Semester-4

(Optimization Techniques)

(According to Purvanchal University Syllabus)

Unit – 1

Linear Programming

Definition of LPP–

- A linear programming problem consists of a linear function to be maximized or minimized subject to certain constraints in the form of linear equations or inequalities.
- Linear Programming Problems (LPP) provide the method of finding such an optimized function along with/or the values which would optimize the required function accordingly. It is one of the most important Operations Research tools.

It consists for four basic components:

- Decision variables represent quantities to be determined
- Objective function represents how the decision variables affect the cost or value to be optimized (minimized or maximized)
- Constraints represent how the decision variables use resources, which are available in limited quantities
- Data quantifies the relationships represented in the objective function and the constraints

All linear programming problems must have following five characteristics:

(a) Objective function:There must be clearly defined objective which can be stated in quantitative way. In business problems the objective is generally profit maximization or cost minimization.

(b) Constraints: All constraints (limitations) regarding resources should be fully spelt out in mathematical form.

(c) Non-negativity:The value of variables must be zero or positive and not negative. For example, in the case of production, the manager can decide about any particular product number in positive or minimum zero, not the negative.

(d) Linearity: The relationships between variables must be linear. Linear proportional relationship between two ‘or more variable, i.e., the degree of variables should be maximum.

(e) Finiteness:

The number of inputs and outputs need one to be finite. In the case of infinite factors, to compute feasible solution is not possible.

Graphical solution of two variable LPP–

Example 1: Solve the given linear programming problems graphically:

Maximize: Z = 8x + y

Constraints are,

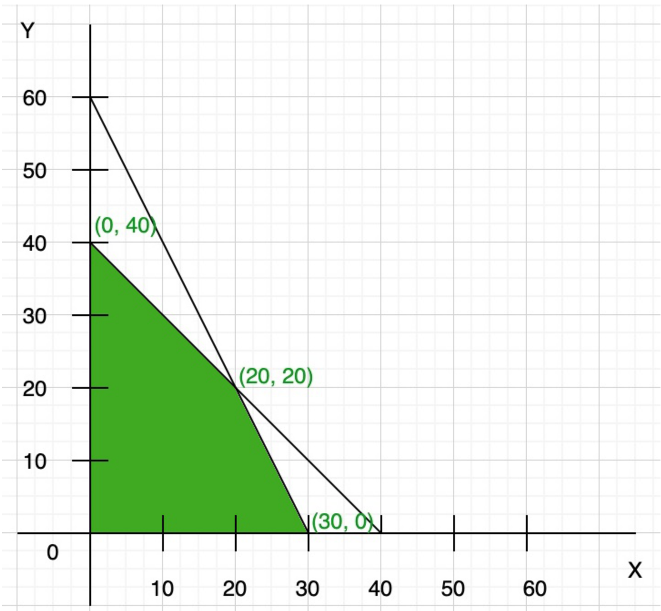
- $x + y \le 40$
- $2x + y \le 60$
- $x \ge 0, y \ge 0$

Solution:

Step 1: Constraints are,

- $x + y \le 40$
- $2x + y \le 60$
- $x \ge 0, y \ge 0$

Step 2: Draw the graph using these constraints.



Here both the constraints are less than or equal to, so they satisfy the below region (towards origin). You can find the vertex of feasible region by graph, or you can calculate using the given constraints:

$x + y = 40$ ...(i)

$2x + y = 60$ ...(ii)

Now put the value of y in any of the equations, we get

$x = 20$

So the third point of the feasible region is (20, 20)

Step 3: To find the maximum value of $Z = 8x + y$. Compare each intersection point of the graph to find the maximum value

Points	**$Z = 8x + y$**
(0, 0)	0
(0, 40)	40
(20, 20)	180
(30, 0)	240

Ans- **So the maximum value of $Z = 240$ at point $x = 30$, $y = 0$.**

General LPP problem–

General Linear Programming problem

The general linear Programming problem (or model) with n decision variables and m constraints can be stated in the following form:

Optimize (Max. or Min.)

