MODULE 2

ERROR CONTROL CODE

Contents

- **Introduction**
- Linear Block codes-G and H matrix
- Generation and Decoding
- Cyclic codes-g(x) and h(x)
- Encoding using an (n-k) bit shift register
- syndrome calculation
- Convolution Codes: Code generation-Time domain and Transfer domain approach and Code Tree

Introduction

The purpose of error control coding is to enable the receiver to detect or even correct the errors by introducing some redundancies in to the data to be transmitted.

There are basically two mechanisms for adding redundancy:

1. Block coding
2. Convolutional coding

Types of codes

i) Block Codes:

Block code consists of **(n-k)** number of check bits(redundant bits) being added to **k** number of information bits to form ‘n’ bit code-words.

i) Convolutional code:

In this code, input databits are fed as streams of data bits which convolve to output bits based upon the logic function of the encoder.

Linear Block codes

- Let C_1 and C_2 be any two code words(n-bits) belonging to a set of (n, k) block code

- If C_1 C_2 , is also a n-bit code word belonging to the same set of(n,k) block code, such a block code is called (n,k)linear block code.

Illustrating the formation of linear block codes

2 distinct message

Matrix description of linear block code

- Let the message block of k-bits(code-words) be represented as a "row-vector" or "k-tuple" called "message vector" is given by

  $[D]=\{ d_1, d_2, ...d_k\}$

- 2^k code-vectors can be represented by

  $C=\{c_1,c_2,.......c_n\}$

- Also $c_i=d_i$ for all i=1,2,......k
- $[C]= \{c_1,c_2,........,c_k, c_{k+1}, c_{k+2},.......c_n \}$

- (n-k) number of check bits $c_{k+1}, c_{k+2}, \ldots\ldots. c_n$ are derived from 'k' message bits using a predetermined rule as below

$$c_{k+1} = p_{11}d_1 + p_{21}d_2 + \ldots\ldots\ldots\ldots\ldots + p_{k1}d_k$$

$$c_{k+2} = p_{12}d_1 + p_{22}d_2 + \ldots\ldots\ldots\ldots\ldots + p_{k2}d_k$$

:

:

$$c_{k+1} = p_{1.\,n-k}d_1 + p_{2.n-k}d_2 + \ldots\ldots\ldots\ldots\ldots + p_{k.n-k}d_k$$

- In matrix form,

$$[c_1, c_2, .., c_k, \; c_{k+1}, c_{k+2}, \ldots c_n] = [d_1, d_2, \ldots.. d_k] \begin{bmatrix} 1\;0\;0 \ldots 0 & p_{11}\; p_{12} \ldots.\; p_{1.\,n-k} \\ 0\;1\;0 \ldots 0 & p_{21}\; p_{22} \ldots.\; p_{2.\,n-k} \\ & \\ 0\;0\;0 \ldots & p_{k1}\; p_{k2} \ldots.\; p_{k.\,n-k} \end{bmatrix}$$

[C]= [D] [G]

- [G] is called as ***generator matrix*** of order (k x n)
- $[G] = [I_k \mid P]_{(k \times n)}$ where I_k unit matrix of order 'k'

  [P]= Parity matrix of order k x (n-k)

- Also $[G] = [P \mid I_k]$

PARITY CHECK MATRIX [H]

- $[H] = [P^T \mid I_{n-k}]$

$$\therefore [H] = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{k1} & 1\ 0\ 0 \ldots\ldots.0 \\ p_{12} & p_{22} & \cdots & p_{k2} & 0\ 1\ 0 \ldots\ldots.0 \\ . & & & & \\ . & & & & \\ p_{1,\ n-k} & p_{2,\ n-k} & \cdots & P_{k,n-k} & 0\ 0\ 0\ \ldots..1 \end{bmatrix}$$

[H] matrix is a (n-k) x (n) matrix.

