For Butterworth filter's poles of normalized filter is given by

$$S_K = \Omega_c \, e^{j\frac{\pi}{2}} \cdot e^{j(2K+1)\frac{\pi}{2N}} \quad K = 0, 1 \ldots N-1$$

ie $$S_K = \Omega_c \, e^{j\left(\frac{\pi}{2} + (2K+1)\frac{\pi}{2N}\right)}$$

$$S_K = \Omega_c \, e^{j\theta_K}, \quad \theta_K = \frac{\pi}{2} + (2K+1)\frac{\pi}{2N}$$

Find θ_K for given N & plot.

$\underline{\Omega_c = 1 \text{ r/s}}$

$$S_K = e^{j\theta_K}$$

poles at S_K give $B_N(s)$

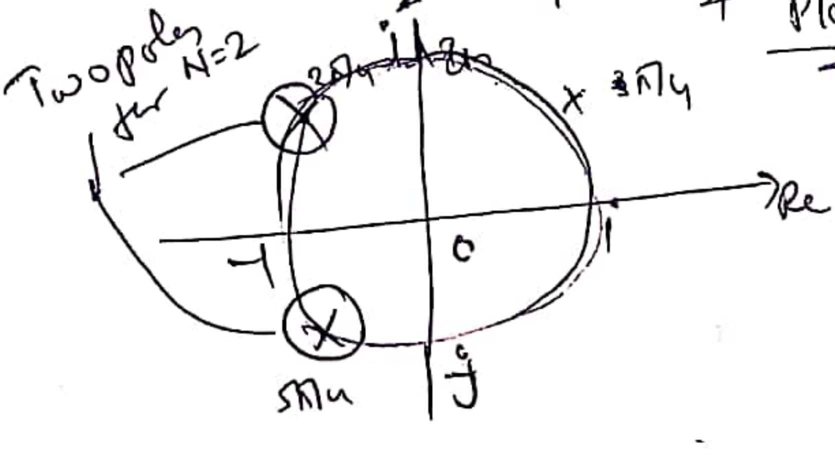
$$B_N(s) = \prod_{K=0}^{N-1} (s - S_K)$$

Ex $N = 2, \; K = 0, 1$

$$\theta_0 = \frac{\pi}{2} + \frac{\pi}{4} = \frac{3\pi}{4} \Rightarrow S_0 = \frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$$

$$\theta_1 = \frac{\pi}{2} + \frac{3\pi}{4} = \frac{5\pi}{4} \Rightarrow S_1 = \frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$$

Plot of θ_K (or) S_K



$\therefore$ for $N=2$, $s_0 = \frac{-1}{\sqrt{2}} + \frac{j}{\sqrt{2}}$, $s_1 = \frac{-1}{\sqrt{2}} - \frac{j}{\sqrt{2}}$

$$B_N(s) = (s - s_0)(s - s_1)$$

$$= \Big(\underbrace{s + \tfrac{1}{\sqrt{2}}}_{a} - \underbrace{\tfrac{j}{\sqrt{2}}}_{b}\Big)\Big(\underbrace{s + \tfrac{1}{\sqrt{2}}}_{a} + \underbrace{\tfrac{j}{\sqrt{2}}}_{b}\Big)$$

$$= (a+b) \quad (a-b)$$

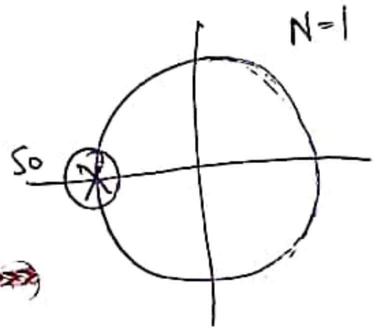
$$= a^2 - b^2$$

$$= \left(s + \frac{1}{\sqrt{2}}\right)^2 - \frac{j^2}{2} = \left(s + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}$$

