Decimation-in-Frequency Algorithm

DIT algorithm is based on the decomposition of the DFT computation by forming smaller and smaller subsequences of the sequence $x(n)$.

In decimation in frequency algorithm the frequency domain sequence X(k) is decimated .

Consider a N-point sequence x(n)

$$
\begin{aligned}
X(k) &= \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{n=N/2}^{N-1} x(n) W_N^{nk} \\
&= \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{(n+N/2)k} \\
&= \sum_{n=0}^{N/2-1} x(n) W_N^{nk} + \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{nk} W_N^{nk/2} \\
&= \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^k x(n+N/2) \right] W_N^{nk}
\end{aligned}
$$

$$X(k) = \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^k x(n+N/2) \right] W_N^{nk}$$

For even values of k, the $X(k)$ can be written as

$$X(2k) = \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^{2k} x(n+N/2) \right] W_N^{2nk}$$

$$= \sum_{n=0}^{N/2-1} \left[x(n) + x(n+N/2) \right] W_{N/2}^{nk} \quad k = 0,1,2,\ldots(N/2-1)$$

For odd values of k, the $X(k)$ can be written as

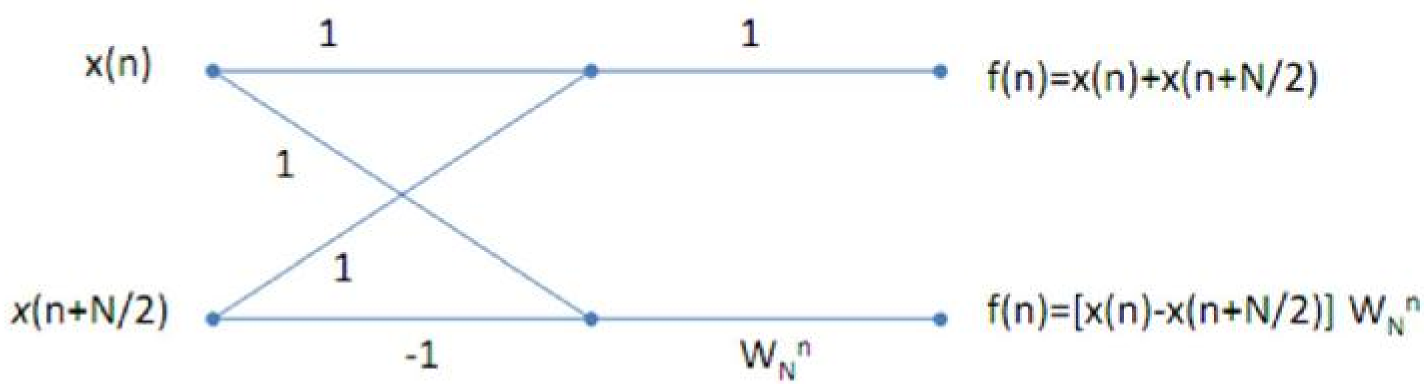
$$X(2k+1) = \sum_{n=0}^{N/2-1} \left[x(n) + (-1)^{(2k+1)} x(n+N/2) \right] W_N^{(2k+1)n}$$

$$= \sum_{n=0}^{N/2-1} \left[x(n) - x(n+N/2) \right] W_N^{2nk} \, W_N^{nk} \quad k = 0,1,2,\ldots(N/2-1)$$

$$= \sum_{n=0}^{N/2-1} \left[x(n) - x(n+N/2) \right] W_N^{n} \, W_{N/2}^{nk}$$

$$f(n)=[x(n)+x(n+N/2)]$$
$$g(n)=[(n)-x(n+N/2)]W_N^n$$

We find that the even and odd samples of the DFT can be obtained from the $N/2$ point DFTs of $f(n)$ and $g(n)$ respectively



Flow graph of basic butterfly diagram for DIF algorithm

Consider N=8,

$$X(0)=\sum_{n=0}^{2} f(n);\quad X(2)=\sum_{n=0}^{3} f(n)W_8^{2n};$$

$$X(4)=\sum_{n=0}^{3} f(n)W_8^{4n}, \sum_{n=0}^{3} f(n)(-1)^n\ ; X(6)=\sum_{n=0}^{3} f(n)W_8^{6n}, \sum_{n=0}^{3} f(n)(-W_8^2)^n$$

