DEPARTMENT
OF
ELECTRONICS & COMMUNICATION ENGINEERING

Digital Communication System

(Theory Notes)

Autonomous Course

Prepared by

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Module – 1 Information Theory
Digital Communication block diagram, Information, Entropy, Shannon's Encoding Algorithm, Huffman coding, Discrete memoryless channels, BSC, channel capacity, Shannon Hartley Theorem and its implications.

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1.1 Digital Communication block diagram

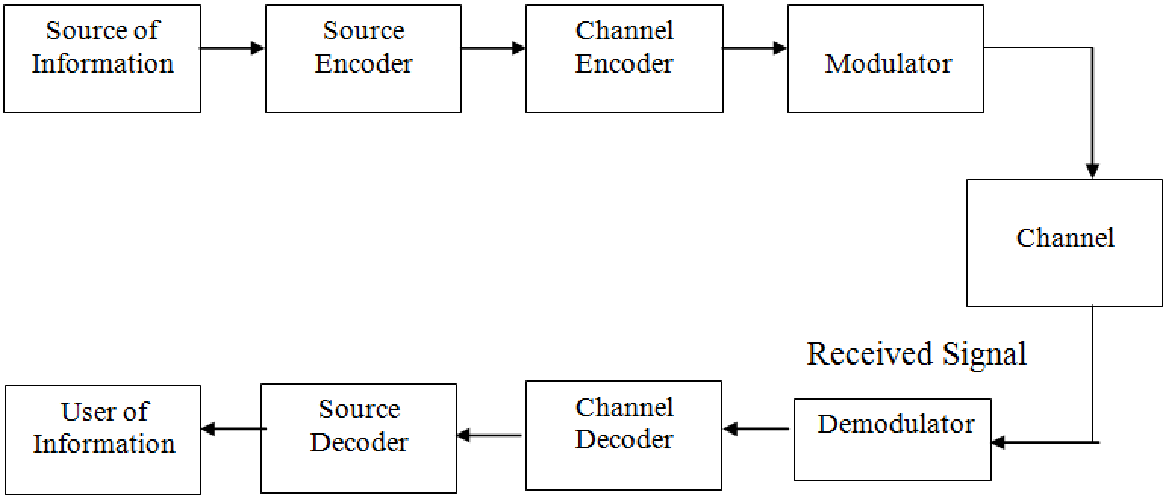


Fig. 1: Elements of digital communication system.

1.1.1 Information source:

The source of information can be analog or digital, e.g. analog: audio or video signal, digital: like teletype signal. In digital communication the signal produced by this source is converted into digital signal which consists of 1′s and 0′s.

1.1.2 Source Encoder

The Source encoder (or Source coder) converts the input i.e. symbol sequence into a binary sequence of 0‟s and 1‟s by assigning code words to the symbols in the input sequence. For eg. :-If a source set is having hundred symbols, then the number of bits used to represent each symbol will be 7 because 2^7=128 unique combinations are available. The important parameters of a source encoder are block size, code word lengths, average data rate and the efficiency of the coder (i.e. actual output data rate compared to the minimum achievable rate)
At the receiver, the source decoder converts the binary output of the channel decoder into a symbol sequence. The decoder for a system using fixed – length code words are quite simple, but the decoder for a system using variable – length code words will be very complex.
Aim of the source coding is to remove the redundancy in the transmitting information, so that bandwidth required for transmission is minimized. Based on the probability of the symbol code word is assigned. Higher the probability, shorter is the code word.
Ex: Huffman coding.

1.1.3 Channel Encoder

Error control is accomplished by the channel coding operation that consists of systematically adding extra bits to the output of the source coder. These extra bits do not convey any information but helps the receiver to detect and / or correct some of the errors in the information bearing bits.

There are two methods of channel coding:

- Block Coding: The encoder takes a block of “k” information bits from the source encoder and adds “r” error control bits, where “r” is dependent on “k” and error control capabilities desired.
- Convolution Coding: The information bearing message stream is encoded in a continuous fashion by continuously interleaving information bits and error control bits.

The Channel decoder recovers the information bearing bits from the coded binary stream. Error detection and possible correction is also performed by the channel decoder.

The important parameters of coder / decoder are: Method of coding, efficiency, error control capabilities and complexity of the circuit.

1.1.4 Modulator

It is performed for the efficient transmission of the signal over the channel. The modulator operates by keying shifts in the amplitude, frequency or phase of a sinusoidal carrier wave to the channel encoder output. The digital modulation techniques are referred to as amplitude-shift keying, frequency- shift keying or phase-shift keying respectively. The Modulator converts the input bit stream into an electrical waveform suitable for transmission over the communication channel. Modulator can be effectively used to minimize the effects of channel noise, to match the frequency spectrum of transmitted signal with channel characteristics, to provide the capability to multiplex many signals.

The detector performs demodulation, thereby producing a signal the follows the time variations in the channel encoder output. The modulator, channel and detector form a discrete channel (because both its input and output signals are in discrete form.

1.1.5 Channel

The Channel provides the electrical connection between the source and destination. The different channels are: Pair of wires, Coaxial cable, Optical fibre, Radio channel, Satellite channel or combination of any of these.

The communication channels have only finite Bandwidth, non-ideal frequency response, the signal often suffers amplitude and phase distortion as it travels over the channel. Also, the signal power decreases due to the attenuation of the channel. The signal is corrupted by unwanted, unpredictable electrical signals referred to as noise.

The important parameters of the channel are Signal to Noise power Ratio (SNR), usable bandwidth, amplitude and phase response and the statistical properties of noise.

1.2. Information

The output of a discrete information source is a message that consists of a sequence of symbols. The actual message that is emitted by the source during a message interval is selected at random from a set of possible messages. The communication system is designed to reproduce at the receiver either exactly or approximately the message emitted by the source.

To measure the information content of a message quantitatively, we are required to arrive at an intuitive concept of the amount of information.

Consider the examples: A trip to Miami, Florida from Minneapolis in the winter time,

- mild and sunny day,
- cold day,
- possible snow flurries.

The amount of information received is obviously different for these messages.

- The first message contains very little information since the weather in Miami is mild and sunny most of the time.
- The forecast of a cold day contains more information since it is not an event that occurs often.
- In contrast, the forecast of snow flurries conveys even more information since the occurrence of snow in Miami is a rare event.

Thus on intuitive basis the amount of information received from the knowledge of occurrence of an event is related to the probability or the likelihood of occurrence of the event. The message associated with an event least likely to occur contains most information.

The information content of a message can be expressed quantitatively in terms of probabilities as follows:

Suppose an information source emits one of 'q' possible messages $m_1, m_2 \ldots\ldots m_q$ with $p_1, p_2 \ldots\ldots p_q$ as their probs. of occurrence. Based on the above intusion, the information content of the k^{th} message, can be written as

$$I\,(m_k) \alpha \frac{1}{p_k}$$

Also to satisfy the intuitive concept, of information.

I (m_k) must → zero as $p_k \to 1$

Therefore,

$$\boxed{\begin{array}{lll} I(m_k) > I(m_j); & \text{if} & p_k < p_j \\ I(m_k) \to O(m_j); & \text{if} & p_k \to 1 \\ I(m_k) \geq O; & \text{when} & O < p_k < 1 \end{array}}$$

Another requirement is that when two independent messages are received, the total information content is – Sum of the information conveyed by each of the messages.

Thus the equation becomes

$$I(m_k \text{ and } m_j) \triangleq I(m_k m_j) = I(m_k) + I(m_j)$$

Whrere m_k and m_j are two independent messages.

A continuous function of p that satisfies the constraints specified in the above equations is the logarithmic function and we can define a measure of information as

$$I(m_k) = \log\left(\frac{1}{p_k}\right)$$

The base for the logarithmic in equation determines the unit assigned to the information content.

Natural logarithm base : 'nat'

Base - 10 : Hartley / decit

Base - 2 : bit

Using the binary digit as the unit of information is based on the fact that if two possible binary digits occur with equal proby ($p_1 = p_2 = ½$) then the correct identification of the binary digit conveys an amount of information. I (m_1) = I (m_2) = – $\log_2$ (½) = 1 bit. Therefore one bit is the amount of information that we gain when one of two possible and equally likely events occurs.

Ex1: A source puts out one of five possible messages during each message interval. The probabilities of these messages are P_1 =1/2, P_2=1/4, P_3=1/4, P_4=1/16, P_5=1/16. What is the information content of these messages?

Solution:

$$I(m_1) = log_2 \frac{1}{(1/2)} = 1\ bits$$

$$I(m_2) = log_2 \frac{1}{(1/4)} = 2\ bits$$

$$I(m_3) = log_2 \frac{1}{(1/4)} = 2\ bits$$

$$I(m_4) = log_2 \frac{1}{(1/16)} = 4\ bits$$

$$I(m_5) = log_2 \frac{1}{(1/16)} = 4\ bits$$

1.3 Entropy

Suppose a source that emits one of M possible symbols $s_1, s_2, \ldots\ldots s_M$ in a statistically independent sequence. Let $p_1, p_2 \ldots\ldots p_q$ be the probabilities of occurrence of the M symbols, respectively. In a long message containing N symbols, the symbol s_1 will occur on the average p_1N times, the symbol s_2 will occur p_2N times, and in general the symbol s_M will occur p_MN times. The information content of the i_{th} symbol is I (s_i)= $\log_2\left(\frac{1}{p_i}\right)$.

Therefore p_1N number of messages of type s_1 contains $p_1N\log_2\left(\frac{1}{p_1}\right)$bits. Similarly p_2N number of messages of type s_1 contains $p_2N\log_2\left(\frac{1}{p_2}\right)$bits.

∴ Total self-information contains all these messages as

$$I_{total} = p_1N\log_2\left(\frac{1}{p_1}\right) + p_2N\log_2\left(\frac{1}{p_2}\right) + \cdots \ldots \ldots \ldots \ldots +p_MN\log_2\left(\frac{1}{p_M}\right)$$

$$I_{total} = N\sum_{i=1}^{M} p_i \log_2\left(\frac{1}{p_i}\right) \text{ bits}$$

The average information *per* symbol is obtained by dividing the total information content of the message by the number of symbols in the message, as

$$Entropy = H = \frac{T_{total}}{N} = \sum_{i=1}^{M} p_i \log_2\left(\frac{1}{p_i}\right) \text{ bits/symbol}$$

1.3.1 Average information rate

If the symbols are emitted by source at a fixed time rate $r_{s,}$ then the average information rate R_s is given by $R_s = r_s * H$ bits/sec

Eamples:

1. Consider a discrete memoryless source with a source alphabet A= ($s_{0,}s_{1,}s_2$) with respective probabilities $p_0 = \frac{1}{4}$, $p_1 = \frac{1}{4}$, $p_2 = \frac{1}{2}$. Find the entropy of the source.

Solution: By definition, the entropy of a source is given by

$$H = \sum_{i=1}^{M} p_i \log \frac{1}{p_i} \quad \text{bits/ symbol}$$

H for this example is

$$H(A) = \sum_{i=0}^{2} p_i \log \frac{1}{p_i}$$

Substituting the values given, we get

$$H(A) = p_o \log \frac{1}{P_o} + P_1 \log \frac{1}{p_1} + p_2 \log \frac{1}{p_2}$$

$$= \tfrac{1}{4} \log_2 4 + \tfrac{1}{4} \log_2 4 + \tfrac{1}{2} \log_2 2$$

$$= \left(\frac{3}{2}\right) = 1.5 \text{ bits}$$

if $r_s = 1$ per sec, then

$$\mathbf{H'(A) = r_s\, H(A) = 1.5\ bits/sec}$$

2. **An analog signal is band limited to B Hz, sampled at the Nyquist rate, and the samples are quantized into 4-levels. The quantization levels Q_1, Q_2, Q_3, and Q_4 (messages) are assumed independent and occur with probs.**

   $P_1 = P_2 = \frac{1}{8}$ and $P_2 = P_3 = \frac{3}{8}$. Find the information rate of the source.

