7.3 Impulse Invariant Transformation

The objective of ***impulse invariant transformation*** is to develop an IIR filter transfer function whose impulse response is the sampled version of the impulse response of the analog filter. The main idea behind this technique is to preserve the frequency response characteristics of the analog filter. It can be stated that the frequency response of digital filter will be identical with the frequency response of the corresponding analog filter if the sampling time period T is selected sufficiently small (or the sampling frequency should be high) to minimize (or avoid completely) the effects of aliasing.

Let, $h(t)$ = Impulse response of analog filter

The Laplace transform of the analog impulse response h(t) gives the transfer function of analog filter.

$\therefore$ Transfer function of analog filter, $H(s) = \mathcal{L}\{h(t)\}$.

When H(s) has N number of distinct poles, it can be expressed as shown in equation (7.1) by partial fraction expansion.

$$H(s) = \sum_{i=1}^{N} \frac{A_i}{s + p_i} = \frac{A_1}{s + p_1} + \frac{A_2}{s + p_2} + \ldots\ldots + \frac{A_N}{s + p_N} \qquad \ldots\ldots(7.1)$$

On taking inverse Laplace transform of equation (7.1) we get,

$$\boxed{\mathcal{L}\left\{e^{-at}\,u(t)\right\} = \frac{1}{s+a}}$$

$$h(t) = \sum_{i=1}^{N} A_i\, e^{-p_i t}\, u(t) = A_1\, e^{-p_1 t}\, u(t) + A_2\, e^{-p_2 t}\, u(t) + \ldots\ldots + A_N\, e^{-p_N t}\, u(t) \qquad \ldots(7.2)$$

where, $u(t)$ = Continuous time unit step function.

Let, T = Sampling period.

$h(n)$ = Impulse response of digital filter.

The impulse response of the digital filter is obtained by uniformly sampling the impulse response of the analog filter.

$$\therefore h(n) = h(t)\big|_{t=nT} = h(nT)$$

Therefore the impulse response h(n) can be obtained from equation (7.2) by replacing t by nT.

$$\therefore h(n) = h(t)\big|_{t=nT} = h(nT) = \sum_{i=1}^{N} A_i\, e^{-p_i nT}\, u(nT)$$

$$= A_1\, e^{-p_1 nT}\, u(nT) + A_2\, e^{-p_2 nT}\, u(nT) + \ldots\ldots + A_N\, e^{-p_N nT}\, u(nT) \qquad \ldots\ldots(7.3)$$

On taking $\mathcal{Z}$-transform of equation (7.3) we get,

$$\boxed{\mathcal{Z}\left\{e^{-anT}\,u(nT)\right\} = \frac{1}{1-e^{-aT}z^{-1}}}$$

$$H(z) = \mathcal{Z}\{h(n)\} = A_1 \frac{1}{1 - e^{-p_1 T}\, z^{-1}} + A_2 \frac{1}{1 - e^{-p_2 T}\, z^{-1}} + \ldots\ldots$$

$$+ A_N \frac{1}{1 - e^{-p_N T}\, z^{-1}} = \sum_{i=1}^{N} A_i \frac{1}{1 - e^{-p_i T}\, z^{-1}} \qquad \ldots\ldots(7.4)$$

Comparing the expression of H(s) and H(z) [i.e., equations (7.1) and (7.4)] we can say that,

$$\boxed{\frac{1}{s+p_i} \xrightarrow{\text{(is transformed to)}} \frac{1}{1-e^{-p_iT}z^{-1}}} \quad .....(7.5)$$

by impulse invariant transformation, where T is the sampling time period.

When a discrete time signal is obtained by sampling analog signal, the frequency spectrum of discrete signal will be scaled by a factor 1/T (Refer section 4.7 of Chapter 4). Due to this fact, the transfer function obtained by impulse invariant method is amplified by the factor 1/T for small values of T. If this amplification is undesirable then the transfer function obtained by impulse invariant tranformation can be multiplied by T to obtain magnitude normalized transfer function $H_N(z)$.