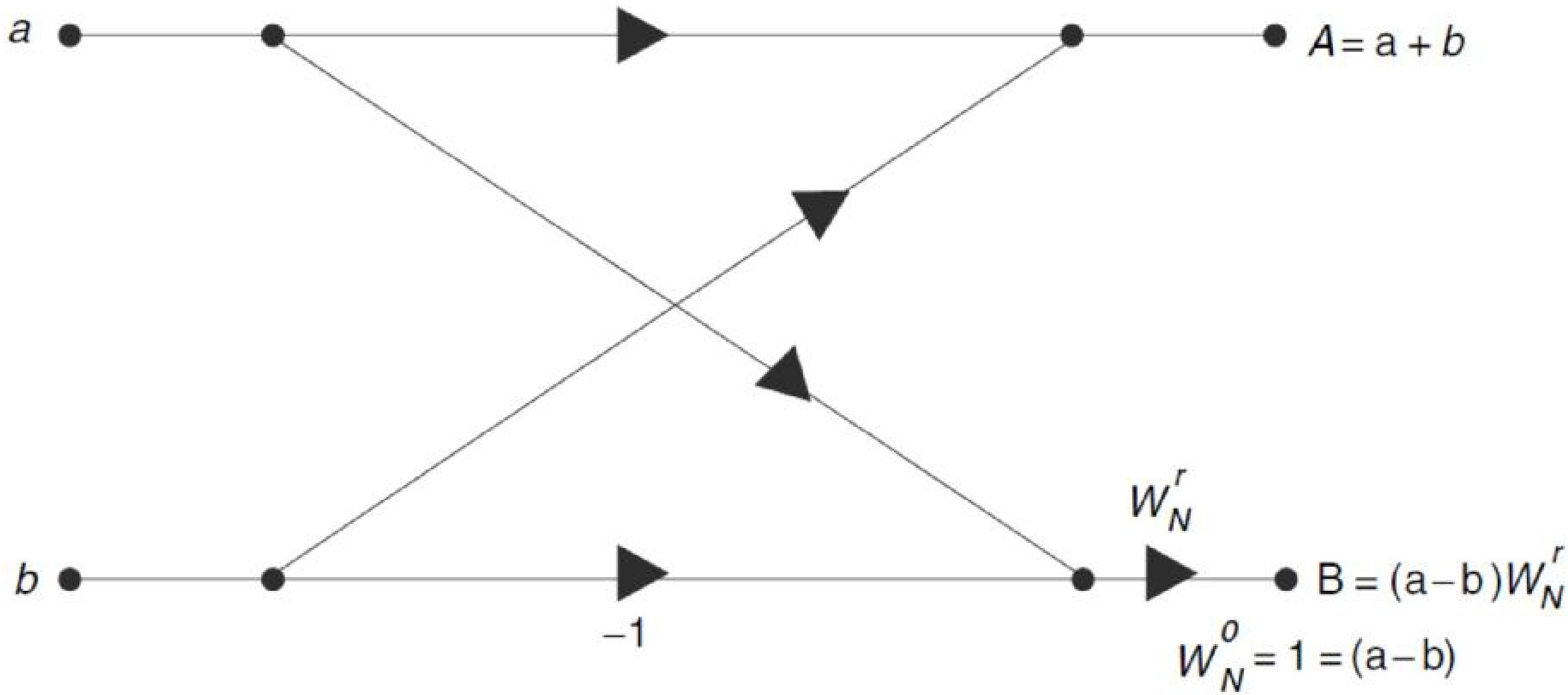


Fig. 6.10 *Basic Butterfly for DIF FFT*

$$X(1)=\sum_{n=0}^{2} g(n);\quad X(3)=\sum_{n=0}^{3} g(n)W_8^{3n};$$

$$X(5)=\sum_{n=0}^{3} g(n)W_8^{5n}, \sum_{n=0}^{3} g(n)(-1)^n\,; X(7)=\sum_{n=0}^{3} g(n)W_8^{7n}, \sum_{n=0}^{3} g(n)(-W_8^2)^n$$

First stage of computation, two number of 4-point sequences $f(n)$ and $g(n)$ are obtained.

$$f(n)=[x(n)+x(n+N/2)]$$
$$g(n)=[(n)-x(n+N/2)]W_N^n$$

Even indexed samples of $X(k)$ can be obtained from the 4-point DFT of the sequence $f(n)$

$$f(0)=x(0)+x(4)$$
$$f(2)=x(1)+x(5)$$
$$f(4)=x(2)+x(6)$$
$$f(6)=x(3)+x(7)$$

- Odd indexed samples of *X(k) can be obtained from the 4-point DFT of the* sequence *g(n)*

$$g(0)=[x(0) - x(4)]W_8^0$$
$$g(1)=[x(1) - x(5)]W_8^0$$
$$g(2)=[x(2) - x(6)]W_8^0$$
$$g(3)=[x(3) - x(7)]W_8^0$$

Steps for radix – 2 DIF FFT algorithm

1. The number of input samples N=2^M, where M is an integer.
2. The input sequence is in natural order.
3. The number of stages in the flowgraph is given by $M = \log_2 N$.
4. Each stage consists of $N/2$ butterflies.
5. Inputs/outputs for each butterfly are separated by 2^{M-m} samples, where m represents the stage index.
6. The number of complex multiplications is given by $N/2 \log_2 N$
7. The number of complex additions is given by $N \log_2 N$
8. The twiddle factor exponents are a function of the stage index m and is given by $k=Nt/2^{M-m+1}$, $t = 0,1,2,\ldots 2^{M-m} - 1$
9. The number of sets or sections of butterflies in each stage is given by the formula 2^{M-1}
10. The exponent repeat factor (ERF), which is the number of times the exponent sequence associated with m is repeated is given by 2^{M-1}

