Chapter 6

The DFT

6.1 INTRODUCTION

In previous chapters, we have seen how to represent a sequence in terms of a linear combination of complex exponentials using the discrete-time Fourier transform (DTFT) and how the sequence values may be used as the coefficients in a power series expansion of a complex-valued function of z. For finite-length sequences there is another representation, called the discrete Fourier transform (DFT). Unlike the DTFT, which is a continuous function of a continuous variable, ω, the DFT is a sequence that corresponds to samples of the DTFT. Such a representation is very useful for digital computations and for digital hardware implementations. In this chapter, we look at the DFT, explore its properties, and see how it may be used to perform such tasks as digital filtering and evaluating the frequency response of a linear shift-invariant system.

6.2 DISCRETE FOURIER SERIES

Let $\tilde{x}(n)$ be a periodic sequence with a period N:

$$\tilde{x}(n) = \tilde{x}(n+N)$$

Although, strictly speaking, $\tilde{x}(n)$ does not have a Fourier transform because it is not absolutely summable, it can be expressed in terms of a discrete Fourier series (DFS) as follows:

$$\tilde{x}(n) = \frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}(k)e^{j2\pi nk/N} \tag{6.1}$$

which is a decomposition of $\tilde{x}(n)$ into a sum of N harmonically related complex exponentials. The values of the discrete Fourier series coefficients, $\tilde{X}(k)$, may be derived by multiplying both sides of this expansion by $e^{-j2\pi nl/N}$, summing over one period, and using the fact that the complex exponentials are orthogonal:

$$\sum_{k=0}^{N-1} e^{j2\pi n(k-l)/N} = \begin{cases} N & k=l \\ 0 & k\neq l \end{cases}$$

The result is

$$\tilde{X}(k) = \sum_{n=0}^{N-1}\tilde{x}(n)e^{-j2\pi nk/N} \tag{6.2}$$

Note that the DFS coefficients are periodic with a period N:

$$\tilde{X}(k+N) = \tilde{X}(k)$$

Equations (*6.1*) and (*6.2*) form a DFS pair, and we write

$$\tilde{x}(n) \overset{DFS}{\Longleftrightarrow} \tilde{X}(k)$$

EXAMPLE 6.2.1 Let us find the discrete Fourier series representation for the sequence

$$\tilde{x}(n) = \sum_{k=-\infty}^{\infty} x(n-10k)$$

where
$$x(n) = \begin{cases} 1 & 0 \le n < 5 \\ 0 & \text{else} \end{cases}$$

Note that $\tilde{x}(n)$ is a periodic sequence with a period $N = 10$. Therefore, the DFS coefficients are

$$\tilde{X}(k) = \sum_{n=0}^{9} \tilde{x}(n) e^{-j2\pi nk/10} = \sum_{n=0}^{4} e^{-j2\pi nk/10} = \frac{1 - e^{-j\pi k}}{1 - e^{-j\pi k/5}}$$

which, for $0 \le k \le 9$, may be simplified to

$$\tilde{X}(k) = \begin{cases} 5 & k = 0 \\ \dfrac{2}{1 - e^{-j\pi k/5}} & k \text{ odd} \\ 0 & k \text{ even} \end{cases}$$

The DFS coefficients for all other values of k may be found from the periodicity of $\tilde{X}(k)$:

$$\tilde{X}(k + N) = \tilde{X}(k)$$

A notational simplification that is often used for the DFS is to define

$$W_N \equiv e^{-j2\pi/N}$$

for the complex exponentials and write the DFS pair as follows:

$$\tilde{x}(n) = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}(k) W_N^{-nk}$$
$$\tilde{X}(k) = \sum_{n=0}^{N-1} \tilde{x}(n) W_N^{nk}$$

The discrete Fourier series has a number of useful and interesting properties. A few of these properties are described below.

