

Theory of Computing

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Reviewing δ

DFA's

- $\delta: Q \times \Sigma \rightarrow Q$
- input: (state, symbol)
- output: state

NFA's

- Not allowing ϵ transitions
 - $\delta: Q \times \Sigma \rightarrow (Q)$
 - input: (state, symbol)
 - output: set of states
- If an automaton is nondeterministic, then the codomain of δ is a power set
- Allowing ϵ transitions,
 - $\delta: Q \times (\cup \{\epsilon\}) \rightarrow (Q)$
 - input: (state, symbol or ϵ)
 - output: set of states
- Note that ϵ is not a symbol and cannot belong to Σ
 - This is because symbols have a length of 1 and ϵ , by definition, has a length of 0

NFA

- General 5-tuple including
 1. Q : set of states
 2. Σ - input alphabet
 3. $\delta: Q \times (\Sigma \cup \{\epsilon\}) \rightarrow (Q)$
 4. $q_0 \in Q$ - starting state

5. $F \subset Q$ - set of final states

- There may exist some inputs for which it is possible that the NFA accepts the sequence if one path is taken and rejects the sequence if another path is taken
- The NFA does not have the ability to look ahead at possible states

Computation

- The NFA runs all possible branches on a given string simultaneously and independently
- A string, w , is rejected by an NFA, N , if every branch of the nondeterminism tree for N on w rejects
 - $\neg \exists$ a branch that accepts
- A string, w , is accepted by an NFA, N , if \exists a branch on the nondeterminism tree for which N on w accepts
- Theorem: Any language recognized by an NFA can be recognized by a DFA
 - Let N be an NFA.
 - Then \exists a DFA, M , with $L(M) = L(N)$
 - The language recognized by a machine, M , ($L(M)$) is the set of exactly all strings accepted by M (no rejected strings allowed)
 - * $\neg L(M)$ = set of all strings rejected by M

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Nondeterministic Finite Automata

DFA Review

- DFA's are 5-tuples with
 - $\delta: Q \times \Sigma \rightarrow Q$
 - The number of transitions, δ , is $|Q||\Sigma|$
- There is no real decision-making here, an input is simply being used alongside a rule to find an output

NFA's

- NFAs are also a 5-tuple
- $\delta: Q \times \Sigma \rightarrow P(Q)$
- $|P(Q)| = 2^{|Q|}$

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Closure Introduction

- The language recognized by a DFA, M , $(L(M))$ is the set of all string accepted by M
- Thus, $M = (Q, \Sigma, \delta, q_0, F)$
- And Σ^* is the universe of all possible inputs to M
- \forall strings $w \in \Sigma^*$, M either accepts or rejects w
- It follows that $L(M) \subset \Sigma^*$
- And $\neg L(M) = \Sigma^* - L(M)$
- Therefore, M accepts every string in $L(M)$ and rejects everything in $\neg L(M)$

Closure Continued

- A set, A , is closed under a binary operation, OP , if $\forall x, y \in A [x OP y \in A]$
- Ex. Natural Numbers (\mathbb{N})
 1. \mathbb{N} is closed under $+$
 2. \mathbb{N} is closed under \times
 3. \mathbb{N} is not closed under $-$
 4. \mathbb{N} is not closed under \setminus
- The class of all languages that are recognized by DFAs is closed under \cup

Closure Properties of DFAs

- Union (\cup)
- Intersection (\cap)
- Complement (\neg)
- Reverse

Applying Closure Properties

- If $L(M_1) \cup L(M_2)$ are DFAs, then \exists DFA, M , with $L(M) = L(M_1) \cup L(M_2)$
- The purpose of a state machine is to make a yes/no decision

Premise:

- $M_1 = \{Q_1, \Sigma, \delta_1, q_{01}, F_1\}$ and $M_2 = \{Q_2, \Sigma, \delta_2, q_{02}, F_2\}$ are DFAs
- M_1 accepts binary strings ending in 1
- M_2 accepts binary strings of odd length
- $L(M) = L(M_1) \cup L(M_2)$
 - Accepts binary strings that either end in 1 OR have odd length (or both)
- M_1 : q_1, q_2 distinguished between ending in 0 and 1
- M_2 : r_1, r_2 distinguished between odd and even length
- Accordingly, M must be able to distinguish between:
 - even length ending in 1
 - even length ending in 0
 - odd length ending in 1
 - odd length ending in 0