For a systematic (6,3) linear block code, the parity matrix P is given by

$$[P] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Find all possible code-vectors

- Solution:

Given n=6, k=3,

Since k=3, 2^k =8 message vectors given by (000)

(001), (010), (001), (011), (100), (101), (111)

- [C]= [D] [G]

where $[G] = [I_k \mid P]$

$[I_3 \mid P]$

$$[G]= \begin{bmatrix} 1 & 0 & 0 & : & 1 & 0 & 1 \\ 0 & 1 & 0 & : & 0 & 1 & 1 \\ 0 & 0 & 1 & : & 1 & 1 & 0 \end{bmatrix}$$

- [C]=[D] [G]

$$=[d_1\ d_2\ d_3]\begin{bmatrix}1&0&0&1&0&1\\0&1&0&0&1&1\\0&0&1&1&1&0\end{bmatrix}$$

$$=[d_1, d_2, d_3, (d_1+d_3), (d_2+d_3), (d_1+d_2)]$$

C=[d1, d2,d3, (d1+d3), (d2+d3),(d1+d2)]

Code name	Message-vector	code-vector for (6,3) linear block code
C_a	000	000000
C_b	001	001110
C_c	010	010011
C_d	011	011101
C_e	100	100101
C_f	101	101011
C_g	110	110110
C_h	111	111000

- $C_c + C_g = (010011) + (110110)$
  $= (100101) = c_e$

For a systematic (7,4) linear block code, generated by

$$G=\left[\begin{array}{cccc|ccc} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{array}\right]$$

Find all possible code vectors.

Solution: $n=7$, $k=4$; $(n-k)= 3$

$\therefore\ 2^k = 2^4 = 16$ message ve

$[C]= [D]\ [G] =$ $\mathbf{[d_1\ \ d_2\ \ d_3\ \ d_4]}$

$$\mathbf{=[d_1\ ,\ d_2\ ,\ d_3\ ,\ d_4\ ,(d_1 + d_2 + d_3), (d_1 + d_2 + d_4), (d_1 + d_3 + d_4)\ ]}$$

Message Vector				Code vector						
d_1	d_2	d_3	d_4	c_1	c_2	c_3	c_4	c_5	c_6	c_7
0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	1	1
0	0	1	0	0	0	1	0	1	0	1
0	0	1	1	0	0	1	1	1	1	0
0	1	0	0	0	1	0	0	1	1	0
0	1	0	1	0	1	0	1	1	0	1
0	1	1	0	0	1	1	0	0	1	1
0	1	1	1	0	1	1	1	0	0	0
1	0	0	0	1	0	0	0	1	1	1
1	0	0	1	1	0	1	0	0	1	0
1	0	1	0	1	0	1	1	0	0	1
1	0	1	1	1	1	0	0	0	0	1
1	1	0	0	1	1	0	1	0	1	0
1	1	0	1	1	1	1	0	1	0	0
1	1	1	0	1	1	1	0	1	0	0
1	1	1	1	1	1	1	1	1	1	1

If C is a valid code vector, namely C=[D G]. Then prove that $CH^T = 0$ where H^T is the transpose of the parity check matrix H.

Wkt

$$\begin{bmatrix} 1\ 0\ 0 \ldots\ldots 0 & p_{11}\ p_{12}\ \ldots\ p_{1.\,n-k} \\ 0\ 1\ 0 \ldots\ldots 0 & p_{21}\ p_{22}\ \ldots\ p_{2.\,n-k} \\ 0\ 0\ 0 \ldots\ldots 1 & p_{k1}\ p_{k2}\ \ldots\ p_{k.\,n-k} \end{bmatrix}$$

i^{th} row of [G] matrix is given by

$$\mathbf{g}_{\mathbf{i}} = [0\ \ 0\ 0 \ldots\ldots \underset{i^{th}\text{ element}}{1} \ldots\ldots 0\ \ p_{i1}\ \ p_{i2}\ \ldots\ \underset{(k+j)^{th}\text{ element}}{p_{ij}}\ \ldots\ldots p_{k.\,n-k}\ ]$$

