Golden

MATHS SERIES

# DIFFERENTIAL EQUATIONS

FIREWALL MEDIA

N.P. Bali

*Published by :*

**FIREWALL MEDIA**

*(An Imprint of Laxmi Publications Pvt. Ltd.)*

113, Golden House, Daryaganj,

New Delhi-110002

*Phone* : 011-43 53 25 00

*Fax* : 011-43 53 25 28

www.laxmipublications.com

info@laxmipublications.com

ISBN : 81-7008-939-5



*Price* : **Rs. 250.00** *Only.*　　*Tenth Edition* : 2006

*Offices :*

**India**

- **Bangalore** (*Phone* : 080-26 61 15 61)
- **Chennai** (*Phone* : 044-24 34 47 26)
- **Cochin** (*Phone* : 0484-239 70 04)
- **Guwahati** (*Phones* : 0361-254 36 69, 251 38 81)
- **Hyderabad** (*Phone* : 040-24 75 02 47)
- **Jalandhar City** (*Phone* : 0181-222 12 72)
- **Kolkata** (*Phones* : 033-22 27 37 73, 22 27 52 47)
- **Lucknow** (*Phone* : 0522-220 95 78)
- **Mumbai** (*Phones* : 022-24 91 54 15, 24 92 78 69)
- **Ranchi** (*Phone* : 0651-230 77 64)

**USA**

- **Boston**
  11, Leavitt Street, Hingham,
  MA 02043, USA
  *Phone* : 781-740-4487

MGA-6706-250-G. DIFF. EQUATIONS　　**C—13019/06/08**

*Typeset at* : Goswami Printers, Delhi.　　*Printed at* : Sanjeev Offset Press, New Delhi.

# CONTENTS

# GET 100 OUT OF 100 MARKS

**The book has been written with a view to guide the students in the examination. It ensures 100% marks**

**The student is advised to take this book only as a help book. He should solve himself as many questions as possible.**

☞ **Special Features :**

1. Solutions to all types of important questions have been given.
2. Questions from up-to-date papers of various Universities.
3. The book is useful to bright as well as weak students alike.
4. Approach to the subject matter is systematic.
5. The Book serves triple purpose :

**Text Book—Help Book—Solved University Papers**

# List of Important Formulae

*[TO BE LEARNT BY HEART]*

## I. VARIABLES SEPARABLE

**To solve $\frac{dy}{dx} = XY$, where X is a function of x only and Y is a function of y only.**

Bring all the terms of $x$ and $dx$ on one side, all the terms of $y$ and $dy$ on the other.

Integrate both sides and add an abritrary constant on one side.

## 2. HOMOGENEOUS EQUATION

**To solve the equation $\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)}$, where $f_1(x, y), f_2(x, y)$ are homogeneous functions of the same degree in x and y.**

Put $y = vx$, so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

Substitute the values of $y$ and $\frac{dy}{dx}$ in the given equation.

Separate the variables $v$ and $x$.

Integrate both sides and add an arbitrary constant on one side.

Substitute back the value of $v\left(= \frac{y}{x}\right)$.

## 3. EQUATIONS REDUCIBLE TO HOMOGENEOUS

(*i*) **To solve $\frac{dy}{dx} = \frac{ax + by + c}{a'x + b'y + c'}$, when $\frac{a}{a'} \neq \frac{b}{b'}$.**

Put $x = X + h$, $y = Y + k$

Equate the constant terms in the numerator and denominator of R.H.S. to zero and find $h$ and $k$.

The equation is now homogeneous in X and Y.

Put $Y = vX$ and proceed as in the case of homogeneous equations. Substitute back the values of $v$, X, Y, $h$ and $k$.

**(ii) To solve the equation $\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}$, when $\frac{a}{a'} = \frac{b}{b'}$.**

Put $ax + by = z$, so that $a + b\frac{dy}{dx} = \frac{dz}{dx}$

*i.e.,* $$\frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right)$$

Put the values of $ax + by$ and $\frac{dy}{dx}$ in the given equation. Separate the variables $z$ and $x$ and proceed as in Variables Separable. Substitute back the value of $z$.

### 4. LINEAR EQUATION

**(i) To solve $\frac{dy}{dx} + Py = Q$, where P, Q are functions of x only.**

Make the co-efficient of $\frac{dy}{dx}$ unity, if not so already.

Find. I.F. $= e^{\int P\,dx}$ and remember that $e^{\log f(x)} = f(x)$

The solution is $y\,.\,(\text{I.F.}) = \int Q.(\text{I.F.})\,dx + c$.

**(ii) To solve $\frac{dx}{dy} + Px = Q$, where P, Q are functions of y only.**

Make the co-efficient of $\frac{dx}{dy}$ unity, if not so already.

Find $\qquad$ I.F. $= e^{\int P\,dy}$

The solution is $x.\,(\text{I.F.}) = \int Q.(\text{I.F.})\,dy + c$.

### 5. EQUATIONS REDUCIBLE TO LINEAR

**(i) To solve $\frac{dy}{dx} + Py = Qy^n$, where P, Q are functions of x only.**

Divide throughout by $y^n$ to make the R.H.S. a function of $x$ only.

Put $\qquad y^{1-n} = z$

The equation is now linear in $z$.

Make the co-efficient of $\frac{dz}{dx}$ unity, if not so already.

Find $\qquad$ I.F. $= e^{\int P\,dx}$

The solution is $z.\,(\text{I.F.}) = \int Q.(\text{I.F.})\,dx + c$

Substitute back the value of $z$.

**(ii) To solve the equation $f'(y)\frac{dy}{dx} + Pf(y) = Q$, where P, Q are functions of x only.**

Put $f(y) = z$, then $f'(y).\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ Given equation becomes $\frac{dz}{dx} + Pz = Q$ which is linear in $z$ and can be solved.

## 6. EXACT DIFFERENTIAL EQUATION

The equation $Mdx + Ndy = 0$ is exact if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

The solution is $\displaystyle\int\limits_{y \text{ constant}} Mdx + \int (\text{terms in N not containing } x)\, dy = c.$

## 7. INTEGRATING FACTORS

(i) If the equation $Mdx + Ndy = 0$ is homogeneous in $x$ and $y$, then

$$\frac{1}{Mx + Ny} \text{ is an integrating factor} \qquad | \; Mx + Ny \neq 0$$

The method is used when $Mx + Ny$ consists of only one term.

(ii) If the equation $Mdx + Ndy = 0$ is of the form

$f_1(xy)y\, dx + f_2(x\, y)\, x\, dy = 0$, then

$$\frac{1}{Mx - Ny} \text{ is an integrating factor} \qquad | \; Mx - Ny \neq 0$$

(iii) If in the equation $Mdx + Ndy = 0$,

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} \text{ is a function of } x \text{ only} = f(x) \quad (say)$$

then $e^{\int f(x)\, dx}$ is an integrating factor.

(iv) If in the equation $Mdx + Ndy = 0$,

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{N} \text{ is a function of } y \text{ only} = f(y) \quad (say)$$

then $e^{\int f(y)\, dy}$ is an integrating factor.

## 8. TO SOLVE THE EQUATION

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = 0$$

**where $a_1, a_2, \ldots, a_n$ are constants.**

Write the given equation in symbolic form

$$\left[\text{by putting } \frac{d}{dx} = D, \frac{d^2}{dx^2} = D^2, \ldots, \frac{d^n}{dx^n} = D^n\right]$$

Thus $\quad (D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n)y = 0$

Write down the auxiliary equation (A.E.) by equating to zero the symbolic co-efficient of $y$

Thus $\quad D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n = 0$

Solve if for D as if it were an ordinary algebraic quantity.

From the roots of the A.E., write down the corresponding part of the complete solution as follows :

| *Roots of A.E.* | *Corresponding part of C.S.* |
|---|---|
| 1. One real root $m_1$ | $c_1 e^{m_1 x}$ |
| 2. Two real and different roots $m_1, m_2$. | $c_1 e^{m_1 x} + c_2 e^{m_2 x}$ |
| 3. Two real and equal roots $m_1, m_1$. | $(c_1 + c_2 x) e^{m_1 x}$ |
| 4. Three real and equal roots $m_1, m_1, m_1$ | $(c_1 + c_2 x + c_3 x^2) e^{m_1 x}$ |
| 5. One pair of complex roots $\alpha \pm i\beta$. | $e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$ |
| 6. Two pairs of complex and equal roots $\alpha \pm i\beta, \alpha \pm i\beta$. | $e^{\alpha x} [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x]$ |

Complete solution is $y$ = corresponding part of C.S.

## 9. RULE TO SOLVE THE EQUATION

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + a_n y = x$$

**where $a_1, a_2, \ldots\ldots, a_n$ are constants and X is a function of x.**

Write down the given equation in the symbolic form

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n) y = 0$$

Auxiliary equation is $\quad D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n = 0$

Solve it for D as if it were an ordinary algebraic quantity.

From the roots of the auxiliary equation, write down the complementary function (C.F.)

$$\text{P.I.} = \frac{1}{D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n} X = \frac{1}{f(D)} X$$

**Case I.** $\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ where $f(a) \neq 0$

**Case II.** $\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax$ where $f(-a^2) \neq 0$

and $\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax$ where $f(-a^2) \neq 0$

**Case III.** $\frac{1}{f(D)} x^m$ where $m$ is a positive integer.

Write $f(D)$ in ascending powers of D and make the first term unity, if not so already.

Let $\quad \frac{1}{f(D)} x^m = \frac{1}{a[1+\phi(D)]} x^m = \frac{1}{a} [1 + \phi(D)]^{-1} x^m$

Expand it by Binomial Theorem, remembering that

$$(1 + t)^{-1} = 1 - t + t^2 - t^3 + \ldots..$$
$$(1 - t)^{-1} = 1 + t + t^2 + t^3 + \ldots\ldots$$
$$(1 + t)^{-2} = 1 - 2t + 3t^2 - 4t^3 + \ldots..$$
$$(1 - t)^{-2} = 1 + 2t + 3t^2 + 4t^3 + \ldots..$$

The expansion is to be carried out upto $D^m$ only.

Operate on $x^m$ with each term of the expansion.

**Case IV.** $\frac{1}{f(D)} e^{ax} X = e^{ax} \frac{1}{f(D+a)} X$

**Case V.** $\frac{1}{f(D)} xV = x \frac{1}{f(D)} V + \frac{d}{dD}\left(\frac{1}{f(D)}\right) V = x \frac{1}{f(D)} V - \frac{f'(D)}{[f(D)]^2} V$

where V is a function of $x$.

## 10. CASES OF FAILURE

If by using the above rules for finding the P.I., we get zero in the denominator, then

$$\frac{1}{f(D)} e^{ax} = x \frac{1}{\frac{d}{dD} f[(D)]} e^{ax} \qquad \text{when } f(a) = 0$$

$$\frac{1}{f(D^2)} \sin ax = x. \frac{1}{\frac{d}{dD}[f(D^2)]} \sin ax \qquad \text{when } f(-a^2) = 0$$

$$\frac{1}{f(D^2)} \cos ax = x . \frac{1}{\frac{d}{dD}[f(D^2)]} \cos ax \qquad \text{when } f(-a^2) = 0$$

*i.e.,* $x . \frac{1}{\text{diff. co-effi. of denom. w.r.t. D}} e^{ax}$ or $\sin ax$ or $\cos ax$ whatever the case is.

If we again get zero in the denominator, repeat the above process.

**11.** $\frac{1}{D} \equiv$ integration.

**12.** $e^{ix} = \cos x + i \sin x$

so that $\cos x$ is the real part of $e^{ix}$ and $\sin x$ is the imaginary part of $e^{ix}$.

## 13. RULE TO SOLVE THE EQUATION

$$\mathbf{a_0x^n \frac{d^ny}{dx^n} + a_1x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots + a_{n-1}x \frac{dy}{dx} + a_ny = X}$$

**where $a_0, a_1, a_2, \ldots, a_n$ are constants and X is a function of x.**

**[Homogeneous Linear Equation of the nth order]**

Put $x = e^z$ so that $z = \log x$ and

$$D = x \frac{d}{dx} = \frac{d}{dz}$$

Then $$x^2 \frac{d^2}{dx^2} = D(D-1)$$

$$x^3 \frac{d^3}{dx^3} = D(D-1)(D-2) \text{ and so on.}$$

The given equation reduces to $f(D)y = Z$, a function of $z$.

Solve it and replace $z$ by $\log x$.

## 14. RULE TO SOLVE THE EQUATION

$$a_0(a + bx)^n \frac{d^n y}{dx^n} + a_1(a + bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2(a + bx)^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + a_{n-1}(a + bx) \frac{dy}{dx} + a_n y = X$$

**where $a_0, a_1, a_2, \ldots\ldots, a_n$ are constants and X is a function of x.**

Put $a + bx = e^z$ so that $z = \log (a + bx)$ and

$$D = \frac{d}{dz}, \text{ then } (a + bx) \frac{d}{dx} = bD,$$

$$(a + bx)^2 \frac{d^2}{dx^2} = b^2 D(D - 1)$$

$$(a + bx)^3 \frac{d^3}{dx^3} = b^3 D(D - 1)(D - 2)$$

The given equation reduces to $f(D)y = Z$, a function of $z$.

Solve it and replace $z$ by $\log (a + bx)$.

# 1

# Differential Equations and Their Formation

## Definitions

**1. Differential Equation.** A **differential equation** is an equation which involves differential co-efficients or differentials.

*e.g.*,
$$\frac{dy}{dx} = x \log x,\ dy = \cos x\, dx,$$
$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} - 12y = 5e^x + \sin x + x^3.$$

**2. Ordinary Differential Equation.** An ordinary differential equation is that which involves only one independent variable and differential co-efficients w.r.t. it.

The above examples are of this type.

**3. Order of a Differential Equation.** The order of a differential equation is the order of the highest differential co-efficient which appears in it.

*e.g.*, the differential equation $\frac{d^2y}{dx^2} + 4y = e^x$ is of second order.

**4. Degree of a Differential Equation.** The degree of a differential equation is the degree of the highest differential co-efficient which appears in it, when the differential co-efficients are free from radicals and fractional powers.

*e.g.*, the differential equation $\left[1+\left(\frac{dy}{dx}\right)\right]^4 = a^2\left(\frac{d^2y}{dx^2}\right)^3$ is of degree 3, the degree of $\frac{d^2y}{dx^2}$.

**Example.** *Find the order and degree of the differential equation :*

(*i*) $y = x\frac{dy}{dx} + a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$ (*ii*) $\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = a\frac{d^2y}{dx^2}$.

**Sol.** (*i*)
$$y = x\frac{dy}{dx} + a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

or
$$y - x\frac{dy}{dx} = a\sqrt{1+\left(\frac{dy}{dx}\right)^2}$$

Squaring [to get rid of radicals]
$$\left(y - x\frac{dy}{dx}\right)^2 = a^2\left[1+\left(\frac{dy}{dx}\right)^2\right]$$

which is of the first order and **second degree.** [$\because$ highest degree of $dy/dx$ is 2] (*ii*) $$\left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} = a\,\frac{d^2y}{dx^2}$$

Squaring [to get rid of radicals]

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = a^2\left[\frac{d^2y}{dx^2}\right]^2$$

which is of the second order and second degree.

$\Big[\because$ order of highest differential co-efficient $\frac{d^2y}{dx^2}$ is 2 and highest degree of $\frac{d^2y}{dx^2}$, the highest differential co-efficient, is 2$\Big]$

**Solution of a Differential Equation.** A solution of a differential equation is a functional relation between the variables involved which satisfies the equation.

**General Solution.** The solution of a differential equation in which the number of arbitrary constants is equal to the order of the differential equation is called the **general solution** [or complete solution or complete integral or complete primitive].

**Particular Solution.** If particular values are given to the arbitrary constants in the general solution, then the solution so obtained is called a **particular solution.**

**Singular Solution.** A solution which does not contain any arbitrary constant and also, is not obtainable from the general solution by giving particular values to the arbitrary constants, is called a **Singular Solution.**

**Example 1.** *Show that*

(*i*) $x^2 + 4y = 0$ *is a solution of* $\left(\frac{dy}{dx}\right)^2 + x\,\frac{dy}{dx} - y = 0.$

(*ii*) $y = \frac{c}{x} + d$ *is a solution of* $\frac{d^2y}{dx^2} + \frac{2}{x}\cdot\frac{dy}{dx} = 0.$

**Sol.** (*i*) The given differential equation is $\left(\frac{dy}{dx}\right)^2 + x\,\frac{dy}{dx} - y = 0$ ...(1)

The given function is $x^2 + 4y = 0$ ...(2)

Differentiating (2) w.r.t. $x$ $\quad 2x + 4.\,\frac{dy}{dx} = 0$

$$\Rightarrow \qquad \frac{dy}{dx} = -\,\frac{x}{2} \qquad ...(3)$$

Substituting for $y$ and $\frac{dy}{dx}$ from (2) and (3) in (1), we get

$$\left(-\,\frac{x}{2}\right)^2 + x\left(-\,\frac{x}{2}\right) - \left(-\,\frac{x^2}{4}\right) = 0$$

or $$\frac{x^2}{4}-\frac{x^2}{2}+\frac{x^2}{4}=0 \qquad \text{or} \qquad x^2-2x^2+x^2=0$$

which is true.

$\therefore$ Equation (2) is a solution of (1).

[Since this solution is free from arbitrary constants, it is a particular solution.]

(*ii*) The given differential equation is $\dfrac{d^2y}{dx^2}+\dfrac{2}{x}\cdot\dfrac{dy}{dx}=0$ ...(1)

The given function is $$y=\frac{c}{x}+d \quad ...(2)$$

Differentiating (2) w.r.t. $x$ $$\frac{dy}{dx}=-\frac{c}{x^2} \quad ...(3)$$

Differentiating again w.r.t. $x$ $$\frac{d^2y}{dx^2}=\frac{2c}{x^3} \quad ...(4)$$

Substituting for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ from (3) and (4) in (1), we get

$$\frac{2c}{x^3}+\frac{2}{x}\left(-\frac{c}{x^2}\right)=0 \qquad \text{or} \qquad \frac{2c}{x^3}-\frac{2c}{x^3}=0$$

which is true.

$\therefore$ Equation (2) is a solution of (1).

[The given diff. eq. is of second order and the solution (2) contains two arbitrary constants *viz.* $c$ and $d$. Hence this solution is the general solution.]

**Example 2.** *Verify that $y=(x^3-x)\log cx$ satisfies the differential equation*

$$(x^3-x)y'-(3x^2-1)y=x^5-2x^3+x$$

*for all values of* c.

**Sol.** The given differential equation is

$$(x^3-x)y'-(3x^2-1)y=x^5-2x^3+x \quad ...(1)$$

The given function is

$$y=(x^3-x)\log cx$$

Differentiating w.r.t. $x$,

$$y'=(x^3-x)\cdot\frac{1}{cx}\cdot c+(3x^2-1)\log cx$$

or $$y'=(x^2-1)+(3x^2-1)\log cx$$

Substituting the values of $y$ and $y'$ in (1), we have

$$(x^3-x)[(x^2-1)+(3x^2-1)\log cx]-(3x^2-1)(x^3-x)\log cx$$
$$=x^5-2x^3+x$$

or $$(x^3-x)(x^2-1)+(x^3-x)(3x^2-1)\log cx-(3x^2-1)(x^3-x)\log cx$$
$$=x^5-2x^3+x$$

or $$x^5-2x^3+x=x^5-2x^3+x$$

which is identically true.

Since the given function satisfies (1), it is a solution of (1).

**Example 3.** *Show that $y = -(1 + x)$ is a solution of the differential equation $(y - x)\, dy - (y^2 - x^2)\, dx = 0$.*

**Sol.** The given differential equation is

$$(y - x)\, dy - (y^2 - x^2)\, dx = 0 \qquad \text{...(1)}$$

The given function is

$$y = -(1 + x) \text{ so that } dy = -dx$$

Substituting the values of $y$ and $dy$ in (1), we have

$$(-1 - x - x)(-dx) - [(1 + x)^2 - x^2]\, dx = 0$$

or

$$(1 + 2x)\, dx - (1 + 2x)\, dx = 0$$

which is identically true.

Since the given function satisfies (1), it is a solution of (1).

**Example 4.** *Show that the function $y = e^{3x}(A + Bx)$ is a solution of the differential equation*

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0.$$

**Sol.** The given differential equation is

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0 \qquad \text{...(1)}$$

The given function is

$$y = e^{3x}(\mathrm{A} + \mathrm{B}x) \qquad \text{...(2)}$$

Differentiating w.r.t. $x$,

$$\frac{dy}{dx} = e^{3x} \,.\, \mathrm{B} + 3e^{3x}(\mathrm{A} + \mathrm{B}x)$$

or

$$\frac{dy}{dx} = \mathrm{B}e^{3x} + 3y \qquad \text{...(3) [Using (2)]}$$

Differentiating again w.r.t. $x$,

$$\frac{d^2y}{dx^2} = 3\mathrm{B}e^{3x} + 3\frac{dy}{dx}$$

or

$$\frac{d^2y}{dx^2} = 3\left[\frac{dy}{dx} - 3y\right] + 3\frac{dy}{dx} \qquad \text{[Using (3)]}$$

or

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$$

which is the same as (1).

Hence, the given function is a solution of (1).

**Example 5.** *Show that the function $y = Ax + \dfrac{B}{x}$ is a solution of the differential equation*

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0.$$

**Sol.** The given differential equation is

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0 \qquad \text{...(1)}$$

The given function is $y = Ax + \dfrac{B}{x}$

so that $$\frac{dy}{dx} = A - \frac{B}{x^2} \quad \text{and} \quad \frac{d^2y}{dx^2} = \frac{2B}{x^3}$$

Substituting the values of $y$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we have

$$x^2\left(\frac{2B}{x^3}\right) + x\left(A - \frac{B}{x^2}\right) - \left(Ax + \frac{B}{x}\right) = 0$$

or $$\frac{2B}{x} + Ax - \frac{B}{x} - Ax - \frac{B}{x} = 0$$

which is identically true.

Hence the given function is a solution of (1).

**Example 6.** *Show that the function $y = ae^{2x} + be^{-x}$ is a solution of the differential equation*

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0.$$

**Sol.** The given differential equation is

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0 \qquad \text{...(1)}$$

The given function is $y = ae^{2x} + be^{-x}$

so that $$\frac{dy}{dx} = 2ae^{2x} - be^{-x} \quad \text{and} \quad \frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x}$$

Substituting the values of $y$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we have

$$(4ae^{2x} + be^{-x}) - (2ae^{2x} - be^{-x}) - 2\,(ae^{2x} + be^{-x}) = 0$$

or $$(4a - 2a - 2a)\,e^{2x} + (b + b - 2b)e^{-x} = 0$$

which is identically true.

Hence the given function is a solution of (1).

**Example 7.** *Show that the function $y = be^{x} + ce^{2x}$ is a solution of the differential equation* $\dfrac{d^2y}{dx^2} - 3\,\dfrac{dy}{dx} + 2y = 0.$

**Sol.** Please try yourself.

## FORMATION OF DIFFERENTIAL EQUATIONS

**Rule to form the differential equation from a given equation is x and y, containing arbitrary constants.**

**1.** *Write down the given equation.*

**2. Differentiate** *w.r.t. x successively* **as many times as the number of arbitrary constants.**

**3.** *Eliminate the arbitrary constants from the equations of the above two steps.*

The resulting equation is the required differential equation.

**Example 1.** *Find the differential equation of all straight lines passing through the origin.*

**Sol.** Equation of any straight line through the origin is

$$y = mx \qquad ...(i)$$

$m$ being the parameter.

Differentiating w.r.t. $x$,

$$\frac{dy}{dx} = m \qquad ...(ii)$$

Now we have to eliminate $m$ between $(i)$ and $(ii)$

From $(i)$, $m = \frac{y}{x}$ $\quad\therefore\quad$ From $(ii)$, $\frac{dy}{dx} = \frac{y}{x}$

or $x\frac{dy}{dx} = y$ which is the required differential equation.

**Example 2.** *Find the differential equation of all non-vertical lines in the xy-plane.*

**Sol.** The equation of any non-vertical line in the $xy$-plane is

$$y = mx + c \qquad ...(i)$$

It contains two arbitrary constants *viz.* $m$ and $c$

Differentiating $(i)$ w.r.t. $x$ $\quad \frac{dy}{dx} = m$

Differentiating again $\quad \frac{d^2y}{dx^2} = 0$

which is the reqd. diff. equation.

**Example 3.** *Obtain a differential equation that should be satisfied by the family of concentric circles $x^2 + y^2 = a^2$.*

**Sol.** $x^2 + y^2 = a^2$. Differentiating w.r.t. $x$, $2x + 2y\frac{dy}{dx} = 0$ [$a$ is eliminated]

or $x + y\frac{dy}{dx} = 0$ which is the required differential equation.

**Example 4.** *Find the differential equation of the system of circles touching the y-axis at the origin.*

**Sol.** Any circle which touches the $y$-axis at the origin must have its centre on the $x$-axis.

Equation of such a circle is

$$x^2 + y^2 + 2gx = 0 \qquad ...(i)$$

Differentiating $(i)$ w.r.t. '$x$'

$$2x + 2y\,.\,\frac{dy}{dx} + 2g = 0 \qquad ...(ii)$$

Now we have to eliminate $g$ between $(i)$ and $(ii)$

From $(ii)$, $\quad 2x + 2y\,.\,\frac{dy}{dx} = -2g = \frac{x^2 + y^2}{x} \qquad | \because$ of $(i)$

or $\quad 2x^2 + 2xy\,.\,\frac{dy}{dx} = x^2 + y^2$

or $\quad 2xy\frac{dy}{dx} + x^2 - y^2 = 0$ which is the required differential equation.

**Example 5.** *Find the differential equation of all circles touching a given straight line at a given point.*

**Sol.** Take the given point as the origin and the given straight line as the axis of $y$.

Now proceed as in Example 4.

**Example 6.** *Find the differential equation of the family of curves* $(x-h)^2 + (y-k)^2 = r^2$, *where $h$ and $k$ are arbitrary constants.*

**Sol.** The equation of all circles of radius $r$ is

$$(x-h)^2 + (y-k)^2 = r^2 \qquad ...(i)$$

where $h$ and $k$, the co-ordinates of the centre are arbitrary.

