SEMESTER-2

DISCRETE MATHMETICS

BCA 204

(According to Purvanchal University Syllabus)

Unit-1

Set Relation & Function

Sets And Subsets-

- ★ A set is an unordered collection of different elements. A set can be written explicitly by listing its elements using set bracket. If the order of the elements is changed or any element of a set is repeated, it does not make any changes in the set.

  Example:

- ★ A set of all positive integers
- ★ A set of all the planets in the solar system
- ★ A set of all the states in India
- ★ A set of all the lowercase letters of the alphabet

Sets can be represented in two ways −

- ★ Roster or Tabular Form
- ★ Set Builder Notation

Roster or Tabular Form-

The set is represented by listing all the elements comprising it. The elements are enclosed within braces and separated by commas.

Example 1 − Set of vowels in English alphabet, A= {a,e,i,o,u}

Example 2 − Set of odd numbers less than 10, B={1,3,5,7,9}

Set Builder Notation-

The set is defined by specifying a property that elements of the set have in common. The set is described as A= {x:p(x)}

Example 1 − The set {a,e,i,o,u} is written as −

A={x:x is a vowel in English alphabet}

Example 2 − The set {1,3,5,7,9} is written as –

B={x:1≤x<10 and (x%2)≠0}

Types of Sets-

★ Sets can be classified into many types. Some of which are finite, infinite, subset, universal, proper, singleton set, etc.

Finite Set -

A set which contains a definite number of elements is called a finite set.

Example − $S=\{x|x\in N \text{ and } 70>x>50\}$

Infinite Set -

A set which contains infinite number of elements is called an infinite set.

Example − $S=\{x|x\in N \text{ and } x>10\}$

Subset-

A set X is a subset of set Y (Written as $X\subseteq Y$) if every element of X is an element of set Y.

Example 1 − Let, $X=\{1,2,3,4,5,6\}$ and $Y=\{1,2\}$. Here set Y is a subset of set X as all the elements of set Y is in set X. Hence, we can write $Y\subseteq X$.

Example 2 − Let, $X=\{1,2,3\}$ and $Y=\{1,2,3\}$. Here set Y is a subset (Not a proper subset) of set X as all the elements of set Y is in set X. Hence, we can write $Y\subseteq X$.

Universal Set -

It is a collection of all elements in a particular context or application. All the sets in that context or application are essentially subsets of this universal set. Universal sets are represented as U.

Example − We may define U as the set of all animals on earth. In this case, set of all mammals is a subset of U, set of all fishes is a subset of U, set of all insects is a subset of U, and so on.

Empty Set or Null Set

An empty set contains no elements. It is denoted by $\varnothing$. As the number of elements in an empty set is finite, empty set is a finite set. The cardinality of empty set or null set is zero.

Example − $S=\{x|x\in N \text{ and } 7<x<8\}=\varnothing$

Singleton Set or Unit Set

Singleton set or unit set contains only one element. A singleton set is denoted by {s}.

Example − S={x|x∈N, 7<x<9} = {8}

Equal Set

If two sets contain the same elements they are said to be equal.

Example − If A={1,2,6} and B={6,1,2}, they are equal as every element of set A is an element of set B and every element of set B is an element of set A.

Equivalent Set

If the cardinalities of two sets are same, they are called equivalent sets.

Example − If A={1,2,6} and B={16,17,22}, they are equivalent as cardinality of A is equal to the cardinality of B. i.e. |A|=|B|=3

Set Operations-

★ Set Operations include Set Union, Set Intersection, Set Difference, Complement of Set, and Cartesian Product.

Set Union- The union of sets A and B (denoted by A∪B) is the set of elements which are in A, in B, or in both A and B.

Example

A={10,11,12,13), B={13,14,15}

Then, (A U B)={10,11,12,13,14,15} (The common element occurs only once)

Set Intersection- The intersection of sets A and B (denoted by A∩BA∩B) is the set of elements which are in both A and B.

Example- A={11,12,13} and B={13,14,15}

then A∩B={13}.

Set Difference/ Relative Complement- The set difference of sets A and B (denoted by A–B) is the set of elements which are only in A but not in B. Hence

Example A={10,11,12,13} B={13,14,15}

(A−B)={10,11,12} and (B−A)={14,15}

Here, we can see (A−B)≠(B−A)

Complement of a Set-The complement of a set A (denoted by A′) is the set of elements which are not in set A.
Hence, A′={x|x∉A}
More specifically, A′=(U−A) where U is a universal set which contains all objects.

Example If A={x|x belongs to set of odd integers}
then A′={y|y does not belong to set of odd integers}

Power Set-

- ★ Power set of a set S is the set of all subsets of S including the empty set. The cardinality of a power set of a set S of cardinality n is 2n. Power set is denoted as P(S).

Example: For a set S={a,b,c,d} let us calculate the subsets

Subsets with 0 elements − {∅} (the empty set)

- ★ Subsets with 1 element − {a},{b},{c},{d}
- ★ Subsets_with_2_elements− {a,b},{a,c},{a,d},{b,c},{b,d},{c,d}

Subsets_with_3_elements− {a,b,c},{a,b,d},{a,c,d},{b,c,d}

- ★ Subsets with 4 elements − {a,b,c,d}

Hence, P(S)=
{{∅},{a},{b},{c},{d},{a,b},{a,c},{a,d},{b,c},{b,d},{c,d},{a,b,c},{a,b

,d},{a,c,d},{b,c,d},{a,b,c,d}}

$|P(S)|=2^4=16$

Note − The power set of an empty set is also an empty set.

$|P(\{\varnothing\})|=2^0=1$

De Morgan's Law-

De Morgan's Laws gives a pair of transformations between union and intersection of two (or more) sets in terms of their complements. The laws are –

$(A\cup B)'=A'\cap B'$

$(A\cap B)'=A'\cup B'$

Example\Let $A=\{1,2,3,4\}, B=\{1,3,5,7\}$ and Universal set $U=\{1,2,3,\ldots,9,10\}$

$A'=\{5,6,7,8,9,10\}$

$B'=\{2,4,6,8,9,10\}$

$A\cup B=\{1,2,3,4,5,7\}$

$A\cap B=\{1,3\}$

$(A\cup B)'=\{6,8,9,10\}$

$A'\cap B'=\{6,8,9,10\}$

Thus, we see that $(A\cup B)'=A'\cap B'$

$(A\cap B)'=\{2,4,5,6,7,8,9,10\}$

$A'\cup B'=\{2,4,5,6,7,8,9,10\}$

Thus, we see that $(A\cap B)'=A'\cup B'$

Cartesion product of two sets-

The Cartesian product of n number of sets $A_1, A_2, \ldots A_n$ denoted as $A_1\times A_2\cdots\times A_n$ can be defined as all possible ordered pairs $(x_1,x_2,\ldots x_n)$

where $x_1 \in A_1$, $x_2 \in A_2, \ldots x_n \in A_n$

Example: If we take two sets $A=\{a, b\}$ and $B=\{1, 2\}$

The Cartesian product of A and B is written as − $A\times B=\{(a,1),(a,2),(b,1),(b,2)\}$ The

Cartesian product of B and A is written as − $B\times A=\{(1,a),(1,b),(2,a),(2,b)\}$

Relation-

★ Let A and B be two non-empty set.

Then, a relation R from A to B is a subset of (AxB)

Thus, R is a relation from A to B ✎ R ∈ (AxB).

If (a,b) ∈ R then we say that A is related to B and we write, ARB.

If (A,B) ∈ R then A is not related to B and we write

A□B. A={x,y}

B={1,2,3}

AxB={(x,1), (x,2), (x,3), (4,1), (4,2), (4,3)}

R={(x,1), (x,3), (y,3)}

Domain and Range -

★ If there are two sets A and B, and relation R have order pair (x, y), then −

★ The domain of R, Dom(R), is the set {x | (x, y) ∈ R for some y in B}

The range of R, Ran(R), is the set {y | (x, y) ∈ R for some x in A}

Examples

Let, A={1,2,9} and B={1,3,7}

★ Case 1 − If relation R is 'equal to' then R={(1,1),(3,3)}

Dom(R) = {1,3},Ran(R)={1,3}

★ Case 2 − If relation R is 'less than' then R={(1,3),(1,7),(2,3),(2,7)}

Dom(R) = {1,2},Ran(R)={3,7}

★ Case 3 − If relation R is 'greater than' then R={(2,1),(9,1),(9,3),(9,7)}

Dom(R) = {2,9},Ran(R)={1,3,7}

Types of Relations –

Empty Relation: A relation R on a set A is called Empty if the set A is empty set.

Full Relation: A binary relation R on a set A and B is called full if AXB.

Reflexive Relation: A relation R on a set A is called reflexive if (a, a) € R holds for every element a € A .i.e. if set A = {a, b} then R = {(a, a), (b, b)} is reflexive relation.

