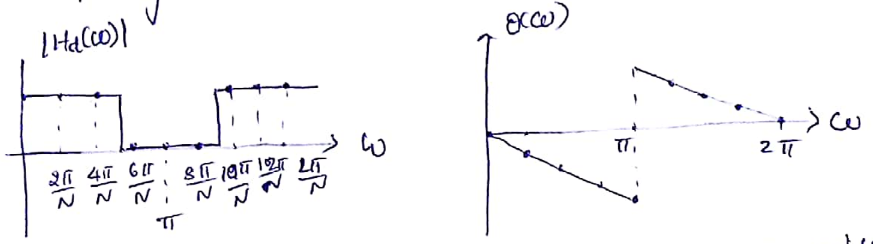
① 7

Frequency Sampling Design of FIR filters :

* Sampling the desired frequency response $H_d(\omega)$ as shown below



Sampling of desired frequency response with samples taken of $\frac{2\pi}{N}$ in the interval $(0, 2\pi)$

* N equally spaced points in the interval
* $\omega = \omega_K = \frac{2\pi K}{N}$

$$H(K) = H_d(\omega)\Big|_{\omega=\omega_K}$$ ie DFT from DTFT.

$$= H_d\left(\frac{2\pi K}{N}\right) \qquad K = 0, 1, \dots N-1$$

* To have real co-efficients for filter impulse response $h(n)$ symmetry is compulsory.

$N = \text{odd} \Rightarrow H(N-K) = H^*(K)$ $\quad K = 1, 2 \dots \frac{N-1}{2}$.

$N = \text{even} \Rightarrow H(N-K) = H^*(K)$ $\quad K = 1, \dots \frac{N}{2} - 1$ & $H\left(\frac{N}{2}\right) = 0$

* FIR coefficient $= h(n) = \text{IDFT}\{H(K)\}$.

$$h(n) = \frac{1}{N} \sum_{K=0}^{N-1} H(K)\, e^{j\frac{2\pi}{N}nK} \qquad n = 0, 1, \dots N-1$$

$$N = \text{odd} \Rightarrow h(n) = \frac{1}{N}\left[H(0) + 2\sum_{K=0}^{\frac{N-1}{2}} Re\left\{H(K)\, e^{j\frac{2\pi nK}{N}}\right\}\right]$$

$$N = \text{even} \Rightarrow h(n) = \frac{1}{N}\left[H(0) + 2\sum_{K=1}^{\frac{N}{2}-1} Re\left\{H(K)\, e^{j\frac{2\pi nK}{N}}\right\}\right]$$

* Z-T of $h(n) \Rightarrow Z\{h(n)\}$

$$H(z) = \sum_{n=0}^{N-1} h(n)\, z^{-n}$$

$$= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{K=0}^{N-1} H(K)\, e^{j\frac{2\pi nK}{N}}\right] z^{-n}$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} H(K) \sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}nK}\, z^{-n}.$$

$$H(z) = \frac{1}{N}\sum_{k=0}^{N-1} H(k) \sum_{n=0}^{N-1}\left[e^{j\frac{2\pi k}{N}} z^{-1}\right]^n$$

$$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a} \quad a \neq 1$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} H(k) \frac{1-\left[e^{j\frac{2\pi k}{N}} z^{-1}\right]^N}{1-e^{j\frac{2\pi k}{N}} z^{-1}}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} H(k) \left[\frac{1-z^{-N}}{1-e^{j\frac{2\pi k}{N}} z^{-1}}\right]$$

