DECIMATION IN TIME FAST FOURIER TRANSFORM

(DIT – FFT)

In the following presentation, the number of samples are assumed as power of 2. i.e, $N = 2^{\vartheta}$, where $\vartheta \rightarrow$ fixed integer.

The decimation in time approach is one of breaking N-point tranform into 2 N/2 transforms, then breaking each N/2 point transforms into N/4 point transforms and continuing this decimation process until 2-point transforms are obtained. This technique is known as divide and conquer approach.

Let $x(n) = x(0), x(1), x(2), \ldots x(N-1)$.

Even indexed sequence : $x(0), x(2) \ldots x(N-2)$
odd indexed sequence : $x(1), x(3) \ldots x(N-1)$

The N-point DFT of $x(n)$ is

$$X(K) = \sum_{n=0}^{N-1} x(n) \, W_N^{kn}$$

$$\Rightarrow X(K) = \sum_{\substack{n=0 \\ n,\ \text{even}}}^{N-2} x(n) \cdot W_N^{kn} + \sum_{\substack{n=1 \\ n,\ \text{odd}}}^{N-1} x(n) \cdot W_N^{kn}$$

For the first decimation, put $n = 2r$ in the first summation and $n = 2r+1$ in the second summation. This gives -

$$X(K) = \sum_{r=0}^{\frac{N}{2}-1} x(2r)\, W_N^{K.2r} + \sum_{r=0}^{\frac{N}{2}-1} x(2r+1)\, W_N^{K(2r+1)}$$

$$= \sum_{r=0}^{\frac{N}{2}-1} x(2r)\, W_N^{K.2r} + W_N^K \sum_{r=0}^{\frac{N}{2}-1} x(2r+1)\, W_N^{K.2r.}$$

Since $W_N^{K2r} = e^{-j\frac{2\pi}{N}\cdot K.2r} = e^{-j\frac{2\pi}{N/2}\cdot Kr} = W_{N/2}^{Kr.}$

the above equation can be written as -

$$X(K) = \sum_{r=0}^{\frac{N}{2}-1} x(2r)\, W_{N/2}^{Kr} + W_N^K \sum_{r=0}^{\frac{N}{2}-1} x(2r+1)\, W_{N/2}^{Kr}$$

$$= G(K) + W_N^K H(K)$$

$$K = 0, 1, 2 \ldots \frac{N}{2} - 1.$$

where $G(K)$ and $H(K)$ $N/2$ - point DFTs, are of even indexed and odd indexed sequences respectively. For computing $X(K)$ for $K = \frac{N}{2}, \frac{N}{2}+1 \ldots N-1$, the periodicity of $G(K)$ and $H(K)$ are exploited. It may be noted that $G(K)$ and $H(K)$ are periodic with a period equal to $N/2$. Thus we can write

$$X(K) = \begin{cases} G(K) + W_N^K H(K), & K = 0, 1, 2 \ldots \frac{N}{2} - 1. \\ G(K+\frac{N}{2}) + W_N^K H(K+\frac{N}{2}), & K = \frac{N}{2}, \frac{N}{2}+1, \ldots N-1. \end{cases}$$

for N = 8 , the above equation becomes –

$$X(K) = \begin{cases} G(K) + W_8^K H(K) & , K = 0, 1, 2, 3. \\ G(K+4) + W_8^K H(K+4) & , K = 4, 5, 6, 7. \end{cases}$$

