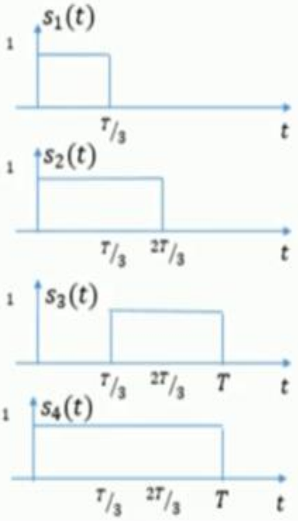
Gram-Schmidt Orthogonalization (GSO) Prodedure

- In Digital communication, we apply input as binary bits which are then converted into symbols and waveforms by a digital modulator.
- These **waveforms should be unique** and different from each other so that receiver can easily identify what symbol/bit is transmitted.
- To **make them unique, Gram-Schmidt Orthogonalization procedure** can be applied.



There are two steps to be applied in GSO procedure

1)Get a reduced set of N linearly independent signal from any given set of M signals

2)Construct a set of N orthonormal basis function from N linearly independent signal

Given set of signals M=4
Reduced set of linearly independent signals N=3
So, it is required to construct N orthonormal basis function

Suppose we have a finite number of M functions $s_i(t) - s_M(t)$. Gram-Schmidt procedure allows us to determine a finite set of N orthogonal fns such that each $s_i(t)$ can be expressed as a linear combination of these N orthogonal functions.

Let $\phi_1(t), \phi_2(t) \cdots \phi_N(t)$ be the N orthogonal fns to be determined such that we can express $s_i(t), i = 1, 2, \cdots M$

$$s_1(t) = s_{11}\phi_1(t) + s_{12}\phi_2(t) + \cdots + s_{1N}\phi_N(t) \quad \text{(1)}$$

$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t) + \cdots + s_{2N}\phi_N(t) \quad \text{(2)}$$

$$s_3(t) = s_{31}\phi_1(t) + s_{32}\phi_2(t) + \cdots + s_{3N}\phi_N(t) \quad \text{(3)}$$

$$\vdots$$

$$s_N(t) = s_{M1}\phi_1(t) + s_{M2}\phi_2(t) + \cdots + s_{MN}\phi_N(t), \quad \text{(4)}$$

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \quad 0 \le t \le T, \quad i = 1, 2, \cdots M$$

$$\int_0^T \phi_i(t)\,\phi_j(t)\,dt = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \ne j \end{cases}$$

Gram Schmidt method proceed as follows

In Eq (1), set all co-efficients $s_{ij} = 0$ except s_{11}. Then

$$s_1(t) = s_{11}\phi_1(t)$$

$$\phi_1(t) = \frac{s_1(t)}{s_{11}}$$

Squaring & integrating both sides.

$$\int_0^T \phi_1^2(t) = \int_0^T \frac{s_1^2(t)}{s_{11}^2}\,dt$$

$$s_{11} = \sqrt{\int_0^T s_1^2(t)\,dt}$$

$$s_{11} = \sqrt{E_1}$$

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

In eq (2), we set all coefficients except s_{21} & s_{22} to zero.

$$s_2(t) = s_{21}\phi_1(t) + s_{22}\phi_2(t)$$

Multiply both sides by $\phi_1(t)$ & $\int_0^T$

$$\int_0^T s_2(t)\,\phi_1(t) = \int_0^T [s_{21}\phi_1(t)\,\phi_1(t)]dt + \int_0^T \underbrace{s_{22}\,\phi_2(t)\,\phi_1(t)\,dt}_{=0}$$

$$\int_0^T s_2(t)\phi_1(t) = s_{21}\underbrace{\int_0^T \phi_1(t)\,\phi_1(t)\,dt}_{=1} + 0$$

$$s_{21} = \int_0^T s_2(t)\,\phi_1(t)$$

→ Rearranging. $s_{22}\phi_2(t) = s_2(t) - s_{21}\phi_1(t)$

Squaring & integrating

$$\int_0^T s_{22}^2\,\phi_2^2(t)\,dt = \int_0^T [s_2(t) - s_{21}\phi_1(t)]^2 dt$$