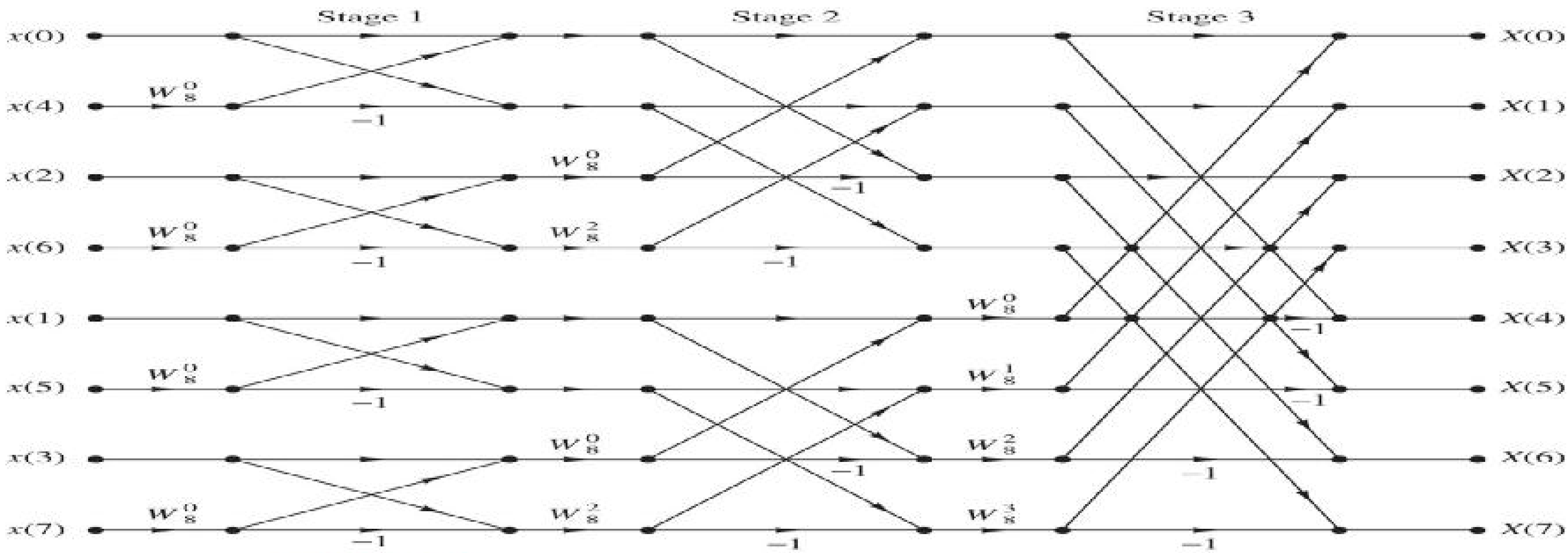


Figure 8.1.6 Eight-point decimation-in-time FFT algorithm.

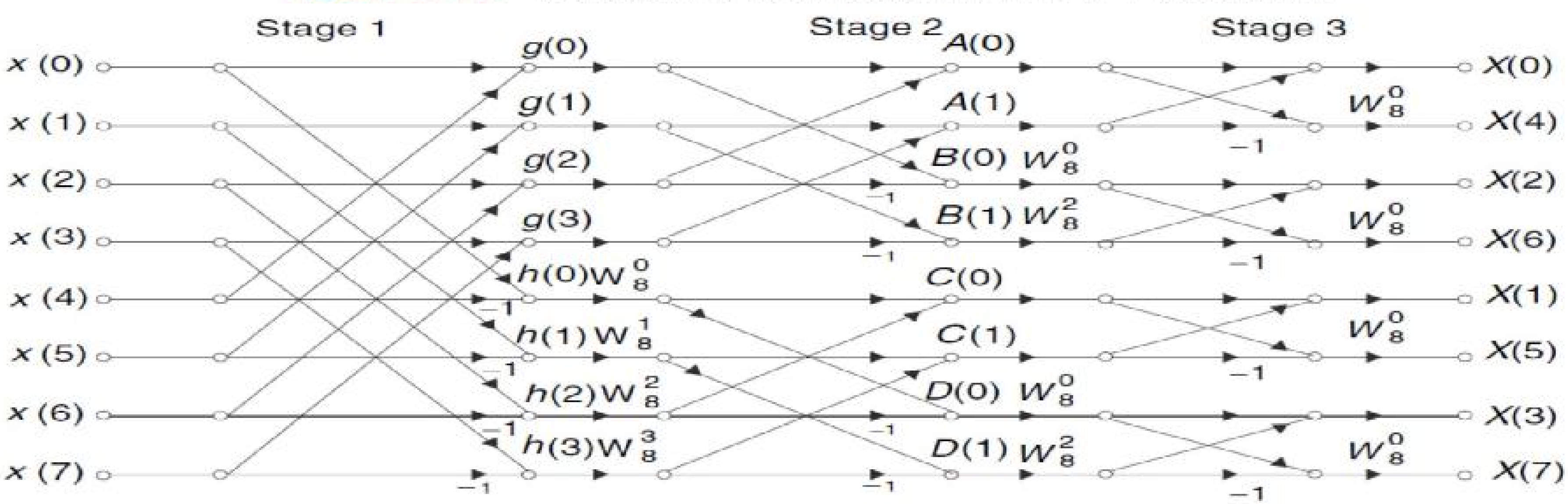


Fig. 6.11 *Reduced Flow Graph of Final Stage DIF FFT for N = 8*

Reduced Flow Graph DIF FFT for N = 8

DIT vs. DIF

Differences

- For DIT, the input is bit-reversed, while the output is in natural order.
- Whereas, for DIF, the input is in natural order while the output is in bit reversed.
- The DIF butterfly is slightly different from the DIT wherein DIF the complex multiplication takes place after the add-subtract operation.

Similarities

- Both algorithms requires Nlog2N operations to compute the DFT.
- Both algorithms can be done in-place and both need to perform bit reversal at some place during computation.