Solution: By definition, the average information H is given by

$$H = p_1 \log \frac{1}{p_1} + p_2 \log \frac{1}{p_2} + p_3 \log \frac{1}{p_3} + p_4 \log \frac{1}{p_4}$$

Substituting the values given, we get

$$H = \frac{1}{8} \log 8 + \frac{3}{8} \log \frac{8}{3} + \frac{3}{8} \log \frac{8}{3} + \frac{1}{8} \log 8$$

$$= 1.8 \text{ bits/ message.}$$

Information rate of the source by definition is

$$R = r_s\, H$$

$$R = 2B, (1.8) = (3.6\, B) \text{ bits/sec}$$

3. **Compute the values of H and R, if in the above example, the quantities levels are so chosen that they are equally likely to occur,**

Solution:

Average information per message is

$H = 4\ (\frac{1}{4} \log_2 4) = 2$ bits/message

and $R = r_s\ H = 2B\ (2) = (4B)$ bits/sec

PROPERTIES OF ENTROPY:

1. Entropy is average self information. It is given by

$$H(S) = \sum_{i=1}^{q} P_i \log (1/P_i)$$

$$H(S) = \sum_{i=1}^{q} P_i I_i \quad \text{bits/symbol}$$

where q represents number of symbols and also $\sum_{i=1}^{q} P_i = 1$.

2. For null event and sure event, the entropy vanishes.

3. The entropy is a symmetrical function of its arguments. The value of $H(S)$ remains the same irrespective of location of probabilities.

4. Extremal property: When all the source symbols become equiprobable, then the entropy attains the maximum value.

$$H(S)_{max} = \log_2 q.$$

5. When source symbols are not equiprobable, then entropy is less than maximum value.

6. The source efficiency, η_s is given by

$$\eta_s = \frac{H(S)}{H(S)_{max}}$$

7. The source redundancy R_{η_s} is given by

$$R_{\eta_s} = 1 - \eta_s$$

Usually efficiency and redundancy are represented in percentage.

Q> A binary source is emitting an independent sequence of 0's and 1's with probabilities P and (1-P) respectively. Plot the entropy of the source versus P.

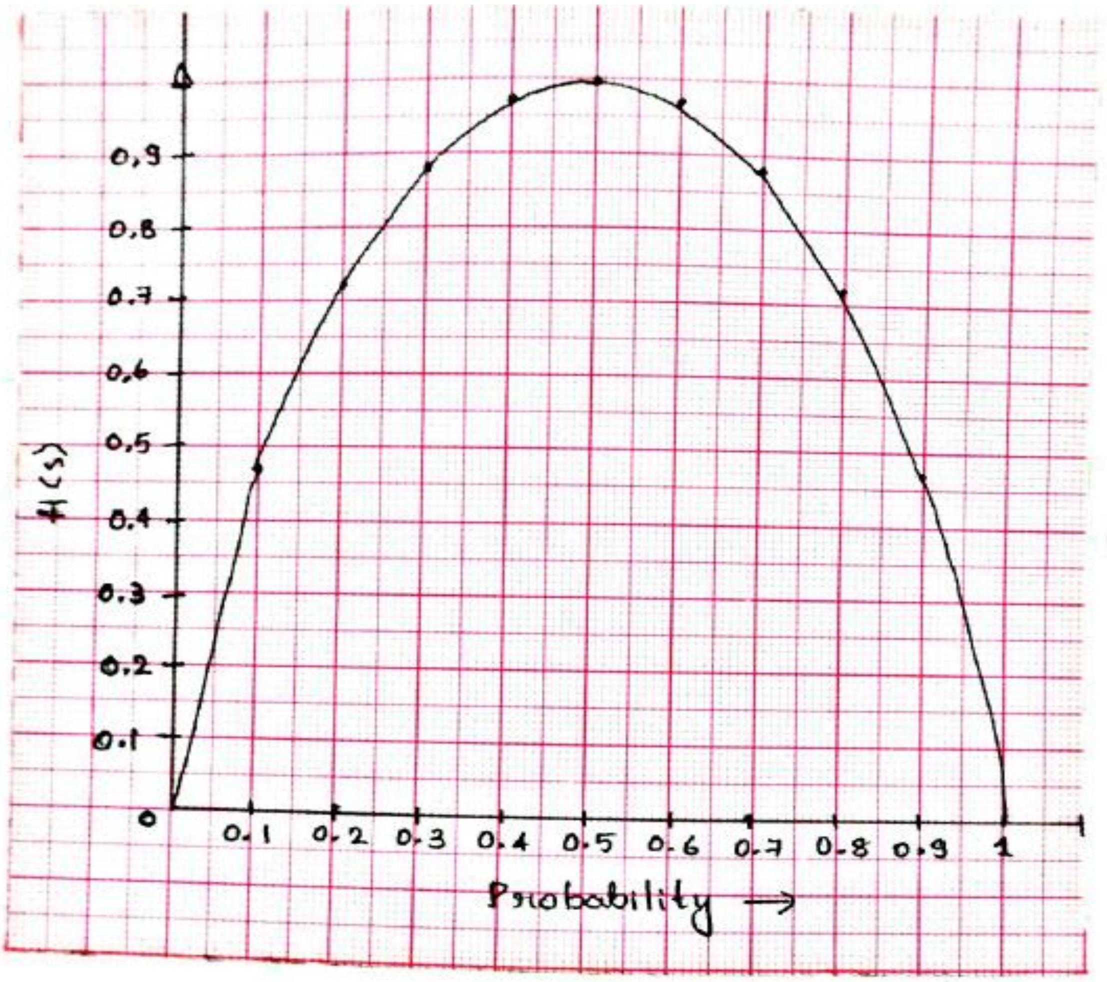
Sol^n: The entropy of the binary source is given by

$$H(S) = \sum_{i=1}^{2} P_i \log_2 \left(\frac{1}{P_i}\right)$$

$$H(S) = P \log_2 \left(\frac{1}{P}\right) + (1-P) \log_2 \left(\frac{1}{1-P}\right) \rightarrow (1)$$

Type eq^n (1) in calculator and 'calculate' the values of H(S) for assumed values of P. P should vary from 0.1 to 1

P	H(S)
0.1	0.469
0.2	0.722
0.3	0.881
0.4	0.971
0.5	1
0.6	0.971
0.7	0.881
0.8	0.722
0.9	0.469
1.0	—



0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
H(s)
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
Probability →

Q5 A discrete message source 's' emits two independent symbols x and y with probability 0.55 and 0.45 respectively. Calculate the efficiency of the source and its redundancy.

Sol^n: $P(X) = P_x = 0.55$; $P(y) = P_y = 0.45$

$$H(S) = \sum_{i=1}^{2} P_i \log 1/P_i$$

$$= 0.55 \log \left(\frac{1}{0.55}\right) + 0.45 \log \left(\frac{1}{0.45}\right)$$

$H(S) = 0.9928$ bits/message symbol

$H(S)_{max} = \log_2 q$ | Here $q=2$

$\therefore H(S)_{max} = \log_2 2 = 1$

$$\eta_s = \frac{H(S)}{H(S)_{max}} = \frac{0.9928}{1} = 0.9928 \ / \ 99.28\%$$

$R_{\eta_s} = 1 - \eta_s = 0.0072 \ / \ 0.72\%$

1.3 Shannon's Encoding algorithm

Source encoding is the process by which the output of an information source is converted in to an r-array sequence. Coding is nothing but transformation of each of the source symbol S=$\{s_1, s_2, s_3 \dots \dots . . s_q\}$ using the symbols from the source alphabet X=$\{q_1, q_2, q_3 \dots \dots . . q_r)$.

In binary coding r represents number of different symbols used in the code alphabet. That is r=2 => X=(0,1). In general if $\{s_1, s_2, s_3 \dots \dots . . s_q)$ are to be transmitted, then q number of different states are required. In binary coding only 2 states are required. Hence the transmission process becomes much easier and efficiency of the system can be increased.

- Let the source symbols in the order of decreasing probabilities

  S=$\{s_1, s_2, s_3 \dots \dots . . s_q)$

  P=$\{p_1, p_2, p_3 \dots \dots . . p_q)$.

  $p_1 \geq p_2 \geq p_3 \dots \dots . \geq p_q$

- Compute the sequence

α_1=0

α_2=$p_1 = p_{1+}\alpha_1$

$\alpha_3 = p_2 + p_1 = p_2 + \alpha_2$

$\alpha_4 = p_3 + p_2 + p_1 = p_3 + \alpha_3$

.

.

$\alpha_{q+1} = p_q + \alpha_q = 1$

- Determine the smallest integer for l_i (length of code word) using the inequality

  $2^{l^i} \geq \frac{1}{p_i}$ for all i=1 to q

- Expand the decimal numbers α_i in binary form up to l_i places neglecting the expansion beyond l_i places.
- Remove the decimal point to get the desired code.

Code efficiency: The average length 'L' of any code is given by $\boldsymbol{L = \sum_{i=1}^{q} p_i L_i}$ where $L_i = l_1 l_2 l_3 l_4 \ldots \ldots l_q$

Code efficiency, $\boldsymbol{\eta_c = \frac{H(S)}{L} * 100}$ for binary codes.

Ex: 1. Construct the Shannon's binary code for the following message symbols S={$s_{1,}s_{2,}s_3, s_4$) with probabilities P=(0.4, 0.3,0.2,0.1).

Solution:

- ***0.4*** > ***0.3*** > ***0.2*** > ***0.1***
- $\alpha_0 = \mathbf{0},$
  $\alpha_1 = \mathbf{0.4}$
  $\alpha_2 = \mathbf{0.4+0.3=0.7}$
  $\alpha_3 = \mathbf{0.7+0.2=0.9}$
  $\alpha_4 = \mathbf{0.9 + 0.1 = 1.0}$
- $2^{-l_1} \leq 0.4 \rightarrow l_1 = 2$
  $2^{-l_2} \leq 0.3 \rightarrow l_2 = 2$
  $2^{-l_3} \leq 0.2 \rightarrow l_3 = 3$
  $2^{-l_4} \leq 0.1 \rightarrow l_4 = 4$
- $\alpha_0 = 0 = 0.00 \,|\underline{0}$
  $\alpha_1 = 0.4 = 0.01 \,|\underline{10}$
  $\alpha_2 = 0.7 = 0.101 \,|\underline{10}$
  $\alpha_3 = 0.9 = 0.1110 \,|\underline{01}$

- The codes are

  $s_1 \rightarrow$ 00, $s_2 \rightarrow$ 01, $s_3 \rightarrow$ 101, $s_4 \rightarrow$ 1110

S_i	P_i	α_i	l_i	binary	Code
S_1	0.4	0	2	$(0.00000\cdots)_2$	00
S_2	0.3	0.4	2	$(0.01100\cdots)_2$	01
S_3	0.2	0.7	3	$(0.101100\cdots)_2$	101
S_4	0.1	0.9	4	$(0.11100\cdots)_2$	1110

0.4×2
0.8×2 → 0
1.6 → 1
0.6×2 →
1.2 → 1
0.2×2
0.4 → 0
⇒ 0.0110....

0.7×2
1.4 → 1
0.4×2
0.8×2 → 0
1.6 → 1
0.6×2 → 1
1.2
0.2×2
0.4×2 → 0
0.8 → 0
⇒ 0.101100...

0.9×2
1.8 → 1
0.8×2
1.6 → 1
0.6×2 → 1
1.2
0.2×2 → 0
0.4×2
0.8×2 → 0
⇒ 0.11100...

The average length of this code is

$$L = 2 \times 0.4 + 2 \times 0.3 + 3 \times 0.2 + 3 \times 0.1 = 2.4 \text{ Binits / message}$$

$$H(S) = 0.4 \log \frac{1}{0.4} + 0.3 \log \frac{1}{0.3} + 0.2 \log \frac{1}{0.2} + 0.1 \log \frac{1}{0.1} = 1.84644 \text{ bits / message;}$$

$$\%\eta_c = \frac{H(S)}{L} * 100 = \frac{1.8464}{2.4} * 100 = 76.93\%$$

Ex: 2: Apply Shannon's binary encoding procedure to the following set of messages and obtain code efficiency and redundancy.

