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ASSIGNMENT - 4

10.34 Number of Concurrent Users follow Normal Distribution with Mean θ and S.D. = 4000 people.

Prior Distribution $\Rightarrow$ Mean(θ) = 14000, S.D. = 2,000.

$\Rightarrow N(14000, 2000)$

From the samples in Ex. 8.2, Mean $\bar{x} = 18000$

$$f(\bar{x}/\theta) \sim N(14000, 2000)$$

Prior distribution of θ, $\pi(\theta) \sim N(\mu, \tau)$

$f(\bar{x}/\theta) \sim N(\theta, \sigma)$

then the Posterior distribution of θ is given by

$$\pi(\theta/\bar{x}) \sim N\left(\frac{n\bar{x}^2/\sigma^2 + \mu/\tau^2}{n/\sigma^2 + 1/\tau^2}, \frac{1}{\sqrt{n/\sigma^2 + 1/\tau^2}}\right)$$

a. Calculate Bayes Estimator (θ)

$$\pi(\theta/\bar{x}) \sim N\left(\frac{50\times18000/4000^2 + 14000/2000^2}{50/4000^2 + 1/2000^2}, \frac{1}{\sqrt{50/4000^2 + 1/2000^2}}\right)$$

$$\sim N\left(\frac{0.05975}{3.375\times10^{-6}}, \frac{1}{\sqrt{3.375\times10^{-6}}}\right)$$

$$\sim N(17703.70, 544.33)$$

$\therefore$ Bayes Estimator of θ

$\hat{\theta} = E(\theta/x)$

$$\boxed{\hat{\theta} = 17704}$$

b. Highest Posterior Density 90%.

$$90\% \text{ HPD} = \left[\mu_x - Z_{\alpha/2}\tau_x \,,\ \mu_x + Z_{\alpha/2}\tau_x\right]$$

Here $Z_{\alpha/2} = 0.05$, since $\alpha = 0.1$ $[1-\alpha = 0.9]$

$$\therefore \left[17704 - Z_{0.05} \times 544 \,,\ 17704 + Z_{0.05} \times 544\right]$$

$[Z_{0.05} = 1.645$, from Table A5$]$

$$= \left[17704 - 1.645 \times 544 \,,\ 17704 + 1.645 \times 544\right]$$

$$\boxed{90\% \text{ HPD} = [16808, 18600]}$$

$\therefore$ We are 90% confident that the number of concurrent users lies between $[16808, 18600]$

c. Since 16000 lies below the credible set [HPD] found above in (b), we can say that at 90% significant level, there is significant evidence that the mean number of concurrent users exceeds 16,000

10.35 Continuing from 10.34, mean $\bar{x} = 18000$ and S.D. $\sigma = 3.16$

a. <u>Non - Bayesian Estimator</u>

For Non-Bayesian estimator, we consider only the information provided by the sample.

$$\therefore \hat{\theta} = \bar{x} = 18,000$$

$$\boxed{\hat{\theta} = 18,000}$$

b. <u>90% Confidence Interval:</u>

$$90\% \ CI = \left[\bar{x} - Z_{\alpha/2} \frac{s}{\sqrt{n}}, \ \bar{x} + Z_{\alpha/2} \frac{s}{\sqrt{n}}\right]$$

Here $\bar{x} = 18000$, $\alpha/2 = 0.1$ [since $1 - \alpha = 0.9$]

$$\therefore Z_{\alpha/2} = 1.645 \quad (Z_{0.05})$$

$s = 3157$, $n = 50$

$$= \left[18000 - 1.645 \times \frac{3157}{\sqrt{50}}, \ 18000 + 1.645 \times \frac{3157}{\sqrt{50}}\right]$$

$$= [17265.66, \ 18735]$$

$$= [17266, \ 18735]$$

Without any prior information on a sample size of 50, we are 90% confident that the number of concurrent users lie between (17266, 18735)

c. Since 16,000 lies below the credible set found in (b), at 90% significant level, there is a significant evidence that the number of concurrent users exceeds 16,000.

d. In the previous problem (10.34)

Credible set was [16808, 18600]

In the current problem (10.35)

Credible set is [17266, 18733]

∴ From the above details, it is clear that adding prior information about the data increases the credibility of the estimation.

