Symmetric and Antisymmetric FIR filters

* If $h(n)$ represents the impulse response of a DT filter, a necessary and sufficient condition for linear phase is that $h(n)$ must have
  1. A finite duration N (0 to $N-1$) and
  2. Is either symmetric or Antisymmetric about its mid-point.

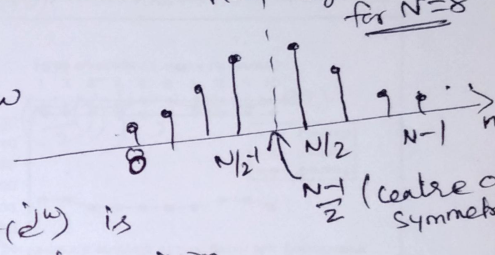
ie $h(n) = h(N-1-n)$; $n = 0, 1, \dots N-1$

(or)

$h(n) = -h(N-1-n)$; $n = 0, 1, \dots N-1$

(1) Freq. response of FIR filter when $h(n)$ is symmetric and N is even with centre of symmetry at $\frac{N-1}{2}$

Eg: for $N=8$



The centre of symmetry lies b/w $\frac{N}{2}-1$ and $N/2$

we know that the freq. resp. $H(e^{j\omega})$ is

$$H(e^{j\omega}) = \sum_{n=0}^{N-1} h(n).\, e^{-j\omega n} = \sum_{n=0}^{N/2-1} h(n)\, e^{-j\omega n} + \sum_{n=N/2}^{N-1} h(n)\, e^{-j\omega n}$$

Let $m = N-1-n$ subs in 2^{nd} Σ

$\Rightarrow n = N-1-m$

$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n).\, e^{-j\omega n} + \sum_{n=0}^{\frac{N}{2}-1} h(N-1-m).\, e^{-j\omega(N-1-m)}$$

For symmetric impulse response $h(n) = h(N-1-n)$

$$\Rightarrow H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} h(n)\, e^{-j\omega n} + \sum_{n=0}^{N/2-1} h(n).\, e^{-j\omega(N-1-m)}$$

$\boxed{m \text{ is a dummy variable } \therefore m = n}$

$$H(e^{j\omega}) = \sum_{n=0}^{N/2-1} h(n) \left[e^{-j\omega n} . e^{j\omega\left(\frac{N-1}{2}\right)} + e^{-j\omega(N-1-n)} . e^{j\omega\left(\frac{N-1}{2}\right)} \right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$\boxed{e^{j\omega\left(\frac{N-1}{2}\right)} . e^{-j\omega\left(\frac{N-1}{2}\right)} = 1}$

$$\sum_{n=0} \Big\} e^{\ldots(\frac{N-1}{2}-n)} + e^{-j\omega(\frac{N-1}{2}-n)}\Big\} e^{-j\omega(\frac{N-1}{2})}$$

$$\boxed{H(e^{j\omega}) = \sum_{n=0}^{N/2-1} 2h(n)\cos\left(\omega\left(\frac{N-1}{2}-n\right)\right) e^{-j\omega(\frac{N-1}{2})}}$$

$e^{j\theta} + e^{-j\theta} = 2\cos\theta$

$\cos(-\theta) = \cos\theta$

(or)

$$H(e^{j\omega}) = \left\{\sum_{n=0}^{\frac{N}{2}-1} 2h(n).\cos\left[\omega\left(n-\frac{N-1}{2}\right)\right]\right\}. e^{-j\omega(\frac{N-1}{2})}$$

$$\Rightarrow \boxed{H(e^{j\omega}) = \sum_{n=0}^{\frac{N}{2}-1} 2h(n)\cos\left[\omega\left(n-\left(\frac{N-1}{2}\right)\right)\right] e^{-j\omega(\frac{N-1}{2})}}$$

We know that $H(e^{j\omega})$ is a complex quantity & can be written as $H(e^{j\omega}) = |H(e^{j\omega})| e^{j\theta(\omega)}$

$$|H(e^{j\omega})| = \sum_{n=0}^{\frac{N}{2}-1} 2h(n).\cos\left[\omega\left(n-\left(\frac{N-1}{2}\right)\right)\right], \text{ N even}$$