Coding M

- Consider $Q = Q_1 \cdot Q_2 = \{q_1r_1, q_2r_1, q_1r_2, q_2r_2\}$
- wherein
 - q_1r_1 = even string ending in 0
 - q_1r_2 = odd string ending in 0
 - q_2r_1 = even string ending in 1
 - q_2r_2 = odd string ending in 1
- Ex. $\delta(q_1r_1, 1) = q_1r_2$
- Applying this logic to a DFA, we know that
 - $Q = \{q_1r_1, q_1r_2, q_2r_1, q_2r_2\}$;
 - $S = \{0, 1\}$;
 - $d: Q \setminus \Sigma^* \rightarrow Q$;
 - $d(q_1r_1, 0) = q_1r_2$;
 - $d(q_1r_1, 1) = q_2r_2$;
 - $d(q_1r_2, 0) = q_1r_1$;
 - $d(q_1r_2, 1) = q_2r_1$;
 - $d(q_2r_1, 0) = q_1r_2$;
 - $d(q_2r_1, 1) = q_2r_2$;
 - $d(q_2r_2, 0) = q_1r_1$;
 - $d(q_2r_2, 1) = q_2r_1$;
 - $q_0 = q_1r_1$;
 - $F = \{q_1r_2, q_2r_1, q_2r_2\}$;

Derivation

- Construct M and show that $L(M) = L(M_1) \cup L(M_2)$.
- $Q = Q_1 \cdot Q_2$
- let $q_i \in Q_1$ and let $r_j \in Q_2$. and let $c \in \Sigma$
- and $q_i r_j \in Q$
- thus, $\delta(q_i r_j, c) = \delta_1(q_i, c) \delta_2(r_j, c)$
- and $q_0 = q_{01}, q_{02}$
- so that $F = \{q_i r_j : q_i \in F_1 \cup r_j \in F_2\}$

Correctness

- Show that M accepts exactly the strings that are accepted by M_1 or M_2
- If $w \in \Sigma^*$. then w is accepted by M and w is accepted by either M_1 or M_2
- To do this, we can organize Σ^* into strings of length 0, length 1, length 2. . .
- We solve with mathematical induction, which is how we prove recurrence relationships

Mathematical Induction

- Need base case and induction hypothesis
- Induction hypothesis says something is true about k , where k is the length of strings
- $\delta^*(q_0, w)$ is the ending state of M on w where
- $\delta^*(q_0, w) = \delta_1^*(q_{01}, w) \delta_2^*(q_{02}, w)$
- Induction Hypothesis: If $|w| = k$, then $\delta^*(q_0, w) \in F \iff \delta_1^*(q_{01}, w) \in F_1 \text{ or } \delta_2^*(q_{02}, w) \in F_2$
- Let $x \in \Sigma$.
- Then $|w \cdot x| = k+1$
- And $\delta^*(q_0, wx) = \delta_1^*(q_{01}, wx) \delta_2^*(q_{02}, wx)$
- If $\delta_1^*(q_{01}, wx) \in F_1$, $\delta^*(q_0, wx) \in F$
- Similarly, if $\delta_2^*(q_{02}, wx) \in F_2$, then $\delta^*(q_0, wx) \in F$

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Deterministic Finite Automata

- We know that $\delta = Q \times \Sigma \rightarrow Q$
- We want a function that takes a starting state and a string, then returns the state after the machine has read that string
- Let's define $\delta^* = Q \times \Sigma^* \rightarrow Q$
 - δ^* takes a state and a string
 - δ takes a state and a symbol
- Now, we need a recursive definition
 - Base case:
 - * Let $q_i \in Q$
 - * $\delta^*(q_i, \epsilon) = q_i$
 - Recursive step:
 - * If $q_i \in Q$, $w \in \Sigma^*$, and $c \in \Sigma$
 - * then $\delta^*(q_i, w \cdot c) = \delta(\delta^*(q_i, w), c)$

Aside: On Σ and Σ^*

- Σ^* is the universe of all strings over Σ
 - $\Sigma = \{0,1\}$
 - $\Sigma^* = \{\epsilon, 0, 1, 00, 01, 11, 10, 000, \dots\}$
- We can see this recursively
 - Base step: $\epsilon \in \Sigma^*$
 - Recursive step:
 - Let $w \in \Sigma^*$, let $c \in \Sigma$
 - Then $w \cdot c \in \Sigma^*$
- We can see this recursion graphically

w	c	w · c	step
ϵ		-	base
ϵ	1	1	recursive
1	0	10	recursive
10	1	101	recursive

Recursively Testing 101

- Solve $\delta^*(q_1, 101)$
- $\delta(\delta^*(q_1, 10), 1)$
- $\delta^*(q_1, 10)$
- $\delta(\delta^*(q_1, 1), 0)$