Differentiating both sides of $(i)$ w.r.t. '$x$'

$$2(x-h) + 2(y-k) \cdot \frac{dy}{dx} = 0$$

or
$$(x-h) + (y-k) \cdot \frac{dy}{dx} = 0 \qquad ...(ii)$$

Differentiating again w.r.t. '$x$'

$$1 + \left(\frac{dy}{dx}\right)^2 + (y-k)\frac{d^2y}{dx^2} = 0 \qquad ...(iii)$$

From $(iii)$,
$$y - k = -\left[1+\left(\frac{dy}{dx}\right)^2\right] \bigg/ \frac{d^2y}{dx^2}$$

$\therefore$ From $(ii)$,
$$x - h = -(y-k)\frac{dy}{dx} = \left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{dy}{dx} \bigg/ \frac{d^2y}{dx^2}$$

Substituting the values of $x-h$ and $y-k$ in $(i)$

$$\frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]\left(\frac{dy}{dx}\right)^2}{\left(\frac{d^2y}{dx^2}\right)^2} + \frac{\left[1+\left(\frac{dy}{dx}\right)^2\right]^2}{\left(\frac{d^2y}{dx^2}\right)^2} = r^2$$

or
$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left(\frac{dy}{dx}\right)^2 + \left[1+\left(\frac{dy}{dx}\right)^2\right]^2 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

or
$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^2\left[1+\left(\frac{dy}{dx}\right)^2\right] = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

or
$$\left[1+\left(\frac{dy}{dx}\right)^2\right]^3 = r^2\left(\frac{d^2y}{dx^2}\right)^2$$

which is the required differential equation.

**Another Form.** *Find the differential equation of all circles of radius 'r'.*

**Example 7.** *Find the differential equation of all circles in the xy-plane.*

**Sol.** The general equation of a circle in the $xy$-plane is

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad ...(i)$$

where $g, f, c$ are arbitrary constants which are to be eliminated.

Differentiating $(i)$ w.r.t. '$x$'

$$2x + 2yy_1 + 2g + 2fy_1 = 0$$

or
$$x + yy_1 + g + fy_1 = 0 \qquad ...(ii)$$

[$c$ is eliminated]

Differentiating $(ii)$, w.r.t. '$x$'

$$1 + yy_2 + y_1^2 + fy_2 = 0 \qquad ...(iii)$$

[$g$ is eliminated]

Differentiating $(iii)$ w.r.t. '$x$'

$$y_1y_2 + yy_3 + 2y_1y_2 + fy_3 = 0 \qquad ...(iv)$$

[to eliminate $f$]

Multiplying $(iii)$ by $y_3$ and $(iv)$ by $y_2$

$$y_3 + yy_2y_3 + y_1^2y_3 + fy_2y_3 = 0 \qquad ...(v)$$

$$yy_2y_3 + y_1y_2^2 + 2y_1y_2^2 + fy_2y_3 = 0 \qquad ...(vi)$$

Subtracting $(vi)$ from $(v)$

$$y_3 + y_1^2y_3 - 3y_1y_2^2 = 0$$

or
$$(1 + y_1^2)y_3 - 3y_1y_2^2 = 0$$

which is the required differential equation.

**Example 8.** (*a*) *Find the differential equation of all circles which pass through the origin and whose centres are on the x-axis.*

(*b*) *Find the differential equation of all circles which pass through the origin and whose centres lie on the y-axis.*

**Sol.** (*a*) The equation of any circle passing through the origin and whose centre is on the $x$-axis is $x^2 + y^2 + 2gx = 0$

Now proceed as in Example 4.

(*b*) Please try yourself.

[**Hint.** $x^2 + y^2 + 2fy = 0$] $\left[\textbf{Ans.}\ \ (x^2 - y^2)\dfrac{dy}{dx} - 2xy = 0\right]$

**Example 9.** *What is the order of the differential equation whose solution is the circle* $(x - \alpha)^2 + y^2 = \alpha^2$, *where* $\alpha$ *is an arbitrary constant ?*

**Sol.** The given equation is $(x - \alpha)^2 + y^2 = \alpha^2$

or
$$x^2 + y^2 - 2\alpha x = 0 \qquad ...(1)$$

Differentiating (1) w.r.t. $x$,

$$2x + 2y\frac{dy}{dx} - 2\alpha = 0$$

or
$$\alpha = x + y\frac{dy}{dx}$$

Substituting this value of $\alpha$ in (1), we get

$$x^2 + y^2 - 2x\left(x + y\frac{dy}{dx}\right) = 0$$

or $$y^2 - x^2 - 2xy\frac{dy}{dx} = 0 \quad \text{or} \quad 2xy\frac{dy}{dx} + x^2 - y^2 = 0$$

which is the differential equation of the family of circles (1) and it is of first order.

**Example 10.** *Show that the differential equation of the family of circles of fixed radius r with centres on y-axis is*

$$(x^2 - r^2)\left(\frac{dy}{dx}\right)^2 + x^2 = 0.$$

**Sol.** Let the center be $(0, k)$, then the equation of circle is

$$x^2 + (y-k)^2 = r^2 \qquad ...(1)$$

where $k$ is an arbitrary constant.

Differentiating w.r.t. $x$,

$$2x + 2(y-k)y' = 0 \text{ where } y' = \frac{dy}{dx}$$

$$\Rightarrow \qquad y - k = -\frac{x}{y'}$$

Substituting the value of $(y - k)$ in (1), we have

$$x^2 + \frac{x^2}{y'^2} = r^2 \quad \text{or} \quad (x^2 - r^2)y'^2 + x^2 = 0$$

or $$(x^2 - r^2)\left(\frac{dy}{dx}\right)^2 + x^2 = 0$$ which is the reqd. differential equation.

**Example 11.** *Find the differential equation of all parabolas whose axes are parallel to y-axis.*

**Sol.** The equation of any parabola whose axis is parallel to $y$-axis is

$$(x-h)^2 = 4a(y-k) \qquad ...(i)$$

where $h, k, a$ are *three* arbitrary constants.

Differentiating (1) w.r.t. $x$ $\quad 2(x-h) = 4a\frac{dy}{dx}$

Differentiating again $\quad 2 = 4a\frac{d^3y}{dx^3}$

Differentiating again $\quad 0 = 4a\frac{d^3y}{dx^3}$

or $\dfrac{d^3y}{dx^3} = 0$ which is the reqd. diff. equation.

**Example 12.** *Find the differential equation of all parabolas with latus rectum "4a" and whose axes are parallel to x-axis.*

**Sol.** The equation of any parabola with latus rectum $4a$ and axis parallel to $x$-axis is

$$(y-k)^2 = 4a(x-h) \qquad ...(1)$$

where $h, k$ are *two* arbitrary constants.

Differentiating (1) w.r.t. $x$ $\quad 2(y-k)\frac{dy}{dx} = 4a \Rightarrow (y-k)\frac{dy}{dx} = 2a \qquad ...(2)$

Differentiating again w.r.t. $x$ $(y-k)\dfrac{d^2y}{dx^2}+\left(\dfrac{dy}{dx}\right)^2=0$ ...(3)

From (2), $$y-k=\frac{2a}{\dfrac{dy}{dx}}$$

Substituting the value of $(y-k)$ in (3)

$$\frac{2a}{\dfrac{dy}{dx}}\cdot\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2=0 \quad \text{or} \quad 2a\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^3=0$$

which is the reqd. diff. equation.

**Example 13.** *Find the differential equation of all conics whose axes coincide with the axes of co-ordinates.* (Indore, 1997 ; Gauhati, 1996)

**Sol.** The equation of any conic whose axes coincide with the axes of co-ordinates is

$$Ax^2+By^2=1 \qquad \text{...(1)}$$

where A, B are *two* arbitrary constants.

Differentiating (1) w.r.t. $x$ $2Ax+2By\dfrac{dy}{dx}=0$

$$\Rightarrow \qquad Ax+By\frac{dy}{dx}=0 \qquad \text{...(2)}$$

Differentiating again w.r.t. $x$

$$A+B\left(\frac{dy}{dx}\right)^2+By\frac{d^2y}{dx^2}=0 \qquad \text{...(3)}$$

From (2), $\dfrac{A}{B}=-\dfrac{y}{x}\cdot\dfrac{dy}{dx}$

From (3), $\dfrac{A}{B}=-\left[\left(\dfrac{dy}{dx}\right)^2+y\dfrac{d^2y}{dx^2}\right]$

$$\therefore \qquad \left(\frac{dy}{dx}\right)^2+y\frac{d^2y}{dx^2}=\frac{y}{x}\cdot\frac{dy}{dx} \quad \text{or} \quad xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$$

which is the reqd. differential equation.

**Example 14.** *Find the differential equation of all ellipses centred at the origin.*

**Sol.** The equation of any ellipse centred at the origin is

$$Ax^2+By^2=1\ (A>0,\ B>0)$$

Now proceed as in Example 13.

**Example 15.** *Prove that the differential equation of the family of parabolas $y^2=4ax$ is $2x\dfrac{dy}{dx}-y=0$.*

**Sol.** Please try yourself.

**Example 16.** *Find the differential equation of all the circles in the first quadrant which touch the co-ordinate axes.*

**Sol.** The equation of all the circles in the first quadrant which touch the co-ordinate axes is

$$(x-a)^2+(y-a)^2=a^2 \qquad \text{...(1)}$$

or $$x^2+y^2-2ax-2ay+a^2=0$$

Differentiating w.r.t. $x$, we get

$$2x+2yy'-2a-2ay'=0, \quad \text{where } y'=\frac{dy}{dx}$$

or $$x+yy'=a(1+y')$$

or $$a=\frac{x+yy'}{1+y'}$$

Substituting the value of $a$ in (1), we get

$$\left(x-\frac{x+yy'}{1+y'}\right)^2+\left(y-\frac{x+yy'}{1+y'}\right)^2=\left(\frac{x+yy'}{1+y'}\right)^2$$

or $$(xy'-yy')^2+(y-x)^2=(x+yy')^2$$

or $$y'^2(x-y)^2+(x-y)^2=(x+yy')^2$$

or $$(x-y)^2[1+y'^2]=(x+yy')^2$$

or $$(x-y)^2\left[1+\left(\frac{dy}{dx}\right)^2\right]=\left(x+y\frac{dy}{dx}\right)^2$$

which is the required differential equation.

**Example 17.** *Show that the differential equation that represents all parabolas having their axis of symmetry coincident with the axis of x is*

$$yy_2+y_1^2=0.$$

**Sol.** The equation of any parabola having its axis of symmetry coincident with the axis of $x$ is

$$y^2=4a(x-h) \qquad \text{...(1)}$$

where $a$ and $h$ are arbitrary.

Differentiating w.r.t. $x$, $\quad 2yy_1=4a$

or $$yy_1=2a$$

Differentiating again w.r.t. $x$,

$$yy_2+y_1^2=0$$

which is the required differential equation.

**Example 18.** *Find the differential equation of a family of central conics represented by* $\dfrac{x^2}{a^2+\lambda}+\dfrac{y^2}{b^2+\lambda}=1$*, where* $\lambda$ *is a parameter.*

**Sol.** Given equation is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1 \qquad \text{...(1)}$$

Differentiating w.r.t. $x$, we get

$$\frac{2x}{a^2+\lambda}+\frac{2yy'}{b^2+\lambda}=0, \quad \text{where } y'=\frac{dy}{dx}$$

or $$x(b^2+\lambda)+yy'(a^2+\lambda)=0$$

or $$\lambda(x+yy')=-(b^2x+a^2yy')$$

or $$\lambda=-\frac{b^2x+a^2yy'}{x+yy'}$$

$\therefore$ $$a^2+\lambda=a^2-\frac{b^2x+a^2yy'}{x+yy'}=\frac{(a^2-b^2)x}{x+yy'}$$

and $$b^2+\lambda=b^2-\frac{b^2x+a^2yy'}{x+yy'}=-\frac{(a^2-b^2)yy'}{x+yy'}$$

Substituting these values of $(a^2+\lambda)$ and $(b^2+\lambda)$ in (1), we get

$$\frac{x^2(x+yy')}{(a^2-b^2)x}-\frac{y^2(x+yy')}{(a^2-b^2)yy'}=1$$

or $$x(x+yy')-\frac{y(x+yy')}{y'}=a^2-b^2$$

or $$x^2y'+xyy'^2-xy-y^2y'=(a^2-b^2)y'$$

or $$xyy'^2+(x^2-y^2-a^2+b^2)y'-xy=0$$

or $$xy\left(\frac{dy}{dx}\right)^2+(x^2-y^2-a^2+b^2)\frac{dy}{dx}-xy=0$$

which is the required differential equation.

**Example 19.** *Form the differential equation corresponding to*

$$y^2=m(a^2-x^2)$$

*by eliminating m and a.*

**Sol.** The given equation is $y^2=m(a^2-x^2)$ ...(1)

Differentiating w.r.t. $x$, we get

$$2yy'=-2mx \qquad \text{or} \qquad yy'=-mx \qquad ...(2)$$

Differentiating again w.r.t. $x$, we get

$$yy''+y'^2=-m$$

$\therefore$ From (2), $$yy'=x(yy''+y'^2)$$

or $$xyy''+xy'^2-yy'=0$$

or $$xy\frac{d^2y}{dx^2}+x\left(\frac{dy}{dx}\right)^2-y\frac{dy}{dx}=0$$

which is the required differential equation.

**Example 20.** *Find the differential equation of the family of curves represented by* $c(y+c)^2=x^3$.

**Sol.** The given equation is $c(y+c)^2=x^3$ ...(1)

Differentiating w.r.t. $x$, we get

$$2c(y+c)y'=3x^2 \qquad ...(2)$$

Dividing (1) by (2), $\dfrac{y+c}{2y'} = \dfrac{x}{3}$

or $y + c = \dfrac{2}{3}xy' \quad \therefore \quad c = \dfrac{2}{3}xy' - y$

Substituting the value of $c$ in (2), we get

$$2\left(\frac{2}{3}xy' - y\right)\left(\frac{2}{3}xy'\right)y' = 3x^2$$

or $4xy'^2\,(2xy'^2 - 3y) = 27x^2$

or $8xy'^3 - 12yy'^2 = 27x$

or $x\,(8y'^3 - 27) = 12yy'^2$

or $$x\left[8\left(\frac{dy}{dx}\right)^3 - 27\right] = 12y\left(\frac{dy}{dx}\right)^2$$

which is the required differential equation.

**Example 21.** *Find the differential equation corresponding to the family of curves $y = c(x - c)^2$, where $c$ is an arbitrary constant.*

**Sol.** The given equation is $y = c(x - c)^2$ ...(1)

Differentiating w.r.t. $x$, $y' = 2c(x - c)$ ...(2)

Dividing (1) by (2), $\dfrac{y}{y'} = \dfrac{x-c}{2}$ or $c = x - \dfrac{2y}{y'}$

Substituting the value of $c$ in (2), we get

$$y' = \left(2x - \frac{4y}{y'}\right)\cdot\frac{2y}{y'} \quad \text{or} \quad y'^3 = 4y\,(xy' - 2y)$$

or $$\left(\frac{dy}{dx}\right)^3 = 4y\left(x\frac{dy}{dx} - 2y\right)$$

which is the required differential equation.

**Example 22.** *Find the differential equation of the family of curves*

$$y = Ae^{2x} + Be^{-2x}.$$ (Punjab, 1996)

**Sol.** The given equation is $y = Ae^{2x} + Be^{-2x}$ ...(1)

It has two arbitrary constants A and B.

Differentiating, $\dfrac{dy}{dx} = 2Ae^{2x} - 2Be^{-2x}$

Differentiating again, $\dfrac{d^2y}{dx^2} = 4Ae^{2x} + 4Be^{-2x}$

$= 4(Ae^{2x} + Be^{-2x})$

$= 4y$ [Using (1)]

or $\dfrac{d^2y}{dx^2} - 4y = 0$ which is the required differential equation.

**Example 23.** *Find the differential equation of the family of curves*

$$y = A\cos(x^2) + B\sin(x^2).$$

**Sol.** The given equation is $y = A \cos x^2 + B \sin x^2$ ...(1)

Differentiating, $\frac{dy}{dx} = A(-\sin x^2) \cdot 2x + B(\cos x^2) \cdot 2x$

$= 2x[-A \sin x^2 + B \cos x^2]$ ...(2)

Differentiating again

$$\frac{d^2y}{dx^2} = 2[-A \sin x^2 + B \cos x^2] + 2x[-A(\cos x^2) \cdot 2x + B(-\sin x^2) \cdot 2x]$$

$$= \frac{1}{x} \cdot \frac{dy}{dx} - 4x^2[A \cos x^2 + B \sin x^2] \quad \text{[Using (2)]}$$

$$= \frac{1}{x} \cdot \frac{dy}{dx} - 4x^2y \quad \text{or} \quad x\frac{dy}{dx} - \frac{dy}{dx} + 4x^3y = 0$$

which is the required differential equation.

**Example 24.** *Find the differential equation of the family of curves*

*$y = Ae^{2x} + Be^{-3x}$, for different values of A and B.*

**Sol.** The given equation is $y = Ae^{-2x} + Be^{-3x}$ ...(*i*)

Differentiating w.r.t. $x$

$$\frac{dy}{dx} = 2Ae^{2x} - 3Be^{-3x} \quad ...(ii)$$

Differentiating again w.r.t. '$x$'

$$\frac{d^2y}{dx^2} = 4Ae^{2x} + 9Be^{-3x} \quad ...(iii)$$

Adding (*ii*) and (*iii*),

$$\frac{d^2y}{dx^2} + \frac{dy}{dx} = 6(Ae^{2x} + Be^{-3x}) = 6y \quad | \because \text{ of } (i)$$

or $$\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$$

which is the reqd. differential equation.

**Example 25.** *Find the differential equation of the family of curves $y = Ae^{3x} + Be^{5x}$, where A, B are arbitrary constants.*

**Sol.** The given equation is $y = Ae^{3x} + Be^{5x}$ ...(1)

Differentiating w.r.t. $x$, we get

$$y' = 3A\,e^{3x} + 5Be^{5x} \quad ...(2)$$

Differentiating again w.r.t. $x$, we get

$$y'' = 9Ae^{3x} + 25Be^{5x} \quad ...(3)$$

[Now we find $Ae^{3x}$ and $Be^{5x}$ from (2) and (3)]

Multiplying (2) by 5 and subtracting (3) from the result

$$5y' - y'' = 6Ae^{3x} \quad \text{or} \quad Ae^{3x} = \frac{1}{6}(5y' - y'')$$

Multiplying (2) by 3 and subtracting the result from (3),

$$y'' - 3y' = 10Be^{5x} \quad \text{or} \quad Be^{5x} = \frac{1}{10}(y'' - 3y')$$

Substituting the values of $Ae^{3x}$ and $Be^{5x}$ in (1), we get

$$y = \frac{1}{6}(5y' - y'') + \frac{1}{10}(y'' - 3y')$$

or $$30y = 25y' - 5y'' + 3y'' - 9y'$$

or $$2y'' - 16y' + 30y = 0 \quad \text{or} \quad y'' - 8y' + 15y = 0$$

or $$\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$$

which is the reqd. differential equation.

**Example 26.** *Find the differential equation of the family of curves*

$$y = ae^{2x} + be^{3x}$$

**Sol.** Please try yourself. $\left[\textbf{Ans. } \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0\right]$

**Example 27.** *Find the differential equation of the family of curves $y = e^{mx}$, where $m$ is an arbitrary constant.*

**Sol.** The given equation is $y = e^{mx}$ ...(1)

Taking logarithms, $\log y = mx$

Differentiating w.r.t. $x$, we get

$$\frac{1}{y} \cdot \frac{dy}{dx} = m$$

Substituting the value of $m$ in (1), we get

$$\log y = \frac{x}{y} \cdot \frac{dy}{dx} \quad \text{or} \quad x\frac{dy}{dx} - y \log y = 0$$

which is the required differential equation.

**Example 28.** *Find the differential equation which has $e^{2y} + 2axe^y + a^2 = 0$ for its complete primitive.*

**Sol.** The given equation is $e^{2y} + 2ax\, e^y + a^2 = 0$ ...(*i*)

It is a **one parameter** family of curves.

Differentiating (*i*) w.r.t. $x$

$$2e^{2y} \cdot y_1 + 2ae^y + 2ax\, e^y \cdot y_1 = 0$$

$$\Rightarrow \quad y_1e^y + a(1 + xy_1) = 0 \qquad \text{[Dividing by } 2e^y\text{]}$$

$$\Rightarrow \quad a = -\frac{y_1e^y}{1 + xy_1}$$

Putting this value of $a$ in (*i*), we get

$$e^{2y} - 2xe^y \cdot \frac{y_1e^y}{1 + xy_1} + \frac{y_1^2\, e^{2y}}{(1 + xy_1)^2} = 0$$

or $$e^{2y}(1 + xy_1)^2 - 2xy_1\, e^{2y}(1 + xy_1) + y_1^2\, e^{2y} = 0$$

or $$1 + 2xy_1 + x^2y_1^2 - 2xy_1 - 2x^2y_1^2 + y_1^2 = 0$$

or $$1 - x^2y_2^2 + y_1^2 = 0 \quad \text{or} \quad (1 - x^2)y_1^2 + 1 = 0$$

or $$(1-x^2)\left(\frac{dy}{dx}\right)^2+1=0$$

which is the reqd. differential equation.

**Example 29.** *Find the differential equation corresponding to*

$$y=ae^x+be^{2x}+ce^{-3x}$$

*where a, b, c are arbitrary constants.*

**Sol.** The given equation is $\quad y=ae^x+be^{2x}+ce^{-3x} \quad$ ...(1)

Differentiating w.r.t. $x$

$$y_1=ae^x+2be^{2x}-3ce^{-3x}$$
$$=(ae^x+be^{2x}+ce^{-3x})+be^{2x}-4ce^{-3x}$$

$\Rightarrow$ $$y_1=y+be^{2x}-4ce^{-3x} \quad \text{...(2)}$$

[Using (1)]

Differentiating again

$$y_2=y_1+2be^{2x}+12ce^{-3x}$$
$$=y_1+2(be^{2x}-4ce^{-3x})+20ce^{-3x}$$
$$=y_1+2(y_1-y)+20ce^{-3x} \quad \text{[Using (2)]}$$

$\Rightarrow$ $$y_2-3y_1+2y=20ce^{-3x} \quad \text{...(3)}$$

Differentiating again w.r.t. $x$

$$y_3-3y_2+2y_1=-60ce^{-3x}$$
$$=-3(20ce^{-3x})$$
$$=-3(y_2-3y_1+2y) \quad \text{[Using (3)]}$$

or $$y_3-7y_1+6y=0$$

which is the reqd. differential equation.

**Example 30.** *Find the differential equation of the family of curves $y = A\cos mx + B\sin mx$ where m is fixed and A, B are arbitrary constants.*

**Sol.** The given equation is

$$y=\text{A}\cos mx+\text{B}\sin mx \quad \text{...(1)}$$

Differentiating w.r.t. $x$

$$\frac{dy}{dx}=-m\text{A}\sin mx+m\text{B}\cos mx$$

Differentiating again

$$\frac{d^2y}{dx^2}=-m^2\text{A}\cos mx-m^2\text{B}\sin mx$$
$$=-m^2(\text{A}\cos mx+\text{B}\sin mx)=-m^2y \quad \text{[Using (1)]}$$

or $$\frac{d^2y}{dx^2}+m^2y=0$$

which is the reqd. differential equation.

**Example 31.** *Form a differential equation of which $y = e^x (A\cos 2x + B\sin 2x)$ is a solution . (A and B being arbitrary constants).*

**Sol.** The given equation is $y=e^x(\text{A}\cos 2x+\text{B}\sin 2x)$ ...(*i*)

Differentiating w.r.t. $x$

$$\frac{dy}{dx}=e^x(\text{A}\cos 2x+\text{B}\sin 2x)+e^x(-2\text{A}\sin 2x+2\text{B}\cos 2x)$$

or $$\frac{dy}{dx} = y + e^x(-2A \sin 2x + 2B \cos 2x) \quad ...(ii)$$

| ∵ of (*i*)

Differentiating again w.r.t. $x$

$$\frac{d^2y}{dx^2} = \frac{dy}{dx} + e^x(-2A \sin 2x + 2B \cos 2x) + e^x(-4A \cos 2x - 4B \sin 2x)$$

$$= \frac{dy}{dx} + \left(\frac{dy}{dx} - y\right) - 4e^x(A \cos 2x + B \sin 2x)$$

| ∵ of (*ii*)

$$= 2\frac{dy}{dx} - y - 4y$$

| ∵ of (*i*)

$$\Rightarrow \quad \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = 0$$

which is the required differential equation.

**Example 32.** *Find the differential equation from the relation $y = e^x(A \cos x + B \sin x)$, where A and B are arbitrary constants.* (Punjab, 1996)

**Sol.** Please try yourself. $\left[\textbf{Ans. } \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = 0\right]$

**Example 33.** *Find the differential equation from the relation $y = a \sin x + b \cos x + x \sin x$, where $a$ and $b$ are arbitrary constants.*

**Sol.** The given equation is $y = a \sin x + b \cos x + x \sin x$ ...(*i*)

Differentiating w.r.t. '$x$'

$$\frac{dy}{dx} = a \cos x - b \sin x + \sin x + x \cos x$$

Differentiating again w.r.t. $x$,

$$\frac{d^2y}{dx^2} = -a \sin x - b \cos x + \cos x + \cos x - x \sin x$$

$$= -(a \sin x + b \cos x + x \sin x) + 2 \cos x$$

$$= -y + 2 \cos x$$

| ∵ of (*i*)

∴ $\frac{d^2y}{dx^2} + y = 2 \cos x$ is the required differential equation.

**Example 34.** *Find the differential equation of the family of curves $xy = Ae^x + Be^{-x} + x^2$ for different values of A and B.*

**Sol.** The given equation is $xy = Ae^x + Be^{-x} + x^2$ ...(*i*)

Differentiating w.r.t. $x$

$$y + x\frac{dy}{dx} = Ae^x - Be^{-x} + 2x$$

Differentiating again w.r.t. $x$

$$\frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2} = Ae^x + Be^{-x} + 2$$

or $$2\frac{dy}{dx} + x\frac{d^2y}{dx^2} = xy - x^2 + 2 \qquad | \because \text{ of } (i)$$

or $$x\frac{d^2y}{dx^2} + 2\frac{dy}{dx} = xy - x^2 + 2$$

is the required differential equation.

