QPSK

* In QPSK System, information is carried by 4 phase of the Sinusoidal carrier i.e $\pi/4$, $3\pi/4$, $5\pi/4$ & $7\pi/4$.

* A QPSK Signal can be represented in time domain as:

$$S_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos\left[2\pi f_c t + (2i-1)\pi/4\right] & 0 \leq t \leq T_b \\ 0 & \text{elsewhere} \end{cases} \longrightarrow (1)$$

Where $i = 1, 2, 3, 4$ and

$E \rightarrow$ Signal energy per Symbol

$T \rightarrow$ Symbol duration.

* There are four message points & associated signal vectors are defined by

$$S_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos\left[2\pi f_c t + \theta_i\right] & ; 0 \leq t \leq T \\ 0 & ; \text{otherwise} \end{cases} \longrightarrow (2)$$

Where $\theta_i = (2i-1)\pi/4$

i	$\theta_i = (2i-1)\pi/4$
1	$\pi/4$
2	$3\pi/4$
3	$5\pi/4$
4	$7\pi/4$

We can write eq (2) as

$S_1(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + \pi/4)$; dibit 10

$S_2(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + 3\pi/4)$; dibit 00

$S_3(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + 5\pi/4)$; dibit 01

$S_4(t) = \sqrt{\frac{2E}{T}} \cos(2\pi f_c t + 7\pi/4)$; dibit 11

$\boxed{\cos(A+B) = \cos A \cdot \cos B - \sin A \cdot \sin B}$

* Eq (1) can be written as

$$S_i(t) = \begin{cases} \sqrt{\frac{2E}{T}} \cos 2\pi f_c t \cdot \cos\left[(2i-1)\pi/4\right] - \sqrt{\frac{2E}{T}} \sin 2\pi f_c t \cdot \sin\left[(2i-1)\pi/4\right] & ; 0 \leq t \leq T \\ 0 & ; \text{otherwise} \end{cases} \longrightarrow (3)$$

* From eq (3), we observe that there are two orthonormal basis functions $\phi_1(t)$ & $\phi_2(t)$ defined by

$$\phi_1(t) = \sqrt{\frac{2}{T}} \cos(2\pi f_c t) \quad ; 0 \leq t \leq T \longrightarrow (4)$$

$$\phi_2(t) = \sqrt{\frac{2}{T}} \sin(2\pi f_c t) \quad ; 0 \leq t \leq T$$

The elements of the Signal vector, namely S_{i1} & S_{i2}.

i	$S_i(t)$	I/p dibit	Phase of QPSK Signal	Co-ordinates of message points	
				S_{i1}	S_{i2}
1	$S_1(t)$	10	$\pi/4$	$+\sqrt{E/2}$	$-\sqrt{E/2}$
2	$S_2(t)$	00	$3\pi/4$	$-\sqrt{E/2}$	$-\sqrt{E/2}$
3	$S_3(t)$	01	$5\pi/4$	$-\sqrt{E/2}$	$+\sqrt{E/2}$
4	$S_4(t)$	11	$7\pi/4$	$+\sqrt{E/2}$	$+\sqrt{E/2}$