Linearity

The DFS pair satisfies the property of linearity. Specifically, if $\tilde{x}_1(n)$ and $\tilde{x}_2(n)$ are periodic with period N, the DFS coefficients of the sum are equal to the sum of the coefficients for $\tilde{x}_1(n)$ and $\tilde{x}_2(n)$ individually,

$$\tilde{x}_1(n) + \tilde{x}_2(n) \overset{DFS}{\Longleftrightarrow} \tilde{X}_1(k) + \tilde{X}_2(k)$$

Shift

If a periodic sequence $\tilde{x}(n)$ is shifted, the DFS coefficients are multiplied by a complex exponential. In other words, if $\tilde{X}(k)$ are the DFS coefficients for $\tilde{x}(n)$, the DFS coefficients for $\tilde{y}(n) = \tilde{x}(n - n_0)$ are

$$\tilde{Y}(k) = W_N^{kn_0} \tilde{X}(k)$$

Similarly, if $\tilde{x}(n)$ is multiplied by a complex exponential,

$$\tilde{y}(n) = W_N^{nk_0} \tilde{x}(n)$$

the DFS coefficients of $\tilde{x}(n)$ are shifted:

$$\tilde{Y}(k) = \tilde{X}(k + k_0)$$

Periodic Convolution

If $\tilde{h}(n)$ and $\tilde{x}(n)$ are periodic with a period N with DFS coefficients $\tilde{H}(k)$ and $\tilde{X}(k)$, respectively, the sequence with DFS coefficients

$$\tilde{Y}(k) = \tilde{H}(k)\tilde{X}(k)$$

is formed by *periodically convolving* $\tilde{h}(n)$ with $\tilde{x}(n)$ as follows:

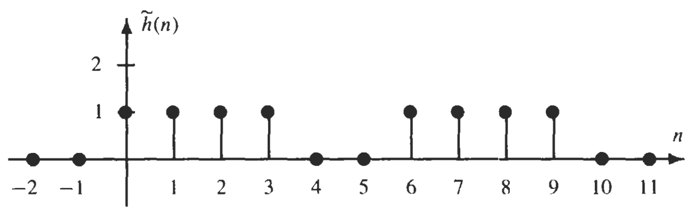
$$\tilde{y}(n) = \sum_{k=0}^{N-1} \tilde{h}(k)\tilde{x}(n-k) \tag{6.3}$$

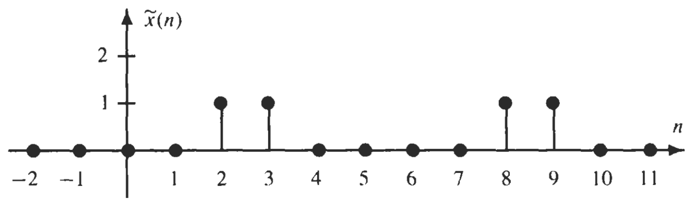
Notationally, the periodic convolution of two sequences is written as

$$\tilde{y}(n) = \tilde{h}(n) \circledast \tilde{x}(n)$$

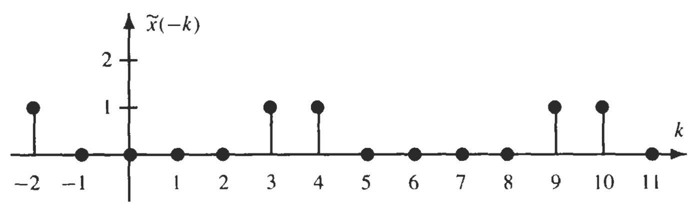
The only difference between periodic and linear convolution is that, with periodic convolution, the sum is only evaluated over a single period, whereas with linear convolution the sum is taken over all values of k.

EXAMPLE 6.2.2 Let us periodically convolve the two sequences pictured below that have a period $N = 6$.