$$\therefore\ H_N(z) = T \times H(z) \quad .....(7.6)$$

7.3.1 Relation Between Analog and Digital Filter Poles in Impulse Invariant Transformation

The analog poles are given by the roots of the term $(s + p_i)$, for i = 1, 2, 3, ..., N. The digital poles are given by the roots of the term $(1-e^{-p_iT}z^{-1})$, for i = 1, 2, 3, ..., N. From equation (7.5) we can say that the analog pole at $s = -p_i$ is transformed into a digital pole at $z = e^{-p_iT}$

Consider the digital pole, $z_i = e^{-p_iT}$ ..... (7.7)

Put, $-p_i = s_i$ in equation (7.7).

$$\therefore\ z_i = e^{-p_iT} = e^{s_iT} \quad ..... (7.8)$$

We know that, "s_i" is a point on s-plane. Let the coordinates of s_i be σ_i and $j\Omega_i$ as shown in fig 7.3.

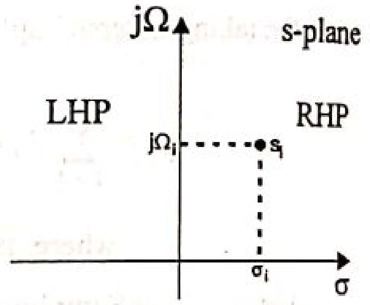

Fig 7.3 : s-plane.

$$\therefore\ s_i = \sigma_i + j\Omega_i \quad ..... (7.9)$$

Using equation (7.9), the equation (7.8) can be written as,

$$z_i = e^{(\sigma_i + j\Omega_i)T} = e^{\sigma_iT}\, e^{j\Omega_iT}$$

We know that "z_i" is a complex number. Hence "z_i" can be expressed in polar coordinates as, $z_i = |z_i| \angle z_i$.

$$\therefore\ |z_i| \angle z_i = e^{\sigma_iT}\, e^{j\Omega_iT} \quad ..... (7.10)$$

On separating the magnitude and phase of equation (7.10) we get,

$$|z_i| = e^{\sigma_iT} \text{ and } \angle z_i = \Omega_iT \quad ..... (7.11)$$

From equation (7.11) the following observations can be made.

1. If $\sigma_i < 0$ (i.e., σ_i is negative), then the analog pole "s_i" lie on Left Half (LHP) of s-plane. In this case, $|z_i| < 1$, hence the corresponding digital pole "z_i" will lie inside the unit circle in z-plane.
2. If $\sigma_i = 0$ (i.e., real part is zero), then the analog pole "s_i" lie on imaginary axis of s-plane. In this case, $|z_i| = 1$, hence the corresponding digital pole "z_i" will lie on the unit circle in z-plane.
3. If $\sigma_i > 0$ (i.e., σ_i is positive), then the analog pole "s_i" lie on the Right Half (RHP) of s-plane. In this case $|z_i| > 1$, hence the corresponding digital pole will lie outside the unit circle in z-plane.

The above discussions are applicable for mapping any point on s-plane to z-plane. In general the impulse invariant transformation maps all points in the s-plane given by,

$$s_i = \sigma_i + j\Omega_i + j\frac{2\pi k}{T}, \quad \text{for } k = 0, \pm 1, \pm 2 \ldots\ldots \qquad \ldots\ldots(7.12)$$

into a single point in the z-plane as

$$z_i = e^{\left(\sigma_i + j\Omega_i + \frac{j2\pi k}{T}\right)T} = e^{\sigma_i T}\, e^{j\Omega_i T}\, e^{j2\pi k} = e^{\sigma_i T}\, e^{j\Omega_i T} \qquad \ldots\ldots(7.13)$$

For integer k, $e^{j2\pi k} = 1$

From equations (7.12) and (7.13) we can say that the strip of width $2\pi/T$ in the s-plane for values of s in the range $-\pi/T \le \Omega \le +\pi/T$ is mapped into the entire z-plane. Similarly the strip of width $2\pi/T$ in the s-plane for values of s in the range $\pi/T \le \Omega \le 3\pi/T$ is also mapped into the entire z-plane. Likewise the strip of width $2\pi/T$ in the s-plane for values of s in the range $-3\pi/T \le \Omega \le -\pi/T$ is also mapped into the entire z-plane.

