5

The Discrete Fourier Transform: Its Properties and Applications

Frequency analysis of discrete-time signals is usually and most conveniently performed on a digital signal processor, which may be a general-purpose digital computer or specially designed digital hardware. To perform frequency analysis on a discrete-time signal $\{x(n)\}$, we convert the time-domain sequence to an equivalent frequency-domain representation. We know that such a representation is given by the Fourier transform $X(\omega)$ of the sequence $\{x(n)\}$. However, $X(\omega)$ is a continuous function of frequency and therefore, it is not a computationally convenient representation of the sequence $\{x(n)\}$.

In this section we consider the representation of a sequence $\{x(n)\}$ by samples of its spectrum $X(\omega)$. Such a frequency-domain representation leads to the discrete Fourier transform (DFT), which is a powerful computational tool for performing frequency analysis of discrete-time signals.

5.1 FREQUENCY DOMAIN SAMPLING: THE DISCRETE FOURIER TRANSFORM

Before we introduce the DFT, we consider the sampling of the Fourier transform of an aperiodic discrete-time sequence. Thus, we establish the relationship between the sampled Fourier transform and the DFT.

5.1.1 Frequency-Domain Sampling and Reconstruction of Discrete-Time Signals

We recall that aperiodic finite-energy signals have continuous spectra. Let us consider such an aperiodic discrete-time signal $x(n)$ with Fourier transform

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \tag{5.1.1}$$

Suppose that we sample $X(\omega)$ periodically in frequency at a spacing of $\delta\omega$ radians between successive samples. Since $X(\omega)$ is periodic with period 2π, only samples in the fundamental frequency range are necessary. For convenience, we take N equidistant samples in the interval $0 \le \omega < 2\pi$ with spacing $\delta\omega = 2\pi/N$, as shown in Fig. 5.1. First, we consider the selection of N, the number of samples in the frequency domain.

If we evaluate (5.1.1) at $\omega = 2\pi k/N$, we obtain

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi kn/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.2}$$

The summation in (5.1.2) can be subdivided into an infinite number of summations, where each sum contains N terms. Thus

$$\begin{aligned} X\left(\frac{2\pi}{N}k\right) &= \cdots + \sum_{n=-N}^{-1} x(n)e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \\ &\quad + \sum_{n=N}^{2N-1} x(n)e^{-j2\pi kn/N} + \cdots \\ &= \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)e^{-j2\pi kn/N} \end{aligned}$$

If we change the index in the inner summation from n to $n - lN$ and interchange the order of the summation, we obtain the result

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{\infty} x(n-lN)\right] e^{-j2\pi kn/N} \tag{5.1.3}$$

for $k = 0, 1, 2, \ldots, N-1$.

The signal

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) \tag{5.1.4}$$

obtained by the periodic repetition of $x(n)$ every N samples, is clearly periodic with fundamental period N. Consequently, it can be expanded in a Fourier

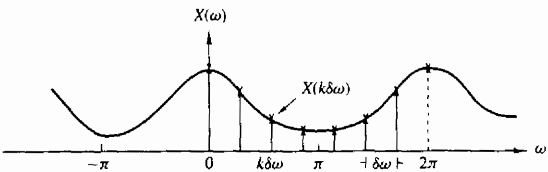


Figure 5.1 Frequency-domain sampling of the Fourier transform.

series as

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi kn/N} \qquad n = 0, 1, \ldots, N-1 \tag{5.1.5}$$

with Fourier coefficients

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) e^{-j2\pi kn/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.6}$$

Upon comparing (5.1.3) with (5.1.6), we conclude that

$$c_k = \frac{1}{N} X\left(\frac{2\pi}{N}k\right) \qquad k = 0, 1, \ldots, N-1 \tag{5.1.7}$$

Therefore,

$$x_p(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N} \qquad n = 0, 1, \ldots, N-1 \tag{5.1.8}$$

The relationship in (5.1.8) provides the reconstruction of the periodic signal $x_p(n)$ from the samples of the spectrum $X(\omega)$. However, it does not imply that we can recover $X(\omega)$ or $x(n)$ from the samples. To accomplish this, we need to consider the relationship between $x_p(n)$ and $x(n)$.

Since $x_p(n)$ is the periodic extension of $x(n)$ as given by (5.1.4), it is clear that $x(n)$ can be recovered from $x_p(n)$ if there is no aliasing in the time domain, that is, if $x(n)$ is time-limited to less than the period N of $x_p(n)$. This situation is illustrated in Fig. 5.2, where without loss of generality, we consider a finite-duration

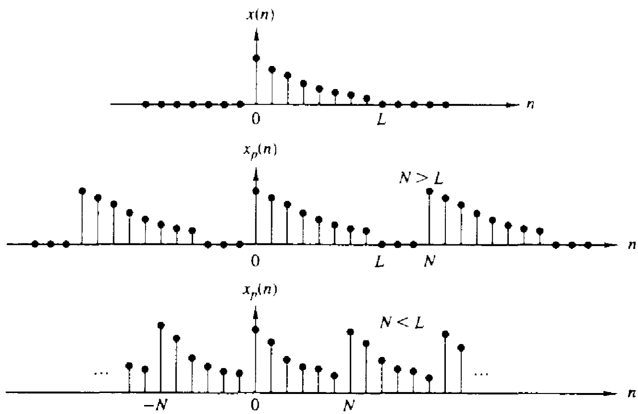


Figure 5.2 Aperiodic sequence $x(n)$ of length L and its periodic extension for $N \geq L$ (no aliasing) and $N < L$ (aliasing).

sequence $x(n)$, which is nonzero in the interval $0 \le n \le L-1$. We observe that when $N \ge L$,

$$x(n) = x_p(n) \qquad 0 \le n \le N-1$$

so that $x(n)$ can be recovered from $x_p(n)$ without ambiguity. On the other hand, if $N < L$, it is not possible to recover $x(n)$ from its periodic extension due to *time-domain aliasing*. Thus, we conclude that the spectrum of an aperiodic discrete-time signal with finite duration L, can be exactly recovered from its samples at frequencies $\omega_k = 2\pi k/N$, if $N \ge L$. The procedure is to compute $x_p(n)$, $n = 0, 1, \ldots, N-1$ from (5.1.8); then

$$x(n) = \begin{cases} x_p(n), & 0 \le n \le N-1 \\ 0, & \text{elsewhere} \end{cases} \tag{5.1.9}$$

and finally, $X(\omega)$ can be computed from (5.1.1).

