## Advantages of FIR filters

1. FIR filters are always stable.
2. FIR filters with exactly linear phase can easily be designed.
3. FIR filters can be realized in both recursive & non recursive structure.
4. FIR filters are free of limit cycle oscillations, when implemented on a finite word length digital systems.
5) Excellent design methods are available for various kinds of FIR filters.

## Disadvantages of FIR filters

1. The implementation of narrow transition band FIR filters are very costly, as it requires considerably more arithmetic operations and hardware components such as multipliers, adders & delay elements.
2. Memory requirement and execution time are very high

## Conditions to Achieve LINEAR PHASE in a FIR filter

The transfer function of FIR causal filter is given as

$$H(z) = \sum_{n=0}^{M-1} h(n)\, z^{-n}$$

where $h(n)$ is the impulse response of the filter.

The DTFT of $h(n)$ is obtained with $z = e^{j\omega}$ in above eqn.

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} h(n)\, e^{-j\omega n}$$

which is periodic in frequency with period $2\pi$.

$H(e^{j\omega})$ can be written in polar form as

$$H(e^{j\omega}) = \pm |H(e^{j\omega})|\, e^{j\angle H(e^{j\omega})} \longrightarrow (1)$$

where,

$H(e^{j\omega})$ : frequency response

$|H(e^{j\omega})|$ : Magnitude response.

$\angle H(e^{j\omega})$ : phase response.

By definition the phase delay & group delay of filter are given as

Phase delay : $$\tau_p = -\frac{\angle H(e^{j\omega})}{\omega} \longrightarrow (2)$$

Group delay : $$\tau_g = -\frac{d}{d\omega}\left(\angle H(e^{j\omega})\right) \longrightarrow (3)$$

An FIR filter to have linear phase.

$$\angle H(e^{j\omega}) = -\alpha\omega \qquad -\pi \le \omega \le \pi \longrightarrow (4)$$

where $\alpha$ is a constant delay in samples.

put Eqn (4) in (2) & (3).

$$\tau_p = -\frac{-\alpha\omega}{\omega} \qquad \tau_g = -\frac{d}{d\omega}(-\alpha\omega)$$

$$\boxed{\tau_p = \alpha} \text{ const} \qquad \boxed{\tau_g = \alpha} \text{ const}$$

i.e., $\tau_p$ & $\tau_g$ are independent of frequency

Consider $H(e^{j\omega}) = H(\omega) = \sum_{n=0}^{M-1} h(n)\, e^{-j\omega n} \longrightarrow (5)$

Also $\quad H(\omega) = \pm |H(\omega)|\, e^{j\angle H(\omega)}$

But $\angle H(\omega) = -\alpha\omega$.

$$H(\omega) = \pm |H(\omega)|\, e^{-j\alpha\omega}. \longrightarrow (6)$$

$\therefore$

comparing Eqn (5) & (6).

$$\sum_{n=0}^{M-1} h(n)\, e^{-j\omega n} = \pm |H(\omega)|\, e^{-j\alpha\omega}.$$

$$\sum_{n=0}^{M-1} h(n) \cos\omega n - j\sum_{n=0}^{M-1} h(n)\sin\omega n = \pm |H(\omega)| \cos\alpha\omega - j|H(\omega)| \sin\alpha\omega]$$

OR

$$\sum_{n=0}^{M-1} h(n) \cos\omega n = \pm |H(\omega)| \cos\alpha\omega \longrightarrow (7)$$

$$\sum_{n=0}^{M-1} h(n) \sin\omega n = \pm |H(\omega)| \sin\alpha\omega. \longrightarrow (8)$$

Eqn (7) $\div$ (8)

$$\frac{\sum_{n=0}^{M-1} h(n) \cos\omega n}{\sum_{n=0}^{M-1} h(n) \sin\omega n} = \frac{\cos\alpha\omega}{\sin\alpha\omega}.$$