First stage of 8-point DIF FFT

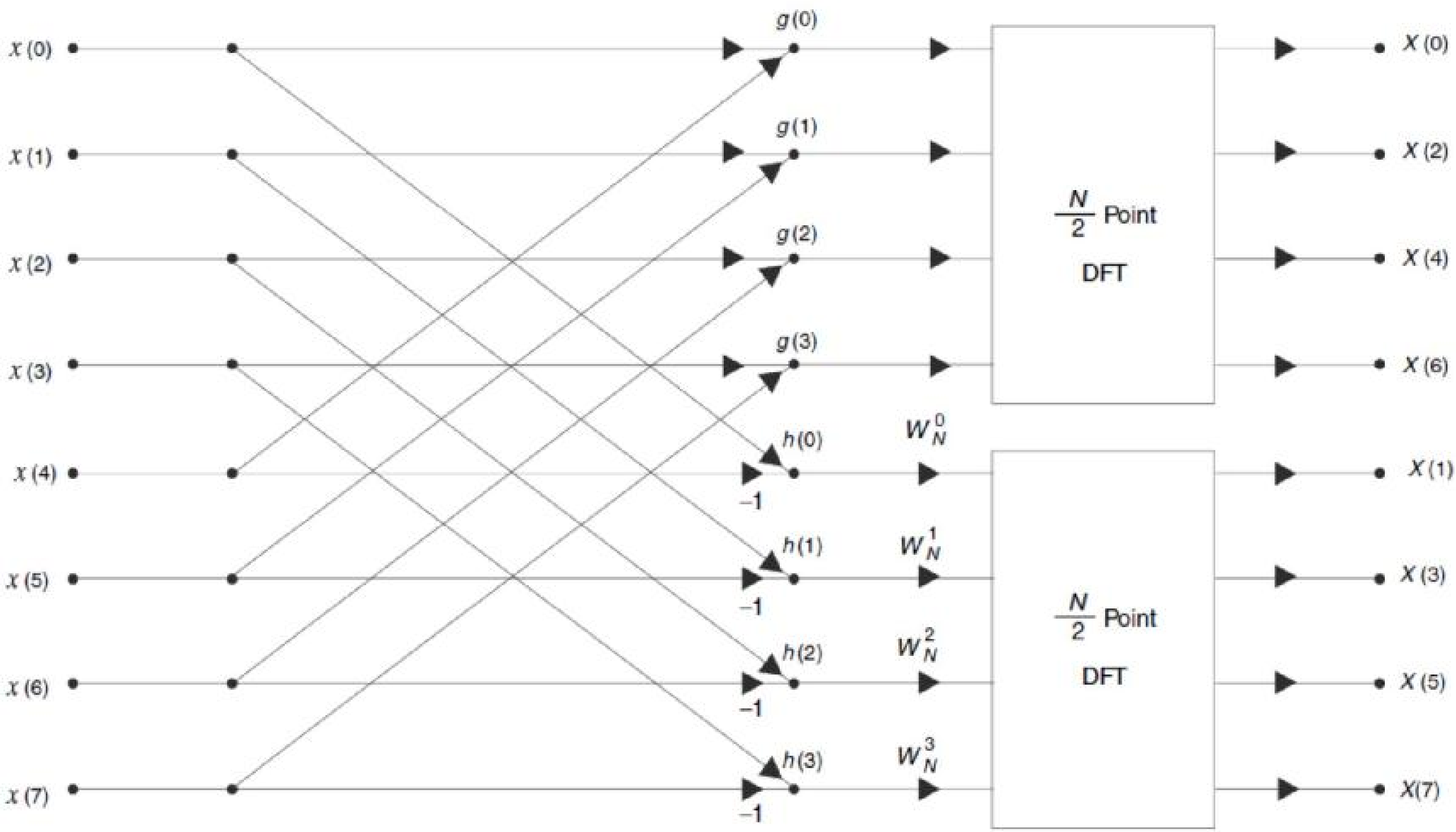


Fig. 6.8 *Flow Graph of the First Stage of Decimation-In-Frequency FFT for N = 8*