As in the case of continuous-time signals, it is possible to express the spectrum $X(\omega)$ directly in terms of its samples $X(2\pi k/N)$, $k = 0, 1, \ldots, N-1$. To derive such an interpolation formula for $X(\omega)$, we assume that $N \ge L$ and begin with (5.1.8). Since $x(n) = x_p(n)$ for $0 \le n \le N-1$,

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N} \qquad 0 \le n \le N-1 \tag{5.1.10}$$

If we use (5.1.1) and substitute for $x(n)$, we obtain

$$\begin{aligned} X(\omega) &= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j2\pi kn/N}\right] e^{-j\omega n} \\ &= \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right)\left[\frac{1}{N}\sum_{n=0}^{N-1} e^{-j(\omega-2\pi k/N)n}\right] \end{aligned} \tag{5.1.11}$$

The inner summation term in the brackets of (5.1.11) represents the basic interpolation function shifted by $2\pi k/N$ in frequency. Indeed, if we define

$$\begin{aligned} P(\omega) &= \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\omega n} = \frac{1}{N}\frac{1-e^{-j\omega N}}{1-e^{-j\omega}} \\ &= \frac{\sin(\omega N/2)}{N\sin(\omega/2)} e^{-j\omega(N-1)/2} \end{aligned} \tag{5.1.12}$$

then (5.1.11) can be expressed as

$$X(\omega) = \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) P\left(\omega - \frac{2\pi}{N}k\right) \qquad N \ge L \tag{5.1.13}$$

The interpolation function $P(\omega)$ is not the familiar $(sin\theta)/\theta$ but instead, it is a periodic counterpart of it, and it is due to the periodic nature of $X(\omega)$. The phase shift in (5.1.12) reflects the fact that the signal $x(n)$ is a causal, finite-duration sequence of length N. The function $\sin(\omega N/2)/(N\sin(\omega/2))$ is plotted in Fig. 5.3 for $N = 5$. We observe that the function $P(\omega)$ has the property

$$P\left(\frac{2\pi}{N}k\right) = \begin{cases} 1, & k = 0 \\ 0, & k = 1, 2, \ldots, N-1 \end{cases} \tag{5.1.14}$$

Consequently, the interpolation formula in (5.1.13) gives exactly the sample values $X(2\pi k/N)$ for $\omega = 2\pi k/N$. At all other frequencies, the formula provides a properly weighted linear combination of the original spectral samples.

The following example illustrates the frequency-domain sampling of a discrete-time signal and the time-domain aliasing that results.

Example 5.1.1

Consider the signal

$$x(n) = a^n u(n) \qquad 0 < a < 1$$

The spectrum of this signal is sampled at frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$. Determine the reconstructed spectra for $a = 0.8$ when $N = 5$ and $N = 50$.

Solution The Fourier transform of the sequence $x(n)$ is

$$X(\omega) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} = \frac{1}{1 - ae^{-j\omega}}$$

Suppose that we sample $X(\omega)$ at N equidistant frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$. Thus we obtain the spectral samples

$$X(\omega_k) \equiv X\left(\frac{2\pi k}{N}\right) = \frac{1}{1 - ae^{-j2\pi k/N}} \qquad k = 0, 1, \ldots, N-1$$

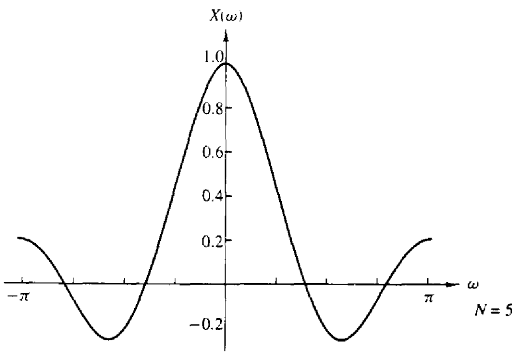


Figure 5.3 Plot of the function $[\sin(\omega N/2)]/[N\sin(\omega/2)]$.

The periodic sequence $x_p(n)$, corresponding to the frequency samples $X(2\pi k/N)$, $k = 0, 1, \ldots, N-1$, can be obtained from either (5.1.4) or (5.1.8). Hence

$$\begin{aligned} x_p(n) &= \sum_{l=-\infty}^{\infty} x(n-lN) = \sum_{l=-\infty}^{0} a^{n-lN} \\ &= a^n \sum_{l=0}^{\infty} a^{lN} = \frac{a^n}{1-a^N} \qquad 0 \le n \le N-1 \end{aligned}$$

where the factor $1/(1-a^N)$ represents the effect of aliasing. Since $0 < a < 1$, the aliasing error tends toward zero as $N \to \infty$.

For $a = 0.8$, the sequence $x(n)$ and its spectrum $X(\omega)$ are shown in Fig. 5.4a and b, respectively. The aliased sequences $x_p(n)$ for $N = 5$ and $N = 50$ and the corresponding spectral samples are shown in Fig. 5.4c and d, respectively. We note that the aliasing effects are negligible for $N = 50$.