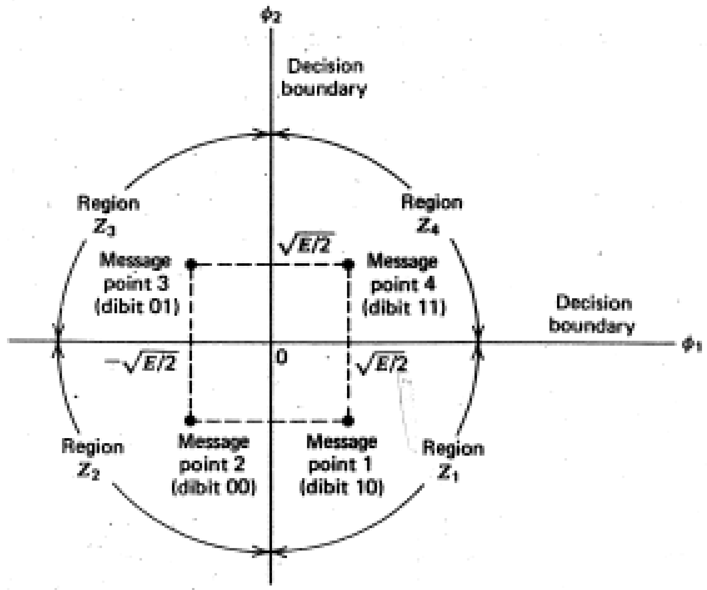
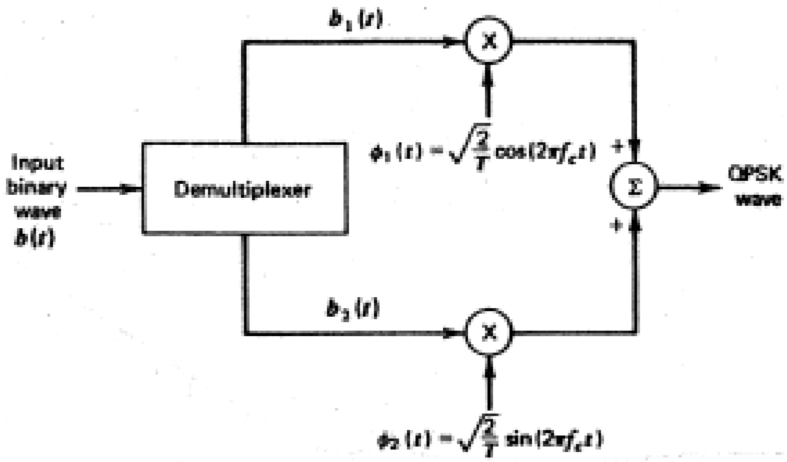


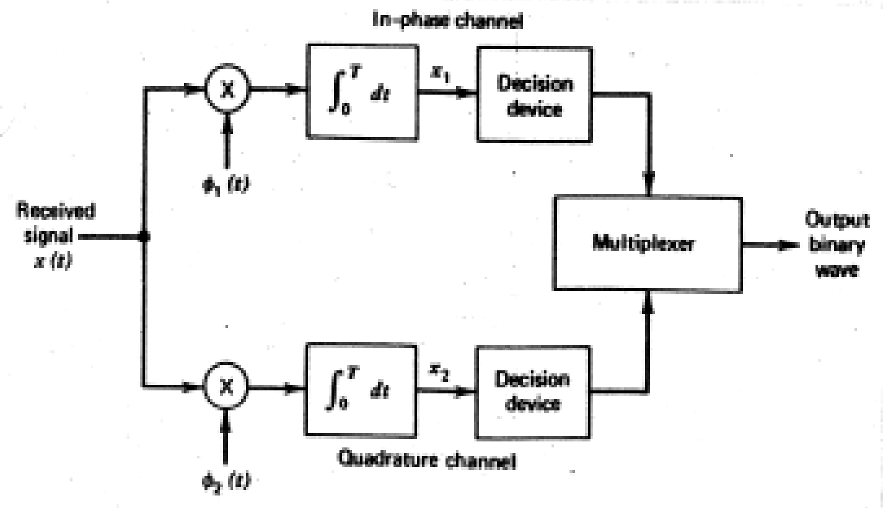
Figure Signal space diagram for coherent QPSK system.

QPSK Transmitter :-



* The I/p binary Sequence b(t) represented in polar form is divided into odd $[b_1(t)]$ & even $[b_2(t)]$ numbered bits by using demultiplexer. They are denoted as $b_1(t)$ & $b_2(t)$.
* These two Sequences, phase modulate two carrier Signal of Same frequency but quadrature in phase.
* Since each Symbol carries two bits, the Signalling rate decreased.
  ∴ BW required is half the bandwidth required compared to BPSK

QPSK Receiver :-



(b)

* The QPSK receiver Consists of a pair of Correlators with locally generated carrier Signals $\phi_1(t)$ & $\phi_2(t)$.
* The o/p of the two Correlators are x_1 & x_2 are Compared with a threshold '0'.