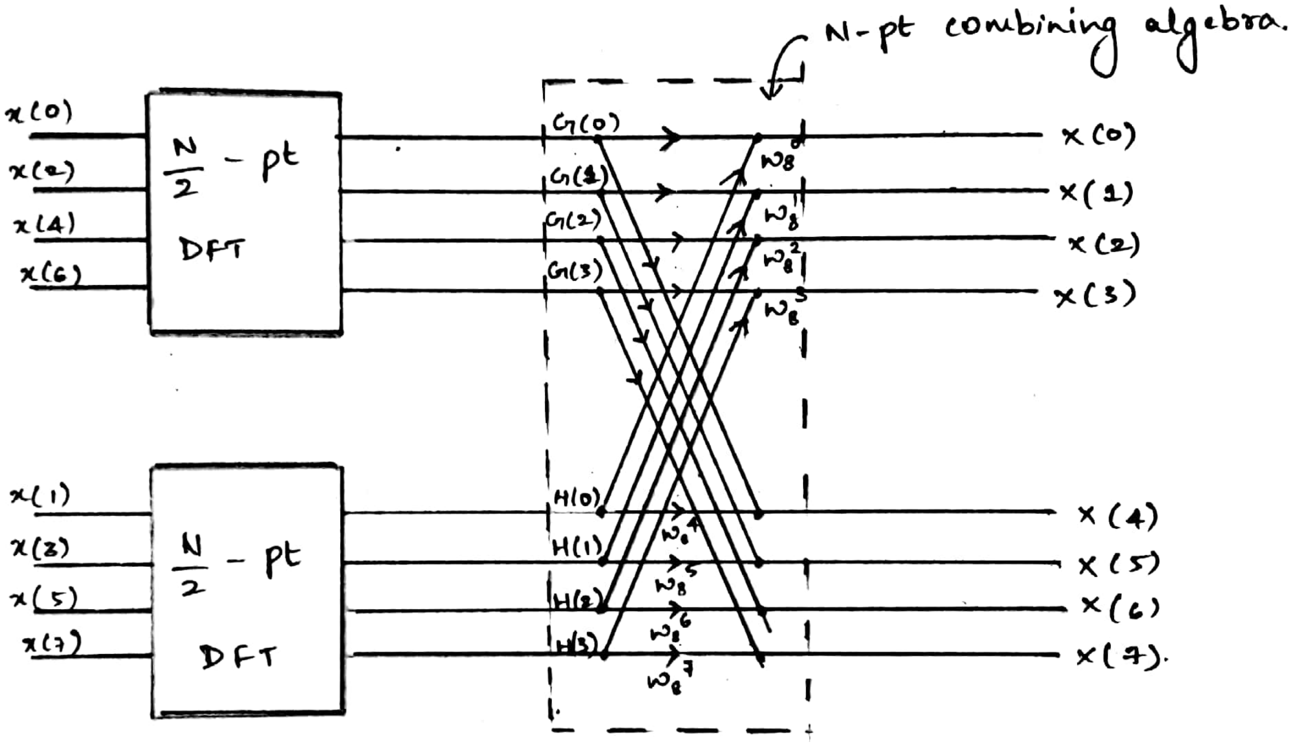


fig 1. Signal flow diagram after first decimation.

Total number of complex multiplications after first decimation is given by –

$$\eta_1 = \left(\frac{N}{2}\right)^2 + \left(\frac{N}{2}\right)^2 + N$$

The first two terms account for the computation of N/2 point DFT's while the last term accounts for N-point combining algebra.

Each of the N/2 - point sequences are further decimated into sequences of length equal to N/4.

$$\boxed{G(K) = \sum_{l=0}^{N/2-1} g(l)\, W_{N/2}^{Kl}}$$

$$= \sum_{\gamma=0}^{\frac{N}{4}-1} g(2\gamma)\, W_{N/2}^{K.2\gamma} + \sum_{\gamma=0}^{\frac{N}{4}-1} g(2\gamma+1)\, W_{N/2}^{K(2\gamma+1)}$$

$$= \sum_{\gamma=0}^{\frac{N}{4}-1} g(2\gamma)\, W_{N/2}^{K.2\gamma} + \sum_{\gamma=0}^{\frac{N}{4}-1} g(2\gamma+1)\, W_{N/2}^{K.2\gamma} \cdot W_{N/2}^{K}$$

Since $W_{N/2}^{K.2\gamma} = e^{-j\frac{2\pi}{N/2} \cdot K \cdot 2\gamma} = e^{-j\frac{2\pi}{N/4} \cdot K\gamma} = W_{N/4}^{K\gamma}$,

we get

$$G(K) = \sum_{\gamma=0}^{\frac{N}{4}-1} g(2\gamma) \cdot W_{N/4}^{K\gamma} + W_{N/2}^{K} \sum_{\gamma=0}^{\frac{N}{4}-1} g(2\gamma+1) \cdot W_{N/4}^{K\gamma}$$

$$G(K) = A(K) + W_{N/2}^{K} \cdot B(K).$$

Since A(K) and B(K) are periodic with a period equal to N/4, we can write –

$$G(K) = \begin{cases} A(K) + W_{N/2}^{K}\, B(K) & K = 0, 1, \cdots \frac{N}{4}-1 \\ A(K+\frac{N}{4}) + W_{N/2}^{K}\, B(K+\frac{N}{4}) & K = \frac{N}{4}, \frac{N}{4}+1, \cdots \frac{N}{2}-1 \end{cases}$$

Similarly, we can write -

$$H(K) = \begin{cases} C(K) + W_{N/2}^{K} D(K) & K = 0, 1, \dots \frac{N}{4} - 1. \\ C(K + \frac{N}{4}) + W_{N/2}^{K} D(K + \frac{N}{4}) & K = \frac{N}{4}, \frac{N}{4}+1 \dots \frac{N}{2} - 1. \end{cases}$$

The above equations for $N = 8$ become the following -

$$G(K) = \begin{cases} A(K) + W_4^K B(K) & K = 0, 1. \\ A(K+2) + W_4^K B(K+2) & K = 2, 3 \end{cases}$$

$$H(K) = \begin{cases} C(K) + W_4^K D(K) & K = 0, 1. \\ C(K+2) + W_4^K D(K+2) & K = 2, 3 \end{cases}$$

Continuing this process, each $\frac{N}{4}$ points transformation is broken into 2 $\frac{N}{8}$ point transforms.

Since $N = 2^{v}$, this process can be continued until there are $\log_2 N$ stages. It may be noted that in each stage there are $N/2$ butterflies and each butterfly has 2 complex multiplications. Therefore, after final decimation, we have

$2 \times \frac{N}{2} \times \log_2 N = N \log_2 N$ complex multiplications.