5.2 U = Standard Uniform Random Variable. Generate the following.

a. Exponential Random Variable with $\lambda = 2.5$

$F(x) = U$ for X [general equation]

where $F(x)$ is the ~~continuous~~ cumulative density function.

For Exponential Random distribution

$F(x) = 1 - e^{-\lambda x}$

$1 - e^{-2.5x} = U$ [Here $\lambda = 2.5$]

$X = \frac{-1}{2.5} \ln(1-U)$

Given $U = 0.3972 \Rightarrow x = \frac{-1}{2.5} \ln(1 - 0.3972)$

$x = \frac{-1}{2.5} \ln(0.6028)$

$= 0.202467$

$\therefore \boxed{x = 0.2025}$

b. Bernoulli Random Variable with probability of success 0.77

For Bernoulli Random Variable,

$$X = \begin{cases} 1, & \text{if } U < 0.77 \rightarrow \text{success} \\ 0, & \text{if } U \geq 0.77 \Rightarrow \text{failure} \end{cases}$$

Here $U = 0.3972$

$\therefore \boxed{X = 1}$

c. Binomial Random Variable with $n=15$, $p=0.4$

Let $F(i)$ be the cumulative distribution function

General form of Binomial Random Variable is

$${}^{n}C_{x}\, p^{x}\, q^{n-x}$$

In this case,

$$15C_0 (0.4)^0 (0.6)^{15-0} + 15C_1 (0.4)^1 (0.6)^{15-1} + \dots + 15C_{i-1} (0.4)^{i-1} (0.6)^{15-i+1}$$

$$\leq U < 15C_0 (0.4)^0 (0.6)^{15-0} + \dots + 15C_{i-1} (0.4)^{i-1} (0.6)^{15-i+1} +$$

$$15C_i (0.4)^i (0.6)^{15-i} \quad \text{— ①}$$

We compute X when $U = 0.3972$

Substitute the value of U in equation ①, we get

$i=5 \Rightarrow \boxed{X=5}$

d. Discrete Random Variable with $P(x)$ where $P(0)=0.2$, $P(2)=0.4$, $P(7)=0.3$, $P(11)=0.1$

Let $F(i)$ be the cumulative distribution function

$F(i-1) \leq U < F(i)$ (or) $P_0 + P_1 + \dots + P_{i-1} \leq U < P_0 + P_1 + \dots + P_{i-1} + P_i$