If we define the aliased finite-duration sequence $x(n)$ as

$$\hat{x}(n) = \begin{cases} x_p(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

then its Fourier transform is

$$\begin{aligned} \hat{X}(\omega) &= \sum_{n=0}^{N-1} \hat{x}(n) e^{-j\omega n} = \sum_{n=0}^{N-1} x_p(n) e^{-j\omega N} \\ &= \frac{1}{1-a^N} \cdot \frac{1-a^N e^{-j\omega n}}{1-ae^{-j\omega}} \end{aligned}$$

Note that although $\hat{X}(\omega) \neq X(\omega)$, the sample values at $\omega_k = 2\pi k/N$ are identical. That is,

$$\hat{X}\left(\frac{2\pi}{N}k\right) = \frac{1}{1-a^N} \cdot \frac{1-a^N}{1-ae^{-j2\pi kN}} = X\left(\frac{2\pi}{N}k\right)$$

5.1.2 The Discrete Fourier Transform (DFT)

The development in the preceding section is concerned with the frequency-domain sampling of an aperiodic finite-energy sequence $x(n)$. In general, the equally spaced frequency samples $X(2\pi k/N)$, $k = 0, 1, \ldots, N-1$, do not uniquely represent the original sequence $x(n)$ when $x(n)$ has infinite duration. Instead, the frequency samples $X(2\pi k/N)$, $k = 0, 1, \ldots, N-1$, correspond to a periodic sequence $x_p(n)$ of period N, where $x_p(n)$ is an aliased version of $x(n)$, as indicated by the relation in (5.1.4), that is,

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN) \tag{5.1.15}$$

When the sequence $x(n)$ has a finite duration of length $L \le N$, then $x_p(n)$ is simply a periodic repetition of $x(n)$, where $x_p(n)$ over a single period is

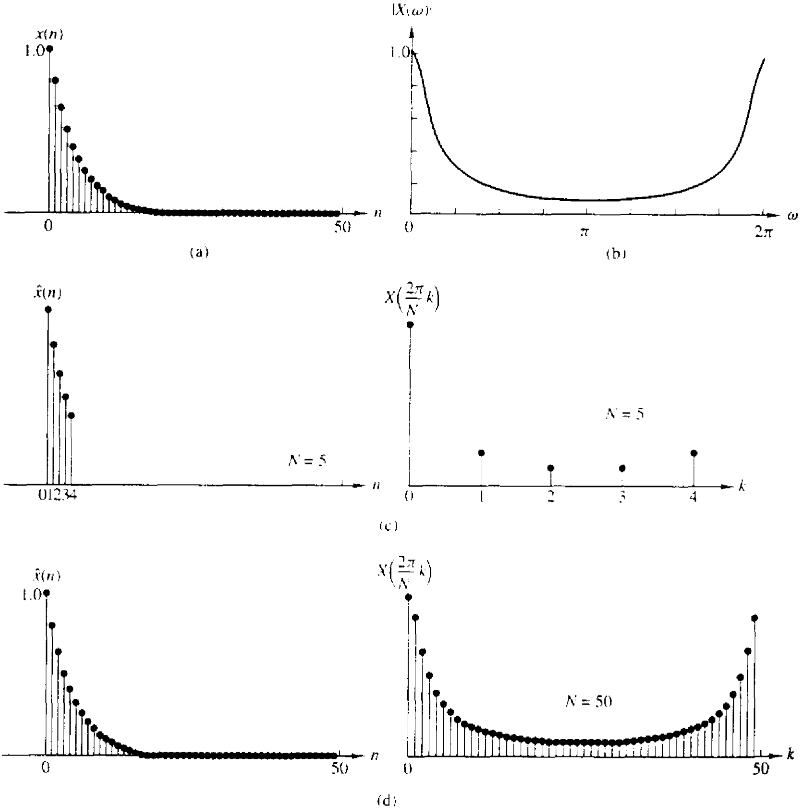


Figure 5.4 (a) Plot of sequence $x(n) = (0.8)^n u(n)$; (b) its Fourier transform (magnitude only); (c) effect of aliasing with $N = 5$; (d) reduced effect of aliasing with $N = 50$.

given as

$$x_p(n) = \begin{cases} x(n), & 0 \le n \le L-1 \\ 0, & L \le n \le N-1 \end{cases} \tag{5.1.16}$$

Consequently, the frequency samples $X(2\pi k/N)$, $k = 0, 1, \ldots, N-1$, uniquely represent the finite-duration sequence $x(n)$. Since $x(n) \equiv x_p(n)$ over a single period (padded by $N - L$ zeros), the original finite-duration sequence $x(n)$ can be obtained from the frequency samples $\{X(2\pi k/N\}$ by means of the formula (5.1.8).

It is important to note that *zero padding* does not provide any additional information about the spectrum $X(\omega)$ of the sequence $\{x(n)\}$. The L equidistant samples of $X(\omega)$ are sufficient to reconstruct $X(\omega)$ using the reconstruction formula (5.1.13). However, padding the sequence $\{x(n)\}$ with $N - L$ zeros and computing an N-point DFT results in a "better display" of the Fourier transform $X(\omega)$.

In summary, a finite-duration sequence $x(n)$ of length L [i.e., $x(n) = 0$ for $n < 0$ and $n \geq L$] has a Fourier transform

$$X(\omega) = \sum_{n=0}^{L-1} x(n)e^{-j\omega n} \qquad 0 \leq \omega \leq 2\pi \tag{5.1.17}$$

where the upper and lower indices in the summation reflect the fact that $x(n) = 0$ outside the range $0 \leq n \leq L - 1$. When we sample $X(\omega)$ at equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, 2, \ldots, N - 1$, where $N \geq L$, the resultant samples are

$$\begin{aligned} X(k) &\equiv X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{L-1} x(n)e^{-j2\pi kn/N} \\ X(k) &= \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \qquad k = 0, 1, 2, \ldots, N - 1 \end{aligned} \tag{5.1.18}$$

where for convenience, the upper index in the sum has been increased from $L - 1$ to $N - 1$ since $x(n) = 0$ for $n \geq L$.

