ZT For Difference Eqs.

Given a difference equation that models a D-T system we may want to solve it:

-with IC's — Apply ZT to the Difference Equation

-with IC's of zero — Use the Transfer Function Approach

Note… the ideas here are very much like what we did with the Laplace Transform for CT systems.

We'll consider the ZT/Difference Eq. approach first…

Solving a First-order Difference Equation using the ZT

Given: $y[n] + ay[n-1] = bx[n]$

$\text{IC} = y[-1]$

$x[n] \quad \text{for } n = 0,\ 1,\ 2,\ \ldots$

Solve for: $y[n]$ for $n = 0, 1, 2, \ldots$

Take ZT of differential equation: $\boxed{Z\{y[n] + ay[n-1]\} = Z\{bx[n]\}}$

Use Linearity of ZT

$$\boxed{\underbrace{Z\{y[n]\}}_{Y(z)} + aZ\{y[n-1]\} = b\underbrace{Z\{x[n]\}}_{X(z)}}$$

Need Right-Shift Property… but which one???

Because of the non-zero IC we need to use the *non*-causal form:

$$\boxed{Z\{y[n-1]\} = z^{-1}Y(z) + y[-1]}$$

Using these results gives: $\boxed{Y(z) + a\left[z^{-1}Y(z) + y[-1]\right] = bX(z)}$

Which is an algebraic equation that can be solved for $Y(z)$:

Using these results gives: $Y(z) + a\left[z^{-1}Y(z) + y[-1]\right] = bX(z)$

Which is an algebraic equation that can be solved for $Y(z)$:

$$Y(z) = \frac{-ay[-1]}{1+az^{-1}} + \frac{b}{1+az^{-1}}X(z)$$

Not the best form for doing Inverse ZT… we want things in terms of z not z^{-1}

Multiply each term by z/z

$$Y(z) = -ay[-1]\frac{z}{z+a} + \frac{bz}{z+a}X(z)$$

On ZT Table

Part due to input signal modified by **Transfer Function**

$$H(z) = \frac{bz}{z+a}$$

$$y[n] = -ay[-1](-a)^n u[n] + Z^{-1}\{H(z)X(z)\}$$

If $|a| < 1$ this dies out as $n \uparrow$, its an IC-driven transient

If the ICs are zero, this is all we have!!!

Ex.: Solving a Difference Equation using ZT: 1st-Order System w/ Step Input

For $x[n] = u[n] \leftrightarrow X(z) = \frac{z}{z-1}$

Then using our general results we just derived we get:

$$Y(z) = \frac{-ay[-1]z}{z+a} + \left(\frac{bz}{z+a}\right)\left(\frac{z}{z-1}\right)$$

For now assume that $a \neq -1$ so we don't have a repeated root.

Then doing Partial Fraction Expansion we get (and we have to do the PFE by hand because we don't know a… but it is not that hard!!!)

$$Y(z) = \frac{-ay[-1]z}{z+a} + \frac{\left(\frac{ab}{a+1}\right)z}{z+a} + \frac{\left(\frac{b}{a+1}\right)z}{z-1}$$

Now using ZT Table we get:

$$y[n] = -ay[-1](-a)^n + \frac{b}{a+1}\left[a(-a)^n + (1)^n\right] \qquad n = 0, 1, 2, \ldots$$

IC-Driven Transient: decays if system is stable

Input-Driven Output… 2 Terms:

1st term decays (Transient)

2nd term persists (Steady State)

Solving a Second-order Difference Equation using the ZT

The Given Difference Equation: $\boxed{y[n]+a_1y[n-1]+a_2y[n-2]=b_0x[n]+b_1x[n-1]}$

Assume that the input is causal

Assume you are given ICs: $y[-1]$ & $y[-2]$

Find the system response $y[n]$ for $n = 0, 1, 2, 3, \ldots$

Take the ZT using the non-causal right-shift property:

$$\boxed{\begin{aligned}Y(z)+a_1\left(z^{-1}Y(z)+y[-1]\right)+a_2\left(z^{-2}Y(z)+z^{-1}y[-1]+y[-2]\right)\\ =b_0X(z)+b_1z^{-1}X(z)\end{aligned}}$$

$$Y(z)=\frac{-a_1y[-1]-a_2y[-1]z^{-1}-a_2y[-2]}{1+a_1z^{-1}+a_2z^{-2}}+\underbrace{\frac{b_0+b_1z^{-1}}{1+a_1z^{-1}+a_2z^{-2}}}_{H(z)}X(z)$$