If $x_1 > 0 \rightarrow$ decision is made in favour of Symbol 1.

If $x_1 < 0 \rightarrow$ decision is made in favour of Symbol 0.

If $x_2 > 0 \rightarrow$ decision is made in favour of Symbol 1.

If $x_2 < 0 \rightarrow$ decision is made in favour of Symbol 0.

* The two o/p's are combined in a multiplier to reproduce original binary Sequence.

Probability of error in QPSK :-

* The Signal points S_1, S_2, S_3 & S_4 are located Symmetrically in two dimensional Signal Space diagram as Shown in below figure.

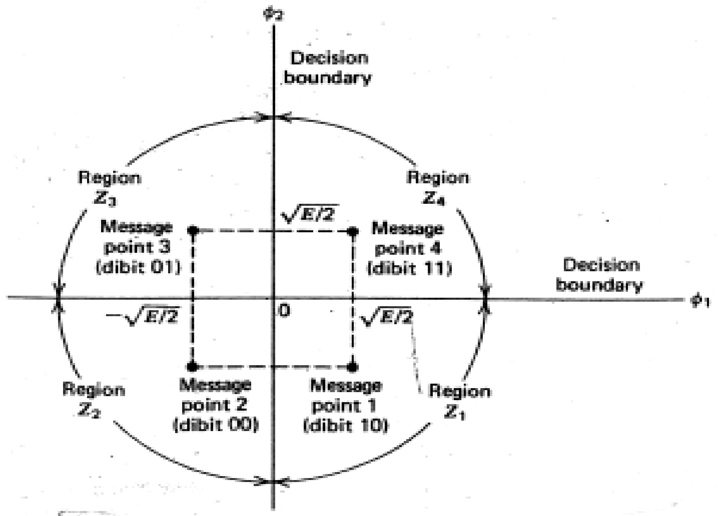


Figure Signal space diagram for coherent QPSK system.

∴ Computing probability for one message point, is remains Same for other three points.

* Consider transmission of Symbol $S_4(t)$ then received Signal $x(t)$ will be

$$x(t) = S_4(t) + w(t) \qquad 0 \leqslant t \leqslant T$$

The Samples x_1 & x_2 are computed as follows

$$x_1 = \int_0^T x(t)\,\phi_1(t)\,dt = \int_0^T \left[S_4(t) + w(t)\right]\phi_1(t)\,dt$$

$$x_1 = \int_0^T S_4(t)\,\phi_1(t)\,dt + \int_0^T w(t)\,\phi_1(t)\,dt$$

$$x_1 = S_{41} + w_1$$

$$\boxed{x_1 = \sqrt{\frac{E}{2}} + W_1}$$

* Similarly

$$x_2 = \int_0^T x(t)\,\phi_2(t) = \int_0^T [S_4(t) + w(t)]\,\phi_2(t)\cdot dt$$

$$x_2 = \int_0^T S_4(t)\cdot\phi_2(t)\,dt + \int_0^T w(t)\,\phi_2(t)\cdot dt$$

$$x_2 = S_{42} + W_2$$

$$\boxed{x_2 = \sqrt{\frac{E}{2}} + W_2}$$

* X_1 & X_2 are gaussian random variables with mean 'μ' $=\sqrt{\frac{E}{2}}$
* W_1 & W_2 are also gaussian random variables with variance $\sigma^2 = \frac{N_0}{2}$
* When Signal $S_4(t)$ is transmitted, the received Signal point lies in the decision region 'Z_4'.

If $\underset{(1)}{x_1 > 0}$ & $\underset{(1)}{x_2 > 0}$, leading to correct decision.

