(1)

Digital Signal Processing

Module - I

Contents : DFT as a linear transformation, its relationship with other transforms, properties of DFT

Introduction : DFT :

we know that the frequency domain analysis can give better and insight analysis compared to time domain analysis. Digital signal processing is the area which is discussing / analysing about the manipulations performed on a digital signal. Digital signal processing can be performed well by a hardware called a digital signal processor, which will accept only signals in digital nature.

$x[n]$ aperiodic discrete

$\downarrow$ DTFT

$X(e^{j\omega})$ periodic continuous

$\downarrow$ Sampling

$X(K)$ - DFT

$\downarrow$

$\boxed{\text{DS Processor}}$

$\downarrow$ $y[n]$ manipulated output s/g

The mentioned flow explains the necessity of DFT. The discrete signal $x[n]$ which is aperiodic can be transformed in to an equivalent frequency domain using DTFT.

DTFT $X(e^{j\omega})$ is continuous and periodic, which can not be processed / accepted by the digital signal processor. Therefore, sampling is performed on $X(e^{j\omega})$ in such a way that 'N' samples are taken within one fundamental period 2π. The no. of samples 'N' should be sufficient to avoid aliasing of frequency spectrum

The samples are DTFT ie DFT are represented as a function of integer k, and so the DFT is a sequence consisting of N complex numbers represented as $X(k)$, $k=0,1,\ldots,N-1$

In general, the equally spaced frequency samples $X\left(\frac{2\pi}{N}k\right)$, $k=0,1,\ldots,N-1$ do not uniquely represent the original sequence $x(n)$, when $x(n)$ has infinite duration. Instead, the frequency samples $X\left(\frac{2\pi}{N}k\right)_{k=0\text{ to }N-1}$ correspond to a periodic sequence $x_p(n)$ of period N where $x_p(n)$ is an aliased version of $x(n)$

$$\text{ie } x_p(n) = \sum_{l=-\infty}^{+\infty} x(n-lN)$$

when the sequence $x(n)$ has a finite duration of length $L \leq N$, then $x_p(n)$ is simply a periodic repetition of $x(n)$, where $x_p(n)$ over a single period is

$$x_p(n) = \begin{cases} x(n) & ; 0 \leq n \leq L-1 \\ 0 & ; L \leq n \leq N-1 \end{cases}$$

Consequently, the frequency samples $X\left(\frac{2\pi}{N}k\right)$, $k=0,1,\ldots,N-1$ uniquely represent the finite duration sequence $x(n)$.

Since $X(k)$ is a sequence consisting of complex numbers, the magnitude and phase of each sample can be computed and listed as magnitude sequence and phase sequence respectively. The plot of $|X(k)|$ versus k is called magnitude spectrum and the plot of $\angle X(k)$ versus k is called phase spectrum.

(2)

From the knowledge of frequency domain sampling and reconstruction, it's clear that the periodic signal $x_p(n)$ from the samples of the spectrum $X(\omega)$. However it does not imply that we can recover $X(\omega)$ or $x(n)$ from the samples. To accomplish this, we need to consider the relationship between $x_p(n)$ and $x(n)$. Since $x_p(n)$ is the periodic extension of $x(n)$, it's clear that $x(n)$ can be recovered from $x_p(n)$, if there is no aliasing in the time domain, that is if $x(n)$ is time limited to less than the period N of $x_p(n)$.

The DFT sequence $X(k) = X(\omega)\Big|_{\omega=\frac{2\pi}{N}k}$ start at k=0

Corresponding to $\omega=0$, but does not include $k=N$ Corresponding to $\omega=2\pi$

In summary, a finite duration sequence $x(n)$ of length L ie $x(n)=0$; $L\le n\le 0$ has a fourier transform $X(\omega)=\sum_{n=0}^{L-1} x(n)e^{-j\omega n}$; $0\le\omega\le 2\pi$