$$s_{22}^2 = \int_0^T [s_2(t) - s_{21}\phi_1(t)]^2 dt$$

$$s_{22} = \sqrt{\int_0^T [s_2(t) - s_{21}\phi_1(t)]^2 dt}$$

→ From this eq

$$\phi_2(t) = \frac{1}{s_{22}}[s_2(t) - s_{21}\phi_1(t)]$$

$$= \frac{1}{s_{22}}\left[s_2(t) - \frac{s_{21}\,s_1(t)}{s_{11}}\right]$$

other representation

$$\phi_2(t) = \frac{\{s_2(t) - s_{21}\phi_1(t)\}}{\sqrt{\int_0^T [s_2(t) - s_{21}\phi_1(t)]^2 dt}} = \frac{s_2(t) - s_{21}\phi_1(t)}{\sqrt{E_2 - s_{21}^2}}$$

$$= \frac{g_2(t)}{\sqrt{\int_0^T g_2(t)^2\,dt}} \quad \Bigg| \quad g_i(t) = s_i(t) - \sum_{j=1}^{j-1} s_{ij}\phi_j(t)$$

$g_2(t) = s_2(t) - s_{21}\varphi_1(t);\ 0 \le t < T$

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\varphi_j(t)$$

$$s_{ij} = \int_0^T s_i(t).\varphi_j(t)dt$$

for i = 1, 2,….., N and j = 1, 2, …, M

$$\int_0^T g_2^{\,2}(t)dt = \int_0^T [s_2(t) - s_{21}\varphi_1(t)]^2 dt$$

$$= \int_0^T s_2^{\,2}(t)dt - 2.s_{21}\int_0^T s_2(t)\varphi_1(t)dt + s_{21}^{\,2}\int_0^T \varphi_1^{\,2}(t)dt$$

$$= E_2 - 2.s_{21}.s_{21} + s_{21}^{\,2} = E_2 - s_{21}^{\,2}$$

From Eq (3)

$S_3(t) = S_{31}\phi_1(t) + S_{32}\phi_2(t) + S_{33}\phi_3(t)$

multiply above Eq with $\phi_1(t)$ & integrate $\int_0^T$

$$\int_0^T S_3(t)\phi_1(t) = \int_0^T S_{31}\phi_1(t)\phi_1(t)\,dt + \cancel{\int_0^T S_{32}\phi_2(t)\phi_1(t)\,dt} + \cancel{\int_0^T S_{33}\phi_3(t)\phi_1(t)\,dt}$$

$$S_{31} = \int_0^T S_3(t)\phi_1(t)\,dt$$

$|||^{ly}$ multiply $S_3(t)$ with $\phi_2(t)$ & integrate both sides

$$S_{32} = \int_0^T S_3(t)\phi_2(t)\,dt$$

~~$|||^{ly}$ multiply $S_3(t)$ with $\phi_3(t)$ & integrate both sides~~

Rearrange the Eq & square, integrate

$$S_{33}\phi_3(t) = S_3(t) - S_{31}\phi_1(t) + S_{32}\phi_2(t)$$

$$\int_0^T S_{33}^2 \underbrace{\phi_3^2(t)}_{=1}\,dt = \int_0^T [S_3(t) - S_{31}\phi_1(t) + S_{32}\phi_2(t)]^2\,dt$$

$$S_{33}^2 = \text{—} \quad ,, \quad \text{————}$$

$$S_{33} = \sqrt{\int_0^T [S_3(t) - S_{31}\phi_1(t) - S_{32}\phi_2(t)]^2\,dt}$$

$$\phi_3(t) = \frac{S_3(t) - S_{31}\phi_1(t) - S_{32}\phi_2(t)}{S_{33}}$$

The procedure will continue till we find N orthonormal fns $\phi_1(t), \phi_2(t) \dots \phi_N(t)$ & coefficients S_{ij}.