**Example 35.** *Find the differential equation which has $y = a \cos(mx + b)$ for its complete integral, a and b being arbitrary constants.*

**Sol.** The given equation is $y = a \cos(mx + b)$ ...(1)

Differentiating w.r.t. $x$ $\dfrac{dy}{dx} = -ma \sin(mx + b)$

Differentiating again $\dfrac{d^2y}{dx^2} = -m^2a \cos(mx + b) = -m^2y$ [Using (1)]

or $$\frac{d^2y}{dx^2} + m^2y = 0$$

which is the reqd. differential equation.

**Example 36.** *Form a differential equation corresponding to*

$$y^2 - 2ay + x^2 = a^2$$

*by eliminating a.*

**Sol.** The given equation is $y^2 - 2ay + x^2 = a^2$ ...(1)

Differentiating (1) w.r.t. $x$ $2yy_1 - 2ay_1 + 2x = 0$

$\Rightarrow$ $$a = \frac{yy_1 + x}{y_1}$$

Substituting this value of $a$ in (1), we get

$$y^2 - 2y\left(\frac{yy_1 + x}{y_1}\right) + x^2 = \left(\frac{yy_1 + x}{y_1}\right)^2$$

or $y^2y_1^2 - 2yy_1(yy_1 + x) + x^2y_1^2 = y^2y_1^2 + 2xyy_1 + x^2$

or $-2y^2y_1^2 - 2xyy_1 + x^2y_1^2 = 2xyy_1 + x^2$

or $(x^2 - 2y^2)y_1^2 - 4xyy_1 - x^2 = 0$

or $$(x^2 - 2y^2)\left(\frac{dy}{dx}\right)^2 - 4xy\frac{dy}{dx} - x^2 = 0$$

which is the reqd. differential equation.

**Example 37.** *Eliminate the arbitrary constants a and b from the equation $y = ax + bx^2$.*

**Sol.** The given equation is $y = ax + bx^2$ ...(1)

Differentiating (1) w.r.t. $x$ $\dfrac{dy}{dx} = a + 2bx$ ...(2)

Differentiating again $\dfrac{d^2y}{dx^2} = 2b$ ...(3)

From (3), $$b = \frac{1}{2}\frac{d^2y}{dx^2}$$

$\therefore$ From (2), $\dfrac{dy}{dx} = a + x\dfrac{d^2y}{dx^2} \Rightarrow a = \dfrac{dy}{dx} - x\dfrac{d^2y}{dx^2}$

Substituting the values of $a$ and $b$ in (1), we get

$$y = \left(\frac{dy}{dx} - x\frac{d^2y}{dx^2}\right)x + \frac{1}{2}x^2\frac{d^2y}{dx^2}$$

$$\Rightarrow \quad 2y = 2x\frac{dy}{dx} - 2x^2\frac{d^2y}{dx^2} + x^2\frac{d^2y}{dx^2}$$

$$\Rightarrow \quad x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

which is the reqd. eliminant.

**Example 38.** *Eliminate m from* $\sqrt{1-x^2} + \sqrt{1-y^2} = m(x-y)$.

**Sol.** The given equation is $\sqrt{1-x^2} + \sqrt{1-y^2} = m(x-y)$ ...(1)

Putting $x = \sin\theta$, $y = \sin\phi$

(1) becomes $\cos\theta + \cos\phi = m(\sin\theta - \sin\phi)$

or $$2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} = m.\ 2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}$$

or $$\cot\frac{\theta-\phi}{2} = m$$

or $\dfrac{\theta-\phi}{2} = \cot^{-1} m$ or $\theta - \phi = 2\cot^{-1} m$

or $\sin^{-1} x - \sin^{-1} y = 2\cot^{-1} m$

Differentiating w.r.t. $x$

$$\frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \quad \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

which is the reqd. eliminant.

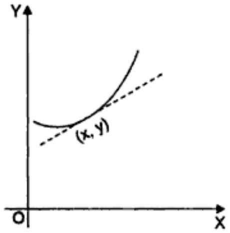
**Geometrical meaning of the differential equation $\dfrac{dy}{dx} = f(x, y)$ of the first order and first degree.**

The given differential equation is $\dfrac{dy}{dx} = f(x, y)$ ...(1)

Let $(x_1, y_1)$ be any point in the $xy$-plane.

Putting the co-ordinates of this point in (1), we get the value of $\dfrac{dy}{dx}$ at $(x_1, y_1)$, say $m_1$ which is the slope (or direction) of the tangent at $(x_1, y_1)$.

Suppose the point moves from $(x_1, y_1)$ in the direction $m_1$ [*i.e.*, along the tangent at $(x_1, y_1)$] for an infinitesimal distance, to a point $(x_2, y_2)$.




Let $m_2$ be the slope of the tangent at $(x_2, y_2)$ determined from (1).

Let the point move in the direction $m_2$ for an infinitesimal distance to a point $(x_3, y_3)$.

Let $m_3$ be the slope of the tangent at $(x_3, y_3)$ determined from (1).

Let the point move in the direction $m_3$ for an infinitesimal distance to a point $(x_4, y_4)$.

Proceeding like this, the point will describe a curve, the co-ordinates of every point of which and the direction of the tangent there at, will satisfy the differential equation (1).

Since $(x_1, y_1)$ is any point in the $xy$-plane, through every point in the plane, there will pass a particular curve, for every point of which $x, y, \frac{dy}{dx}$ will satisfy (1).

The equation of each curve is a particular solution of (1). The equation of the system of such curves is the general solution of (1). All the curves represented by the general solution, taken together, form the locus of the differential equation (1).

# 2

# Solution of Differential Equations of the First Order and First Degree

**Definition.** A differential equation of first order and first degree is an equation of the form

$$\frac{dy}{dx} = f(x, y) \quad \text{or} \quad \mathrm{M}dx + \mathrm{N}dy = 0$$

where M, N are functions of $x$ and $y$.

## TYPE I. VARIABLES SEPARABLE

**If in an equation, it is possible to get all the functions of x and dx to one side, and all the functions of y and dy to the other, the variables are said to be separable.**

**Rule to solve an equation in which the variables are separable**

Consider the equation $\frac{dy}{dx} = \mathrm{XY}$, where X is a function of $x$ only and Y is a function of $y$ only.

1. Given differential equation is $\frac{dy}{dx} = \mathrm{XY}$.

2. $\frac{dy}{\mathrm{Y}} = \mathrm{X}\, dx$ *i.e.,* variables have been separated.

3. Integrating both sides, $\int \frac{dy}{\mathrm{Y}} = \int \mathrm{X}\, dx + c$, where $c$ is an arbitrary constant, is the required solution.

**Note 1. Never forget to add an arbitrary constant on one side (only). A solution without this constant is wrong, for it is not the general solution.**

**Note 2. The nature of the arbitrary constant depends upon the nature of the problem.**

**Note 3. A constant is, after all, a constant, in whatever form it may be taken.**

**Note 4. The solution of a differential equation must be put in a form as simple as possible.**

**Note 5. Remember :**

(*i*) $\log x + \log y = \log xy$

(*ii*) $\log x - \log y = \log \frac{x}{y}$

(*iii*) $n \log x = \log x^n$

[From Logarithms]

(*iv*) $\tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x + y}{1 - xy}$

(*v*) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x - y}{1 + xy}$

[From Trigonometry]

**Example 1.** *Solve the following differential equations :*

(*i*) $\dfrac{dy}{dx} = e^{x+y} + x^2e^y$ (*ii*) $\dfrac{dy}{dx} = e^{x-y} + e^{2\log x - y}$

(*iii*) $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$.

**Sol.** (*i*) $$\frac{dy}{dx} = e^{x+y} + x^2 . e^y = e^x . e^y + x^2 . e^y$$

or $$\frac{dy}{dx} = e^y(e^x + x^2)$$ | **Note**

Separating the variables, $e^{-y}dy = (e^x + x^2)\,dx$

Integrating both sides, $\int e^{-y}dy = \int (e^x + x^2)\,dx + c$

or $$\frac{e^{-y}}{-1} = e^x + \frac{x^3}{3} + c$$

or $e^x + e^{-y} + \dfrac{x^3}{3} + c = 0$, which is the required solution.

(*ii*) $$\frac{dy}{dx} = e^{x-y} + e^{2\log x - y} = e^{x-y} + e^{2\log x} . e^{-y}$$

$$= e^{x-y} + e^{\log x^2} . e^{-y} = e^{x-y} + x^2 e^{-y}$$ $| \because \ e^{\log f(x)} = f(x)$

Now proceed as in part (*i*).

(*iii*) $y - x\dfrac{dy}{dx} = ay^2 + a\dfrac{dy}{dx}$ or $y - ay^2 = (a + x) . \dfrac{dy}{dx}$

Separating the variables,

$$\frac{1}{y - ay^2}\,dy = \frac{dx}{a + x} \quad \text{or} \quad \frac{1}{y(1 - ay)}\,dy = \frac{dx}{a + x}$$

or $$\left[\frac{1}{y} + \frac{a}{1 - ay}\right] dy = \frac{dx}{a + x}$$ [Partial fractions]

Integrating both sides,

$$\int \left(\frac{1}{y} + \frac{a}{1 - ay}\right) dy = \int \frac{dx}{a + x} + c_1$$

or $$\log y + a\left(-\frac{1}{a}\right)\log(1 - ay) = \log(a + x) + c_1$$

or $$\log y - \log(1 - ay) = \log(a + x) + \log c$$ | taking $c_1 = \log c$ [See Note ...]

or $$\log\frac{y}{1 - ay} = \log c\,(a + x)$$

$$\left[\because \ \log m - \log n = \log\frac{m}{n}; \log m + \log n = \log mn\right]$$

or $$\frac{y}{1 - ay} = c(a + x)$$

or $y = c(a + x)(1 - ay)$, which is the required solution.

**Example 2.** *Solve the following differential equations :*

(*i*) $\sec^2 x \tan y \, dx + \sec^2 y \tan x \, dy = 0$

(*ii*) $\dfrac{dy}{dx} = \dfrac{x(2 \log x + 1)}{\sin y + y \cos y}$ (*iii*) $\left(y - x \dfrac{dy}{dx}\right) x = y$

(*iv*) $3e^x \tan y \, dx + (1 - e^x) \sec^2 y \, dy = 0.$

**Sol.** (*i*) $\sec^2 y \tan x \, dy = - \sec^2 x \tan y \, dx$

Separating the variables,

$$\frac{\sec^2 y}{\tan y} dy = - \frac{\sec^2 x}{\tan x} dx$$

Integrating both sides,

$$\int \frac{\sec^2 y}{\tan y} dy = - \int \frac{\sec^2 x}{\tan x} dx + c_1$$

or $\log \tan y = - \log \tan x + c_1$ $\quad \left| \because \int \dfrac{f'(x)}{f(x)} dx = \log f(x) \right.$

or $\log \tan x + \log \tan y = \log c$ $\quad |$ taking $c_1 = \log c$

or $\log \tan x \tan y = \log c$ $\quad | \because \log m + \log n = \log mn$

or $\tan x \tan y = c$ is the required solution.

(*ii*) $$\frac{dy}{dx} = \frac{x(2 \log x + 1)}{\sin y + y \cos x}$$

Separating the variables,

$$(\sin y + y \cos y) \, dy = x(2 \log x + 1) \, dx$$

Integrating both sides,

$$\int (\sin y + y \cos y) \, dy = \int (2x \log x + x) \, dx + c = 2 \int x \log x \, dx + \int x \, dx + c$$

or $- \cos y + y \sin y - \int 1. \sin y \, dy$

$$= 2 \left[ \log x . \frac{x^2}{2} - \int \frac{1}{x} . \frac{x^2}{2} dx \right] + \frac{x^2}{2} + c$$

$$= 2 \left[ \frac{x^2}{2} \log x - \frac{1}{2} \int x \, dx \right] + \frac{x^2}{2} + c$$

or $- \cos y + y \sin y + \cos y = 2 \left[ \dfrac{x^2}{2} \log x - \dfrac{x^2}{4} \right] + \dfrac{x^2}{2} + c$

or $y \sin y = x^2 \log x + c$ is the required solution.

(*iii*) $$\left(y - x . \frac{dy}{dx}\right) x = y$$

or $xy - x^2 . \dfrac{dy}{dx} = y$ or $x^2 . \dfrac{dy}{dx} = y(x - 1)$

Separating the variables,

$$\frac{dy}{y} = \frac{x - 1}{x^2} dx \quad \text{or} \quad \frac{dy}{y} = \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$$

Integrating both sides,

$$\int \frac{dy}{y} = \int \left(\frac{1}{x} - \frac{1}{x^2}\right) dx + c$$

or $$\log y = \log x + \frac{1}{x} + c \text{ which is the required solution.}$$

*(iv)* $$(1 - e^x) \sec^2 y \, dy = - 3e^x \tan y \, dx$$

or $$(e^x - 1) \sec^2 y \, dy = 3e^x \tan y \, dx$$

Separating the variables $\dfrac{\sec^2 y}{\tan y} dy = \dfrac{3e^x}{e^x - 1} dx$

Integrating both sides,

$$\int \frac{\sec^2 y}{\tan y} dy = 3 \int \frac{e^x}{e^x - 1} dx + c_1$$

or $$\log \tan y = 3 \log (e^x - 1) + c_1$$

or $$\log \tan y = \log (e^x - 1)^3 + \log c \qquad | \text{ taking } c_1 = \log c$$

or $$\log \tan y = \log c \, (e^x - 1)^3$$

or $$\tan y = c(e^x - 1)^3$$

which is the required solution.

**Example 3.** *Solve the following differential equations :*

*(i)* $(x^2 - yx^2) \, dy + (y^2 + xy^2) \, dx = 0$ *(ii)* $(x - y^2x) \, dx - (y - x^2y) \, dy = 0.$

**Sol.** *(i)* $(x^2 - yx^2) \, dy + (y^2 + xy^2) \, dx = 0$

or $$x^2(1 - y) \, dy + y^2 (1 + x) dx = 0$$

Separating the variables,

$$\frac{1 - y}{y^2} dy + \frac{1 + x}{x^2} dx = 0$$

or $$\left(\frac{1}{y^2} - \frac{1}{y}\right) dy + \left(\frac{1}{x^2} + \frac{1}{x}\right) dx = 0$$

Integrating both sides,

$$\int \left(\frac{1}{y^2} - \frac{1}{y}\right) dy + \int \left(\frac{1}{x^2} + \frac{1}{x}\right) dx = c$$

or $$- \frac{1}{y} - \log y - \frac{1}{x} + \log x = c$$

or $$\log \frac{x}{y} - \left(\frac{1}{x} + \frac{1}{y}\right) = c \text{ is the reqd. solution.}$$

*(ii)* $(x - y^2x) \, dx - (y - x^2y) \, dy = 0$

or $$y (1 - x)^2 \, dy = x(1 - y^2) \, dx$$

Separating the variables, $\dfrac{y}{1 - y^2} dy = \dfrac{x}{1 - x^2} dx$

Integrating both sides, $\int \frac{y}{1-y^2}\, dy = \int \frac{x}{1-x^2}\, dx + c_1$

or $$-\frac{1}{2}\int \frac{-2y}{1-y^2}\, dy = -\frac{1}{2}\int \frac{-2x}{1-x^2}\, dx + c_1$$

or $$-\tfrac{1}{2}\log(1-y^2) = -\tfrac{1}{2}\log(1-x^2) + c_1$$

or $$\log(1-y^2) = \log(1-x^2) - 2c_1$$

or $$\log(1-y^2) = \log(1-x^2) + \log c \qquad | \text{ taking } -2c_1 = \log c$$

or $$1 - y^2 = c(1-x^2).$$

**Example 4.** *Solve the following differential equations :*

(i) $\frac{dy}{dx} = xy + x + y + 1$ (ii) $x(1+y^2)\, dx + y(1+x^2)\, dy = 0$

(iii) $\frac{dy}{dx} = e^{2x-3y} + 4x^2e^{-3y}$ (iv) $(x+1)\frac{dy}{dx} + 1 = 2e^{-y}$.

**Sol.** (i) $\frac{dy}{dx} = xy + x + y + 1 = (x+1)(y+1)$

Separating the variables $\frac{dy}{y+1} = (x+1)\, dx$

Integrating both sides, $\log(y+1) = \frac{x^2}{2} + x + c$ is the reqd. solution.

(ii) $x(1+y^2)\, dx + y(1+x^2)\, dy = 0$

Separating the variables, $\frac{y}{1+y^2}\, dy + \frac{x}{1+x^2}\, dx = 0$

Integrating both sides, $\int \frac{y}{1+y^2}\, dy + \int \frac{x}{1+x^2}\, dx = c_1$

$$\frac{1}{2}\int \frac{2y}{1+y^2}\, dy + \frac{1}{2}\int \frac{2x}{1+x^2}\, dx = c_1$$

or $$\frac{1}{2}\log(1+y^2) + \frac{1}{2}\log(1+x^2) = c_1$$

or $$\log(1+y^2) + \log(1+x^2) = 2c_1$$

or $$\log(1+y^2)(1+x^2) = \log c$$

$\therefore$ $$(1+y^2)(1+x^2) = c$$

is the required solution.

(iii) The given equation can be written as

$$\frac{dy}{dx} = (e^{2x} + 4x^2)\, e^{-3y} \quad \Rightarrow \quad e^{3y}dy = (e^{2x} + 4x^2)dx$$

Integrating $$\frac{e^{3y}}{3} = \left(\frac{e^{2x}}{2} + \frac{4x^3}{3}\right) + c$$

or $$2e^{3y} = 3e^{2x} + 8x^3 + C \qquad \text{where} \quad C = 6c$$

(*iv*) The given equation can be written as

$$(x+1)\frac{dy}{dx} = 2e^{-y} - 1$$

$$\Rightarrow \qquad \frac{dy}{2e^{-y}-1} = \frac{dx}{x+1} \qquad \Rightarrow \qquad \frac{e^y}{2-e^y}\,dy = \frac{dx}{x+1}$$

Integrating $\qquad -\log(2-e^y) = \log(x+1) + c$

$$\Rightarrow \qquad \log(x+1) + \log(2-e^y) = -c$$

$$\Rightarrow \qquad \log(x+1)(2-e^y) = -c$$

$$\Rightarrow \qquad (x+1)(2-e^y) = e^{-c}$$

$$\Rightarrow \qquad (x+1)(2-e^y) = C \qquad \text{where} \quad C = e^{-c}.$$

**Example 5.** *Solve the following differential equations :*

(*i*) $\dfrac{dy}{dx} = \dfrac{xy+y}{xy+x}$ (*ii*) $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$

(*iii*) $x\cos^2 y\,dx = y\cos^2 x\,dy$ (*iv*) $\sec^2 x \tan y\,dy + \sec^2 y \tan x\,dx = 0$

(*v*) $\dfrac{dy}{dx}\,.\,\tan y = \sin(x+y) + \sin(x-y)$.

**Sol.** (*i*) $\dfrac{dy}{dx} = \dfrac{y(x+1)}{x(y+1)}$

Separating the variables,

$$\frac{y+1}{y}\,dy = \frac{x+1}{x}\,dx \qquad \text{or} \qquad \left(1+\frac{1}{y}\right)dy = \left(1+\frac{1}{x}\right)dx$$

Integrating both sides,

$$\int\left(1+\frac{1}{y}\right)dy = \int\left(1+\frac{1}{x}\right)dx + c$$

or $\qquad y + \log y = x + \log x + c$

or $\qquad y - x = \log\dfrac{x}{y} + c$ is the required solution.

(*ii*) $(e^y+1)\cos x\,dx + e^y\sin x\,dy = 0$

Separating the variables, $\dfrac{e^y}{e^y+1}\,dy + \dfrac{\cos x}{\sin x}\,dx = 0$

Integrating both sides, $\log(e^y+1) + \log\sin x = \log c$

or $\qquad (e^y+1)\sin x = c$ is the required solution.

(*iii*) $x\cos^2 y\,dx = y\cos^2 x\,dy$

Separating the variables $\dfrac{y}{\cos^2 y}\,dy = \dfrac{x}{\cos^2 x}\,dx$

Integrating both sides $\int y\sec^2 y\,dy = \int x\sec^2 x\,dx + c$

Integrating by parts, we get

$$y\tan y - \int\tan y\,dy = x\tan x - \int\tan x\,dx + c$$

or $\qquad y\tan y + \log\cos y = x\tan x + \log\cos x + c$

is the required solution.

(*iv*) $\sec^2 x \tan y \, dy + \sec^2 y \tan x \, dx = 0$

Separating the variables $\dfrac{\tan y}{\sec^2 y} dy + \dfrac{\tan x}{\sec^2 x} dx = 0$

or $\sin y \cos y \, dy + \sin x \cos x \, dx = 0$

or $\sin 2y \, dy + \sin 2x \, dx = 0$

Integrating both sides, $-\frac{1}{2} \cos 2y - \frac{1}{2} \cos 2x = -\frac{1}{2} c$

or $\cos 2x + \cos 2y = c$ is the required solution.

(*v*) $\dfrac{dy}{dx} \tan y = \sin (x + y) + \sin (x - y) = 2 \sin x \cos y$

$\left| \because \; \sin C + \sin D = 2 \sin \dfrac{C+D}{2} \cos \dfrac{C-D}{2} \right.$

Separating the variables, $\sec y \tan y \, dy = 2 \sin x \, dx$

Integrating both sides

$\sec y = -2 \cos x + c$ is the required solution.

**Example 6.** *Solve the following differential equations :*

(*i*) $\dfrac{dy}{dx} + \sqrt{\dfrac{1-y^2}{1-x^2}} = 0$ (*ii*) $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$

(*iii*) $y - x \dfrac{dy}{dx} = 3 \left( 1 + x^2 \dfrac{dy}{dx} \right)$ (*iv*) $y \sqrt{1+x^2} \, dx + x \sqrt{1+y^2} \, dy = 0$

(*v*) $(1 + x)y \, dx + (1 + y)x \, dy = 0.$ (Bangalore, 1996)

**Sol.** (*i*) $$\frac{dy}{dx} + \sqrt{\frac{1-y^2}{1-x^2}} = 0$$

or $$\frac{dy}{\sqrt{1-y^2}} + \frac{dx}{\sqrt{1-x^2}} = 0$$

Integrating both sides, $\sin^{-1} y + \sin^{-1} x = \sin^{-1} c$

or $\sin^{-1} [x \sqrt{1-y^2} + y \sqrt{1-x^2}] = \sin^{-1} c$

or $x \sqrt{1-y^2} + y \sqrt{1-x^2} = c$ which is the reqd. solution.

(*ii*) $\dfrac{dy}{dx} = \dfrac{1+y^2}{1+x^2}$ or $\dfrac{dy}{1+y^2} = \dfrac{dx}{1+x^2}$

Integrating both sides, $\tan^{-1} y = \tan^{-1} x + c_1$

or $\tan^{-1} y - \tan^{-1} x = c_1$

or $\tan^{-1} \dfrac{y-x}{1-yx} = \tan^{-1} c$ | taking $c_1 = \tan^{-1} c$

or $\dfrac{y-x}{1+xy} = c$ or $y - x = c(1 + xy)$

which is the reqd. solution.

(*iii*) $y - x \dfrac{dy}{dx} = 3 \left( 1 + x^2 \dfrac{dy}{dx} \right)$ or $y - 3 = (3x^2 + x) \dfrac{dy}{dx}$

or $$\frac{dy}{y-3}=\frac{dx}{x(3x+1)} \quad \text{or} \quad \frac{dy}{y-3}=\left(\frac{1}{x}-\frac{3}{3x+1}\right)dx \qquad \text{[Partial fractions]}$$

Integrating both sides,

$$\log(y-3)=\log x-3\cdot\tfrac{1}{3}\log(3x+1)+c_1$$

$$=\log\frac{x}{3x+1}+\log c \qquad |\ \text{taking } c_1=\log c$$

or $$\log(y-3)=\log\frac{cx}{3x+1} \quad \text{or} \quad y-3=\frac{cx}{3x+1}$$

or $$(y-3)(3x+1)=cx$$

which is the reqd. solution,

*(iv)* $$y\sqrt{1+x^2}\,dx+x\sqrt{1+y^2}\,dy=0$$

or $$\frac{\sqrt{1+y^2}}{y}dy+\frac{\sqrt{1+x^2}}{x}dx=0$$

Integrating both sides

$$\int\frac{\sqrt{1+y^2}}{y}dy+\int\frac{\sqrt{1+x^2}}{x}dx=c \qquad \text{...(1)}$$

Now $$\int\frac{\sqrt{1+y^2}}{y}dy=\int\frac{1+y^2}{y\sqrt{1+y^2}}dy=\int\frac{1}{y\sqrt{1+y^2}}dy+\int\frac{y}{\sqrt{1+y^2}}dy$$

$$=\int\frac{-\frac{1}{t^2}dt}{\frac{1}{t}\sqrt{1+\frac{1}{t^2}}}+\tfrac{1}{2}\int(1+y)^{-\frac{1}{2}}\,.\,2ydy \qquad \text{where } t=\frac{1}{y}$$

$$=-\int\frac{dt}{\sqrt{t^2+1}}+\tfrac{1}{2}\cdot\frac{(1+y^2)^{1/2}}{\frac{1}{2}}=-\sinh^{-1}t+\sqrt{1+y^2}$$

$$=-\log(t+\sqrt{1+t^2})+\sqrt{1+y^2}=-\log\left(\frac{1}{y}+\sqrt{1+\frac{1}{y^2}}\right)+\sqrt{1+y^2}$$

$$=-\log\frac{1+\sqrt{1+y^2}}{y}+\sqrt{1+y^2}$$

Similarly $$\int\frac{\sqrt{1+x^2}}{x}dx=-\log\frac{1+\sqrt{1+x^2}}{x}+\sqrt{1+x^2}$$

$\therefore$ From (1),

$$-\log\frac{1+\sqrt{1+y^2}}{y}+\sqrt{1+y^2}-\log\frac{1+\sqrt{1+x^2}}{x}+\sqrt{1+x^2}=c$$

or $$\sqrt{1+x^2} + \sqrt{1+y^2} = \log \frac{1+\sqrt{1+x^2}}{x} + \log \frac{1+\sqrt{1+y^2}}{xy} + c$$

or $$\sqrt{1+x^2} + \sqrt{1+y^2} = \log \frac{(1+\sqrt{1+x^2})(1+\sqrt{1+y^2})}{xy} + c$$

which is the reqd. solution.