Irreflexive relation: A relation R on a set A is called reflexive if no (a, a) € R holds for every element a € A.
i.e. if set A = {a, b}
then R = {(a, b), (b, a)} is irreflexive relation.

Symmetric Relation: A relation R on a set A is called symmetric if (b,a) € R holds when (a, b) € R.
i.e. The relation R={(4,5),(5,4),(6,5),(5,6)} on set A={4,5,6} is symmetric.

Anti-Symmetric Relation: A relation R on a set A is called anti-symmetric if (a, b)€ R and (b, a) € R then a = b is called anti-symmetric.
i.e. The relation R = {(a, b)→R |a ≤ b} is anti-symmetric since a ≤ b and b ≤ a implies a = b.

Transiitive Relation: A relation R on a set A is called transitive if (a, b) € R and (b, c) € R then (a, c) € R for all a, b, c € A.
i.e. Relation R={(1,2),(2,3),(1,3)} on set A={1,2,3} is transitive.

Equivalence Relation: A relation is an Equivalence Relation if it is reflexive, symmetric, and transitive. i.e. relation R={(1,1),(2,2),(3,3),(1,2),(2,1),(2,3),(3,2),(1,3),(3,1)} on set A={1,2,3} is equivalence relation as it is reflexive, symmetric, and transitive.

Asymmetric relation: Asymmetric relation is opposite of symmetric relation. A relation R on a set A is called asymmetric if no (b, a) € R when (a, b) € R.

Function-

- ★ A function or mapping (Defined as $f:X\rightarrow Y$) is a relaƟonship from elements of one set X to elements of another set Y (X and Y are non-empty sets). X is called Domain and Y is called Codomain of function 'f'.
- ★ Function 'f' is a relation on X and Y such that for each x X, there exists a unique y Y such that (x,y) R. 'x' is called pre-image and 'y' is called image of function f.
- ★ A function can be one to one or many to one but not one to many.

Injective / One-to-one function-

- ★ A function $f:A\rightarrow B$ is injective or one-to-one function if for every $b\in B$, there exists at most one $a\in A$ such that $f(s)=t$.
- ★ This means a function f is injective if $a_1\neq a_2$ implies $f(a1)\neq f(a2)$.

Example

- ★ $f:N\rightarrow N, f(x)=5x$ is injective.
- ★ $f:N\rightarrow N, f(x)=x^2$ is injective.
- ★ $f:R\rightarrow R, f(x)=x^2$ is not injective as $(-x)^2=x^2$

Surjective / Onto function-

- ★ A function $f:A\rightarrow B$ is surjective (onto) if the image of f equals its range. Equivalently, for every $b\in B$, there exists some $a\in A$ such that $f(a)=b$.This means that for any y in B, there exists some x in A such that $y=f(x)$.

Example

- ★ $f:N\rightarrow N, f(x)=x+2$ is surjective.
- ★ $f:R\rightarrow R, f(x)=x2$ is not surjective since we cannot find a real number whose square is negative

Bijective / One-to-one Correspondent-

- ★ A function $f:A\rightarrow B$ is bijective or one-to-one correspondent if and only if f is both injective and surjective.

Problem Prove that a function $f:R\rightarrow R$ defined by $f(x)=2x-3$ is a bijective .

Explanation – We have to prove this function is both injective and surjective.

If $f(x_1)=f(x_2)$,then $2x_1–3=2x_2–3$ and it implies that $x_1=x_2$.

Hence, f is injective.

Here, $2x–3=y$

So, $x=(y+5)/3$ which belongs to R and $f(x)=y$.

Hence, f is surjective.

Since f is both surjective and injective, we can say f is bijective.

Inverse of a Function

The inverse of a one-to-one corresponding function $f:A\rightarrow B$, is the function $g:B\rightarrow A$, holding the following property –

$f(x)=y\Leftrightarrow g(y)=x$

The function f is called invertible, if its inverse function g exists.

Example

- A Function $f:Z\rightarrow Z,f(x)=x+5$, is invertible since it has the inverse function $g:Z\rightarrow Z,g(x)=x-5$.
- A Function $f:Z\rightarrow Z,f(x)=x^2$ is not invertiable since this is not one-to-one as $(-x)^2=x^2$.

Group and Field

Groups-

★a group is an algebraic structure consisting of a set of elements equipped with an operation that combines any two elements to form a third element and that satisfies four conditions called the group axioms, namely closure, associativity, identity and invertibility.

★One of the most familiar examples of a group is the set of integers together with the addition operation, but the abstract formalization of the group axioms, detached as it is from the concrete nature of any particular group and its operation, applies much more widely.

Subgroup-

★A subgraph S of a graph G is a graph whose set of vertices and set of edges are all subsets of G. (Since every set is a subset of itself, every graph is a subgraph of itself.) ... An edge-induced subgraph consists of some of the edges of the original graph and the vertices that are at their endpoints.

★A subgroup of a group is a subset of that forms a group with the same law of composition. For example, the even numbers form a subgroup of the group of integers with group law of addition.

★Any group has at least two subgroups.

Finite and infinite group-

★a finite group is a mathematical group with a finite number of elements. A group is a set of elements together with an operation which associates, to each ordered pair of elements, an element of the set.

★ In group theory, an area of mathematics, an infinite group is a group, of which the underlying set contains an infinite number of elements.

Cyclic Group-

★ A cyclic group is a group that can be generated by a single element X (the group generator). Cyclic groups are Abelian.

★ A cyclic group of finite group order n is denoted C_n, Z_n, Z_n, or C_n; Shanks 1993, p. 75), and its generator X satisfies

X^n=I,

(1)

where I is the identity element.

★The ring of integers Z form an infinite cyclic group under addition, and the integers 0, 1, 2, ..., n-1 (Z_n) form a cyclic group of order n under addition (mod n). In both cases, 0 is the identity element.

Permutation group-

★A permutation group is a group G whose elements are permutations of a given set M and whose group operation is the composition of permutations in G (which are thought of as bijective functions from the set M to itself).

★The group of all permutations of a set M is the symmetric group of M, often written as Sym(M).[1] The term permutat group thus means a subgroup of the symmetric group. If M = {1,2,...,n} then, Sym(M), the symmetric group on n letters is usually denoted by Sn.

Homomorphism-

★ A homomorphism is a map between two groups which respects the group structure. More formally, let G and H be two group, and f a map from G to H (for every g ∈G, f(g) ∈H). Then f is a

homomorphism if for every g1,g2∈G, f(g1g2)=f(g1)f(g2). For example, if H<G, then the inclusion map i(h)=h ∈G is a

homomorphism. Another example is a homomorphism from Z to Z given by multiplication by 2, f(n)=2n. This map is a homomorphism since f(n+m)=2(n+m)=2n+2m=f(n)+f(m).

Isomorphism-

★ Isomorphism, in modern algebra, a one-to-one correspondence (mapping) between two sets that preserves binary relationships between elements of the sets. For example, the set of natural numbers can be mapped onto the set of even natural numbers by multiplying each natural number by 2. The binary operation of adding two numbers is preserved—that is, adding two natural numbers and then multiplying the sum by 2 gives the same result as multiplying each natural number by 2 and then adding the products together—so the sets are isomorphic for addition.

Automorphism-

An automorphism of a group is any of the following equivalent things:

- ★ An isomorphism from the group to itself
- ★ A bijective endomorphism of the group
- ★ A homomorphism that is both an endomorphism and an isomorphism

Endomorphism-

- ★ An endomorphism is a morphism (or homomorphism) from a mathematical object to itself. For example, an endomorphism of a vector space V is a linear map $f: V \to V$, and an endomorphism of a group G is a group homomorphism $f: G \to G$. In general, we can talk about endomorphisms in any category. In the category of sets, endomorphisms are functions from a set S to itself.

Coset-

★ if G is a group, and H is a subgroup of G, and g is an element of G, then:

$gH = \{ gh : h$ an element of $H \}$ is the left coset of H in G with respect to g, and

$Hg = \{ hg : h$ an element of $H \}$ is the right coset of H in G with respect to g.

Field-

★ In mathematics, a field is a set on which addition, subtraction, multiplication, and division are defined, and behave as when they are applied to rational and real numbers. A field is thus a fundamental algebraic structure, which is widely used in algebra, number theory and many other areas of mathematics.

Subfield and Ring-

★ The definition still holds for a field, by dint of the fact that a field is also a ring with unity.
★ Let (F,+,o) be a field.
★ Let K be a subset of F such that (K,+,o) is also a field.
Then (K,+,o) is a subfield of (F,+,o).

Ring- A ring is a set of elements with two operations, one of which is like addition, the other of which is like multiplication, which we will call add and mul. It has the following properties:

1. The elements of the ring, together with the addition operation, form a group.
2. Addition is commutative. That is, for any two elements of the set p and q, p add q = q add p. (The word Abelian is also used for "commutative", in honor of the mathematician Niels Henrik Abel.)
3. The multiplication operation is associative.
4. Multiplication distributes over addition: that is, for any three elements of the group a, b, and c, a mul (b add c) = (a mul b) add (a mul c).