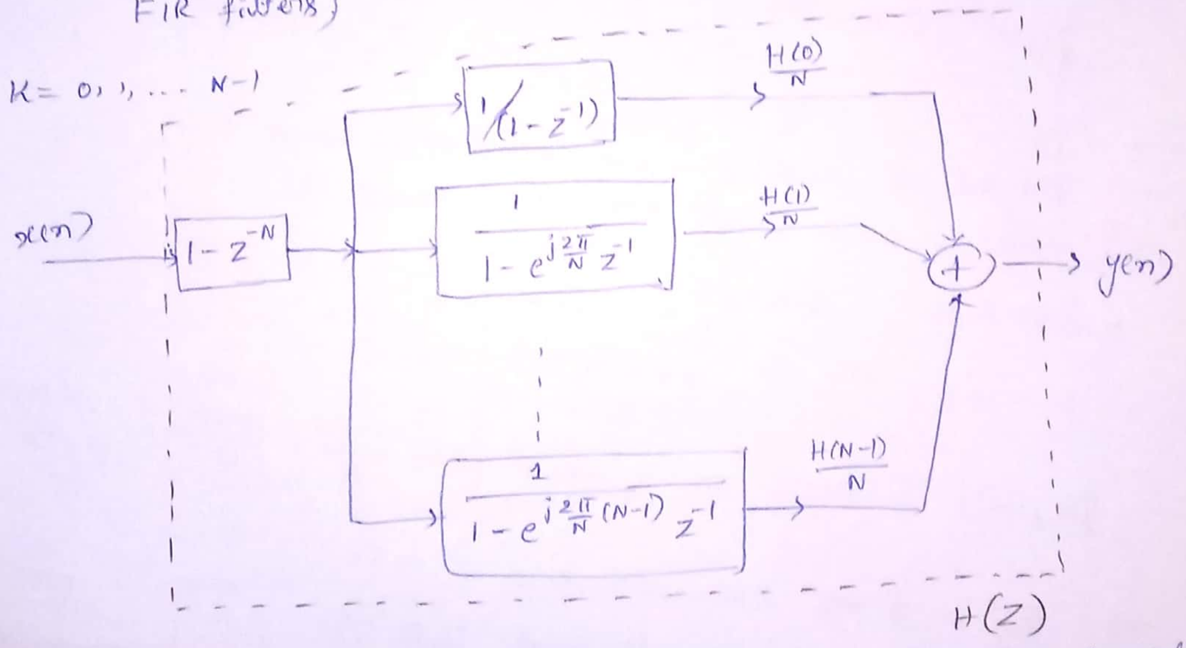
$1-z^{-1}$ contain $1-e^{j\frac{2\pi k}{N}}z^{-1}$

OBSERVATION :-

⇒ pole-zero cancellation takes place.

⇒ only zeros will be in $H(z)$

(necessary & sufficient condition for an FIR filters)



Realization of an FIR filter based on frequency sampling.

To find the frequency response :-

$$z = e^{j\omega}$$

$$H(z) = H(e^{j\omega}) = H(\omega) = \frac{1}{N}\sum_{k=0}^{N-1} H(k)\left(\frac{1-e^{-j\omega N}}{1-e^{j\frac{2\pi k}{N}}e^{-j\omega}}\right)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} H(k) \frac{e^{-j\omega\frac{N}{2}}\left(e^{j\omega\frac{N}{2}} - e^{-j\omega\frac{N}{2}}\right)}{e^{j\frac{2\pi}{N}\frac{k}{2}} e^{-j\frac{\omega}{2}}\left(e^{-j\frac{2\pi}{N}\frac{k}{2}} e^{j\frac{\omega}{2}} - e^{j\frac{2\pi k}{N2}} e^{-j\frac{\omega}{2}}\right)}$$

$$= e^{-j\omega\left(\frac{N-1}{2}\right)} \sum_{K=0}^{N-1} \frac{H(k)}{N} e^{-j\frac{\pi K}{N}} \frac{\sin \omega N/2}{\sin\left(\frac{\omega}{2} - \frac{\pi K}{N}\right)}$$

(*) Design a 17- tap linear - phase FIR filter with a cutoff frequency $W_c = \pi/2$. use frequency Sampling technique.

Soln $\omega_c = \pi/2 \Rightarrow$ LPF

$$H_d(e^{j\omega}) = H_d(\omega) = \begin{cases} e^{-j\left(\frac{N-1}{2}\right)\omega} & 0 < \omega < \pi/2 \\ 0 & \pi/2 < \omega < \pi \end{cases}$$

$\Rightarrow$ magnitude response = even symmetric about π

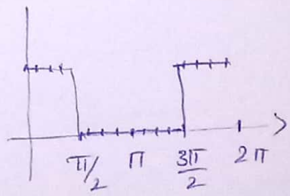
$\Rightarrow$ phase response = odd symmetric about π.