(*v*) $$(1 + x)y\,dx + (1 + y)x\,dy = 0$$

or $$\frac{1+y}{y}\,dy + \frac{1+x}{x}\,dx = 0$$

or $$\left(\frac{1}{y}+1\right)dy + \left(\frac{1}{x}+1\right)dx = 0$$

Integrating both sides

$$\log y + y + \log x + x = c$$

or $$x + y + \log xy = c$$

which is the reqd. solution.

**Example 7.** *Solve the following differential equations*

(*i*) $(1 - x^2)(1 - y)\,dx = xy(1 + y)dy$ (*ii*) $a(xdy + 2ydx) = xydy$

(*iii*) $\frac{ds}{dx} + x^2 = x^2e^{3s}$ (*iv*) $\log\left(\frac{dy}{dx}\right) = ax + by$ (Magadh 1998)

(*v*) $ydx + (1 + x^2)\tan^{-1} x\,dy = 0.$

**Sol.** (*i*) $$(1 - x^2)(1 - y)\,dx = xy\,(1 + y)\,dy$$

or $$\frac{y(1+y)}{1-y}\,dy = \frac{1-x^2}{x}\,dx$$

or $$\left(-y-2+\frac{2}{1-y}\right)dy = \left(\frac{1}{x}-x\right)dx$$

Integrating both sides

$$-\frac{y^2}{2} - 2y - 2\log(1-y) = \log x - \frac{x^2}{2} + c.$$

or $$\frac{x^2}{2} - \frac{y^2}{2} - 2y = \log x + 2\log(1-y) + c$$

which is the reqd. solution.

(*ii*) $a(xdy + 2ydx) = xydy$ or $2aydx = x(y - a)\,dy$

or $$\frac{y-a}{y}\,dy = \frac{2a}{x}\,dx \quad \text{or} \quad \left(1-\frac{a}{y}\right)dy = \frac{2a}{x}\,dx$$

Integrating both sides

$$y - a\log y = 2a\log x + c$$

which is the reqd. solution.

(*iii*) $$\frac{ds}{dx} + x^2 = x^2e^{3s} \quad \text{or} \quad \frac{ds}{dx} = x^2\,(e^{3s} - 1)$$

or $$\frac{ds}{e^{3s}-1} = x^2dx \quad \text{or} \quad \frac{e^{-3s}}{1-e^{-3s}}\, ds = x^2dx$$ | **Note carefully**

or $$\frac{1}{3}\cdot\frac{3e^{-3s}}{1-e^{-3s}}\, ds = x^2dx$$

Integrating both sides,

$$\frac{1}{3}\log(1-e^{-3s}) = \frac{x^3}{3} + c_1$$

or $$\log(1-e^{-3s}) = x^3 + 3c_1 = \log e^{x^3} + \log c = \log ce^{x^3}$$

or $$1 - e^{-3s} = ce^{x^3}$$

which is the reqd. solution.

(*iv*) $\log\left(\frac{dy}{dx}\right) = ax + by \quad \Rightarrow \quad \frac{dy}{dx} = e^{ax+by}$ [$\because$ $\log_a m = x \Rightarrow m = a^x$

Also, when the base is not mentioned, it is understood to be $e$]

or $$\frac{dy}{dx} = e^{ax}\,.\,e^{by} \quad \text{or} \quad e^{-by}\,dy = e^{ax}\,dx$$

Integrating both sides,

$$\frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + c_1$$

or $$be^{ax} + ae^{-by} = -abc_1 \quad \text{or} \quad be^{ax} + ae^{-by} = c$$

which is the reqd. solution.

(*v*) $$ydx + (1+x^2)\tan^{-1} xdy = 0$$

or $$\frac{dy}{y} + \frac{dx}{(1+x^2)\tan^{-1} x} = 0$$

or $$\frac{dy}{y} + \frac{\frac{1}{1+x^2}}{\tan^{-1} x}\, dx = 0$$

Integrating both sides,

$$\log y + \log(\tan^{-1} x) = \log c$$

or $$\log(y\tan^{-1} x) = \log c$$

or $$y\tan^{-1} x = c$$

which is the reqd. solution.

**Example 8.** *Solve the following differential equations :*

(*i*) $(1+x^2)\,dy + x\sqrt{1-y^2}\,dx = 0$ (*ii*) $e^x\sqrt{1-y^2}\,dx + \frac{y}{x}\,dy = 0$

(*iii*) $x^2\frac{dy}{dx} + y = 1$ (*iv*) $\frac{dy}{dx} + \frac{y^2+y+1}{x^2+x+1} = 0$

(*v*) $\sqrt{1+x^2+y^2+x^2y^2} + xy\frac{dy}{dx} = 0.$

**Sol.** (*i*) $$(1 + x^2)\, dy + x\sqrt{1-y^2}\, dx = 0$$

or $$\frac{dy}{\sqrt{1-y^2}} + \frac{x}{\sqrt{1+x^2}}\, dx = 0$$

or $$\frac{dy}{\sqrt{1-y^2}} + \frac{1}{2} \cdot \frac{2x}{1+x^2}\, dx = 0$$

Integrating both sides,

$$\sin^{-1} y + \frac{1}{2} \log (1 + x^2) = c$$

which is the reqd. solution.

(*ii*) $$e^x \sqrt{1-y^2}\, dx + \frac{y}{x}\, dy = 0$$

or $$\frac{y}{\sqrt{1-y^2}}\, dy + xe^x\, dx = 0$$

Integrating both sides,

$$\int \frac{y}{\sqrt{1-y^2}}\, dy + \int xe^x\, dx = c$$

or $$-\frac{1}{2} \int (1 - y^2)^{-1/2} . (-2y) dy + x . e^x - \int 1 . e^x\, dx = c$$

or $$-\frac{1}{2}\, \frac{(1-y^2)^{1/2}}{\frac{1}{2}} + xe^x - e^x = c$$

or $$(x - 1)e^x = \sqrt{1-y^2} + c$$

which is the reqd. solution.

(*iii*) $$x^2 \frac{dy}{dx} + y = 1$$

or $$x^2 \frac{dy}{dx} = 1 - y$$

or $$\frac{dy}{1-y} = \frac{dx}{x^2}$$

Integrating both sides,

$$-\log (1 - y) = -\frac{1}{x} - c$$

$$\log (1 - y) = \frac{1}{x} + c$$

which is the reqd. solution.

(*iv*) $$\frac{dy}{dx} + \frac{y^2 + y + 1}{x^2 + x + 1} = 0$$

or $$\frac{dy}{y^2+y+1}+\frac{dx}{x^2+x+1}=0$$

Integrating both sides,

$$\int\frac{dy}{y^2+y+1}+\int\frac{dx}{x^2+x+1}=c_1$$

or $$\int\frac{dy}{\left(y+\frac{1}{2}\right)^2+\frac{3}{4}}+\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\frac{3}{4}}=c_1$$

or $$\frac{2}{\sqrt{3}}\tan^{-1}\frac{y+\frac{1}{2}}{\frac{\sqrt{3}}{2}}+\frac{2}{\sqrt{3}}\tan^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}=c_1$$

or $$\tan^{-1}\frac{2y+1}{\sqrt{3}}+\tan^{-1}\frac{2x+1}{\sqrt{3}}=\frac{\sqrt{3}}{2}c_1=\tan^{-1}c$$ [replacing $c_1$ by $\tan^{-1}c$]

or $$\tan^{-1}\frac{\frac{2y+1}{\sqrt{3}}+\frac{2x+1}{\sqrt{3}}}{1-\frac{2y+1}{\sqrt{3}}\cdot\frac{2x+1}{\sqrt{3}}}=\tan^{-1}c \quad \Bigg|\because\ \tan^{-1}a+\tan^{-1}b=\tan^{-1}\frac{a+b}{1-ab}$$

or $$\frac{2x+2y+2}{\sqrt{3}}\cdot\frac{3}{3-(2x+1)(2y+1)}=c$$

or $$2\sqrt{3}\,(x+y+1)=c\,(2-2x-2y-4xy)$$

or $$(x+y+1)=C(1-x-y-2xy)$$

which is the reqd. solution.

(*v*) The given equation can be written as

$$\sqrt{(1+x^2)(1+y^2)}+xy\frac{dy}{dx}=0$$

or $$\frac{y}{\sqrt{1+y^2}}dy+\frac{\sqrt{1+x^2}}{x}dx=0$$

Integrating both sides,

$$\int\frac{y}{\sqrt{1+y^2}}dy+\int\frac{\sqrt{1+x^2}}{x}dx=c$$

or $$\sqrt{1+y^2}-\log\frac{1+\sqrt{1+x^2}}{x}+\sqrt{1+x^2}=c$$

[For integration, see example 6 (*iv*)]

or $$\sqrt{1+x^2}+\sqrt{1+y^2}=\log\frac{1+\sqrt{1+x^2}}{x}+c$$

which is the reqd. solution.

**Example 9.** (*i*) *If* $\dfrac{dy}{dx} = e^{x+y}$ *and it is given that for* $x = 1, y = 1$ *; find* $y$ *when* $x = -1$.

(*ii*) *Find the equation of the curve represented by*

$$(y - yx)\,dx + (x + xy)\,dy = 0$$

*and passing through the point (1, 1).*

**Sol.** (*i*) $$\frac{dy}{dx} = e^{x+y} = e^x \cdot e^y$$

Separating the variables, $e^{-y}\,dy = e^x\,dx$

Integrating both sides, $-e^{-y} = e^x + c$ ...(*i*)

when $x = 1, y = 1 \quad \Rightarrow \quad -e^{-1} = e + c$

or $$c = -\frac{1}{e} - e = -\frac{1+e^2}{e}$$

$\therefore$ From (*i*), $$-e^{-y} = e^x - \frac{1+e^2}{e}$$

Now when $x = -1$, we have

$$-e^{-y} = e^{-1} - \frac{1+e^2}{e} = \frac{1}{e} - \frac{1+e^2}{e} = -e$$

or $e^{-y} = e^1 \;\therefore\; -y = 1$ or $y = -1$.

(*ii*) $y(1 - x)\,dx + x(1 + y)\,dy = 0$

Separating the variables, $\dfrac{1+y}{y}\,dy + \dfrac{1-x}{x}\,dx = 0$

or $$\left(\frac{1}{y} + 1\right) dy + \left(\frac{1}{x} - 1\right) dx = 0$$

Integrating both sides, $\log y + y + \log x - x = c$ ...(*i*)

If it passes through (1, 1) $c = 0$ | $\because \log 1 = 0$

$\therefore$ The equation of the curve is

$$\log y + y + \log x - x = 0 \quad \text{or} \quad \log xy = x - y.$$

**Example 10.** *Solve the following differential equations :*

(*i*) $(1 - x^2)\,dy + xy\,dx = xy^2 dx$ (*ii*) $(1 + x)(1 + y^2)\,dx + (1 + y)(1 + x^2)\,dy = 0$

(*iii*) $x \cos y\,dy = e^x (x \log x + 1)\,dx$.

**Sol.** (*i*) The given equation can be written as

$$(1 - x^2)\,dy + xy\,(1 - y)\,dx = 0$$

or $$\frac{dy}{y(1-y)} + \frac{x}{1-x^2}\,dx = 0$$

Integrating both sides,

$$\int \frac{dy}{y(1-y)} + \int \frac{x}{1-x^2}\,dx = c'$$

or $$\int \left(\frac{1}{y} + \frac{1}{1-y}\right) dy - \frac{1}{2}\int \frac{2x}{1-x^2}\,dx = c'$$

or $$\log y - \log(1-y) - \frac{1}{2}\log(1-x^2) = c'$$

or $$\log \frac{y}{(1-y)\sqrt{1-x^2}} = \log c \text{ where } \log c = c'$$

or $$\frac{y}{(1-y)\sqrt{1-x^2}} = c \qquad \text{or} \qquad y = c(1-y)\sqrt{1-x^2}$$

which is the required solution.

(*ii*) The given equation is

$$(1+x)(1+y^2)\,dx + (1+y)(1+x^2)\,dy = 0$$

or $$\frac{1+x}{1+x^2}\,dy + \frac{1+y}{1+y^2}\,dy = 0$$

Integrating both sides,

$$\int\left(\frac{1}{1+x^2} + \frac{1}{2}\cdot\frac{2x}{1+x^2}\right)dx + \int\left(\frac{1}{1+y^2} + \frac{1}{2}\cdot\frac{2y}{1+y^2}\right)dy = c$$

or $$\tan^{-1}x + \frac{1}{2}\log(1+x^2) + \tan^{-1}y + \frac{1}{2}\log(1+y^2) = c$$

or $$\tan^{-1}\frac{x+y}{1-xy} + \frac{1}{2}\log(1+x^2)(1+y^2) = c$$

which is the required solution.

(*iii*) The given equation can be written as

$$\cos y\,dy = e^x\left(\log x + \frac{1}{x}\right)dx$$

Integrating both sides,

$$\sin y = \int \log x\,.\,e^x\,dx + \int \frac{1}{x}\,e^x\,dx + c$$

$$= \log x\,.\,e^x - \int \frac{1}{x}\,.\,e^x\,dx + \int \frac{1}{x}\,.\,e^x\,dx + c$$

or $$\sin y = e^x \log x + c$$

which is the required solution.

**Example 11.** *Find the particular solution of the differential equation*

*$\log\left(\frac{dy}{dx}\right) = 3x + 4y$, given that $y = 0$ when $x = 0$.*

**Sol.** The given equation is

$$\log\left(\frac{dy}{dx}\right) = 3x + 4y$$

or $$\frac{dy}{dx} = e^{3x+4y} = e^{3x}\,.\,e^{4y}$$

or $$e^{-4y}\,dy = e^{3x}\,dx$$

Integrating both sides,

$$\frac{e^{-4y}}{-4} = \frac{e^{3x}}{3} + c \qquad ...(1)$$

Since $y = 0$ when $x = 0$, we have

$$-\frac{1}{4} = \frac{1}{3} + c \qquad \text{or} \qquad c = -\frac{7}{12}$$

Putting this particular value of $c$ in (1), the required particular solution is

$$-\frac{1}{4}e^{-4y} = \frac{1}{3}e^{3x} - \frac{7}{12} \qquad \text{or} \qquad 4e^{3x} + 3e^{-4y} = 7.$$

**Example 12.** *The line normal to a given curve at each point (x, y) on the curve passes through the point (2, 0). If the curve contains the point (2, 3), find its equation.*

**Sol.** The normal at $(x, y)$ passes through the point $(2, 0)$

$\therefore$ Slope of normal $= \dfrac{y-0}{x-2} = \dfrac{y}{x-2}$

Since the slope of tangent at $(x, y)$ is $\dfrac{dy}{dx}$

$\therefore$ Slope of normal $= -\dfrac{dx}{dy}$

Thus $$-\frac{dx}{dy} = \frac{y}{x-2} \qquad \text{or} \qquad (x-2)\,dx + y\,dy = 0 \qquad ...(1)$$

Integrating both sides

$$\frac{x^2}{2} - 2x + \frac{y^2}{2} = c' \qquad \text{or} \qquad x^2 + y^2 - 4x = c \quad \text{where } c = 2c'$$

This is the equation of the family of curves satisfying (1).

The particular member contains the point (2, 3).

$$\therefore \qquad 2^2 + 3^2 - 4 \times 2 = c \qquad \text{or} \qquad c = 5$$

Thus, the equation of the particular member is

$$x^2 + y^2 - 4x = 5 \qquad \text{or} \qquad (x-2)^2 + y^2 = 9.$$

**Example 13.** *Solve the differential equation $(1 + y^2)\,dx + (1 + x^2)\,dy = 0$ given that $y = -1$ when $x = 0$.*

**Sol.** The given equation is

$$(1 + y^2)\,dx + (1 + x^2)\,dy = 0$$

or $$\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0$$

Integrating both sides,

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}c$$

or $$\tan^{-1}\frac{x+y}{1-xy} = \tan^{-1}c$$

or $$\frac{x+y}{1-xy} = c \qquad \text{or} \qquad x + y = c(1 - xy) \qquad ...(1)$$

Since $y = -1$ when $x = 0$, we have

$$0 - 1 = c\,(1 - 0) \quad \text{or} \quad c = -1$$

$\therefore$ The required particular solution is

$$x + y = -(1 - xy) \quad \text{or} \quad x + y + 1 = xy.$$

**Example 14.** *Solve the differential equation*

$$\sqrt{1-x^2}\, dy + \sqrt{1-y^2}\, dx = 0.$$

*Also find the solution which passes through* $\left(\dfrac{\sqrt{3}}{2}, 0\right)$.

**Sol.** Please try yourself. $\left[\textbf{Ans.}\ \ x\sqrt{1-y^2} + y\sqrt{1-x^2} = \dfrac{\sqrt{3}}{2}\right]$

[**Hint.** See Example 6(*i*)]

**Example 15.** *Solve the following differential equations :*

(*i*) $\operatorname{cosec} x \log y\, dy + x^2 y^2\, dx = 0$

(*ii*) $\cos y \log(\sec x + \tan x)\, dx = \cos x \log(\sec y + \tan y)\, dy$

(*iii*) $ydx - xdy = xydx$ (*iv*) $x^2(y + 1)\, dx + y^2(x - 1)dy = 0$

(*v*) $e^{2x-3y}\, dx + e^{2y-3x}\, dy = 0.$

**Sol.** (*i*) The given equation is

$$\operatorname{cosec} x \log y dy + x^2y^2\, dx = 0$$

or $$\frac{\log y}{y^2}\, dy + x^2 \sin x\, dx = 0$$

Integrating both sides,

$$\int (\log y)\, y^{-2}\, dy + \int x^2 \sin x\, dx = c$$

or $$(\log y)\,.\,\frac{y^{-1}}{-1} - \int \frac{1}{y}\,.\,\frac{y^{-1}}{-1}\, dy + x^2(-\cos x) - \int 2x(-\cos x)\, dx = c$$

or $$-\frac{\log y}{y} + \int y^{-2}\, dy - x^2 \cos x + 2\int x \cos x\, dx = c$$

or $$-\frac{\log y}{y} + \frac{y^{-1}}{-1} - x^2 \cos x + 2\left[x \sin x - \int 1\,.\sin x\, dx\right] = c$$

or $$-\frac{\log y}{y} - \frac{1}{y} - x^2 \cos x + 2x \sin x + 2 \cos x = c$$

or $$-\frac{1}{y}(1 + \log y) + (2 - x^2)\cos x + 2x \sin x = c$$

which is the required solution.

(*ii*) The given equation is

$$\cos y \log(\sec x + \tan x)\, dx = \cos x \log(\sec y + \tan y)\, dy$$

or $$\sec x \log(\sec x + \tan x)\, dx = \sec y \log(\sec y + \tan y)\, dy$$

Integrating both sides

$$\frac{1}{2}[\log(\sec x + \tan x)]^2 = \frac{1}{2}[\log(\sec y + \tan y)]^2 + c'$$

$$\left[\because \int \sec x\,dx = \log(\sec x + \tan x)\right.$$

$$\therefore \frac{d}{dx}(\log(\sec x + \tan x)) = \sec x$$

$$\left.\text{and} \int f(x)\,f'(x)\,dx = \frac{1}{2}[f(x)]^2\right]$$

or $$[\log(\sec x + \tan x)]^2 = [\log(\sec y + \tan y)]^2 + c \qquad \text{where } c = 2c'$$

(*iii*) The given equation is

$$y(1-x)\,dx - x dy = 0$$

or $$\left(\frac{1-x}{x}\right)dx - \frac{dy}{y} = 0$$

Integrating both sides

$$\int\left(\frac{1}{x} - 1\right)dx - \int\frac{dy}{y} = c$$

or $$\log x - x - \log y = c$$

or $$\log\left(\frac{x}{y}\right) = x + c$$

(*iv*) The given equation is

$$\frac{x^2}{x-1}\,dx + \frac{y^2}{y+1}\,dy = 0$$

Integrating both sides

$$\int\frac{(x^2-1)+1}{x-1}\,dx + \int\frac{(y^2-1)+1}{y+1}\,dy = c$$

or $$\int\left(x + 1 + \frac{1}{x-1}\right)dx + \int\left(y - 1 + \frac{1}{y+1}\right)dy = c$$

or $$\frac{x^2}{2} + x + \log(x-1) + \frac{y^2}{2} - y + \log(y+1) = c$$

or $$\frac{1}{2}(x^2 + y^2) + (x - y) + \log[(x-1)(y+1)] = c$$

which is the required solution.

(*v*) The given equation is

$$\frac{e^{2x}}{e^{3y}}\,dx + \frac{e^{2y}}{e^{3x}}\,dy = 0 \qquad \text{or} \qquad e^{5x}\,dx + e^{5y}\,dy = 0$$

Integrating both sides

$$\frac{e^{5x}}{5} + \frac{e^{5y}}{5} = c' \qquad \text{or} \qquad e^{5x} + e^{5y} = c \qquad \text{where } c = 5c'.$$

**Example 16.** *Solve the following differential equations :*

(*i*) $\dfrac{dy}{dx} = e^{x+y} + x^2 e^{x^3} + y$ (*ii*) $(e^x + 1)\, y dy = (y + 1)\, e^x\, dx$

(*iii*) $(2x + e^y)\, dx + x\, e^y\, dy = 0$ (*iv*) $(3 + 2 \sin x + \cos x)\, dy = (1 + 2 \sin y + \cos y)\, dx$

(*v*) $\dfrac{xdx + ydy}{xdy - ydx} = \sqrt{\dfrac{a^2 - x^2 - y^2}{x^2 + y^2}}$ (I.A.S. 1999)

**Sol.** (*i*) The given equation is

$$\frac{dy}{dx} = (e^x + x^2 e^{x^3}) \,.\, e^y \qquad \text{or} \qquad e^{-y}\, dy = (e^x + x^2 e^{x^3})\, dx$$

Integrating both sides,

$$-e^{-y} = e^x + \int x^2 e^{x^3}\, dx + c$$

$$= e^x + \frac{1}{3} \int e^t\, dt + c, \text{ where } t = x^3$$

$$= e^x + \frac{1}{3} e^t + c \qquad \text{or} \qquad -e^{-y} = e^x + \frac{1}{3} e^{x^3} + c$$

which is the required solution.

(*ii*) The given equation is

$$\frac{y}{y+1}\, dy = \frac{e^x}{e^x + 1}\, dx$$

Integrating both sides,

$$\int \left(1 - \frac{1}{y+1}\right) dy = \log (e^x + 1) + c$$

or $$y - \log (y + 1) = \log (e^x + 1) + c.$$

(*iii*) The given equation is

$$2x\, dx + (e^y\, dx + x\, e^y\, dy) = 0 \qquad \text{or} \qquad d\,(x^2 + x\, e^y) = 0$$

Integrating $x^2 + xe^y = c$.

(*iv*) The given equation is

$$\frac{dy}{1 + 2 \sin y + \cos y} = \frac{dx}{3 + 2 \sin x + \cos x}$$

Integrating both sides,

$$\int \frac{dy}{(1 + \cos y) + 2 \sin y} = \int \frac{dx}{2 + (1 + \cos x) + 2 \sin x} + c$$

or $$\int \frac{dy}{2 \cos^2 \frac{y}{2} + 4 \sin \frac{y}{2} \cos \frac{y}{2}} = \int \frac{dx}{2 + 2 \cos^2 \frac{x}{2} + 4 \sin \frac{x}{2} \cos \frac{x}{2}} + c$$

or $$\int \frac{\sec^2 \frac{y}{2}}{2 + 4 \tan \frac{y}{2}}\, dy = \int \frac{\sec^2 \frac{x}{2}}{2 \sec^2 \frac{x}{2} + 2 + 4 \tan \frac{x}{2}}\, dx + c$$

or $$\int \frac{\frac{1}{2}\sec^2\frac{y}{2}}{1+2\sec^2\frac{y}{2}}dy = \int \frac{\frac{1}{2}\sec^2\frac{x}{2}}{1+\tan^2\frac{x}{2}+1+2\tan\frac{x}{2}}\,dx + c$$

or $$\int \frac{dt}{1+2t} = \int \frac{dz}{z^2+2z+2} + c$$

where $t = \tan\frac{y}{2}$ and $z = \tan\frac{x}{2}$

or $$\frac{1}{2}\log(1+2t) = \int \frac{dz}{(z+1)^2+1} + c = \tan^{-1}(z+1) + c$$

or $$\frac{1}{2}\log\left(1+2\tan\frac{y}{2}\right) = \tan^{-1}\left(\tan\frac{x}{2}+1\right) + c$$

which is the required solution.

($v$) The given equation is

$$\frac{xdx+ydy}{xdy-ydx} = \sqrt{\frac{a^2-x^2-y^2}{x^2+y^2}} \qquad ...(1)$$

Changing to polar co-ordinates by putting

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta$$

so that $$r = \sqrt{x^2+y^2},\ \theta = \tan^{-1}\frac{y}{x}$$

Also $$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$$

$$\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$$

Now equation (1) can be written as

$$\frac{x\frac{dx}{d\theta}+y\frac{dy}{d\theta}}{x\frac{dy}{d\theta}-y\frac{dx}{d\theta}} = \sqrt{\frac{a^2-(x^2+y^2)}{x^2+y^2}}$$

or $$\frac{r\cos\theta\left(\frac{dr}{d\theta}\cos\theta-r\sin\theta\right)+r\sin\theta\left(\frac{dr}{d\theta}\sin\theta+r\cos\theta\right)}{r\cos\theta\left(\frac{dr}{d\theta}\sin\theta+r\cos\theta\right)-r\sin\theta\left(\frac{dr}{d\theta}\cos\theta-r\sin\theta\right)} = \sqrt{\frac{a^2-r^2}{r^2}}$$

or $$\frac{r\frac{dr}{d\theta}(\cos^2\theta+\sin^2\theta)}{r^2(\cos^2\theta+\sin^2\theta)} = \frac{\sqrt{a^2-r^2}}{r}$$

or $$\frac{dr}{d\theta} = \sqrt{a^2-r^2} \quad \text{or} \quad \frac{dr}{\sqrt{a^2-r^2}} = d\theta$$

Integrating both sides, $\sin^{-1}\frac{r}{a} = \theta + c$

$\therefore \qquad r = a \sin(\theta + c)$

where $\qquad r = \sqrt{x^2 + y^2} \qquad$ and $\qquad \theta = \tan^{-1} \dfrac{y}{x}$.

