PROPERTIES OF DFT.

1. Periodicity Property

If $x(n)$ & $X(K)$ are the DFT pair.

Then: $x(n+N) = x(n)$. for all n.

& $X(K+N) = X(K)$ for all K.

Proof:-

Consider DFT.

$$X(K) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}$$

replace $K \rightarrow K+N$.

$$X(K+N) = \sum_{n=0}^{N-1} x(n) . W_N^{(K+N)n}.$$

$$= \sum_{n=0}^{N-1} x(n) W_N^{Kn} . W_N^{Nn}.$$

Since $W_N^{Nn} = 1$ $\left[W_N^{Nn} = e^{-j\frac{2\pi}{N}.Nn} = e^{-j2\pi n} = 1 \right]$

$$X(K+N) = \sum_{n=0}^{N-1} x(n) W_N^{Kn}$$

$$X(K+N) = X(K).$$

Similarly:- $x(n) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-Kn}.$

$$x(n+N) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-K(n+N)}$$

$$= \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-Kn} . W_N^{-KN}$$

Since $W_N^{-KN} = 1$.

$$x(n+N) = \frac{1}{N} \sum_{K=0}^{N-1} X(K) W_N^{-Kn}.$$

$$x(n+N) = x(n).$$

2. Linearity Property.

If $x_1(n) \xleftrightarrow{DFT} x_1(K)$.

$x_2(n) \xleftrightarrow{DFT} x_2(K)$

then, $a_1x_1(n) + a_2x_2(n) \xleftrightarrow{DFT} a_1X_1(K) + a_2X_2(K)$.

where a_1 & a_2 are constants.

Proof:-

Consider DFT.

$$x(K) = \sum_{n=0}^{N-1} x(n) W_N^{kn}.$$

Let $x(n) = a_1x_1(n) + a_2x_2(n)$.

$$x(K) = \sum_{n=0}^{N-1} [a_1x_1(n) + a_2x_2(n)] W_N^{kn}$$

$$= \sum_{n=0}^{N-1} a_1 x_1(n) W_N^{kn} + \sum_{n=0}^{N-1} a_2x_2(n) W_N^{kn}.$$

$$= a_1 x_1(K) + a_2x_2(K).$$

$\therefore$ $DFT\{a_1x_1(n) + a_2x_2(n)\} = a_1x_1(K) + a_2x_2(K)$.

3. Circular symmetry of Sequence :-

$$xp(n) = \sum_{L=-\infty}^{\infty} x(n-LN)$$ [$xp(n) \rightarrow$ periodic repetion of $x(n)$].

Shifting periodic sequence: (K shifts towards right)

$$x'p(n) = xp(n-K) = \sum_{L=-\infty}^{\infty} x(n-K-LN)$$

Circular even sequence : $x(N-n) = x(n)$.

Circular odd sequence: $x(N-n) = -x(n)$

Circular folded sequence $x(-n)_N = x(N-n)$.

4. ## Symmetry Property:

N point sequence $x(n)$ & its DFT $x(k)$ be complex valued & expressed as.

$$x(n) = x_R(n) + jx_I(n) \quad -(1).$$

$$x(k) = x_R(k) + jx_I(k) \quad -(2)$$

Substituting eq 2 in DFT expression we get:

$$x(k) = \sum_{n=0}^{N-1} x_R(n) + jx_I(n)\, W_N^{kn} \qquad \left[W_N^{kn} = e^{-j\frac{2\pi}{N}.kn}\right]$$

$$= \cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)$$

$$= \sum_{n=0}^{N-1}\left(x_R(n) + jx_I(n)\right)\left(\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right).$$

$$X_R(k) = \sum_{n=0}^{N-1} x_R(n)\cos\left(\frac{2\pi kn}{N}\right) + X_I(n).\sin\left(\frac{2\pi kn}{N}\right) \quad -(3).$$

$$X_I(k) = -\sum_{n=0}^{N-1} x_R(n)\sin\left(\frac{2\pi kn}{N}\right) - X_I(n)\cos\left(\frac{2\pi kn}{N}\right) \quad -(4).$$

Similarly; substituting eq (1) in expression of IDFT.