The Conditional PDF is given by

$$f_{X1}(x_1/S_4(t)) = \frac{1}{\sqrt{2\pi\sigma^2}}\; e^{-\left[\frac{(x_1-\mu)^2}{2\sigma^2}\right]} \quad \text{and}$$

$$f_{X2}(x_2/S_4(t)) = \frac{1}{\sqrt{2\pi\sigma^2}}\; e^{-\left[\frac{(x_2-\mu)^2}{2\sigma^2}\right]}$$

$$f_{X_1}(x_1|S_4(t)) = \frac{1}{\sqrt{\cancel{2}\pi\frac{N_0}{\cancel{2}}}}\cdot e^{-\left[\frac{\left(x_1-\sqrt{\frac{E}{2}}\right)^2}{\cancel{2}\frac{N_0}{\cancel{2}}}\right]} \quad \text{and}$$

$$f_{X2}(x_2|S_4(t)) = \frac{1}{\sqrt{\cancel{2}\pi\frac{N_0}{\cancel{2}}}}\cdot e^{-\left[\frac{\left(x_2-\sqrt{\frac{E}{2}}\right)^2}{\cancel{2}\frac{N_0}{\cancel{2}}}\right]}$$

$$\boxed{f_{X1}(x_1|S_4(t)) = \frac{1}{\sqrt{\pi N_0}}\; e^{-\left[\frac{x_1-\sqrt{\frac{E}{2}}}{\sqrt{N_0}}\right]^2}} \longrightarrow (1) \quad \text{and}$$

$$\boxed{f_{X2}(x_2|S_4(t)) = \frac{1}{\sqrt{\pi N_0}}\; e^{-\left[\frac{x_2-\sqrt{\frac{E}{2}}}{\sqrt{N_0}}\right]^2}} \longrightarrow (2)$$

* Let us assume $S_4(t)$ is transmitted. If the received Signal 'x' Should fall in region Z_4 i.e. both x_1 & x_2 Should be +ve.

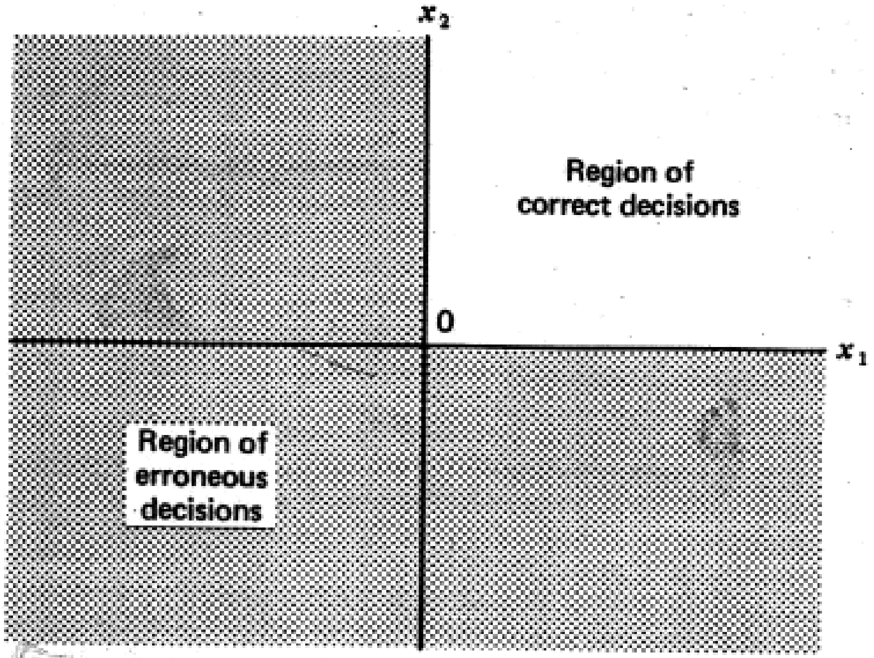


Figure Illustrating the region of correct decisions and the region of erroneous decisions, given that signal $s_4(t)$ was transmitted.