$$\phi_1 = \frac{S_1(t)}{\sqrt{E_1}}$$

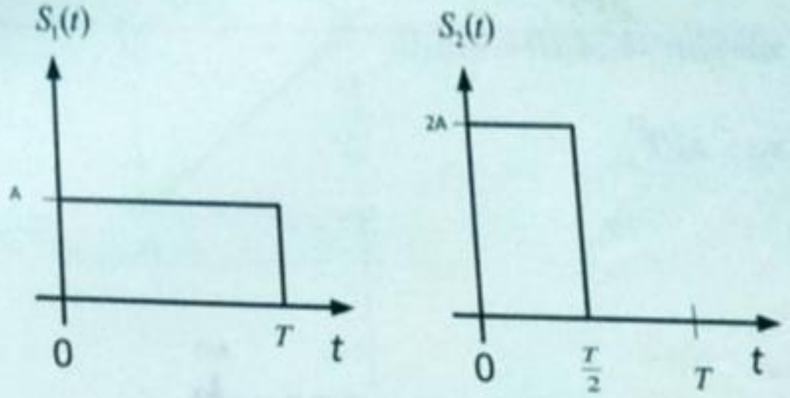
$$\phi_2 = \frac{S_2(t) - S_{21}\phi_1(t)}{\sqrt{E_2 - S_{21}^2}}$$

$$\phi_3 = \frac{S_3(t) - S_{31}\phi_1(t) - S_{32}\phi_2(t)}{\sqrt{E_3 - S_{32}^2}}$$

$$E_1 = \int_0^T s_1^2(t)\,dt$$

$$E_2 = \int_0^T s_2^2(t)\,dt$$

Two functions $S_1(t)$&$S_2(t)$ are given in the Figure.The interval is $0 \le t \le T$ seconds. Using Gram-Schmidt Procedure,express these functions in terms of orthonormal functins.Also Sketch $\phi_1(t)$ and $\phi_2(t)$



Solution:

To Find $\phi_1(t)$

$$S_{11} = \left[\int_0^T S_1^2(t)dt\right]^{\frac{1}{2}} = \left[\int_0^T A^2 dt\right]^{\frac{1}{2}} = A\sqrt{T}$$

and

$$\phi_1(t) = \frac{S_1(t)}{S_{11}} = \frac{A}{A\sqrt{T}} = \frac{1}{\sqrt{T}} \quad \text{for} 0 \le t \le T$$

To Find $\phi_2(t)$

$$S_{21} = \int_0^T S_2(t)\phi_1(t)dt = \int_0^{\frac{T}{2}} 2A\left(\frac{1}{\sqrt{T}}\right)dt = \frac{2A}{\sqrt{T}}\frac{T}{2} = A\sqrt{T}$$

We Know that

$$S_{22} = \left[\int_0^T [S_2(t) - S_{21}\phi_1(t)]^2 dt\right]^{\frac{1}{2}}$$

$$= \left[\int_0^{\frac{T}{2}} (2A - A)^2 dt + \int_{\frac{T}{2}}^T (0 - A)^2 dt\right]^{\frac{1}{2}}$$

$$= [A^2 T]^{\frac{1}{2}} = A\sqrt{T}$$

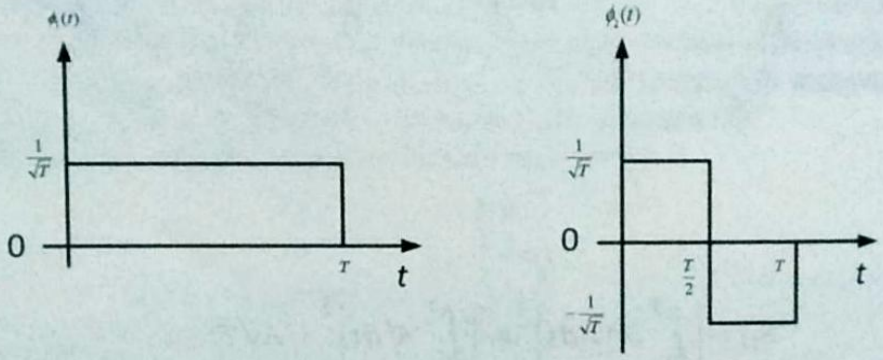
Also We Know that

$$\phi_2(t) = \frac{1}{S_{22}}[S_2(t) - S_{21}\phi_1(t)]$$
$$= \frac{1}{S_{22}}\left[S_2(t) - S_{21}\frac{S_1(t)}{S_{11}}\right]$$