$$x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_R(k)\cos\left(\frac{2\pi kn}{N}\right) - X_I(k)\sin\left(\frac{2\pi kn}{N}\right) -(5)$$

$$X_I(n) = \frac{1}{N}\sum_{k=0}^{N-1} X_R(k)\sin\left(\frac{2\pi kn}{N}\right) + X_I(k)\cos\left(\frac{2\pi kn}{N}\right) -(6).$$

i) ### Real valued sequence:-

If $x(n)$ is real then:

$$x(N-k) = x^*(k) = x(-k).$$

Consider DFT sequence:

$$x(k) = \sum_{n=0}^{N-1} x(n).W_N^{kn}.$$

$$x(N-k) = \sum_{n=0}^{N-1} x(n)W_N^{(N-k)n}.$$

$$= \sum_{n=0}^{N-1} x(n).W_N^{-kn}.W_N^{Nn}. \qquad \left[W_N^{Nn} = 1\right]$$

$$= \sum_{n=0}^{N-1} x(n).W_N^{-kn} = x(-k). = x^*(k).$$

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ii) Real & even sequence:

If $x(n)$ is real & even sequence then:

$$x(k) = \sum_{n=0}^{N-1} x(n) \cdot \cos\left(\frac{2\pi kn}{N}\right).$$

W.K.T. $x(k) = x_R(k) + j\, x_I(k)$.

Considering only real part.

$$x(k) = x_R(k) = \sum_{n=0}^{N-1} x_R(n) \cos\left(\frac{2\pi kn}{N}\right) \mp j x_R(n) \sin\left(\frac{2\pi kn}{N}\right)$$

Since sine is odd function, eliminating 2nd term we get:

$$x(k) = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi kn}{N}\right)$$

iii) Real & odd sequence:

Let $x(n)$ be real & odd sequence then.

$$x(k) = -j\sum_{n=0}^{N-1} x(n) \sin\left(\frac{2\pi kn}{N}\right).$$

W.K.T.

$$x(K) = x_R(K) + j\, x_I(K).$$

Considering only real part.

$$x(k) = \sum_{n=0}^{N-1} x_R(n)\left[\cos\left(\frac{2\pi kn}{N}\right) - j\sin\left(\frac{2\pi kn}{N}\right)\right].$$

Eliminating even terms.

$$x(k) = -j \sum_{n=0}^{N-1} x(n) \sin\left(\frac{2\pi kn}{N}\right).$$

iv) Imaginary & Even sequence.

If $x(n)$ is imaginary & even:

$$x(k) = j\sum_{n=0}^{N-1} x(n) \cos\left(\frac{2\pi kn}{N}\right).$$

Now: $x(k) = x_R(k) + j\, x_I(k)$

Consider only imaginary part.

$$x(k) = \sum_{n=0}^{N-1} j x_I(k)\left[\cos\left(\frac{2\pi kn}{N} - j\sin\frac{2\pi kn}{N}\right)\right].$$

eliminating odd terms we get:

$$x(k) = j\sum_{n=0}^{N-1} x(n) \cdot \cos\left(\frac{2\pi kn}{N}\right).$$

4) Imaginary & odd sequence.

If $x(n)$ is imaginary & odd sequence.

$$x(k) = \sum_{n=0}^{N-1} x(n) \sin\left(\frac{2\pi kn}{N}\right).$$

Now: $x(k) = x_R(k) + j x_I(k)$.

Consider only imaginary terms.

$$x(k) = \sum_{n=0}^{N-1} j x_I(k) \left[\cos\frac{2\pi kn}{N} - j\sin\frac{2\pi kn}{N}\right]$$

eliminating even terms.

$$x(k) = \sum_{n=0}^{N-1} x(n) \cdot \sin\left(\frac{2\pi kn}{N}\right)$$

5) <u>Circular Convolution:</u>

If $x_1(n) \xleftrightarrow{DFT} x_1(k)$.

$x_2(n) \xleftrightarrow{DFT} x_2(k)$

then: $x_1(n) \; Ⓝ \; x_2(n) \xleftrightarrow[N]{DFT} x_1(k) x_2(k)$.

Proof:- consider DFT of a sequence.

$$x_1(k) = \sum_{n=0}^{N-1} x_1(n) w_N^{kn}.$$

$$x_2(k) = \sum_{k=0}^{N-1} x_2(k) w_N^{kk}$$

$$x_3(k) = x_1(k) x_2(k).$$

Let $x_3(m)$ be the sequence whose DFT is $x_3(k)$.