In general any strip of width $2\pi/T$ in the s-plane for values of s in the range, $(2k-1)\pi/T \le \Omega \le (2k+1)\pi/T$ (where k is an integer), is mapped into the entire z-plane. The left half portion of each strip in s-plane maps into the interior of the unit circle in z-plane, right half portion of each strip in s-plane maps into the exterior of the unit circle in z-plane and the imaginary axis of each strip in s-plane maps into the unit circle in z-plane as shown in fig 7.4. Therefore we can say that the impulse invariant mapping is many-to-one mapping (and does not provide one-to-one mapping).

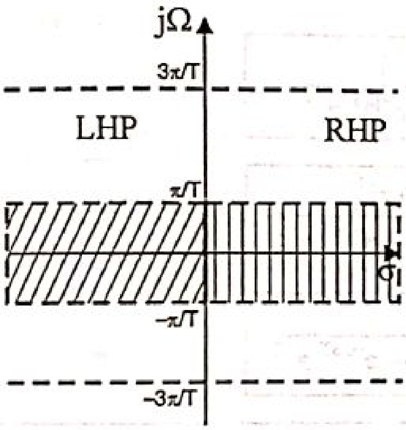

Fig 7.4 a : s-plane.

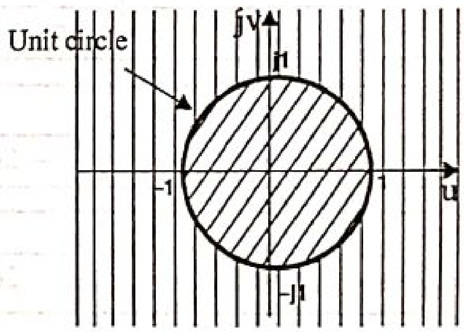

Fig 7.4 b : z-plane.

Fig 7.4 : Mapping of s-plane into z-plane in impulse invariant transformation.

The stability of a filter (or system) is related to the location of the poles. For a stable analog filter the poles should lie on the left half of the s-plane. Since the left half of s-plane maps inside the unit circle in z-plane we can say that, for a stable digital filter the poles should lie inside the unit circle in z-plane.

7.3.2 Relation Between Analog and Digital Frequency in Impulse Invariant Transformation

Let, Ω = Analog frequency in rad/second.

ω = Digital frequency in rad/sample.

Let, $z = re^{j\omega}$ be a point on z-plane,

and $s = \sigma + j\Omega$ be the corresponding point in s-plane.

Then by impulse invariant transformation,

$$z = e^{sT} \qquad \ldots\ldots(7.14)$$

Put, $z = r\,e^{j\omega}$ and $s = \sigma + j\Omega$ in equation (7.14).

$$\therefore\ r\,e^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$r\,e^{j\omega} = e^{\sigma T}\,e^{j\Omega T} \qquad \text{.....(7.15)}$$

On equating the phase on either side of equation (7.15) we get,

$$\boxed{\text{Digital frequency, } \omega = \Omega T} \quad \text{or} \quad \boxed{\text{Analog frequency, } \Omega = \frac{\omega}{T}} \qquad \text{.....(7.16)}$$

When impulse invariant transformation is employed the equation (7.16) can be used to compute the digital frequency for a given analog frequency and vice versa.

The mapping of analog to digital frequency is not one-to-one. Since ω is unique over the range ($-\pi$ to $+\pi$), the mapping $\omega = \Omega T$ implies that the interval $-\pi/T \le \Omega \le +\pi/T$ maps into the corresponding values of $-\pi \le \omega \le +\pi$. In general the interval $(2k-1)\,\pi/T \le \Omega \le (2k+1)\,\pi/T$ (where k is an integer) maps into the corresponding values of $-\pi \le \omega \le +\pi$. Thus the mapping from the analog frequency Ω to the digital frequency ω is many-to-one. This reflects the effects of aliasing due to sampling.