$$Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

Subject to the linear constraints

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n(\leq, =, \geq)b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n(\leq, =, \geq)b_2 \\
\vdots \qquad\qquad \vdots \qquad \vdots \qquad \vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n(\leq, =, \geq)b_m
\end{aligned}$$

and $x_1, x_2, \ldots, x_n \geq 0$

Canonical and standard form of LPP–

Canonical form of standard LPP

- Canonical form of standard LPP is a set of equations consisting of the 'objective function and all the 'equality constraints' (standard form of LPP) expressed in *canonical form.*
- Understanding the canonical form of LPP is necessary for studying *simplex method, the most* popular method of solving LPP
- Let us consider a set of three equations with three variables for ease of discussion.

$$3x + 2y + z = 10 \qquad (A_0)$$

$$x - 2y + 3z = 6 \qquad (B_0)$$

$$2x + y - z = 1 \qquad (C_0)$$

- The system of equations can be transformed in such a way that a new set of three different equations are obtained, each having only one variable with nonzero coefficient. This can be achieved by some *elementary operations*

Simplex methods and artificial variable–

- Simplex method is an iterative procedure for solving LPP in a finite number of steps.
- This method provides an algorithm which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more as the case may be than at the previous vertex.
- This procedure is repeated and since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of unbounded solution.

Artificial Variable :

- In order to use the simplex method on problems with mixed constraints, we turn to a device called an artificial variable.
- This variable has no physical meaning in the original problem and is introduced solely for the purpose of obtaining a basic feasible solution so that we can apply the simplex method.
- An artificial variable is a variable introduced into each equation that has a surplus variable. To ensure that we 7 q p consider only basic feasible solutions, an artificial variable is required to satisfy the nonnegative constraint.

Sensitivity analysis–

- Sensitivity Analysis deals with finding out the amount by which we can change the input data for the output of our linear programming model to remain comparatively unchanged.
- This helps us in determining the sensitivity of the data we supply for the problem. If a small change in the input (for example in the change in the availability of some raw material) produces a large change in the optimal solution for some model, and a corresponding small change in the input for some other model doesn't affect its optimal solution as much, we can

conclude that the second problem is more robust then the first. The second model is less sensitive to the changes in the input data.

Problem of degeneracy & Concept of duality–

Every LPP (called the primal) is associated with another LPP (called its dual).

Either of the problem can be considered as primal with the other as dual. The importance of the duality concept is due to two main reasons:

(i) If the primal contains a large number of constraints and a smaller number of variables, the labour of computation can be considerably reduced by converting it in to the dual problem and then solving it.

(ii) (ii) The interpretation of the dual variables from the cost or economic point of view, proves extremely useful in making future decisions in the activities being programmed.

Formulation of Dual Problem

For formulating a dual problem, first we bring the problem in the canonical form. The following changes are used in formulating the dual problem.

(1) Change the objective function of maximization in the primal into that of minimization in the dual and vice versa.

(2) The number of variables in the primal will be the number of constraints in the dual and vice versa.

(3) The cost coefficients C1, C2 ... Cn in the objective function of the primal will be the RHS constant of the constraints in the dual and vice versa.

(4) In forming the constraints for the dual, we consider the transpose of the body matrix of the primal problem.

(5) The variables in both problems are non-negative.

(6) If the variable in the primal is unrestricted in sign, then the corresponding constraint in the dual will be an equation and vice versa.

Unit – 2

Transportation

Introduction to transportation model-

- Transportation problems are one of the subclasses of LPPs in which the objective is to transport various quantities of a single homogenous commodity that are initially stored at various origins, to different destinations in such a way that the transportation cost is minimum.

Matrix form of TP–

		Destination j			
		D_1	D_2	D_3	
Source i	S_1	10	30	20	10
	S_2	10	30	20	30
	S_3	10	30	20	40
	S_4	10	30	20	20
		20	50	30	