Second stage of 8-point DIF FFT

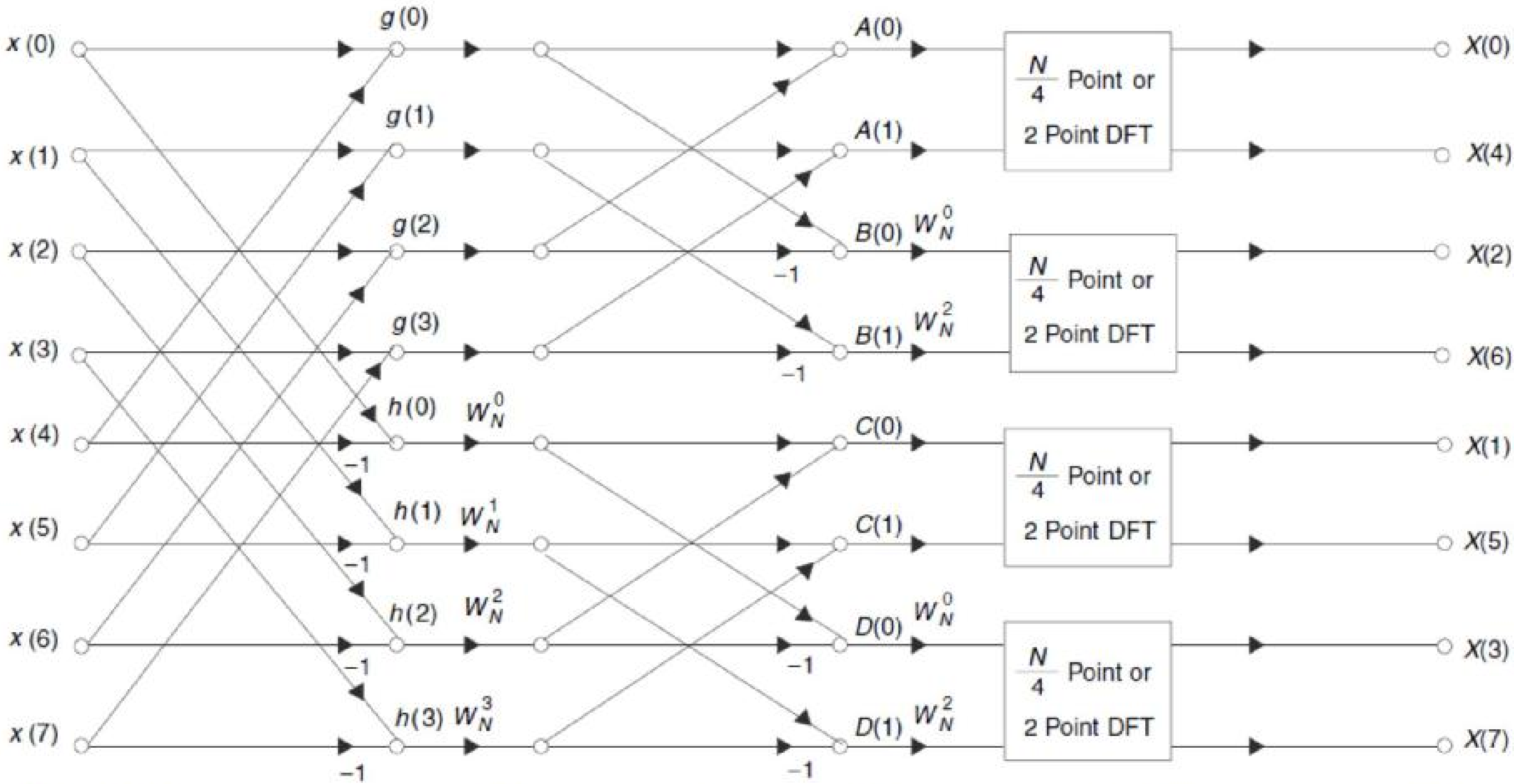


Fig. 6.9 *Flow Graph of the Second Stage of Decimation-In-Frequency FFT for* $N = 8$