1/8, 1/16, 3/16, 1/4, 3/8

Solution:

$\alpha_1 = 0$

$\alpha_2 = P_1 = 3/8 = 0.375$

$\alpha_3 = P_2 + \alpha_2 = 1/4 + 3/8 = 0.625$

$\alpha_4 = P_3 + \alpha_3 = 3/16 + 0.625 = 0.8125$

$\alpha_5 = P_4 + \alpha_4 = 1/8 + 0.8125 = 0.9375$

S_i	P_i	α_i	l_i	binary	Code
S_1	3/8	0	2	0.00	00
S_2	1/4	0.375	2	0.01	01
S_3	3/16	0.625	3	0.101	101
S_4	1/8	0.8125	3	0.110	110
S_5	1/16	0.9375	4	0.1111	1111

0.375×2
$0.75 \times 2 \rightarrow 0$
$1.5 \rightarrow 1$
0.5×2
1
$\Rightarrow \boxed{0.01}$

0.625×2
$1.25 \rightarrow 1$
0.25×2
$0.5 \times 2 \rightarrow 0$
1
$\boxed{0.101}$

0.8125×2
$1.625 \rightarrow 1$
0.625×2
$1.25 \rightarrow 1$
0.25×2
$0.5 \times 2 \rightarrow 0$
1
$\Rightarrow \boxed{0.110}$

0.9375×2
$1.875 \rightarrow 1$
0.875×2
$1.75 \rightarrow 1$
0.75×2
$1.5 \rightarrow 1$
0.5×2
1
$\Rightarrow \boxed{0.1111}$

$$H(S) = \frac{1}{4}log_2 4 + \frac{3}{8}log_2\frac{8}{3} + \frac{1}{8}log_2 8 + \frac{3}{16}log_2\frac{16}{3} + \frac{1}{4}log_2 4 + \frac{1}{16}log_2 16$$

$$H(S) = 2.1085\ bits/symbol$$

$$L = \sum_{i=1}^{q} P_i l_i = \frac{1}{4}(2) + \frac{3}{8}(2) + \frac{1}{8}(3) + \frac{3}{16}(3) + \frac{1}{16}(4)$$

$$L = 2.4375\ bits/\text{symbol}$$

$$\%\eta = \frac{H(S)}{L} * 100 = 86.5\%$$

Redundancy=1- η=100-86.5=13.5%

Ex: 3: Repeat the above messages **(**$x_{1,}x_{2,}x_3$) with P= (1/2, 1/5, 3/10)

Solution:

x_i	P_i	α_i	l_i	binary α_i	Code
x_1	1/2 (0.5)	0	1	0	0
x_3	3/10 (0.3)	0.5	2	0.1	10
x_2	1/5 (0.2)	0.8	3	0.11001	110

$$H(S) = \frac{1}{2}log_2 2 + \frac{3}{10}log_2\frac{10}{3} + \frac{1}{5}log_2 5$$

$$H(S) = 1.4855\ bits/symbol$$

$$L = \sum_{i=1}^{3} P_i L_i = \frac{1}{2}(1) + \frac{3}{10}(2) + \frac{1}{5}(3)$$

$$L = 1.7 bits/symbol$$

$$\%\eta = \frac{H(S)}{L} * 100 = 87.38\%$$

1.4 Huffman Coding

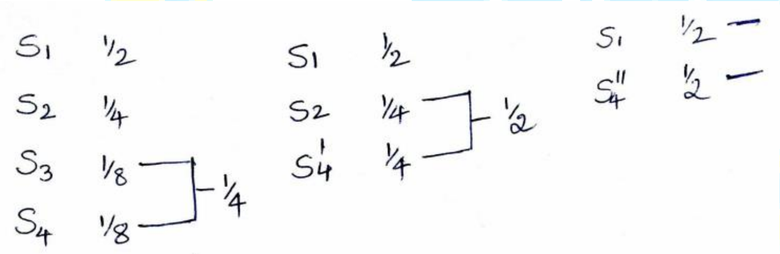
- The source symbols are listed in the decreasing order of probabilities.

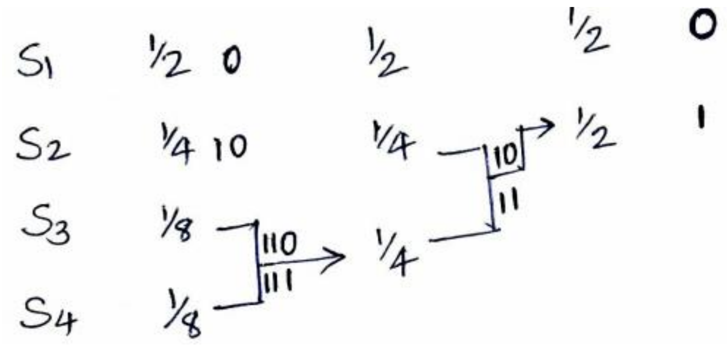
- Check if *q* = *r* + a*(r-1)* is satisfied and find the integer 'a', where q is number of source symbols and r is number of symbols used in code alphabets. 'a' values is calculated and it should be an integer, otherwise add suitable number of dummy symbols of zero probability of occurrence to satisfy the equation. This step is not required if we are to determine binary codes.
- Combine the last 'r' symbols into a single composite symbol whose probability of occurrence is equal to the sum of the probabilities of occurrence of the last ***r*** – symbols involved in the step.
- Repeat the above three steps respectively on the resulting set of symbols until in the final step exactly ***r***- symbols are left.
- The last source with 'r' symbols are encoded with 'r' different codes 0,1,2,3,….r-1
- In binary coding the last source are encoded with 0 and 1
- As we pass from source to source working backward, decomposition of one code word each time is done in order to form 2 new code words.
- This procedure is repeated till we assign the code words to all the source symbols of alphabet of source 's' discarding the dummy symbols.

Ex:1. Construct a Huffman Code for symbols having probabilities$\left\{\frac{1}{2},\frac{1}{4},\frac{1}{8},\frac{1}{8}\right\}$. Also find efficiency and redundancy.

$q = r + \alpha(r-1)$

4=2+α(1) => α=2$\in Z$





Symbols	Codes	Probabilities	Length
S_1	0	½	1
S_2	10	¼	2
S_3	110	1/8	3
S_4	111	1/8	3

$$H(S) = \frac{1}{2}log_2 2 + \frac{8}{8}log_2 8 + \frac{1}{4}log_2 4$$

$$H(S) = 1.75\ bits/symbol$$

$$L = \sum_{i=1}^{4} P_i L_i = \frac{1}{2}(1) + \frac{1}{4}(2) + \frac{1}{8}(3) + \frac{1}{8}(3)$$

$$L = 1.75\ bits/symbol$$

$$\%\eta = \frac{H(S)}{L} * 100 = 100\%$$

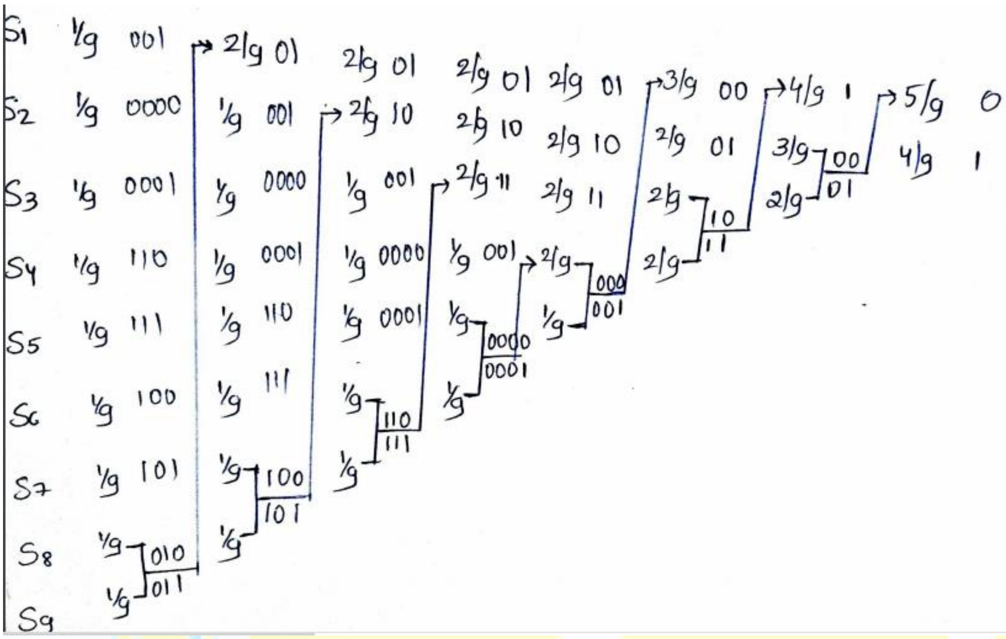
Redundancy=0%

Ex. 2: A source has 9 symbols and each occur with a probability of 1/9. Construct a binary Huffman code. Find efficiency and redundancy of coding.

Solution:

$$q = r + \alpha(r - 1)$$

9=2+α(1) => α=7∈ Z



Symbols	Codes	Probabilities	Length
S_1	001	1/9	3
S_2	0000	1/9	4
S_3	0001	1/9	4
S_4	110	1/9	3
S_5	111	1/9	3
S_6	100	1/9	3
S_7	101	1/9	3
S_8	010	1/9	3
S_9	011	1/9	3

$$H(S) = \frac{9}{9}log_2 9$$

$$H(S) = 3.17\ bits/symbol$$

$$L = \sum_{i=1}^{9} P_i L_i = \frac{1}{9}(3+4+4+3+3+3+3+3+3)$$

$$L = 3.22\ bits/symbol$$

$$\%\eta = \frac{H(S)}{L} * 100 = 98.45\%$$

Redundancy=100-% η=1.55%

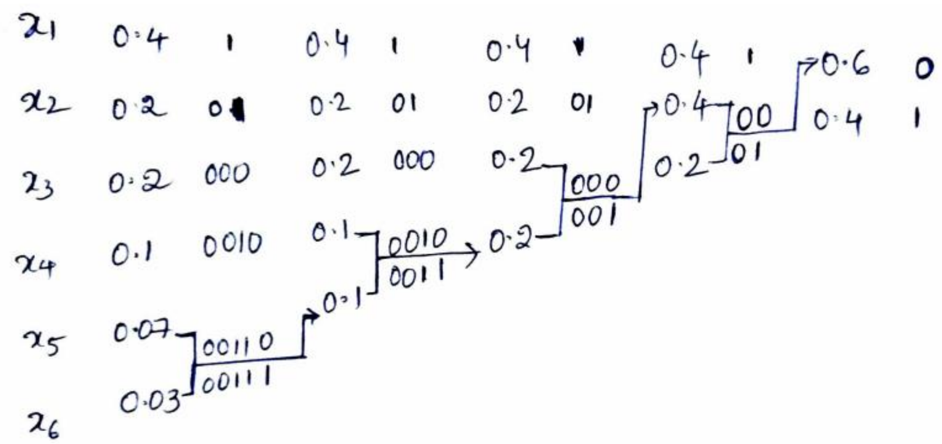
Ex. 3: Given the messages x_1, x_2, x_3, x_4, x_5 &x_6 with probabilities 0.4, 0.2, 0.2, 0.1, 0.07, 0.03. Construct binary and trinary code by applying Huffman encoding procedure. Also find efficiency and redundancy.

Solution:

(i) Binary

$q = r + \alpha(r - 1)$

6=2+α(1) => α=4∈ Z



Symbols	Codes	Probabilities	Length
x_1	1	0.4	1
x_2	01	0.2	2
x_3	000	0.2	3
x_4	0010	0.1	4
x_5	00110	0.07	5
x_6	00111	0.03	5

$$H(S) = 0.4\, log_2\left(\frac{1}{0.4}\right) + 0.2\, log_2\left(\frac{1}{0.2}\right) + 0.2\, log_2\left(\frac{1}{0.2}\right) + 0.1\, log_2\left(\frac{1}{0.1}\right) + 0.07 log_2\left(\frac{1}{0.07}\right) + 0.03 log_2\left(\frac{1}{0.03}\right)$$

$H(S) = \quad bits/symbol$

$$L = \sum_{i=1}^{6} P_i L_i = 0.4(1) + 0.2(2) + 0.2(3) + 0.1(4) + 0.07(5) + 0.03(5)$$

$L = \quad bits/symbol$

$$\%\eta = \frac{H(S)}{L} * 100 =$$

Redundancy=

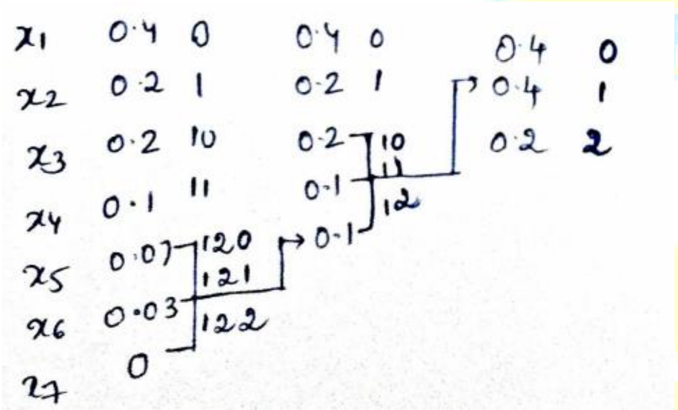
(ii) Trinary

$q = r + \alpha(r - 1)$

6=3+α(2) => α=3/2 ; Let α=2

If α=2 => $q = 3 + 2(2)$= 7

Hence add a symbol x_7 with probability '0'.



