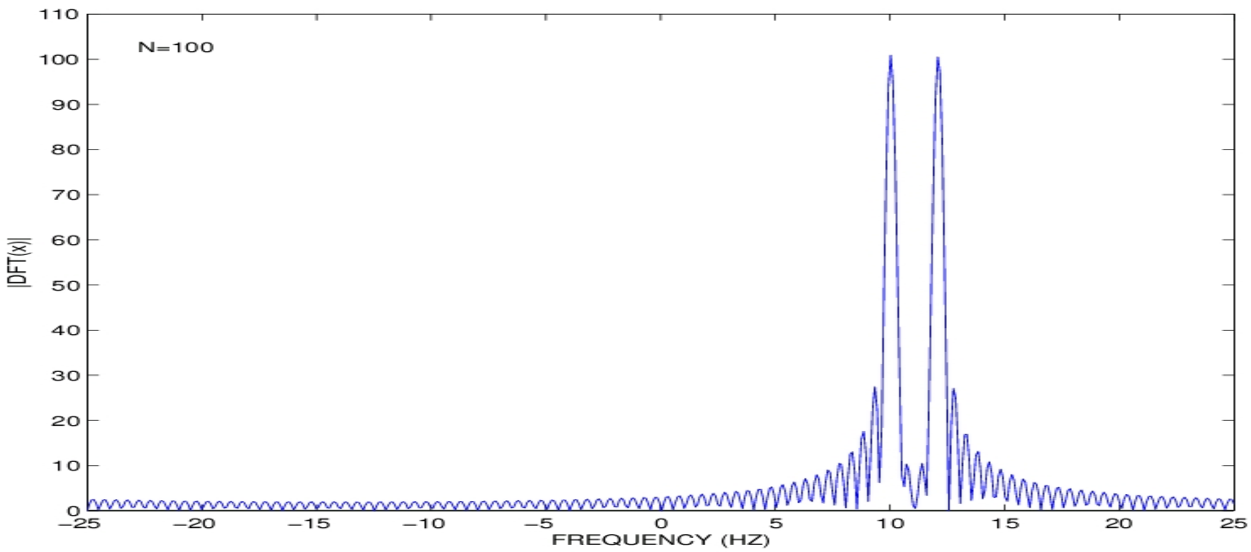
DFT with $N = 10$ and zero padding to 512 points.
Not resolved: $F_2 - F_1 = 2$ Hz $< 1/(NT) = 5$ Hz.



DFT with $N = 15$ and zero padding to 512 points. Not resolved: $F_2 - F_1 = 2$ Hz $< 1/(NT) \approx 3.3$ Hz.



DFT with $N = 30$ and zero padding to 512 points. Resolved: $F_2 - F_1 = 2$ Hz $> 1/(NT) \approx 1.7$ Hz.



DFT with $N = 100$ and zero padding to 512 points. Resolved: $F_2 - F_1 = 2$ Hz $> 1/(NT) = 0.5$ Hz.

DFT Interpretation Using Discrete Fourier Series

Construct a periodic sequence by periodic repetition of $x(n)$ every N samples:

$$\{\widetilde{x}(n)\} = \{\ldots, \underbrace{x(0), \ldots, x(N-1)}_{\{x(n)\}}, \underbrace{x(0), \ldots, x(N-1)}_{\{x(n)\}}, \ldots\}$$

The discrete version of the Fourier Series can be written as

$$\widetilde{x}(n) = \sum_k X_k e^{j2\pi \frac{kn}{N}} = \frac{1}{N} \sum_k \widetilde{X}(k) e^{j2\pi \frac{kn}{N}} = \frac{1}{N} \sum_k \widetilde{X}(k) W^{-kn},$$

where $\widetilde{X}(k) = N X_k$. Note that, for integer values of m, we have

$$W^{-kn} = e^{j2\pi \frac{kn}{N}} = e^{j2\pi \frac{(k+mN)n}{N}} = W^{-(k+mN)n}.$$

As a result, the summation in the Discrete Fourier Series (DFS) should contain only N terms:

$$\widetilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \widetilde{X}(k) e^{j2\pi \frac{kn}{N}} \quad \text{DFS.}$$

Inverse DFS

The DFS coefficients are given by

$$\widetilde{X}(k) = \sum_{n=0}^{N-1} \widetilde{x}(n) e^{-j2\pi \frac{kn}{N}} \quad \text{inverse DFS.}$$