The relation in (5.1.18) is a formula for transforming a sequence $\{x(n)\}$ of length $L \leq N$ into a sequence of frequency samples $\{X(k)\}$ of length N. Since the frequency samples are obtained by evaluating the Fourier transform $X(\omega)$ at a set of N (equally spaced) discrete frequencies, the relation in (5.1.18) is called the *discrete Fourier transform* (DFT) of $x(n)$. In turn, the relation given by (5.1.10), which allows us to recover the sequence $x(n)$ from the frequency samples

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \qquad n = 0, 1, \ldots, N - 1 \tag{5.1.19}$$

is called the *inverse DFT* (IDFT). Clearly, when $x(n)$ has length $L < N$, the N-point IDFT yields $x(n) = 0$ for $L \leq n \leq N - 1$. To summarize, the formulas for the DFT and IDFT are

DFT

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N} \qquad k = 0, 1, 2, \ldots, N - 1 \tag{5.1.18}$$

IDFT

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N} \qquad n = 0, 1, 2, \ldots, N - 1 \tag{5.1.19}$$

Example 5.1.2

A finite-duration sequence of length L is given as

$$x(n) = \begin{cases} 1, & 0 \le n \le L-1 \\ 0, & \text{otherwise} \end{cases}$$

Determine the N-point DFT of this sequence for $N \ge L$.

Solution The Fourier transform of this sequence is

$$\begin{aligned} X(\omega) &= \sum_{n=0}^{L-1} x(n)e^{-j\omega n} \\ &= \sum_{n=0}^{L-1} e^{-j\omega n} = \frac{1-e^{-j\omega L}}{1-e^{-j\omega}} = \frac{\sin(\omega L/2)}{\sin(\omega/2)} e^{-j\omega(L-1)/2} \end{aligned}$$

The magnitude and phase of $X(\omega)$ are illustrated in Fig. 5.5 for $L = 10$. The N-point DFT of $x(n)$ is simply $X(\omega)$ evaluated at the set of N equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$. Hence

$$\begin{aligned} X(k) &= \frac{1-e^{-j2\pi kL/N}}{1-e^{-j2\pi k/N}} \qquad k = 0, 1, \ldots, N-1 \\ &= \frac{\sin(\pi kL/N)}{\sin(\pi k/N)} e^{-j\pi k(L-1)/N} \end{aligned}$$

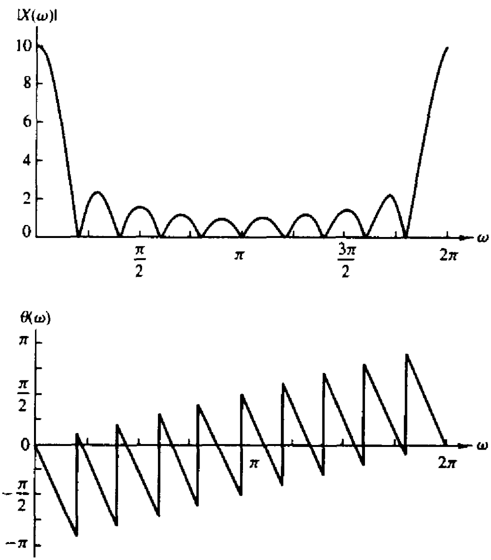


Figure 5.5 Magnitude and phase characteristics of the Fourier transform for signal in Example 5.1.2.

If N is selected such that $N = L$, then the DFT becomes

$$X(k) = \begin{cases} L, & k = 0 \\ 0, & k = 1, 2, \ldots, L-1 \end{cases}$$

Thus there is only one nonzero value in the DFT. This is apparent from observation of $X(\omega)$, since $X(\omega) = 0$ at the frequencies $\omega_k = 2\pi k/L$, $k \neq 0$. The reader should verify that $x(n)$ can be recovered from $X(k)$ by performing an L-point IDFT.

Although the L-point DFT is sufficient to uniquely represent the sequence $x(n)$ in the frequency domain, it is apparent that it does not provide sufficient detail to yield a good picture of the spectral characteristics of $x(n)$. If we wish to have better picture, we must evaluate (interpolate) $X(\omega)$ at more closely spaced frequencies, say $\omega_k = 2\pi k/N$, where $N > L$. In effect, we can view this computation as expanding the size of the sequence from L points to N points by appending $N - L$ zeros to the sequence $x(n)$, that is, zero padding. Then the N-point DFT provides finer interpolation than the L-point DFT.

Figure 5.6 provides a plot of the N-point DFT, in magnitude and phase, for $L = 10$, $N = 50$, and $N = 100$. Now the spectral characteristics of the sequence are more clearly evident, as one will conclude by comparing these spectra with the continuous spectrum $X(\omega)$.