Symbols	Codes	Probabilities	Length
x_1	0	0.4	1
x_2	1	0.2	1
x_3	10	0.2	2
x_4	11	0.1	2
x_5	120	0.07	3
x_6	121	0.03	3

x_7 must be ignored as it is a dummy symbol.

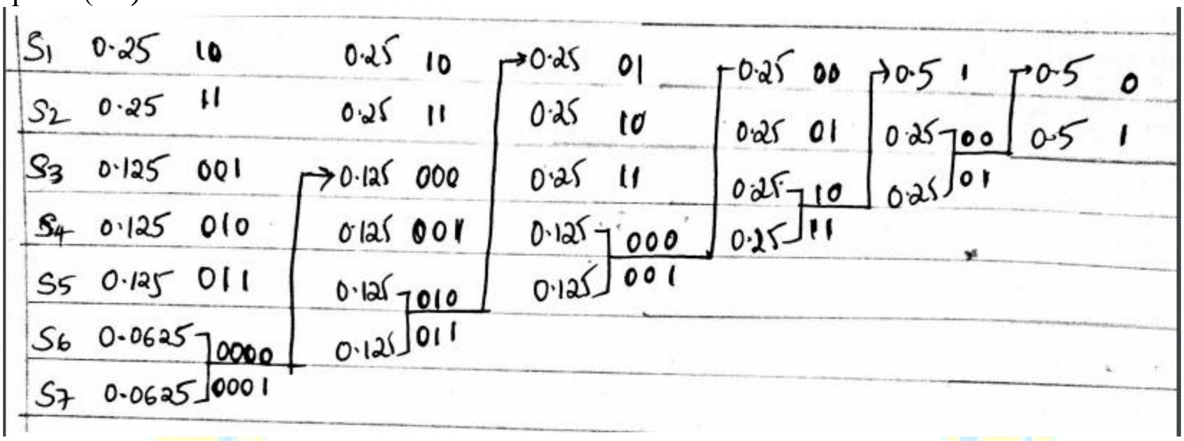
$$H(S) = 0.4\, log_3\left(\frac{1}{0.4}\right) + 0.2\, log_3\left(\frac{1}{0.2}\right) + 0.2\, log_3\left(\frac{1}{0.2}\right) + 0.1\, log_3\left(\frac{1}{0.1}\right) + 0.07 log_3\left(\frac{1}{0.07}\right) + 0.03 log_3\left(\frac{1}{0.03}\right)$$

H(S)= bits/symbol

Ex. 4: Consider a zero memory source has an alphabet of 7 symbols whose probability of occurrence of (0.25, 0.25, 0.125, 0.125, 0.125, 0.0625, 0.0625). Compute the Huffman code for this source moving a combined symbol **as high as possible**. Evaluate the code efficiency. And also construct code tree.

Solution:

q=r+α(r-1) => α=5



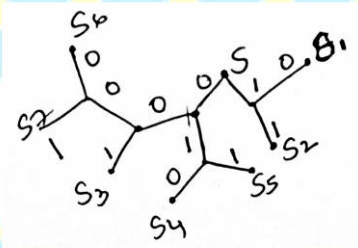
Symbols	Pi	Code	length
S1	0.25	10	2
S2	0.25	11	2
S3	0.125	001	3
S4	0.125	010	3
S5	0.125	011	3
S6	0.0625	0000	4
S7	0.0625	0001	4

H(S)=2.625 bits/symbol

L=2.625 bits/symbol

%η=H(s)/L=100%

Code tree



1.6 Discrete memoryless Channel:

A channel is defined as the medium through which the coded signals are generated by an information source are transmitted. In general, the input to the channel is a symbol belonging to an alphabet 'A' with 'r' symbols, the output of a channel is a symbol belonging to an alphabet 'B' with 's' symbols.

Due to errors in the channel, the output symbols may differ from input symbols.

1.6.1 Representation of a channel:

A communication channel may be represented by a set of input alphabets A=$(a_{1,}a_{2,}a_3 \ldots \ldots .. a_r)$ consisting of 'r' symbols and set of output alphabets B=$(b_{1,}b_{2,}b_3 \ldots \ldots .. b_s)$ consisting of s symbols and a set of conditional probability P$({}^{b_j}/_{a_i})$

with i=1,2,….r and j=1,2,…….s

$$A \left.\begin{matrix} a_1 \\ a_2 \\ \vdots \\ a_r \end{matrix}\right\} \rightarrow \mathrm{P}({}^{b_j}/_{a_i}) \rightarrow \left\{\begin{matrix} b_1 \\ b_2 \\ \vdots \\ b_s \end{matrix}\right. B$$

The conditional probabilities come in to the existence due to the presence of noise in the channel. Because of noise there will be some amount of uncertainty in the reception of any symbols. For this reason there are 's' number of symbols at the receiver from 'r' symbols at transmitter. Totally there are r * s conditional probabilities represented in a form of matrix which is called as Channel Matrix or Noise Matrix.

$$P(b_j/a_i) = \begin{array}{c c} & \begin{matrix} b_1 & b_2 & b_3 & & b_s \end{matrix} \\ a_1 & \begin{bmatrix} P(b_1/a_1) & P(b_2/a_1) & P(b_3/a_1) & \cdots & P(b_s/a_1) \\ P(b_1/a_2) & P(b_2/a_2) & P(b_3/a_2) & \cdots & P(b_s/a_2) \\ P(b_1/a_3) & P(b_2/a_3) & P(b_3/a_3) & \cdots & P(b_s/a_3) \\ P(b_1/a_4) & P(b_2/a_4) & P(b_3/a_4) & \cdots & P(b_s/a_4) \\ \vdots & & & & \\ P(b_1/a_r) & P(b_2/a_r) & P(b_3/a_r) & \cdots & P(b_s/a_r) \end{bmatrix} \end{array}$$

When a_1 is transmitted, it can be received as any one of the output symbols $(b_{1,}b_{2,}b_3 \ldots \ldots .. b_s)$

Therefore $P_{11} + P_{12} + P_{13} + \cdots \ldots P_S$=1

=>P$({}^{b_1}/_{a_1})$+ P$({}^{b_2}/_{a_1})$+ P$({}^{b_3}/_{a_1})$+……………… P$({}^{b_s}/_{a_1})$=1

In general, $\sum_{j=1}^{s} \mathrm{P}({}^{b_j}/_{a_i}) = 1$ for i= 1 to r

Thus the sum of all the elements in any row of the channel matrix is equal to UNITY.

1.6.2 Joint Probability:

Joint probability between any input symbol a_i and any output symbol $b_j$is given by

$\mathrm{P}(a_i \cap b_j) = \boldsymbol{P(a_i, b_j)} = \mathbf{P}({}^{\boldsymbol{b_j}}/_{\boldsymbol{a_i}})\boldsymbol{P(a_i)}$

$$\boldsymbol{P(a_i, b_j)} = \mathbf{P}({}^{\boldsymbol{a_i}}/_{\boldsymbol{b_j}})\boldsymbol{P(b_j)}$$

Properties:

Consider the source alphabet A=$(a_{1,}a_{2,}a_3 \dots \dots . . a_r)$ and output alphabet B=$(b_{1,}b_{2,}b_3 \dots \dots . . b_s)$

- The source entropy is given by $H(A) = \sum_{i=1}^{r} P_{a_i} log_2(\frac{1}{P_{a_i}})$
- The entropy of the receiver or output is given by $H(B) = \sum_{j=1}^{s} P_{b_j} log_2(\frac{1}{P_{b_j}})$
- If all the symbols are equi-probable, then maximum source entropy is $H(A)_{max} = log_2 r$
- Conditional Entropy: The entropy of input symbols $a_{1,}a_{2,}a_3 \dots \dots . . a_r$ after the transmission and reception of particular output symbol b_j is defined as conditional entropy, denoted by $H({}^{\boldsymbol{A}}/_{\boldsymbol{b_j}})$

$$H\left({}^{A}/_{b_j}\right) = \sum_{i=1}^{r} \mathrm{P}\left({}^{a_i}/_{b_j}\right) log_2 \frac{1}{\mathrm{P}\left({}^{a_i}/_{b_j}\right)}.$$

- If the average value of all the conditional probability is taken as j varies from 1 to s denoted by $H({}^{A}/_{B}) = \sum_{j=1}^{s} P(b_j)\ H\left({}^{A}/_{b_j}\right)$

$$=\sum_{j=1}^{s}\sum_{i=1}^{r} P(b_j)\ \mathrm{P}\left({}^{a_i}/_{b_j}\right) log_2 \frac{1}{\mathrm{P}\left({}^{a_i}/_{b_j}\right)}$$

$\boldsymbol{H({}^{A}/_{B}) = \sum_{j=1}^{s}\sum_{i=1}^{r} P(a_i, b_j)\ log_2 \frac{1}{P({}^{a_i}/_{b_j})}}$ is conditional entropy of transmitter

Similarly $\boldsymbol{H({}^{B}/_{A}) = \sum_{i=1}^{r}\sum_{j=1}^{s} P(a_i, b_j)\ log_2 \frac{1}{P({}^{b_j}/_{a_i})}}$ is conditional entropy of receiver.

- $\boldsymbol{H(A, B) = \sum_{i=1}^{r}\sum_{j=1}^{s} P(a_i, b_j)\ log_2 \frac{1}{P(a_i, b_j)}}$ is joint conditional probability.

1.6.3 Mutual Information:

When an average amount of information H(x) is transmitted over a noisy channel, then an amount of information $H\left({}^{x}/_{y}\right)$ is last in the channel. The balance of the information at the receiver is defined as Mutual Information I(x.y)

I(x,y)= H(x)- $H\left({}^{x}/_{y}\right)$

= H(y)- $H\left({}^{y}/_{x}\right)$

I(x_i) = $\log(\frac{1}{P(x_i)})$ and $I\left({}^{x_i}/_{y_j}\right) = \log(\frac{1}{P\left({}^{x_i}/_{y_j}\right)})$

The difference between the above 2 is the information gained through the channel.

$$I\,(x_i, y_j) = \log\left(\frac{1}{P(x_i)}\right) - \log(\frac{1}{P\left({}^{x_i}/_{y_j}\right)})$$

$$I\,(x_i, y_j) = \log\frac{P\left({}^{x_i}/_{y_j}\right)}{P(x_i)}$$

$$I\,(x_i, y_j) = \log\frac{P(x_i, y_j)}{P(x_i)P(y_j)}$$

Properties:

- The Mutual Information is symmetric. $I(x_i, y_j) = I(y_j, x_i)$
- $I(X,Y) = H(X) + H(Y) - H(X,Y)$
- $I(X,Y) = H(X) - H\left({}^{X}/_{Y}\right)$
- $I(X,Y) = H(Y) - H\left({}^{Y}/_{X}\right)$

1.6.4 Channel Capacity

It is known that average information content of the source is $H(X) = \sum_{i=1}^{M} p(x_i) \log_2\left(\frac{1}{p(x_i)}\right)$. Average information per symbol going in to the channel is $R_{in} = r_s * H(X)$. Due to the error, it is not possible to reconstruct the input symbol sequence with certainity on the recovered sequence. Therefore source information is lost due to the errors.

- Therefore average rate of information transmission is given by $R_t = I(X,Y)\,.\,r_s$. Bits/sec.

- The capacity of a discrete memoryless noisy channel is defined a s maximum possible rate of maximum rate of transmission occurs when the source is matched to the channel.
- $\therefore C = Max(R_t)$
- $= \text{Max}[I(X,Y) \,.\, r_s]$
- $C = \text{Max}\{[H(X) - H(X/Y)]\, r_s\}$

1.6.5 Channel Efficiency

$\%\eta_{ch} = \frac{R_t}{C} * 100$

$= \frac{I(X,Y) \,.\, r_s}{\text{Max}[I(X,Y) \,.\, r_s]} * 100$

$\%\eta_{ch} = \frac{H(X) - H(X/Y)}{Max[H(X) - H(X/Y)]} * 100$

Redundancy =1-η_{ch}

1.6.6 Symmetry Channel

Symmetry channel is defined as the channel in which the channel matrix has 2nd and subsequent rows, the same elements as the first row, but in different order.