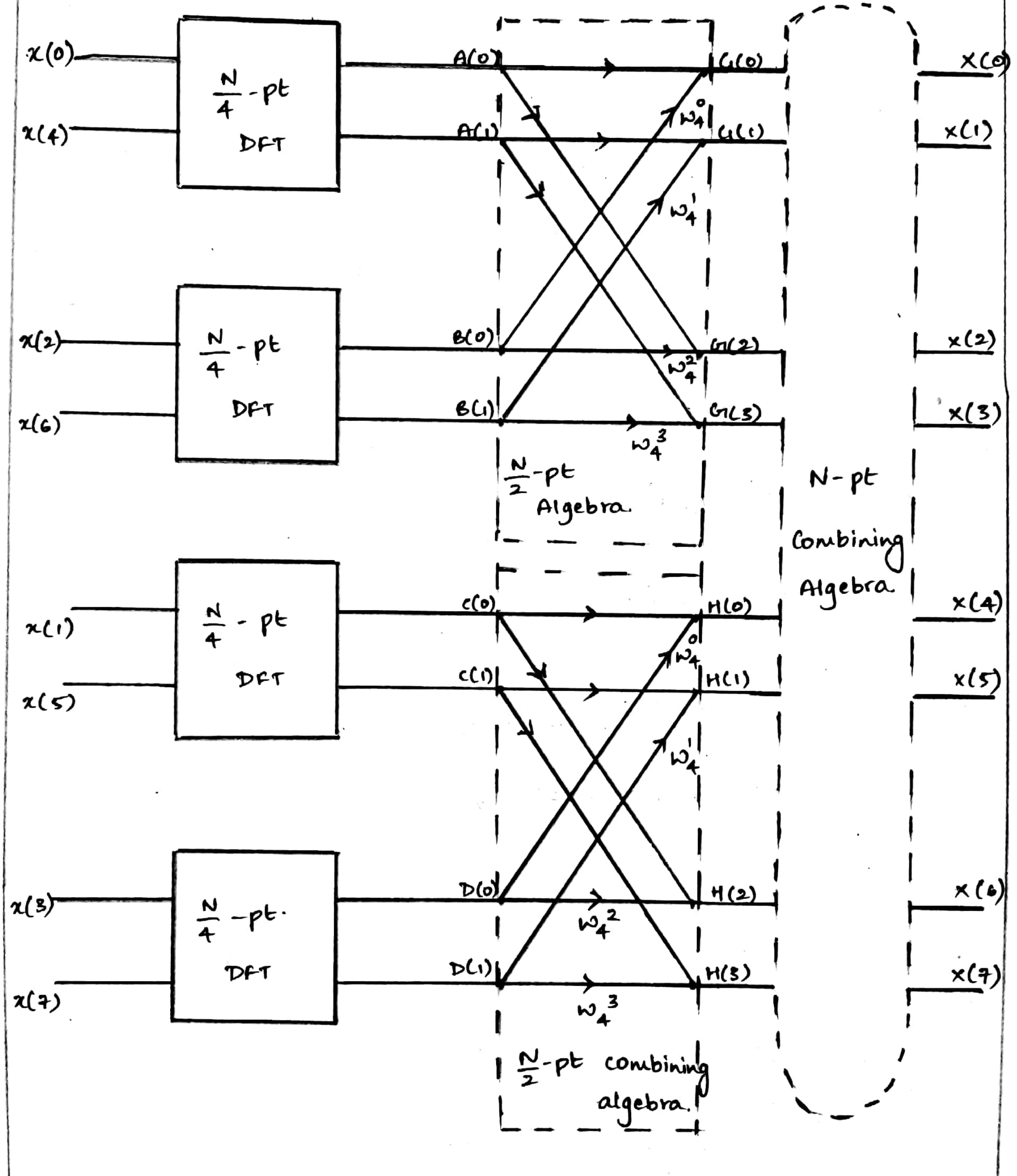


fig 2 : Signal flow diagram after second decimation.