$N \Rightarrow 17$ taps $\therefore N = 17$

$\omega = \omega_K = \frac{2\pi K}{N}$ $\quad K = 0$ to $N-1$
$= 0, 1, \cdots 16$.

$|H_d(\omega)|$ & $\theta(\omega)$ on ω axis. $\omega = \frac{2\pi(1)}{17}, \frac{2\pi \times 2}{17} \cdots \frac{2\pi(16)}{17}$

$\Rightarrow \frac{N-1}{2} = 8 \Rightarrow$ 8 samples b/w 0 to π.
4 samples b/w 0 to $\pi/2$.



$$|H(K)| = \begin{cases} 1 , & 0 \le K \le 4 \\ 0 , & 5 \le K \le 12 \\ 1 . & 13 \le K \le 16 \end{cases}$$

$$\theta_K = -8\omega_K = -8 \times \frac{2\pi k}{N} = -\frac{8 \times 2\pi K}{17} = -\frac{16\pi K}{17}$$

$$\& \ \theta_K = -\frac{16\pi}{17}(K - 17) \qquad 9 \le K \le 16.$$

$$H(K) = |H(K)| \underline{/H(K)} = |H(K)| e^{j\theta_k}.$$

$$H(K) = \begin{cases} e^{-j\frac{16\pi K}{17}} & 0 \le K \le 4 \\ 0 & 5 \le K \le 12 \\ e^{-j\frac{16\pi(K-17)}{17}} & 13 \le K \le 16. \end{cases}$$

IDFT of $H(k)$

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left\{ H(k)\, e^{j\frac{2\pi nk}{N}} \right\}\right]$$

$$= \frac{1}{17}\left[H(0) + 2\sum_{k=1}^{8} Re\left\{ H(k)\, e^{j\frac{2\pi nk}{17}} \right\}\right]$$

$$= \frac{1}{17}\left[H(0) + 2\left[Re\left\{ \underbrace{H(1)}_{1} e^{j\frac{2\pi n}{17}} + \underbrace{H(2)}_{1} e^{j4\pi n/17} + \underbrace{H(3)}_{1} e^{j6\pi n/17} + \underbrace{H(4)}_{1} e^{j\frac{8\pi n}{17}} \right\}\right]\right]$$

$\because$ 5 to 12 = 0. ie 5, 6, 7 & 8 = 0.

$$= \frac{1}{17}\left[1 + 2\,Re\left\{ e^{-j\frac{16\pi}{17}} e^{j\frac{2\pi n}{17}} + e^{-j\frac{32\pi}{17}} e^{j\frac{4\pi n}{17}} + e^{-j\frac{48\pi}{17}} e^{j\frac{6\pi n}{17}} + e^{-j\frac{64\pi}{17}} e^{j\frac{8\pi n}{17}} \right\}\right]$$

$$h(n) = \frac{1}{17}\left[1 + 2\cos\left[\frac{2\pi}{17}(n-8)\right] + 2\cos\left[\frac{4\pi}{17}(n-8)\right] + 2\cos\left[\frac{16\pi}{17}(n-8)\right] + \left[2\cos\frac{8\pi}{17}(n-8)\right]\right]$$

$h(n) \Rightarrow \quad n = 0$ to $N-1 \Rightarrow 0, 1, 2$ to 16.

$\frac{N-1}{2} = \frac{16}{2} = 8$

Symmetry point

n	$h(n)$	n	$h(n)$
0	0.0398	9	0.31876
1	-0.0488	10	-0.0299
2	-0.03459	11	-0.10747
3	0.06598	12	0.03154
4	0.03154	13	0.06598
5	-0.10747	14	-0.03459
6	-0.0299	15	-0.0488
7	0.31876	16	0.0398.
8	0.5294		

Example 6.15 Determine the filter coefficients $h(n)$ obtained by samping

$$H_d(e^{j\omega}) = e^{-j(N-1)\omega/2} \quad 0 \le |\omega| \le \frac{\pi}{2}$$
$$= 0 \quad \frac{\pi}{2} \le |\omega| \le \pi$$

for $N = 7$.

Solution

The ideal magnitude response with samples for the given specification is shown in Fig. 6.59.