Let $x_3(m)$ be $x_1(k) \cdot x_2(k) e^{j\frac{2\pi km}{N}}$

Then $x_3(m) = \frac{1}{N}\sum_{K=0}^{N-1} x_1(k) \cdot x_2(k) e^{j\frac{2\pi km}{N}}$

$$x_3(m) = \frac{1}{N}\sum_{K=0}^{N-1}\left(\sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi kn}{N}}\right)\left(\sum_{k=0}^{N-1} x_2(k) e^{-j\frac{2\pi kk}{N}}\right) e^{j\frac{2\pi km}{N}}$$

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$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \sum_{k=0}^{N-1} e^{\frac{j2\pi k}{N}(m-n-l)}$$

we have: $\sum_{k=0}^{N-1} a^k = \frac{1-a^N}{1-a} = \begin{cases} N, & \text{multiples of } N. \\ 0, & \text{otherwise.} \end{cases}$

$$\therefore x_3(m) = \frac{1}{N}\sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l) \cdot N.$$

$$= \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(l)$$

In above eq. $(m-n-l)$ multiple of N can be written as:

$$m-n-l = -PN$$

$$l = m-n+PN.$$

Substituting we get:

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(m-n+PN)$$

$$= \sum_{n=0}^{N-1} x_1(n) \cdot x_2(m-n+PN)$$

According to circular symmetry we have:

$$(m-n+PN) = ((m-N))_N$$

$$\therefore x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N.$$

Property of circular convolution is proved.

$$x_3(n) = x_1(n) \, (N) \, x_2(n) = \sum_{n=0}^{N-1} x_1(n) \cdot x_2((m-n))_N$$

6) <u>Time reversal of Sequence</u>.

If $x_1(n) \xleftrightarrow[N]{DFT} x(k)$.

then $x((-n))_N = x(N-n) \xleftrightarrow[N]{DFT} x((-k))_N = x(N-k).$

Proof:-

Consider DFT of $x(n)$.

$$DFT\{x(n)\} = \sum_{n=0}^{N-1} x(n) W_N^{kn}.$$

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Shifting the sequence by N positions we get:

$$DFT\{x(n-N)\} = \sum_{n=0}^{N-1} x(N-n) W_N^{kn}.$$

Let $l = N-n$.

When $n=0$ $l=N$.

$n=N-1$ $l=1$.

$$DFT\{x(N-n)\} = \sum_{l=N}^{1} x(l) W_N^{k(N-l)}$$

$$= \sum_{l=1}^{N} x(l) W_N^{-kl} \cdot 1.$$

Since $W_N^{NL} = 1$.

$$DFT\{x(N-n)\} = \sum_{l=1}^{N} x(l) W_N^{-kl} \cdot W^{NL}.$$

$$= \sum_{l=1}^{N} x(l) W^{l(N-k)}.$$

$$= x(n-k).$$

7) <u>Circular time shift of sequence.</u>

If $x(n) \xleftrightarrow[n]{DFT} x(k)$

$$x((n-l))_N \xleftrightarrow[n]{DFT} W_N^{kl} \cdot x(k).$$

Proof:-

Consider DFT of the sequence $x(n)$.

$$DFT\{x(n)\} = \sum_{n=0}^{N-1} x(n) \cdot W_N^{kn}.$$

$$DFT\{x((n-l))_N\} = \sum_{n=0}^{N-1} x((n-l))_N W_N^{kn}.$$

we know: $x((n-l))_N = x(n-l+PN)$

Let $P=1$.

$$DFT\{x((n-l))_N\} = \sum_{n=0}^{N-1} x(n-l+N) W_N^{kn}.$$

Splitting the sequence:

$$x(n-l+N) = \begin{cases} x(n-l+N) ; & 0 \text{ to } l-1 \\ x(n-l) ; & l \text{ to } N-1. \end{cases}$$

$$DFT\{x((n-L))_N\} = \sum_{n=0}^{L-1} x(n-L+N)\, \omega_N^{km} + \sum_{n=L}^{N-1} x(n-m)\, \omega_N^{kn}.$$

Let $m = n-L+N$.

$n=0 \quad m = N-L.$

$n = L-1 \quad m = N-1$

Let $m = n-L$

When $n=L \quad m=0.$

$n = N-1. \quad m = N-L-1.$

Substituting we get:

$$DFT\{x((n-L))_N\} = \sum_{m=N-L}^{N-1} x(m)\, \omega_N^{k(m+L-N)} + \sum_{m=0}^{N-L-1} x(m)\, \omega_N^{k(m+L)}.$$

$$= \sum_{m=0}^{N-1} x(m)\, \omega_N^{k(m+L)}.$$

$$= \sum_{m=0}^{N-1} x(m)\, \omega_N^{km} \cdot \omega_N^{kL}.$$

$$\therefore DFT\{x((n-L))_N\} = \omega_N^{kL} \cdot x(k).$$

8) **Circular Frequency shift.**

If $x(n) \xleftrightarrow[n]{DFT} x(k)$.

then $\omega_N^{-mn}\, x(n) \xleftrightarrow[n]{DFT} x((k-m))_N$.