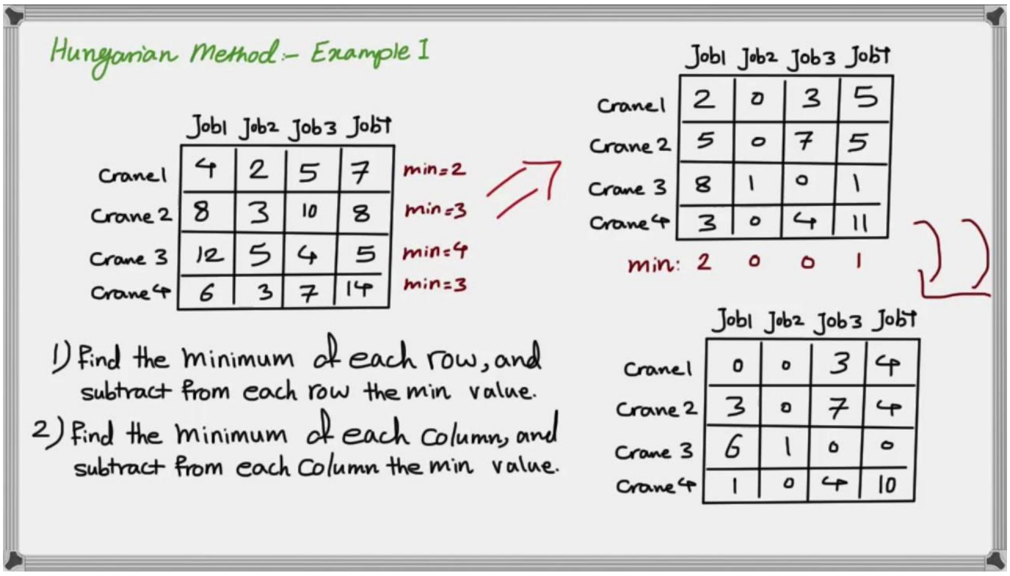
Table 1(b) Transportation matrix with variable costs shown inside the matrix

		Destination j			
		D_1	D_2	D_3	
Source i	S_1	2	3	4	10
	S_2	3	2	1	30
	S_3	1	4	3	40
	S_4	4	5	2	20
		20	50	30	

Application of TP model

- Application of Transportation Problem:
  - o Minimize shipping costs
  - o Determine low cost location
  - o Find minimum cost production schedule
  - o Military distribution system

Assignment problems–



Mathematical Formulation–

Mathematical formulation

- A common choice to find the model parameters is by minimizing the regularized training error given by

$$E(w, b) = \frac{1}{n}\sum_{i=1}^{n} L(y_i, f(x_i)) + \alpha R(w)$$

- L is a loss function that measure model fit.
- R is a regularization term (aka penalty) that penalizes model complexity
- $\alpha > 0$ is a non-negative hyperparameter

Finding IBFS–

Find the Initial Basic Feasible Solution (IBFS) for the following Transportation Model using:

- North-West Corner method
- Least Cost Cell method
- Vogel's Approximation method

Problem 1:

	D1	D2	D3	D4	D5	Supply
S1	10	2	3	15	9	*35*
S2	5	10	15	2	4	*40*
S3	15	5	14	7	15	*20*
S4	20	15	13	25	8	*30*
Demand	*20*	*20*	*40*	*10*	*35*	

1)North-West Corner Rule

Definition: The North-West Corner Rule is a method adopted to compute the initial feasible solution of the transportation problem. The name North-west corner is given to this method because the basic variables are selected from the extreme left corner.

The concept of North-West Corner can be well understood through a transportation problem given below:

To / Source	D	E	F	Supply
A	5	8	4	50
B	6	6	3	40
C	3	9	6	60
Demand	20	95	35	150

solution-

To / Source	D	E	F	Supply
A	5 20	8 30	4	50
B	6	6 40	3	40
C	3	9 25	6 35	60
Demand	20	95	35	150

Total Cost = 20*5+ 30*8+ 40*6+ 25*9+ 35*6 = Rs 1015

2)Least Cost Method

Definition: The Least Cost Method is another method used to obtain the initial feasible solution for the transportation problem. Here, the allocation begins with the cell which has the minimum cost. The lower cost cells are chosen over the higher-cost cell with the objective to have the least cost of transportation.Let's understand the concept of Least Cost method through a problem given below:

To / Source	D	E	F	Supply
A	5	8	4	50
B	6	6	3	40
C	3	9	6	60
Demand	20	95	35	150

solution -

To / Source	D	E	F	Supply
A	5	8 50	4	50
B	6	6 5	3 35	40
C	3 20	9 40	6	60
Demand	20	95	35	150

Total Cost = 50*8 + 5*6 + 35*3 +20*3 +40*9 = Rs 955.