$\therefore H(Y/X) = h$, where →entropy of any single row. The channel capacity with r_s=1 bits/sec is given by,

$C = Max(R_t)$

$= \text{Max}[I(X,Y)]\, r_s$

$= \text{Max}[I(X,Y)]$

$= \text{Max}\left(H(Y) - H(Y/X)\right)$

$= \text{Max}[H(Y)] - \text{Max}[H(Y/X)]$

$= \text{Max}[H(Y)]$ - $\text{Max}(h)$

$C = \text{Max}[H(Y)] - h$

H(Y) is the entropy of symbol which becomes maximum if and only if all the receive symbols become equi-probable.

Since there are ‘s’ output symbols

$\text{Max}[H(Y)] = log_2 s$

$\therefore$ $\boldsymbol{C = log_2 s - h}$

Ex.1: A transmitter has an alphabet containing of 5 letters $\{a_1, a_2, a_3, a_4, a_5\}$ and the receiver has an alphabet of four letters$\{b_1, b_2, b_3, b_4\}$. The joint probabilities of the system are given below. Compute different entropies of this channel.

$$P(A, B) = \begin{array}{c c} & \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} & \begin{bmatrix} 0.25 & 0 & 0 & 0 \\ 0.10 & 0.30 & 0 & 0 \\ 0 & 0.05 & 0.10 & 0 \\ 0 & 0 & 0.05 & 0.1 \\ 0 & 0 & 0.05 & 0 \end{bmatrix} \end{array}$$

Solution:

$P(b_1) = 0.35, P(b_2) = 0.35, P(b_3) = 0.2$ $P(b_4) = 0.1$

$P(a_1) = 0.25, P(a_2) = 0.4$ $P(a_3) = 0.15, P(a_4) = 0.15,$ $P(a_5) = 0.05$

$$H(A) = \sum_{i=1}^{5} P(a_i) \log \frac{1}{P(a_i)}$$

$$= 0.25 \log \frac{1}{0.25} + 0.4 \log \frac{1}{0.4} + 0.15 \log \frac{1}{0.15} + 0.15 \log \frac{1}{0.15} + 0.05 \log \frac{1}{0.05}$$

$\mathbf{H(A) = 2.066}$ **bits/message-symbol**

$$H(B) = \sum_{j=1}^{4} P(b_j) \log \frac{1}{P(b_j)}$$

$$= 0.35 \log \frac{1}{0.35} + 0.35 \log \frac{1}{0.35} + 0.2 \log \frac{1}{0.2} + 0.1 \log \frac{1}{0.1}$$

$\mathbf{H(B) = 1.857}$ **bits/message-symbol**

$$H(A, B) = \sum_{i=1}^{5} \sum_{j=1}^{4} P(a_i, b_j) \log \frac{1}{P(a_i, b_j)}$$

$$H(A, B) = 0.25 \log \frac{1}{0.25} + 0.1 \log \frac{1}{0.1} + 0.3 \log \frac{1}{0.3} + 0.05 \log \frac{1}{0.05} + 0.1 \log \frac{1}{0.1} + 0.05 \log \frac{1}{0.05} + 0.05 \log \frac{1}{0.05} + 0.1 \log \frac{1}{0.1}$$

$\mathbf{H(A, B) = 2.666}$ **bits/message-symbol**

$$H(B/A) = H(A, B) - H(A)$$
$$= 2.666 - 2.066$$
$$\mathbf{H(B/A) = 0.6 \text{ bits/message-symbol}}$$

$$H(A/B) = H(A, B) - H(B)$$
$$= 2.666 - 1.857$$
$$\mathbf{H(A/B) = 0.809 \text{ bits/message-symbol}}$$

$$I(A, B) = H(A) - H(A/B)$$
$$= 2.066 - 0.809$$
$$\mathbf{I(A, B) = 1.257 \text{ bits/message-symbol}}$$

$$I(A, B) = H(B) - H(B/A)$$
$$= 1.857 - 0.6$$
$$\mathbf{I(A, B) = 1.257 \text{ bits/message-symbol}}$$

Ex.2: A transmitter transmits 5 symbols with probabilities 0.2, 0.3, 0.2, 0.1 and 0.2. Given the channel matrix $P\left({}^{B}/_{A}\right)$, Calculate H(B) and H(A,B)

$$P(B/A) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1/4 & 3/4 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 \\ 0 & 0 & 1/3 & 2/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Solution:

We know that, $P(a_i, b_j)=P(a_i)P(b_j/a_i)$

The JPM may now be constructed by multiplying 1st row elements by $P(a_1) = 0.2 = \frac{1}{5}$, 2nd row by $P(a_2) = 0.3 = \frac{3}{10}$, 3rd row by $P(a_3) = 0.2 = \frac{1}{5}$, 4th row by $P(a_4) = 0.1 = \frac{1}{10}$ and 5th row by $P(a_5) = 0.2 = \frac{1}{5}$.

$$P(A, B) = \begin{array}{c} \\ a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{array} \begin{array}{c} \begin{array}{cccc} b_1 & b_2 & b_3 & b_4 \end{array} \\ \begin{bmatrix} 1/5 & 0 & 0 & 0 \\ 3/40 & 9/40 & 0 & 0 \\ 0 & 1/15 & 2/15 & 0 \\ 0 & 0 & 1/30 & 1/15 \\ 0 & 0 & 1/5 & 0 \end{bmatrix} \end{array}$$

Adding the element pf each column, we get

$$P(b_1) = \frac{1}{5} + \frac{3}{40} = \frac{11}{40}$$

$$P(b_2) = \frac{9}{40} + \frac{1}{15} = \frac{7}{24}$$

$$P(b_3) = \frac{2}{15} + \frac{1}{30} + \frac{1}{5} = \frac{11}{30}$$

$$P(b_4) = \frac{1}{15}$$

$$H(B) = \sum_{j=1}^{4} P(b_j) \log \frac{1}{P(b_j)}$$

$$= \frac{11}{40} \log \frac{40}{11} + \frac{7}{24} \log \frac{24}{7} + \frac{11}{30} \log \frac{30}{11} + \frac{1}{15} \log 15$$

$$\mathbf{H(B) = 1.822 \text{ bits/message-symbol}}$$

$$H(A, B) = \sum_{i=1}^{5} \sum_{j=1}^{4} P(a_i, b_j) \log \frac{1}{P(a_i, b_j)}$$

$$= \frac{1}{5} \log 5 + \frac{3}{40} \log \frac{40}{3} + \frac{9}{40} \log \frac{40}{9} + \frac{1}{15} \log 15 + \frac{2}{15} \log \frac{15}{2}$$

$$+ \frac{1}{30} \log 30 + \frac{1}{5} \log 5 + \frac{1}{15} \log 15$$

$$\mathbf{H(A, B) = 2.7653 \text{ bits/message-symbol}}$$

Ex.3: For the JPM given below, compute individually H(X), H(Y), H(X,Y), H(X/Y), H(Y/X), I(X,Y) and channel Capacity if r=1000 symbols/sec. Verify the relationship among these entropies.

$$P(X, Y) = \begin{bmatrix} 0.05 & 0 & 0.20 & 0.05 \\ 0 & 0.10 & 0.10 & 0 \\ 0 & 0 & 0.20 & 0.10 \\ 0.05 & 0.05 & 0 & 0.10 \end{bmatrix}$$

Solution:

The given JPM can be rewritten with input and output symbols as below:

$$P(X, Y) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 0.05 & 0 & 0.20 & 0.05 \\ 0 & 0.10 & 0.10 & 0 \\ 0 & 0 & 0.20 & 0.10 \\ 0.05 & 0.05 & 0 & 0.10 \end{bmatrix} \end{array}$$

From property-1 of JPM, addition of elements of JPM columnwise results in probabil of output symbols.

$$\therefore \quad P(y_1) = 0.05 + 0.05 = 0.10$$
$$P(y_2) = 0.10 + 0.05 = 0.15$$
$$P(y_3) = 0.2 + 0.1 + 0.2 = 0.50$$
$$P(y_4) = 0.05 + 0.1 + 0.1 = 0.25$$

From property-2 of JPM, addition of elements of JPM rowwise results in probability of input symbols.

$$\therefore \quad P(x_1) = 0.05 + 0.2 + 0.05 = 0.30$$
$$P(x_2) = 0.10 + 0.10 = 0.20$$
$$P(x_3) = 0.20 + 0.10 = 0.30$$
$$P(x_4) = 0.05 + 0.05 + 0.1 = 0.20$$

$$H(X) = \sum_{i=1}^{4} P(x_i) \log \frac{1}{P(x_i)}$$

$$= \left(0.3 \log \frac{1}{0.3}\right)(2) + \left(0.2 \log \frac{1}{0.2}\right)(2)$$

H(X) = 1.971 bits/message-symbol

$$H(Y) = \sum_{j=1}^{4} P(y_j) \log \frac{1}{P(y_j)}$$

$$= 0.1 \log \frac{1}{0.1} + 0.15 \log \frac{1}{0.15} + 0.5 \log \frac{1}{0.5} + 0.25 \log \frac{1}{0.25}$$

H(Y) = 1.743 bits/message-symbol

$$H(X/Y) = \sum_{i=1}^{4} \sum_{j=1}^{4} P(x_i, y_j) \log \frac{1}{P(x_i/y_j)} \text{ bits/message symbol.}$$

Using the relationship $P(x_i/y_j) = \dfrac{P(x_i, y_j)}{P(y_j)}$, the matrix P(X/Y) is constructed as below:

$$P(X/Y) = \begin{bmatrix} \frac{0.05}{0.10} & 0 & \frac{0.2}{0.5} & \frac{0.05}{0.25} \\ 0 & \frac{0.10}{0.15} & \frac{0.1}{0.5} & 0 \\ 0 & 0 & \frac{0.2}{0.5} & \frac{0.10}{0.25} \\ \frac{0.05}{0.10} & \frac{0.05}{0.15} & 0 & \frac{0.10}{0.25} \end{bmatrix}$$

$\therefore$

$$P(X/Y) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1/2 & 0 & 2/5 & 1/5 \\ 0 & 2/3 & 1/5 & 0 \\ 0 & 0 & 2/5 & 2/5 \\ 1/2 & 1/3 & 0 & 2/5 \end{bmatrix} \end{array}$$

$\therefore \quad H(X/Y) = 0.05 \log 2 + 0.05 \log 2 + 0.1 \log 3/2 + 0.05 \log 3 + 0.2 \log 5/2$
$\qquad + 0.1 \log 5 + 0.2 \log 5/2 + 0.05 \log 5 + 0.1 \log 5/2 + 0.1 \log 5/2$

$\therefore$ **$H(X/Y) = 1.379$ bits/message symbol**

$$H(Y/X) = \sum_{i=1}^{4}\sum_{j=1}^{4} P(x_i, y_j) \log \frac{1}{P(y_j/x_i)} \text{ bits/message-symbol.}$$

Using the relationship $P(y_j/x_i) = \frac{P(x_i, y_j)}{P(x_i)}$, the channel matrix $P(Y/X)$ is constructed below:

$$P(Y/X) = \begin{bmatrix} \frac{0.05}{0.30} & 0 & \frac{0.20}{0.30} & \frac{0.05}{0.30} \\ 0 & \frac{0.10}{0.20} & \frac{0.10}{0.20} & 0 \\ 0 & 0 & \frac{0.20}{0.30} & \frac{0.10}{0.30} \\ \frac{0.05}{0.20} & \frac{0.05}{0.20} & 0 & \frac{0.10}{0.20} \end{bmatrix}$$

$$P(Y/X) = \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{array} \begin{array}{c} \begin{array}{cccc} y_1 & y_2 & y_3 & y_4 \end{array} \\ \begin{bmatrix} 1/6 & 0 & 2/3 & 1/6 \\ 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 2/3 & 1/3 \\ 1/4 & 1/4 & 0 & 1/2 \end{bmatrix} \end{array}$$

$H(Y/X) = 0.05 \log 6 + 0.2 \log 3/2 + 0.05 \log 6 + 0.1 \log 2 + 0.1 \log 2$
$\qquad + 0.2 \log 3/2 + 0.1 \log 3 + 0.05 \log 4 + 0.05 \log 4 + 0.1 \log 2$

$H(Y/X) = 1.151$ bits/message-symbol

Verification:

$H(Y/X) = H(X, Y) - H(X)$
$\qquad = 3.122 - 1.971$

$H(Y/X) = 1.151$ bits/message-symbol as before

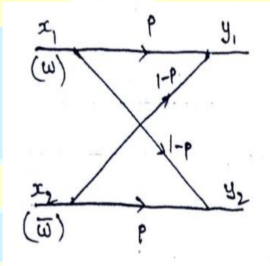
$$H(X/Y) = H(X, Y) - H(Y)$$
$$= 3.122 - 1.743$$
$$\mathbf{H(Y/X) = 1.379 \text{ bits/message-symbol as before}}$$

$$I(X, Y) = H(X) - H(X/Y)$$
$$= 1.971 - 1.379$$
$$\mathbf{I(X, Y) = 0.592 \text{ bits/message-symbol}}$$

$$I(X, Y) = H(Y) - H(Y/X)$$
$$= 1.743 - 1.151$$
$$\mathbf{I(X, Y) = 0.592 \text{ bits/message-symbol}} \quad \text{– verified.}$$

1.7 Binary Symmetric Channel:

The binary symmetric channel is one of the most commonly and widely used channel whose channel diagram is given below



From the above diagram, channel matrix can be written as

$$P\left(X/_Y\right) = \begin{bmatrix} P & 1-P \\ 1-P & P \end{bmatrix} = \begin{bmatrix} P & \bar{P} \\ \bar{P} & P \end{bmatrix}$$

The matrix is a symmetric matrix. Hence the channel is binary symmetric channel.