* Probability of correct decision 'P_c' is equal to the product of Conditional probabilities of events $x_1 > 0$ & $x_2 > 0$, both given that $S_4(t)$ was transmitted.

Region Z_4 : $\boxed{\begin{array}{l} 0 \leq x_1 \leq \infty \\ 0 \leq x_2 \leq \infty \end{array}}$

Sub eq ① & ② in eq ③

$$P_c = \int_0^\infty f_{x_1}(x_1/s_4(t))\, dx_1 \times \int_0^\infty f_{x_2}(x_2/s_4(t)) \cdot dx_2 \longrightarrow ③$$

$$P_c = \int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\left[\frac{x_1 - \sqrt{\frac{E}{2}}}{\sqrt{N_0}}\right]^2} dx_1 \cdot \int_0^\infty \frac{1}{\sqrt{\pi N_0}} e^{-\left[\frac{x_2 - \sqrt{\frac{E}{2}}}{\sqrt{N_0}}\right]} \cdot dx_2$$

Let

$Z = \frac{x_1 - \sqrt{\frac{E}{2}}}{\sqrt{N_0}}$	$Z = \frac{x_2 - \sqrt{\frac{E}{2}}}{\sqrt{N_0}}$
$dZ = \frac{dx_1 - 0}{\sqrt{N_0}}$	$dZ = \frac{dx_2 - 0}{\sqrt{N_0}}$
$\boxed{dx_1 = \sqrt{N_0}\, dZ}$	$\boxed{dx_2 = \sqrt{N_0}\, dZ}$

Limits

$x_1 = \infty$	$x_2 = \infty$
then, $Z = \infty$	then, $Z = \infty$

$x_1 = 0, \quad Z = \dfrac{x_1 - \sqrt{\frac{E}{2}}}{\sqrt{N_0}}$	$x_2 = 0, \quad Z = \dfrac{x_2 - \sqrt{\frac{E}{2}}}{\sqrt{N_0}}$
$Z = \dfrac{0 - \sqrt{\frac{E}{2}}}{\sqrt{N_0}}$	$Z = \dfrac{0 - \sqrt{\frac{E}{2}}}{\sqrt{N_0}}$
$\boxed{Z = -\sqrt{\dfrac{E}{2N_0}}}$	$\boxed{Z = -\sqrt{\dfrac{E}{2N_0}}}$

$$P_c = \int_{-\sqrt{\frac{E}{2N_0}}}^{\infty} \frac{1}{\sqrt{\pi \cancel{N_0}}} \cdot e^{-Z^2} \cdot \sqrt{\cancel{N_0}}\, dZ \cdot \int_{-\sqrt{\frac{E}{2N_0}}}^{\infty} \frac{1}{\sqrt{\pi \cancel{N_0}}} \cdot e^{-Z^2} \sqrt{\cancel{N_0}}\, dZ$$

$$P_c = \int_{-\sqrt{\frac{E}{2N_0}}}^{\infty} \frac{1}{\sqrt{\pi}} \cdot e^{-Z^2} dZ \cdot \int_{-\sqrt{\frac{E}{2N_0}}}^{\infty} \frac{1}{\sqrt{\pi}} \cdot e^{-Z^2} . dZ$$

$$\boxed{P_c = \left[\frac{1}{\sqrt{\pi}} \int_{\sqrt{\frac{E}{2N_0}}}^{\infty} e^{-Z^2} . dZ \right]^2} \longrightarrow 4$$

$$\because \boxed{P_e' = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)}$$

From the definition of the complementary error function we have

$$\frac{1}{\sqrt{\pi}} \int_{-\sqrt{\frac{E}{2N_0}}}^{\infty} e^{-Z^2} dZ = 1 - \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) \longrightarrow 5$$

$$\because \boxed{P_c = 1 - P_e}$$

Substituting eq 5 in eq 4

$$\therefore \quad P_c = \left[1 - \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) \right]^2$$