Scanned by CamScanner

$$\sum_{n=0}^{M-1} h(n)\sin\alpha\omega\cos\omega n = \sum_{n=0}^{M-1} h(n)\cos\alpha\omega\sin\omega n$$

$$\sum_{n=0}^{M-1} h(n)\left[\sin\alpha\omega\cos\omega n - \cos\alpha\omega\sin\omega n\right] = 0.$$

$$\sum_{n=0}^{M-1} h(n)\sin(\alpha\omega - n\omega) = 0$$

$$\sum_{n=0}^{M-1} h(n)\sin(\alpha - n)\,\omega = 0. \longrightarrow (9)$$

In Eqn (9) LHS is equal to zero. only if

$$\boxed{h(n) = h(M-1-n)} \rightarrow (10) \quad \& \quad \boxed{\alpha = \frac{M-1}{2}} \rightarrow (11)$$

Eqn (10) & (11) give condition for linear phase.

**Verification:**

Using (10) & (11) in (9)

$$\sum_{n=0}^{M-1} h(M-1-n)\sin\left(\frac{M-1}{2} - n\right) = 0$$

with $M = 5$

$$\sum_{n=0}^{4} h(4-n)\sin(2-n) = 0.$$

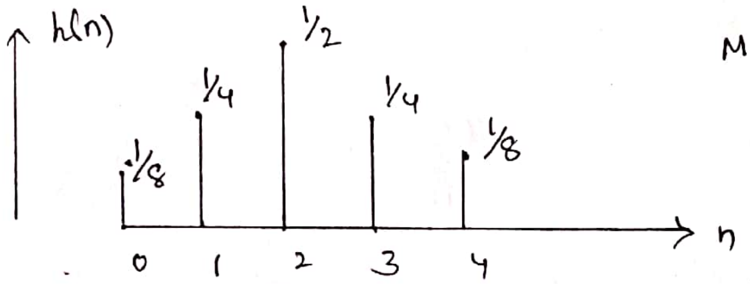
If $h(n) = h(M-1-n)$ ; $M=5$

| $n$ | $h(n) = h(4-n)$ |
|---|---|
| 0 | $h(0) = h(4)$ |
| 1 | $h(1) = h(3)$ |
| 2 | $h(2) = h(2)$ |

on expanding.

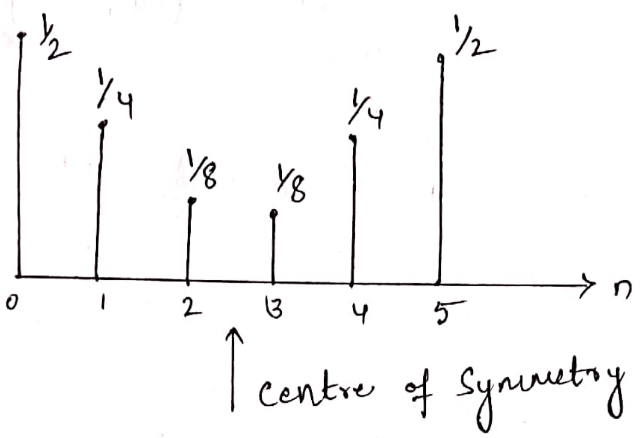
$$h(4)\sin(2) + h(3)\sin(1) + h(2)\sin(0) + h(1)\sin(-1) + h(0)\sin(-2) = 0$$

$$\cancel{h(4)\sin 2} + \cancel{h(3)\sin(1)} + h(2)\underbrace{\sin(0)}_{\to 0} - \cancel{h(1)\sin(1)} - \cancel{h(0)\sin 2} = 0.$$




M = 5 (odd)

↑ centre of symmetry is $\alpha = \frac{M-1}{2} = \frac{5-1}{2} = 2$




$M = 6$ Even

$\alpha = \dfrac{M-1}{2}$

$\alpha = \dfrac{6-1}{2}$

$\alpha = \dfrac{5}{2}$

If only constant group delay is required, and not the phase delay we can write.