TODO Complete Recursion Sequence

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Tuples & DFAs

- Tuples are sequences which are always finite in length
- The deterministic finite automaton shown is a 5-tuple:
 1. Q : finite nonempty set of states
 - state: configuration of logic of a machine
 2. Σ (Sigma) - input alphabet
 - alphabet: a finite, nonempty set of symbols where symbols are an object of length 1
 3. δ (Delta) - transition function
 4. $Q_0 \in Q$ - starting state
 5. $F \subset Q$ - set of final states
- For this deterministic finite automaton,
 - $\delta: Q \times \Sigma \rightarrow Q_2$

Represented as a table,

Step	State	Input	Transition
1	Q_1	1	$Q_1 \rightarrow Q_2$
2	Q_2	0	$Q_2 \rightarrow Q_1$
3	Q_1	1	$Q_1 \rightarrow Q_2$
4	Q_2	1	$Q_2 \rightarrow Q_2$

Domains & Codomains

- Domain: set of all possible function inputs
- Codomain: set of all possible outputs

Strings

- In computer science, strings are character arrays
- In mathematics, strings are sequences of symbols
- Specifically a string over an alphabet, Σ , is a sequence of symbols belonging to Σ
- ϵ is the empty string
- Concatenation: If $w_1, w_2 \in \Sigma$, $w_1 \cdot w_2 = w_1w_2$
- If $c \in \Sigma$, then $\epsilon \cdot c = c \cdot \epsilon = c$

TODO Review Recursive Definitions

- Base step: a step that can not be broken down any further, a fact that is always true regardless of the input
- Recursive step:
- Defining the length of a string over Σ
 - Base: $|\epsilon| = 0$
 - Recursive:
 - * let w be a string over Σ , and $c \in \Sigma$
 - * then $|w \cdot c| = |w| + 1$
- Using this to define $|1011|$,
 1. $|1011| = |101 \cdot 1| = |101| + 1 =$
 2. $|10 \cdot 1| + 1 = |10| + 1 + 1 =$
 3. $|1 \cdot 0| + 1 + 1 = |1| + 1 + 1 + 1 =$
 4. $|\epsilon \cdot 1| + 1 + 1 + 1 =$
 5. $|\epsilon| + 1 + 1 + 1 + 1 =$
 6. $0 + 1 + 1 + 1 + 1 = 4$

Languages

- Languages over Σ - a set of finite strings over Σ
- Languages recognized by an automaton, M , $L(M)$ is the language accepted by M
- \emptyset is the empty language
- $\epsilon \neq \emptyset$
- $\epsilon \neq \{\epsilon\}$
- ϵ is not a symbol in any alphabet

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Automaton (automata)

- Self running machine requiring a continuous power source
 - Historically used power sources include water, steam, and electricity
- Course revolves around defining the mathematics powering machines

The Mathematics of Automata

Mathematicians & History

- Cantor defines sets as collections of objects
- Cantor also argues that infinities can be of different magnitudes - there are infinitely more real numbers than natural numbers
- Goedel eventually derives his incompleteness theorem
 - No logical system that contains the natural numbers can prove its own soundness
 - Every sound logical system containing the natural numbers contains valid statements that cannot be proved or disproved
- In 1936, Turing proves The Halting Problem is not decidable, it is impossible
 - The Halting Problem is an algorithm that can analyze any other algorithm and determine whether or not it goes into an infinite loop
- Turing creates the Turing machine as an object consisting of sets and processes wherein the object can use any finite process to complete an action.
- Turing machine sets the basis for a computer, which leads to a series of important questions:
 - What can & can't a machine do?
 - What does it mean for a problem to be harder than another?
 - What does it mean for a machine to be more powerful than another?

Sequential Logic

- Sentential Logic- based on boolean results
 - Predicated on AND, OR, NOT
 - XOR, XAND, etc. can be derived using the above

Necessary Review

- Textbook Ch. 0
- Logic Statements
- Set Theory
- Functions

Functions

- Functions - something that maps objects from one set to another
- Given $f: a \rightarrow b$;
 - Everything in a is mapped to something in b

- * For every x , such that x is an element of a , there exists a y , such that y is an element of b
- No one point in the domain can be mapped to two different points in the codomain
 - * Logically, you can't have a function that takes in one input and returns two different outputs
 - * If f maps $x \rightarrow y_1$ and $x \rightarrow y_2$, $y_1 = y_2$
- $\neg \forall x \in A \ y_1, y_2 \in B \ [f(x)=y_1 \wedge f(x)=y_2 \rightarrow y_1 = y_2]$

TODO Types of Functions - Definition & Logical Statement

- Injective Functions
- Surjective Functions
- Proof by Induction (\forall)
- Proof by Contradiction ($\neg \exists$)

Finite Automaton (Finite State Machine)

- States are logical configurations
- States are generally based upon input
- Purpose of a state machine is to make a yes/no decision