5.1.3 The DFT as a Linear Transformation

The formulas for the DFT and IDFT given by (5.1.18) and (5.1.19) may be expressed as

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.20}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad n = 0, 1, \ldots, N-1 \tag{5.1.21}$$

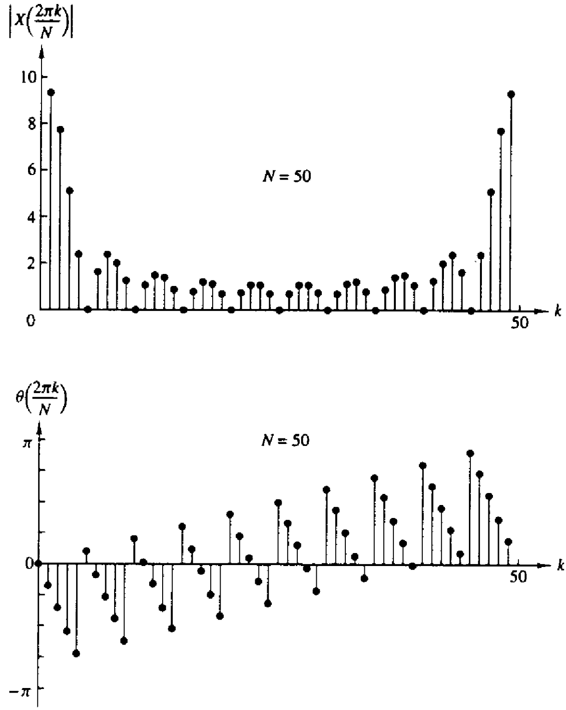
where, by definition,

$$W_N = e^{-j2\pi/N} \tag{5.1.22}$$

which is an Nth root of unity.

We note that the computation of each point of the DFT can be accomplished by N complex multiplications and $(N-1)$ complex additions. Hence the N-point DFT values can be computed in a total of N^2 complex multiplications and $N(N-1)$ complex additions.

It is instructive to view the DFT and IDFT as linear transformations on sequences $\{x(n)\}$ and $\{X(k)\}$, respectively. Let us define an N-point vector $\mathbf{x}_N$ of the signal sequence $x(n)$, $n = 0, 1, \ldots, N-1$, an N-point vector $\mathbf{X}_N$ of frequency



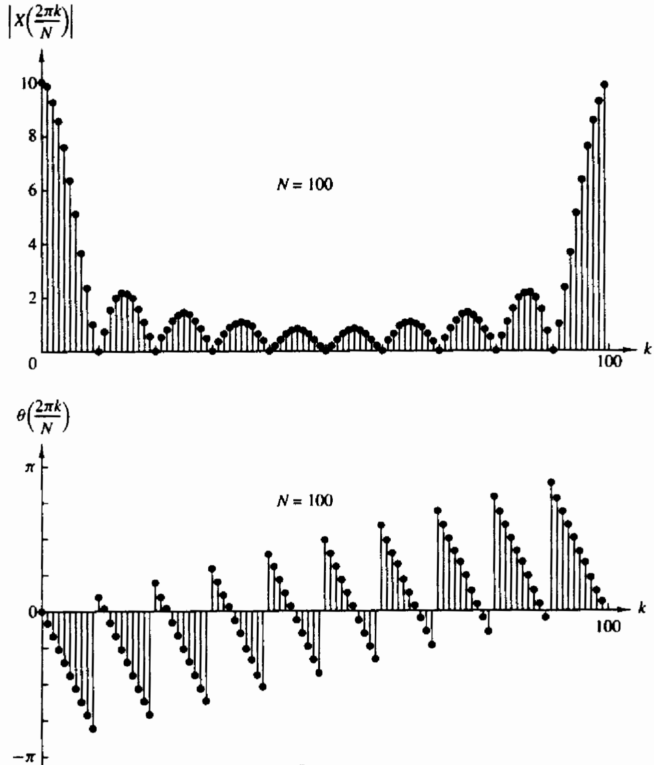
(a)

Figure 5.6 Magnitude and phase of an N-point DFT in Example 6.4.2; (a) $L = 10$, $N = 50$; (b) $L = 10$, $N = 100$.

samples, and an $N \times N$ matrix $\mathbf{W}_N$ as

$$\mathbf{x}_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}, \qquad \mathbf{X}_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$\mathbf{W}_N = \begin{bmatrix} 1 & 1 & 1 \cdots & 1 & \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix} \tag{5.1.23}$$



(b)

Figure 5.6 *continued*

With these definitions, the N-point DFT may be expressed in matrix form as

$$\mathbf{X}_N = \mathbf{W}_N \mathbf{x}_N \tag{5.1.24}$$

where $\mathbf{W}_N$ is the matrix of the linear transformation. We observe that $\mathbf{W}_N$ is a symmetric matrix. If we assume that the inverse of $\mathbf{W}_N$ exists, then (5.1.24) can be inverted by premultiplying both sides by $\mathbf{W}_N^{-1}$. Thus we obtain

$$\mathbf{x}_N = \mathbf{W}_N^{-1} \mathbf{X}_N \tag{5.1.25}$$

But this is just an expression for the IDFT.

In fact, the IDFT as given by (5.1.21), can be expressed in matrix form as

$$\mathbf{x}_N = \frac{1}{N} \mathbf{W}_N^* \mathbf{X}_N \tag{5.1.26}$$

where $\mathbf{W}_N^*$ denotes the complex conjugate of the matrix $\mathbf{W}_N$. Comparison of (5.1.26) with (5.1.25) leads us to conclude that

$$\mathbf{W}_N^{-1} = \frac{1}{N}\mathbf{W}_N^* \tag{5.1.27}$$

which, in turn, implies that

$$\mathbf{W}_N \mathbf{W}_N^* = N\mathbf{I}_N \tag{5.1.28}$$

where $\mathbf{I}_N$ is an $N \times N$ identity matrix. Therefore, the matrix $\mathbf{W}_N$ in the transformation is an orthogonal (unitary) matrix. Furthermore, its inverse exists and is given as $\mathbf{W}_N^*/N$. Of course, the existence of the inverse of $\mathbf{W}_N$ was established previously from our derivation of the IDFT.