$$= s^2 + \frac{1}{2} + \sqrt{2}s + \frac{1}{2}$$

$$\boxed{B_N(s) = s^2 + 1.414s + 1}$$

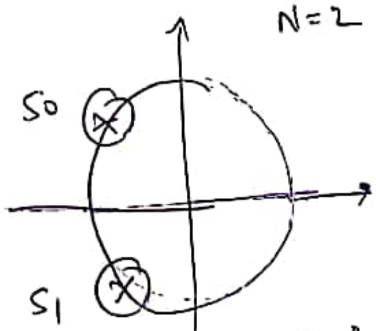
N=1



Single pole on Real axis

$B_N(s) = (s+1)$

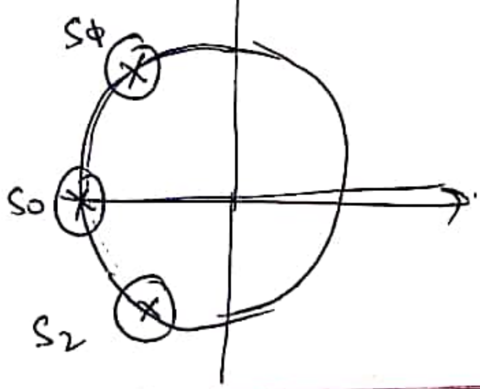
N=2



* $N \Rightarrow$ even $\Rightarrow \frac{N}{2}$ poles on each quadrant of left side
* $\frac{N}{2}$ complex conjugate poles

$B_N(s) = s^2 + \sqrt{2}s + 1$

N=3



* N = even $\Rightarrow$ one pole on Real axis

in Remaing $(N-1)$ poles

$\frac{N-1}{2}$ on each quadrant.

* $\frac{N-1}{2}$ complex conjugate poles

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Design a LP filter which has flat pass band & monotonic transition band. for the following specifications.

i) Pass band & SB frequencies (cal^r ω) of 500 r/s & 1000 r/s.

ii) Min pass band attenuation of −3 dB & max stop — " — " — −15 dB.

Soln. **Method − I**

using $H_N(s) \longrightarrow H_p(s) \longrightarrow H_a(s)$

step 1) Given specification

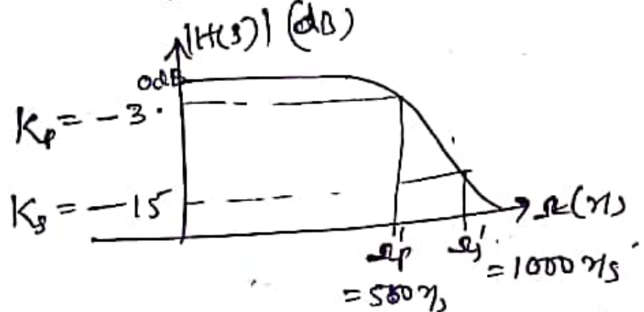

fig ①

Normalized resp

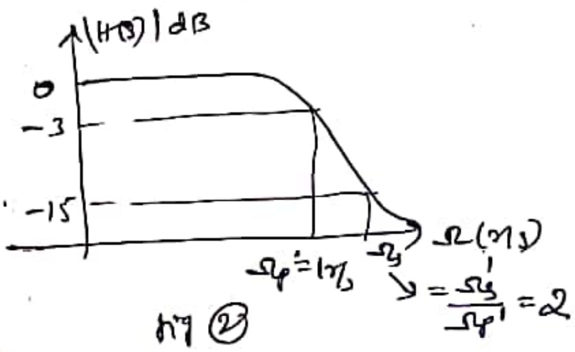

fig ②

step 2) Find order (N) & normalized cutoff freq (Ω_{cn}) using fig ②

$$N = \log\left[\frac{10^{-0.1k_p} - 1}{10^{-0.1k_s} - 1}\right] \div 2\log\left(\frac{\Omega_p}{\Omega_s}\right) =$$

$$= \log\left[\frac{10^{+0.3} - 1}{10^{1.5} - 1}\right] \div 2\log\left(\frac{1}{2}\right) = 2.4717$$