Proof.

$$\begin{aligned}
\sum_{n=0}^{N-1} \widetilde{x}(n) e^{-j2\pi \frac{kn}{N}} &= \sum_{n=0}^{N-1} \left\{ \frac{1}{N} \sum_{p=0}^{N-1} \widetilde{X}(p) e^{j2\pi \frac{pn}{N}} \right\} e^{-j2\pi \frac{kn}{N}} \\
&= \sum_{p=0}^{N-1} \widetilde{X}(p) \underbrace{\left\{ \frac{1}{N} \sum_{n=0}^{N-1} e^{j2\pi \frac{(p-k)n}{N}} \right\}}_{\delta(p-k)} = \widetilde{X}(k).
\end{aligned}$$

$\square$

The DFS coefficients are given by

$$\begin{aligned}
\widetilde{X}(k) &= \sum_{n=0}^{N-1} \widetilde{x}(n) e^{-j2\pi \frac{kn}{N}} \quad \text{analysis,} \\
\widetilde{x}(n) &= \frac{1}{N} \sum_{k=0}^{N-1} \widetilde{X}(k) e^{j2\pi \frac{kn}{N}} \quad \text{synthesis.}
\end{aligned}$$

- DFS and DFT pairs are identical, except that
  - DFT is applied to finite sequence $x(n)$,
  - DFS is applied to periodic sequence $\widetilde{x}(n)$.

- Conventional (continuous-time) FS vs. DFS
  - CFS represents a continuous periodic signal using an infinite number of complex exponentials,
    whereas
  - DFS represents a discrete periodic signal using a finite number of complex exponentials.

DFT: Properties

Linearity

Circular shift of a sequence: if $X(k) = \mathcal{DFT}\{x(n)\}$ then

$$X(k)e^{-j2\pi\frac{km}{N}} = \mathcal{DFT}\{x((n-m) \bmod N)\}$$

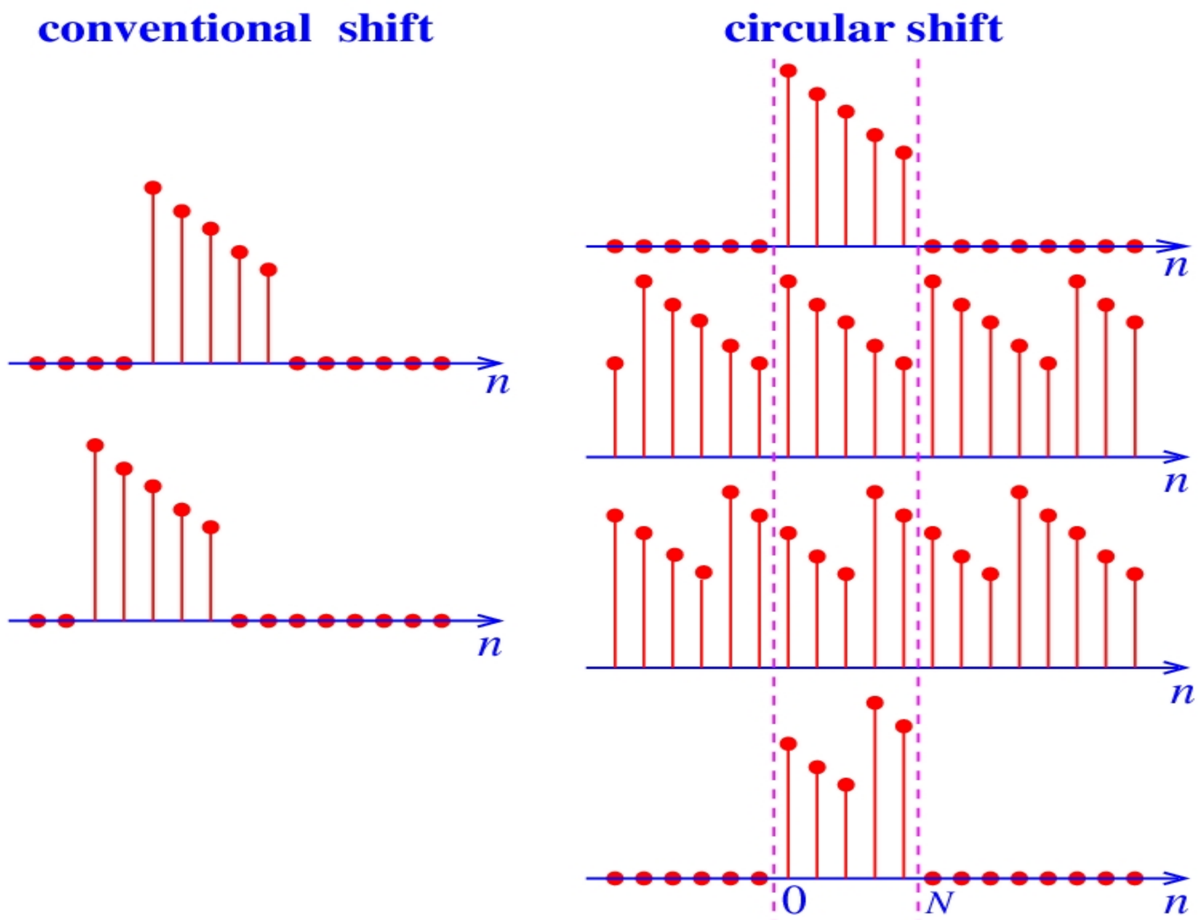
Also if $x(n) = \mathcal{DFT}^{-1}\{X(k)\}$ then