Proof:

Consider IDFT of the sequence.

$$IDFT\{x((k-m))_N\} = \frac{1}{N}\sum_{k=0}^{N-1} x((k-m))_N\, \omega_N^{-km}.$$

Now: $x((k-m))_N = x(k-m+PN)$

Assume $P=1$.

$x((k-m))_N = x(k-m+N)$

$$IDFT\{(x(k-m))_N\} = \frac{1}{N}\sum_{k=0}^{N-1} x(k-m+N)\, \omega_N^{-km}.$$

Splitting the sequence.

$$x(k-m+N) = \begin{cases} x(k-m+N) & ;\ 0 \text{ to } m-1. \\ x(k-m) & ;\ m \text{ to } N-1 \end{cases}$$

Substituting :-

$$IDFT\{x((K-m))_N\} = \frac{1}{N}\left[\sum_{K=0}^{m-1} x(K-m+N)\,\omega_N^{-Kn} + \sum_{K=m}^{N-1} x(K-m)\,\omega_N^{-Km}\right]$$

Let $L = K-m+N$

$K=0 \quad L = N-m.$

$K=m-1 \quad L = N-1.$

Let $L = K-m.$

$K=m \quad L=0.$

$K=N-1 \quad L = N-m-1.$

$$IDFT\{x(K-n)_N\} = \frac{1}{N}\left[\sum_{L=N-m}^{N-1} x(L)\,\omega_N^{-n(L+m-N)} + \sum_{L=0}^{N-m-1} x(L)\,\omega_N^{-n(L+m)}\right]$$

$$= \frac{1}{N}\left[\sum_{L=N-m}^{N-1} x(L)\,\omega_N^{-(L+m)n} + \sum_{L=0}^{N-m-1} x(L)\,\omega_N^{-n(L+m)}\right]$$

$$= \frac{1}{N}\sum_{L=0}^{N-1} x(L)\,\omega_N^{-Ln}\,.\,\omega^{-mn}.$$

$$IDFT\{x((K-m))_N\} = \omega^{-mn}.\,x(n).$$

9) <u>Complex Conjugate Property.</u>

If $x(n) \xleftrightarrow[n]{DFT} X(K)$ then:

$$x^*(n) \xleftrightarrow[N]{DFT} X^*(N-K) = X^*((-K))_N.$$

& $$x^*(N-n) \xleftrightarrow[n]{DFT} X^*(K).$$

Proof :-

Consider DFT of $x^*(n)$.

$$DFT\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n).\,\omega^{kn}.$$

we know $e^{\frac{j2\pi Nn}{N}} = e^{j2\pi n} = 1.$

$$\therefore DFT\{x^*(n)\} = \sum_{n=0}^{N-1} x^*(n)\,e^{-\frac{j2\pi kn}{N}}\,e^{\frac{j2\pi Nn}{N}}$$

$$= \sum_{n=0}^{N-1} x^*(n)\,e^{\frac{j2\pi n}{N}(N-K)}$$

$$= \left[\sum_{n=0}^{N-1} x(n) \cdot e^{-\frac{j2\pi n(N-k)}{N}}\right]^*$$

$$= \left[X(N-K)\right]^* = X^*(N-K).$$

1.

10) Modulation property.

If $x_1(n) \xleftrightarrow[n]{DFT} X_1(K)$ &

$x_2(n) \xleftrightarrow[n]{DFT} X_2(K)$ then.

$$x_1(n) . x_2(n) \xleftrightarrow[n]{DFT} \frac{1}{N}\left[X_1(K) (N) X_2(K)\right].$$