1.8 Channel Capacity

It is known that C = Max$\left\{\left[H(Y) - H\left(Y/_X\right)\right] r_s\right\}$.

For symmetry channel, $H\left(Y/_X\right) = h = P\log\frac{1}{P} + \bar{P}\log\frac{1}{\bar{P}}$

Since it is a binary symmetric channel, $H(Y)_{max} = log_2 s = log_2 2 = 1$

$\therefore C = 1 - h$ bits/sec.

Ex.1: A binary symmetric channel has the following noise matrix with source probabilities of $P(x_1)$=2/3 and $P(x_2)$=1/3. $P\left({}^{Y}/_{X}\right) = \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix}$. Determine H(X), H(Y), H(X,Y), H(Y/X), H(X/Y), I(X,Y), Channel Capacity, Channel efficiency and redundancy.

Solution:

$$p = \frac{1}{4},\ \bar{p} = \frac{3}{4},\ w = \frac{2}{3} \text{ and } \bar{w} = \frac{1}{3}$$

$$H(X) = \sum_{i=1}^{2} P(x_i) \log \frac{1}{P(x_i)}$$

$$= \frac{2}{3} \log \frac{3}{2} + \frac{1}{3} \log 3$$

$$\therefore \quad \mathbf{H(X) = 0.9183 \text{ bits/message-symbol}}$$

We have $\bar{p}w + p\bar{w} = \left(\frac{3}{4}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{3}\right) = \frac{7}{12}$

and $pw + \bar{p}\bar{w} = \left(\frac{1}{4}\right)\left(\frac{2}{3}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{3}\right) = \frac{5}{12}$

H(Y)=7/12 log(12/7)+5/12 log(12/5)

H(Y)=0.9799 bits/symbol

$$H\left({}^{Y}/_{X}\right) = h = P\log\frac{1}{P} + \bar{P}\log\frac{1}{\bar{P}}$$

= ¾ log(4/3)+1/4 log(4/1)

=0.8113 bits/symbols

$$H(X, Y) = H(X) + H(Y/X)$$
$$= 0.9183 + 0.8113$$
$$\mathbf{H(X, Y) = 1.7296 \text{ bits/message-symbol}}$$

$$H(Y, X) = H(X, Y) = H(Y) + H(X/Y)$$
$$H(X/Y) = H(X, Y) - H(Y)$$
$$= 1.7296 - 0.9799$$
$$\mathbf{H(X/Y) = 0.7497 \text{ bits/message-symbol}}$$

$$I(X, Y) = H(X) - H(X/Y)\ [\text{or } H(Y) - H(Y/X)]$$
$$= 0.9183 - 0.7497$$
$$\mathbf{I(X, Y) = 0.1686 \text{ bits/message-symbol}}$$

$$C = 1 - h = 1 - H(Y/X)$$
$$= 1 - 0.8113$$
$$\mathbf{C = 0.1887 \text{ bits/message-symbol}}$$

$$\text{Channel efficiency} = \frac{I(X,Y)}{C}$$
$$= \frac{0.1686}{0.1887}$$
$$\therefore \quad \eta_{ch} = \mathbf{89.35\%}$$

Channel Redundancy =η_{ch} = 10.65%

Problem : Consider a source with 8 alphabet A to H with respective probabilities of 0.22, 0.20, 0.18, 0.15, 0.10, 0.08, 0.05, 0.02.

i] Construct a binary compact code and determine the code efficiency.

ii] Construct a ternary compact code and determine efficiency of the code

iii] Construct a quaternary compact code and determine the code efficiency. Compare and comment on the result. Draw code trees for all three cases.

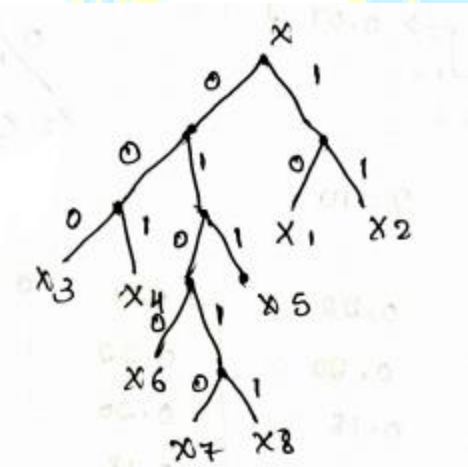
iv] Decode the messages using appropriate code trees

a) 0101001000001101011001…

b) 12111011020012002 ….

c) 031132020300100231 …..

Solution:



0

(i) $q = r + \alpha(r-1)$

$q = 2 + \alpha(2-1)$

$q = 2$ ✗

$q = 8$, $r = 2$, $\alpha = 6$

Symbol	Prob						
X_1	0.22 10	0.22 10	0.22 10	0.25 01	0.33 00	0.42 1	0.58 0
X_2	0.20 11	0.20 11	0.20 11	0.22 10	0.25 01	0.33 00	0.42 1
X_3	0.18 000	0.18 000	0.18 000	0.20 11	0.22 10	0.25 01	
X_4	0.15 001	0.15 001	0.15 001	0.18 000	0.20 11		
X_5	0.10 011	0.10 011	0.15 010	0.15 001			
X_6	0.08 0100	0.08 0100	0.10 011				
X_7	0.05 01010	0.07 0101					
X_8	0.02 01011						

$H(S) = \sum_{i=1}^{q} P_i \log_2 \frac{1}{P_i}$

$H(S) = 2.75$ bits / symbol

$L = \sum_{i=1}^{q} P_i l_i$

$L = 2.8$

$\eta_c = \frac{H(S)}{L}$

$\eta_c = 98.25\%$

$R_{\eta_c} = 1.79\%$

iv a) 0101001000001101011001

$= X_7\ X_6\ X_3\ X_2\ X_8\ X_4$

(ii) $q = 3 + \alpha(2)$

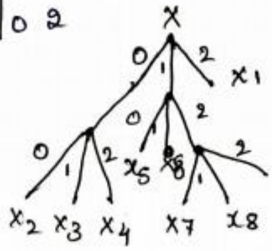
$\alpha = 2.5$

$\alpha = 3 \Rightarrow q = 9$

Symbol	Prob			
X_1	0.22 2	0.22 2	0.25 1	0.53 0
X_2	0.20 00	0.20 00	0.22 2	0.25 1
X_3	0.18 01	0.18 01	0.20 00	0.22 2
X_4	0.15 02	0.15 02	0.18 01	
X_5	0.10 10	0.10 10	0.15 02	
X_6	0.08 11	0.08 11		
X_7	0.05 120	0.07 12		
X_8	0.02 121			
X_9	0 122			

iv) b) 121 11 01 10 2 00 120 02

$X_8\ X_6\ X_3\ X_1\ X_2\ X_7\ X_4$



$H_r(S) = \frac{H(S)}{\log_2 r}$

$H_r(S) = \frac{H(S)}{\log_2(3)} = 1.735$

$L = \sum_{i=1}^{q} P_i l_i$

$L = 1.85$

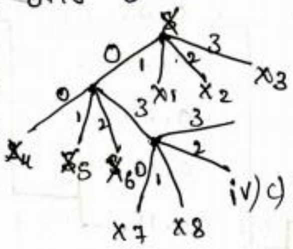
$\eta_c = \frac{H(S)}{L} = 93.73\%$

$R_{\eta_c} = 1 - \eta_c = 6.21\%$

Decoding Alphabets: X8, X6, X3, X5, X1, X2, X7,X4

iii) $\alpha = 2$ $q = 10$

Symbol	Prob						
x_1	0.22	1	0.22	1	0.4	0	
x_2	0.20	2	0.20	2	0.22	1	
x_3	0.18	3	0.18	3	0.20	2	
x_4	0.15	00	0.15	00	0.18	3	
x_5	0.10	01	0.10	01			
x_6	0.08	02	0.08	02			
x_7	0.05	030	0.07	03			
x_8	0.02	031					
x_9	0	032					
x_{10}	0	033					



$H(S) = 1.375$

$L = \sum_{i=1}^{q} P_i l_i$

$L = 1.47$

$\eta_c = \frac{H(S)}{L} = 93.53\%$

$R_{\eta_c} = 1 - \eta_c = 6.47\%$

iv) c) 0 3 1 1 3 2 0 2 0 3 0 0 1 0 0 2 3 1

$x_8 \; x_1 \; x_3 \; x_2 \; x_6 \; x_7 \; x_5 \; x_4 \; x_2 \; x_3 \; x_1$

Problem : Consider a Zero memory source with S=[S1,S2,S3,S4,S5,S6,S7] and Probabilities P=[0.4,0.2,0.1,0.1,0.1,0.05,0.05]

i. Construct a binary Huffman code by placing the composite symbol as low as possible.
ii. Repeat (i) by moving a composite symbol as high as possible.
iii. In each of the cases (i) and (ii) above,
   - Compute the variances of the word lengths and comment on the result.
   - Find Efficiency and Redundancy.
iv. Considering Case(ii) table,
   - Write the code tree and decode the message 01110110011000100……
   - Determine probabilities of 0's and 1's.