**Equations reducible to the form in which variables can be separated.**

Equations of the form $\dfrac{dy}{dx} = f(ax + by + c)$ can be reduced to the form in which the variables are separable.

Put $ax + by + c = z$, we have

$$a + b \cdot \frac{dy}{dx} = \frac{dz}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{b}\left(\frac{dz}{dx} - a\right)$$

$\therefore$ Given equation becomes $\dfrac{1}{b}\left(\dfrac{dz}{dx} - a\right) = f(z)$

or
$$\frac{dz}{dx} - a = bf(z) \quad \text{or} \quad \frac{dz}{dx} = a + bf(z)$$

Separating the variables $\dfrac{dz}{a + bf(z)} = dx$

which can now be integrated.

**Example 1.** *Solve the following differential equations :*

(*i*) $(x + y)^2 \dfrac{dy}{dx} = a^2$ (Indore, 1998 ; Meerut, 1997)

(*ii*) $(x - y)^2 \dfrac{dy}{dx} = a^2$ (Delhi, 1999) (*iii*) $\dfrac{dy}{dx} = (4x + y + 1)^2$

(*iv*) $\dfrac{dy}{dx} = (x + y)^2$.

**Sol.** (*i*) $(x + y)^2 \dfrac{dy}{dx} = a^2$

Put $x + y = z$, then $1 + \dfrac{dy}{dx} = \dfrac{dz}{dx}$ or $\dfrac{dy}{dx} = \dfrac{dz}{dx} - 1$

$\therefore$ Given equation becomes $z^2\left(\dfrac{dz}{dx} - 1\right) = a^2$

or
$$\frac{dz}{dx} - 1 = \frac{a^2}{z^2} \quad \text{or} \quad \frac{dz}{dx} = \frac{z^2 + a^2}{z^2}$$

Separating the variables, $\dfrac{z^2}{z^2 + a^2}\, dz = dx$

or
$$\left[1 - \frac{a^2}{z^2 + a^2}\right] dz = dx$$

Integrating both sides, $z - a^2 \cdot \dfrac{1}{a} \tan^{-1} \dfrac{z}{a} = x + c$

or $$x + y - a \tan^{-1} \frac{x+y}{a} = x + c$$

or $$y = a \tan^{-1} \frac{x+y}{a} + c \text{ is the required solution.}$$

(*ii*) Put $x - y = z$, then $1 - \frac{dy}{dx} = \frac{dz}{dx}$ or $\frac{dy}{dx} = 1 - \frac{dz}{dx}$

$\therefore$ Given equation becomes $z^2 \left(1 - \frac{dz}{dy}\right) = a^2$

or $$1 - \frac{dz}{dx} = \frac{a^2}{z^2} \quad \text{or} \quad \frac{dz}{dx} = 1 - \frac{a^2}{z^2} = \frac{z^2 - a^2}{z^2}$$

Separating the variables,

$$\frac{z^2}{z^2 - a^2}\, dz = dx \quad \text{or} \quad \left(1 + \frac{a^2}{z^2 - a^2}\right) dz = dx$$

Integrating both sides, $z + a^2 \frac{1}{2a} \log \frac{z-a}{z+a} = x + c$

or $$x - y + \frac{a}{2} \log \frac{x-y-a}{x-y+a} = x + c$$

or $$\frac{a}{2} \log \frac{x-y-a}{x-y+a} = y + c \text{ is the reqd. solution.}$$

(*iii*) Put $4x + y + 1 = z$, then

$$4 + \frac{dy}{dx} = \frac{dz}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{dz}{dx} - 4$$

$\therefore$ Given equation becomes

$$\frac{dz}{dx} - 4 = z^2 \quad \text{or} \quad \frac{dz}{dx} = z^2 + 4$$

Separating the variables, $\frac{dz}{z^2 + 4} = dx$

Integrating both sides $\frac{1}{2} \tan^{-1} \frac{z}{2} = x + c$

or $$\frac{1}{2} \tan^{-1} \frac{4x+y+1}{2} = x + c \text{ is the required solution.}$$

(*iv*) Put $x + y = z$, then $1 + \frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ Given equation becomes

$$\frac{dz}{dx} - 1 = z^2 \quad \text{or} \quad \frac{dz}{dx} = 1 + z^2$$

Separating the variables, $\frac{dz}{z^2 + 1} = dx$

Integrating both sides, $\tan^{-1} z = x + c$

or $\quad \tan^{-1}(x + y) = x + c \quad$ or $\quad x + y = \tan(x + c)$

is the required solution.

**Example 2.** *Solve (i)* $(x + y + 1)\dfrac{dy}{dx} = 1$

*(ii)* $(x + y + 1)^2 \dfrac{dy}{dx} = 1$ *(iii)* $\sin^{-1}\left(\dfrac{dy}{dx}\right) = x + y.$ (Meerut, 1996)

**Sol.** (*i*) Put $x + y + 1 = z$, then $1 + \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ Given equation becomes, $z\left(\dfrac{dz}{dx} - 1\right) = 1$

or $$\frac{dz}{dx} - 1 = \frac{1}{z} \quad \text{or} \quad \frac{dz}{dx} = 1 + \frac{1}{z} = \frac{z+1}{z}$$

Separating the variables,

$$\frac{z}{z+1}\, dz = dx \quad \text{or} \quad \left(1 - \frac{1}{z+1}\right) dz = dx$$

Integrating both sides, $\quad z - \log(z + 1) = x + c_1$

or $$x + y + 1 - \log(x + y + 1 + 1) = x + c_1$$

or $$y - \log(x + y + 2) = c_1 - 1$$

or $y - \log(x + y + 2) = c$ is the required solution.

(*ii*) Proceeding as in part (*i*), we have

$$z^2\left(\frac{dz}{dx} - 1\right) = 1 \quad \text{or} \quad z^2 \frac{dz}{dx} = z^2 + 1$$

or $$\frac{z^2}{z^2+1}\, dz = dx \quad \text{or} \quad \left(1 - \frac{1}{z^2+1}\right) dz = dx$$

Integrating, we have $\quad z - \tan^{-1} z = x + c$

or $$x + y + 1 = \tan^{-1}(x + y + 1) + x + c$$

or $y = \tan^{-1}(x + y + 1) + C.\quad$ where $\quad C = c - 1.$

(*iii*) $$\sin^{-1}\left(\frac{dy}{dx}\right) = x + y \quad \Rightarrow \quad \frac{dy}{dx} = \sin(x + y)$$

Put $x + y = z$, then $1 + \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ Given equation becomes $\dfrac{dz}{dx} - 1 = \sin z$

or $$\frac{dz}{dx} = 1 + \sin z.$$

Separating the variables, $\quad \dfrac{dz}{1 + \sin z} = dx$

Integrating both sides, $\quad \displaystyle\int \frac{dz}{1 + \sin z} = x + c$

or $$\int \frac{1-\sin z}{1-\sin^2 z}\, dz = x + c \qquad \text{or} \qquad \int \frac{1-\sin z}{\cos^2 z}\, dz = x + c$$

or $$\int (\sec^2 z - \sec z \tan z)\, dz = x + c$$

or $$\tan z - \sec z = x + c$$

or $\tan (x + y) - \sec (x + y) = x + c$ is the reqd. solution.

**Example 3.** *Solve the following* :

(*i*) $\dfrac{dy}{dx} = \sin (x + y) + \cos (x + y)$ (*ii*) $\dfrac{dy}{dx} = \dfrac{2}{x + 2y - 3}$.

**Sol.** (*i*) Put $x + y = z$, then $1 + \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ Given equation becomes,

$$\frac{dz}{dx} - 1 = \sin z + \cos z \qquad \text{or} \qquad \frac{dz}{dx} = 1 + \cos z + \sin z$$

Separating the variables,

$$\frac{dz}{1+\cos z+\sin z} = dx$$

or $$\frac{dz}{2\cos^2 \frac{z}{2} + 2 \sin \frac{z}{2} \cos \frac{z}{2}} = dx$$

or $$\frac{\sec^2 \frac{z}{2}\, dz}{2\left(1+\tan \frac{z}{2}\right)} = dx$$ | Dividing the num. and denom. by $\cos^2 \dfrac{z}{2}$

Integrating both sides, $$\int \frac{\frac{1}{2}\sec^2 \frac{z}{2}\, dz}{1+\tan \frac{z}{2}} = x + c$$

or $$\log \left(1+\tan \frac{z}{2}\right) = x + c$$

or $\log \left[1+\tan \dfrac{x+y}{2}\right] = x + c$ is the required solution.

(*ii*) Put $x + 2y - 3 = z$, then

$$1 + 2 \cdot \frac{dy}{dx} = \frac{dz}{dx} \qquad \therefore \quad \frac{dy}{dx} = \frac{1}{2}\left(\frac{dz}{dx} - 1\right)$$

$\therefore$ Given equation becomes $\dfrac{1}{2}\left(\dfrac{dz}{dx} - 1\right) = \dfrac{2}{z}$

or $$\frac{dz}{dx} - 1 = \frac{4}{z} \qquad \text{or} \qquad \frac{dz}{dx} = 1 + \frac{4}{z} = \frac{z+4}{z}$$

Separating the variables,

$$\frac{z}{z+4}\, dz = dx \qquad \text{or} \qquad \left(1 - \frac{4}{z+4}\right) dz = dx$$

Integrating both sides, $z - 4 \log (z + 4) = x + c_1$

or $$x + 2y - 3 - 4 \log (x + 2y - 3 + 4) = x + c_1$$

or $$2y - 4 \log (x + 2y + 1) = c_1 + 3$$

or $$2y - 4 \log (x + 2y + 1) = 2c$$

or $$y - 2 \log (x + 2y + 1) = c$$ is the required solution.

**Example 4.** *Solve the following equations :*

(i) $\dfrac{dy}{dx} = \cos (x + y)$ (ii) $\cos (x + y)\, dy = dx$

(iii) $\dfrac{dy}{dx} = \dfrac{1}{x - y} + 1$ (iv) $\dfrac{dy}{dx} = \dfrac{x + y + 1}{x + y}$

(v) $(x + 2y)(dx - dy) = dx + dy$.

**Sol.** (i) Put $x + y = z$, then $1 + \dfrac{dy}{dx} = \dfrac{dz}{dx}$

or $$\frac{dy}{dx} = \frac{dz}{dx} - 1$$

Given equation becomes

$$\frac{dz}{dx} - 1 = \cos z$$

or $$\frac{dz}{dx} = 1 + \cos z = 2 \cos^2 \frac{z}{2}$$

or $$\frac{1}{2} \sec^2 \frac{z}{2}\, dz = dx$$

Integrating both sides,

$$\frac{1}{2} \cdot \frac{\tan \frac{z}{2}}{\frac{1}{2}} = x + c \qquad \text{or} \qquad \tan \frac{x + y}{2} = x + c$$

is the required solution.

(ii) The given equation is

$$\frac{dx}{dy} = \cos (x + y)$$

Put $$x + y = z$$

Differentiating w.r.t. $y$

$$\frac{dx}{dy} + 1 = \frac{dz}{dx} \qquad \text{or} \qquad \frac{dx}{dy} = \frac{dz}{dy} - 1$$

$\therefore$ Given equation becomes

$$\frac{dz}{dy} - 1 = \cos z \qquad \text{or} \qquad \frac{dz}{dy} = 1 + \cos z = 2 \cos^2 \frac{z}{2}$$

or $$\frac{1}{2} \sec^2 \frac{z}{2}\, dz = dy$$

Integrating both sides,

$$\frac{1}{2} \cdot \frac{\tan \frac{z}{2}}{\frac{1}{2}} = y + c$$

or $$\tan \frac{x+y}{2} = y + c \text{ is the required solution.}$$

(*iii*) Put $x - y = z$, then $1 - \frac{dy}{dx} = \frac{dz}{dx}$ or $\frac{dy}{dx} = 1 - \frac{dz}{dx}$

$\therefore$ Given equation becomes

$$1 - \frac{dz}{dx} = \frac{1}{z} + 1 \qquad \text{or} \quad -z dz = dx$$

Integrating both sides,

$$-\frac{z^2}{2} = x + c_1$$

or $$2x + (x - y)^2 + 2c_1 = 0 \qquad \text{or} \qquad 2x + (x - y)^2 = c$$

is the required solution.

(*iv*) Put $$x + y = z$$

then $$1 + \frac{dy}{dx} = \frac{dz}{dx} \qquad \text{or} \qquad \frac{dy}{dx} = \frac{dz}{dx} - 1$$

$\therefore$ Given equation becomes

$$\frac{dz}{dx} - 1 = \frac{z+1}{z} \qquad \text{or} \qquad \frac{dz}{dx} = \frac{1+2z}{z}$$

or $$\frac{z}{1+2z}\, dz = dx \qquad \text{or} \qquad \frac{1}{2}\left(1 - \frac{1}{1+2z}\right) dz = dx$$

Integrating both sides,

$$\frac{1}{2}\,[z - \tfrac{1}{2}\log(1 + 2z)] = x + c_1$$

or $$\frac{z}{2} - \frac{1}{4}\log(1 + 2z) = x + c_1$$

or $$2z - \log(1 + 2z) = 4x + 4c_1$$

or $$2(x + y) - \log(1 + 2x + 2y) = 4x + c$$

or $$2(y - x) = \log(2x + 2y + 1) + c$$

is the required solution.

(*v*) $$(x + 2y)(dx - dy) = dx + dy$$

or $$(x + 2y - 1)\, dx = (x + 2y + 1)\, dy$$

or $$\frac{dy}{dx} = \frac{x + 2y - 1}{x + 2y + 1} \qquad \text{...(1)}$$

Put $x + 2y = z$, then $1 + 2\frac{dy}{dx} = \frac{dz}{dx}$

or $$\frac{dy}{dx} = \frac{1}{2}\left(\frac{dz}{dx} - 1\right)$$

$\therefore$ (1) becomes $\quad \frac{1}{2}\left(\frac{dz}{dx}-1\right)=\frac{z-1}{z+1}$

or $$\frac{dz}{dx}-1=\frac{2z-2}{z+1} \quad \text{or} \quad \frac{dz}{dx}=\frac{3z-1}{z+1}$$

or $$\frac{z+1}{3z-1}\,dz=dx \quad \text{or} \quad \frac{1}{3}\left(1+\frac{4}{3z-1}\right)dz=dx$$

Integrating both sides, $\frac{1}{3}\left[z+\frac{4}{3}\log(3z-1)\right]=x+c_1$

or $$3z+4\log(3z-1)=9x+9c_1$$

or $$3(x+2y)+4\log(3x+6y-1)=9x+9c_1$$

or $$6(y-x)+4\log(3x+6y-1)=9c_1$$

or $$3(y-x)+2\log(3x+6y-1)=\frac{9}{2}c_1=c$$

which is the required solution.

**Example 5.** *Solve the following differential equations :*

*(i)* $\frac{dy}{dx}+1=e^{x+y}$ $\qquad$ *(ii)* $\left(\frac{x+y-a}{x+y-b}\right)\frac{dy}{dx}=\left(\frac{x+y+a}{x+y+b}\right)$

**Sol.** *(i)* The given equation is

$$\frac{dy}{dx}+1=e^{x+y} \qquad \text{...(1)}$$

Put $x+y=z$ so that $\quad 1+\frac{dy}{dx}=\frac{dz}{dx}$

$\therefore$ Equation (1) becomes $\quad \frac{dz}{dx}=e^z \quad$ or $\quad e^{-z}\,dz=dx$

Integrating $-e^{-z}=x+c$

or $$-1=(x+c)e^z \quad \text{or} \quad 1+(x+c)e^{x+y}=0$$

*(ii)* Put $x+y=z$ so that $1+\frac{dy}{dx}=\frac{dz}{dx}$

or $$\frac{dy}{dx}=\frac{dz}{dx}-1$$

$\therefore$ The given equation becomes

$$\left(\frac{z-a}{z-b}\right)\left(\frac{dz}{dx}-1\right)=\frac{z+a}{z+b}$$

or $$\frac{dz}{dx}-1=\frac{(z+a)(z-b)}{(z+b)(z-a)}=\frac{z^2+az-bz-ab}{z^2-az+bz-ab}$$

or $$\frac{dz}{dx}=\frac{2(z^2-ab)}{z^2+(b-a)z-ab}$$

or $$\frac{z^2+(b-a)z-ab}{z^2-ab}\,dz=2dx$$

or $$\left[1+\frac{(b-a)\,z}{z^2-ab}\right]dz = 2dx$$

Integrating both sides,

$$z+\frac{b-a}{2}\int\frac{2a}{z^2-ab}\,dz = 2x+c$$

or $$z+\tfrac{1}{2}(b-a)\log(z^2-ab) = 2x+c$$

or $$x+y+\tfrac{1}{2}(b-a)\log[(x+y)^2-ab] = 2x+c$$

or $$\tfrac{1}{2}(b-a)\log[(x+y)^2-ab] = x-y+c$$

which is the required solution.

**Example 6.** *Solve the following differential equations :*

(*i*) $x^4 \cdot \dfrac{dy}{dx} + x^3y + \text{cosec}\,(xy) = 0$ (*ii*) $\dfrac{y}{x}\cdot\dfrac{dy}{dx}+\dfrac{2(x^2+y^2)-1}{x^2+y^2+1} = 0.$

**Sol.** (*i*) The given equation can be written as

$$x^3\left(x\frac{dy}{dx}+y\right)+\text{cosec}\,(xy) = 0$$

Putting $xy = t$, we get $x\dfrac{dy}{dx}+y = \dfrac{dt}{dx}$

$\therefore$ The given equation becomes

$$x^3\frac{dt}{dx}+\text{cosec}\,t = 0 \qquad \text{or} \qquad \sin t\,dt+\frac{dx}{x^3} = 0$$

Integrating, we have $-\cos t+\dfrac{x^{-2}}{-2} = c$

or $$\cos(xy)+\frac{1}{2x^2} = C \qquad \text{where } C = -c$$

(*ii*) The given equation is

$$\frac{y}{x}\cdot\frac{dy}{dx}+\frac{2(x^2+y^2)-1}{x^2+y^2+1} = 0$$

Putting $x^2+y^2 = t$, we get $2x+2y\dfrac{dy}{dx} = \dfrac{dt}{dx}$

or $$1+\frac{y}{x}\cdot\frac{dy}{dx} = \frac{1}{2x}\cdot\frac{dt}{dx} \qquad \text{or} \qquad \frac{y}{x}\cdot\frac{dy}{dx} = \frac{1}{2x}\cdot\frac{dt}{dx}-1$$

$\therefore$ The given equation becomes

$$\frac{1}{2x}\cdot\frac{dt}{dx}-1+\frac{2t-1}{t+1} = 0$$

$$\Rightarrow \quad \frac{1}{2x}\cdot\frac{dt}{dx}+\frac{t-2}{t+1} = 0 \quad \Rightarrow \quad \frac{t+1}{t-2}\,dt+2x\,dx = 0$$

$$\Rightarrow \quad \left(1+\frac{3}{t-2}\right)dt+2x\,dx = 0$$

Integrating, we have $$t + 3 \log (t-2) + x^2 = c$$

or $$x^2 + y^2 + 3 \log (x^2 + y^2 - 2) + x^2 = c$$

or $$2x^2 + y^2 + 3 \log (x^2 + y^2 - 2) = c.$$

## TYPE II. HOMOGENEOUS EQUATIONS

**Homogeneous Function. Definition.** A function is said to be homogeneous of the $n$th degree in $x$ and $y$ if it can be put in the form

$$x^n f\left(\frac{y}{x}\right).$$

**Homogeneous Equation. Definition.** A differential equation in $x$ and $y$ is said to be homogeneous if it can be put in the form $\dfrac{dy}{dx} = \dfrac{f_1(x, y)}{f_2(x, y)}$, where $f_1(x, y)$ and $f_2(x, y)$ are homogeneous functions of the same degree in $x, y$.

**Art. To solve the equation $\dfrac{dy}{dx} = \dfrac{f_1(x, y)}{f_2(x, y)}$, where $f_1(x, y), f_2(x, y)$ are homogeneous functions of the same degree in x and y.**

Given equation is $$\frac{dy}{dx} = \frac{f_1(x, y)}{f_2(x, y)} \qquad \text{...}(i)$$

$\because$ $f_1(x, y), f_2(x, y)$ are homogeneous functions of the same degree in $x$ and $y$.

Let $$f_1(x, y) = x^n \phi_1\left(\frac{y}{x}\right); f_2(x, y) = x^n \phi_2\left(\frac{y}{x}\right)$$

$\therefore$ From $(i)$, $$\frac{dy}{dx} = \frac{x^n \phi_1\left(\frac{y}{x}\right)}{x^n \phi_2\left(\frac{y}{x}\right)} = \frac{\phi_1\left(\frac{y}{x}\right)}{\phi_2\left(\frac{y}{x}\right)} \quad \text{or} \quad \frac{dy}{dx} = f\left(\frac{y}{x}\right) (say) \qquad \text{...}(ii)$$

Put $y = vx$, then $\dfrac{dy}{dx} = v + x \cdot \dfrac{dv}{dx}$

$\therefore$ From $(ii)$, $$v + x \cdot \frac{dv}{dx} = f(v) \quad \text{or} \quad x\frac{dv}{dx} = f(v) - v$$

Separating the variables, $$\frac{dv}{f(v) - v} = \frac{dx}{x}$$

which can be easily integrated. In the solution, putting $v = \dfrac{y}{x}$, we get the required solution.

**Method** : 1. Put $y = vx$, then $\dfrac{dy}{dx} = v + x \cdot \dfrac{dv}{dx}$

2. Put the above values of $y$ and $\dfrac{dy}{dx}$ in the given equation.

3. Separate the variables and integrate.

4. Replace $v$ by $\dfrac{y}{x}$ to get the required solution.

**Example 1.** *Solve the following* :

(*i*) $(x^2 - y^2)\,dx + 2xy\,dy = 0$ (*ii*) $(x^2 + y^2)\,dx - 2xy\,dy = 0$

(*iii*) $(x^2 + y^2)\,dx + 2xy\,dy = 0$ (*iv*) $(x^2 + y^2)\dfrac{dy}{dx} = xy.$ (Kerala, 2001)

**Sol.** (*i*) Given equation is

$$(x^2 - y^2)\,dx + 2xy\,dy = 0$$

or $$2xy\,dy = -(x^2 - y^2)\,dx$$

or $$\frac{dy}{dx} = -\frac{x^2 - y^2}{2xy} \qquad ...(i)$$

Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ From (*i*), $$v + x\frac{dv}{dx} = -\frac{x^2 - v^2x^2}{2vx^2} = -\frac{1-v^2}{2v}$$

or $$x\frac{dv}{dx} = -\frac{1-v^2}{2v} - v = -\frac{1-v^2+2v^2}{2v} = -\frac{1+v^2}{2v}$$

Separating the variables, $\dfrac{2v}{1+v^2}\,dv = -\dfrac{1}{x}\,dx$

Integrating both sides, $\log(1 + v^2) = -\log x + \log c$

or $$\log(1 + v^2) = \log\frac{c}{x} \quad \text{or} \quad (1 + v^2) = \frac{c}{x}$$

or $$1 + \frac{y^2}{x^2} = \frac{c}{x} \quad \text{or} \quad x^2 + y^2 = cx$$

which is the required solution.

(*ii*) Given equation is $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2xy}$ ...(*i*)

Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ From (*i*), $$v + x\frac{dv}{dx} = \frac{x^2 + v^2x^2}{2vx^2} = \frac{1+v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{1+v^2}{2v} - v = \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$$

Separating the variables,

$$\frac{2v}{1-v^2}\,.\,dv = \frac{dx}{x} \quad \text{or} \quad -\frac{-2v}{1-v^2}\,dv = \frac{dx}{x}$$

Integrating both sides, $-\log(1 - v^2) = \log x + \log c$

or $$\log(1 - v^2)^{-1} = \log cx \qquad \therefore \quad (1 - v^2)^{-1} = cx$$

or $$\frac{1}{1-v^2} = cx \quad \text{or} \quad \frac{1}{1-\dfrac{y^2}{x^2}} = cx$$

or $$\frac{x^2}{x^2 - y^2} = cx \qquad \text{or} \qquad x = c(x^2 - y^2)$$

is the required solution.

(*iii*) Given equation is $$\frac{dy}{dx} = -\frac{x^2 + y^2}{2xy} \qquad ...(i)$$

Put $y = vx$, then $$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ From (*i*), $$v + x.\frac{dv}{dx} = -\frac{x^2 + v^2x^2}{2vx^2} = -\frac{1+v^2}{2v}$$

$$x\frac{dv}{dx} = -\frac{1+v^2}{2v} - v = -\frac{1+v^2+2v^2}{2v} = -\frac{1+3v^2}{2v}$$

Separating the variables,

$$\frac{2v}{1+3v^2}\,dv = -\frac{dx}{x} \qquad \text{or} \qquad \frac{1}{3}\cdot\frac{6v}{1+3v^2}\,dv = -\frac{dx}{x}$$

Integrating both sides, $\frac{1}{3}\log(1 + 3v^2) = -\log x + c_1$

or $$\log(1 + 3v^2) = -3\log x + 3c_1$$

or $$\log\left(1 + \frac{3y^2}{x^2}\right) = \log x^{-3} + \log c \qquad \Big|\text{taking } 3c_1 = \log c$$

or $$\log\frac{x^2 + 3y^2}{x^2} = \log cx^{-3} \qquad \text{or} \qquad \frac{x^2+3y^2}{x^2} = \frac{c}{x^3}$$

or $x(x^2 + 3y^2) = c$ is the required solution.