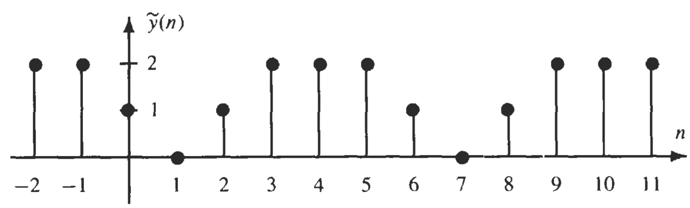




The periodic convolution of two sequences may be performed graphically, analytically, or using the DFS. In this problem, we will use the graphical approach. We begin by plotting $\tilde{x}(n-k)$ versus k. This sequence, for $n = 0$, is illustrated below.



The value of $\tilde{y}(0)$ is then found by summing the product $\tilde{h}(k)\tilde{x}(-k)$ from $k = 0$ to $k = 5$. The result is $\tilde{y}(0) = 1$. Next, $\tilde{x}(-k)$ is shifted to the right by one and multiplied by $\tilde{h}(k)$. Because the only two nonzero values of $\tilde{x}(1-k)$ are at $k = 4, 5$, the product $\tilde{h}(k)\tilde{x}(1-k)$ is equal to zero, and $\tilde{y}(1) = 0$. This process is continued until we have one period of $\tilde{y}(n)$. The result is illustrated below.



6.3 DISCRETE FOURIER TRANSFORM

The DFT is an important decomposition for sequences that are finite in length. Whereas the DTFT is a mapping from a sequence to a function of a continuous variable, ω,

$$x(n) \overset{DTFT}{\Longleftrightarrow} X(e^{j\omega})$$

the DFT is a mapping from a sequence, $x(n)$, to another sequence, $X(k)$,

$$x(n) \overset{DFT}{\Longleftrightarrow} X(k)$$

The DFT may be easily developed from the discrete Fourier series representation for periodic sequences. Let $x(n)$ be a finite-length sequence of length N that is equal to zero outside the interval $[0, N-1]$. A periodic sequence $\tilde{x}(n)$ may be formed from $x(n)$ as follows:

$$\tilde{x}(n) = \sum_{k=-\infty}^{\infty} x(n+kN)$$

This periodic extension may be expressed as follows:

$$\tilde{x}(n) = x(n \bmod N) \equiv x((n))_N$$

where $(n \bmod N)$ and $((n))_N$ are taken to mean "n modulo N." That is to say, if n is written in the form $n = kN + l$ where $0 \le l < N$,

$$(n \bmod N) = ((n))_N = l$$

For example, $((13))_8 = 5$ and $((-6))_8 = 2$.

A periodic sequence may be expanded using the DFS as in Eq. *(6.1)*. Because $x(n) = \tilde{x}(n)$ for $n = 0, 1, \ldots, N-1$, $x(n)$ may similarly be expanded as follows:

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} \tilde{X}(k) e^{j2\pi nk/N} \qquad 0 \le n < N$$

Because the DFS coefficients are periodic, if we let $X(k)$ be one period of $\tilde{X}(k)$ and replace $\tilde{X}(k)$ in the sum with $X(k)$, then we have

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N} \qquad 0 \le n < N \tag{6.4}$$

The sequence $X(k)$ is called the N-point DFT of $x(n)$. These coefficients are related to $x(n)$ as follows:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \qquad 0 \le k < N \tag{6.5}$$

Equations *(6.4)* and *(6.5)* form a DFT pair, and we write

$$x(n) \overset{DFT}{\Longleftrightarrow} X(k)$$

This expansion is valid for *complex-valued* as well as real-valued sequences.

Comparing the definition of the DFT of $x(n)$ to the DTFT, it follows that the DFT coefficients are *samples* of the DTFT:

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi nk/N} = X(e^{j\omega})\big|_{\omega=2\pi k/N}$$

Alternatively, the DFT coefficients correspond to N samples of $X(z)$ that are taken at N equally spaced points around the unit circle:

$$X(k) = X(z)\Big|_{z=\exp\{j2\pi k/N\}}$$

6.4 DFT PROPERTIES

In this section, we list some of the properties of the DFT. Because each sequence is assumed to be finite in length, some care must be exercised in manipulating DFTs.