&

$$\theta(\omega) = \begin{cases} -\omega\left(\frac{N-1}{2}\right); & \text{if } H_r(\omega) > 0 \\ -\omega\left(\frac{N-1}{2}\right) + \pi; & \text{if } H_r(\omega) < 0 \end{cases}$$

ie

$$\theta(\omega) = \begin{cases} -\omega\alpha; & H_r(\omega) > 0 \\ -\omega\alpha + \pi; & H_r(\omega) < 0 \end{cases}$$

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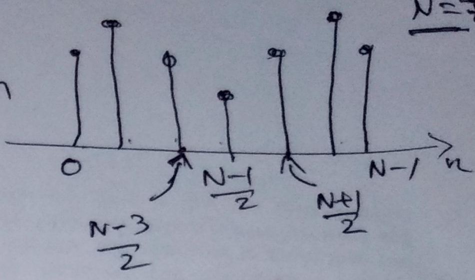
Frequency response of FIR filter when $h(n)$ is symmetric and N is odd with centre of symmetry at $\frac{N-1}{2}$

$N=7$



$$H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n).e^{-j\omega n} + h\left(\tfrac{N-1}{2}\right) e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=\frac{N+1}{2}}^{N-1} h(n)\, e^{-j\omega n}$$

Let $m = N-1-n$

$n = N-1-m$

$$\Rightarrow H(e^{j\omega}) = \sum_{n=0}^{\frac{N-3}{2}} h(n).e^{-j\omega n} + h\left(\tfrac{N-1}{2}\right).e^{-j\omega\left(\frac{N-1}{2}\right)} + \sum_{n=}^{\frac{N-3}{2}} h(N-1-m).e^{-j\omega(N-1-m)}$$

$h(n) = h(N-1-n)$ for symmetry condition

$$\Rightarrow H(e^{j\omega}) = \left[\cancel{H}h\left(\tfrac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h[n] \left\{ e^{-j\omega n}.e^{+j\omega\left(\frac{N-1}{2}\right)} + e^{-j\omega(N-1-n)}.e^{j\omega\left(\frac{N-1}{2}\right)} \right\}\right] e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$= \left\{ h\left(\tfrac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} h[n] \left[e^{j\omega\left(\frac{N-1}{2}-n\right)} + e^{-j\omega\left(\frac{N-1}{2}-n\right)} \right] \right\} e^{-j\omega\left(\frac{N-1}{2}\right)}\ e^{j\omega\left(\frac{N-1}{2}\right)}$$

$$= \left\{ h\left(\tfrac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} 2h(n) \cos\left[\omega\left(n-\tfrac{N-1}{2}\right)\right] \right\} e^{-j\omega\left(\frac{N-1}{2}\right)}$$

$$\boxed{H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ h\left(\tfrac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} 2h(n).\cos\left[\omega\left(n-\tfrac{N-1}{2}\right)\right] \right\}}$$

N odd

$$|H(e^{j\omega})| = h\left(\tfrac{N-1}{2}\right) + \sum_{n=0}^{\frac{N-3}{2}} 2h(n) \cos\left[\omega\left(n-\tfrac{N-1}{2}\right)\right]$$

&

$$\theta(\omega) = \begin{cases} -\omega\left(\frac{N-1}{2}\right), & \text{if } H_r(\omega) > 0 \\ -\omega\left(\frac{N-1}{2}\right) + \pi; & \text{if } H_r(\omega) < 0 \end{cases}$$

when $h(n) = -h(N-1-n)$, the impulse response is said to be asymmetric. (8)

For N odd, the mid point of the antisymmetric $h(n)$ is $n = \frac{N-1}{2}$ and consequently $h\left(\frac{N-1}{2}\right) = 0$

$$\therefore \boxed{H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ \sum_{n=0}^{\frac{N-3}{2}} 2h(n) \sin\omega\left(n - \left(\frac{N-1}{2}\right)\right) \right\}}$$

For N even, each term in $h(n)$ has a matching term of opposite sign

$$\boxed{H(e^{j\omega}) = e^{-j\omega\left(\frac{N-1}{2}\right)} \left\{ \sum_{n=0}^{\frac{N}{2}-1} 2h(n) \cdot \sin\left(\omega\left(n - \frac{N-1}{2}\right)\right) \right\}}$$