(*iv*) Given equation is $$\frac{dy}{dx} = \frac{xy}{x^2 + y^2} \qquad ...(i)$$

Put $y = vx$ then $$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ From (*i*), $$v + x.\frac{dv}{dx} = \frac{vx^2}{v^2 + v^2x^2} = \frac{v}{1+v^2}$$

$$x\frac{dv}{dx} = \frac{v}{1+v^2} - v = \frac{v - v - v^3}{1+v^2} = -\frac{v^3}{1+v^2}$$

Separating the variables,

$$\frac{1+v^2}{v^3}\,dv = -\frac{dx}{x} \qquad \text{or} \qquad \left(v^3 + \frac{1}{v}\right)dv = -\frac{1}{x}\,dx$$

Integrating both sides, $$\frac{v^{-2}}{-2} + \log v = -\log x + c$$

or $$-\frac{1}{2v^2} + \log v = -\log x + c$$

or $$-\frac{x^2}{2y^2} + \log\frac{y}{x} = -\log x + c$$

or $$-\frac{x^2}{2y^2} + \log y - \log x = -\log x + c$$

or $$-\frac{x^2}{2y^2} + \log y = c \text{ is the required solution.}$$

**Example 2.** *Solve the following :*

(*i*) $x\frac{dy}{dx} + \frac{y^2}{x} = y$ (Delhi, 1997) (*ii*) $x^2y\,dx - (x^3 + y^3)dy = 0$

(*iii*) $x\frac{dy}{dx} - y = \sqrt{x^2 + y^2}$ (*iv*) $y^2 + x^2\frac{dy}{dx} = xy\frac{dy}{dx}$.

**Sol.** (*i*) Given equation is $x\frac{dy}{dx} = y - \frac{y^2}{x} = \frac{xy - y^2}{x}$ or $\frac{dy}{dx} = \frac{xy - y^2}{x^2}$ ...(*i*)

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ From (*i*), $v + x.\frac{dv}{dx} = \frac{vx^2 - v^2x^2}{x^2} = v - v^2$

or $$x\frac{dv}{dx} = -v^2$$

Separating the variables, $\frac{dv}{v^2} = -\frac{dx}{x}$

Integrating both sides,

$$-\frac{1}{v} = -\log x + c \qquad \text{or} \qquad -\frac{x}{y} + \log x = c$$

is the required solution.

(*ii*) Given equation is $\frac{dy}{dx} = \frac{x^2y}{x^3 + y^3}$ ...(*i*)

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ From (*i*), $v + x\frac{dv}{dx} = \frac{vx^3}{x^3 + v^3x^3} = \frac{v}{1+v^3}$

$$x\frac{dv}{dx} = \frac{v}{1+v^3} - v = \frac{-v^4}{1+v^3}$$

Separating the variables,

$$\frac{1+v^3}{v^4}dv = -\frac{dx}{x} \qquad \text{or} \qquad \left(v^{-4} + \frac{1}{v}\right)dv = -\frac{dx}{x}$$

Integrating both sides, $-\frac{1}{3v^3} + \log v = -\log x + c$

or $$-\frac{x^3}{3y^3} + \log\frac{y}{x} = -\log x + c$$

or $$-\frac{x^3}{3y^3} + \log y - \log x = -\log x + c$$

or $$-\frac{x^3}{3y^3} + \log y = c \text{ is the required solution.}$$

(*iii*) Given equation is $x\frac{dy}{dx} = y + \sqrt{x^2+y^2}$

or $$\frac{dy}{dx} = \frac{y+\sqrt{x^2+y^2}}{x} \qquad ...(i)$$

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ From (*i*), $$v + x\frac{dv}{dx} = \frac{vx + \sqrt{x^2+v^2x^2}}{x} = v + \sqrt{1+v^2}$$

or $$x\frac{dv}{dx} = \sqrt{1+v^2}$$

Separating the variables $\frac{dv}{\sqrt{1+v^2}} = \frac{dx}{x}$

Integrating both sides, $\sinh^{-1} v = \log x + \log c$

or $$\log(v + \sqrt{1+v^2}) = \log cx \qquad \therefore \quad v + \sqrt{1+v^2} = cx$$

or $$\frac{y}{x} + \sqrt{1+\frac{y^2}{x^2}} = cx \qquad \text{or} \qquad y + \sqrt{x^2+y^2} = cx^2$$

is the required solution.

**Another Form of the Above Question**

*Solve* $xdy - ydx = \sqrt{x^2+y^2}\, dx$ (Delhi, 1999)

(*iv*) Given equation is $y^2 = (xy - x^2)\frac{dy}{dx}$ or $\frac{dy}{dx} = \frac{y^2}{xy - x^2}$ ...(*i*)

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ From (*i*), $v + x\frac{dv}{dx} = \frac{v^2x^2}{vx^2 - x^2} = \frac{v^2}{v-1}$

or $$x\frac{dv}{dx} = \frac{v^2}{v-1} - v = \frac{v^2 - v^2 + v}{v-1} = \frac{v}{v-1}$$

Separating the variables,

$$\frac{v-1}{v}\, dv = \frac{dx}{x} \qquad \text{or} \qquad \left(1 - \frac{1}{v}\right) dv = \frac{1}{x}\, dx$$

Integrating both sides, $v - \log v = \log x + c$

or $$\frac{y}{x} - \log \frac{y}{x} = \log x + c$$

or $$\frac{y}{x} - (\log y - \log x) = \log x + c$$

or $$\frac{y}{x} = \log y + c \text{ is the required solution.}$$

**Example 3.** *Solve the following :*

(*i*) $x^2\, dy + y\,(x + y)dx = 0$ (*ii*) $\dfrac{dy}{dx} = \dfrac{y}{x} + \sin \dfrac{y}{x}$

(*iii*) $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan \dfrac{y}{x}$ (*iv*) $x \dfrac{dy}{dx} = y(\log y - \log x + 1)$

(*v*) $x \dfrac{dy}{dx} = y - x \cos^2 \left(\dfrac{y}{x}\right)$.

**Sol.** (*i*) Given equation is $$\frac{dy}{dx} = -\frac{y(x+y)}{x^2} \qquad ...(i)$$

Put $y = vx$, then $$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ From (*i*), $$v + x\frac{dv}{dx} = -\frac{vx(x+vx)}{x^2} = -v - v^2$$

or $$x\frac{dv}{dx} = -(2v + v^2)$$

Separating the variables,

$$\frac{dv}{v^2 + 2v} = -\frac{dx}{x} \qquad \text{or} \qquad \frac{1}{v(v+2)}\, dv = -\frac{dx}{x}$$

or $$\frac{1}{2}\left[\frac{1}{v} - \frac{1}{v+2}\right] dv = -\frac{dx}{x}$$ | Partial fractions

Integrating both sides, $\frac{1}{2}[\log v - \log (v + 2)] = -\log x + c_1$

or $$\log \frac{v}{v+2} = -2\log x + 2c_1$$

or $$\log \frac{v}{v+2} = -\log x^2 + \log c$$

or $$\log \frac{y}{y+2x} = \log \frac{c}{x^2} \qquad \text{or} \qquad \frac{y}{2x+y} = \frac{c}{x^2}$$

or $$x^2y = c\,(2x + y) \text{ is the required solution.}$$

(*ii*) Given equation is $$\frac{dy}{dx} = \frac{y}{x} + \sin\frac{y}{x} \qquad ...(i)$$

Put $y = vx$, then $$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$\therefore$ From (*i*), $$v + x.\frac{dv}{dx} = v + \sin v$$

or $$x\frac{dv}{dx} = \sin v$$

Separating the variables $\operatorname{cosec} v\,dv = \frac{1}{x}\,dx$

Integrating both sides, $\log \tan \frac{v}{2} = \log x + \log c$

or $$\log \tan \frac{y}{2x} = \log cx \quad \text{or} \quad \tan \frac{y}{2x} = cx$$

which is the required solution.

(*iii*) Please try yourself. $\left[\textbf{Ans.}\ \sin \frac{y}{x} = cx\right]$

(*iv*) Given equation is $x\frac{dy}{dx} = y(\log y - \log x + 1)$

or $$\frac{dy}{dx} = \frac{y}{x}\left(\log \frac{y}{x} + 1\right) \qquad ...(i)$$

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ From (*i*), $v + x\frac{dv}{dx} = v(\log v + 1) = v \log v + v$

or $$x\frac{dv}{dx} = v \log v$$

Separating the variables, $\frac{1}{v \log v}\,dv = \frac{dx}{x}$

or $$\frac{\frac{1}{v}}{\log v}\,dv = \frac{1}{x}\,dx$$

Integrating both sides, $\log(\log v) = \log x + \log c = \log cx$

or $\log v = cx$ or $\log \frac{y}{x} = cx$ is the reqd. solution.

(*v*) Given equation is $x\frac{dy}{dx} = y - x\cos^2\left(\frac{y}{x}\right)$

or $$\frac{dy}{dx} = \frac{y}{x} - \cos^2\left(\frac{y}{x}\right) \qquad ...(i)$$

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ From (*i*), $v + x.\frac{dv}{dx} = v - \cos^2 v$ or $x\frac{dv}{dx} = -\cos^2 v$

Separating the variables, $\sec^2 v\,dv = -\frac{dx}{x}$

Integrating both sides, $\tan v = -\log x + c$

or $\tan\left(\frac{y}{x}\right) + \log x = c$ is the required solution.

**Example 4.** *Solve the following :*

(*i*) $\dfrac{dy}{dx} = \dfrac{3xy + y^2}{3x^2}$ (*ii*) $(x + y)\dfrac{dy}{dx} + (x - y) = 0$

(*iii*) $y^2 dx + (xy + x^2)\, dy = 0.$

**Sol.** (*i*) Given equation is $\dfrac{dy}{dx} = \dfrac{3xy + y^2}{3x^2}$ ...(*i*)

Put $y = vx$, then $\dfrac{dy}{dx} = v + x.\dfrac{dv}{dx}$

$\therefore$ From (*i*), $v + x.\dfrac{dv}{dx} = \dfrac{3vx^2 + v^2x^2}{3x^2} = \dfrac{3v + v^2}{3}$

or $$x\dfrac{dv}{dx} = \dfrac{3v + v^2}{3} - v = \dfrac{v^2}{3}$$

Separating the variables, $\dfrac{dv}{v^2} = \dfrac{1}{3}.\dfrac{dx}{x}$

Integrating both sides, $-\dfrac{1}{v} = \dfrac{1}{3}\log x + c$

or $-\dfrac{x}{y} = \dfrac{1}{3}\log x + c$ which is the required solution.

(*ii*) Given equation is $(x + y).\dfrac{dy}{dx} = y - x$

or $$\dfrac{dy}{dx} = \dfrac{y - x}{x + y} \quad \text{...}(i)$$

Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ From (*i*), $v + x\dfrac{dv}{dx} = \dfrac{vx - x}{x + vx} = \dfrac{v - 1}{v + 1}$

or $$x\dfrac{dv}{dx} = \dfrac{v - 1}{v + 1} - v = \dfrac{v - 1 - v^2 - v}{v + 1} = -\dfrac{v^2 + 1}{v + 1}$$

Separating the variables, $\dfrac{v + 1}{v^2 + 1}dv = -\dfrac{dx}{x}$

or $$\left(\dfrac{1}{2}.\dfrac{2v}{v^2 + 1} + \dfrac{1}{v^2 + 1}\right)dv = -\dfrac{dx}{x}$$

Integrating both sides,

$$\dfrac{1}{2}\log(v^2 + 1) + \tan^{-1} v = -\log x + c$$

or $$\dfrac{1}{2}\log\left(\dfrac{y^2}{x^2} + 1\right) + \tan^{-1}\dfrac{y}{x} = -\log x + c$$

or $$\dfrac{1}{2}\log\dfrac{x^2 + y^2}{x^2} + \tan^{-1}\dfrac{y}{x} = -\log x + c$$

or $$\frac{1}{2}\log(x^2+y^2)-\frac{1}{2}\log x^2+\tan^{-1}\frac{y}{x}=-\log x+c$$

or $$\frac{1}{2}\log(x^2+y^2)-\frac{1}{2}\,.\,2\log x+\tan^{-1}\frac{y}{x}=-\log x+c$$

or $$\frac{1}{2}\log(x^2+y^2)+\tan^{-1}\frac{y}{x}=c \text{ is the required solution.}$$

(*iii*) Given equation is $y^2dx+(xy+x^2)\,dy=0$

or $$\frac{dy}{dx}=-\frac{y^2}{xy+x^2} \qquad ...(i)$$

Put $y = vx$, then $$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

$\therefore$ From (*i*), $$v+x\frac{dv}{dx}=-\frac{v^2x^2}{vx^2+x^2}=-\frac{v^2}{v+1}$$

or $$x\frac{dv}{dx}=-\frac{v^2}{v+1}-v=-\frac{v^2+v^2+v}{v+1}=-\frac{v(2v+1)}{v+1}$$

Separating the variables, $$\frac{v+1}{v(2v+1)}\,dv=-\frac{dx}{x}$$

or $$\left(\frac{1}{v}-\frac{1}{2v+1}\right)dv=-\frac{dx}{x} \qquad \mid \text{ Partial fractions}$$

Integrating both sides, $\log v-\frac{1}{2}\log(2v+1)=-\log x+c_1$

or $$2\log v-\log(2v+1)=-2\log x+c_1$$

or $$\log v^2-\log(2v+1)=-\log x^2+\log c$$

or $$\log\frac{v^2}{2v+1}=\log\frac{c}{x^2} \quad \text{or} \quad \frac{v^2}{2v+1}=\frac{c}{x^2}$$

or $$\frac{y^2}{x^2}\Big/\frac{2y}{x}+1=\frac{c}{x^2} \quad \text{or} \quad \frac{y^2}{x(2y+x)}=\frac{c}{x^2}$$

or $$xy^2=c(x+2y) \text{ is the required solution.}$$

**Example 5.** *Solve the following* :

(*i*) $x^2\dfrac{dy}{dx}=\dfrac{y(x+y)}{2}$ (*ii*) $\dfrac{dy}{dx}=\dfrac{\sqrt{x^2-y^2}+y}{x}$

(*iii*) $(1+e^{x/y})\,dx+e^{x/y}\left(1-\dfrac{x}{y}\right)dy=0.$

**Sol.** (*i*) Given equation is $$\frac{dy}{dx}=\frac{y(x+y)}{2x^2} \qquad ...(i)$$

Put $y = vx$, then $$\frac{dy}{dx}=v+x\frac{dv}{dx}$$

$\therefore$ From (*i*), $$v+x\frac{dv}{dx}=\frac{vx(x+vx)}{2x^2}=\frac{v(1+v)}{2}$$

or $$x\frac{dv}{dx} = \frac{v+v^2}{2} - v = \frac{v^2-v}{2}$$

Separating the variables, $\frac{2}{v^2-v}\,dv = \frac{dx}{x}$

or $$\frac{2}{v(v-1)}\,dy = \frac{dx}{x} \quad \text{or} \quad 2\left[\frac{1}{v-1} - \frac{1}{v}\right] dv = \frac{dx}{x}$$

Integrating both sides,

$$2[\log(v-1) - \log v] = \log x + \log c$$

or $$2\log\frac{v-1}{v} = \log cx \quad \text{or} \quad \log\left(\frac{v-1}{v}\right)^2 = \log cx$$

or $$\left(\frac{v-1}{v}\right)^2 = cx \quad \text{or} \quad \left\{\frac{\frac{y}{x}-1}{y/x}\right\}^2 = cx$$

or $$\left[\frac{y-x}{y}\right]^2 = cx \quad \text{or} \quad (y-x)^2 = cxy^2$$

is the required solution.

(*ii*) Given equation is $\frac{dy}{dx} = \frac{\sqrt{x^2-y^2}+y}{x}$ ...(*i*)

Put $y = vx$, then $\frac{dy}{dx} = v + x.\frac{dv}{dx}$

$\therefore$ From (*i*), $v + x\,.\,\frac{dv}{dx} = \frac{\sqrt{x^2-v^2x^2}+vx}{x} = \sqrt{1-v^2}+v$

or $$x\frac{dv}{dx} = \sqrt{1-v^2}$$

Separating the variables, $\frac{dv}{\sqrt{1-v^2}} = \frac{dx}{x}$

Integrating both sides, $\sin^{-1} v = \log x + c$

or $\sin^{-1}\frac{y}{x} = \log x + c$ is the required solution.

(*iii*) Given equation is $(1+e^{x/y})\,dx + e^{x/y}\left(1-\frac{x}{y}\right)dy = 0$ ...(*i*)

Put $\frac{x}{y} = v$ *i.e.,* $x = vy$ | **Not carefully**

$$\therefore \quad \frac{dx}{dy} = v + y\frac{dv}{dy}$$

Now from (*i*), $(1+e^{x/y})\,dx = -\,e^{x/y}\left(1-\frac{x}{y}\right)dy$

or $$\frac{dx}{dy} = -\frac{e^{x/y}\left(1-\frac{x}{y}\right)}{1+e^{x/y}} \quad \text{or} \quad v + y\,.\,\frac{dv}{dy} = -\frac{e^{v}(1-v)}{1+e^{v}}$$

or $$y\,.\,\frac{dv}{dy} = -\frac{e^{v}-ve^{v}}{1+e^{v}} - v = -\frac{e^{v}-ve^{v}+v+ve^{v}}{1+e^{v}}$$

or $$y.\,\frac{dv}{dy} = -\frac{v+e^{v}}{1+e^{v}}$$

Separating the variables, $\frac{1+e^{v}}{v+e^{v}}\,dv = -\frac{dy}{y}$

Integrating both sides, $\log(v+e^{v}) = -\log y + \log c$

or $$\log(v+e^{v}) = \log\frac{c}{y} \quad \text{or} \quad v+e^{v} = \frac{c}{y}$$

or $$\frac{x}{y} + e^{x/y} = \frac{c}{y} \quad \text{or} \quad x + ye^{x/y} = c$$

is the required solution.

**Example 6.** *Solve the following :*

*(i)* $\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y - \left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x\,\frac{dy}{dx} = 0$ (Lucknow, 1997 ; Kanpur, 1996)

*(ii)* $x\,\frac{dy}{dx} - y = x\sqrt{x^2+y^2}$ *(iii)* $\frac{dy}{dx} + \frac{x^2+3y^2}{3x^2+y^2} = 0.$

*(iv)* $x\sin\left(\frac{y}{x}\right)dy = \left[y\sin\left(\frac{y}{x}\right) - x\right]dx$ *(v)* $\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx + x\sec^2\frac{y}{x}\,dy = 0.$

**Sol.** *(i)* Given equation is

$$\frac{dy}{dx} = \frac{\left(x\cos\frac{y}{x} + y\sin\frac{y}{x}\right)y}{\left(y\sin\frac{y}{x} - x\cos\frac{y}{x}\right)x} \qquad \text{...}(i)$$

Put $y = vx$, then $\frac{dy}{dx} = v + x.\,\frac{dv}{dx}$

$\therefore$ From *(i)*, $v + x\,.\,\frac{dv}{dx} = \frac{(x\cos v + vx\sin v)\,vx}{(vx\sin v - x\cos v)\,x} = \frac{(\cos v + v\sin v)\,v}{(v\sin v - \cos v)}$

$$x\,\frac{dv}{dx} = \frac{v\cos v + v^2\sin v}{v\sin v - \cos v} - v$$

$$= \frac{v\cos v + v^2\sin v - v^2\sin v + v\cos v}{v\sin v - \cos v} = \frac{2v\cos v}{v\sin v - \cos v}$$

Separating the variables, $\frac{v\sin v - \cos v}{v\cos v}\,dv = 2\,\frac{dx}{x}$

or $$\left(\tan v - \frac{1}{v}\right) dv = 2\frac{dx}{x}$$

Integrating both sides, $-\log \cos v - \log v = 2 \log x + c_1$

or $$\log \cos v + \log v + 2 \log x = -c_1$$

or $$\log v \cos v + \log x^2 = \log c$$

or $\log vx^2 \cos v = \log c$ or $vx^2 \cos v = c$

or $\frac{y}{x} . x^2 \cos \frac{y}{x} = c$ or $xy \cos \frac{y}{x} = c$

is the required solution.

**Another Form of Above Question**

*Solve* $x\left(\cos \frac{y}{x}\right)(ydx + xdy) = y\left(\sin \frac{y}{x}\right)(xdy - ydx)$

(*ii*) Given equation is $x \frac{dy}{dx} = y + x\sqrt{x^2 + y^2}$

or $$\frac{dy}{dx} = \frac{y + x\sqrt{x^2 + y^2}}{x} \quad ...(i)$$

[**Note.** Equation (*i*) though not homogeneous in $x$ and $y$, can still be solved by the same method.]

Put $y = vx$, then $\frac{dy}{dx} = v + x . \frac{dv}{dx}$

$\therefore$ From (*i*), $$v + x\frac{dv}{dx} = \frac{vx + x\sqrt{x^2 + v^2x^2}}{x} = v + x\sqrt{1+v^2}$$

or $x\frac{dv}{dx} = x\sqrt{1+v^2}$ or $\frac{dv}{dx} = \sqrt{1+v^2}$

Separating the variables, $\frac{dv}{\sqrt{1+v^2}} = dx$

Integrating both sides, $\sinh^{-1} v = x + c_1$

or $$\log [v + \sqrt{1+v^2}\,] = \log e^x + \log c \qquad | \because \log e^x = x \log e = x$$

or $$v + \sqrt{1+v^2} = ce^x$$

or $\frac{y}{x} + \sqrt{1 + \frac{y^2}{x^2}} = ce^x$ or $y + \sqrt{x^2 + y^2} = cxe^x$

which is the required solution.

(*iii*) Given equation is $$\frac{dy}{dx} + \frac{x^2 + 3y^2}{3x^2 + y^2} = 0 \quad ...(i)$$

Put $y = vx$, then $\frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ From (*i*), $v + x . \frac{dv}{dx} + \frac{x^2 + 3v^2x^2}{3x^2 + v^2x^2} = 0$

or $$v + x\frac{dv}{dx} + \frac{1+3v^2}{3+v^2} = 0$$

or $$x\frac{dv}{dx} + \frac{1+3v^2+3v+v^3}{3+v^2} = 0$$

or $$x\frac{dv}{dx} + \frac{(v+1)^3}{v^2+3} = 0$$

Separating the variables, $\dfrac{v^2+3}{(v+1)^3}\,dv + \dfrac{dx}{x} = 0$

Integrating both sides, $\displaystyle\int \frac{v^2+3}{(v+1)^3}\,dv + \log x = c$ ...(*ii*)

Put $v + 1 = t$, then

$$\therefore \qquad \frac{v^2+3}{(v+1)^3} = \frac{(t-1)^2+3}{t^3} = \frac{t^2-2t+4}{t^3} = \frac{1}{t} - \frac{2}{t^2} + \frac{4}{t^3}$$

$$= \frac{1}{v+1} - \frac{2}{(v+1)^2} + \frac{4}{(v+1)^3}$$

| Note the method of Partial Fractions

$\therefore$ From (*ii*), $\displaystyle\int \left[\frac{1}{v+1} - \frac{2}{(v+1)^2} + \frac{4}{(v+1)^3}\right] dv + \log x = c$

or $$\log(v+1) + \frac{2}{v+1} - \frac{2}{(v+1)^2} + \log x = c$$

or $$\log\left(\frac{y}{x}+1\right) + \frac{2}{\frac{y}{x}+1} - \frac{2}{\left(\frac{y}{x}+1\right)^2} + \log x = c$$

or $$\log(x+y) - \log x + \frac{2x}{x+y} - \frac{2x^2}{(x+y)^2} + \log x = c$$

or $$\log(x+y) + \frac{2x}{x+y} - \frac{2x^2}{(x+y)^2} = c$$

or $$\log(x+y) + \frac{2x}{x+y}\left[1 - \frac{x}{x+y}\right] = c$$

or $$\log(x+y) + \frac{2xy}{(x+y)^2} = c \text{ is the required solution.}$$

(*iv*) Given equation is

$$x\sin\left(\frac{y}{x}\right)dy = \left[y\sin\left(\frac{y}{x}\right) - x\right]dx$$

or $$\frac{dy}{dx} = \frac{y\sin\frac{y}{x} - x}{x\sin\frac{y}{x}} = \frac{\frac{y}{x}\sin\frac{y}{x} - 1}{\sin\frac{y}{x}} \qquad \text{...(i)}$$

Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ From $(i)$, $v + x\dfrac{dv}{dx} = \dfrac{v\sin v - 1}{\sin v}$

or $$x\frac{dv}{dx} = \frac{v\sin v - 1}{\sin v} - v = \frac{-1}{\sin v}$$

Separating the variables $-\sin v\, dv = \dfrac{dx}{x}$

Integrating both sides, $\cos v = \log x + c$

or $\cos\dfrac{y}{x} = \log x + c$ is the required solution.

$(v)$ The given equation is

$$\left(x\tan\frac{y}{x} - y\sec^2\frac{y}{x}\right)dx + x\sec^2\frac{y}{x}\,dy = 0$$

or $$x\sec^2\frac{y}{x}\,dy = \left(y\sec^2\frac{y}{x} - x\tan\frac{y}{x}\right)dx$$

or $$\frac{dy}{dx} = \frac{y\sec^2\frac{y}{x} - x\tan\frac{y}{x}}{x\sec^2\frac{y}{x}} = \frac{y}{x} - \frac{\tan\frac{y}{x}}{\sec^2\frac{y}{x}} \qquad ...(i)$$

Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ Equation $(i)$ becomes $v + x\dfrac{dv}{dx} = v - \dfrac{\tan v}{\sec^2 v}$

$\Rightarrow$ $x\dfrac{dv}{dx} = -\dfrac{\tan v}{\sec^2 v}$ $\Rightarrow$ $\dfrac{\sec^2 v}{\tan v}\,dv = -\dfrac{dx}{x}$

Integrating, we get $\log\tan v + \log x = c$

or $\log(x\tan v) = c$ or $x\tan v = e^c$

or $x\tan\dfrac{y}{x} = C$ where $C = e^c$.