Unit-2

Mathematical Logic

Statement and Notations-

- ★ A statement (or proposition) is a sentence that is either true or false (both not both). So '3 is an odd integer' is a statement. But 'π is a cool number' is not a (mathematical) statement. Note that '4 is an
- ★ odd integer' is also a statement, but it is a false statement. A mathematical notation is a writing system used for recording concepts in mathematics. The notation uses symbols or symbolic expressions which are intended to have a precise semantic meaning. In the history of mathematics, these symbols have denoted numbers, shapes, patterns, and change.

Connectives-

- ★ A function, or the symbol representing a function, which corresponds to English conjunctions such as "and," "or," "not," etc. that takes one or more truth values as input and returns a single truth value as output. The terms "logical connective" and "propositional connective" (Mendelson 1997, p. 13) are also used. The following table summarizes some common connectives and their notations

connective symbol

AND	$A \wedge B$, $A \cdot B$, $A.B$, $A\,B$, $A \,\&\, B$, $A \,\&\&\, B$

equivalent , ,

implies , ,

NAND , ,

nonequivalent , ,

NOR , ,

NOT , , ,

OR , , ,

XNOR XNOR

XOR	,

Normal Forms-

Normal form (scientific notation) is a way to write very large or very small numbers in a more compact form. It has two parts:

- ★ A number, usually in the range 0 - 10, called the coefficient.
- ★ A power of ten to multiply it by called the exponent.

Theory of inference for the-statement calculus-

★ The method of derivation involving predicate formulas uses the rules of inference given for the statement calculus and

certain additional rules which are required to deal with the formulas involving quantifiers.

- ★ Using of rules P and T remain the same. If the conclusion is given in the form of a conditional, we shall use the rule of conditional proof called CP.
- ★ In order to use the equivalences and implications, we need some rules on how to eliminate quantifiers during the course of derivation.
- ★ This elimination is done by the rules of specification, called rules US and ES. Once the quantifiers are eliminated and the conclusion is reached. It may happen that the desired conclusion is quantified. In this case we need rules of generalization called rules UG and EG which can be used to attach a quantifier.

Rule US (Universal Specification) : From (x)A(x) one can conclude A(y).

Rule ES (Existential Specification) : From ($x)A(x) one can conclude A(y) provided that y is not free in any given premise and also not free in any prior step of the derivation. These requirements can easily be met by choosing a new variable each time ES is used.

Rule EG (Existential Generalization) : From A(x) one can conclude ($ y)A(y).

Rule UG (Universal Generalization) : From A(x) one can conclude (y)A(y) provided that x is not free in any of the given premises and provided that if x is free in a prior step which resulted from use of ES, then no variables introduced by that use of ES appear free in A(x).

Example1: Show that (x) (H(x)® M(x)) Ù H(s)Þ M(s).

This is a well - known argument known as the "Socrates argument" which is given by

All men are mortal.

Socrates is a man.

Therefore Socrates is mortal.

If we denote H(x) : x is a man, M(x) : x is a mortal and s : Socrates, we can put the argument in the above form.

Solution:

{1} (1) (x)(H(x)® M(x)) Rule P

{1} (2) H(s)® M(s) Rule US, (1)

{3} (3) H(s) Rule P

{1, 3} (4) M(s) Rule T, (2), (3), I11

Example 2:

Show that (x)(P(x)® Q(x)) Ù (x)(Q(x)® R(x)) Þ

(x)(P(x)® R(x)). Solution:

{1} (1) (x)(P(x)® Q(x)) Rule P

{1} (2) P(y)® Q(y) Rule US, (1)

{3} (3) (x)(Q(x)® R(x)) Rule P

{4} (4) Q(y)® R(y) Rule US, (3)

{1,3} (5) P(y)® R(y) Rule T, (2), (4), I13

{1, 3} (6) (x)(P(x)® R(x)) Rule UG, (5)

Predicate calculus-

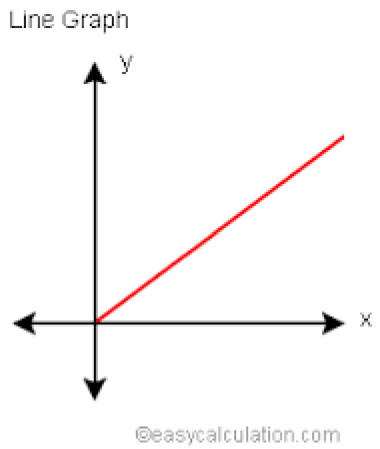
- ★Predicate calculus, also called Logic Of Quantifiers, that part of modern formal or symbolic logic which systematically exhibits the logical relations between sentences that hold purely in virtue of the manner in which predicates or noun expressions are distributed through ranges of subjects by means of quantifiers such as “all” and “some” without regard to the meanings or conceptual contents of any predicates in particular.
- ★Such predicates can include both qualities and relations; and, in a higher-order form called the functional calculus, it also includes functions, which are “framework” expressions with one or with several variables that acquire definite truth-values only when the variables are replaced by specific terms.
- ★The predicate calculus is to be distinguished from the propositional calculus, which deals with unanalyzed whole propositions related by connectives (such as “and,” “if . . . then,” and “or”).

Unit-3

Basic concept of Graph

Basics of Graph-

★ A graph is a diagram of points and lines connected to the points. It has at least one line joining a set of two vertices with no vertex connecting itself.



Point- A point is a particular position in a one-dimensional, two dimensional, or three-dimensional space. For better understanding, a point can be denoted by an alphabet. It can be represented with a dot.

Example Here, the dot is a point named ‘a’.

Line- A Line is a connection between two points. It can be represented with a solid line.

Example Here, ‘a’ and ‘b’ are the points. The link between these two points is called a line.

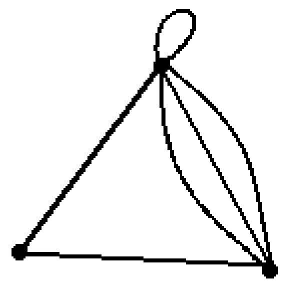
Vertex- A vertex is a point where multiple lines meet. It is also called a node. Similar to points, a vertex is also denoted by an alphabet.

Example Here, the vertex is named with an alphabet 'a'.

Edge- An edge is the mathematical term for a line that connects two vertices. Many edges can be formed from a single vertex. Without a vertex, an edge cannot be formed.

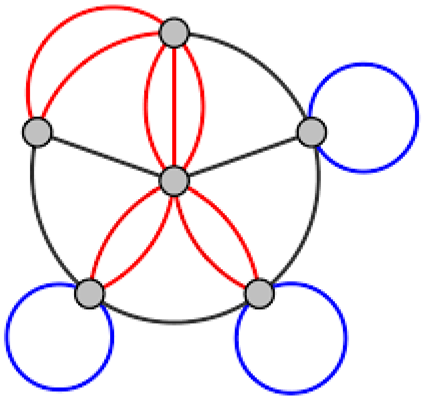
Pseudograph-

★ A pseudograph is a non-simple graph in which both graph loops and multiple edges are permitted



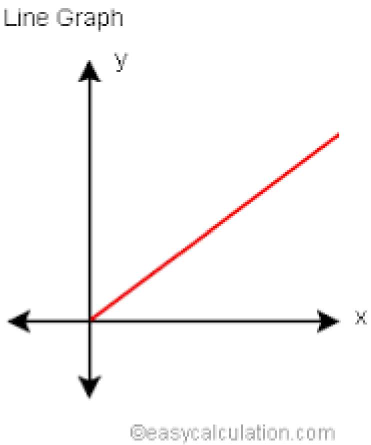
Multigraph-

★ In mathematics, and more specifically in graph theory, a multigraph (in contrast to a simple graph) is a graph which is permitted to have multiple edges (also called parallel edges), that is, edges that have the same end nodes. Thus two vertices may be connected by more than one edge.



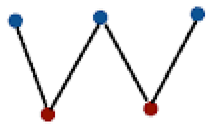
Simple graph-

★ A simple graph, also called a strict graph is an un-weighted, undirected graph containing no graph loops or multiple edges . A simple graph may be either connected or disconnected.

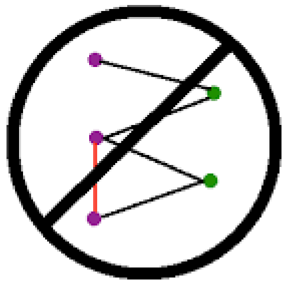


Bipartite graph and Complete Bipartite graph-

Bipartite Graph- A Bipartite Graph is a graph whose vertices can be divided into two independent sets, U and V such that every edge (u, v) either connects a vertex from U to V or a vertex from V to U. In other words, for every edge (u, v), either u belongs to U and v to V, or u belongs to V and v to U. We can also say that there is no edge that connects vertices of same set.