WKT

$(a-b)^2 = a^2 + b^2 - 2ab$

$a = 1, \quad b = \frac{1}{2} \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)$

$$P_c = 1 + \frac{1}{4} \operatorname{erfc}^2\left(\sqrt{\frac{E}{2N_0}}\right) - \cancel{2}(1) \cdot \frac{1}{\cancel{2}} \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)$$

$$\boxed{P_c = 1 + \frac{1}{4} \operatorname{erfc}^2\left(\sqrt{\frac{E}{2N_0}}\right) - \operatorname{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)}$$

Thus average probability of symbol error

$$P_e = 1 - P_c$$

$$P_e = 1 - \left[1 + \frac{1}{4}\,\text{erfc}^2\left(\sqrt{\frac{E}{2N_0}}\right) - \text{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)\right]$$

$$= \cancel{1} - \cancel{1} - \frac{1}{4}\,\text{erfc}^2\left(\sqrt{\frac{E}{2N_0}}\right) + \text{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)$$

$$\boxed{P_e = \text{erfc}\left(\sqrt{\frac{E}{2N_0}}\right) - \frac{1}{4}\,\text{erfc}^2\left(\sqrt{\frac{E}{2N_0}}\right)}$$

* In the region $Z_4 : \left(\frac{E}{2N_0}\right) \gg 1$, hence we can ignore second term

$$\boxed{P_e \approx \text{erfc}\left(\sqrt{\frac{E}{2N_0}}\right)}$$

* In QPSK two bits are transmitted per symbol

thus $\boxed{E = 2E_b}$

$$P_e \approx \text{erfc}\left(\sqrt{\frac{\cancel{2}E_b}{\cancel{2}N_0}}\right)$$

$$\boxed{P_e \approx \text{erfc}\left(\sqrt{\frac{E_b}{N_0}}\right)}$$

FORMULAE

	Coherent detection	Non Coherent detection
PSK	$P_e = \frac{1}{2}\, erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$	
FSK	$P_e = \frac{1}{2}\, erfc\left(\sqrt{\frac{E_b}{2N_0}}\right)$	$P_e = \frac{1}{2} e^{\left(-E_b/2N_0\right)}$
ASK	$P_e = \frac{1}{2}\, erfc\left[\frac{1}{2}\sqrt{\frac{E_b}{N_0}}\right]$	$P_e = \frac{1}{2} e^{\left(-E_b/4N_0\right)}$
DPSK	—	$P_e = \frac{1}{2} e^{\left(\frac{-E_b}{N_0}\right)}$
QPSK	$P_e = erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$	—
MFSK	$P_e = erfc\left(\sqrt{\frac{E_b}{N_0}}\right)$	—

* $E_b = PT_b$
* $P = \frac{A_c^2}{2}$
* $T_b = \frac{1}{R_b}$
* $erf(u) = 1 - erfc(u)$
* Channel bandwidth $B_T = \frac{1}{T_b} = R_b$

Error Function Table :

Table Error Function

u	erf(u)	u	erf(u)
0.00	0.00000	1.10	0.88021
0.05	0.05637	1.15	0.89612
0.10	0.11246	1.20	0.91031
0.15	0.16800	1.25	0.92290
0.20	0.22270	1.30	0.93401
0.25	0.27633	1.35	0.94376
0.30	0.32863	1.40	0.95229
0.35	0.37938	1.45	0.95970
0.40	0.42839	1.50	0.96611
0.45	0.47548	1.55	0.97162
0.50	0.52050	1.60	0.97635
0.55	0.56332	1.65	0.98038
0.60	0.60386	1.70	0.98379
0.65	0.64203	1.75	0.98667
0.70	0.67780	1.80	0.98909
0.75	0.71116	1.85	0.99111
0.80	0.74210	1.90	0.99279
0.85	0.77067	1.95	0.99418
0.90	0.79691	2.00	0.99532
0.95	0.82089	2.50	0.99959
1.00	0.84270	3.00	0.99998
1.05	0.86244	3.30	0.999998