Tips: Variance= $\sum_{i=1}^{i=q} P_i \; (l_i - L)^2$

Probability of 0's : P(0)= $\frac{1}{L}\sum_{i=1}^{i=q}(No. of 0' s \text{ in } the \;\; code\; for\; Si)\;\; P_i$

(i)

As low as possible.

s_1	0.4	1	0.4	1	0.4	1	0.4	1	0.4	1	0.6	0
s_2	0.2	01	0.2	01	0.2	01	0.2	01	0.4	00	0.4	1
s_3	0.1	0010	0.1	0010	0.2	000	0.2	000	0.2	01		
s_4	0.1	0011	0.1	0011	0.1	0010	0.2	001				
s_5	0.1	0000	0.1	0000	0.1	0011						
s_6	0.05	00010	0.1	0001								
s_7	0.05	00011										

(i) as low as possible

S_1	1
S_2	01
S_3	0010
S_4	0011
S_5	0000
S_6	00010
S_7	00011

$H(S) = 2.421$

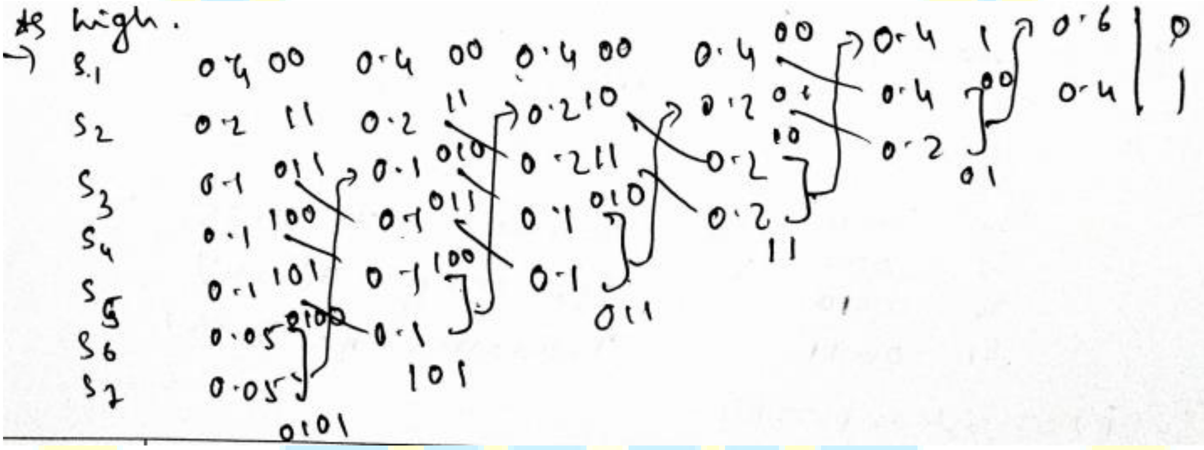
$L = 2.5$

Variance = ~~[illegible]~~ 2.25

Efficiency $\eta = 96.84\%$

Redundancy $R_\eta = 3.16\%$

(ii)



(iii) as high as possible

S_1	00
S_2	11
S_3	011
S_4	100
S_5	101
S_6	0100
S_7	0101

$H(S) = 2.421$

$L = 2.5$

Variance = 0.45

Probability of 0's = 0.58

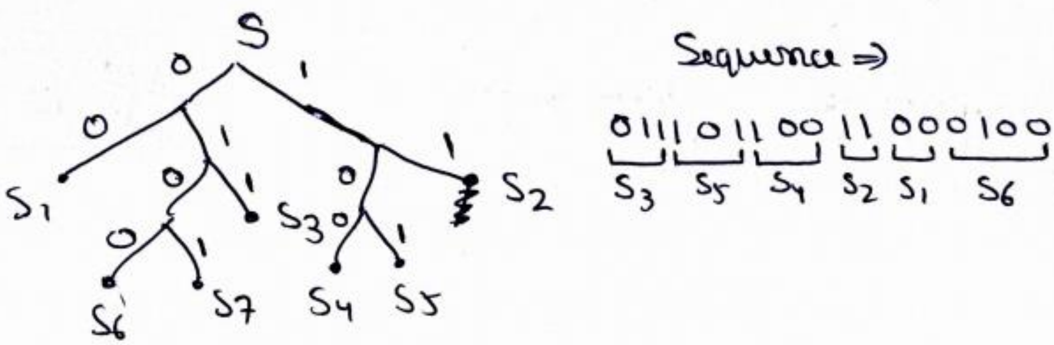
Probability of 1's = 0.42

Efficiency $\eta = 96.84\%$

Redundancy $R_\eta = 3.16\%$

(iv)

Code tree using code (in case (ii))



1. 9 Shannon Hartley Theorem and its Implication

The waveform received at the receiver may be accompanied by a waveform which varies with respect to time in an entirely unpredictable manner. This unpredictable voltage waveform is a random process called "noise".

The noise introduced due to thermal motion of electrons is called ***"Johnson Noise"*** or ***"White Noise"*** and the noise resulting due to the flow of electrons across semiconductor junction is called ***"Shot Noise"***. When the noise adds to the signal it is called ***"Additive Noise"*** and if it multiplies, then it is called ***"Fading"***.

In channels, the noise is almost white and it has a distribution which resembles Gaussian (or normal) distribution with zero mean and some variance. Hence, this noise is also called "additive white Gaussian-noise (AWGN)".

Statement of Shannon-Hartley Law :

Shannon-Hartley law also called Shannon's third theorem, states that the capacity of a band-limited Gaussian channel with AWGN is given by

$$C = B \log\left(1+\frac{S}{N}\right) \text{ bits/sec} \quad \ldots\ldots (4.99)$$

Where B = Channel bandwidth in Hz

S = Signal power in watts

N = Noise power in watts $= \eta B$

where the two sided power spectral density of noise is $(\eta/2)$ watts/Hz.

Proof :

From equation (4.85), the channel capacity C is given by

$$C = [H(Y) - H(N)]_{max} \quad \text{...... (4.100)}$$

If the noise is additive, white and Gaussian having a power N in a bandwidth of B Hz, then from equation (4.66), we have

$$H(N)_{max} = B \log 2\pi e N \text{ bits/sec} \quad \text{...... (4.101)}$$

When the input signal is limited to an average power S, over the same bandwidth of B Hz, and when the signal at the receiver Y = X + N with X and N being independent, then the received signal power is nothing but variance given by

$$\sigma_Y^2 = (S + N) \quad \text{...... (4.102)}$$

We have seen in the derivation of equation (4.65) that for a given mean square value, the entropy will be maximum if the signal is Gaussian and the maximum entropy is given by

$$H(Y)_{max} = B \log 2\pi e \sigma_Y^2 \text{ bits/sec} \quad \text{...... (4.103)}$$

Using equation (4.102) in (4.103), we get

$$H(Y)_{max} = B \log 2\pi e(S + N) \text{ bits/sec} \quad \text{...... (4.104)}$$

Using equations (4.101) and (4.104) in equation (4.100), we get

$$C = B \log 2\pi e(S + N) - B \log 2\pi e N$$

$$= B \log \frac{2\pi e(S+N)}{2\pi e N}$$

$$\text{or} \quad \mathbf{C = B \log \left(1 + \frac{S}{N}\right) \text{ bits/sec}} \quad \text{...... (4.105)}$$

1st Implication :

From Shannon-Hartley law, we have

$$C = B \log_2 \left(1 + \frac{S}{N}\right) \text{ bits/sec} \quad \text{...... (4.107)}$$

It looks, from the above equation that when B is increased, channel capacity C also increases and since $R_{max} = C$, the maximum rate of information transmission can be enhanced to any large value as we please. However, the channel capacity does not become infinite when the bandwidth is made infinite. This is because, as B increases, the noise power N which is dependent on B, also increases thereby reducing $\left(\frac{S}{N}\right)$. Thus the product of B and $\log_2\left(1 + \frac{S}{N}\right)$ will increase only upto a certain value and becomes constant with increasing B. This value is denoted by C_∞. Let us calculate that value.

Substituting $N = \eta B$ in equation (4.107), we get

$$C = B \log_2 \left(1 + \frac{S}{\eta B}\right)$$

$$= \frac{S}{\eta}\left(\frac{\eta B}{S}\right)\log_2\left(1+\frac{S}{\eta B}\right)$$

$$= \frac{S}{\eta}\log_2\left(1+\frac{S}{\eta B}\right)^{(\eta B/S)}$$

Let $x = \frac{S}{\eta B}$

Then $C = \frac{S}{\eta}\log_2(1+x)^{(1/x)}$

Accordingly when $B \to \infty$, $x \to 0$

$$\therefore \lim_{\substack{B\to\infty \\ x\to 0}} C = \lim_{\substack{B\to\infty \\ x\to 0}} \frac{S}{\eta}\log_2(1+x)^{(1/x)}$$

$$\therefore C_\infty = \frac{S}{\eta}\log_2 \lim_{x\to 0}\left[(1+x)^{(1/x)}\right]$$

$$\therefore C_\infty = \frac{S}{\eta}\log_2 e$$

$$\text{or} \quad \mathbf{C_\infty = B\frac{S}{N}\log_2 e \text{ bits/sec}} \qquad \text{...... (4.108)}$$

SHANNON'S LIMIT :

We define an ***"ideal system"*** as one that transmits data at a bit rate R_t equal to the channel capacity C. We may then express the average transmitted power as

$$S = E_b C \qquad \text{...... (4.109)}$$

Where E_b = transmitted energy per bit in joules.

Using $N = \eta B$ and $S = E_b C$ in equation (4.107), we get for an ideal system

$$C = B\log_2\left(1+\frac{E_b}{\eta}\frac{C}{B}\right)$$

$$\text{or} \quad \frac{C}{B} = \log_2\left(1+\frac{E_b}{\eta}\frac{C}{B}\right) \qquad \text{...... (4.110)}$$

The quantity $\left(\frac{C}{B}\right)$ is called ***"Bandwidth-efficiency"*** and the quantity (E_b/η) is given by

$$\frac{E_b}{\eta} = \frac{2^{C/B}-1}{(C/B)} \qquad \text{...... (4.111)}$$

When (R_t/B) is plotted as a function of (E_b/η), we get the bandwidth-efficiency diagram which is shown in figure 4.3. The resulting curve represents the capacity boundary for which

$R = C$ corresponding to equation (4.111). Based on this diagram, the following observations are made:

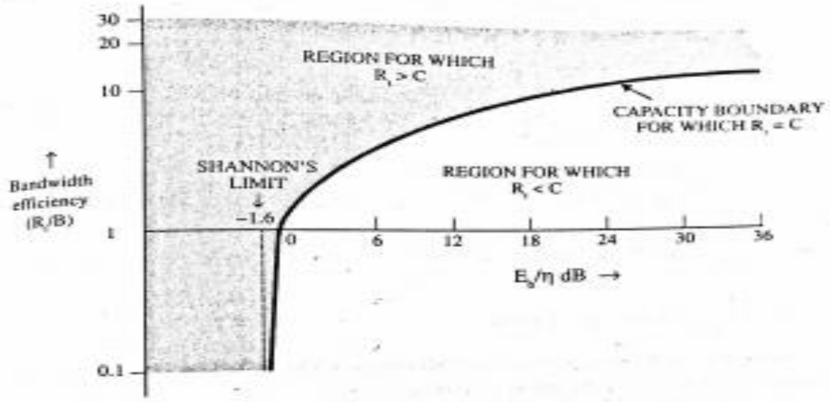


Fig. 4.3 : Illustrating Bandwidth-efficiency diagram

1. For infinite bandwidth, the signal energy-to-noise ratio E_b/η approaches the limiting value.

$$\left(\frac{E_b}{\eta}\right)_\infty = \lim_{B\to\infty}\left(\frac{E_b}{\eta}\right) = \lim_{B\to\infty}\left[\frac{2^{C/B}-1}{(C/B)}\right]$$

Let $\frac{C}{B} = x$. As $B \to \infty$, $x \to 0$

$$\therefore \quad \left(\frac{E_b}{\eta}\right)_\infty = \lim_{x\to 0}\left[\frac{2^x-1}{x}\right] \qquad \text{...... (4.112)}$$

Using L'Hospital Rule, the above limit can be evaluated as below:

Let $y = 2^x$

Taking ln on both sides

$$ln\, y = x\, ln2$$

Differentiating, $\frac{1}{y}\, dy = (ln2)\, dx$ ...... (4.113)

$$\therefore \quad \frac{dy}{dx} = y\,(ln2) = 2^x\,(ln2)$$

Differentiating both numerator and denominator of the RHS of equation (4.112) with respect to 'x', we get

$$\left(\frac{E_b}{\eta}\right)_\infty = \lim_{x \to 0}\left[\frac{\frac{d}{dx}(2^x - 1)}{\frac{d}{dx}(x)}\right]$$

$$= \lim_{x \to 0}\left[\frac{2^x (ln2)}{1}\right] \text{ by using equation (4.113)}$$

$$= 2^0\, ln2$$

$$\therefore \left(\frac{E_b}{\eta}\right)_\infty = ln2 = 0.693$$

$$\text{or} \quad \left(\frac{E_b}{\eta}\right)_\infty \text{ in dB} = 10 \log_{10}(0.693)$$

$$\therefore \left(\frac{E_b}{\eta}\right)_\infty \text{ in dB} \cong -1.6 \text{ dB} \qquad \text{...... (4.114)}$$

This value of – 1.6 dB is called the **"Shannon's Limit"**. The corresponding value of channel capacity is given by equation (4.108) as

$$C_\infty = B\frac{S}{N}\log_2 e \text{ bits/sec}$$

2. The capacity boundary, defined by the curve for critical bit rate $R_t = C$, separates combinations of system parameters that have the potential for supporting error free transmission ($R_t < C$) from those for which error-free transmission is not possible ($R_t > C$). The latter region is shown using dots in figure 4.3.
3. The diagram of figure 4.3 highlights trade-offs between (E_b/η) and (R_t/B). This is discussed in a difficult aspect in the 2nd implication of Shannon-Hartley law.