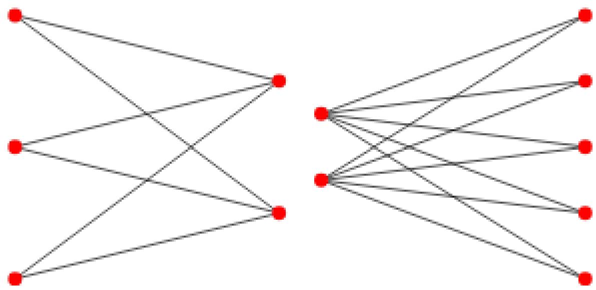
Bipartite



NOT Bipartite

Complete Bipartite graph-

★ A complete bipartite graph, sometimes also called a complete bicolored graph (ErdḦ os et al. 1965) or complete bigraph, is a bipartite graph (i.e., a set of graph vertices decomposed into two disjoint sets such that no two graph vertices within the same set are adjacent) such that every pair of graph vertices in the two sets are adjacent.



Hand Shaking Lemma-

★ Handshaking lemma is about undirected graph. In every finite undirected graph number of vertices with odd degree is always even. The handshaking lemma is a consequence of the degree sum formula (also sometimes called the handshaking lemma) \sum_{u \epsilon v} deg(v) = 2\left | E \right |

Handshaking Theorem - Handshaking theorem states that the sum of degrees of the vertices of a graph is twice the number of edges.

If G=(V,E) be a graph with E edges,then-

Σ degG(V) = 2E

Proof-

Since the degree of a vertex is the number of edges incident with that vertex, the sum of degree counts the total number of times an edge is incident with a vertex. Since every edge is incident with exactly two vertices,each edge gets counted twice,once at each end. Thus the sum of the degrees is equal twice the number of edges.

★ This theorem applies even if multiple edges and loops are present.The theorem holds this rule that if several people shake hands, the total number of hands shake must be even that is why the theorem is called handshaking theorem.

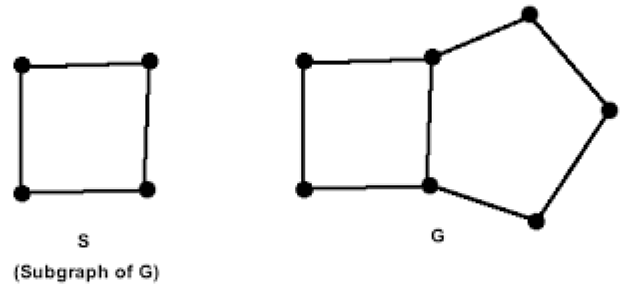
Sub graphs-

★ A spanning subgraph is a subgraph that contains all the vertices of the original graph. A spanning tree is a spanning subgraph that is often of interest. A cycle in a graph that contains all the vertices of the graph would be called a spanning cycle.

★ A subgraph S of a graph G is a graph whose set of vertices and set of edges are all subsets of G. (Since every set is a subset of itself, every graph is a subgraph of itself.)

★ All the edges and vertices of G might not be present in S; but if a vertex is present in S, it has a corresponding vertex in G and any edge that connects two vertices in S will also connect the

corresponding vertices in G. All of these graphs are subgraphs of the first graph.



<u>Operations on graph</u> –

- ★ Graph Operations – Extracting sub graphs
- ★ In this section we will discuss about various types of sub graphs we can extract from a given Graph.

Sub graph

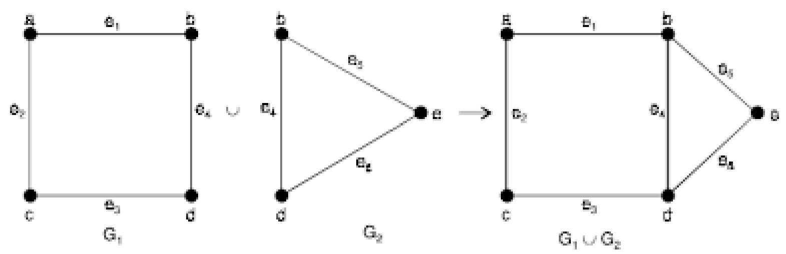
Getting a sub graph out of a graph is an interesting operation. A sub graph of a graph G(V,E) can be obtained by the following **means:**

- ★ Removing one or more vertices from the vertex set.
- ★ Removing one or more edges from the edge family.
- ★ Removing either vertices or edges from the graph.
- ★ Points worth noting

The vertices of sub graphs are subsets of the original vertices The edges of sub graphs are subsets of the original edges

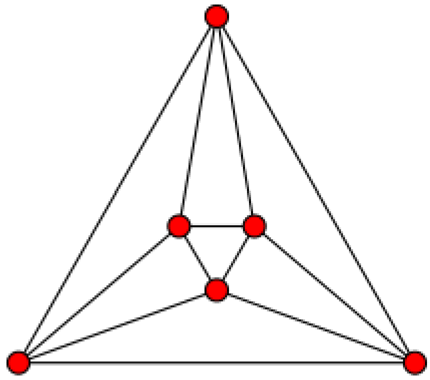
We can extract sub graphs for simple graphs, directed graphs, multi edge graphs and all types of graphs.

The Null Graph is always a sub graph of all the graphs. There can be many sub graphs for a graph.



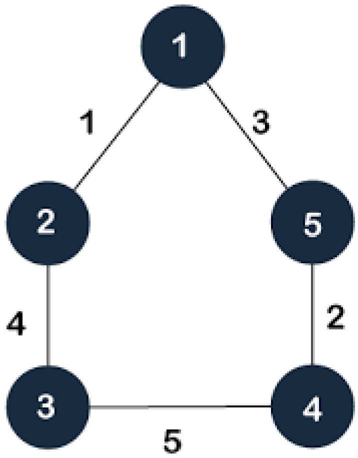
Neighbourhood graph-

★ The neighbourhood graph of a graph G(V,E) only makes sense when we mention it with respect to a given vertex set. For e.g. if V = {1,2,3,4,5} then we can find out the Neighbourhood graph of G(V,E) for vertex set {1}.



Spanning Tree-

A spanning tree of a connected graph G(V,E) is a sub graph that is also a tree and connects all vertices in V. For a disconnected graph the spanning tree would be the spanning tree of each component respectively.



Walk–

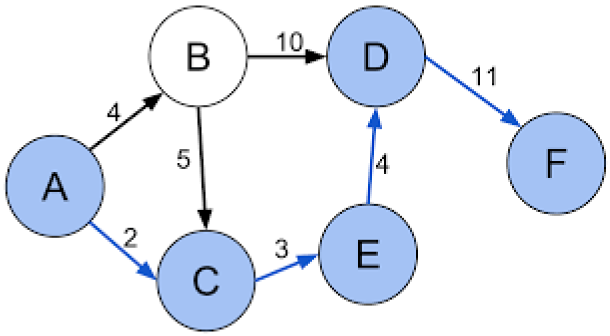
★ For a graph G=(V(G),E(G)), a Walk is defined as a sequence of alternating vertices and edges such as v0,e1,v1,e2,...,ek,vk where each edge ei={vi−1,vi}. The Length of this walk is k.

★ For example, the graph below outlines a possibly walk (in blue).

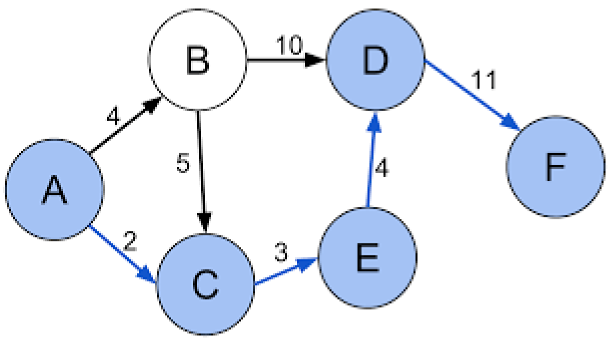
The walk is denoted as abcdb. Note that walks can have repeated edges. For example, if we had the walk abcdcbce, then that would be perfectly fine even though some edges are repeated.

Shortest Path problem–

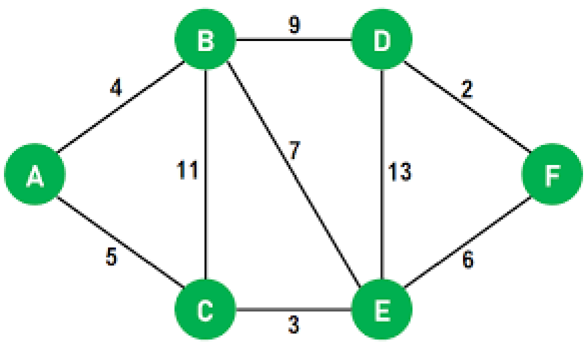
★ Shortest path can be calculated only for the weighted graphs. The edges connecting two vertices can be assigned a nonnegative real number, called the weight of the edge. A graph with such weighted edges is called a weighted graph.