3) Vogel's Approximation Method

1. First of all the difference between two least cost cells are calculated for each row and column, which can be seen in the iteration given for each row and column. Then the largest difference is selected, which is 4 in this case. So, allocate 20 units to cell BD, since the minimum cost is to be chosen for the allocation. Now, only 20 units are left with the source B.

To / From	D	E	F	Supply	Iteration-I
A	6	4	1	50	3
B	3 (20)	8	7	40	4
C	4	4	2	60	2
Demand	20	95	35	150	
Iteration-I	1	0	1		

2. Column D is deleted, again the difference between the least cost cells is calculated for each row and column, as seen in the iteration below. The largest difference value comes to be 3, so allocate 35 units to cell AF and 15 units to the cell AE. With this, the Supply and demand of source A and origin F gets saturated, so delete both the row A and Column F.

To / From	E	F	Supply	Iteration-II
A	4 (15)	1 (35)	50	3
B	8	7	20	1
C	4	2	60	2
Demand	95	35	150	
Iteration-II	0	1		

3. Now, single column E is left, since no difference can be found out, so allocate 60 units to the cell CE and 20 units to cell BE, as only 20 units are left with source B. Hence the demand

To / From	E	Supply
B	8 (20)	20
C	4 (60)	60
Demand	80	150

.

Now the total cost can be computed, by multiplying the units assigned to each cell with the cost concerned. Therefore,

Total Cost = 20*3 + 35*1 + 15*4 + 60*4 + 20*8 = Rs 555

Degeneracy–

In a TP, if the number of non-negative independent allocations is less than m]n[1,
m is the number of origins (rows) and n is the number of destinations where m is the
where m is the number of origins (rows) and n is the number of destinations (columns)
(columns) there exists a degeneracy. This may occur either at the initial stage or at the subsequent iteration.

To resolve this degeneracy, we adopt the following steps.

(1) Among the empty cell, we choose an empty cell having the least cost which is of an

independent position. If this cell is more than one, choose any one

of an independent position. If this cell is more than one, choose any one
of an independent position. If this cell is more than one, choose any one arbitrarily.

(2) To the cell as chosen in step (1) we allocate a small positive quantity ell as chosen in step (1)

Unit – 3

Sequencing models & Related problems

Sequencing problem–

- When a number of jobs are given to be done and they require processing on two or more machines, the main concern of a manager is to find the order or sequence to perform these jobs. We shall consider the sequencing problems in respect of the jobs to be performed in a factory and study the method of their solution. Such sequencing problems can be broadly divided in two groups. In the first one, there are n jobs to be done, each of which requires processing on some or all of the k different machines.

Processing n jobs through two machine–

Example

Six jobs go first over machine I and then over machine II. The order of the completion of jobs has no significance. The table shows the machine times in hours for six jobs and the two machines.

Job	J_1	J_2	J_3	J_4	J_5	J_5
Machine I	2	4	9	6	7	4
Machine II	6	7	4	3	3	11

Find the sequence of the jobs that minimizes the total elapsed time to complete the jobs. Also find the idle time for Machine I and Machine II.

Sol:

Establish a sequence table containing six job cells. Find the least time available for both Machine I and Machine II. Job 1 has the least processing time, i.e., 1. Place the sequence from left to right (or the first cell) as shown in Table, since it occurs Machine I.

J_1					

Deleting job 1, we get the reduced table as shown in Table below

Job	J_2	J_3	J_4	J_5	J_6
Machine I	3	8	5	6	3
Machine II	6	3	2	2	10

The least time available in the reduced table is 2, which is on Job 4 and Job 5 for Machine II. Now, compare the adjacent time available for Machine I. Here, Job 4 time is less than that for Job 5.