7.3.3 Useful Impulse Invariant Transformation

The following transformations are given without proof. The equation (7.17) can be used when the analog real poles has a multiplicity of m. The equations (7.18) and (7.19) can be used when the analog poles are complex conjugate.

$$\boxed{\frac{1}{(s + p_i)^m} \longrightarrow \frac{(-1)^{m-1}}{(m-1)!}\,\frac{d^{m-1}}{dp_i^{m-1}}\,\frac{1}{1 - e^{-p_i T}\,z^{-1}}} \qquad \text{.....(7.17)}$$

$$\boxed{\frac{(s + a)}{(s + a)^2 + b^2} \longrightarrow \frac{1 - e^{-aT}(\cos bT)\,z^{-1}}{1 - 2e^{-aT}(\cos bT)\,z^{-1} + e^{-2aT}\,z^{-2}}} \qquad \text{....(7.18)}$$

$$\boxed{\frac{b}{(s + a)^2 + b^2} \longrightarrow \frac{e^{-aT}(\sin bT)\,z^{-1}}{1 - 2e^{-aT}(\cos bT)\,z^{-1} + e^{-2aT}\,z^{-2}}} \qquad \text{....(7.19)}$$

7.4 Bilinear Transformation

The ***bilinear transformation*** is a conformal mapping that transforms the imaginary axis of s-plane into the unit circle in the z-plane only once, thus avoiding aliasing of frequency components. In this mapping all points in the left half of s-plane are mapped inside the unit circle in the z-plane and all points in the right half of s-plane are mapped outside the unit circle in the z-plane.

The bilinear transformation can be linked to the trapezoidal formula for numerical integration. Any analog system is governed by a differential equation in time domain. Consider the first order differential equation of an analog system as shown in equation (7.20).

Let, $$\frac{dy(t)}{dt} = x(t) \qquad .....(7.20)$$

On integrating both sides of equation (7.20) we get,

$$\int_{(n-1)T}^{nT} \frac{dy(t)}{dt}\,dt = \int_{(n-1)T}^{nT} x(t)\,dt$$

$$[y(t)]_{(n-1)T}^{nT} = \int_{(n-1)T}^{nT} x(t)\,dt$$

$$y(nT) - y((n-1)T) = \int_{(n-1)T}^{nT} x(t)\,dt \qquad .....(7.21)$$

> The trapezoidal rule when integration is approximated by two trapezoids is,
>
> $$\int_a^b f(x)\,dx = \frac{b-a}{2}[f(a) + f(b)]$$

The integral on the right side of equation (7.21) can be approximated by the trapezoidal rule, so that,

$$y(nT) - y[(n-1)T] \approx \left(\frac{T}{2}\right)[x(nT) + x((n-1)T)] \qquad .....(7.22)$$

For discrete time system, the equation (7.22) can be written as,

$$y(n) - y(n-1) = \frac{T}{2}[x(n) + x(n-1)] \qquad .....(7.23)$$

On taking Z-transform of equation (7.23) we get,

$$Y(z) - z^{-1}Y(z) = \frac{T}{2}\left[X(z) + z^{-1}X(z)\right]$$

$$\left[1 - z^{-1}\right]Y(z) = \frac{T}{2}\left[1 + z^{-1}\right]X(z)$$

$$\frac{2(1 - z^{-1})}{T(1 + z^{-1})}Y(z) = X(z) \qquad .....(7.24)$$

On taking Laplace transform of equation (7.20) we get,

$$s\,Y(s) = X(s) \qquad .....(7.25)$$

On comparing equations (7.24) and (7.25) we can say that,

$$\boxed{s\,Y(s) \xrightarrow[\text{(is transformed to)}]{} \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}Y(z)} \qquad .....(7.26)$$

by bilinear transformation, where T is the sampling time period.

Hence in the s-domain transfer function, if "s" is substituted by the term $\frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}$ the resulting transfer function will be z-domain transfer function.

7.4.1 Relation Between Analog and Digital Filter Poles in Bilinear Transformation

The mapping of s-domain function to z-domain function by bilinear transformation is a one to one mapping, that is, for every point in z-plane, there is exactly one corresponding point in s-plane and vice versa. The transformation is accomplished when,

$$\boxed{s = \frac{2}{T}\frac{1-z^{-1}}{1+z^{-1}}} \qquad .....(7.27)$$

The equation (7.27) can be rearranged as shown below to express "z" in terms of "s".