(6, 4, 5, 1) and (6, 4, 3, 2, 1) are both paths between vertices 6 and 1



Shortest path (A, C, E, D, F) between vertices A and F in the weighted directed graph



Dijkstra algorithm-

Unit-4

Eulerian & Hamiltonian Graph

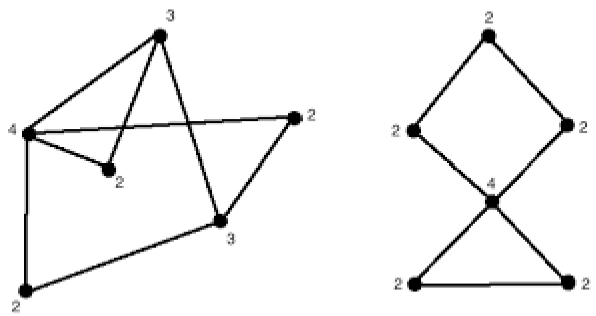
Unicursal and Eulerian graph

Euler Graphs –

★ A closed walk in a graph G containing all the edges of G is called an Euler line in G. A graph containing an Euler line is called an Euler graph. We know that a walk is always connected. Since the Euler line (which is a walk) contains all the edges of the graph, an Euler graph is connected except for any isolated vertices the graph may contain. As isolated vertices do not contribute anything to the understanding of an Euler graph, it is assumed now onwards that Euler graphs do not have any isolated vertices and are thus connected.

Example

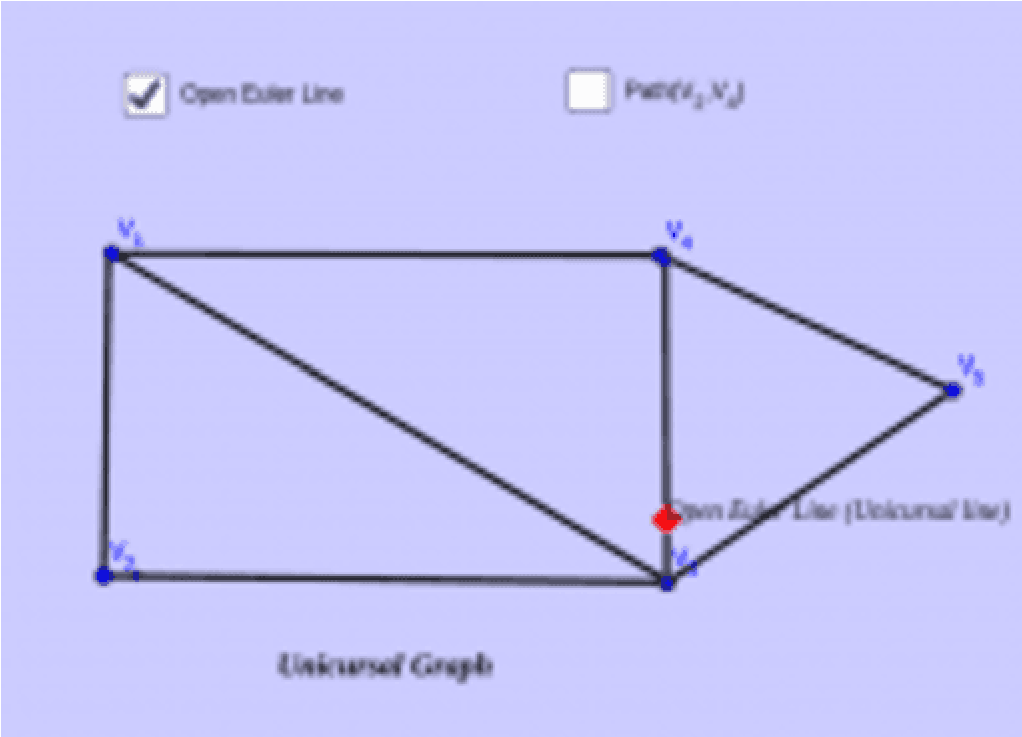
Consider the graph shown in Figure Clearly, v1 e1 v2 e2 v3 e3 v4 e4 v5 e5 v3 v6 e7 v1 in (a) is an Euler line, whereas the graph shown in (b) is non-Eulerian.



Unicursal Graphs -

★ An open walk that includes (or traces) all edges of a graph without retracing any edge is called a unicursal line or open Euler line. A connected graph that has a unicursal line is called a unicursal graph. Figure 3.6 shows a unicursal graph.

★ A connected graph that has a Unicursal line is called Unicursal Graph.
In this figure show a Unicursal Graph



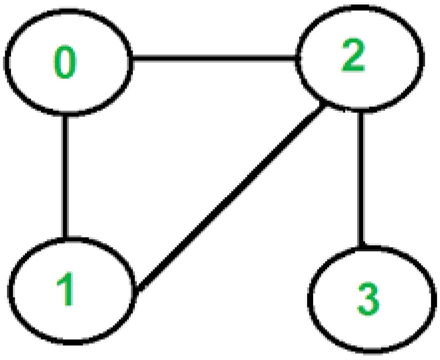
Randomly Eulerian graph-

★ A graph is defined to be randomly matchable if every matching of G can be extended to a perfect matching. ... Ore proved that an Eulerian graph is randomly Eulerian from u if and only if u lies on every cycle of G. Moreover, the only graphs that are randomly Eulerian from every vertex are q(p23).

Fleury’s Algorithm-

★ Fleury's algorithm shows you how to find an Euler path or circuit. It begins with giving the requirement for the graph. The graph must have either 0 or 2 odd vertices. An odd vertex is one where the number of edges connecting the vertex to other vertices is odd.

★ If we have 2 odd vertices, then we will have an Euler path. If we have 0 odd vertices, then we will have an Euler circuit. If we have 2 odd vertices, then we start at one of those two vertices. If we have 0 odd vertices, then we can start anywhere.

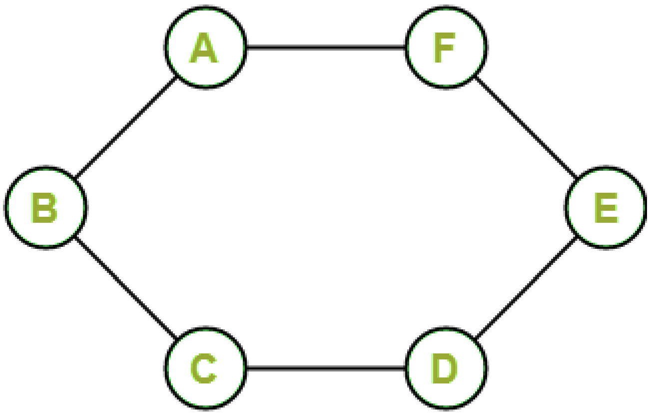


Chinese Postman Problem-

★ Chinese Postman Problem is a variation of Eulerian circuit problem for undirected graphs. An Euler Circuit is a closed walk that covers every edge once starting and ending position is same.
★ Chiniese Postman problem is defined for connected and undirected graph. The problem is to find shortest path or circuity that visits every edge of the graph at least once.
★ If input graph contains Euler Circuit, then a solution of the problem is Euler Circuit.
★ An undirected and connected graph has Eulerian cycle if “all vertices have even degree”.

Hamiltonian Graph-

★ A Hamiltonian graph, also called a Hamilton graph, is a graph possessing a Hamiltonian cycle. A graph that is not Hamiltonian is said to be non-hamiltonian.
★ A Hamiltonian graph on n nodes has graph circumference
★ While it would be easy to make a general definition of "Hamiltonian" that goes either way as far as the singleton graph K_1 is concerned, defining "Hamiltonian" to mean "has a Hamiltonian cycle" and taking "Hamiltonian cycles" to be a subset of "cycles" in general would lead to the convention that the singleton graph is non-hamiltonian.



Example of Hamiltonian Graph

Necessary and Sufficient conditions

Necessary condition-

- ★ A condition A is said to be necessary for a condition B, if (and only if) the falsity (/nonexistence /non-occurrence) [as the case may be] of A guarantees (or brings about) the falsity (/nonexistence /non occurrence) of B.
- ★ So common is this notion of necessary condition that there are, not surprisingly, a great many ways to express that something is a necessary condition. Here are a number of examples, all - more or less - saying the same thing:
    - o "Air is necessary for human life."
    - o "Human beings must have air to live."
    - o "Without air, human beings die (i.e. do not live)."
    - o "If a human being is alive, then that human being has air (to breathe)."

Sufficient condition-

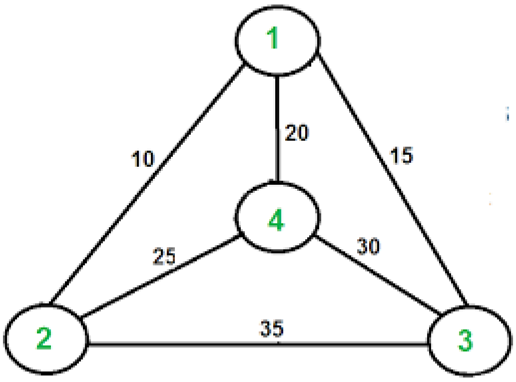
- ★ A condition A is said to be sufficient for a condition B, if (and only if) the truth (/existence /occurrence) [as the case may be] of A guarantees (or brings about) the truth (/existence /occurrence) of B.