Linearity

If $x_1(n)$ and $x_2(n)$ have N-point DFTs $X_1(k)$ and $X_2(k)$, respectively,

$$ax_1(n) + bx_2(n) \stackrel{DFT}{\Longleftrightarrow} aX_1(k) + bX_2(k)$$

In using this property, it is important to ensure that the DFTs are the same length. If $x_1(n)$ and $x_2(n)$ have different lengths, the shorter sequence must be *padded* with zeros in order to make it the same length as the longer sequence. For example, if $x_1(n)$ is of length N_1 and $x_2(n)$ is of length N_2 with $N_2 > N_1$, $x_1(n)$ may be considered to be a sequence of length N_2 with the last $N_2 - N_1$ values equal to zero, and DFTs of length N_2 may be taken for both sequences.

Symmetry

If $x(n)$ is real-valued, $X(k)$ is *conjugate symmetric*,

$$X(k) = X^*((-k)) = X^*((N-k))_N$$

and if $x(n)$ is imaginary, $X(k)$ is *conjugate antisymmetric*,

$$X(k) = -X^*((-k)) = -X^*((N-k))_N$$

Circular Shift

The circular shift of a sequence $x(n)$ is defined as follows:

$$x((n-n_0))_N\mathcal{R}_N(n) = \tilde{x}(n-n_0)\mathcal{R}_N(n)$$

where n_0 is the amount of the shift and $\mathcal{R}_N(n)$ is a rectangular window:

$$\mathcal{R}_N(n) = \begin{cases} 1 & 0 \le n < N \\ 0 & \text{else} \end{cases}$$

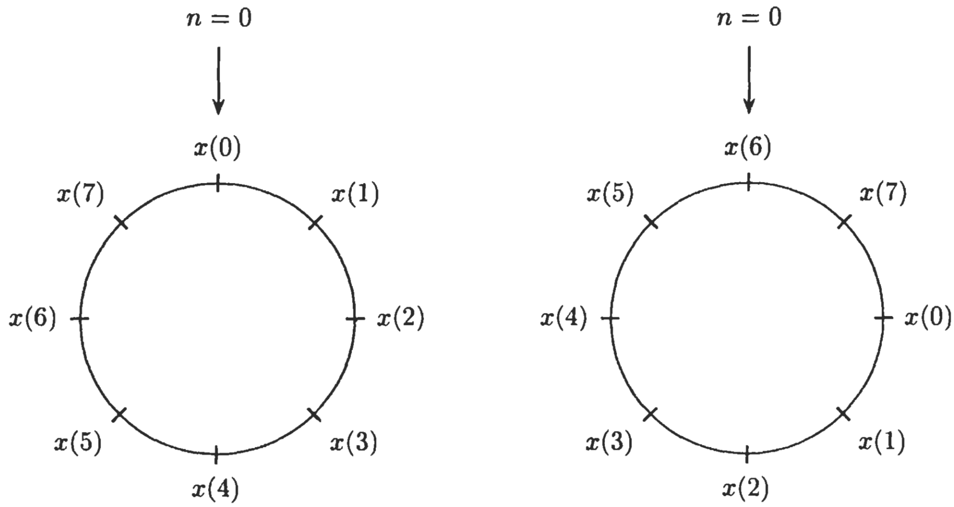
A circular shift may be visualized as follows. Suppose that the values of a sequence $x(n)$, from $n = 0$ to $n = N-1$, are marked around a circle as illustrated in Fig. 6-1 or in an eight-point sequence. A circular shift to the right by n_0 corresponds to a rotation of the circle n_0 positions in a clockwise direction. An example illustrating the circular shift of a four-point sequence is shown in Fig. 6-2. Another way to circularly shift a sequence is to form the periodic sequence $\tilde{x}(n)$, perform a linear shift, $\tilde{x}(n-n_0)$, and then extract one period of $\tilde{x}(n-n_0)$ by multiplying by a rectangular window.