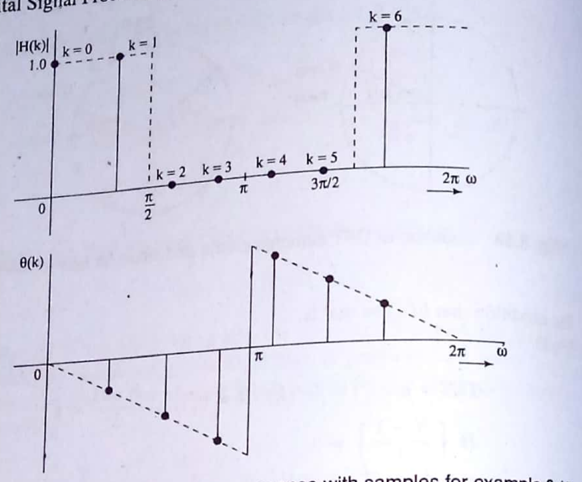


Fig. 6.59 Ideal magnitude and phase response with samples for example 6.15

Given $N = 7$

$$H(k) = H_d(e^{j\omega})\Big|_{\omega=\frac{2\pi k}{7}} \quad k = 0, 1, 2, \ldots 6$$

From Fig. 6.59 we have

$$\begin{aligned} |H(k)| &= 1 \quad \text{for} \quad k = 0, 1, 6 \\ &= 0 \quad \text{for} \quad k = 2, 3, 4, 5 \end{aligned} \tag{6.140}$$

Using Eq.(6.126) we have

$$\begin{aligned} \theta(k) &= -\left(\frac{N-1}{N}\right)\pi k = -\frac{6}{7}\pi k \quad \text{for} \quad k = 0, 1, 2, 3 \\ &= (N-1)\pi - \left(\frac{N-1}{N}\right)\pi k = 6\pi - \frac{6\pi k}{7} = \frac{6\pi}{7}(7-k) \quad \text{for} \quad k = 4, 5, 6 \end{aligned} \tag{6.141}$$

Now the frequency response of the linear phase filter can be wirtten by substituting Eq.(6.140) and Eq.(6.141) in Eq.(6.120)

$$\begin{aligned} H(k) &= e^{-j6\pi k/7} \quad k = 0, 1 \\ &= \quad 0 \quad \text{for} \quad k = 2, 3, 4, 5 \\ &= e^{-j6\pi(k-7)/7} \quad \text{for} \quad k = 6 \end{aligned}$$

The filter coefficients for N odd are given by

$$\begin{aligned} h(n) &= \frac{1}{N}\left\{H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left[H(k)e^{j2\pi kn/7}\right]\right\} \quad n = 0, 1, \ldots N-1 \\ &= \frac{1}{7}\left\{1 + 2Re(e^{-j6\pi/7}e^{j2\pi kn/7})\right\} \\ &= \frac{1}{7}\left\{1 + 2Re(e^{j2\pi(n-3)/7})\right\} \\ &= \frac{1}{7}\left\{1 + 2\cos\frac{2\pi}{7}(n-3)\right\} \end{aligned}$$

$$h(0) = h(6) = \frac{1}{7}\left(1 + 2\cos\frac{6\pi}{7}\right) = -0.11456$$

$$h(1) = h(5) = \frac{1}{7}\left(1 + 2\cos\frac{4\pi}{7}\right) = 0.07928$$

$$h(2) = h(4) = \frac{1}{7}\left(1 + 2\cos\frac{2\pi}{7}\right) = 0.321$$

$$h(3) = \frac{1}{7}(1+2) = 0.42857$$

Example 6.16 Determine the coefficients of a linear phase FIR filter of length $M = 15$ has a symmetric unit sample response and a frequency response that satisfies the conditions

$$\begin{aligned} H\left(\frac{2\pi k}{15}\right) &= 1 \quad k = 0, 1, 2, 3 \\ &= 0 \quad k = 4, 5, 6, 7 \end{aligned}$$

Solution

$$\begin{aligned} |H(k)| &= 1 \quad \text{for} \quad 0 \le k \le 3 \quad \text{and} \quad 12 \le k \le 14 \\ &= 0 \quad \text{for} \quad 4 \le k \le 11 \end{aligned}$$