Select Job 4 first and sequence it as shown in Table below:

J_1					J_4

Now select Job 5 and sequence it as shown in Table below

J_1				J_5	J_4

The reduced table is shown in Table below

Job	J_2	J_3	J_6
Machine I	3	8	3
Machine II	6	3	10

Now we have Job 2, Job 3 and Job 6 having the least time which is 3. Compare these times with the adjacent machine time and select the least time. Here we have Job 2 with least adjacent time and hence sequence is as shown below in Table below

J_1	J_2			J_5	J_4

Now select Job 3 and sequence it, as shown in Table below

J_1	J_2		J_3	J_5	J_4

Finally, select Job 6 and sequence it, as shown in Table below

J_1	J_2	J_6	J_3	J_5	J_4

The optimal sequence thus obtained is

The total elapsed time and idle time for Machine I and Machine II is calculated as shown in Table give below

$$J_1 \rightarrow J_2 \rightarrow J_6 \rightarrow J_3 \rightarrow J_5 \rightarrow J_4$$

Job Sequence	Machine I		Machine II		Idle Time A	Idle Time B
	Time In	Time Out	Time In	Time Out		
J_1	0	2	2	8	0	2
J_2	2	6	8	15	0	0
J_6	6	10	15	26	0	0
J_3	10	19	26	30	0	0
J_5	19	26	30	33	0	0
J_4	26	32	33	36	0	0
					36-32=4	

The total elapsed time is 36 hours

Idle time for Machine I is 4 hours

Idle time for Machine II is 2 hours

<u>Processing n jobs through three machine–</u>

Jobs	Machine M_1	$T_{i,1\to2}$	Machine M_2	$T_{i,2\to3}$	Machine M_3	w_i
i	In – Out		In – Out		In - Out	
5	0 – 7	5	12 – 18	1	19 - 28	**5**
1	7 – 12	5	18 – 26	3	**29 – 37**	4
2	12 – 20	3	**26 – 35**	2	37 – 43	3
4	20 – 29	4	35 – 42	5	47 – 52	1
3	**29 – 39**	1	42 – 46	4	52 – 59	2

40

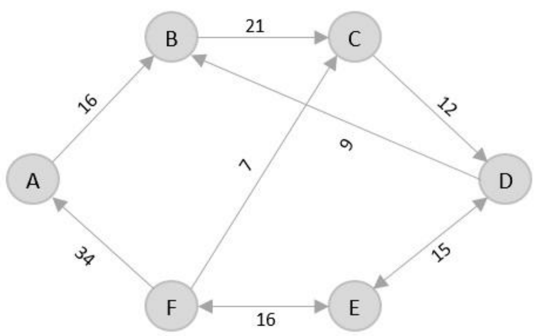
Min. total elapsed time is 40 hours.
Idle time for Machine A is 8 hrs. (32-40)
Idle time for Machine B is 25 hours (0-7, 8-12, 15-21, 26-27, 31-32 and 34-40) Idle time for Machine C is 12 hours (0-8, 22-26.)

Traveling salesman problem–

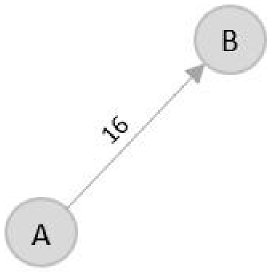
- The Traveling Salesman Problem (TSP) is a problem in combinatorial optimization studied in operations research and theoretical computer science. Given a list of cities and their pair wise distances, the task is to find a shortest possible tour that visits each city exactly once.
- The TSP has several applications even in its purest formulation, such as planning, logistics, and the manufacture of microchips.

Examples

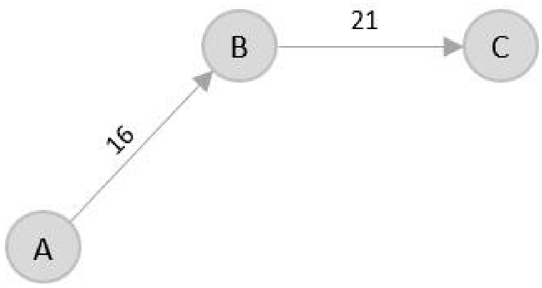
Consider the following graph with six cities and the distances between them −



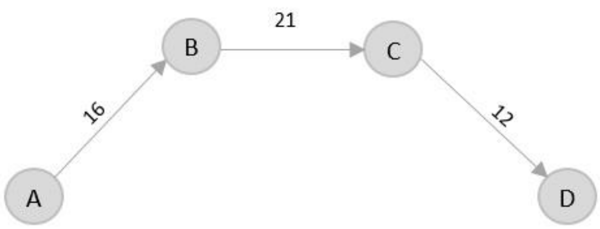
From the given graph, since the origin is already mentioned, the solution must always start from that node. Among the edges leading from A, A → B has the shortest distance.