Let $x_3(m) = x_1(n) . x_2(n).$

$$DFT\{x_3(m)\} = \sum_{n=0}^{N-1} x_3(m) . e^{-\frac{j2\pi km}{N}}$$

$$= \sum_{n=0}^{N-1} x_1(n) . x_2(n) . e^{-\frac{j2\pi km}{N}}.$$

Let $x_1(n) = \frac{1}{N}\sum_{K=0}^{N-1} x_1(K) . e^{\frac{j2\pi kn}{N}}$

$$x_2(n) = \frac{1}{N}\sum_{L=0}^{N-1} x_2(L) \; e^{\frac{j2\pi nL}{N}}.$$

$$DFT\{x_3(m)\} = \frac{1}{N^2}\sum_{n=0}^{N-1}\left[\sum_{K=0}^{N-1} x_1(K) . e^{\frac{j2\pi kn}{N}}\right]\left[\sum_{L=0}^{N-1} x_2(L) e^{\frac{j2\pi nL}{N}}\right] . e^{-\frac{j2\pi km}{N}}$$

$$= \frac{1}{N^2}\left(\sum_{K=0}^{N-1} x_1(K)\right)\left(\sum_{L=0}^{N-1} x_1(L)\right)\sum_{n=0}^{N-1} e^{-\frac{j2\pi n}{N}(m-K-L)}$$

We have $\sum_{n=0}^{N-1} a^K = \frac{1-a^N}{1-a} = \begin{cases} N ; & \text{multiple of } N \\ 0 & \text{otherwise} \end{cases}$

$$\therefore DFT\{x_3(m)\} = \frac{1}{N^2}\sum_{K=0}^{N-1} x_1(K) \sum_{L=0}^{N-1} x_1(L) . N$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} x_1(K) \sum_{L=0}^{N-1} x_1(L).$$

In above eq. $m-K-L$ is multiple of N.

$\therefore m-K-L = -PN$

$L = m-K+PN.$

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Substituting we get:

$$DFT\{x_3(m)\} = \frac{1}{N}\sum_{K=0}^{N-1} x_1(K) \sum_{L=0}^{N-1} x_2(m-K+PN).$$

$$= \frac{1}{N}\sum_{K=0}^{N-1} x_1(K) \cdot x_2(m-K+PN).$$

According to circular symmetry:

$$m-K+PN = ((m-K))_N.$$

$$\therefore DFT\{x_3(m)\} = \frac{1}{N}\sum_{K=0}^{N-1} x_1(K) \cdot x_2((m-K))_N.$$

$$= \frac{1}{N} \quad x_1(K) \; (N) \; x_2(K).$$

11. <u>Circular co-relation.</u>

If $x(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} x(K).$

$y(n) \underset{N}{\overset{DFT}{\longleftrightarrow}} Y(K)$

then: $\tilde{r}_{xy}(l) \underset{N}{\overset{DFT}{\longleftrightarrow}} \tilde{R}_{xy}(K) = X(K)Y^*(K).$

where $\tilde{r}_{xy}(l) = \sum_{n=0}^{N-1} x(n)\, y^*((n-l))_N.$

lly $\tilde{r}_{xx}(l) \underset{N}{\overset{DFT}{\longleftrightarrow}} \tilde{R}_{xx}(K) = X(K)X^*(K) = |X(K)|^2.$

Proof: Consider DFT of the sequence.

$$DFT\{\tilde{r}_{xy}(l)\} = \sum \tilde{r}_{xy}(l) \cdot W_N^{KL}$$

$$= \sum x(n)\, y^*((n-l))_N.$$

From circular convolution property.

$$x_3(m) = \sum x(n)\, y((m-n))_N = x_1(K)y(K)$$

$$\therefore DFT\{\tilde{r}_{xy}(l)\} = x(K) \cdot y^*(K).$$

12. Parseval's Theorem.

If $x(n) \xleftrightarrow[n]{DFT} X(k)$.

$y(n) \xleftrightarrow[n]{DFT} Y(K)$

then $\sum_{n=0}^{N-1} x(n) \cdot y^*(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(K) \cdot Y^*(K)$.

From circular correlation property.

$DFT\{\tilde{r}_{xy}(l)\} = X(K) \cdot Y^*(K)$.

$IDFT\{X(K) \cdot Y^*(K)\} = \frac{1}{N} \sum_{k=0}^{N-1} X(K) \cdot Y^*(K) \cdot e^{\frac{j2\pi kl}{N}}$.

Let $l = 0$.

$IDFT\{X(K) \cdot Y^*(K)\} = \frac{1}{N} \sum_{K=0}^{N-1} Y^*(K) \cdot X(K)$.

Also .

$IDFT\{X(K) \cdot Y^*(K)\} = \tilde{r}_{xy}(l)$

$= \sum_{n=0}^{N-1} x(n) \cdot y^*((n-l))_N$.

Let $L = 0$.

$= \sum_{n=0}^{N-1} x(n) \cdot y^*(n)$.

$\therefore \sum_{n=0}^{N-1} x(n) \cdot y^*(n) = \frac{1}{N} \sum_{K=0}^{N-1} Y^*(K) \cdot X(K)$.