The total signal flow diagram after final decimation is as shown below. –

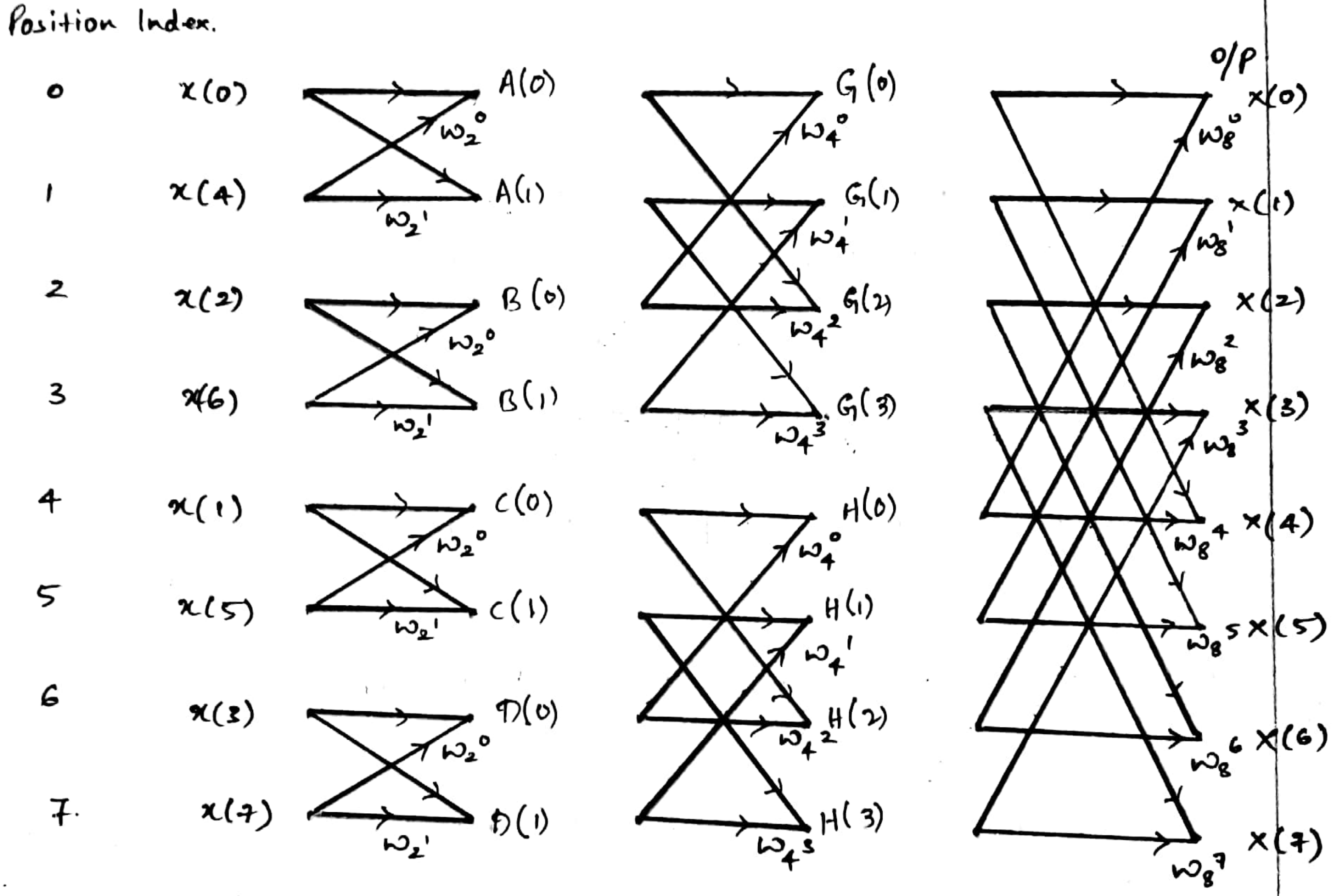


fig 3. : Signal flow diagram after final decimation.

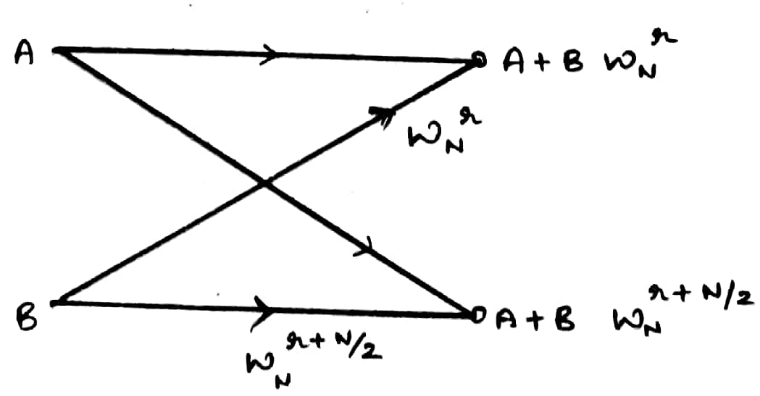


fig 4 : A sample Butterfly

The following observations are made from the signal flow graph shown in fig 3 :

1) Input data appears in bit - reversed order

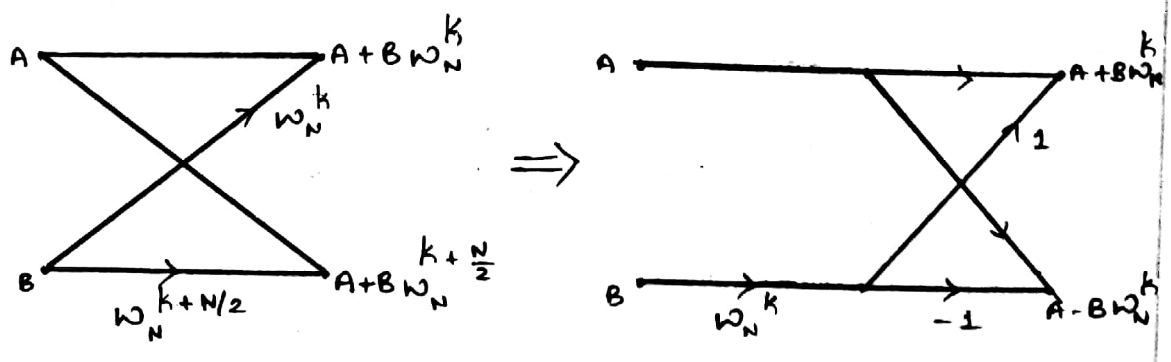
2) Basic computational block in the signal flow diagram is called a Butterfly and is as shown in fig 4. The power 'r' of W_N is a variable and depends upon the position of the butterfly.