If a sequence is circularly shifted, the DFT is multiplied by a complex exponential,

$$x((n-n_0))_N\mathcal{R}_N(n) \stackrel{DFT}{\Longleftrightarrow} W_N^{n_0k}X(k) \tag{6.6}$$

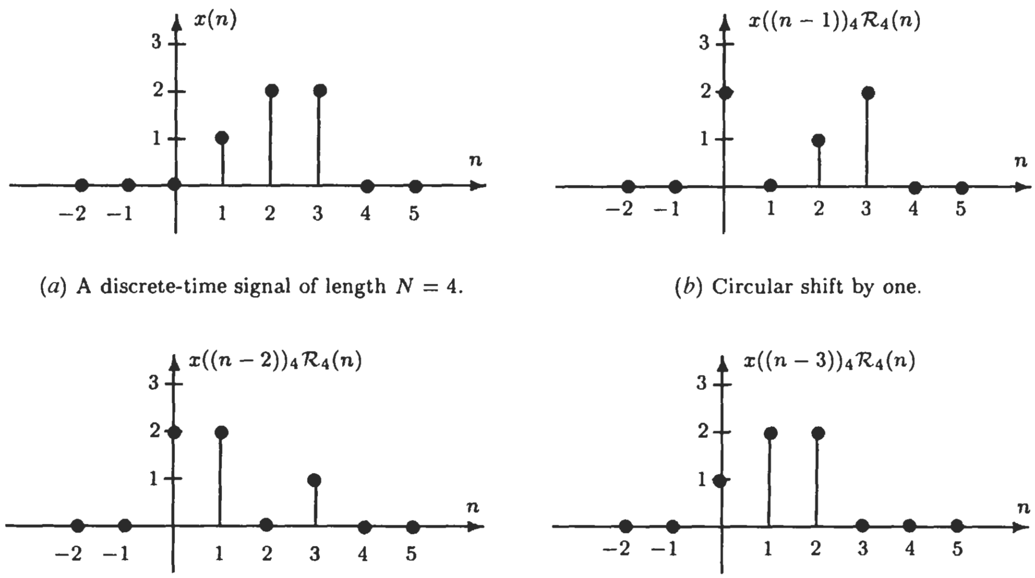
Similarly, with a circular shift of the DFT, $X((k-k_0))_N$, the sequence is multiplied by a complex exponential,

$$W_N^{nk_0}x(n) \stackrel{DFT}{\Longleftrightarrow} X((k+k_0))_N \tag{6.7}$$



(*a*) An eight-point sequence. (*b*) Circular shift by two.

Fig. 6-1. Visualizing a circular shift by rotating a circle that has the sequence values written around the circle.



(*a*) A discrete-time signal of length $N = 4$. (*b*) Circular shift by one.

(*c*) Circular shift by two. (*d*) Circular shift by three.

Fig. 6-2. The circular shift of a four-point sequence.

Circular Convolution

Let $h(n)$ and $x(n)$ be finite-length sequences of length N with N-point DFTs $H(k)$ and $X(k)$, respectively. The sequence that has a DFT equal to the product $Y(k) = H(k)X(k)$ is

$$y(n) = \left[\sum_{k=0}^{N-1} \tilde{h}(k)\tilde{x}(n-k)\right]\mathcal{R}_N(n) = \left[\sum_{k=0}^{N-1} \tilde{h}(n-k)\tilde{x}(k)\right]\mathcal{R}_N(n) \tag{6.8}$$

where $\tilde{x}(n)$ and $\tilde{h}(n)$ are the periodic extensions of the sequences $x(n)$ and $h(n)$, respectively. Because $\tilde{h}(n) = h(n)$ for $0 \le n < N$, the sum in Eq. (6.8) may also be written as

$$y(n) = \left[\sum_{k=0}^{N-1} h(k)\tilde{x}(n-k)\right]\mathcal{R}_N(n) \tag{6.9}$$

The sequence $y(n)$ in Eq. (6.9) is the *N-point circular convolution* of $h(n)$ with $x(n)$, and it is written as