J^{th} row of [H] matrix is given by

$$h_j = [p_{1j}\quad p_{2j} \ldots\ldots p_{ij}\quad \ldots\ p_{kj}\quad 0\ 0\ 0 \ldots\ldots 1 \ldots\ldots 0\ ]$$

$$g_i.h_j^T = [0\ \ 0\ 0 \ldots\ldots 1 \ldots\ldots 0\ \ p_{i1}\ \ p_{i2} \ldots\ p_{ij} \ldots\ldots p_{k.\,n-k}\ ] . [p_{1j}\ \ p_{2j} \ldots\ldots p_{ij}\ \ldots\ p_{kj}\ \ 0\ 0\ 0 \ldots\ldots 1 \ldots\ldots 0\ ]^T$$

$$= [0\ \ 0\ 0 \ldots\ldots 1 \ldots\ldots 0\ \ p_{i1}\ \ p_{i2} \ldots\ p_{ij} \ldots\ldots p_{k.\,n-k}\ ] \begin{bmatrix} p_{1j} \\ p_{2j} \\ : \\ p_{ij} \\ : \\ p_{kj} \\ 0 \\ 0 \\ : \\ 1 \\ : \\ 0 \end{bmatrix}$$

ith element

(k+j)th element

Modulo-2 multiplication yields ; $g_i.h_j^T$

Error correction and Syndrome

- Let us suppose that

  $C=(c_1\ c_2\ ....c_n)$ be a valid code-vector transmitted over a noisy communication channel belonging to a (n,k) linear block code.
- Let $R=\{r_1\ r_2........r_n\ \}$ be the received vector.
- Error-vector or error pattern E is defined as difference between R and C.

  E=R-C ....**(1)**

  $E=(\ e_1\ \ e_2\ \ ....e_n\ )$ ....**(2)**

  $e_i=1$ if $R\neq C$

  $e_i=0$ if $R=C$

(the 1's present in the error-vector 'E' represent the errors caused by noise in the channel)

- Receiver does the decoding operation by determining an (n-k) vector S defined as $S=RH^T$ ....**(3)**

  $= (s_1\ \ s_2\ \ \ ....s_{n-k}\ )$

  The (n-k) vector S is called **'error syndrome'** of R .

  From eqn (1) ; $R=C+E$

  $\therefore\ S=(C+E)\,H^T$

  $= C\,H^T + E\,H^T$

  $S = E\,H^T$

note: when $R\neq C$ then $S \neq 0$.

1.For a systematic (6,3) code, find all the transmitted code vector , draw the encoding circuit. If received vector R=[110010], detect and correct the error that has occurred due to noise. Given P= $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

Solution:

(Refer notes)

(problem has been solved during the class also)

Video link

- https://youtu.be/qI1M6UzdyQw

- (The above video can be watched for linear block code concept)

2.For the systematic (6,3) code , parity matrix P is given by $[P]= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ The received vector $R=[r_1\ r_2\ r_3\ r_4\ r_5\ r_6\ ]$. Construct the corresponding syndrome calculation circuit.

Solution:(refer previous example for steps to find H^T)

$$H^T = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S=[s_1\ s_2\ s_3]= RH^T= =[r_1\ r_2\ r_3\ r_4\ r_5\ r_6\ ] \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ The syndrome bits are

$$S_1 = r_1 + r_3 + r_4$$

$$S_2 = r_2 + r_3 + r_5$$

$$S_3 = r_1 + r_2 + r_6$$

Syndrome calculation circuit is

(refer notes)

3]For a systematic (7,4) linear block code, the parity matrix P is given by

$$[P] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

i)Find all possible valid-code vectors

ii)Draw the corresponding encoding circuit.

iii)A single error has occurred in each of these received vectors. Detect and correct those errors.

a) R_A=[0111110] b)R_B = [1011100] c)R_C =[1010000]

iv) Draw the syndrome calculation circuit.