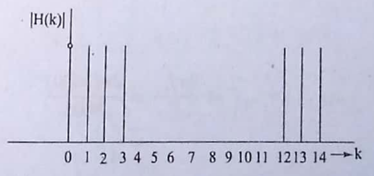


Fig. 6.60 Ideal magnitude response with samples for example 6.16

$$\theta(k) = -\left(\frac{N-1}{N}\right)\pi k$$
$$= \frac{-14}{15}\pi k \quad 0 \le k \le 7$$

and

$$\theta(k) = 14\pi - \frac{14\pi k}{15} \quad \text{for } 8 \le k \le 14$$

$$H(k) = e^{-j14\pi k/15} \quad \text{for} \quad k = 0, 1, 2, 3$$
$$= 0 \quad \text{for} \quad 4 \le k \le 11$$
$$= e^{-j14\pi(k-15)/15} \quad \text{for} \quad 12 \le k \le 14$$

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} Re\left(H(k)e^{j2\pi nk/15}\right)\right]$$
$$= \frac{1}{15}\left[1 + 2\sum_{k=1}^{7} Re\left(e^{-j14\pi k/15}e^{j2\pi nk/15}\right)\right]$$
$$= \frac{1}{15}\left[1 + 2\sum_{k=1}^{3}\cos\frac{2\pi k(7-n)}{15}\right]$$
$$= \frac{1}{15}\left[1 + 2\cos\frac{2\pi(7-n)}{15} + 2\cos\frac{4\pi(7-n)}{15} + 2\cos\frac{6\pi(7-n)}{15}\right]$$

$h(0) = h(14) = -0.05;\quad h(1) = h(3) = 0.041 \quad h(4) = h(10) = -0.1078$
$h(2) = h(12) = 0.0666;\quad h(3) = h(11) = -0.0365 \quad h(5) = h(9) = 0.034$
$h(6) = h(8) = 0.3188 \quad h(7) = 0.466$

Example 6.17 Using frequency sampling method, design a bandpass filter with the following specifications.

sampling frequency F = 8000Hz
cut off frequencies f_{c1} = 1000Hz
f_{c2} = 3000Hz

Determine the filter coefficients for $N = 7$.

Solution

$$\omega_{c_1} = 2\pi f_{c_1}T = \frac{2\pi f_{c_1}}{F} = \frac{2\pi(1000)}{8000}$$
$$= \frac{\pi}{4}$$

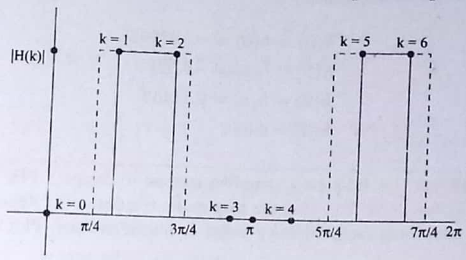


Fig. 6.61 Ideal magnitude response with samples for example 6.17

$$\omega_{c_2} = 2\pi f_{c_2}T = \frac{2\pi f_{c_2}}{F} = \frac{2\pi(3000)}{8000}$$
$$= \frac{3\pi}{4}$$

$$H(k) = H_d(e^{j\omega})\Big|_{\omega=\frac{2\pi}{7}k} \quad k = 0, 1, \ldots, 6$$

$$|H(k)| = 0 \quad \text{for } k = 0, 3$$
$$= 1 \quad \text{for } k = 1, 2$$
$$\theta(k) = -\left(\frac{N-1}{N}\right)\pi \quad \text{for } 0 \le k \le \frac{N-1}{2}$$
$$= -\frac{6}{7}\pi k \quad \text{for } 0 \le k \le 3$$
$$H(k) = 0 \quad \text{for } k = 0, 3$$
$$= e^{-j6\pi k/7} \quad \text{for } k = 1, 2$$

The filter coefficients are given by

$$h(n) = \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \text{Re}(H(k)e^{j2\pi kn/N})\right]$$
$$= \frac{1}{7}\left[2\sum_{k=1}^{3}\text{Re}(e^{-j6\pi k/7}e^{j2\pi kn/7})\right]$$
$$= \frac{1}{7}\left[2\sum_{k=1}^{2}\cos\frac{2\pi k}{7}(3-n)\right]$$
$$= \frac{2}{7}\left[\cos\frac{2\pi}{7}(3-n) + \cos\frac{4\pi}{7}(3-n)\right]$$

$$h(0) = h(6) = -0.07928$$
$$h(1) = h(5) = -0.321$$
$$h(2) = h(4) = 0.11456$$
$$h(3) = 0.57$$