Example 5.1.3

Compute the DFT of the four-point sequence

$$x(n) = (0 \quad 1 \quad 2 \quad 3)$$

Solution The first step is to determine the matrix $\mathbf{W}_4$. By exploiting the periodicity property of $\mathbf{W}_4$ and the symmetry property

$$W_N^{k+N/2} = -W_N^k$$

the matrix $\mathbf{W}_4$ may be expressed as

$$\mathbf{W}_4 = \begin{bmatrix} W_4^0 & W_4^0 & W_4^0 & W_4^0 \\ W_4^0 & W_4^1 & W_4^2 & W_4^3 \\ W_4^0 & W_4^2 & W_4^4 & W_4^6 \\ W_4^0 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^0 & W_4^2 \\ 1 & W_4^3 & W_4^2 & W_4^1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Then

$$\mathbf{X}_4 = \mathbf{W}_4\mathbf{x}_4 = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

The IDFT of $\mathbf{X}_4$ may be determined by conjugating the elements in $\mathbf{W}_4$ to obtain $\mathbf{W}_4^*$ and then applying the formula (5.1.26).

The DFT and IDFT are computational tools that play a very important role in many digital signal processing applications, such as frequency analysis (spectrum analysis) of signals, power spectrum estimation, and linear filtering. The importance of the DFT and IDFT in such practical applications is due to a large extent on the existence of computationally efficient algorithms, known collectively as fast

Fourier transform (FFT) algorithms, for computing the DFT and IDFT. This class of algorithms is described in Chapter 6.

5.1.4 Relationship of the DFT to Other Transforms

In this discussion we have indicated that the DFT is an important computational tool for performing frequency analysis of signals on digital signal processors. In view of the other frequency analysis tools and transforms that we have developed, it is important to establish the relationships between the DFT to these other transforms.

Relationship to the Fourier series coefficients of a periodic sequence. A periodic sequence $\{x_p(n)\}$ with fundamental period N can be represented in a Fourier series of the form

$$x_p(n) = \sum_{k=0}^{N-1} c_k e^{j2\pi nk/N} \qquad -\infty < n < \infty \tag{5.1.29}$$

where the Fourier series coefficients are given by the expression

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j2\pi nk/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.30}$$

If we compare (5.1.29) and (5.1.30) with (5.1.18) and (5.1.19), we observe that the formula for the Fourier series coefficients has the form of a DFT. In fact, if we define a sequence $x(n) = x_p(n)$, $0 \le n \le N-1$, the DFT of this sequence is simply

$$X(k) = Nc_k \tag{5.1.31}$$

Furthermore, (5.1.29) has the form of an IDFT. Thus the N-point DFT provides the exact line spectrum of a periodic sequence with fundamental period N.

Relationship to the Fourier transform of an aperiodic sequence. We have already shown that if $x(n)$ is an aperiodic finite energy sequence with Fourier transform $X(\omega)$, which is sampled at N equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$, the spectral components

$$X(k) = X(\omega)|_{\omega=2\pi k/N} = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi nk/N} \qquad k = 0, 1, \ldots, N-1 \tag{5.1.32}$$

are the DFT coefficients of the periodic sequence of period N, given by

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \tag{5.1.33}$$

Thus $x_p(n)$ is determined by aliasing $\{x(n)\}$ over the interval $0 \le n \le N-1$. The finite-duration sequence

$$\hat{x}(n) = \begin{cases} x_p(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{5.1.34}$$

bears no resemblance to the original sequence $\{x(n)\}$, unless $x(n)$ is of finite duration and length $L \leq N$, in which case

$$x(n) = \hat{x}(n) \qquad 0 \leq n \leq N-1 \tag{5.1.35}$$

Only in this case will the IDFT of $\{X(k)\}$ yield the original sequence $\{x(n)\}$.

Relationship to the z-transform. Let us consider a sequence $x(n)$ having the z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{5.1.36}$$

with a ROC that includes the unit circle. If $X(z)$ is sampled at the N equally spaced points on the unit circle $z_k = e^{j2\pi k/N}$, 0, 1, 2, ..., $N-1$, we obtain

$$\begin{aligned} X(k) &\equiv X(z)|_{z=e^{j2\pi nk/N}} \qquad k = 0, 1, \ldots, N-1 \\ &= \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi nk/N} \end{aligned} \tag{5.1.37}$$

The expression in (5.1.37) is identical to the Fourier transform $X(\omega)$ evaluated at the N equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$, which is the topic treated in Section 5.1.1.

If the sequence $x(n)$ has a finite duration of length N or less, the sequence can be recovered from its N-point DFT. Hence its z-transform is uniquely determined by its N-point DFT. Consequently, $X(z)$ can be expressed as a function of the DFT {X(k)} as follows

$$\begin{aligned} X(z) &= \sum_{n=0}^{N-1} x(n)z^{-n} \\ X(z) &= \sum_{n=0}^{N-1}\left[\frac{1}{N}\sum_{k=0}^{N-1} X(k)e^{j2\pi kn/N}\right]z^{-n} \\ X(z) &= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\sum_{n=0}^{N-1}\left(e^{j2\pi k/N}z^{-1}\right)^n \\ X(z) &= \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1}\frac{X(k)}{1-e^{j2\pi k/N}z^{-1}} \end{aligned} \tag{5.1.38}$$

When evaluated on the unit circle, (5.1.38) yields the Fourier transform of the finite-duration sequence in terms of its DFT, in the form

$$X(\omega) = \frac{1-e^{-j\omega N}}{N}\sum_{k=0}^{N-1}\frac{X(k)}{1-e^{-j(\omega-2\pi k/N)}} \tag{5.1.39}$$

This expression for the Fourier transform is a polynomial (Lagrange) interpolation formula for $X(\omega)$ expressed in terms of the values $\{X(k)\}$ of the polynomial at a set of equally spaced discrete frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, \ldots, N-1$. With some algebraic manipulations, it is possible to reduce (5.1.39) to the interpolation formula given previously in (5.1.13).