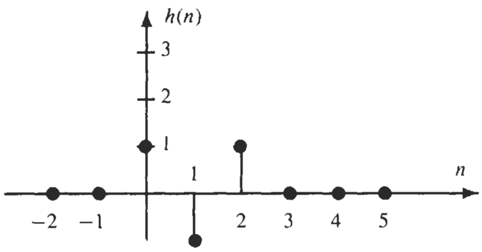
$$y(n) = h(n) \circledN x(n) = x(n) \circledN h(n)$$

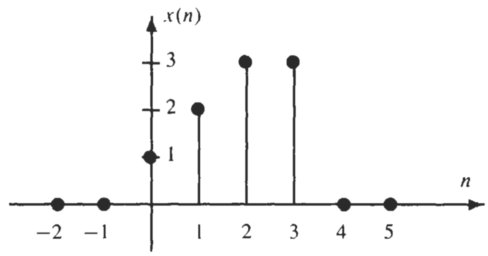
The circular convolution of two finite-length sequences $h(n)$ and $x(n)$ is equivalent to one period of the periodic convolution of the periodic sequences $\tilde{h}(n)$ and $\tilde{x}(n)$,

$$y(n) = h(n) \circledN x(n) = [\tilde{h}(n) \circledast \tilde{x}(n)]\mathcal{R}_N(n)$$

In general, circular convolution is not the same as *linear* convolution, and N-point circular convolution is different, in general, from M-point circular convolution when $M \neq N$.

EXAMPLE 6.4.1 Let us perform the four-point circular convolution of the two sequences $h(n)$ and $x(n)$ shown below.





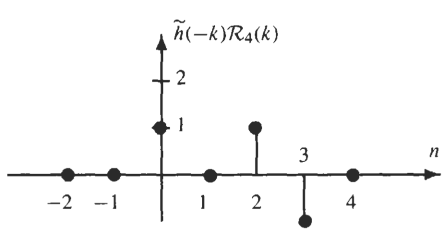
The four-point circular convolution is

$$y(n) = \left[\sum_{k=0}^{3} \tilde{h}(n-k)\tilde{x}(k)\right]\mathcal{R}_4(n)$$

which may be performed graphically, as follows. The value of $y(n)$ at $n = 0$ is

$$y(0) = \sum_{k=0}^{3} \tilde{h}(-k)\tilde{x}(k)$$

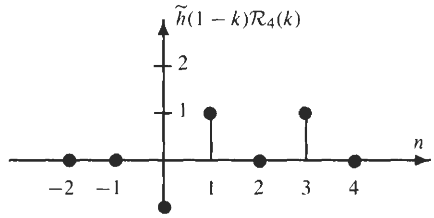
Shown in the figure below is a plot of the sequence $\tilde{h}(-k)\mathcal{R}_4(k)$.



To evaluate $y(0)$, we multiply this sequence by $x(k)$ and sum the product from $k = 0$ to $k = 3$. The result is $y(0) = 1$. Next, to find the value of $y(1)$, we evaluate the sum

$$y(1) = \sum_{k=0}^{3} \tilde{h}(1-k)\tilde{x}(k)$$

Shown in the figure below is a plot of $\tilde{h}(1-k)\mathcal{R}_4(n)$.



Multiplying by $\tilde{x}(k)$ and summing from $k=0$ to $k=3$, we find that $y(1)=4$. Repeating for $n=2$ and $n=3$, we have

$$y(2)=\sum_{k=0}^{3}\tilde{h}(2-k)\tilde{x}(k)=2$$

$$y(3)=\sum_{k=0}^{3}\tilde{h}(3-k)\tilde{x}(k)=2$$

Therefore, $$y(n)=h(n)\,④\,x(n)=\delta(n)+4\delta(n-1)+2\delta(n-2)+2\delta(n-3)$$

By comparison, the linear convolution of $h(n)$ with $x(n)$ is the following six-length sequence:

$$h(n)*x(n)=\delta(n)+\delta(n-1)+2\delta(n-2)+2\delta(n-3)+3\delta(n-5)$$

Another way to perform circular convolution is to compute the DFTs of each sequence, multiply, and compute the inverse DFT.