$$\angle H(\omega) = \beta - \alpha\omega.$$

then $$H(\omega) = \pm |H(\omega)| e^{j(\beta - \alpha\omega)}$$

$$\sum_{n=0}^{M-1} h(n) e^{-j\omega n} = \pm |H(\omega)| e^{j(\beta - \alpha\omega)}$$

$$\sum_{n=0}^{M-1} h(n) \cos \omega n - j \sum_{n=0}^{M-1} h(n) \sin \omega n =$$

$$\pm |H(\omega)| \cos(\beta - \alpha\omega) \pm |H(\omega)| \sin(\beta - \alpha\omega)$$

$$\sum_{n=0}^{M-1} h(n) \cos \omega n = \pm |H(\omega)| \cos(\beta - \alpha\omega) \rightarrow (12)$$

$$-\sum_{n=0}^{M-1} h(n) \sin \omega n = \pm |H(\omega)| \sin(\beta - \alpha\omega) \rightarrow (13)$$

Taking the ratio of (12) & (13).

$$\frac{\sum_{n=0}^{M-1} h(n) \cos \omega n}{-\sum_{n=0}^{M-1} h(n) \sin \omega n} = \frac{\cos(\beta - \alpha\omega)}{\sin(\beta - \alpha\omega)}$$

$$\sum_{n=0}^{M-1} h(n) \cos wn \sin(\beta - \alpha w) = \sum_{n=0}^{M-1} h(n) \sin(wn) \cos(\beta - \alpha w)$$

$$\sum_{n=0}^{M-1} h(n) \left[ \sin(\beta - \alpha w) \cos wn - \cos(\beta - \alpha w) \sin wn \right] = 0$$

$$\sum_{n=0}^{M-1} h(n) \sin\left[\beta - \alpha w - wn\right] = 0$$

$$\sum_{n=0}^{M-1} h(n) \sin\left[\beta - (\alpha - n) w\right] = 0 \longrightarrow (14)$$

If $\beta = \frac{\pi}{2}$ Eqn (14) becomes

$$\sum_{n=0}^{M-1} h(n) \cos(\alpha - n) w = 0. \longrightarrow (15)$$

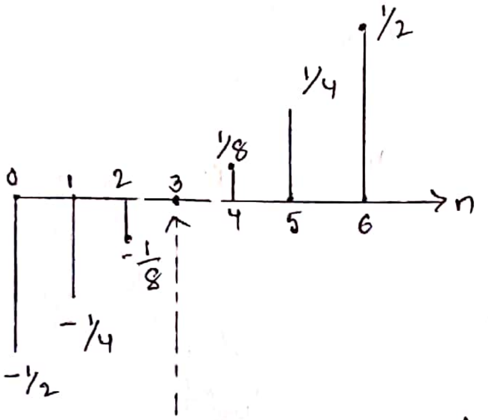
Eqn (15) will be satisfied when

$$h(n) = -h(M-1-n) \quad \& \quad \alpha = \frac{M-1}{2}$$

Therefore FIR filters have constant group delay $\tau_g$ and not constant phase delay when the impulse response is antisymmetrical about $\alpha = \frac{M-1}{2}$

Examples:

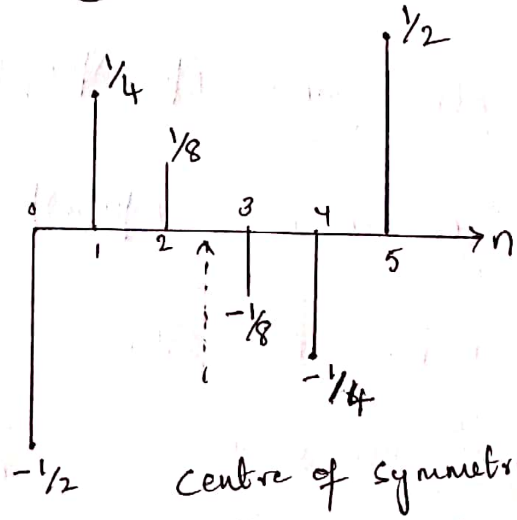
(1) $M = 7$ (odd)




Centre of symmetry

$\alpha = \frac{M-1}{2} = \frac{7-1}{2} = 3$

(2) $M = 6$ (Even)