Solution:

i) Generator matrix $[G] = [I_k \mid P] = [I_4 \mid P]$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & : & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & : & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & : & 0 & 1 & 1 \end{bmatrix}$$

- C= [D] [G] =

$$[d_1\ d_2\ d_3\ d_4\ ] \begin{bmatrix} 1 & 0 & 0 & 0 & : & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & : & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & : & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & : & 0 & 1 & 1 \end{bmatrix}$$

$[d1,\ d_2, d_3, d_4, (d_1+d_2+d_3)\ \ , (\ d_1+d_2+d_4)\ , (d_1+d_3+d_4)\ ]$

Message Vector				Code vector						
d_1	d_2	d_3	d_4	c_1	c_2	c_3	c_4	c_5	c_6	c_7
0	0	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	1	0	1	1
0	0	1	0	0	0	1	0	1	0	1
0	0	1	1	0	0	1	1	1	1	0
0	1	0	0	0	1	0	0	1	1	0
0	1	0	1	0	1	0	1	1	0	1
0	1	1	0	0	1	1	0	0	1	1
0	1	1	1	0	1	1	1	0	0	0
1	0	0	0	1	0	0	0	1	1	1
1	0	0	1	1	0	0	1	1	0	0
1	0	1	0	1	0	1	0	0	1	0
1	0	1	1	1	0	1	1	0	0	1
1	1	0	0	1	1	0	0	0	0	1
1	1	0	1	1	1	0	1	0	1	0
1	1	1	0	1	1	1	0	1	0	0
1	1	1	1	1	1	1	1	1	1	1

ii)encoding circuit (refer notes)

iii)Given $R_A = [0\ 1\ 1\ 1\ 1\ 1\ 0]$

Parity check matrix H is given by $[H] = [P^T \mid I_{n-k}] = [P^T \mid I_3]$

$$= \begin{bmatrix} 1 & 1 & 1 & 0 & : & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & : & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & : & 0 & 0 & 1 \end{bmatrix}$$

Syndrome $S_A = R_A H^T$

$$= [0\ 1\ 1\ 1\ 1\ 1\ 0] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [1\ 1\ 0]$$

b)Given R_B= [1 0 1 1 1 0 0]

Syndrome $S_B = R_B H^T$

$$= [1\ 0\ 1\ 1\ 1\ 0\ 0] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

= [1 01], which is located in 3rd row of H^T matrix. Hence the 3rd bit counting from left is in error.

∴ Corresponding error vector is given by

E_B=[0 0 1 0 0 0 0]

∴ Corrected code-vector which is the transmitted vector is given by

$C_B = R_B + E_B$=[1 0 1 1 1 0 0] +[0 0 1 0 0 0 0]

=[1 0 0 1 1 0 0]

This is the valid code vector corresponding to message vector 1001 (refer table)

c)Given $R_C = [1\ 0\ 1\ 0\ 0\ 0\ 0]$

Syndrome $S_C = R_C H^T$

$$= [1\ 0\ 1\ 0\ 0\ 0\ 0] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$= [0\ 1\ 0]$, which is located in 6^{th} row of H^T matrix. Hence the 6^{th} bit counting from left is in error.

$\therefore$ Corresponding error vector is given by

$E_C=[0\ 0\ 0\ 0\ 0\ 1\ 0]$

$\therefore$ Corrected code-vector which is the transmitted vector is given by

$C_C=R_C+E_C=[1\ 0\ 1\ 0\ 0\ 0\ 0] +[0\ 0\ 0\ 0\ 0\ 1\ 0]$

$=[1\ 0\ 1\ 0\ 0\ 1\ 0]$

This is the valid code vector corresponding to message vector 1010 (refer table)

iv) Syndrome calculation circuit :

Let $R = [r_1\ r_2\ r_3\ r_4\ r_5\ r_6\ ]$

Syndrome

$$S = [s_1\ s_2\ s_3] = RH^T = = [r_1\ r_2\ r_3\ r_4\ r_5\ r_6\ ] \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [(r_1 + r_2 + r_3 + r_5), (r_1 + r_2 + r_4 + r_6), (r_1 + r_3 + r_4 + r_7)\ ]$$

(Refer notes to write syndrome calculation circuit)

This syndrome is located in second row of H^T matrix. Hence the 2^{nd} bit counting from left is in error.

