Equations (4.73) and (4.72) are identical. Therefore, we can conclude that when the input symbols become equiprobable, the mutual information maximizes and becomes equal to channel capacity C.

Example 4.11 : A binary symmetric channel has the following noise matrix with source probabilities of $P(x_1) = \frac{2}{3}$ and $P(x_2) = \frac{1}{3}$.

$$P(Y/X) = \begin{array}{c} \\ x_1 \\ x_2 \end{array} \begin{array}{c} \begin{array}{cc} y_1 & y_2 \end{array} \\ \begin{bmatrix} 3/4 & 1/4 \\ 1/4 & 3/4 \end{bmatrix} \end{array}$$

(i) Determine H(X), H(Y), H(X, Y), H(Y/X), H(X/Y) and I(X, Y).

(ii) Find the channel capacity C

(iii) Find channel efficiency and redundancy.

Solution

The channel diagram of the given channel is as shown in figure 4.7 with

$$p = \frac{1}{4},\ \bar{p} = \frac{3}{4},\ w = \frac{2}{3} \text{ and } \bar{w} = \frac{1}{3}$$

(i) From equation (4.30),

$$H(X) = \sum_{i=1}^{2} P(x_i) \log \frac{1}{P(x_i)}$$

$$= \frac{2}{3} \log \frac{3}{2} + \frac{1}{3} \log 3$$

$$\therefore \quad \mathbf{H(X) = 0.9183\ bits/message\text{-}symbol}$$

We have $\bar{p}w + p\bar{w} = \left(\frac{3}{4}\right)\left(\frac{2}{3}\right) + \left(\frac{1}{4}\right)\left(\frac{1}{3}\right) = \frac{7}{12}$

and $pw + \bar{p}\bar{w} = \left(\frac{1}{4}\right)\left(\frac{2}{3}\right) + \left(\frac{3}{4}\right)\left(\frac{1}{3}\right) = \frac{5}{12}$

From equation (4.70),

$$H(Y) = \frac{7}{12} \log \frac{12}{7} + \frac{5}{12} \log \frac{12}{5}$$

$$\therefore \quad \mathbf{H(Y) = 0.9799\ bits/message\text{-}symbol}$$

From equation (4.68),

$$H(Y/X) = h = \bar{p} \log \frac{1}{\bar{p}} + p \log \frac{1}{p}$$

$$= \frac{3}{4} \log \frac{4}{3} + \frac{1}{4} \log 4$$

$$\therefore \quad \mathbf{H(Y/X) = 0.8113\ bits/message\text{-}symbol}$$

The joint entropy H(X, Y) is given by equation (4.10) as

$$H(X, Y) = H(X) + H(Y/X)$$

$$= 0.9183 + 0.8113$$

$$\therefore \quad \mathbf{H(X, Y) = 1.7296\ bits/message\text{-}symbol}$$

From equation (4.42), we have

$$H(Y, X) = H(X, Y) = H(Y) + H(X/Y)$$

$$\therefore \quad H(X/Y) = H(X, Y) - H(Y)$$

$$= 1.7296 - 0.9799$$

$$\therefore \quad \mathbf{H(X/Y) = 0.7497\ bits/message\text{-}symbol}$$

From equation (4.34), we have

$$I(X, Y) = H(X) - H(X/Y) \text{ [or } H(Y) - H(Y/X)]$$

$$= 0.9183 - 0.7497$$

$$\mathbf{I(X, Y) = 0.1686\ bits/message\text{-}symbol}$$

(ii) From equation (4.72), the channel capacity 'C' is given by

$$C = 1 - h = 1 - H(Y/X)$$

$$= 1 - 0.8113$$

$$\therefore \quad C = \mathbf{0.1887\ bits/message\text{-}symbol}$$

(iii) ∴ Channel efficiency $= \dfrac{I(X, Y)}{C}$

$$= \frac{0.1686}{0.1887}$$

$$\therefore \quad \eta_{ch} = \mathbf{89.35\%}$$

And channel redundancy $= \eta_{ch} = 10.65\%$