$\boxed{N = 3}$ Always approximate to higher integer

$$\Omega_{cn} = \frac{\Omega_p}{\left(10^{-0.1K_p} - 1\right)^{1/2N}} = \frac{1}{\left(10^{+0.3} - 1\right)^{1/6}}$$

$$\Omega_{cn} = 1 \text{ r/s}$$

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Step 3) To find $H_{N}(s)$

Since $N=3$ $\quad B_N(s) = s^3 + 2s^2 + 2s + 1$

$$\therefore H_N(s) = \frac{1}{s^3+2s^2+2s+1}$$

Step 4) Analog to analog transformation.

i) $H_n(s) \rightarrow H_p(s)$

$$H_p(s) = H_n(s) \Big|_{s = \frac{s}{\Omega_{cn}}} = \frac{s}{1 \text{ r/s}}$$

$$H_p(s) = \frac{1}{s^3+2s^2+2s+1}$$

ii) $H_p(s) \longrightarrow H_a(s) \longrightarrow$ LP to LP transformation

$$H_a(s) = H_p(s) \Big|_{s = \frac{s}{\Omega'_p}} = \frac{s}{500}$$

$$\therefore H_a(s) = \frac{1}{\left(\frac{s}{500}\right)^3 + 2\left(\frac{s}{500}\right)^2 + 2\left(\frac{s}{500}\right) + 1}$$

$$H_a(s) = \frac{(500)^3}{s^3 + 1000\,s + 2(500)^2 s + (500)^3}$$ //

Compare with Direct method.

Direct method. (Only for LP filter).

Step 1) Find order N & Ω_c by using given spec.

i.e. $$N = \log\left[\frac{10^{-0.1k_p}-1}{10^{0.1k_s}-1}\right] \div 2\log\left(\Omega'_p/\Omega'_s\right)$$

$$= \log\left[\frac{10^{-0.3}-1}{10^{-4.5}-1}\right] \div 2\log\left[\frac{500}{1000}\right]$$

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$$N = 2.4717 \simeq 3$$

$$\Omega_c = \frac{\Omega_p}{\left(10^{0.1K_p} - 1\right)^{1/2N}} = 500 \text{ r/s.}$$

step 2) Find $B_N(s)$ & $H_N(s)$

$$H_N(s) = \frac{1}{s^3 + 2s^2 + 2s + 1}$$

step 3) Transform $H_N(s)$ to $H_a(s)$

$$H_a(s) = H_N(s)\Big|_{s = \frac{s}{\Omega_c} = \frac{s}{500 \text{ r/s}}}$$

$$H_a(s) = \frac{500^3}{s^3 + 2(500)s^2 + 2(500)^2 s + 500^3} \;//$$

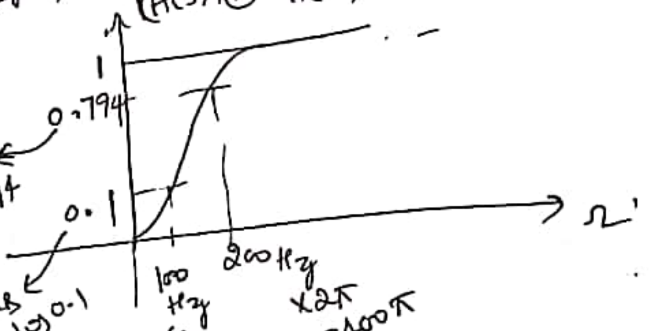
2) Design a HPF for the following specification.

i) Max passband attenuation not less than 0.794

ii) Stop band attenuation of not more than 0.1

iii) PB & SB edge freq of 200 & 100 Hz rly.