Relationship to the Fourier series coefficients of a continuous-time signal. Suppose that $x_a(t)$ is a continuous-time periodic signal with fundamental period $T_p = 1/F_0$. The signal can be expressed in a Fourier series

$$x_a(t) = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0} \tag{5.1.40}$$

where $\{c_k\}$ are the Fourier coefficients. If we sample $x_a(t)$ at a uniform rate $F_s = N/T_p = 1/T$, we obtain the discrete-time sequence

$$\begin{aligned} x(n) \equiv x_a(nT) &= \sum_{k=-\infty}^{\infty} c_k e^{j2\pi k F_0 n T} = \sum_{k=-\infty}^{\infty} c_k e^{j2\pi kn/N} \\ &= \sum_{k=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} c_{k-lN} \right] e^{j2\pi kn/N} \end{aligned} \tag{5.1.41}$$

It is clear that (5.1.41) is in the form of an IDFT formula, where

$$X(k) = N \sum_{l=-\infty}^{\infty} c_{k-lN} \equiv N\tilde{c}_k \tag{5.1.42}$$

and

$$\tilde{c}_k = \sum_{l=-\infty}^{\infty} c_{k-lN} \tag{5.1.43}$$

Thus the $\{\tilde{c}_k\}$ sequence is an aliased version of the sequence $\{c_k\}$.

5.2 PROPERTIES OF THE DFT

In Section 5.1.2 we introduced the DFT as a set of N samples $\{X(k)\}$ of the Fourier transform $X(\omega)$ for a finite-duration sequence $\{x(n)\}$ of length $L \leq N$. The sampling of $X(\omega)$ occurs at the N equally spaced frequencies $\omega_k = 2\pi k/N$, $k = 0, 1, 2, \ldots, N-1$. We demonstrated that the N samples $\{X(k)\}$ uniquely represent the sequence $\{x(n)\}$ in the frequency domain. Recall that the DFT and inverse DFT (IDFT) for an N-point sequence $\{x(n)\}$ are given as

$$\text{DFT: } X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \qquad k = 0, 1, \ldots, N-1 \tag{5.2.1}$$

$$\text{IDFT: } x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-kn} \qquad n = 0, 1, \ldots, N-1 \tag{5.2.2}$$

where W_N is defined as

$$W_N = e^{-j2\pi/N} \tag{5.2.3}$$

In this section we present the important properties of the DFT. In view of the relationships established in Section 5.1.4 between the DFT and Fourier series, and Fourier transforms and z-transforms of discrete-time signals, we expect the properties of the DFT to resemble the properties of these other transforms and series. However, some important differences exist, one of which is the circular convolution property derived in the following section. A good understanding of these properties is extremely helpful in the application of the DFT to practical problems.

The notation used below to denote the N-point DFT pair $x(n)$ and $X(k)$ is

$$x(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X(k)$$

5.2.1 Periodicity, Linearity, and Symmetry Properties

Periodicity. If $x(n)$ and X(k) are an N-point DFT pair, then

$$x(n+N) = x(n) \quad \text{for all } n \tag{5.2.4}$$

$$X(k+N) = X(k) \quad \text{for all } k \tag{5.2.5}$$

These periodicities in $x(n)$ and $X(k)$ follow immediately from formulas (5.2.1) and (5.2.2) for the DFT and IDFT, respectively.

We previously illustrated the periodicity property in the sequence $x(n)$ for a given DFT. However, we had not previously viewed the DFT $X(k)$ as a periodic sequence. In some applications it is advantageous to do this.

Linearity. If

$$x_1(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X_1(k)$$

and

$$x_2(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} X_2(k)$$

then for any real-valued or complex-valued constants a_1 and a_2,

$$a_1x_1(n) + a_2x_2(n) \underset{N}{\overset{\text{DFT}}{\longleftrightarrow}} a_1X_1(k) + a_2X_2(k) \tag{5.2.6}$$

This property follows immediately from the definition of the DFT given by (5.2.1).

Circular Symmetries of a Sequence. As we have seen, the N-point DFT of a finite duration sequence, $x(n)$ of length $L \leq N$ is equivalent to the N-point DFT of a periodic sequence $x_p(n)$, of period N, which is obtained by periodically extending $x(n)$, that is,

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN) \tag{5.2.7}$$

Now suppose that we shift the periodic sequence $x_p(n)$ by k units to the right. Thus we obtain another periodic sequence

$$x'_p(n) = x_p(n-k) = \sum_{l=-\infty}^{\infty} x(n-k-lN) \tag{5.2.8}$$

The finite-duration sequence

$$x'(n) = \begin{cases} x'_p(n), & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases} \tag{5.2.9}$$

is related to the original sequence $x(n)$ by a circular shift. This relationship is illustrated in Fig. 5.7 for $N = 4$.

In general, the circular shift of the sequence can be represented as the index modulo N. Thus we can write

$$\begin{aligned} x'(n) &= x(n-k, \text{modulo } N) \\ &\equiv x((n-k))_N \end{aligned} \tag{5.2.10}$$

For example, if $k = 2$ and $N = 4$, we have

$$x'(n) = x((n-2))_4$$

which implies that

$$\begin{aligned} x'(0) &= x((-2))_4 = x(2) \\ x'(1) &= x((-1))_4 = x(3) \\ x'(2) &= x((0))_4 = x(0) \\ x'(3) &= x((1))_4 = x(1) \end{aligned}$$

Hence $x'(n)$ is simply $x(n)$ shifted circularly by two units in time, where the counterclockwise direction has been arbitrarily selected as the positive direction. Thus we conclude that a circular shift of an N-point sequence is equivalent to a linear shift of its periodic extension, and vice versa.