Reduced Flow Graph DIF FFT for N = 8

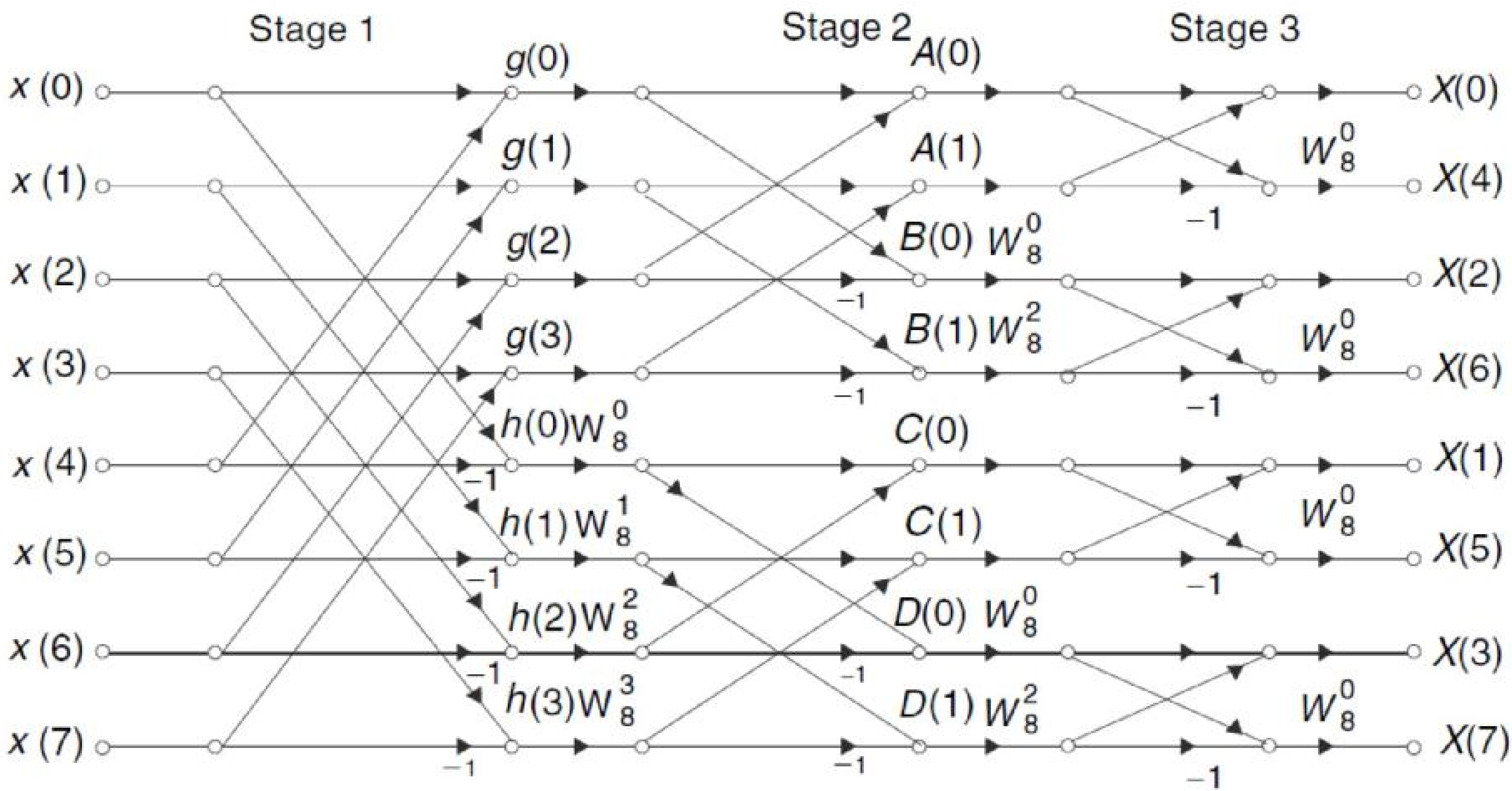
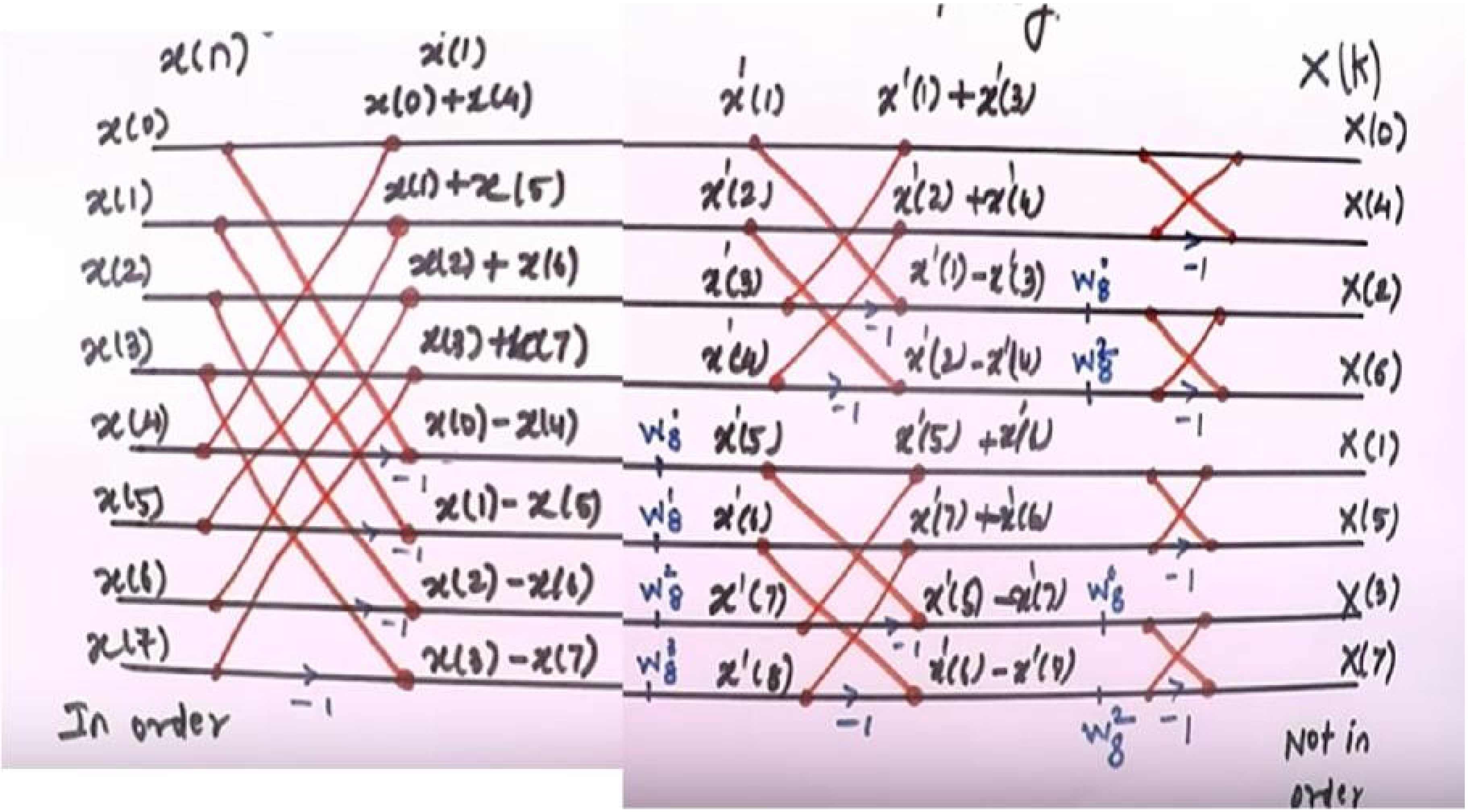


Fig. 6.11 *Reduced Flow Graph of Final Stage DIF FFT for N = 8*

DIF FFT



x(n)
x(0)
x(1)
x(2)
x(3)
x(4)
x(5)
x(6)
x(7)
In order
x'(1)
x(0)+x(4)
x(1)+x(5)
x(2)+x(6)
x(3)+x(7)
x(0)-x(4)
x(1)-x(5)
x(2)-x(6)
x(3)-x(7)
-1
x'(1)
x'(2)
x'(3)
x'(4)
x'(5)
x'(6)
x'(7)
x'(8)
x'(1)+x'(3)
x'(2)+x'(4)
x'(1)-x'(3)
x'(2)-x'(4)
x'(5)+x'(7)
x'(6)+x'(8)
x'(5)-x'(7)
x'(6)-x'(8)
X(k)
X(0)
X(4)
X(2)
X(6)
X(1)
X(5)
X(3)
X(7)
Not in order

Question

- Given $x(n)=\{1,2,-1,2,4,2,-1,2\}$ find $X(k)$ using DIF FFT.