EXAMPLE 6.4.2 Let us perform the N-point circular convolution of $x_1(n)$ and $x_2(n)$ where

$$x_1(n)=x_2(n)=\begin{cases}1 & 0\le n\le N-1\\ 0 & \text{else}\end{cases}$$

Because the N-point DFTs of $x_1(n)$ and $x_2(n)$ are

$$X_1(k)=X_2(k)=\sum_{n=0}^{N-1}W_N^{nk}=\begin{cases}N & k=0\\ 0 & \text{else}\end{cases}$$

then $$X(k)=X_1(k)X_2(k)=\begin{cases}N^2 & k=0\\ 0 & \text{else}\end{cases}$$

Therefore, the N-point circular convolution of $x_1(n)$ with $x_2(n)$ is the inverse DFT of $X(k)$, which is

$$x(n)=\begin{cases}N & 0\le n\le N-1\\ 0 & \text{else}\end{cases}$$

Circular Versus Linear Convolution

In general, circular convolution is not the same as linear convolution. However, there is a simple relationship between circular and linear convolution that illustrates what steps must be taken in order to ensure that they are the same. Specifically, let $x(n)$ and $h(n)$ be finite-length sequences and let $y(n)$ be the linear convolution

$$y(n)=x(n)*h(n)$$

The N-point circular convolution of $x(n)$ with $h(n)$ is related to $y(n)$ as follows:

$$h(n)\,Ⓝ\,x(n)=\left[\sum_{k=-\infty}^{\infty}y(n+kN)\right]\mathcal{R}_N(n) \tag{6.10}$$

In other words, the circular convolution of two sequences is found by performing the linear convolution and *aliasing* the result.

An important property that follows from Eq. (*6.10*) is that if $y(n)$ is of length N or less, $y(n-kN)\mathcal{R}_N(n)=0$ for $k \neq 0$ and

$$h(n) \circledN x(n) = h(n) * x(n)$$

that is, circular convolution is equivalent to linear convolution. Thus, if $h(n)$ and $x(n)$ are finite-length sequences of length N_1 and N_2, respectively, $y(n) = h(n) * x(n)$ is of length $N_1 + N_2 - 1$, and the N-point circular convolution is equivalent to linear convolution provided $N \geq N_1 + N_2 - 1$.

EXAMPLE 6.4.3 Let us find the four-point circular convolution of the sequences $h(n)$ and $x(n)$ in Example 6.4.1. Because the linear convolution is

$$y(n) = \delta(n) + \delta(n-1) + 2\delta(n-2) + 2\delta(n-3) + 3\delta(n-5)$$

we may set up a table to evaluate the sum

$$h(n) \circledN x(n) = \left[\sum_{k=-\infty}^{\infty} y(n+kN)\right]\mathcal{R}_N(n)$$

This is done by listing the values of the sequence $y(n+kN)$ in a table and summing these values for $n = 0, 1, 2, 3$. Note that the only sequences that have nonzero values in the interval $0 \leq n \leq 3$ are $y(n)$ and $y(n+4)$, and these are the only sequences that need be listed. Thus, we have

n	0	1	2	3	4	5	6	7
$y(n)$	1	1	2	2	0	3	0	0
$y(n+4)$	0	3	0	0	0	0	0	0
$h(n) \circled{4} x(n)$	1	4	2	2	–	–	–	–

Summing the columns for $0 \leq n \leq 3$, we have

$$h(n) \circled{4} x(n) = \delta(n) + 4\delta(n-1) + 2\delta(n-2) + 3\delta(n-3)$$

which is the same as computed in Example 6.4.1.