But $S_{21} = S_{11}$, therefore

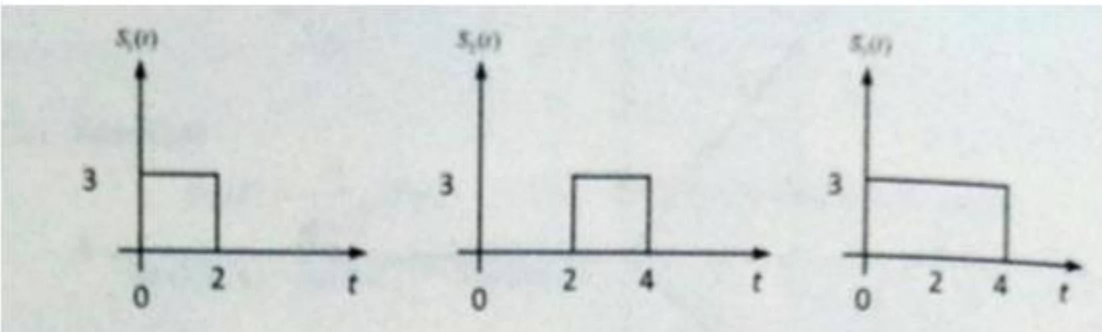
$$\phi_2(t) = \frac{1}{A\sqrt{T}}[S_2(t) - S_1(t)] \quad \text{for} 0 \le t \le T S_1(t) = S_{11}\phi_1(t)$$
$$\text{and} S_2(t) = S_{21}\phi_1(t) + S_{22}\phi_2(t)$$

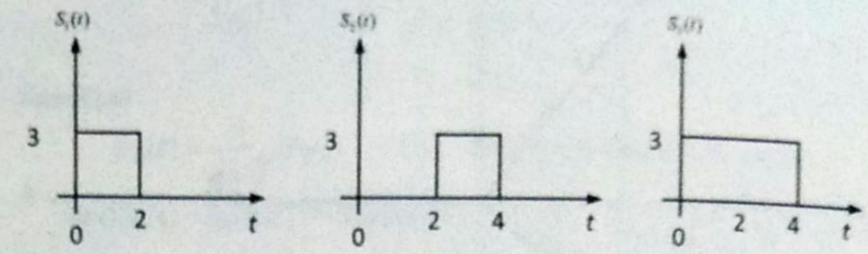
Where $S_{11} = S_{21} = S_{22} = A\sqrt{T}$



Problem

3 signals $S_1(t)$, $S_2(t)$, $S_3(t)$ are as shown in the figure. Apply GSO to obtain orthonormal basis functions for signals. Express the signals in terms of set of basis functions





To Obtain $\phi_1(t)$ Energy of $S_1(t)$ is,

$$E_1 = \int_0^T S_1^2(t)dt = \int_0^2 (3)^2 dt = 18$$

$$\phi_1(t) = \frac{S_1(t)}{\sqrt{E_1}}$$

$$= \begin{cases} \frac{3}{\sqrt{18}} & \text{for} \quad 0 \le t \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

$$\phi_1(t) = \begin{cases} \frac{1}{\sqrt{2}} & \text{for} \quad 0 \le t \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

To Obtain $\phi_2(t)$

$S_{21} = \int_0^T S_2(t)\phi_1(t)dt$

There is no overlap between $S_2(t)$and $\phi_1(t)$.Hence the product $S_2(t)\phi_1(t) = 0$

$$g_2(t) = S_2(t) - S_{21}\phi_1(t)$$

$$= S_2(t), \text{Since} S_{21}(t) = 0$$

$$E_{g2} = \int_0^T S_2^2(t)dt = \int_2^4 (3)^2 dt = 18$$

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{E_{g2}}}$$

$$= \begin{cases} \frac{3}{\sqrt{18}} & \text{for} \quad 2 \le t \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

$$= \begin{cases} \frac{1}{\sqrt{2}} & \text{for} \quad 2 \le t \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