3) Frequency domain output $X(k)$ appears in Normal order.

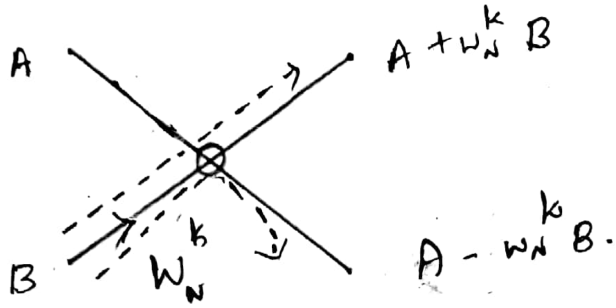
FURTHER REDUCTION. (Cooley - Tukey Algorithm).

Basic Butterfly configuration can be further simplified to reduce the number of complex multiplications per butterfly by one.

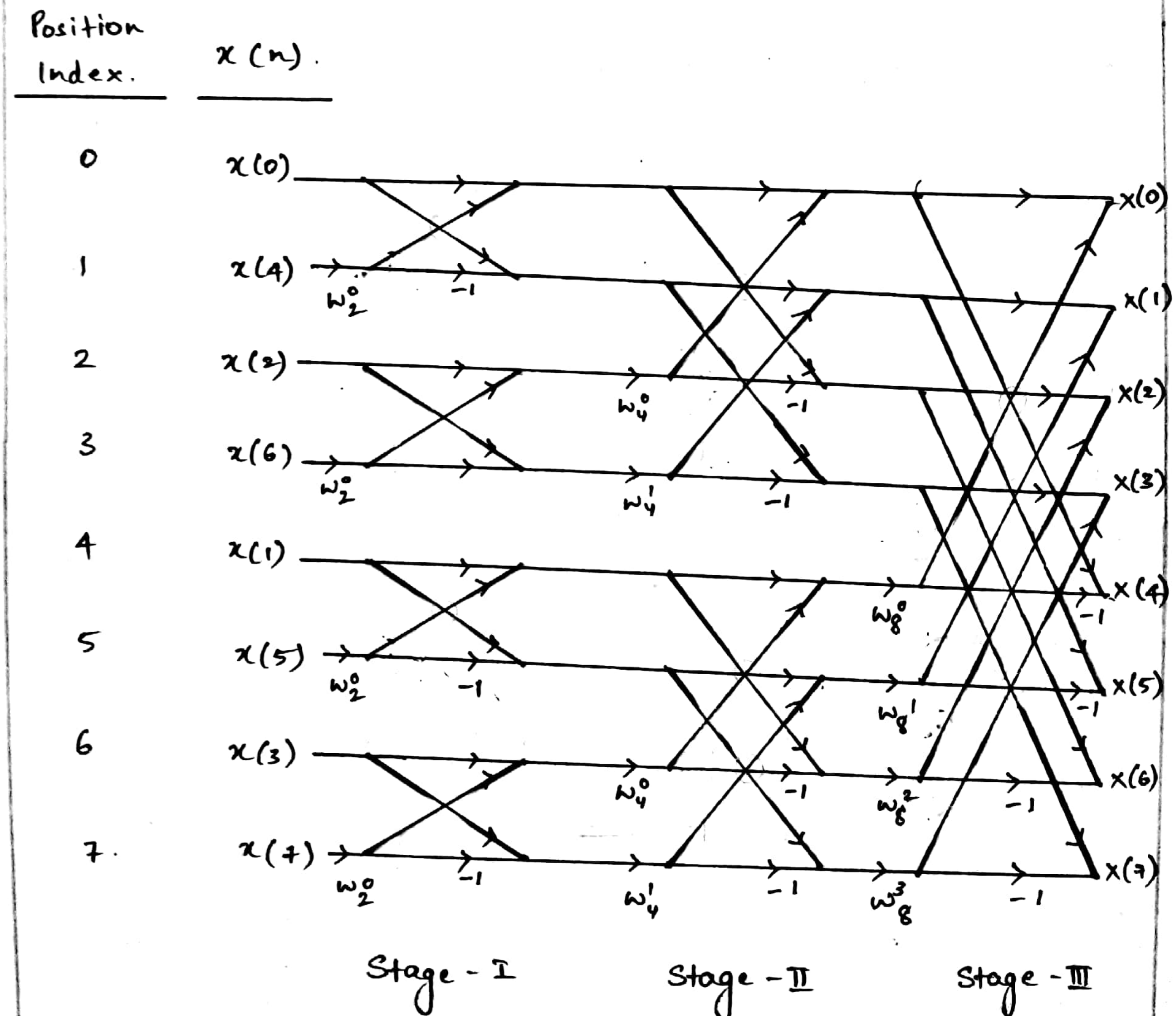
$$W_N^{k+\frac{N}{2}} = e^{-j\frac{2\pi}{N}(k+\frac{N}{2})} = e^{-j\frac{2\pi k}{N}} \cdot e^{-j\pi} = -W_N^k$$



$\Rightarrow$



The total signal flow diagram based on Cooley - Tukey Algorithm for $N = 8$ is as shown below.