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \quad \Rightarrow \quad \frac{T}{2}s = \frac{1 - z^{-1}}{1 + z^{-1}} \quad \Rightarrow \quad \frac{T}{2}s = \frac{z^{-1}(z-1)}{z^{-1}(z+1)}$$

$$\therefore\ \frac{T}{2}s = \frac{z-1}{z+1} \qquad .....(7.28)$$

On cross multiplying equation (7.28) we get,

$$\frac{T}{2}s(z+1) = z-1 \quad \Rightarrow \quad \frac{T}{2}s\,z + \frac{T}{2}s = z-1 \quad \Rightarrow \quad \frac{T}{2}s\,z - z = -1 - \frac{T}{2}s$$

$$\therefore\ -z\left(1 - \frac{T}{2}s\right) = -\left(1 + \frac{T}{2}s\right)$$

$$\therefore\ z = \frac{1 + \frac{T}{2}s}{1 - \frac{T}{2}s} \qquad .....(7.29)$$

In equation (7.29), the variable "s" represent a point on s-plane and "z" is the corresponding point in z-plane.

Let, $s_i = \sigma_i + j\Omega_i$.

On substituting, $s_i = \sigma_i + j\Omega_i$ in equation (7.29) we get,

$$z_i = \frac{1+\frac{T}{2}(\sigma_i + j\Omega_i)}{1-\frac{T}{2}(\sigma_i + j\Omega_i)} = \frac{1+\frac{T}{2}\sigma_i + j\frac{T}{2}\Omega_i}{1-\frac{T}{2}\sigma_i - j\frac{T}{2}\Omega_i} \qquad .....(7.30)$$

The magnitude of equation (7.30) is given by,

$$|z_i| = \left[\frac{\left(1+\frac{T}{2}\sigma_i\right)^2 + \left(\frac{T}{2}\Omega_i\right)^2}{\left(1-\frac{T}{2}\sigma_i\right)^2 + \left(-\frac{T}{2}\Omega_i\right)^2}\right]^{\frac{1}{2}} \qquad .....(7.31)$$

From equation (7.31) the following observations can be made,

1. If $\sigma_i < 0$ (i.e., σ_i is negative), then the point $s_i = \sigma_i + j\Omega_i$, lie on the left half of s-plane. In this case, $|z_i| < 1$, hence the corresponding point in z-plane will lie inside the unit circle in z-plane.
2. If $\sigma_i = 0$ (i.e., real part is zero), then the point $s_i = \sigma_i + j\Omega_i$ lie on the imaginary axis in the s-plane. In this case, $|z_i| = 1$, hence the corresponding point in z-plane will lie on the unit circle in z-plane.
3. If $\sigma_i > 0$ (i.e., σ_i is positive), then the point $s_i = \sigma_i + j\Omega_i$ lie on the right half of s-plane. In this case, $|z_i| > 1$, hence the corresponding point in z-plane will lie outside the unit circle in z-plane.

The above discussions are applicable for mapping poles and zeros from s-plane to z-plane. The stability of the filter is associated with location of poles. We know that for a stable analog filter the poles should lie on the left half of s-plane. In bilinear transformation, the points on left half of s-plane are mapped as points inside unit circle in z-plane. Hence for stability of digital filter the digital poles should lie inside the unit circle in z-plane.

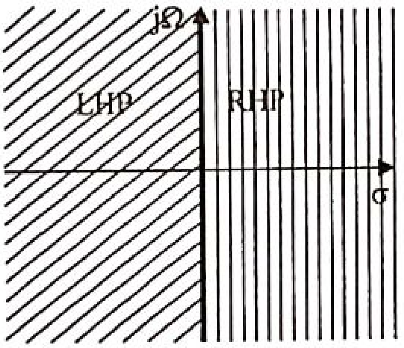

Fig 7.5a : s-plane.

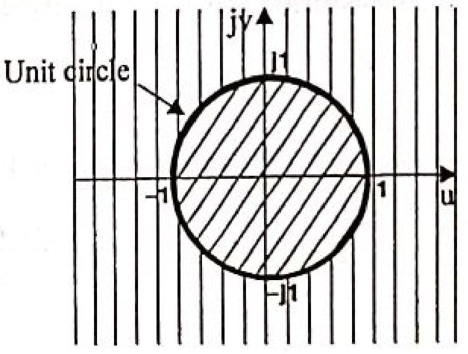

Fig 7.5b : z-plane.

Fig 7.5 : Mapping of s-plane into z-plane in bilinear transformation.