  For example, while air is a necessary condition for human life, it is by no means a sufficient condition, i.e. it does not, by itself, i.e. alone, suffice for human life. While someone may have air to breathe, that person will still die if s/he lacks water (for a number of days), has taken poison, is exposed to extremes of cold or heat, etc.

Traveling Salesman Problem-

★ Given a set of cities and distance between every pair of cities, the problem is to find the shortest possible route that visits every city exactly once and returns to the starting point.

★ Note the difference between Hamiltonian Cycle and TSP. The Hamiltoninan cycle problem is to find if there exist a tour that visits every city exactly once. Here we know that Hamiltonian Tour exists (because the graph is complete) and in fact many such tours exist, the problem is to find a minimum weight



Hamiltonian Cycle.

For example, consider the graph shown in figure. A TSP tour in the graph is 1-2-4-3-1.

The cost of the tour is 10+25+30+15 which is 80.

Unit-5

Trees & Spanning Trees

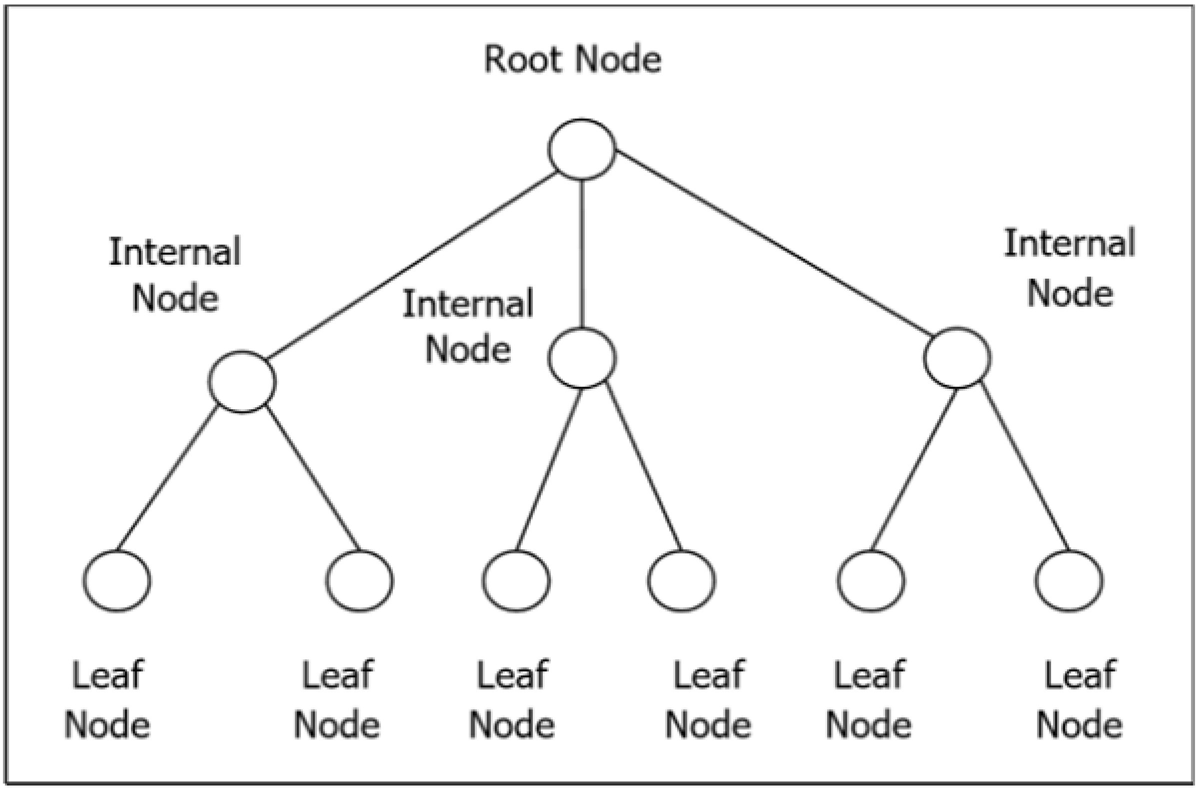
Tree-

- ★ is non-linear type of data structure.
- ★ Tree is a discrete structure that represents hierarchical relationships between individual elements or nodes. A tree in which a parent has no more than two children is called a binary tree.

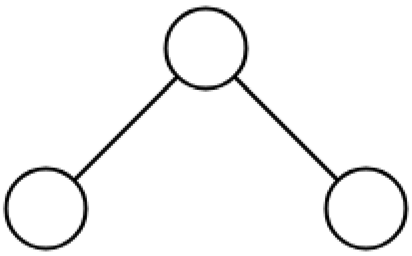
Properties of Tree-

- ★ There are various properties of tree such as –

Labeled Trees - A labeled tree is a tree the vertices of which are assigned unique numbers from 1 to n. We can count such trees for small values of n by hand so as to conjecture a general formula. The number of labeled trees of n number of vertices is nn−2. Two labeled trees are isomorphic if their graphs are isomorphic and the corresponding points of the two trees have the same labels.

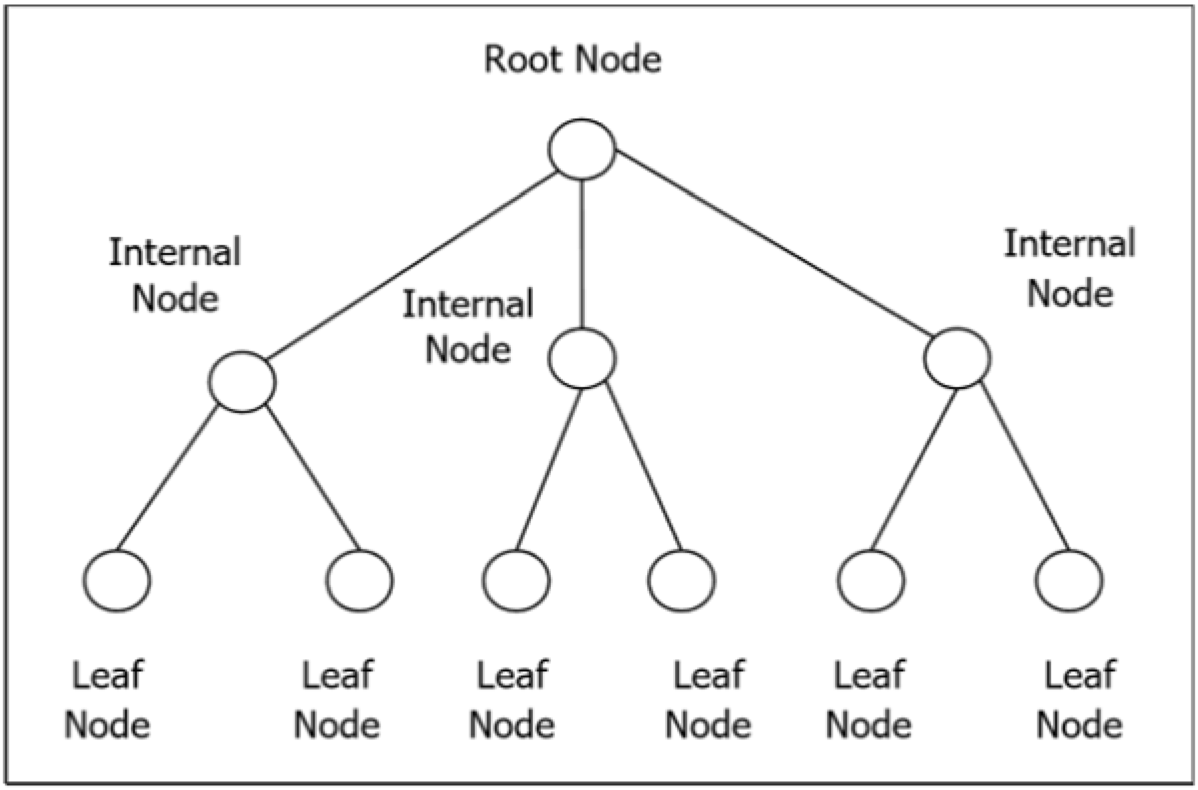


Unlabeled Trees - An unlabeled tree is a tree the vertices of which are not assigned any numbers.



Unlabeled Binary Tree

Rooted Tree - A rooted tree G is a connected acyclic graph with a special node that is called the root of the tree and every edge directly or indirectly originates from the root. An ordered rooted tree is a rooted tree where the children of each internal vertex are ordered.