Inverse Fast Fourier Transform (IFFT)

Inverse Fast Fourier Transform (IFFT)

- An FFT algorithm for calculating the DFT samples can also be used to evaluate efficiently the inverse DFT (IDFT).

- The inverse DFT is given by

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) W_N^{-nk}, \; n = 0,1, \ldots N-1$$

- Taking this complex conjugate of above equation.

$$N x^*(n) = \sum_{k=0}^{N-1} X^*(k) W^{nk}, \; n = 0,1, \ldots N-1$$

- RHS is DFT of sequence *X*(k).* Therefore,

$$x^*(n) = (1/N)DFT[X^*(k)]$$

- Taking the complex conjugate of both side, the desired output sequence x(n) which is given by

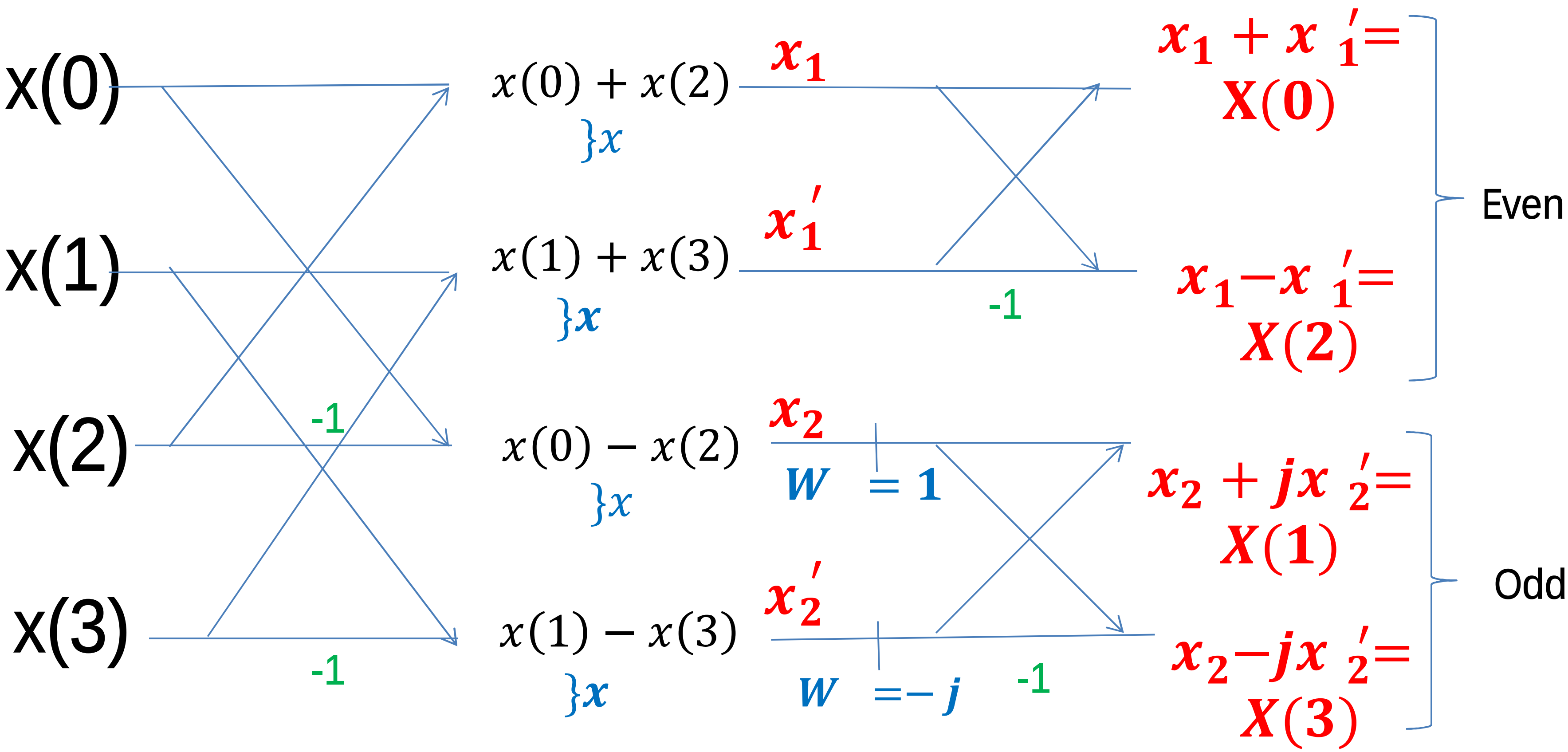
$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N-1} X^*(k)W^{nk}\right]^*$$

$$x(n) = \frac{1}{N}[FFT[X^*(k)]]^*$$

- Use to compute IDFT
  - if the output is divided by N
  - The “twiddle factors” are negative power of W_N
- IFFT flow graph obtained from an FFT flow graph by
  - replacing all the x(n) by X(k),
  - dividing the input data by N, or
  - dividing each stage by 2 when N is power of 2
  - Changing the exponents of W_N to negative values.
- DIT IFFT and DIF IFFT are same.