If we sample $X(\omega)$ in such a way that N equidistant samples within 2π ie $X(k)=X(\omega)\Big|_{\omega=\frac{2\pi}{N}k}$, the resultant is called as DFT and denoted by $X(k)$ given by

$$\boxed{X(k)=\sum_{n=0}^{N-1} x(n).e^{-j\frac{2\pi}{N}k} \;;\; k=0,1,\cdots,N-1}$$

Similarly

$$\boxed{x[n]=\frac{1}{N}\sum_{k=0}^{N-1} X(k).e^{+j\frac{2\pi}{N}n}}$$

Example 1:

1. Find the 4 point DFT of signal $x[n] = \{1,1,1,1\}$

we know that DFT $X(k)$ is given by

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n).\, e^{-j\frac{2\pi}{N}kn} \;;\; k=0,1,\cdots,N-1}$$

Given $N=4$ sub $x(n)$ values for $n=0,1,2,3$

$$\Rightarrow \boxed{X(k) = x(0).\, e^{0} + x(1) e^{-j\frac{\pi}{2}k} + x(2).\, e^{-j\pi k} + x(3) e^{-j\frac{3\pi}{2}k} \;;\; k=0,1,2,3}$$

sub various values of k ranging from 0, 1, 2, 3

k=0

$X(0) = 1.1 + 1.1 + 1.1 + 1.1$ for $k=0$ all the $e^{-j\frac{2\pi}{N}k}$ values $=1$

$$\boxed{X(0) = 4}$$

k=1

$X(1) = 1.1 + 1.(-j) + 1.(-1) + 1.(+j)$

$X(1) = \cancel{1} - \cancel{j} - \cancel{1} + \cancel{j}$

$$\boxed{X(1) = 0}$$

k=2

$X(2) = 1.1 + 1.(-1) + 1.1 + 1.(-1)$

$$\boxed{X(2) = 0}$$

k=3

$X(3) = 1.1 + 1.(j) + 1(-1) + 1.(-j)$

$$\boxed{X(3) = 0}$$

$X(k) = \{4, 0, 0, 0\}$

since $X(k)$ is a complex number can be written as

$X(k) = |X(k)|.\, \angle X(k)$, where $|X(k)| = \sqrt{(\text{Real part})^2 + (\text{Imag. Part})^2}$

and $\angle X(k) = \tan^{-1}\left(\frac{\text{Img part}}{\text{Real part}}\right)$

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At this point, in the calculation of DFT a new factor called twiddle factor W_N is introduced, which is defined as $W_N = e^{-j\frac{2\pi}{N}}$ - an N^{th} root of unity.

Also from the previous example for the computation of DFT, we note that the computation of each point of the DFT involves N complex multiplications and (N-1) complex additions. Hence the N-point DFT values can be computed in a total of N^2 complex multiplications and N(N-1) complex additions.

Properties of Twiddle factor:

W_N^k has got periodicity and symmetry property which enhances the computation of DFT easy.

Periodicity property

$$W_N^{k+N} = W_N^k$$

proof:

WKT $W_N^{k+N} = e^{-j\frac{2\pi}{N}(k+N)} = e^{-j\frac{2\pi}{N}k} \cdot e^{-j\frac{2\pi}{N}\cdot N}$

$e^{-j2\pi} = 1$

$e^{-j\frac{2\pi}{N}k} = W_N^k$

$\Rightarrow \boxed{W_N^{k+N} = W_N^k}$ Thus proved.

Symmetry property:

$$W_N^{k+\frac{N}{2}} = -W_N^k$$

WKT $W_N^{k+N/2} = e^{-j\frac{2\pi}{N}(k+N/2)} = e^{-j\frac{2\pi}{N}k} \cdot e^{-j\frac{2\pi}{N}\frac{N}{2}}$

$e^{-j\pi} = -1$

$e^{-j\frac{2\pi}{N}k} = W_N^k$

$\Rightarrow \boxed{W_N^{k+\frac{N}{2}} = -W_N^k}$ Thus proved.

DFT as a linear transformation matrix

We know that

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn} \; ; \; k = 0, 1, \ldots, N-1$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot W_N^{-kn} \; ; \; n = 0, 1, \ldots, N-1$$

It is instructive to view the DFT and IDFT as linear transformations on sequences $\{x[n]\}$ and $\{X(k)\}$ respectively.