Example 6.18 (a) Use frequency sampling method to design a FIR lowpass filter with $\omega_c = \frac{\pi}{4}$, for $N = 15$. Plot the magnitude response. (b) Repeat part (a) by selecting an additional sample $|H(k)| = 0.5$ in transition band. Plot the magnitude response.

Solution

(a) From Fig. 6.62 the frequency samples can be obtained as

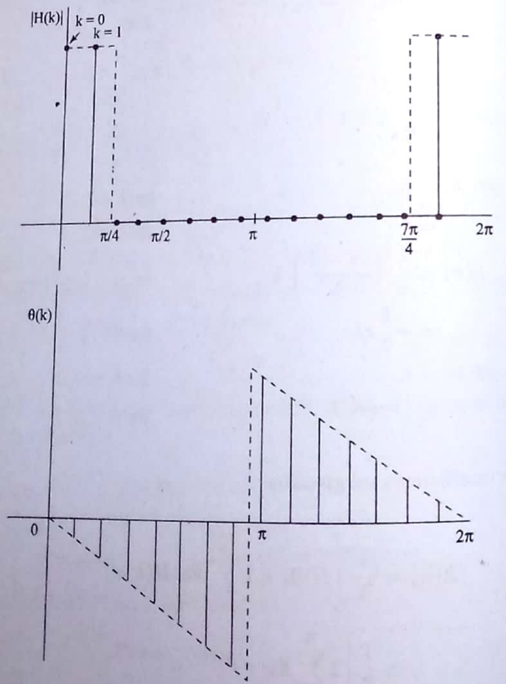


Fig. 6.62 Ideal magnitude and phase response of example 6.18.

$$\begin{aligned} H(k) &= e^{-j14\pi k/15} && \text{for } k = 0, 1 \\ &= 0 && \text{for } 2 \le k \le 13 \\ &= e^{-j14\pi(k-15)/15} && \text{for } k = 14 \end{aligned}$$

$$\begin{aligned} h(n) &= \frac{1}{N}\left[H(0) + 2\sum_{k=1}^{\frac{N-1}{2}} \text{Re}\left(H(k)e^{j2\pi kn/N}\right)\right] \\ &= \frac{1}{15}\left[1 + 2\sum_{k=1}^{7} \text{Re}(e^{-j14\pi k/15}e^{j2\pi kn/15})\right] \\ &= \frac{1}{15}\left[1 + 2\cos\frac{2\pi}{15}(7-n)\right] \end{aligned}$$

$$h(0) = h(14) = -0.0637$$
$$h(1) = h(13) = -0.0412$$
$$h(2) = h(12) = 0$$
$$h(3) = h(11) = 0.05273$$
$$h(4) = h(10) = 0.1078$$
$$h(5) = h(9) = 0.156$$
$$h(6) = h(8) = 0.188$$
$$h(7) = 0.2$$

The frequency response is given by

$$\overline{H}(e^{j\omega}) = \sum_{n=0}^{\frac{N-1}{2}} a(n)\cos\omega n$$

where

$$a(0) = h\left(\frac{N-1}{2}\right)$$
$$a(n) = 2h\left(\frac{N-1}{2} - n\right)$$

$$\Rightarrow \overline{H}(e^{j\omega}) = 0.2 + 0.376\cos\omega + 0.312\cos 2\omega + 0.2156\cos 3\omega + 0.10546\cos 4\omega - 0.0824\cos 6\omega - 0.1274\cos 7\omega$$

ω (in degrees)	0	15	30	60	75	105	135	165	180
$\overline{H}(e^{j\omega})$	0.999	1.083	0.8216	−0.1824	0.0504	−0.0854	0.0712	−0.025	0.07086
$\lvert H(e^{j\omega})\rvert_{dB}$	−0.0064	0.69	−1.7	−14	−26	−21.37	−23	−32.04	−23