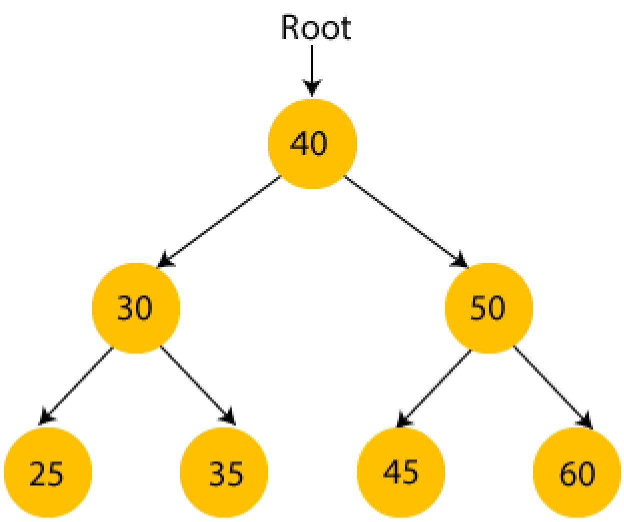
Binary Search Tree - Binary Search tree is a binary tree which satisfies the following property −

X in left sub-tree of vertex V, Value (X) ≤ Value (V)

Y in right sub-tree of vertex V, Value (Y) ≥ Value

(V)

So, the value of all the vertices of the left sub-tree of an internal node V are less than or equal to V and the value of all the vertices of the right sub-tree of the internal node V are greater than or equal to V.



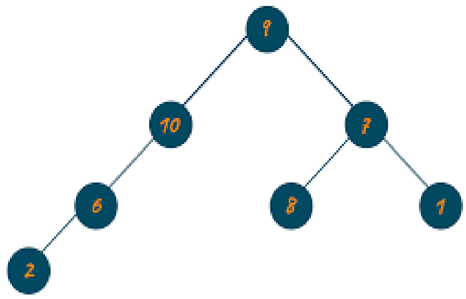
Distance-

★ Distance between two nodes is the minimum number of edges to be traversed to reach one node from other.

Diameter of a Tree-

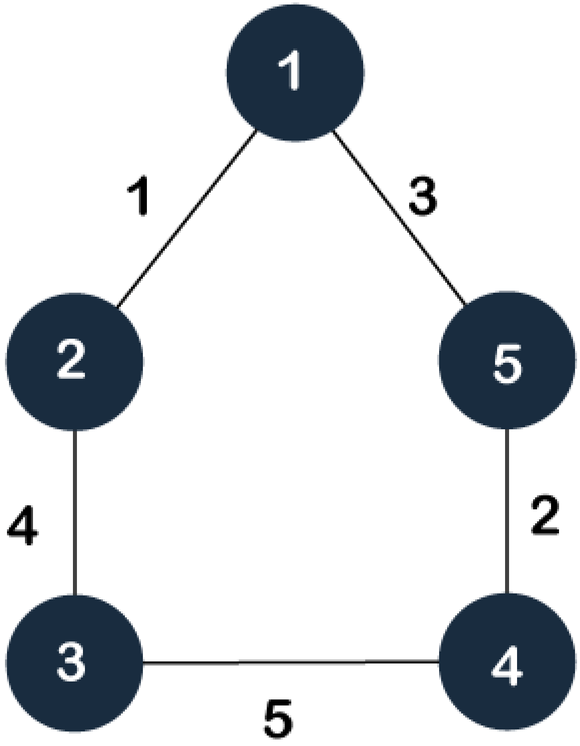
★ The diameter of a tree (sometimes called the width) is the number of nodes on the longest path between two end nodes. The diagram below shows two trees each with diameter nine, the leaves that form the ends of a longest path are shaded (note

that there is more than one path in each tree of length nine, but no path longer than nine nodes).



Spanning Tree-

★ A spanning tree of a connected undirected graph G is a tree that minimally includes all of the vertices of G. A graph may have many spanning trees.



Fundamenta circuit –

- ★ A circuit formed by adding a chord to spanning tree is called a fundamental circuit.

Cayley's Formula for number of Spanning Tree-

- ★ Cayley's formula counts the number of labeled trees on n vertices. Put another way, it counts the number of spanning trees of a complete graph Kn. Note that it does not count the number of nonisomorphic trees on n vertices.
- ★ For comparison, there are 6 nonisomorphic trees on 6 vertices, while there are 64 = 1296 labeled trees on 6 vertices.

Let Tn denote the number of trees on n labeled vertices.

Cayley’s formula states:

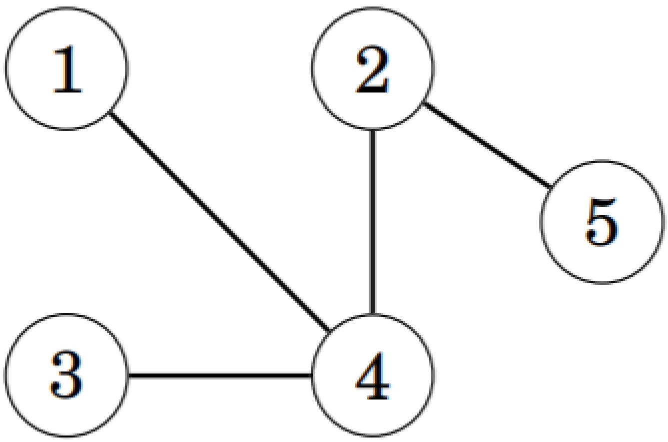
Tn = n

n−2

In total, there are Tn · n · (n − 1)! different sequences of directed edges to add in a graph so as to form a directed rooted tree; so

τ = Tn · n · (n − 1)!

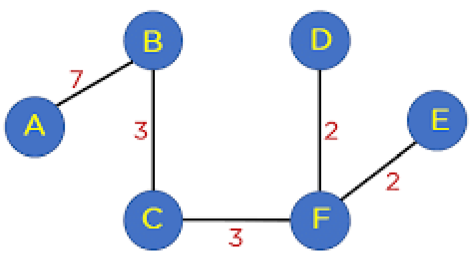
These are called spanning trees on n vertices, and we will denote the set of these spanning trees by Tn. The following is a diagram of all of elements of T4:



Minimum Spanning Tree-

★ A spanning tree with assigned weight less than or equal to the weight of every possible spanning tree of a weighted, connected and undirected graph GG, it is called minimum spanning tree (MST). The weight of a spanning tree is the sum of all the weights assigned to each edge of the spanning tree.

★ In a weighted graph, a minimum spanning tree is a spanning tree that has minimum weight than all other spanning trees of the same graph.



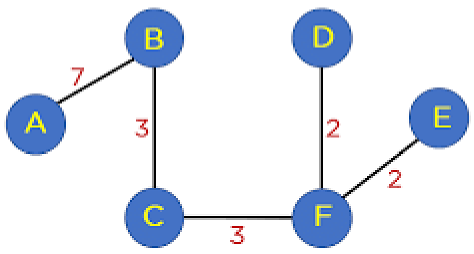
Minimum Spanning Tree.

Kruskal's Algorithm-

★ Kruskal's algorithm is a minimum-spanning-tree algorithm which finds an edge of the least possible weight that connects any two trees in the forest. It is a greedy algorithm in graph theory as it finds a minimum spanning tree for a

connected weighted graph adding increasing cost arcs at each step.

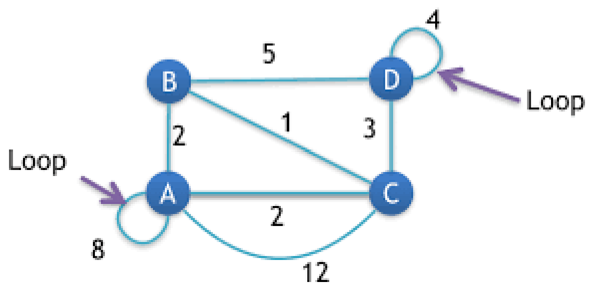
To understand Kruskal's algorithm let us consider the following example –



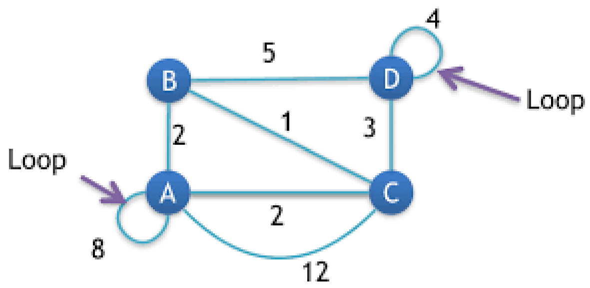
Minimum Spanning Tree.

Step 1 - Remove all loops and Parallel Edges

Remove all loops and parallel edges from the given graph.