7.4.2 Relation Between Analog and Digital Frequency in Bilinear Transformation

Let, $s = j\Omega$ be points on imaginary axis and the corresponding points on the z-plane on unit circle are given by $z = e^{j\omega}$. For bilinear transformation,

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}} \qquad .....(7.32)$$

Put, $s = j\Omega$ and $z = e^{j\omega}$ in equation (7.32)

$$\therefore\ j\Omega = \frac{2}{T}\frac{1 - e^{-j\omega}}{1 + e^{-j\omega}} = \frac{2}{T}\frac{\left(e^{\frac{j\omega}{2}}e^{\frac{-j\omega}{2}} - e^{-j\omega}\right)}{\left(e^{\frac{j\omega}{2}}e^{\frac{-j\omega}{2}} + e^{-j\omega}\right)}$$

$\boxed{e^{j\theta}\,e^{-j\theta} = 1}$

$$j\Omega = \frac{2}{T}\frac{e^{\frac{-j\omega}{2}}\left(e^{\frac{j\omega}{2}} - e^{\frac{-j\omega}{2}}\right)}{e^{\frac{-j\omega}{2}}\left(e^{\frac{j\omega}{2}} + e^{\frac{-j\omega}{2}}\right)} = \frac{2}{T}\frac{2j\sin\frac{\omega}{2}}{2\cos\frac{\omega}{2}}$$

$$\boxed{\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}}$$

$$\boxed{\cos\theta = \frac{e^{j\theta} + e^{-j\theta}}{2}}$$

$$\therefore \Omega = \frac{2}{T}\frac{\sin\frac{\omega}{2}}{\cos\frac{\omega}{2}} = \frac{2}{T}\tan\frac{\omega}{2} \qquad .....(7.33)$$

$$\boxed{\therefore \text{ Analog frequency, } \Omega = \frac{2}{T}\tan\frac{\omega}{2}}$$

The equation (7.33) relates the analog frequency, Ω and digital frequency, ω.

From equation (7.33) we get,

$$\frac{\Omega T}{2} = \tan\frac{\omega}{2} \quad \Rightarrow \quad \frac{\omega}{2} = \tan^{-1}\frac{\Omega T}{2}$$

$$\boxed{\therefore \text{ Digital frequency, } \omega = 2\tan^{-1}\frac{\Omega T}{2}} \qquad .....(7.34)$$

The equation (7.34) can be used to estimate the digital frequency ω for a given analog frequency, Ω. The equation (7.33) is used to calculate the analog frequency for a given digital frequency. From the above analysis it is evident that the analog frequency Ω and digital frequency ω has a nonlinear relationship, because the entire negative imaginary axis in the s-plane (from $\Omega = -\infty$ to 0) is mapped into the lower half of unit circle in z-plane (from $\omega = -\pi$ to 0) and the entire positive imaginary axis in the s-plane (from $\Omega = 0$ to $+\infty$) is mapped into the upper half of unit circle in z-plane (from $\omega = 0$ to $+\pi$). This nonlinear mapping introduces a distortion in the frequency axis, which is called frequency warping.

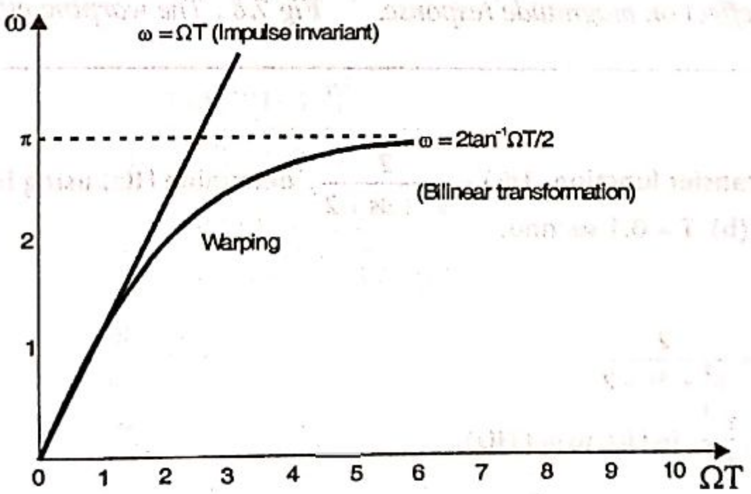

Fig 7.6 : Correspondence between analog and digital frequencies resulting from the bilinear transformation.