6.5 SAMPLING THE DTFT

Let $x(n)$ be a sequence with a DTFT $X(e^{j\omega})$, and consider the finite-length sequence $y(n)$ of length N whose DFT coefficients are obtained by sampling $X(e^{j\omega})$ at $\omega_k = 2\pi k/N$:

$$Y(k) = X(e^{j\omega})|_{\omega=2\pi k/N} \qquad k = 0, 1, \ldots, N-1 \tag{6.11}$$

Because the DTFT is equal to the z-transform evaluated around the unit circle, these DFT coefficients may also be obtained by sampling $X(z)$ at N equally spaced points around the unit circle at $z_k = \exp\{j2\pi k/N\}$:

$$Y(k) = X(z)|_{z=\exp\{j2\pi k/N\}} \qquad k = 0, 1, \ldots, N-1$$

These sampling points are illustrated in Fig. 6-3 for $N = 8$. To express the sequence values $y(n)$ in terms of $x(n)$, we begin by finding the inverse DFT of $Y(k)$:

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k)e^{j2\pi nk/N} \tag{6.12}$$

Because the DFT coefficients $Y(k)$ are samples of the DTFT of $x(n)$,

$$Y(k) = X(e^{j\omega})|_{\omega=2\pi k/N} = \sum_{l=-\infty}^{\infty} x(l)e^{-j2\pi lk/N}$$

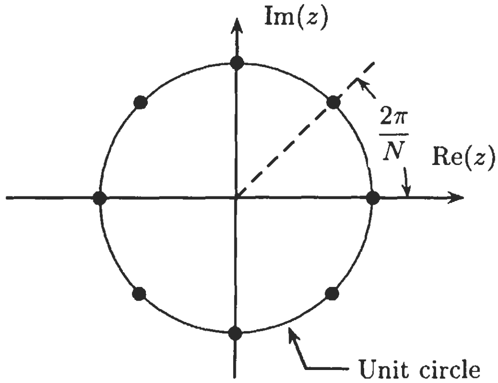


Fig. 6-3. Sampling the z-transform at eight equally spaced points around the unit circle.

Substituting this expression for $Y(k)$ into Eq. (6.12) gives

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left\{\sum_{l=-\infty}^{\infty} x(l)e^{-j2\pi lk/N}\right\}e^{j2\pi nk/N} = \frac{1}{N}\sum_{l=-\infty}^{\infty} x(l)\left\{\sum_{k=0}^{N-1} e^{j2\pi(n-l)k/N}\right\}$$

The term in brackets is equal to N when $l = n + mN$ where m is an integer, and it is equal to zero otherwise. Therefore,

$$y(n) = \left[\sum_{m=-\infty}^{\infty} x(n - mN)\right]\mathcal{R}_N(n) \tag{6.13}$$

and it follows that $y(n)$ is formed by *aliasing* $x(n)$ in time.

6.6 LINEAR CONVOLUTION USING THE DFT

The DFT provides a convenient way to perform convolutions without having to evaluate the convolution sum. Specifically, if $h(n)$ is N_1 points long and $x(n)$ is N_2 points long, $h(n)$ may be linearly convolved with $x(n)$ as follows:

1. Pad the sequences $h(n)$ and $x(n)$ with zeros so that they are of length $N \geq N_1 + N_2 - 1$.
2. Find the N-point DFTs of $h(n)$ and $x(n)$.
3. Multiply the DFTs to form the product $Y(k) = H(k)X(k)$.
4. Find the inverse DFT of $Y(k)$.

It would appear that there is considerably more effort involved in performing convolutions using DFTs. However, significant computational savings may be realized with this approach if the DFTs are computed efficiently. As we will see in Chap. 7, the fast Fourier transform (FFT) provides such an algorithm.

In spite of its computational advantages, there are some difficulties with the DFT approach. For example, if $x(n)$ is *very long*, we must commit a significant amount of time computing very long DFTs and in the process accept very long processing delays. In some cases, it may even be possible that $x(n)$ is *too long* to compute the DFT. The solution to these problems is to use *block convolution*, which involves segmenting the signal to be filtered, $x(n)$, into sections. Each section is then filtered with the FIR filter $h(n)$, and the filtered sections are pieced together to form the sequence $y(n)$. There are two block convolution techniques. The first is overlap-add, and the second is overlap-save.