compute x when $U = 0.3972$

$\therefore P(0) \leq U < P(0) + P(2)$ — ①

From equation ① $\Rightarrow$ $\boxed{X=2}$

e. Continuous Random Variable $f(x) = 3x^2, \quad 0 < x < 1$

For Continuous Random Variable,

$$F(x) = \int_0^x f(x)\,dx$$

$$= \int_0^x 3x^2\,dx$$

$$= \left[x^3\right]_0^x = X^3$$

$$\therefore X^3 = U \Rightarrow X = U^{1/3}$$

$$X = (0.3972)^{1/3}$$

$$\boxed{\therefore X = 0.7351}$$

f. Continuous Random Variable $f(x) = 1.5x^2, \quad -1 < x < 1$

$$F(x) = \int_0^x 1.5x^2\,dx$$

$$= \left[\frac{1.5x^3}{3}\right]_0^X = \left[0.5x^3\right]_0^x = 0.5X^3 + 0.5$$

$$0.5x^3 + 0.5 = U$$

Multiply both sides by 2

$$x^3 + 1 = 2U$$

$$x^3 = 2U - 1$$

$$x = (2U-1)^{1/3}$$

$$x = \left[2(0.3972) - 1\right]^{1/3}$$

$$= -0.59$$

$$\boxed{\therefore X = -0.59}$$

9. Continuous Random Variable, $f(x) = \frac{1}{12}\sqrt[3]{x}$, $0 \le x \le 8$

$$F(x) = \int_0^{x} f(x)\,dx$$

$$= \int_0^{x} \frac{1}{12}(x)^{1/3}\,dx$$

$$= \left[\frac{1}{12} \cdot \frac{x^{4/3}}{4/3}\right]_0^{x}$$

$$= \left[\frac{1}{16} x^{4/3}\right]$$

$$\frac{1}{16} x^{4/3} = U$$

$$\Rightarrow \quad X = 8U^{3/4}$$

Here $U = 0.3972 \Rightarrow X = 8 \times (0.3972)^{3/4}$

$= 4.00$

$$\boxed{\therefore \quad X = 4.00}$$

5.6 Generate Random Variable X :

find out the density function and the cumulative density function (cdf) of X

Density function of the variable X is given by $\lambda e^{-\lambda x}$

For first mechanic, $f_1(x) = 5e^{-5x}$, $0 \le x < \infty$

For second mechanic, $f_2(x) = 20e^{-20x}$, $0 \le x < \infty$

For first mechanic, Service time probability is 1/5 and
For second mechanic, probability is 4/5

$$f(x) = \frac{1}{5} 5 e^{-5x} + \frac{4}{5} 20 e^{-20x}$$

$$\boxed{\therefore \ f(x) = e^{-5x} + 16 e^{-20x}}$$

cdf of f(x) is given by
$$F(x) = \int_0^x f(x)\,dx = \int (e^{-5x} + 16 e^{-20x})\,dx$$

$$= \left[\frac{1}{5} - \frac{1}{5} e^{-5x}\right]_0^x + 16\left[\frac{1}{20} + \frac{1}{20} e^{-20x}\right]_1^x$$

$$\boxed{\therefore \ F(X) = \left[1 - \frac{1}{5} e^{-5X} - \frac{4}{5} e^{-20X}\right]}$$

It has a complicated form Random variables can be generated by rejecting method as density is available

From A1 table, row 12, column 8 => $U = 0.6649$

equating $F(X) = U$

$$0.6649 = 1 - \frac{1}{5} e^{-5X} - \frac{4}{5} e^{-20X}$$

$$e^{-5X} = (e^X)^{-5} ; \quad e^{-20X} = (e^X)^{-20}$$

$$0.6649 = \cancel{1} - \frac{1}{5}(U)^{-5} - \frac{4}{5}(U)^{-20} \quad \Rightarrow \text{take } e^X = U$$

Solving for U => $U = \sqrt[5]{1.42385}$

$$U = e^X$$

$$e^X = \sqrt[5]{1.42385}$$

$$x = \frac{1}{5} \ln(1.42385)$$

$$\boxed{X = 0.07067289}$$

From 5.9

i) Method to generate sample for number of days

Let x be no. of days

$n \rightarrow$ number of ~~infected~~ computers

count $= 1 \rightarrow$ number (infected)

Let i be infected, ni be non-infected

```
for Cj in i
    for Ci in ni
        u = sample from Bernoulli (0.1) [from 5.9]
        if u < 0.1
            count ++
            i. add (Ci)
            ni. remove (Cj)
        end if
    end for
end for

for i=1 to min (count, 5) // choose 5 computers
    rand = randomly selected from infected
    i. remove (rand)
    ni. add (rand)
    count --
end for
X ++      // increase the number of days
```

iii) Method to generate X, where X is the number of ~~infected days~~ computers infected

```
X = 1
Let i be infected , ni be non-infected

for Ci in i
    for Cj in ni
        u = sample from Bernoulli (0.1)
        if u < 0.1
            X ++
            i. add (Ci)
            ni. remove (Cj)
        end if
    end for
end for

for i = 0 to min (i.length , 5)  // choose 5 at random
    rand = randomly select from infected
    i. remove (rand)
    ni. add (rand)
    X --
end for
```