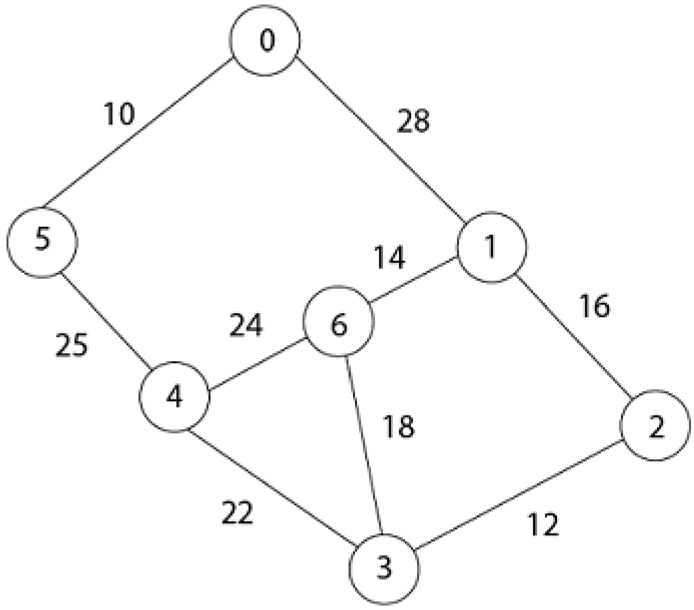
In case of parallel edges, keep the one which has the least cost associated and remove all others.



Prim's Spanning Tree Algorithm-

★ Prim's algorithm to find minimum cost spanning tree (as Kruskal's algorithm) uses the greedy approach. Prim's algorithm shares a similarity with the shortest path first algorithms.

★ Prim's algorithm, in contrast with Kruskal's algorithm, treats the nodes as a single tree and keeps on adding new nodes to the spanning tree from the given graph.

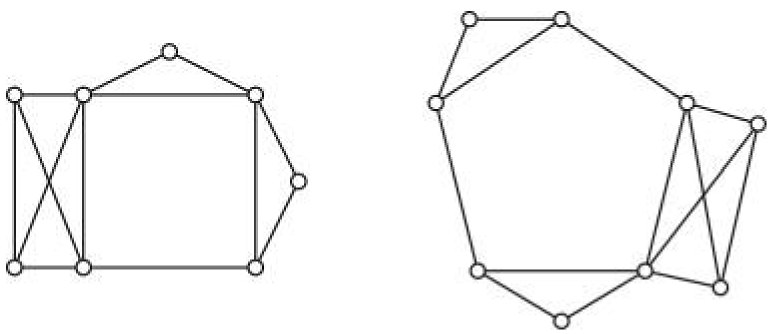


Connectivity and Seperability-

★ A graph is said to be connected if there is a path between every pair of vertex. From every vertex to any other vertex, there should be some path to traverse. That is called the connectivity of a graph. A graph with multiple disconnected vertices and edges is said to be disconnected.

Example 1

In the following graph, it is possible to travel from one vertex to any other vertex. For example, one can traverse from vertex 'a' to vertex 'e' using the path 'a-b-e'.



Example 2

In the following example, traversing from vertex 'a' to vertex 'f' is not possible because there is no path between them directly or indirectly. Hence it is a disconnected graph.

<u>Seperability</u> -

★ A graph G is said to be separable if it is either disconnected or can be disconnected by removing one vertex, called articulation. A graph that is not separable is said to be biconnected (or nonseparable).

Network Flow

★ In graph theory, a flow network (also known as a transportation network) is a directed graph where each edge has a capacity and each edge receives a flow. The amount of flow on an edge cannot exceed the capacity of the edge.

★ Supply = sourse

★ Use for commodity = sink

Cuts in a Network-

★ In a flow network, an s–t cut is a cut that requires the source and the sink to be in different subsets, and its cut-set only consists of edges going from the source's side to the sink's side.

The capacity of an s–t cut is defined as the sum of capacity of each edge in the cut-set.

★ If s and t are specified vertices of the graph G, then an s–t cut is a cut in which s belongs to the set S and t belongs to the set T.

Max flow min cut theory-

★ The max-flow min-cut theorem is a network flow theorem. This theorem states that the maximum flow through any network from a given source to a given sink is exactly the sum of the edge weights that, if removed, would totally disconnect the source from the sink. In other words, for any network graph and a selected source and sink node, the max-flow from source to sink = the min-cut necessary to separate source from sink. Flow can apply to anything. For instance, it could mean the amount of water that can pass through network pipes. Or, it could mean the amount of data that can pass through a computer network like the Internet.

★ Max-flow min-cut has a variety of applications. In computer science, networks rely heavily on this algorithm. Network reliability, availability, and connectivity use max-flow min-cut.

In mathematics, matching in graphs (such as bipartite matching) uses this same algorithm. In less technical areas, this algorithm can be used in scheduling. For example, airlines use this to decide when to allow planes to leave airports to maximize the "flow" of flights.

<u>Augmenting path</u>-

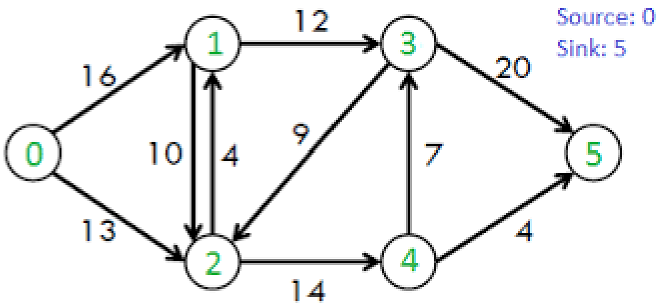
- ★ Augmenting path is a directed path from a source node s to a sink node t in the residual network. The residual capacity of an augmenting path is the minimum residual capacity of any arc in the path. Obviously, we can send additional flow from the source to the sink along an augmenting path.
- ★ An augmenting path is a simple path - a path that does not contain cycles - through the graph using only edges with positive capacity from the source to the sink.
- ★ So the statement above is somehow obvious - if you can not find a path from the source to the sink that only uses positive capacity edges, then the flow can not be increased.

<u>Ford and Fulkerson algorithm</u>-

- ★ It was discovered in 1956 by Ford and Fulkerson. This algorithm is sometimes referred to as a method because parts of its protocol are not fully specified and can vary from implementation to implementation. An algorithm typically refers to a specific protocol for solving a problem, whereas a method is a more general approach to a problem.
- ★ The Ford-Fulkerson algorithm assumes that the input will be a graph, , along with a source vertex, , and a sink vertex, . The graph is any representation of a weighted graph where vertices are connected by edges of specified weights. There must also be a source vertex and sink vertex to understand the beginning and end of the flow network.
- ★ Ford-Fulkerson has a complexity of where is the maximum flow of the network. The Ford-Fulkerson algorithm was

eventually improved upon by the Edmonds-Karp algorithm, which does the same thing in time, independent of the maximum flow value.

★ The intuition behind the algorithm is quite simple (even though the implementation details can obscure this). Imagine a flow network that is just a traffic network of cars. Each road can hold a certain number of cars. This can be illustrated by the following graphic-




```
Ford-Fulkerson Algorithm(Graph , source , sink )-
initialize flow to 0
path = findAugmentingPath(G, s, t)
while path exists:
augment flow along path #This is purposefully  4
ambiguous for now
G_f = createResidualGraph()
path = findAugmentingPath(G_f, s, t)
return flow
```

Edonds and Karp algorithm-

★ Edmonds-Karp improves the runtime of Ford-Fulkerson, which is , to . This improvement is important because it makes the runtime of Edmonds-Karp independent of the maximum flow of the network.

★ Finding the maximum flow for a network was first solved by the Ford-Fulkerson algorithm. A network is often abstractly defined as a graph, , that has a set of vertices, , connected by a set of edges, . There is a source, , and a sink, , which represent where the flow is coming from and where it is going to. Finding the maximum flow through a network was solved via the max-flow min-cut theorem, which then was used to create the

Ford Fulkerson algorithm.

- ★ Edmonds-Karp is identical to Ford-Fulkerson except for one very important trait. The search order of augmenting paths is well defined. As a refresher from the Ford-Fulkerson wiki, augmenting paths, along with residual graphs, are the two important concepts to understand when finding the max flow of a network.
- ★ Augmenting paths are simply any path from the source to the sink that can currently take more flow. Over the course of the algorithm, flow is monotonically increased. So, there are times when a path from the source to the sink can take on more flow, and that is an augmenting path.

The following graph shows a set of vertices and edges. Each edge shows two numbers: its current flow divided by its capacity

Menger's Theorems –

- ★ In the mathematical discipline of graph theory and related areas, Menger's theorem is a characterization of the connectivity in finite undirected graphs in terms of the maximum number of disjoint paths that can be found between any pair of vertices.
- ★ Let G be a directed graph. The minimum number of edges whose removal disconnects s from t (the minimum-cut between s and t) is equal to the maximum number of edge-disjoint paths in G between s and t.

Proof : Max flow-min cut theorem and integrality of flow.

- ★ Menger proved his theorem before Max flow-Min cut theorem! Max flow-Min cut theorem is a generalization of Menger's theorem to capacitated graphs.