To express the signals interms of othonormal functions as follows

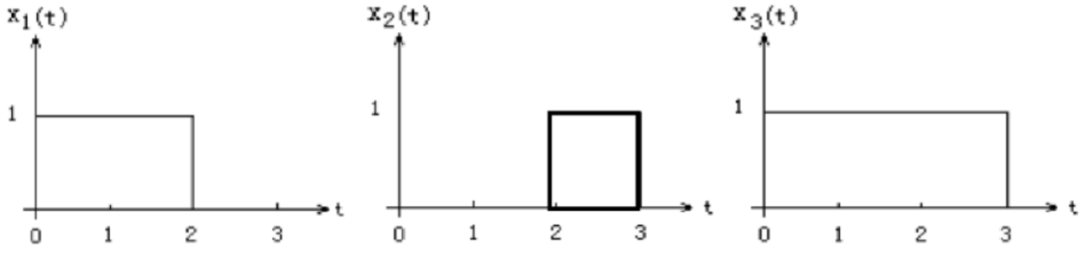
$$S_1(t) = 3\sqrt{2}\phi_1(t)$$

$$S_2(t) = 3\sqrt{2}\phi_2(t)$$

$$S_3(t) = 3\sqrt{2}\phi_1(t) + 3\sqrt{2}\phi_2(t)$$

Problem:

Use the Gram-Schmidt procedure to find a set orthonormal basis functions corresponding to the signals show below: Express x1, x2, and x3in terms of the orthonormal basis functions. Draw the constellation diagram for this signal set



Step 1: Eliminate dependent signals from the set.

Stage1: Here $x_3(t)=x_1(t)+x_2(t)$, so $x_3(t)$ is dependent signal and eliminate it from the set.

Remaining signals $\{x_1(t),x_2(t)\}$

Hence two basis functions exits for this signal set. First two basis functions are calculated as shown in step1 and step 2.

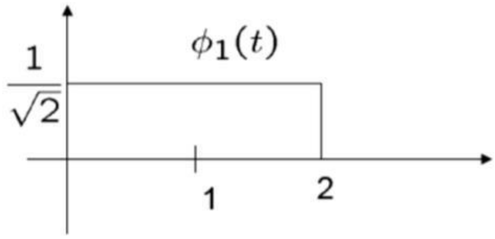
Then received signal point in signal space is calculated by finding coefficients S_{ij}

Step 1:

$$E_1 = \int_{-\infty}^{\infty} x_1^2(t)dt = 2$$

$$\phi_1(t) = \frac{1}{\sqrt{2}} x_1(t)$$

$$x_{11} = \sqrt{2}$$



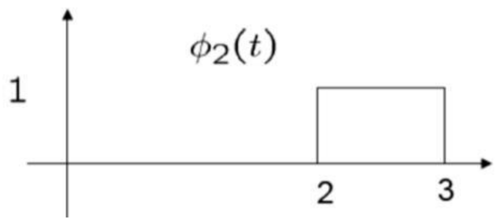
Step 2:

$$x_{21} = \int_{-\infty}^{\infty} x_2(t)\phi_1(t)dt = 0$$

$g_2(t) = x_2(t)$ and $E_{g_2} = 1$

$$\phi_2(t) = x_2(t)$$

$$x_{22} = 1$$



Step 3:

$$x_{31} = \int_{-\infty}^{\infty} x_3(t)\phi_1(t)dt = \sqrt{2}$$

$$x_{32} = \int_{-\infty}^{\infty} x_3(t)\phi_2(t)dt = 1$$

$$g_3(t) = x_3(t) - x_{31}f_1(t) - x_{32}f_2(t) = 0$$

Express x_1, x_2, x_3 in basis functions

$$x_1(t) = \sqrt{2}\phi_1(t)\,, \quad x_2(t) = \phi_2(t)$$

$$x_3(t) = \sqrt{2}\phi_1(t) + \phi_2(t)$$

Constellation diagram