Vehicle Routing Problem as a Mixed Integer Linear Program

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1 Introduction

The Vehicle Routing Problem (VRP) is a well-known optimization problem in supply chain and logistics management. A standard approach to model this problem is through a Mixed-Integer Linear Program (MILP). This repository presents a simple example of formulating and solving the VRP using MILP techniques.

1.1 Problem Overview

We consider a distribution center that operates a fleet of vehicles, all identical in size and capacity. A set of customers, each located at a distinct geographic location, has known commodity demands. Each customer specifies a preferred *time window* during which deliveries must occur. That is, a vehicle must arrive within the specified window to make a delivery.

Upon arrival, each delivery requires a fixed amount of service time. For instance, if a customer specifies a delivery window of 2:00–3:00 PM, the vehicle must arrive no earlier than 2:00 PM and no later than 3:00 PM.

The objective is to determine vehicle routes that minimize the total travel distance while ensuring that:

- All customer demands are met,
- Each vehicle operates within its capacity, and
- All deliveries occur within the respective time windows.

2 Problem Formulation

To formulate the VRP described in Subsection 1.1, we begin by defining the relevant notation in Table 1.

V	Number of vehicles available at the depot
I	Number of customers, with the customer set defined as $\{1,\dots,I\}$ and indexed by i
(x_i, y_i)	Geographical coordinates of customer $i \in \{1,\dots,I\}$
(x_0, y_0)	Geographical coordinates of the depot, denoted as node $i = 0$
$d_{i,j}$	Distance between nodes i and j, for all $i, j \in \{0, 1, \dots, I\},\ i \neq j$
q	Capacity of each vehicle (identical for all $v = 1, \dots, V$)
e_i	Earliest allowable service start time for customer $i \in \{1,\dots,I\}$
l_i	Latest allowable service start time for customer $i \in \{1,\dots,I\}$
δ	Fixed service duration at each customer location
M	A sufficiently large constant used for time-based constraints

Table 1: Model Notation

2.1 Decision Variables

Based on the notation in Table 1, we define the decision variables of the MILP model in Table 2.

$x_{i,j} \in \{0,1\}$	1 if a vehicle travels from $i = 0, 1, \cdots, I$ to $j = 0, 1, \cdots, I : j \neq i$, 0 o/w
$y_i \geq 0$	Load of the vehicle just before servicing customer $i = 1, \cdots, I$
$s_i \geq 0$	Start time of service at customer $i = 1, \cdots, I$

Table 2: Decision Variables

This formulation assumes a *homogeneous fleet* in which all vehicles have the same capacity and characteristics. In the case of a *heterogeneous fleet*, the load variable y_i would need to be vehicle-specific (e.g., $y_{i,v}$) to capture varying capacities and routing choices per vehicle. The binary decision variables can also be modified accordingly.

2.2 MILP Model

The MILP formulation for the VRP is given by equations in Model (1).

$$\min \quad \sum_{i=0}^{I} \sum_{\substack{j=0 \\ j \neq i}}^{I} d_{i,j} x_{i,j} \tag{1a}$$

$$\text{s.t.} \quad \sum_{i=1}^{I} x_{0,i} \leq V, \tag{1b}$$

$$\sum_{i=1}^{I} x_{i,0} = \sum_{i=1}^{I} x_{0,i}, \tag{1c}$$

$$\sum_{\substack{i=0 \\ i \neq j}}^{I} x_{i,j} = 1, \quad \forall j \in \{1, \ldots, I\}, \tag{1d}$$

$$\sum_{\substack{j=0 \\ j \neq i}}^{I} x_{i,j} = 1, \quad \forall i \in \{1, \ldots, I\}, \tag{1e}$$

$$s_j \geq s_i + t_{i,j} x_{i,j} + \delta - M(1 - x_{i,j}), \quad \forall i \neq j, \tag{1f}$$

$$y_j \geq y_i + d_i x_{i,j} - M(1 - x_{i,j}), \quad \forall i \neq j, \tag{1g}$$

$$y_i \leq q, \quad \forall i \in \{0, 1, \ldots, I\}, \tag{1h}$$

$$e_i \leq s_i \leq l_i, \quad \forall i \in \{1, \ldots, I\}, \tag{1i}$$

$$y_i \geq 0, \ s_i \geq 0, \quad \forall i \in \{1, \ldots, I\}, \tag{1j}$$

$$x_{i,j} \in \{0, 1\}, \quad \forall i, j \in \{0, \ldots, I\}, \ i \neq j. \tag{1k}$$

The equations in the MILP model (1) serve the following purposes:

- **The objective function** (1a) minimizes the total travel distance of all vehicles.
- **Constraint** (1b): Limits the number of vehicles that can leave the depot to at most V, the total fleet size.
- **Constraint** (1c): Ensures that every vehicle that departs from the depot eventually returns to it, maintaining route continuity.
- **Constraint** (1d): Guarantees that each customer is visited exactly once by one vehicle.
- **Constraint** (1e): Ensures that after servicing a customer, the vehicle continues its route to the next location (except for the last node on a route which is the depot itself).

- **Constraint** (1f): Enforces correct sequencing of service times. If a vehicle travels directly from customer i to j, then service at j must start after finishing service at i and traveling to j. The big-M term deactivates the constraint when arc (i, j) is not used.
- **Constraint** (1g): Tracks cumulative demand served along the route. If a vehicle moves from i to j, the load at j must account for the demand already served at i.
- **Constraint** (1h): Ensures that the cumulative demand on any vehicle does not exceed its capacity q.
- **Constraint** (1i): Enforces that service at each customer starts within their preferred time window, between e_i and l_i.
- **Constraint** (1j): Declares the non-negativity of the continuous decision variables y_i and s_i.
- **Constraint** (1k): Defines the binary nature of routing decisions: $x_{i,j}$ equals 1 if a vehicle travels from i to j, and 0 otherwise.