2nd Implication :

Bandwidth - (S/N) Trade Off :

An important implication of Shannon-Hartley law is the exchange of bandwidth with signal to noise power ratio and vice-versa as given below :

$$\text{Suppose} \left(\frac{S_1}{N_1}\right) = 7 \text{ and } B_1 = 4 \text{ KHz.}$$

$$\therefore \text{ Channel capacity } C_1 = B_1 \log\left(1 + \frac{S_1}{N_1}\right)$$

$$= 4 \times 10^3 \log(1 + 7)$$

$$= 12 \times 10^3 \text{ bits/sec.}$$

Keeping the channel capacity C_2 same as C_1 and if signal-to-noise ratio is increased to 15, then

$$C_2 = C_1 = 12 \times 10^3 = B_2 \log\left(1 + \frac{S_2}{N_2}\right)$$
$$= B_2 \log(1 + 15)$$
$$\therefore \quad \mathbf{B_2 = 3\ KHz}$$

Since the noise power $N = \eta B$, as the bandwidth gets reduced from 4 to 3 KHz, the noise also decreases indicating an increase in signal power as shown below.

We have $N_1 = \eta B_1 = (\eta)\,(4\text{ KHz})$

and $N_2 = \eta B_2 = (\eta)\,(3\text{ KHz})$

Consider $\dfrac{(S_2/N_2)}{(S_1/N_1)} = \dfrac{15}{7}$

$$\therefore \quad \frac{S_2}{S_1} = \frac{15N_2}{7N_1} = \frac{(15)(\eta)(3\text{KHz})}{(7)(\eta)(4\text{KHz})} = \frac{45}{28} = 1.6$$

Thus a 25% reduction in bandwidth from 4 KHz to 3 KHz requires a 60% approximate increase in signal power for maintaining the same channel capacity. Let us look into the exact significance by drawing the ***"trade-off curve"***.

From Shannon-Hartley law

$$\frac{B}{C} = \frac{1}{\log_2\left(1 + \frac{S}{N}\right)} \qquad \text{...... (4.115)}$$

The values of (B/C) for different values of (S/N) are listed in table 4.1 below :

$\frac{S}{N}$	0.5	1	2	5	10	15	20	30
$\frac{B}{C}$	1.71	1	0.63	0.37	0.289	0.25	0.23	0.2

Table 4.1 : Table of values of (B/C) for different values of (S/N)

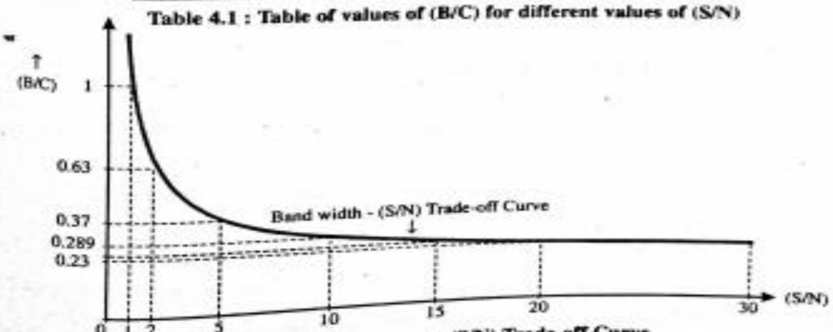


Fig. 4.4 : Bandwidth-to-(S/N) Trade-off Curve

Figure 4.4 above shows a plot of (B/C) as a function of (S/N). Using this trade-off curve the same channel capacity may be obtained by increasing bandwidth if (S/N) is small. Furthermore, the curve also indicates that there exists a threshold point at around (S/N) = 10 up to which the exchange rate of bandwidth with (S/N) is advantageous. Beyond (S/N) = 10, the reduction in B with increasing (S/N) is very poor. FM, PM and PCM systems including DM and ADM systems require larger bandwidths with reasonably good (S/N) ratio.

Ex.1:

Alphanumeric data are entered into a computer from a remote terminal through a voice-grade telephone channel. The channel has a bandwidth of 3.4 KHz, and output signal-to-noise ratio of 20 dB. The terminal has a total of 128 symbols. Assume that the symbols are equiprobable and the successive transmissions are statistically independent.

(a) Calculate channel capacity.

(b) Find the average information content per character.

(c) Calculate the maximum symbol rate for which error-free transmission over the channel is possible.

Solution

Given $B = 3.4 \text{ KHz} = 3400 \text{ Hz}$

$$10 \log_{10} \frac{S}{N} = 20 \text{ dB} \quad \therefore \quad \frac{S}{N} = 100$$

Number of characters $= q = 128$ equiprobable characters.

(a) Channel capacity from equation (4.107)

$$C = B \log_2\left(1+\frac{S}{N}\right)$$

$$= 3400 \log_2(1 + 100)$$

$$\therefore \quad \mathbf{C = 22638 \text{ bits/sec}}$$

(b) Average information content per character [it is maximum since all the characters are equiprobable] is

$$H = H_{max} = \log_2 q = \log_2 128$$

$$\therefore \quad \mathbf{H = 7 \text{ bits/character}}$$

(c) Average information rate $= R_s = r_s H$

For error-free transmission we must have $R_s < C$

$$\therefore \quad r_s H < C$$

$$\therefore \quad r_s < \frac{C}{H}$$

$$\therefore \quad r_s < \frac{22638}{7}$$

$$\therefore \quad \mathbf{r_s < 3234 \text{ symbols/sec}}$$

$\therefore$ The maximum symbol rate for which error-free transmission over the channel is possible = 3234 symbols/sec.

Ex.2: A CRT terminal is used to enter alphanumeric data into a chamber. The CRT is connected through a voice-grade telephone line having usable band width of 3KHz and an output (S/N) of 10 dB. Assume that the terminal has 128 characters and data is sent in an independent manner with equal peobability.

(i) Find the average information per characters.

(ii) Find capacity of the channel.

(iii) Find maximum rate at which data can be sent from terminal to the computer error.

(i) Since all the characters are equiprobable, the average information content per character is

$$H = H_{max} = \log_2 q = \log_2 128$$

or $$\mathbf{H = 7\ bits/character}$$

(ii) Given, $$B = 3\ KHz = 3000\ Hz$$

$$10 \log_{10} \frac{S}{N} = 10 \quad \therefore \quad \frac{S}{N} = 10$$

From equation (4.107), channel capacity is given by

$$C = B \log_2 \left(1 + \frac{S}{N}\right)$$

$$= 3000 \log_2 (1 + 10)$$

$$\therefore \quad \mathbf{C = 10378.295\ bits/sec}$$

(iii) Average information rate = $R_s = r_s H$

For error-free transmission we must have

$$R_s = C$$

i.e., $$r_s H < C$$

$$\therefore \quad r_s < \frac{C}{H}$$

$$\therefore \quad r_s < \frac{10378.295}{7}$$

$$\therefore \quad \mathbf{r_s < 1482.614\ symbols/sec}$$

$\therefore$ The maximum rate at which data can be sent from terminal to the computer without error = 1482.614 symbols/sec.

Ex.3: A voice-grade channel of the telephone network has a bandwidth of 3.4 KHz.

(a) Calculate channel capacity of the telephone channel foe a signal-to-noise ratio of 30dB.

(b) Calculate the minimum signal-to-noise ratio required to support information transmission through the telephone channel at the rate of 4800 bits/sec.

Solution

$$\text{Given} \quad B = 3.4 \text{ KHz} = 3400 \text{ Hz}$$

$$10 \log_{10} \frac{S}{N} = 30 \text{ dB} \quad \therefore \frac{S}{N} = 1000$$

(a) Channel capacity from equation (4.107)

$$C = B \log_2\left(1+\frac{S}{N}\right)$$

$$= 3400 \log_2(1 + 1000)$$

$$\therefore \quad \mathbf{C = 33889 \text{ bits/sec}}$$

(b) $$\text{Given} \quad C = 4800 \text{ bits/sec}, \frac{S}{N} = ?$$

$$\text{We have } C = B \log_2\left(1+\frac{S}{N}\right)$$

$$\therefore \quad 4800 = 3400 \log_2\left(1+\frac{S}{N}\right)$$

$$\therefore \quad \frac{S}{N} = 2^{48/34} - 1$$

$$\therefore \quad \frac{S}{N} = \mathbf{1.66}$$

$$\text{or} \quad \frac{S}{N} \text{ in dB} = 10 \log_{10} 1.66 = 2.2 \text{ dB}$$

$$\therefore \quad \frac{S}{N} = \mathbf{2.2 \text{ dB}}$$

Ex.4:

A black and white television picture may be viewed as consisting of approximately 3×10^5 elements, each one of which may occupy one of 10 distinct brightness levels with equal probability. Assume (a) the rate of transmission is 30 picture frames per second and (b) the signal-to-noise ratio is 30 dB.

Using the channel capacity theorem (Shannon-Hartley law), calculate the minimum bandwidth required to support the transmission of the resultant video signal.

Solution:

Given number of elements/picture frame $= 3 \times 10^5$

Number of brightness levels $= 10$

$\therefore$ Number of different frames possible $= 10^{3 \times 10^5}$ frames.

Since all the levels are equiprobable, the maximum average information content per frame is given by

$$I = \log_2 10^{3\times10^5} \text{ bits/frame}$$
$$= 3 \times 10^5 \log_2 10 \text{ bits/frame}$$
$$\mathbf{I = 9.96 \times 10^5 \text{ bits/frame}}$$

The maximum rate of information is given by

$$R_{s_{max}} = r_s I$$
$$= (30 \text{ frames/sec}) (9.96 \times 10^5 \text{ bits/frame})$$
$$= 29.88 \times 10^6 \text{ bits/sec}$$

According to Shannon's second theorem, $R_{s_{max}}$ is equal to channel capacity C. And according to Shannon-Hartley law.

$$C = B \log_2\left(1+\frac{S}{N}\right)$$

Given $$10 \log_{10}\frac{S}{N} = 30 \text{ dB}$$

$$\therefore \frac{S}{N} = 1000$$

$$\therefore B = \frac{C}{\log_2\left(1+\frac{S}{N}\right)} = \frac{29.88\times10^6}{\log_2(1+1000)}$$

$$\therefore \mathbf{B = 3 \text{ MHz}}$$

Ex.5:

An analog signal has a 4 KHz bandwidth. The signal is sampled at 2.5 times the Nyquist rate and each sample quantized into 256 equally likely levels. Assume that the successive samples are statistically independent.

(i) Find the information rate of this source.

(ii) Can the output of this source be transmitted without errors over a Gaussian channel of bandwidth 50 KHz and (S/N) ratio of 20 dB?

(iii) If the output of this source is to be transmitted without errors over an analog channel having (S/N) of 10 dB, compute the bandwidth requirement of the channel.

Solution

Given $B = 4000$ Hz

$\therefore$ Nyquist rate $= 2B = 8000$ Hz

$\therefore$ Sampling rate r_s = (2.5) (Nyquist rate)

= (2.5) (8000)

= 20, 000 samples/sec

Since all the 256 quantization levels are equally likely, the maximum information content is

$$I = \log_2 q = \log_2 256 = 8 \text{ bits/sample}$$

(i) Information Rate $R_s = r_s I$

$$= (20{,}000 \text{ samples/sec}) (8 \text{ bits/sample})$$

$$\therefore \mathbf{R_s = 1{,}60{,}000 \text{ bits/sec}}$$

(ii) From Shannon-Hartley law, we have

$$C = B \log\left(1+\frac{S}{N}\right)$$

Given $10 \log_{10}\frac{S}{N} = 20 \text{ dB}$

$$\therefore \frac{S}{N} = 100 \text{ and } B = 50 \times 10^3 \text{ Hz}$$

$$\therefore C = 50 \times 10^3 \log(1 + 100)$$

$$\therefore \mathbf{C = 332.91 \times 10^3 \text{ bits/sec}}$$

Since $R_s < C$, according to Shannon's second theorem, it is possible to transmit over t given channel without errors.

(iii) Given $10 \log_{10}\frac{S}{N} = 10 \text{ dB} \quad \therefore \quad \frac{S}{N} = 10$

From Shannon-Hartley law,

$$C = B \log\left(1+\frac{S}{N}\right) = R_s$$

$$\therefore B = \frac{R_s}{\log\left(1+\frac{S}{N}\right)} = \frac{160\times10^3}{\log(1+10)}$$

$$\therefore \mathbf{B = 46.25 \text{ KHz}}$$