Due to IC's… decays if system is stable

Due to input – will have transient part and steady-state part

Let's take a look at the IC-Driven transient part:

$$\boxed{Y_{zi}(z)=\frac{-a_1y[-1]-a_2y[-1]z^{-1}-a_2y[-2]}{1+a_1z^{-1}+a_2z^{-2}}=\frac{A-Bz^{-1}}{1+a_1z^{-1}+a_2z^{-2}}}$$

Multiply top and bottom by z^2: $\boxed{Y_{zi}(z)=\frac{Az^2+Bz}{z^2+a_1z+a_2}}$

Now to do an inverse ZT on this requires a bit of trickery…
Take the bottom two entries on the ZT table and form a linear combination:

$$\boxed{\begin{aligned}&C_1a^n\cos(\Omega_o n)u[n]\\ &\quad+C_2a^n\sin(\Omega_o n)u[n]\end{aligned}\quad\leftrightarrow\quad\frac{C_1z^2+a\left(C_2\sin(\Omega_o)-C_1\cos(\Omega_o)\right)z}{z^2-2a\cos(\Omega_o)z+a^2}}$$

$$\boxed{\begin{aligned}&a=\sqrt{a_2} &&\Omega_0=\cos^{-1}\left[\frac{-a_1}{2\sqrt{a_2}}\right]\\ &C_1=A &&C_2=\frac{B}{a\sin(\Omega_0)}-C_1\frac{\cos(\Omega_0)}{\sin(\Omega_0)}\end{aligned}}$$

Compare & Identify

Finally, by a trig ID we know that

$$C_1 a^n \cos(\Omega_o n)u[n] + C_2 a^n \sin(\Omega_o n)u[n] = Ca^n \cos(\Omega_o n + \theta)u[n]$$

So… all of this machinery leads to the insight that the IC-Driven transient of a second-order system will look like this:

$$\boxed{y_{zi}[n] = Ca^n \cos(\Omega_o n + \theta)u[n]}$$

…where:

1. The <u>frequency Ω_0</u> and <u>exponential a</u> are set by the Characteristic Eq. $\left\} \boxed{a = \sqrt{a_2} \quad \Omega_0 = \cos^{-1}\left[\frac{-a_1}{2\sqrt{a_2}}\right]} \right.$
2. The <u>amplitude C</u> and the <u>phase θ</u> are set by the ICs

Note: If $|a_2| < 1$ then we get a <u>decaying</u> response!!

<u>Solving a N^{th}-order Difference Equation using the ZT</u>

$$\underbrace{y[n] + \sum_{i=1}^{N} a_i y[n-i]}_{} = \underbrace{\sum_{i=0}^{M} b_i x[n-1]}_{}$$

Contains $x[n]$, $x[n-1]$,…

If this system is causal, we won't have $x[n+1]$, $x[n+2]$, etc. here

$$A(z) = z^N + a_1 z^{N-1} + \ldots + a_{N-1} z + a_N$$

$$B(z) = b_0 z^N + b_1 z^{N-1} + \ldots + b_M z^{N-M}$$

$$C(z) = \text{depends on the IC's}$$

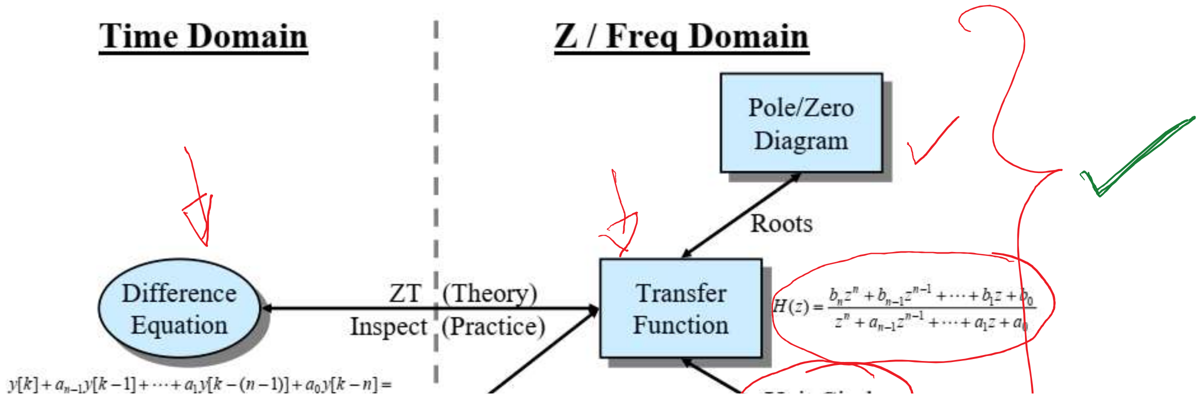
Transforming gives:

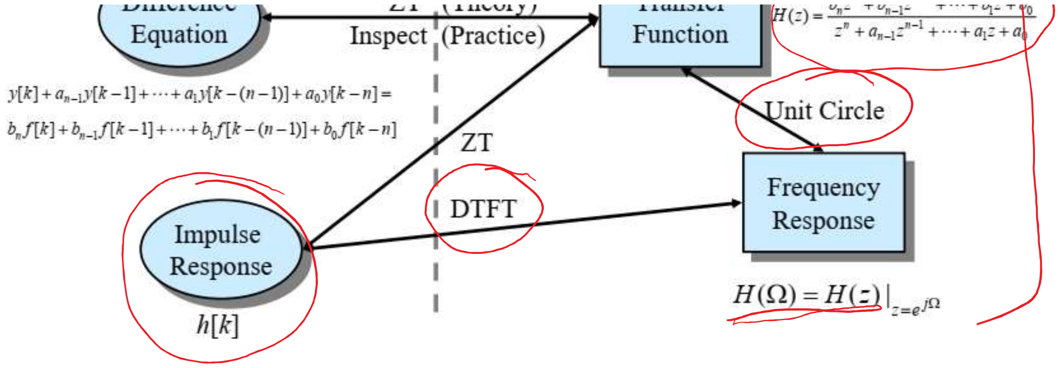
$$Y(z) = \frac{C(z)}{A(z)} + \underbrace{\frac{B(z)}{A(z)}}_{H(z)\text{ – transfer function}} X(z)$$

Transient and steady state part due to input

Transient part due to ICs

<u>Discrete-Time System Relationships</u>





Example System Relationships

Time Domain | **Z / Freq Domain**

Difference Equation $\xrightarrow[\text{Inspect (Practice)}]{\text{ZT (Theory)}}$ Transfer Function

Input-Output Form

$$\boxed{y(n)-\alpha y(n-1)=\beta x(n)}$$

$$\boxed{Y(z)\left[1-\alpha z^{-1}\right]=\beta X(z)}$$

Recursion Form

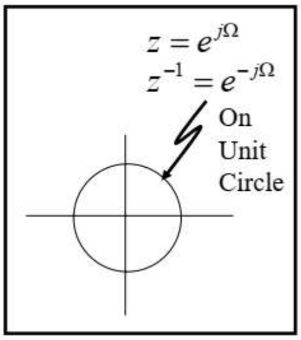
$$\boxed{y(n)=\beta x(n)+\alpha y(n-1)}$$

$$\boxed{H(z)=\frac{Y(z)}{X(z)}=\frac{\beta}{1-\alpha z^{-1}}}$$

$$\boxed{H(z)=\frac{\beta z}{z-\alpha}}$$

Example System (cont.)

Z / Freq Domain



$$\boxed{H(z)=\frac{\beta}{1-\alpha z^{-1}}}$$

Transfer Function

$z^{-1}=e^{-j\Omega}$

Unit Circle

$$\boxed{H(\Omega)=\frac{\beta}{1-\alpha e^{-j\Omega}} \quad \Omega\in(-\pi,\pi]}$$

Frequency Response

Example System (cont.)

Plotting Transfer Function

$$H(\Omega) = \frac{\beta}{1-\alpha e^{-j\Omega}} = \frac{\beta}{1-\left[\alpha\cos(\Omega) - j\alpha\sin(\Omega)\right]}$$

$$= \frac{\beta}{\left[1-\alpha\cos(\Omega)\right] + j\alpha\sin(\Omega)}$$

$$\left|H(\Omega)\right| = \frac{\beta}{\sqrt{\left[1-\alpha\cos(\Omega)\right]^2 + \left[\alpha\sin(\Omega)\right]^2}}$$

$$\angle H(\Omega) = -\tan^{-1}\left[\frac{\alpha\sin(\Omega)}{1-\alpha\cos(\Omega)}\right]$$

Euler's Equation

Group into Real & Imag

Standard Eqs for Mag. & Angle

Example System (cont.)

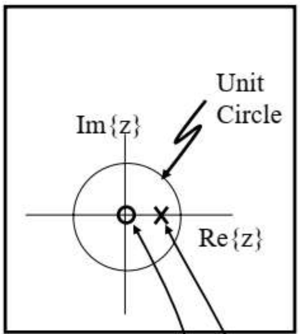
Z/Freq Domain

Pole/Zero Diagram

Roots

Transfer Function

$$H(z) = \frac{\beta}{1-\alpha z^{-1}} = \frac{\beta z}{z-\alpha}$$



Zero
Num. = 0
When z = 0

Pole
Den. = 0
When z = α

If α < 1: **In**side UC
If α > 1: **Out**side UC

Example System (cont.)