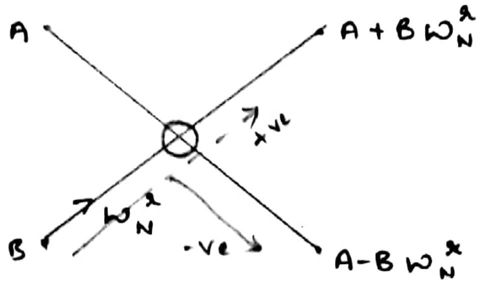
Total number of complex multiplications with Cooley - Tukey Algorithm is

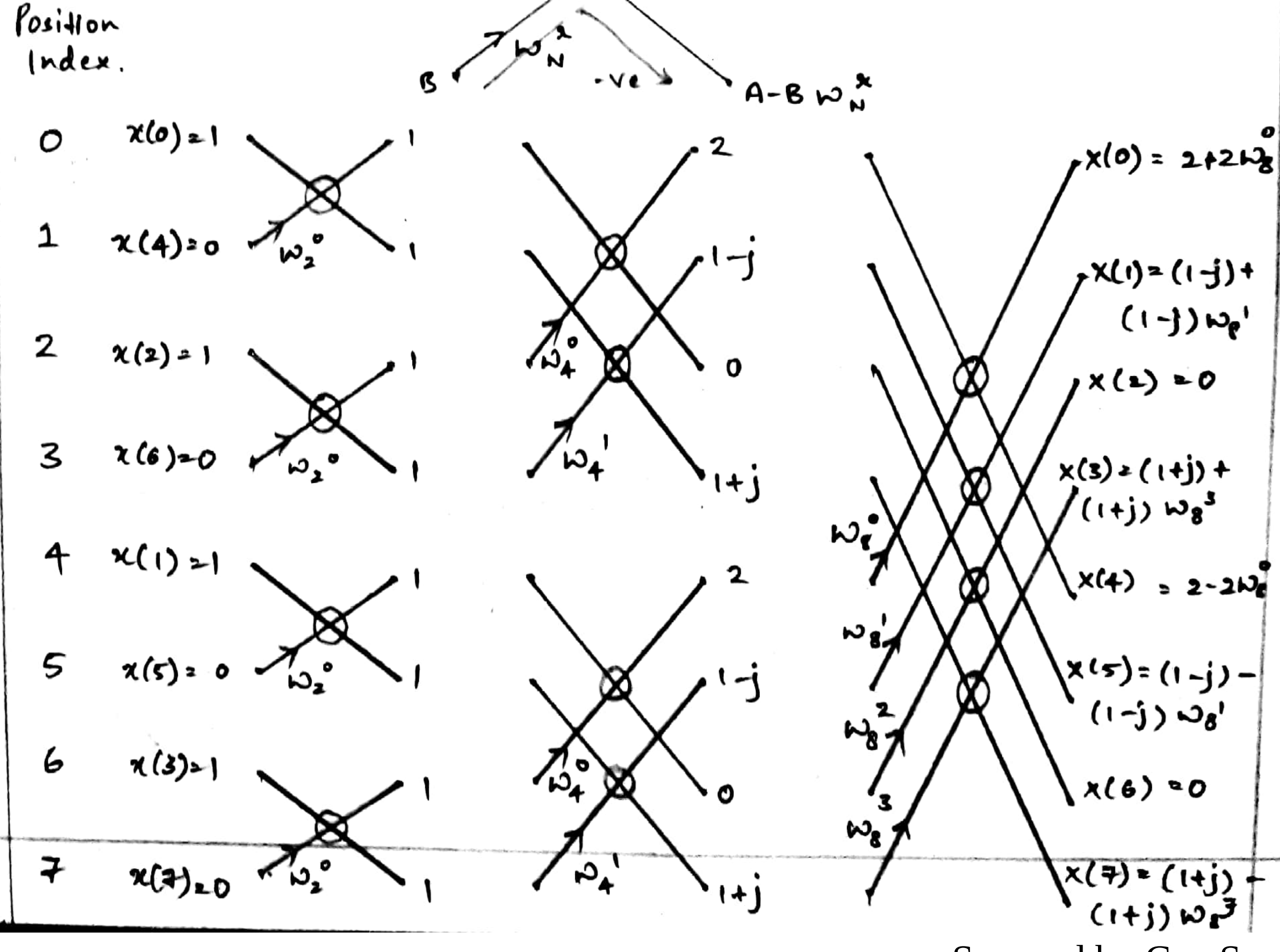
$$\boxed{1 \times \frac{N}{2} \times \log_2 N}$$