Soln: Step 1) given specifications



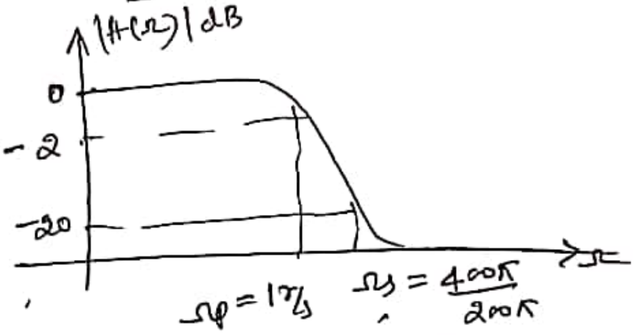
K_p in dB $= 20\log 0.794 = -2$dB

K_s in dB $= +20\log 0.1 = -20$ dB

$100 \text{ Hz} \times 2\pi = 200\pi$

$200 \text{ Hz} \times 2\pi = 400\pi$

Normalized LP spec



$\Omega_p = 1$ r/s

$\Omega_s = \frac{400\pi}{200\pi} = 2$

step 2} Find order N & cut off freq of normalized LPF using eq (2).

$$N = \log\left[\frac{10^{-0.1K_p}-1}{10^{-0.1K_s}-1}\right] \div 2\log\left(\frac{\Omega_p}{\Omega_s}\right)$$

$$= \log\left[\frac{10^{-0.1(-2)}-1}{10^{-0.1(-20)}-1}\right] \div 2\log\left(\frac{1}{2}\right)$$

$$= 3.7016$$

$\boxed{N \simeq 4}$

$$\Omega_{cn} = \frac{\Omega_p}{(10^{-0.1K_p}-1)^{1/8}} = 1.069 \text{ r/s}$$

step 3} To find $H_n(s)$ (using analog to analog transformation)

$$H_n(s) = \frac{1}{B_n(s)} = \frac{1}{(s^2+0.7653s+1)(s^2+1.8477s+1)}$$

$$H_p(s) = H_n(s)\Big|_{s=\frac{s}{\Omega_{cn}}} \rightarrow \text{normalized LP to proto-type LP}$$

$$H_a(s) = H_p(s)\Big|_{s=\frac{\Omega'_p}{s}} = H_n(s)\Big|_{s=\frac{\Omega_p'}{\Omega_{cn}s}} \text{ r/s}$$

$\rightarrow$ LP to HP transformation.

$$\therefore H_a(s) = \frac{s^4}{s^4+488.713s^3+119.42\times10^3 s^2+17.095\times10^6 s+1.2236\times10^8}$$

//

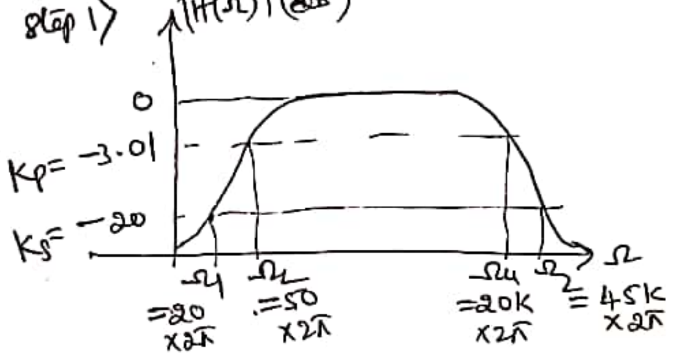
3) Design analog BP butterworth filter to meet following spec.

i) −3.01dB upper & lower cutoff freq. is 20KHz & 50Hz rly.

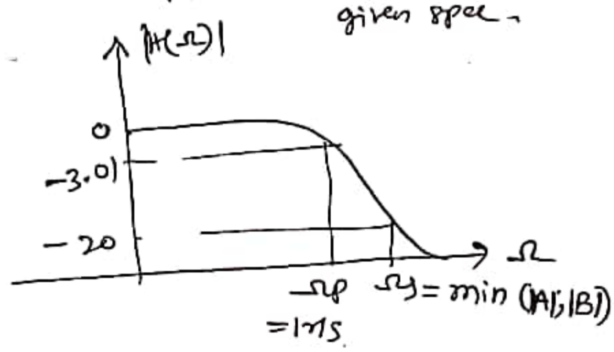
ii) A stop band attenuation is at least 20dB at 20Hz & 45 kHz.

Sol. Given spec.

step 1)



normalized LPF from given spec.