$$x((n-m) \bmod N) = \mathcal{DFT}^{-1}\{X(k)e^{-j2\pi\frac{km}{N}}\}$$

where the operation $\bmod N$ denotes the periodic extension $\widetilde{x}(n)$ of the signal $x(n)$:

$$\widetilde{x}(n) = x(n \bmod N).$$

DFT: Circular Shift



$$
\begin{aligned}
& \sum_{n=0}^{N-1} x((n-m)\mathrm{mod}N)W^{kn} \\
= \quad & W^{km} \sum_{n=0}^{N-1} x((n-m)\mathrm{mod}N)W^{k(n-m)}
\end{aligned}
$$

$$
\begin{aligned}
&= W^{km} \sum_{n=0}^{N-1} x((n-m)\mathrm{mod}N) W^{k(n-m)\mathrm{mod}N} \\
&= W^{km} X(k),
\end{aligned}
$$

where we use the facts that $W^{k(l \mathrm{mod} N)} = W^{kl}$ and that the order of summation in DFT does not change its result.

Similarly, if $X(k) = \mathcal{DFT}\{x(n)\}$, then

$$
X((k-m)\mathrm{mod}N) = \mathcal{DFT}\{x(n) e^{j2\pi \frac{mn}{N}}\}.
$$

DFT: Parseval's Theorem

$$
\sum_{n=0}^{N-1} x(n) y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} \mathbf{X}(k) \mathbf{Y}^*(k)
$$

Using the matrix formulation of the DFT, we obtain

$$
\begin{aligned}
\boldsymbol{y}^H \boldsymbol{x} &= \left(\frac{1}{N} W^H \mathbf{Y} \right)^H \left(\frac{1}{N} W^H \mathbf{Y} \right) \\
&= \frac{1}{N^2} \mathbf{Y}^H \underbrace{W W^H}_{NI} \mathbf{X} = \frac{1}{N} \mathbf{Y}^H \mathbf{X}.
\end{aligned}
$$

DFT: Circular Convolution

If $X(k) = \mathcal{DFT}\{x(n)\}$ and $Y(k) = \mathcal{DFT}\{y(n)\}$, then

$$X(k)Y(k) = \mathcal{DFT}\left\{\{x(n)\} \circledast \{y(n)\}\right\}$$

Here, $\circledast$ stands for circular convolution defined by

$$\{x(n)\} \circledast \{y(n)\} = \sum_{m=0}^{N-1} x(m)y((n-m) \bmod N).$$

$$\begin{aligned}
&\mathcal{DFT}\left\{\{x(n)\} \circledast \{y(n)\}\right\} \\
&= \sum_{n=0}^{N-1} \underbrace{\left[\textstyle\sum_{m=0}^{N-1} x(m)y((n-m) \bmod N)\right]}_{\{x(n)\}\circledast\{y(n)\}} W^{kn} \\
&= \sum_{m=0}^{N-1} \underbrace{\left[\textstyle\sum_{n=0}^{N-1} y((n-m) \bmod N)W^{kn}\right]}_{Y(k)W^{km}} x(m) \\
= \quad & Y(k) \underbrace{\sum_{m=0}^{N-1} x(m)W^{km}}_{X(k)} = X(k)Y(k).
\end{aligned}$$