1) Compute 8-point DFT of the sequence -

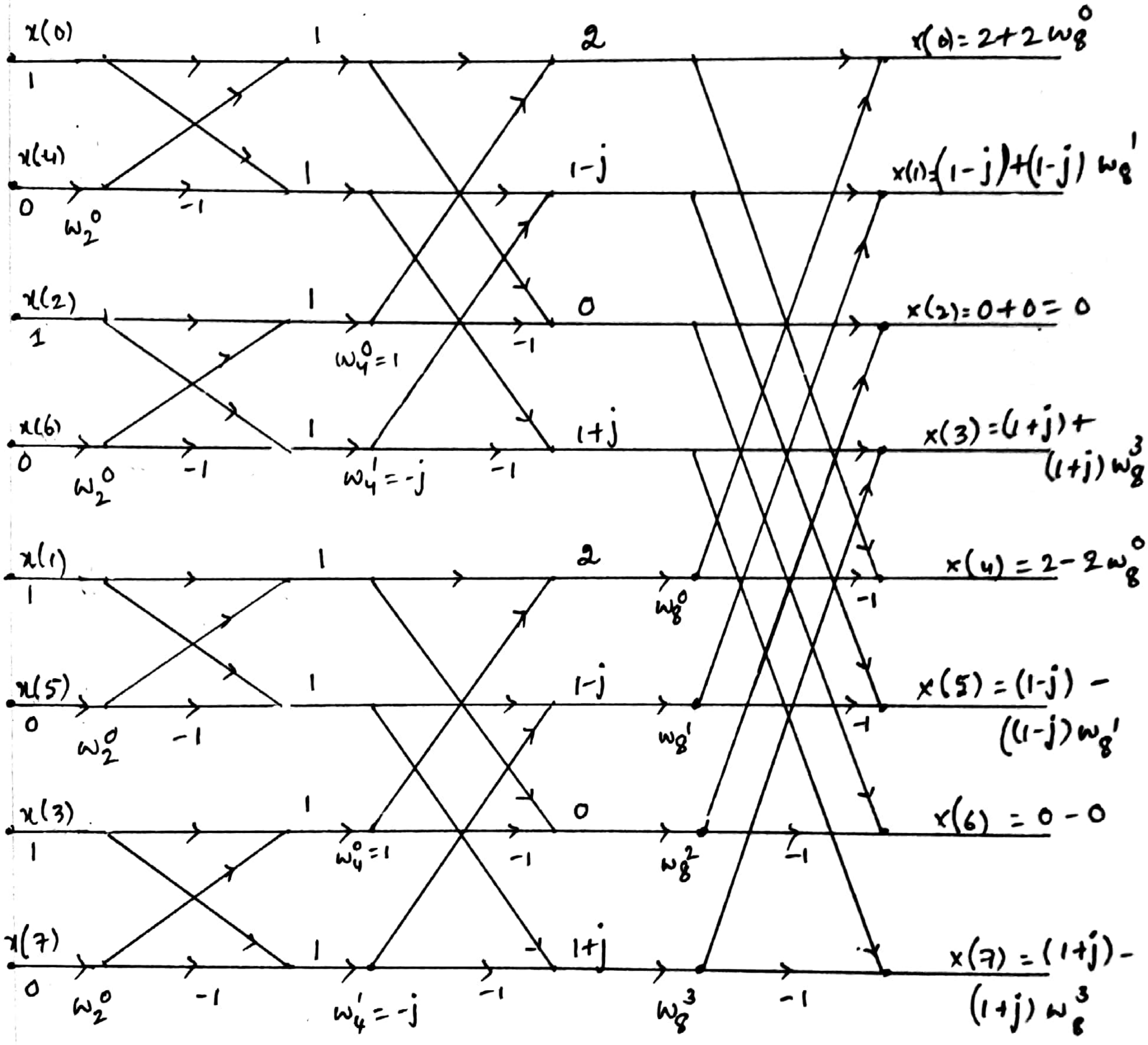
$x(n) = (1,1,1,1,0,0,0,0)$ using decimation in time radix-2 FFT Algorithm.

Let the sample Butterfly be as shown in the figure below -





Compute 8 point DFT of the sequence $x(n) = (1, 1, 1, 1, 0, 0, 0, 0)$ using decimation in time radix-2 FFT algorithm.



$X(0) = 2 + 2W_8^0 = 2 + 2 = \underline{\underline{4}}$

$X(1) = (1-j) + (1-j)W_8^1 = 1 - j + (1-j)\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

$X(1) = \underline{\underline{1 - j\,2.414}}$

$X(2) = \underline{\underline{0}}$

$X(3) = (1+j) + (1+j)W_8^3 = 1 + j + (1+j)\left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

$= 1 + j - \cancel{\frac{1}{\sqrt{2}}} - j\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} + \cancel{\frac{1}{\sqrt{2}}}$

$X(3) = \underline{\underline{1 + j\,0.414}}$

$X(4) = 2 - 2W_8^0 = 2 - 2 = 0$

$X(5) = (1-j) - (1-j)W_8^1 = 1-j - (1-j)\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