The inherent periodicity resulting from the arrangement of the N-point sequence on the circumference of a circle dictates a different definition of even and odd symmetry, and time reversal of a sequence.

An N-point sequence is called circularly *even* if it is symmetric about the point zero on the circle. This implies that

$$x(N-n) = x(n) \qquad 1 \le n \le N-1 \tag{5.2.11}$$

An N-point sequence is called circularly *odd* if it is antisymmetric about the point zero on the circle. This implies that

$$x(N-n) = -x(n) \qquad 1 \le n \le N-1 \tag{5.2.12}$$

The time reversal of an N-point sequence is attained by reversing its samples about the point zero on the circle. Thus the sequence $x((-n))_N$ is simply given as

$$x((-n))_N = x(N-n) \qquad 0 \le n \le N-1 \tag{5.2.13}$$

This time reversal is equivalent to plotting $x(n)$ in a clockwise direction on a circle.

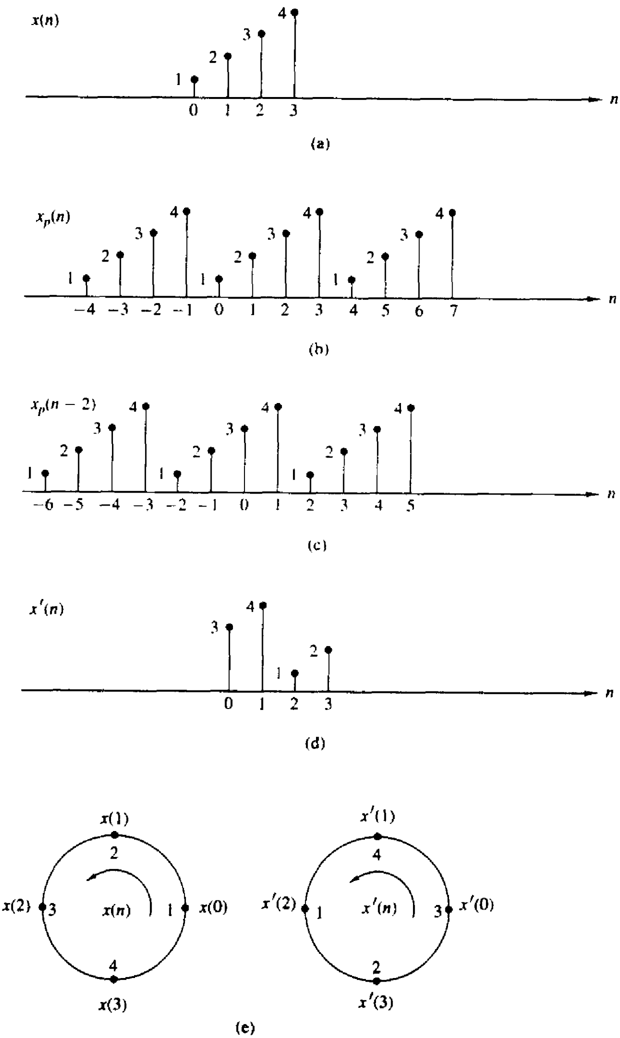


Figure 5.7 Circular shift of a sequence.

An equivalent definition of even and odd sequences for the associated periodic sequence $x_p(n)$ is given as follows

$$\begin{aligned} \text{even:}\quad x_p(n) &= x_p(-n) = x_p(N-n) \\ \text{odd:}\quad x_p(n) &= -x_p(-n) = -x_p(N-n) \end{aligned} \tag{5.2.14}$$

If the periodic sequence is complex-valued, we have

$$\begin{aligned} \text{conjugate even:}\quad x_p(n) &= x_p^*(N-n) \\ \text{conjugate odd:}\quad x_p(n) &= -x_p^*(N-n) \end{aligned} \tag{5.2.15}$$

These relationships suggest that we decompose the sequence $x_p(n)$ as

$$x_p(n) = x_{pe}(n) + x_{po}(n) \tag{5.2.16}$$

where

$$\begin{aligned} x_{pe}(n) &= \tfrac{1}{2}[x_p(n) + x_p^*(N-n)] \\ x_{po}(n) &= \tfrac{1}{2}[x_p(n) - x_p^*(N-n)] \end{aligned} \tag{5.2.17}$$

Symmetry properties of the DFT. The symmetry properties for the DFT can be obtained by applying the methodology previously used for the Fourier transform. Let us assume that the N-point sequence $x(n)$ and its DFT are both complex valued. Then the sequences can be expressed as

$$x(n) = x_R(n) + jx_I(n) \qquad 0 \le n \le N-1 \tag{5.2.18}$$

$$X(k) = X_R(k) + jX_I(k) \qquad 0 \le k \le N-1 \tag{5.2.19}$$

By substituting (5.2.18) into the expression for the DFT given by (5.2.1), we obtain

$$X_R(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\frac{2\pi kn}{N} + x_I(n)\sin\frac{2\pi kn}{N}\right] \tag{5.2.20}$$

$$X_I(k) = -\sum_{n=0}^{N-1}\left[x_R(n)\sin\frac{2\pi kn}{N} - x_I(n)\cos\frac{2\pi kn}{N}\right] \tag{5.2.21}$$

Similarly, by substituting (5.2.19) into the expression for the IDFT given by (5.2.2), we obtain

$$x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\cos\frac{2\pi kn}{N} - X_I(k)\sin\frac{2\pi kn}{N}\right] \tag{5.2.22}$$

$$x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\sin\frac{2\pi kn}{N} + X_I(k)\cos\frac{2\pi kn}{N}\right] \tag{5.2.23}$$

Real-valued sequences. If the sequence $x(n)$ is real, it follows directly from (5.2.1) that

$$X(N-k) = X^*(k) = X(-k) \tag{5.2.24}$$