$\therefore$ Corresponding error vector is given by

$E_A = [0\ 1\ 0\ 0\ 0\ 0\ 0\ ]$

$\therefore$ Corrected code-vector which is the transmitted vector is given by

$C_A = R_A + E_A = [\ 0\ 1\ 1\ 1\ 1\ 1\ 0] + [0\ 1\ 0\ 0\ 0\ 0\ 0]$

$= [0\ 0\ 1\ 1\ 1\ 1\ 0]$

This is the valid code vector corresponding to message vector 0011 (refer table)

4]The generator matrix of a (5,1) repetition code(represent simplest type of linear block code) is given by

$[G] = [\ 1\ 1\ 1\ 1\ \vdots\ 1\ ]$

i) Write its parity check matrix.

ii) Evaluate the syndrome for all five possible single error patterns and also for all ten possible double error patters.

Solution: Given n=5, k=1

$[I_k] = [I_1] = [1]$

And $[G] = [P : I_k] = [P \vdots I_1] = [1\ 1\ 1\ 1 \vdots 1]$

$\therefore [P] = [1\ 1\ 1\ 1]$

$\therefore [H] = [I_{n-k} \vdots P^T] = [I_4 \vdots P^T]$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & \vdots & 1 \\ 0 & 1 & 0 & 0 & \vdots & 1 \\ 0 & 0 & 1 & 0 & \vdots & 1 \\ 0 & 0 & 0 & 1 & \vdots & 1 \end{bmatrix}$$

ii) Since k=1, message vector [D] can be either [0] or [1]

Wkt [C]=[D] [G]

When [D]=[0], [C]= [0] [1 1 1 1 ¦ 1] = [0 0 0 0 0]

When [D]=[1], [C]= [1] [1 1 1 1 ¦ 1] = [1 1 1 1 1]

Let the transmitted vector be [0 0 0 0 0];

Then there are 5 single –error patterns given by

[1 0 0 0 0], [0 1 0 0 0], [0 0 1 0 0], [0 0 0 1 0], [0 0 0 0 1].

Syndrome for all these 5 single error received vectors can be found using equation

$[S]=[R]\ [H^T]$

$\therefore$ For [1 0 0 0 0], $S_A = [1\ 0\ 0\ 0\ 0] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

= [1 0 0 0]

For [0 1 0 0 0], S_B= [0 1 0 0 0] $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

= [0 1 0 0]

For [0 0 1 0 0], S_C= [0 0 1 0]

For [0 0 0 1 0], S_D=[0 0 0 1]

For [0 0 0 0 1], S_E=[1 1 1 1]

- There are 10 double error patterns given by

[1 1 0 0 0], [1 0 1 0 0], [1 0 0 1 0], [1 0 0 0 1], [0 1 1 0 0], [0 1 0 1 0], [0 1 0 0 1], [0 0 1 1 0], [0 0 1 0 1] and [0 0 0 1 1]

For [1 1 0 0 0], S_F = [1 1 0 0]

For [1 0 1 0 0], S_G = [1 0 1 0]

For [1 0 0 1 0], S_H = [1 0 0 1]

For [1 0 0 0 1], S_I = [0 1 1 1]

For [0 1 1 0 0], S_J = [0 1 1 0]

For [0 1 0 1 0], S_K = [0 1 0 1]

For [0 1 0 0 1], S_L = [1 0 1 1]

For [0 0 1 1 0], S_M = [0 0 1 1]

For [0 0 1 0 1], S_N = [1 1 0 1]

For [0 0 0 1 1], S_0 = [1 1 1 0]