**Example 7.** *Solve the following :*

*(i)* $(x^3 - 3xy^2)dx = (y^3 - 3x^2y)dy$ *(ii)* $\dfrac{dy}{dx} + \dfrac{x - 2y}{2x - y} = 0$

*(iii)* $(2\sqrt{xy} - x)\dfrac{dy}{dx} + y = 0.$

**Sol.** $(i)$ Given equation is $$\frac{dy}{dx} = \frac{x^3 - 3xy^2}{y^3 - 3x^2y} \qquad ...(i)$$

Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ From (i), $$v + x \cdot \frac{dv}{dx} = \frac{x^3 - 3v^2x^3}{v^3x^3 - 3vx^3} = \frac{1-3v^2}{v^3 - 3v}$$

or $$x\frac{dv}{dx} = \frac{1-3v^2}{v^3-3v} - v = \frac{1-3v^2-v^4+3v^2}{v^3-3v} = \frac{1-v^4}{v^3-3v}$$

Separating the variables, $\dfrac{v^3-3v}{1-v^4}\,dv = \dfrac{dx}{x}$

Integrating both sides, $\displaystyle\int \frac{v^3-3v}{1-v^4}\,dv = \log x + c_1$ ...(ii)

Now $$\int \frac{v^3-3v}{1-v^4}\,dv = -\frac{1}{4}\int \frac{-4v^3}{1-v^4}\,dv - \frac{3}{2}\int \frac{2v}{1-v^4}\,dv$$

$$= -\frac{1}{4}\log(1-v^4) - \frac{3}{2}\int \frac{dt}{1-t^2} \text{ where } t = v^2$$

$$= -\frac{1}{4}\log(1-v^4) - \frac{3}{2}\cdot\frac{1}{2}\log\frac{1+t}{1-t}$$

$$= -\frac{1}{4}\left[\log(1-v^4) + 3\log\frac{1+v^2}{1-v^2}\right]$$

$$= -\frac{1}{4}\log\left[(1-v^4)\left(\frac{1+v^2}{1-v^2}\right)^3\right]$$

$$= -\frac{1}{4}\log\left[\frac{(1+v^2)^4}{(1-v^2)^2}\right] = -\frac{1}{2}\log\left[\frac{(1+v^2)^2}{1-v^2}\right]$$

$\therefore$ From (ii), $-\dfrac{1}{2}\log\dfrac{(1+v^2)^2}{1-v^2} = \log x + c_1$

or $$\log\frac{(1+v^2)^2}{1-v^2} = -2\log x - 2c_1 = -\log x^2 + \log c = \log\frac{c}{x^2}$$

or $$\frac{(1+v^2)^2}{1-v^2} = \frac{c}{x^2} \quad \text{or} \quad \frac{\left(1+\dfrac{y^2}{x^2}\right)^2}{1-\dfrac{y^2}{x^2}} = \frac{c}{x^2}$$

or $$\frac{(x^2+y^2)^2}{x^4}\cdot\frac{x^2}{x^2-y^2} = \frac{c}{x^2}$$

or $(x^2+y^2)^2 = c(x^2-y^2)$ which is the required solution.

(ii) Given equation is $\dfrac{dy}{dx} + \dfrac{x-2y}{2x-y} = 0$ ...(i)

Put $y = vx$, then $\dfrac{dy}{dx} = v + x\cdot\dfrac{dv}{dx}$

$\therefore$ From (*i*), $$v + x \cdot \frac{dv}{dx} + \frac{x - 2vx}{2x - vx} = 0$$

or $$x \frac{dv}{dx} + \frac{1 - 2v}{2 - v} + v = 0$$

or $$x \frac{dv}{dx} + \frac{1 - 2v + 2v - v^2}{2 - v} = 0$$

or $$x \frac{dv}{dx} + \frac{1 - v^2}{2 - v} = 0$$

Separating the variables, $$\frac{2 - v}{1 - v^2} dv + \frac{dx}{x} = 0$$

Integrating both sides, $$\int \frac{2 - v}{1 - v^2} dv + \log x = c_1$$

or $$2 \int \frac{1}{1 - v^2} dv + \frac{1}{2} \int \frac{-2v}{1 - v^2} dv + \log x = c_1$$

or $$2 \cdot \frac{1}{2} \log \frac{1 + v}{1 - v} + \frac{1}{2} \log (1 - v^2) + \log x = c_1$$

or $$\log \frac{1 + v}{1 - v} + \frac{1}{2} \log (1 - v^2) + \log x = c_1$$

or $$\log \frac{1 + v}{1 - v} + \log (1 - v^2)^{1/2} + \log x = c_1$$

or $$\log \left[ \frac{1 + v}{1 - v} \cdot \sqrt{1 - v^2} \cdot x \right] = \log c_2$$

or $$\frac{1 + v}{1 - v} \cdot \sqrt{(1 + v)(1 - v)} \cdot x = c_3$$

or $$\frac{(1 + v)^{3/2}}{\sqrt{1 - v}} \cdot x = c_2$$

Squaring and cross-multiplying, $(1 + v)^3 x^2 = c_2{}^2 (1 - v)$

or $$\left(1 + \frac{y}{x}\right)^3 \cdot x^2 = c \left(1 - \frac{y}{x}\right)$$

or $(x + y)^3 = c(x - y)$ which is the required solution.

(*iii*) Given equation is $$\frac{dy}{dx} + \frac{y}{2\sqrt{xy} - x} = 0 \qquad ...(i)$$

Put $y = vx$, then $\frac{dy}{dx} = v + x \frac{dv}{dx}$

$\therefore$ From (*i*), $$v + x \cdot \frac{dv}{dx} + \frac{vx}{2\sqrt{vx^2} - x} = 0$$

or $$v + x \,.\, \frac{dv}{dx} + \frac{v}{2\sqrt{v} - 1} = 0$$

or $$x\,\frac{dv}{dx} + \frac{v + 2v\sqrt{v} - v}{2\sqrt{v} - 1} = 0 \qquad \text{or} \qquad x\,\frac{dv}{dx} + \frac{2v\sqrt{v}}{2\sqrt{v} - 1} = 0$$

Separating the variables, $$\frac{2\sqrt{v} - 1}{2v\sqrt{v}}\,dv + \frac{dx}{x} = 0$$

or $$\left[\frac{1}{v} - \frac{1}{2v\sqrt{v}}\right] dv + \frac{dx}{x} = 0$$

Integrating both sides, $$\int \left(\frac{1}{v} - \frac{1}{2}\,v^{-3/2}\right) dv + \log x = c$$

or $$\log v - \frac{1}{2}\,\frac{v^{-1/2}}{-\frac{1}{2}} + \log x = c \qquad \text{or} \qquad \log v + \frac{1}{\sqrt{v}} + \log x = c$$

or $$\log \frac{y}{x} + \sqrt{\frac{x}{y}} + \log x = c \qquad \text{or} \qquad \log y - \log x + \sqrt{\frac{x}{y}} + \log x = c$$

or $$\log y + \sqrt{\frac{x}{y}} = c \text{ is the reqd. solution.}$$

**Example 8.** *Solve the following :*

(*i*) $x dx + y dy = m(x dy - y dx)$ (*ii*) $(x^2 + 2xy - y^2)\,dx + (y^2 + 2xy - x^2)\,dy = 0$

(*iii*) $2\,\dfrac{dy}{dx} = \dfrac{x}{y} - 1.$

**Sol.** (*i*) Given equation is $x dx + y dy = m(x dy - y dx)$

or $$(x + my)\,dx = (mx - y)\,dy$$

or $$\frac{dy}{dx} = \frac{x + my}{mx - y} \qquad \text{...(i)}$$

Put $y = vx$, then $\dfrac{dy}{dx} = v + x\,\dfrac{dv}{dx}$

$\therefore$ From (*i*), $$v + x\,\frac{dv}{dx} = \frac{x + mvx}{mx - vx} = \frac{1 + mv}{m - v}$$

or $$x\,\frac{dv}{dx} = \frac{1 + mv}{m - v} - v = \frac{1 + v^2}{m - v}$$

Separating the variables $$\frac{m - v}{1 + v^2}\,dv = \frac{dx}{x}$$

or $$\left[\frac{m}{1 + v^2} - \frac{1}{2}\,.\,\frac{2v}{1 + v^2}\right] dv = \frac{dx}{x}$$

Integrating both sides

$$m \tan^{-1} v \,.\, \tfrac{1}{2} \log (1 + v^2) = \log x + c$$

or $$m \tan^{-1} \frac{y}{x} - \frac{1}{2} \log \left(1+\frac{y^2}{x^2}\right) = \log x + c$$

or $$m \tan^{-1} \frac{y}{x} - \frac{1}{2} [\log (x^2 + y^2) - \log x^2] = \log x + c$$

or $$m \tan^{-1} \frac{y}{x} - \frac{1}{2} [\log (x^2 + y^2) - 2 \log x] = \log x + c$$

or $$m \tan^{-1} \frac{y}{x} - \frac{1}{2} \log (x^2 + y^2) = c \text{ is the reqd. solution.}$$

(*ii*) Given equation is $$\frac{dy}{dx} + \frac{x^2 + 2xy - y^2}{y^2 + 2xy - x^2} = 0 \quad ...(1)$$

Put $y = vx$, then $\dfrac{dy}{dx} = v + x \dfrac{dv}{dx}$

$\therefore$ From (1), $$v + x \frac{dv}{dx} + \frac{x^2 + 2vx^2 - v^2x^2}{v^2x^2 + 2vx^2 - x^2} = 0$$

or $$v + x \frac{dv}{dx} + \frac{1 + 2v - v^2}{v^2 + 2v - 1} = 0$$

or $$x \frac{dv}{dx} + \frac{1 + 2v - v^2 + v^3 + 2v^2 - v}{v^2 + 2v - 1} = 0$$

or $$x \frac{dv}{dx} + \frac{v^3 + v^2 + v + 1}{v^2 + 2v - 1} = 0$$

Separating the variables, $$\frac{v^2 + 2v - 1}{v^3 + v^2 + v + 1}\, dv + \frac{dx}{x} = 0$$

or $$\frac{v^2 + 2v - 1}{(v + 1)(v^2 + 1)}\, dv + \frac{dx}{x} = 0$$

Integrating both sides, $$\int \frac{v^2 + 2v - 1}{(v + 1)(v^2 + 1)}\, dv + \int \frac{dx}{x} = c_1 \quad ...(2)$$

Let $$\frac{v^2 + 2v - 1}{(v + 1)(v^2 + 1)} = \frac{A}{v + 1} + \frac{Bv + C}{v^2 + 1}$$

Multiplying both sides by $(v + 1)(v^2 + 1)$,

$$v^2 + 2v - 1 = A(v^2 + 1) + (Bv + C)(v + 1) \quad ...(3)$$

Putting $v = -1$ in (3), $1 - 2 - 1 = A(1 + 1)$ $\quad \therefore A = -1$

Equating co-effi. of $v^2$, $1 = A + B$ $\quad \therefore B = 1 - A = 2$

Equating constant terms, $-1 = A + C$ $\quad \therefore C = -1 - A = 0$

$\therefore$ $$\frac{v^2 + 2v - 1}{(v + 1)(v^2 + 1)} = \frac{-1}{v + 1} + \frac{2v}{v^2 + 1}$$

$\therefore$ From (2), $\int \left( \frac{-1}{v+1} + \frac{2v}{v^2+1} \right) dv + \log x = c_1$

or $\quad -\log (v+1) + \log (v^2+1) + \log x = c_1$

or $\quad \log \frac{x(v^2+1)}{v+1} = \log c \quad \therefore \quad \frac{x(v^2+1)}{v+1} = c$

or $\quad x\left(\frac{y^2}{x^2}+1\right) = c\left(\frac{y}{x}+1\right)$

or $\quad x^2 + y^2 = c(x+y)$ is the reqd. solution.

(*iii*) Given equation is $\quad \frac{dy}{dx} = \frac{x-y}{2y}$ ...(1)

Put $y = vx$, then $\quad \frac{dy}{dx} = v + x\frac{dv}{dx}$

$\therefore$ From (1), $\quad v + x\frac{dv}{dx} = \frac{x - vx}{2vx} = \frac{1-v}{2v}$

or $\quad x\frac{dv}{dx} = \frac{1-v}{2v} - v = \frac{1-v-2v^2}{2v}$

Separating the variables, $\quad \frac{2v}{1-v-2v^2}\, dv = \frac{dx}{x}$

or $\quad \frac{2v}{(1+v)(1-2v)}\, dv = \frac{dx}{x}$

or $\quad \frac{2}{3}\left[\frac{1}{1-2v} - \frac{1}{1+v}\right] dv = \frac{dx}{x}$ | Partial fractions

Integrating both sides,

$$\tfrac{2}{3}\left[-\tfrac{1}{2}\log(1-2v) - \log(1+v)\right] = \log x + c_1$$

or $\quad -\frac{1}{3}\log(1-2v) - \frac{2}{3}\log(1+v) = \log x + c_1$

or $\quad \log(1-2v) + 2\log(1+v) = -3\log x - 3c_1$

or $\quad \log(1-2v) + \log(1+v)^2 = -\log x^3 + \log c$

or $\quad \log(1-2v)(1+v)^2 = \log\frac{c}{x^3}$

or $\quad (1-2v)(1+v)^2 = \frac{c}{x^3}$

or $\quad \left(1-\frac{2y}{x}\right)\left(1+\frac{y}{x}\right)^2 = \frac{c}{x^3}$

or $\quad (x-2y)(x+y)^2 = c$ is the reqd. solution.

**Example 9.** *Solve the following* :

(*i*) $2x^2 dy = (x^2+y^2)\, dx$ (*ii*) $(x^2+xy)dy = (x^2+y^2)\, dx$

(*iii*) $(y^3 - 2yx^2)dx + (2xy^2 - x^3)\, dy = 0.$

**Sol.** (*i*) Given equation is $\dfrac{dy}{dx} = \dfrac{x^2 + y^2}{2x^2}$ ...(*i*)

Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ From (*i*), $v + x\dfrac{dv}{dx} = \dfrac{x^2 + v^2x^2}{2x^2} = \dfrac{1+v^2}{2}$

or $x\dfrac{dv}{dx} = \dfrac{1+v^2}{2} - v = \dfrac{1+v^2-2v}{2} = \dfrac{(v-1)^2}{2}$

Separating the variables, $\dfrac{dv}{(v-1)^2} = \dfrac{dx}{2x}$

Integrating both sides, $-\dfrac{1}{v-1} = \dfrac{1}{2}\log x + c_1$

or $-\dfrac{1}{\dfrac{y}{x}-1} = \dfrac{1}{2}\log x + c_1$

or $-\dfrac{x}{y-x} = \dfrac{1}{2}\log x + c_1$ or $\dfrac{2x}{x-y} = \log x + 2c_1$

or $2x = (x-y)(\log x + c)$ which is the reqd. solution.

(*ii*) Given equation is $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{x^2+xy}$ ...(*i*)

Put $y = vx$, then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$\therefore$ From (*i*), $v + x\dfrac{dv}{dx} = \dfrac{x^2+v^2x^2}{x^2+vx^2}$

or $v + x\dfrac{dv}{dx} = \dfrac{1+v^2}{1+v}$

or $x\dfrac{dv}{dx} = \dfrac{1+v^2}{1+v} - v = \dfrac{1-v}{1+v}$

or $\dfrac{1+v}{1-v}\,dv = \dfrac{dx}{x}$

or $\left(-1+\dfrac{2}{1-v}\right)dv = \dfrac{dx}{x}$

Integrating both sides, $-v - 2\log(1-v) = \log x + c$

or $-\dfrac{y}{x} - 2\log\left(1-\dfrac{y}{x}\right) = \log x + c$

or $-\dfrac{y}{x} - 2\log\dfrac{x-y}{x} = \log x + c$

or $$-\frac{y}{x}-2\,[\log(x-y)-\log x]=\log x+c$$

or $$\log x = 2\log(x-y)+\frac{y}{x}+c$$

which is the reqd. solution.

(*iii*) Given equation is $\dfrac{dy}{dx}+\dfrac{y^3-2yx^2}{2xy^2-x^3}=0$ ...(*i*)

Putting $y = vx$, $\dfrac{dy}{dx}=v+x\dfrac{dv}{dx}$

$\therefore$ From (*i*), $$v+x\frac{dv}{dx}+\frac{v^3x^3-2\,.\,vx\,.\,x^2}{2\,.\,x\,.\,v^2x^2-x^3}=0$$

or $$x\frac{dv}{dx}+\frac{v^3-2v}{2v^2-1}+v=0$$

or $$x\frac{dv}{dx}+\frac{3v^3-3v}{2v^2-1}=0$$

Separating the variables, $$\frac{2v^2-1}{3v^3-3v}\,dv+\frac{dx}{x}=0$$

Integrating both sides, $$\int\frac{2v^2-1}{3v(v^2-1)}\,dv+\log x=c_1$$

or $$\frac{1}{6}\int\frac{(2v^2-1)\,.\,2v}{v^2(v^2-1)}\,dv+\log x=c_1$$ | Note this step

or $$\frac{1}{6}\int\frac{2t-1}{t(t-1)}\,dt+\log x=c_1,\text{ where } t=v^2$$

or $$\frac{1}{6}\int\left(\frac{1}{t}+\frac{1}{t-1}\right)dt+\log x=c_1$$ | Partial fractions

or $$\tfrac{1}{6}\,[\log t+\log(t-1)]+\log x=c_1$$

or $$\log t(t-1)+6\log x=6c_1$$

or $$\log v^2(v^2-1)+\log x^6=6c_1$$

or $$\log[v^2(v^2-1)\,.\,x^6]=\log c_2$$

or $$v^2(v^2-1)\,.\,x^6=c_2$$

or $$\frac{y^2}{x^2}\left(\frac{y^2}{x^2}-1\right)x^6=c_2$$

or $$x^2y^2(y^2-x^2)=c_2$$

or $$xy\sqrt{y^2-x^2}=\sqrt{c_2}=c$$

which is the reqd. solution.

**Example 10.** *Solve the following equations :*

(*i*) $\dfrac{dy}{dx} + \dfrac{y}{x} = \dfrac{y^2}{x^2}$ (*ii*) $(x^3 + y^3)\, dx = (x^2y + xy^2)\, dy$

(*iii*) $\dfrac{dy}{dx} = \dfrac{6x^2 + 2y^2}{x^2 + 4xy}$ (*iv*) $\dfrac{dy}{dx} + \dfrac{2(x^2 + xy)}{x^2 + y^2} = 0$

(*v*) $x(x - y) \dfrac{dy}{dx} = y(x + y)$ (*vi*) $x(x - y)\, dy + y^2\, dx = 0.$

**Sol.** (*i*) **Hint.** Putting $y = vx$, the given equation becomes

$$\frac{dv}{v(v-2)} = \frac{dx}{x}$$

Integrating $\displaystyle\int \frac{1}{2}\left[\frac{1}{v-2} - \frac{1}{v}\right] dv = \log x + c'$

or $$\log \frac{v-2}{v} = 2 \log x + 2c' = 2 \log x + \log c$$

or $$\frac{v-2}{v} = cx^2 \qquad \text{or} \qquad 1 - \frac{2x}{y} = cx^2$$

$\therefore$ $$y - 2x = cx^2y$$

(*ii*) **Hint.** Putting $y = vx$, the given equation reduces to

$$\frac{v}{1-v}\, dv = \frac{dx}{x}.$$

Integrating $\displaystyle\int \left(-1 + \frac{1}{1-v}\right) dv = \log x + c$

or $$-v - \log (1 - v) = \log x + c$$

or $$-\frac{y}{x} - \log\left(1 - \frac{y}{x}\right) = \log x + c$$

or $$-\frac{y}{x} - \log (x - y) + \log x = \log x + c.$$

(*iii*) **Hint.** Putting $y = vx$, the given equation reduces to

$$x\frac{dv}{dx} = \frac{6 - v - 2v^2}{1 + 4v}$$

or $$\frac{1 + 4v}{2v^2 + v - 6}\, dv = -\frac{dx}{x}$$

Integrating $\log (2v^2 + v - 6) = -\log x + \log c$

or $$2\frac{y^2}{x^2} + \frac{y}{x} - 6 = \frac{c}{x} \qquad \text{or} \qquad 2y^2 + xy - 6x^2 = cx.$$

(*iv*) **Hint.** Putting $y = vx$, the given equation reduces to

$$x\frac{dv}{dx} + \frac{2 + 3v + v^3}{1 + v^2} = 0 \qquad \text{or} \qquad \frac{v^2 + 1}{v^3 + 3v + 2}\, dv + \frac{dx}{x} = 0$$

Integrating $\frac{1}{3}\int \frac{3v^2+3}{v^3+3v+2}\,dv + \log x = c'$

or $\quad \frac{1}{3}\log(v^3+3v+2) + \log x = c'$

or $\quad \log(v^3+3v+2) + 3\log x = 3c' = \log c$

or $\quad \log x^3(v^3+3v+2) = \log c$

or $\quad x^3\left(\frac{y^3}{x^3}+\frac{3y}{x}+2\right) = c \quad$ or $\quad y^3+3x^2y+2x^3 = c.$

(*v*) Please try yourself. $\left[\textbf{Ans. } \frac{x}{y}+\log(xy) = c\right]$

(*vi*) Please try yourself. [**Ans.** $y = ce^{y/x}$]

## TYPE III. EQUATIONS REDUCIBLE TO HOMOGENEOUS

**Art. To solve the equation** $\mathbf{\frac{dy}{dx} = \frac{ax+by+c}{a'x+b'y+c'}}$

**Case I.** When $\quad \frac{a}{a'} \neq \frac{b}{b'}$

Put $x = X + h, \qquad y = Y + k \qquad \ldots(i)$

where X, Y are new variables and $h$, $k$ are constants.

Then $dx = dX$, $dy = dY$

$\therefore$ Given equation becomes $\quad \frac{dY}{dX} = \frac{a(X+h)+b(Y+k)+c}{a'(X+h)+b'(Y+k)+c'}$

or $$\frac{dY}{dX} = \frac{(aX+bY)+(ah+bk+c)}{(a'X+b'Y)+(a'h+b'k+c')} \qquad \ldots(ii)$$

Choose $h$, $k$ such that

$$ah+bk+c = 0$$
$$a'h+b'k+c' = 0$$

| **Why ?** To get rid of the constant terms.

Solving by cross multiplication,

$$\frac{h}{bc'-b'c} = \frac{k}{ca'-c'a} = \frac{1}{ab'-a'b}$$

$\therefore$ $$h = \frac{bc'-b'c}{ab'-a'b};\ k = \frac{ca'-c'a}{ab'-a'b} \qquad \ldots(iii)$$

$\therefore$ Eqn. (*ii*) becomes, $\quad \frac{dY}{dX} = \frac{aX+bY}{a'X+b'Y}$

**which is homogeneous in X and Y** and can be solved by putting $Y = vX$. In the solution, putting $X = x - h$, $Y = y - k$ [where $h$ and $k$ have the values given by (*iii*)], we get the reqd. solution.

If, however, $\frac{a}{a'} = \frac{b}{b'}$ *i.e.*, $ab' - a'b = 0$

$h$ and $k$ both become infinite. Hence the method fails.

**Case II.** When $\dfrac{a}{a'} = \dfrac{b}{b'}$

Let $\dfrac{a}{a'} = \dfrac{b}{b'} = \dfrac{1}{m}$ (say)

then $a' = ma,\ b' = mb$

$\therefore$ Given equation becomes, $\dfrac{dy}{dx} = \dfrac{ax + by + c}{max + mby + c'}$

or $$\frac{dy}{dx} = \frac{ax + by + c}{m(ax + by) + c'} \qquad ...(i)$$

Put $ax + by = z$

then $a + b \cdot \dfrac{dy}{dx} = \dfrac{dz}{dx}$ or $\dfrac{dy}{dx} = \dfrac{1}{b}\left(\dfrac{dz}{dx} - a\right)$

$\therefore$ From (*i*), $\dfrac{1}{b}\left(\dfrac{dz}{dx} - a\right) = \dfrac{z + c}{mz + c'}$ or $\dfrac{dz}{dx} - a = \dfrac{b(z + c)}{mz + c'}$

or $$\frac{dz}{dx} = \frac{b(z + c)}{mz + c'} + a = \frac{b(z + c) + a(mz + c')}{mz + c'}$$

Separating the variables, $\dfrac{mz + c'}{b(z + c) + a(mz + c')}\, dz = dx$

which can be easily integrated.

In the solution, putting $z = ax + by$, we get the required solution.

**Example 1.** *Solve the following* :

(*i*) $\dfrac{dy}{dx} = \dfrac{x + 2y - 3}{2x + y - 3}$ (Agra, 1996 ; Bangalore 1996)

(*ii*) $\dfrac{dy}{dx} = \dfrac{y - x + 1}{y + x - 5}$ (*iii*) $\dfrac{dy}{dx} = \dfrac{x - y + 1}{x + y - 2}$.

**Sol.** (*i*) Given equation is $\dfrac{dy}{dx} = \dfrac{x + 2y - 3}{2x + y - 3}$ ...(*i*)

$\left[\text{Here } a = 1, a' = 2, b = 2, b' = 1, \dfrac{a}{a'} \neq \dfrac{b}{b'}\right]$

Put $x = X + h,\ y = Y + k$

$\therefore$ $dx = dX,\ dy = dY$

$\therefore$ From (*i*), $\dfrac{dY}{dX} = \dfrac{X + h + 2(Y + k) - 3}{2(X + h) + (Y + k) - 3}$

or $$\frac{dY}{dX} = \frac{(X + 2Y) + (h + 2k - 3)}{(2X + Y) + (2h + k - 3)} \qquad ...(ii)$$

Choose $h$ and $k$ such that

$$h + 2k - 3 = 0$$

and $$2h + k - 3 = 0$$

Solving by cross-multiplication, $\dfrac{h}{-6 + 3} = \dfrac{k}{-6 + 3} = \dfrac{1}{1 - 4}$

$\Rightarrow \qquad h = 1, k = 1 \qquad ...(iii)$

Eqn. (*iii*) becomes, $\dfrac{dY}{dX} = \dfrac{X+2Y}{2X+Y} \qquad ...(iv)$

Put $Y = vX$, then $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$\therefore$ From (*iv*), $v + X \cdot \dfrac{dv}{dX} = \dfrac{X+2vX}{2X+vX} = \dfrac{1+2v}{2+v}$

$$X\frac{dv}{dX} = \frac{1+2v}{2+v} - v = \frac{1+2v-2v-v^2}{2+v} = \frac{1-v^2}{2+v}$$

Separating the variables, $\dfrac{2+v}{1-v^2}\,dv = \dfrac{dX}{X}$

Integrating both sides, $\displaystyle\int \frac{2+v}{1-v^2}\,dv = \int \frac{dX}{X} + c_1$

or $\displaystyle\int \frac{2+v}{(1+v)(1-v)}\,dv = \log X + c_1$

or $\displaystyle\int \left[\frac{2-1}{(1+v)(1+1)} + \frac{2+1}{(1+1)(1-v)}\right] dv = \log X + c_1$

or $\displaystyle\int \left[\frac{1}{2(1+v)} + \frac{3}{2(1-v)}\right] dv = \log X + c_1$

or $\frac{1}{2}\log(1+v) - \frac{3}{2}\log(1-v) = \log X + c_1$

or $\log(1+v) - 3\log(1-v) = 2\log X + 2c_1$

or $\log(1+v) - \log(1-v)^3 = \log X^2 + \log c$

or $\log\dfrac{1+v}{(1-v)^3} = \log cX^2 \qquad \therefore \qquad \dfrac{1+v}{(1-v)^3} = cX^2$

or $\dfrac{1+\dfrac{Y}{X}}{\left(1-\dfrac{Y}{X}\right)^3} = cX^2 \qquad$ or $\qquad \dfrac{X^2\,(X+Y)}{(X-Y)^3} = cX^2$

or $X + Y = c(X-Y)^2$

or $x - 1 + y - 1 = c[(x-1) - (y-1)]^2$

or $x + y - 2 = c(x-y)^2 \qquad \because \begin{array}{l} X = x - h = x - 1 \\ Y = y - k = y - 1 \end{array}$

which is the required solution.