Let us define an N-point vector x_N of the signal sequence $x[n]$, where $n = 0, 1, \ldots N-1$, an N-point vector $X[k]$ of frequency samples and an $N \times N$ matrix W_N as

$$x[n] = \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}, \quad X[k] = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}, \quad W_N = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & W_N^1 & W_N^2 & \cdots & W_N^{N-1} \\ \cdot & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & & & & \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

An N pt DFT may be expressed in matrix form as

$$\boxed{X[k] = W_N \cdot x[n]}$$

where W_N - the matrix of linear transformation | Symmetric matrix | twiddle factor matrix

If inverse of W_N exists then IDFT

$$\boxed{\begin{gathered} x[n] = W_N^{-1} X[k] \\ \text{(or)} \\ x[n] = \frac{1}{N} W_N^{*} \cdot X[k] \end{gathered}}$$

$$\Rightarrow W_N^{-1} = \frac{1}{N} W_N^{*}$$

$W_N \cdot W_N^{*} = N \cdot I_N$ where I_N - Identity matrix

The above relation proves that W_N is an orthogonal (unitary) matrix.

Calculation of twiddle factor matrix:

	k=0	k=1	k=2	k=3
n=0	$W_N^{0,0}$	$W_N^{0,1}$	$W_N^{0,2}$	$W_N^{0,3}$
n=1	$W_N^{1,0}$	$W_N^{1,1}$	$W_N^{1,2}$	$W_N^{1,3}$
n=2	$W_N^{2,0}$	$W_N^{2,1}$	$W_N^{2,2}$	$W_N^{2,3}$
n=3	$W_N^{3,0}$	$W_N^{3,1}$	$W_N^{3,2}$	$W_N^{3,3}$

$$W_4^0 = e^{-j\frac{2\pi(0)}{4}} = 1$$
$$W_4^1 = e^{-j\frac{2\pi}{4}(1)} = -j$$
$$W_4^2 = e^{-j\frac{2\pi}{4}(2)} = -1$$
$$W_4^3 = e^{-j\frac{2\pi}{4}(3)} = +j$$

$N=4$
$k=0,1,2,3$
&
$n=0,1,2,3$

$$\Rightarrow \quad W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix}$$

* Twiddle factor matrix exhibits symmetry about the point $\frac{N}{2}$ ie for W_4 about either $k=2$ or $n=2$

Similarly W_8 matrix can be calculated as

$$W_8^0 = 1$$
$$W_8^1 = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$
$$W_8^2 = -j$$
$$W_8^3 = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$
$$W_8^4 = -1$$
$$W_8^5 = -\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$
$$W_8^6 = +j$$
$$W_8^7 = \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}$$

Example 1.

Find 4 point DFT of the sequence $x[n] = \cos\left(\frac{\pi n}{4}\right)$

~~For~~ ~~To solve the~~ To determine 4 point DFT, the very first step is to write $x(n)$ sequence by substituting values of n in $x(n)$ equation

$$\Rightarrow x(n) = \{\underset{\uparrow}{1}, \tfrac{1}{\sqrt{2}}, 0, -\tfrac{1}{\sqrt{2}}\}$$

We know that DFT $X(k)$

$$X(k) = W_N \cdot x[n]$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & +j \\ 1 & -1 & 1 & -1 \\ 1 & +j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$$

$$\begin{bmatrix} X[0] \\ X[1] \\ X[2] \\ X[3] \end{bmatrix} = \begin{bmatrix} 1 \\ 1-j1.4142 \\ 1 \\ 1+j1.4142 \end{bmatrix}$$

$$\boxed{X[k] = \{1, 1-j1.4142, 1, 1+j1.4142\}}$$

Example 2:

Determine the time domain sequence $x(n)$ from the $X(k)$ given. $X(k) = \{2, 0, 2, 0\}$ using matrix method

$$x[n] = \frac{1}{N}\left[W_N^* \cdot X[k]\right]$$

$$= \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$