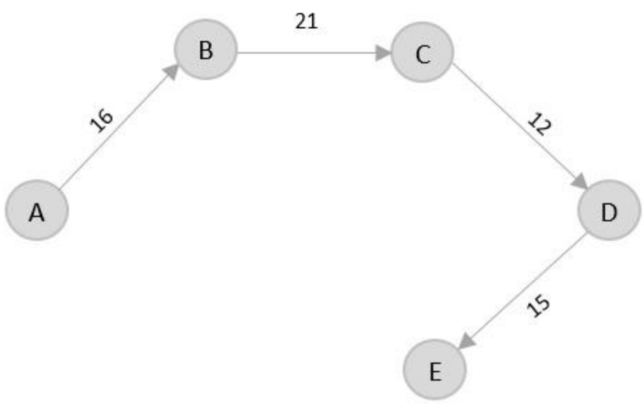
Then, B → C has the shortest and only edge between, therefore it is included in the output graph.



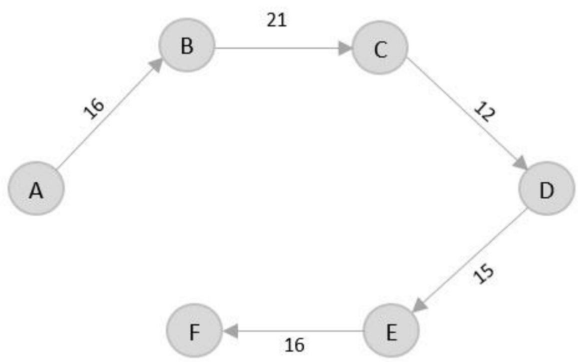
There’s only one edge between C → D, therefore it is added to the output graph.



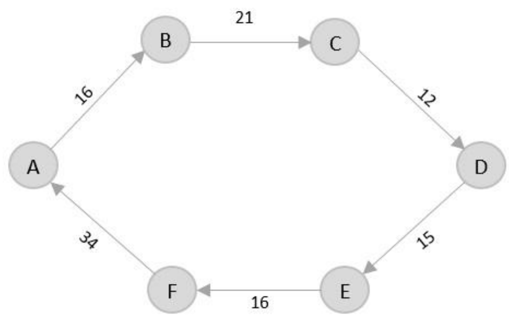
There’s two outward edges from D. Even though, D → B has lower distance than D → E, B is already visited once and it would form a cycle if added to the output graph. Therefore, D → E is added into the output graph.



There’s only one edge from e, that is E → F. Therefore, it is added into the output graph.



Again, even though F → C has lower distance than F → A, F → A is added into the output graph in order to avoid the cycle that would form and C is already visited once.



The shortest path that originates and ends at A is A → B → C → D → E → F → A

The cost of the path is: 16 + 21 + 12 + 15 + 16 + 34 = 114.

Even though, the cost of path could be decreased if it originates from other nodes but the question is not raised with respect to that.

Unit – 4

PERT & CPM

PERT–

- PERT is an acronym for Program (Project) Evaluation and Review Technique, in which planning, scheduling, organizing, coordinating and controlling uncertain activities take place. The technique studies and represents the tasks undertaken to complete a project, to identify the least time for completing a task and the minimum time required to complete the whole project. It was developed in the late 1950s. It is aimed to reduce the time and cost of the project.

CPM–

- Developed in the late 1950s, Critical Path Method or CPM is an algorithm used for planning, scheduling, coordination and control of activities in a project. Here, it is assumed that the activity duration is fixed and certain. CPM is used to compute the earliest and latest possible start time for each activity.

PERT	Vs	CPM
Its Full-Form Project Evaluation & Review Technique.		CPM Full form Critical Path Method.
It is Event oriented technique.		It is activity oriented technique
PERT manages unpredictable activities.		CPM manages the predictable activities
It is focused on time control.		It focus on cost optimization
It was developed in 1958.		It was developed in 1957.
It is a three-time estimate.		It is single time estimates.
It is a probability model.		It is a deterministic model.