4-point IFFT Flow Graph

X(n) = { x(0), x(1), x(2), x(3)}



Example: 4-point IFFT

Question:

Given $X(k) = \{10, -2+2j, -2, -2-2j\}$, find x(n) using DIT IFFT algorithm.

Solution:

N = 4

$$W_N^{-k} = e^{j\frac{2\pi}{N}k}$$

$$W_4^{-0} = 1$$

$$W_4^{-1} = e^{j\frac{2\pi}{4}(1)} = e^{j\frac{\pi}{2}} = j$$

Answer: $x(n) = \{1\ 2\ 3\ 4\}$

8 point IFFT

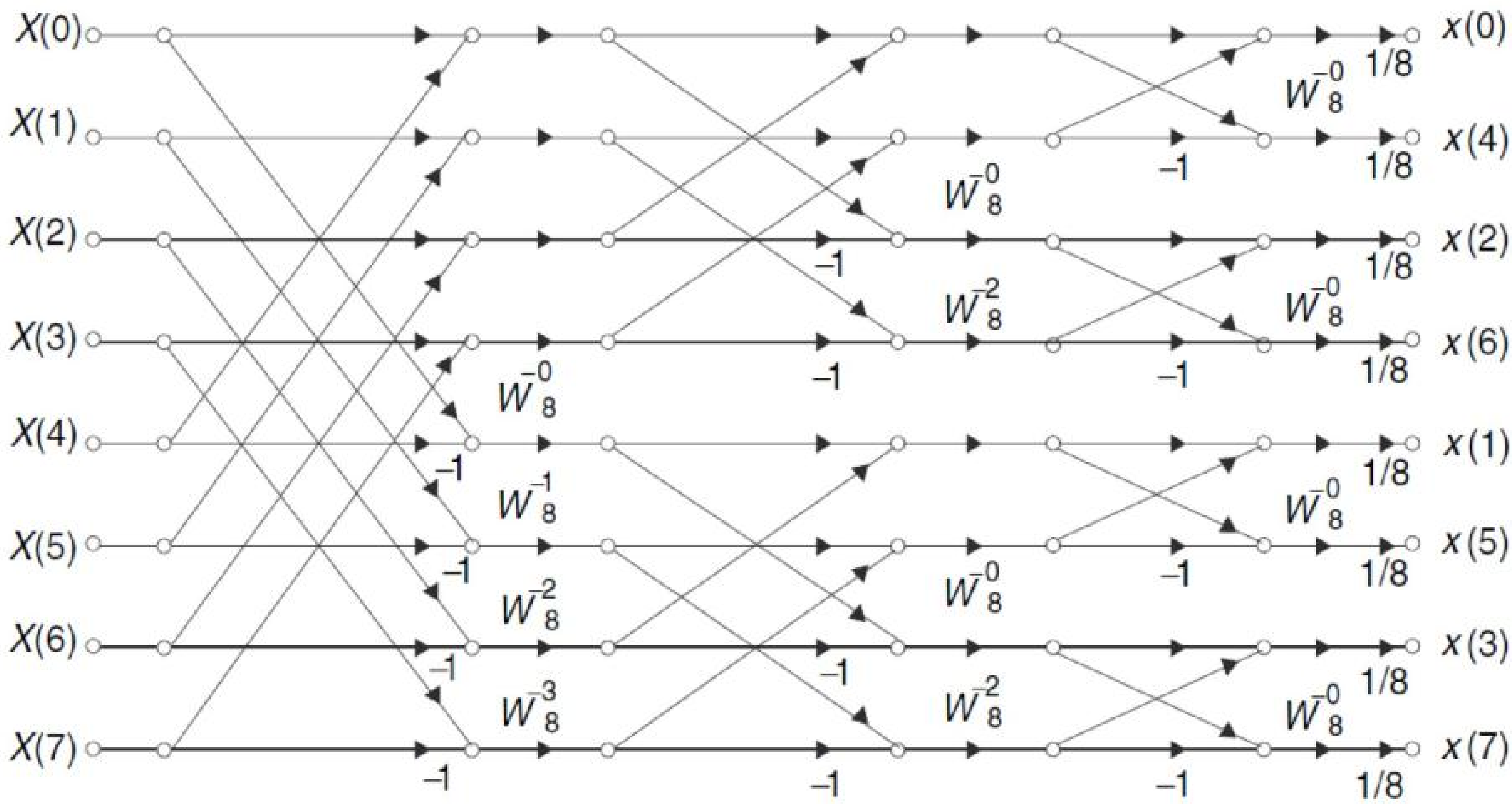


Fig. 6.12

Example

Given X(k)={-1,3,-9,3,-1,3,-9,3}, find x(n) using DIT IFFT algorithm.

Solution:

N = 8

Answer: $x(n) = \{-1,0,2,0,-4,0,2,0\}$

Example (from DIF)

Given X(k)={11,-3,7,-3,-5,-3,7,-3}, find x(n) using DIF IFFT algorithm.

Solution:

N = 8

Answer: $x(n) = \{1,2,-1,2,4,2,-1,2\}$