The effect of warping on the magnitude response can be explained by considering an analog filter with a number of passbands as shown in fig 7.7. The corresponding digital filter will have same number of passbands, but with disproportionate bandwidth, as shown in fig 7.7.

In designing digital filter using bilinear transformation the effect of warping on amplitude response can be eliminated by prewarping the analog filter. In this method, the specified digital frequencies are converted to analog equivalent using equation (7.33). This analog frequencies are called prewarp frequencies. Using the prewarp frequencies, the analog filter transfer function is designed and then it is transformed to digital filter transfer function.

The effect of warping on the phase response can be explained by considering an analog filter with linear phase response as shown in fig 7.8. The phase response of corresponding digital filter will be nonlinear.

From the above discussions it can be stated that the bilinear transformation preserves the magnitude response of an analog filter only if the specification requires piecewise constant magnitude, but the phase response of the analog filter is not preserved. Therefore the bilinear transformation can be used only to design digital filters with prescribed magnitude response with piecewise constant values. A linear phase analog filter cannot be transformed to a linear phase digital filter using bilinear transformation.

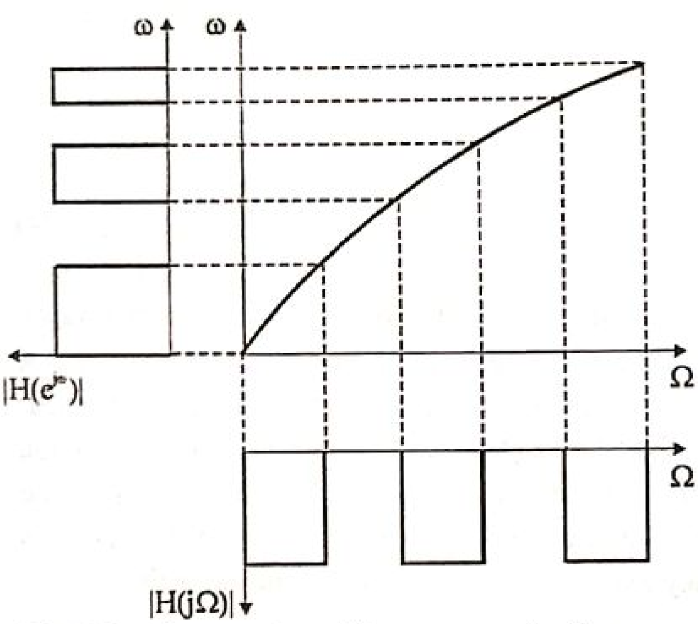

Fig 7.7 : The warping effect on magnitude response.

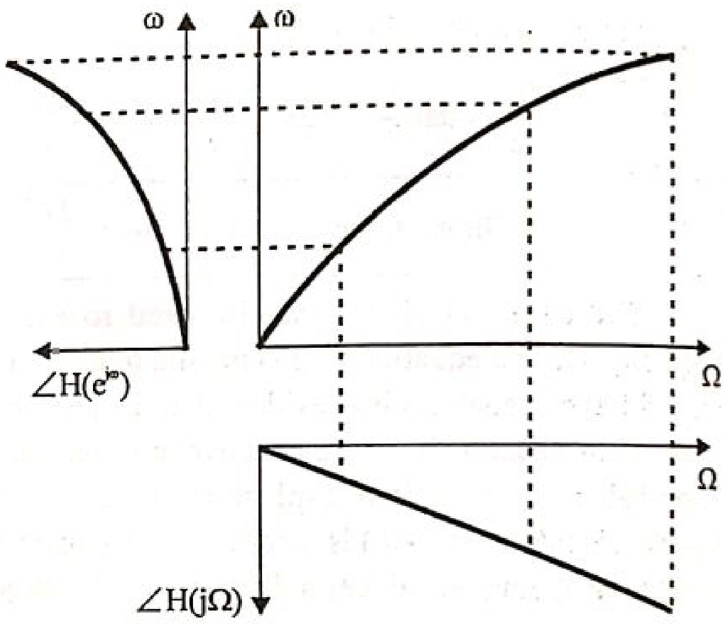

Fig 7.8 : The warping effect on phase response.