$\Omega_1 = 40\pi$ r/s $\Omega_2 = 90\pi$ r/s, $\Omega_l = 50\times2\pi$, $\Omega_u = 400\pi$ r/s

$B_0 = \Omega_u - \Omega_l = 39900\pi$

$\Omega_0^2 = \Omega_l \Omega_u = 4\pi^2\times10^6$

$$A = \frac{-\Omega_1^2 + \Omega_0^2}{B_0 \Omega_1} = 2.5053$$

$$B = \frac{\Omega_2^2 - \Omega_0^2}{B_0 \Omega_2} = 2.2545$$

$\therefore \Omega_s = \min(|A|, |B|) = 2.2545$ r/s.

step 2) Find order ~~N~~ 6

$$N = \log\left[\frac{10^{-0.1K_p} - 1}{10^{-0.1K_s} - 1}\right] \div 2\log(\Omega_p/\Omega_s)$$

$$= \log\left[\frac{10^{+0.301} - 1}{10^{+2} - 1}\right] \div 2\log(1/2.2545) = 2.8363.$$

$$\boxed{N = 3}$$

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step 3: To find $B_n(s)$

For $N=3$, $B_N(s) = s^3+2s^2+2s+1$

$$\therefore H_N(s) = \frac{1}{s^3+2s^2+2s+1}$$

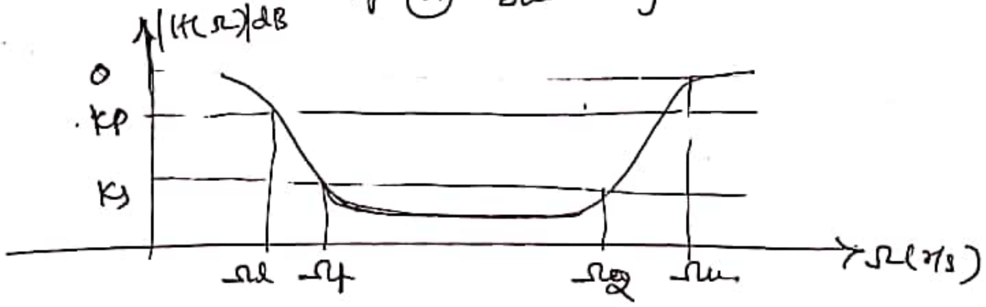
step.4: Find $H_a(s)$ from $H_N(s)$ (Analog to analog transf)

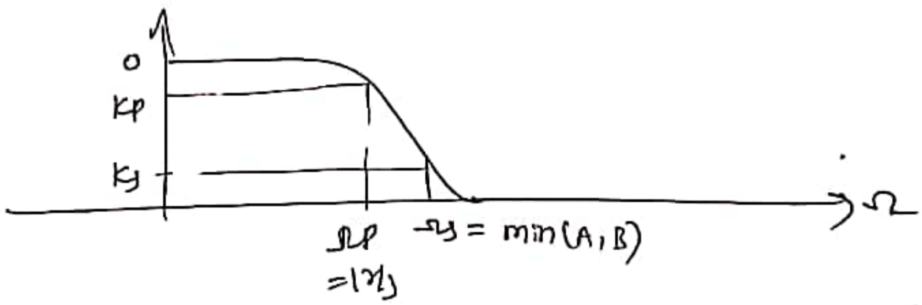
ie $$H_a(s) = H_n(s)\Big|_{s=\frac{s^2+\Omega_0^2}{B_0 s}} = \frac{s^2+4\pi^2\times10^6}{39900\pi s}$$

$$= \frac{s^2+39.47\times10^6}{125.34\times10^{-3}\, s}$$

$H_a(s)$ = simplify

IIIly for Band stop @ band reject filters





$$A = \frac{B_0\Omega_1}{-\Omega_1^2+\Omega_0^2}$$

$$B = \frac{B_0\Omega_2}{\Omega_2^2-\Omega_0^2}$$

$\Omega_0^2 = \Omega_l\Omega_u$

$B_0 = (\Omega_u-\Omega_l)$

Transformation (LP to BP)

replace $s = \dfrac{B_0 s}{s^2+\Omega_0^2}$ in $B_n(s)$.

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