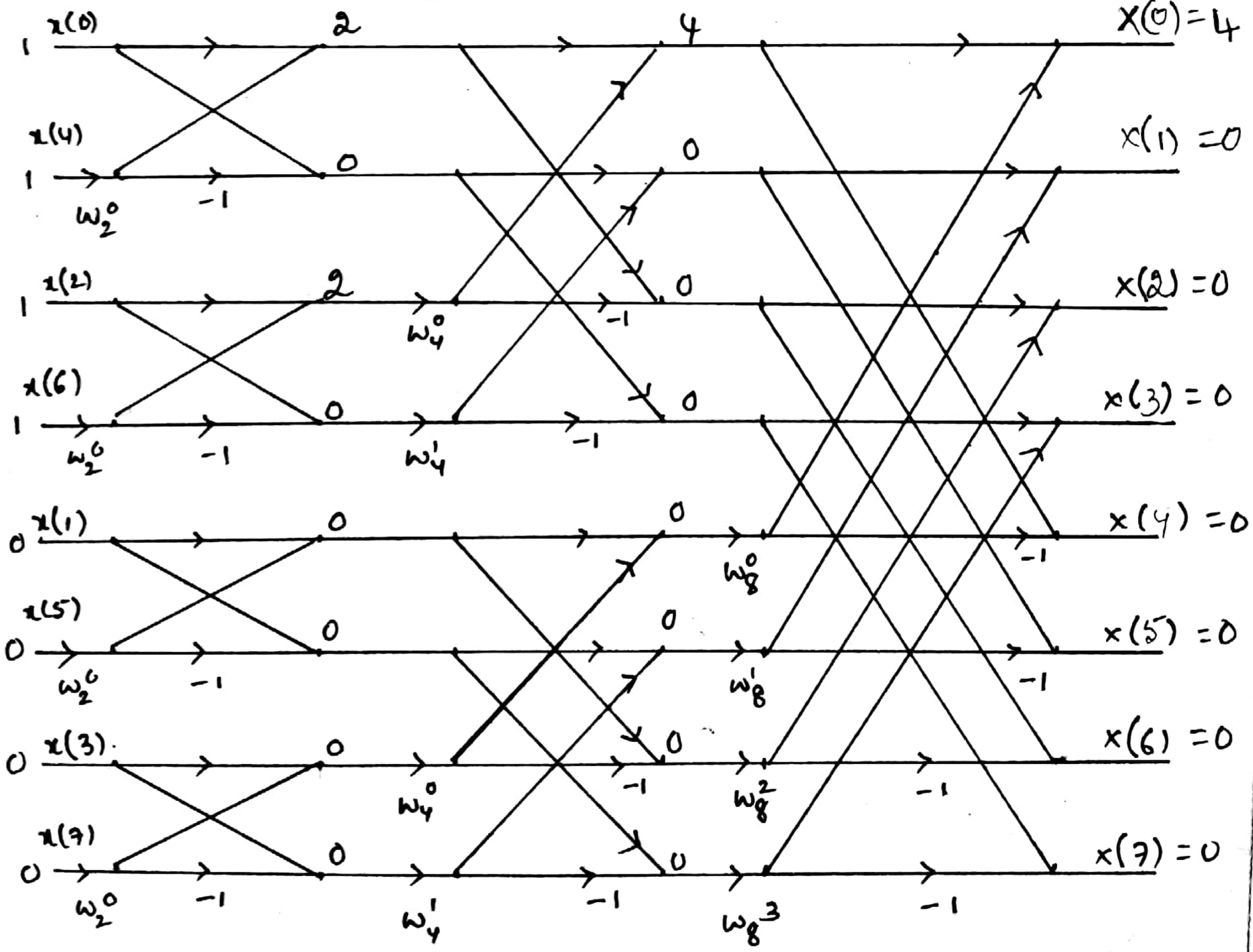
$= 1-j - \frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= 1 - j0.414$

$X(6) = 0$

$X(7) = (1+j) - (1+j)W_8^3 = 1 + j\,2.414$

2) Compute 8-point DFT of the sequence $x(n) = (1, 0, 1, 0, 1, 0, 1, 0)$ using decimation time radix-2 FFT algorithm.



$x(0) = 2+2W_8^0 = 2+2 = 4$

$x(1) = (1-j)+(1-j)W_8^1 = 1-j+(1-j)\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right) = 1-j2.414.$

$x(2) = 0$

$x(3) = (1+j)+(1+j)W_8^3 = 1+j+(1+j)\left(-\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

$= 1+j-\cancel{\frac{1}{\sqrt{2}}} - j\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}} + \cancel{\frac{1}{\sqrt{2}}}$

$= 1+j0.414.$

$x(4) = 2-2W_8^0 = 2-2 = 0.$

$x(5) = (1-j)-(1-j)W_8^1 = 1-j-(1-j)\left(\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}\right)$

$= 1-j-\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}$

$= 1-j0.414.$

$x(6) = 0$

$x(7) = (1+j)-(1+j)W_8^3 = \underline{\underline{1+j2.414.}}$

2) Compute 8-point DFT of the sequence

$x(n) = (1,0,1,0,1,0,1,0)$ using decimation in time radix - 2 FFT Algorithm.

Let the sample butterfly be as shown in the figure below -