Centre of symmetry

$\alpha = \frac{5}{2}$

# LINEAR PHASE FIR filter Transfer functions

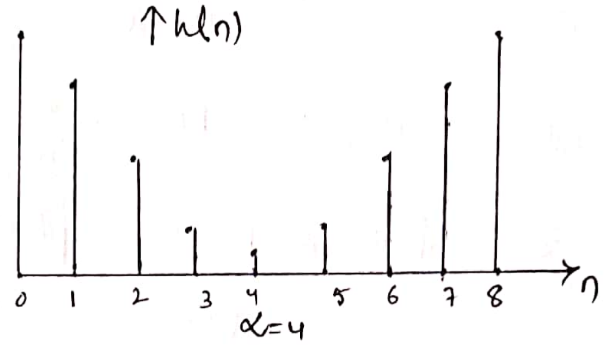
**Type 1:** Symmetric Impulse Response with ODD length

Let $M = 9$ $\qquad \alpha = \frac{M-1}{2} \qquad \alpha = \frac{9-1}{2} \qquad \boxed{\alpha = 4}$

The transfer function

$$H(z) = \sum_{n=0}^{M-1} h(n)\, z^{-n}$$

$$H(z) = \sum_{n=0}^{8} h(n)\, z^{-n}$$




$$H(z) = h(0) + h(1) z^{-1} + h(2) z^{-2} + h(3) z^{-3}$$
$$h(4) z^{-4} + h(5) z^{-5} + h(6) z^{-6} + h(7) z^{-7} + h(8) z^{-8}$$

Because of symmetry $h(n) = h(M-1-n)$

$h(0) = h(8) \qquad h(2) = h(6)$

$h(1) = h(7) \qquad h(3) = h(5)$

$$H(z) = h(0)(1 + z^{-8}) + h(1)(z^{-1} + z^{-7})$$
$$h(2)(z^{-2} + z^{-6}) + h(3)(z^{-3} + z^{-5}) + h(4) z^{-4}$$

Since $\alpha = 4$ taking $z^{-4}$ common

$$H(z) = z^{-4}\left[h(0)(z^{4} + z^{-4}) + h(3)(z^{3} + z^{-3})\right.$$
$$\left. h(2)(z^{2} + z^{-2}) + h(3)(z^{1} + z^{-1}) + h(4)\right]$$

Since $z = e^{j\omega}$, $\quad z + z^{-1} = e^{j\omega} + e^{-j\omega} = 2\cos\omega$ and so on

$$H(e^{j\omega}) = e^{-j4\omega}\left[2h(0)\cos 4\omega + 2h(3)\cos 3\omega\right.$$
$$\left. + 2h(2)\cos 2\omega + 2h(1)\cos\omega + h(4)\right]$$

Generalizing with $\alpha = \frac{M-1}{2} = 4$

$$\boxed{H(e^{j\omega}) = e^{-j\alpha\omega}\left[h(\alpha) + 2\sum_{n=1}^{\alpha} h(\alpha-n)\cos\omega n\right]} \rightarrow (1) \quad \alpha = \frac{M-1}{2}$$

$$H(e^{j\omega}) = e^{-j\alpha\omega} H_R(\omega)$$

where $H_R(\omega) = h(\alpha) + \sum_{n=1}^{\alpha} h(\alpha-n)\cos\omega n$, called zero phase response/amplitude response.

(OR)

$$\boxed{H(e^{j\omega}) = e^{-j\omega\left(\frac{M-1}{2}\right)}\left[h\left(\frac{M-1}{2}\right) + 2\sum_{n=1}^{\frac{M-1}{2}} h\left(\frac{M-1}{2} - n\right)\cos\omega n\right]} \rightarrow (2)$$

In Eqn (2), the quantity inside the braces is a real function of $\omega$ & can assume +ve or −ve values in the range $0 \leq |\omega| \leq \pi$.

The phase function in Eqn (1) is given as

$$\angle H(e^{j\omega}) = \beta - \alpha\omega.$$

In particular

$$\angle H(e^{j\omega}) = \beta - 4\omega.$$

where $\beta$ is either 0 or $\pi$ & hence it is a linear function of $\omega$.

The group delay is given by

$$\tau_g = -\frac{d\,\angle H(e^{j\omega})}{d\omega}$$

$$\tau_g = -\frac{d}{d\omega}(\beta - \alpha\omega) = -\frac{d}{d\omega}(\beta - 4\omega)$$

$$\tau_g = 4$$

indicating a constant group delay of 4 samples.