(*ii*) Given equation is $\dfrac{dy}{dx} = \dfrac{y-x+1}{y+x-5} \qquad ...(i)$

$\left[\text{Here } a = -1, v = 1, a' = 1, b' = 1, \dfrac{a}{a'} \neq \dfrac{b}{b'}\right]$

Put $x = X + h, y = Y + k$

then $dx = dX, dy = dY$

$\therefore$ From (*i*), $$\frac{dY}{dX} = \frac{Y + k - (X + h) + 1}{Y + k + X + h - 5} = \frac{(-X + Y) + (-h + k + 1)}{(X + Y) + (h + k - 5)} \quad ...(ii)$$

Choose $h$ and $k$ such that

$$-h + k + 1 = 0$$

and $$h + k - 5 = 0 \quad ...(iii)$$

whence $h = 3, k = 2$

$\therefore$ Equation (*ii*) becomes, $$\frac{dY}{dX} = \frac{-X + Y}{X + Y} \quad ...(iv)$$

Put $Y = vX$, then $$\frac{dY}{dX} = v + X \frac{dv}{dX}$$

$\therefore$ From (*iv*), $$v + X \cdot \frac{dv}{dX} = \frac{-X + vX}{X + vX} = \frac{-1 + v}{1 + v}$$

$$X \cdot \frac{dv}{dX} = \frac{-1 + v}{1 + v} - v = \frac{-1 + v - v - v^2}{1 + v} = -\frac{1 + v^2}{1 + v}$$

Separating the variables, $$\frac{1 + v}{1 + v^2} dv = -\frac{dX}{X}$$

Integrating both sides, $$\int \frac{1 + v}{1 + v^2} dv = -\int \frac{dX}{X} + c$$

or $$\int \frac{1}{1 + v^2} dv + \frac{1}{2} \int \frac{2v}{1 + v^2} dv = -\log X + c$$

or $$\tan^{-1} v + \tfrac{1}{2} \log (1 + v^2) = -\log X + c$$

or $$\tan^{-1} \frac{Y}{X} + \frac{1}{2} \log \left(1 + \frac{Y^2}{X^2}\right) = -\log X + c$$

or $$\tan^{-1} \frac{Y}{X} + \frac{1}{2} \log \frac{X^2 + Y^2}{X^2} = -\log X + c$$

or $$\tan^{-1} \frac{Y}{X} + \frac{1}{2} [\log (X^2 + Y^2) - \log X^2] = -\log X + c$$

or $$\tan^{-1} \frac{Y}{X} + \frac{1}{2} [\log (X^2 + Y^2) - 2 \log X] = -\log X + c$$

or $$\tan^{-1} \frac{Y}{X} + \frac{1}{2} \log (X^2 + Y^2) = c$$

or $$\tan^{-1} \frac{y - 2}{x - 3} + \frac{1}{2} \log [(x - 3)^2 + (y - 2)^2] = c$$

$\left| \because \; X = x - h = x - 3 \right.$

$\quad Y = y - k = y - 2$

or $$\tan^{-1} \frac{y - 2}{x - 3} + \frac{1}{2} \log (x^2 + y^2 - 6x - 4y + 13) = c$$

which is the required solution.

(*iii*) Given equation is $\dfrac{dy}{dx} = \dfrac{x - y + 1}{x + y - 2}$ ...(*i*)

$\left[\text{Here } a = 1, a' = 1, b = -1, b' = 1, \dfrac{a}{a'} \neq \dfrac{b}{b'}\right]$

Put $\quad x = X + h, y = Y + k$

$\therefore \quad dx = dX, dy = dY$

$\therefore$ From (*i*), $\dfrac{dY}{dX} = \dfrac{(X + h) - (Y + k) + 1}{(X + h) + (Y + k) - 2} = \dfrac{(X - Y) + (h - k + 1)}{(X + Y) + (h + k - 2)}$ ...(*ii*)

Choose $h$ and $k$ such that

$$h - k + 1 = 0$$

and $\quad h + k - 2 = 0$

whence $\quad h = \frac{1}{2}, k = \frac{3}{2}$ ...(*iii*)

$\therefore$ From (*ii*), $\dfrac{dY}{dX} = \dfrac{X - Y}{X + Y}$ ...(*iv*)

Put $Y = vX$, then $\dfrac{dY}{dX} = v + X \cdot \dfrac{dv}{dX}$

$\therefore$ From (*iv*), $v + X \cdot \dfrac{dv}{dX} = \dfrac{X - vX}{X + vX} = \dfrac{1 - v}{1 + v}$

$$X \cdot \frac{dv}{dX} = \frac{1 - v}{1 + v} - v = \frac{1 - v - v - v^2}{1 + v} = \frac{1 - 2v - v^2}{1 + v} = -\frac{v^2 + 2v - 1}{v + 1}$$

Separating the variables, $\dfrac{v + 1}{v^2 + 2v - 1}\, dv = -\dfrac{dX}{X}$

or $\quad \dfrac{1}{2} \cdot \dfrac{2v - 2}{v^2 + 2v - 1}\, dv = -\dfrac{dX}{X}$

Integrating both sides, $\frac{1}{2} \log (v^2 + 2v - 1) = -\log X + c_1$

or $\quad \log (v^2 + 2v - 1) = -2 \log X + 2c_1 = -2 \log X + \log c$

or $\quad \log (v^2 + 2v - 1) = -\log X^2 + \log c = \log \dfrac{c}{X^2}$

or $\quad v^2 + 2v - 1 = \dfrac{c}{X^2}$ or $\dfrac{Y^2}{X^2} + \dfrac{2Y}{X} - 1 = \dfrac{c}{X^2}$

or $\quad Y^2 + 2XY - X^2 = c$

or $\quad (y - \frac{3}{2})^2 + 2(x - \frac{1}{2})(y - \frac{3}{2}) - (x - \frac{1}{2})^2 = c$

or $\quad y^2 - 3y + \frac{9}{4} + 2\,(xy - \frac{3}{2}\,x - \frac{1}{2}\,y + \frac{3}{4}) - (x^2 - x + \frac{1}{4}) = c$

or $\quad y^2 - 3y + \frac{9}{4} + 2xy - 3x - y + \frac{3}{2} - x^2 + x - \frac{1}{4} = c$

or $\quad y^2 + 2xy - x^2 - 2x - 4y = c - \frac{7}{2}$

or $\quad y^2 + 2xy - x^2 - 2x - 4y = A$

where A is an arbitrary constant, is the required solution.

**Example 2.** *Solve the following :*

*(i)* $(x - y)\, dy = (x + y + 1)\, dx$ *(ii)* $\dfrac{dy}{dx} = \dfrac{(x+y-1)^2}{4(x-2)^2}$

*(iii)* $(2x + 3y - 5)\dfrac{dy}{dx} + (3x + 2y - 5) = 0.$

**Sol.** (*i*) Given equation is $\dfrac{dy}{dx} = \dfrac{x+y+1}{x-y}$ ...(*i*)

$$\left[\text{Here } a = 1, a' = 1, b = 1, b' = -1, \therefore\ \frac{a}{a'} \neq \frac{b}{b'}\right]$$

Put $x = X + h, y = Y + k$

then $dx = dX, dy = dY$

$\therefore$ From (*i*), $\dfrac{dY}{dX} = \dfrac{X+h+Y+k+1}{(X+h)-(Y+k)} = \dfrac{(X+Y)+(h+k+1)}{(X-Y)+(h-k)}$ ...(*ii*)

Choose $h$ and $k$ such that

$$h + k + 1 = 0 \quad \text{and} \quad h - k = 0$$

whence $h = -\frac{1}{2}, k = -\frac{1}{2}$

$\therefore$ (*ii*) becomes $\dfrac{dY}{dX} = \dfrac{X+Y}{X-Y}$ ...(*iii*)

Put $Y = vX$, then $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$\therefore$ From (*iii*), $v + X\dfrac{dv}{dX} = \dfrac{X+vX}{X-vX} = \dfrac{1+v}{1-v}$

$$X\,.\,\frac{dv}{dX} = \frac{1+v}{1-v} - v = \frac{1+v-v+v^2}{1-v} = \frac{1+v^2}{1-v}$$

Separating the variables, $\dfrac{1-v}{1+v^2}\,dv = \dfrac{dX}{X}$

or $$\left[\frac{1}{1+v^2} - \frac{1}{2}\cdot\frac{2v}{1+v^2}\right] dv = \frac{dX}{X}$$

Integrating both sides, $\tan^{-1} v - \frac{1}{2}\log(1 + v^2) = \log X + c$

or $$\tan^{-1}\frac{Y}{X} - \frac{1}{2}\log\left(1 + \frac{Y^2}{X^2}\right) = \log X + c$$

or $$\tan^{-1}\frac{Y}{X} - \frac{1}{2}[\log(X^2 + Y^2) - \log X^2] = \log X + c$$

or $$\tan^{-1}\frac{Y}{X} - \frac{1}{2}[\log(X^2 + Y^2) - 2\log X] = \log X + c$$

or $$\tan^{-1}\frac{Y}{X} - \frac{1}{2}\log(X^2 + Y^2) = c$$

or $$\tan^{-1}\frac{y-k}{x-h}-\frac{1}{2}\log[(x-h)^2+(y-k)^2]=c$$

or $$\tan^{-1}\frac{y+\frac{1}{2}}{x+\frac{1}{2}}-\frac{1}{2}\log[(x+\tfrac{1}{2})^2+(y+\tfrac{1}{2})^2]=c$$

or $$\tan^{-1}\frac{2y+1}{2x+1}-\frac{1}{2}\log(x^2+y^2+x+y+\tfrac{1}{2})=c$$

which is the required solution.

(*ii*) Given equation is $$\frac{dy}{dx}=\frac{(x+y-1)^2}{4(x-2)^2} \quad ...(i)$$

Put $x=X+h,\ y=Y+k$

then $dx=dX,\ dy=dY$

$\therefore$ From (*i*), $$\frac{dY}{dX}=\frac{(X+h+Y+k-1)^2}{4(X+h-2)^2}=\frac{(X+Y+h+k-1)^2}{4(X+h-2)^2} \quad ...(ii)$$

Choose $h$ and $k$ such that

$$h+k-1=0 \quad \text{and} \quad h-2=0$$

when $h=2,\ k=-1$

$\therefore$ (*ii*) becomes, $$\frac{dY}{dX}=\frac{(X+Y)^2}{4X^2} \quad ...(iii)$$

Put $Y=vX$, then $\dfrac{dY}{dX}=v+X\,.\,\dfrac{dv}{dX}$

$\therefore$ From (*iii*), $$v+X\,.\frac{dv}{dX}=\frac{(X+vX)^2}{4X^2}=\frac{(1+v)^2}{4}$$

$$X\frac{dv}{dX}=\frac{(1+v)^2}{4}-v=\frac{(1+v)^2-4v}{4}=\frac{(1-v)^2}{4}$$

Separating the variables, $\dfrac{4}{(1-v)^2}\,dv=\dfrac{dX}{X}$

Integrating both sides, $4\int(1-v)^{-2}\,dv=\log X+c$

or $$4\,.\,\frac{(1-v)^{-1}}{(-1)(-1)}=\log X+c$$

or $$\frac{4}{1-v}=\log X+c \quad \text{or} \quad \frac{4}{1-\frac{Y}{X}}=\log X+c$$

or $$\frac{4X}{X-Y}=\log X+c \quad \text{or} \quad \frac{4(x-h)}{(x-h)-(y-k)}=\log(x-h)+c$$

or $$\frac{4(x-2)}{(x-2)-(y+1)}=\log(x-2)+c$$

or $\dfrac{4(x-2)}{x-y-3}=\log(x-2)+c$ is the required solution.

(*iii*) Given equation is $\frac{dy}{dx} = -\frac{3x+2y-5}{2x+3y-5}$ ...(*i*)

$\left[\text{Here } a = 3, a' = 2, b = 2, b' = 3 \quad \therefore \quad \frac{a}{a'} \neq \frac{b}{b'}\right]$

Put $x = X + h$, $y = Y + k$ then $dx = dX$, $dy = dY$

$\therefore$ From (*i*), $\frac{dY}{dX} = -\frac{3(X+h)+2(Y+k)-5}{2(X+h)+3(Y+k)-5}$

or $\frac{dY}{dX} = -\frac{(3X+2Y)+(3h+2k-5)}{(2X+3Y)+(2h+3k-5)}$ ...(*ii*)

Choose $h$ and $k$ such that

$$3h + 2k - 5 = 0 \quad \text{and} \quad 2h + 3k - 5 = 0 \quad \Rightarrow \quad h = 1, k = 1.$$

$\therefore$ (*ii*) becomes $\frac{dY}{dX} = -\frac{3X+2Y}{2X+3Y}$ ...(*iii*)

Put $Y = vX$, then $\frac{dY}{dX} = v + X \cdot \frac{dv}{dX}$

$\therefore$ From (*iii*), $v + X \cdot \frac{dv}{dX} = -\frac{3X+2vX}{2X+3vX} = -\frac{3+2v}{2+3v}$

$$X \cdot \frac{dv}{dX} = -\frac{3+2v}{2+3v} - v = -\frac{3+2v+2v+3v^2}{2+3v} = -\frac{3v^2+4v+3}{3v+2}$$

Separating the variables, $\frac{3v+2}{3v^2+4v+3}\, dv = -\frac{dX}{X}$

or $$\frac{6v+4}{3v^2+4v+3}\, dv + \frac{2dX}{X} = 0$$

Integrating, $\log(3v^2 + 4v + 3) + 2\log X = \log c$

or $\log(3v^2 + 4v + 3) + \log X^2 = \log c$

or $\log(3v^2 + 4v + 3)X^2 = \log c$

or $(3v^2 + 4v + 3)X^2 = c$ or $\left(\frac{3Y^2}{X^2} + \frac{4Y}{X} + 3\right)X^2 = c$

or $3Y^2 + 4XY + 3X^2 = c$ or $3(y-k)^2 + 4(x-h)(y-k) + 3(x-h)^2 = c$

or $3(y-1)^2 + 4(x-1)(y-1) + 3(x-1)^2 = c$

or $3x^2 + 4xy + 3y^2 - 10x - 10y = c - 10 = A$

where A is a constant, is the required solution.

**Example 3.** *Solve the following* :

(*i*) $\frac{dy}{dx} = \frac{2x-y+1}{x+2y-3}$ (*ii*) $(12x + 21y - 9)dx + (47x + 40y + 7)dy = 0.$

**Sol.** (*i*) Given equation is

$$\frac{dy}{dx} = \frac{2x-y+1}{x+2y-3} \quad ...(i)$$

$\left[\text{Here } a = 2, a' = 1, b = -1, b' = 2 \therefore \frac{a}{a'} \neq \frac{b}{b'}\right]$

Put $x = X + h, y = Y + k$

then $dx = dX, dy = dY$

$\therefore$ From (*i*), $$\frac{dY}{dX} = \frac{(X+h)-(Y+k)+1}{(X+h)+2(Y+k)-3} = \frac{(2X-Y)+(2h-k+1)}{(X+2Y)+(h+2k-3)} \quad ...(ii)$$

Choose $h$ and $k$ such that

$$2h - k + 1 = 0 \quad \text{and} \quad h + 2k - 3 = 0$$

$\Rightarrow$ $$h = \frac{1}{5}, k = \frac{7}{5}$$

$\therefore$ (*ii*) becomes $$\frac{dY}{dX} = \frac{2X - Y}{X + 2Y} \quad ...(iii)$$

Put $Y = vX$, then $$\frac{dY}{dX} = v + X\frac{dv}{dX}$$

$\therefore$ From (*iii*), $$v + X\frac{dv}{dX} = \frac{2X - vX}{X + 2vX} = \frac{2-v}{1+2v}$$

or $$X\frac{dv}{dX} = \frac{2-v}{1+2v} - v = \frac{2(1-v-v^2)}{1+2v}$$

or $$\frac{2v+1}{v^2+v-1}\,dv = -\frac{2}{X}\,dX$$

Integrating both sides,

$$\log(v^2 + v - 1) = -2\log X + \log c$$

or $$\log(v^2 + v - 1) = -\log X^2 + \log c = \log\frac{c}{X^2}$$

or $$v^2 + v - 1 = \frac{c}{X^2}$$

or $$\frac{Y^2}{X^2} + \frac{Y}{X} - 1 = \frac{c}{X^2} \quad \text{or} \quad Y^2 + XY - X^2 = c$$

or $$(y-k)^2 + (x-h)(y-k) - (x-h)^2 = c$$

or $$\left(y - \frac{7}{5}\right)^2 + \left(y - \frac{1}{5}\right)\left(y - \frac{7}{5}\right) - \left(x - \frac{1}{5}\right)^2 = c$$

or $$y^2 - \frac{14}{5}y + \frac{49}{25} + xy - \frac{7}{5}x - \frac{1}{5}y + \frac{7}{25} - x^2 + \frac{2}{5}x - \frac{1}{25} = c$$

or $$y^2 + xy - x^2 - x - 3y = c - \frac{11}{5}$$

or $$y^2 + xy - x^2 - x - 3y = A$$

where A is an arbitrary constant, is the reqd. solution.

(*ii*) The given equation is

$$\frac{dy}{dx} = -\frac{12x + 21y - 9}{47x + 40y + 7} \quad ...(i)$$

$$\left[\text{Here } a = 12, a' = 47, b = 21, b' = 40 \therefore \frac{a}{a'} \neq \frac{b}{b'}\right]$$

Put $\qquad x = X + h, y = Y + k$

then $\qquad dx = dX, dy = dY$

$\therefore$ From (*i*), $\dfrac{dY}{dX} = -\dfrac{12(X+h)+21(Y+k)-9}{47(X+h)+40(Y+k)+7} = -\dfrac{(12X+21Y)+(12h+21k-9)}{(47X+40Y)+(47h+40k+7)}$ ...(*ii*)

Choose $h$ and $k$ such that

$$12h + 21k - 9 = 0 \quad \text{and} \quad 4h + 40k + 7 = 0$$

$\Rightarrow \qquad h = -1, k = 1$

$\therefore$ (*ii*) becomes $\qquad \dfrac{dY}{dX} = -\dfrac{12X+21Y}{47X+40Y}$ ...(*iii*)

Put $\qquad Y = vX$

then $\qquad \dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$\therefore$ From (*iii*), $\qquad v + X\dfrac{dv}{dX} = -\dfrac{12X+21vX}{47X+40vX} = -\dfrac{12+21v}{47+40v}$

or $\qquad X\dfrac{dv}{dX} = -\dfrac{12+21v}{47+40v} - v = -\dfrac{12+68v+40v^2}{47+40v}$

or $\qquad \dfrac{47+40v}{12+68v+40v^2}\, dv = -\dfrac{dX}{X}$

or $\qquad \dfrac{40v+47}{10v^2+17v+3}\, dv = -\dfrac{4dX}{X}$

or $\qquad \dfrac{40v+47}{(2v+3)(5v+1)}\, dv = -\dfrac{4dX}{X}$

or $\qquad \left(\dfrac{2}{2v+3} + \dfrac{15}{5v+1}\right) dv = -\dfrac{4dX}{X}$ (Partial fractions)

Integrating both sides, we get

$$\log(2v+3) + 3\log(5v+1) = -4\log X + \log c$$

$\Rightarrow \qquad \log(2v+3)(5v+1)^3 = \log\dfrac{c}{X^4}$

$\Rightarrow \qquad (2v+3)(5v+1)^3 = \dfrac{c}{X^4} \qquad \Rightarrow \qquad \left(2\dfrac{Y}{X}+3\right)\left(5\dfrac{Y}{X}+1\right)^3 = \dfrac{c}{X^4}$

$\Rightarrow \qquad (3X + 2Y)(X + 5Y)^3 = c$

$\Rightarrow \qquad [3(x+1) + 2(y-1)]\,[(x+1) + 5(y-1)]^3 = c$

$\Rightarrow \qquad (3x + 2y + 1)(x + 5y - 4)^3 = c$

is the reqired solution.

**Example 4.** *Solve the following differential equations* :

(*i*) $(2x - y + 1)\, dx + (2y - x - 1)\, dy = 0$ $\quad$ (*ii*) $(3y - 7x - 3)\, dx + (7y - 3x - 7)\, dy = 0$

(*iii*) $\dfrac{dy}{dx} = \dfrac{2x+9y-20}{6x+2y-10}$.

**Sol.** (*i*) The given equation is

$$(2x - y + 1)\, dx + (2y - x - 1)\, dy = 0$$

or $$\frac{dy}{dx} = \frac{2x - y + 1}{x - 2y + 1} \qquad ...(1)$$

$$\left[\text{Here } a = 2, b = -1, a' = 1, b' = -2 \therefore \ \frac{a}{a'} \neq \frac{b}{b'}\right]$$

Put $$x = X + h \text{ and } y = Y + k$$

so that $$dx = dX \text{ and } dy = dY$$

$\therefore$ from (1), $$\frac{dY}{dX} = \frac{2(X+h) - (Y+k) + 1}{(X+h) - 2(Y+k) + 1} = \frac{(2X - Y) + (2h - k + 1)}{(X - 2Y) + (h - 2k + 1)} \qquad ...(2)$$

Choose $h$ and $k$ such that

$$2h - k + 1 = 0 \quad \text{and} \quad h - 2k + 1 = 0$$

$\Rightarrow$ $$h = -\tfrac{1}{3}, \ k = \tfrac{1}{3}$$

$\therefore$ From (2), $$\frac{dY}{dX} = \frac{2X - Y}{X - 2Y} \qquad ...(3)$$

Put $y = vX$ so that $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$\therefore$ From (3), $$v + X\frac{dv}{dX} = \frac{2X - vX}{X - 2vX}$$

or $$X\frac{dv}{dX} = \frac{2 - v}{1 - 2v} - v = \frac{2 - 2v + 2v^2}{1 - 2v}$$

or $$\frac{2v - 1}{v^2 - v + 1}\, dv = -2\, \frac{dX}{X}$$

Integrating both sides,

$$\log (v^2 - v + 1) = -2 \log X + \log c$$

or $$\log\left(\frac{Y^2}{X^2} - \frac{Y}{X} + 1\right) = \log \frac{c}{X^2} \quad \text{or} \quad \frac{Y^2 - XY + X^2}{X^2} = \frac{c}{X^2}$$

or $$(y - k)^2 - (x - h)(y - k) + (x - h)^2 = c$$

or $$(y - \tfrac{1}{3})^2 - (x + \tfrac{1}{3})(y - \tfrac{1}{3}) + (x + \tfrac{1}{3})^2 = c$$

or $$x^2 - xy + y^2 + x - y + \tfrac{1}{3} = c$$

or $$x^2 - xy + y^2 + x - y + C = 0, \qquad \text{where } C = \tfrac{1}{3} - c.$$

(*ii*) The given equation is

$$(3y - 7x - 3)\, dx + (7y - 3x - 7)\, dy = 0$$

or $$\frac{dy}{dx} = -\frac{7x - 3y + 3}{3x - 7y + 7} \qquad ...(1)$$

$$\left[\text{Here } a = 7, b = -3, a' = 3, b' = -7 \therefore \ \frac{a}{a'} \neq \frac{b}{b'}\right]$$

Put $x = X + h$ and $y = Y + k$

so that $dx = dX$ and $dy = dY$

$\therefore$ From (1), $$\frac{dY}{dX} = -\frac{7(X+h)-3(Y+k)+3}{3(X+h)-7(Y+k)+7} = -\frac{(7X-3Y)+(7h+3k+3)}{(3X-7Y)+(3h-7k+7)} \quad ...(2)$$

Choose $h$ and $k$ such that

$$7h - 3k + 3 = 0 \quad \text{and} \quad 3h - 7k + 7 = 0 \quad \Rightarrow \quad h = 0, k = 1$$

$\therefore$ From (2), $$\frac{dY}{dX} = -\frac{7X-3Y}{3X-7Y} \quad ...(3)$$

Put $Y = vX$ so that $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$\therefore$ From (3), $v + X\dfrac{dv}{dX} = -\dfrac{7X-3vX}{3X-7vX}$

or $$X\frac{dv}{dX} = -\frac{7-3v}{3-7v} - v = -\frac{7-7v^2}{3-7v}$$

or $$\frac{7v-3}{(v+1)(v-1)}\,dv = -7\frac{dX}{X}$$

or $$\left[\frac{5}{v+1} + \frac{2}{v-1}\right] dv = -7\frac{dX}{X}$$

Integrating $5 \log (v+1) + 2 \log (v-1) = -7 \log X + \log c$

or $\log (v+1)^5(v-1)^2 = \log \dfrac{c}{X^7}$ or $\left(\dfrac{Y}{X}+1\right)^5 \left(\dfrac{Y}{X}-1\right)^2 = \dfrac{c}{X^7}$

or $(Y+X)^5 (Y-X)^2 = c$ or $(y-k+x-h)^5 [(y-k)-(x-h)]^2 = c$

or $(x+y-1)^5 (x-y+1)^2 = c.$

(*iii*) The given equation is

$$\frac{dy}{dx} = \frac{2x+9y-20}{6x+2y-10} \quad ...(1)$$

$\left[\text{Here } a = 2, b = 9, a' = 6, b' = 2 \therefore \dfrac{a}{a'} \neq \dfrac{b}{b'}\right]$

Put $x = X + h$ and $y = Y + k$

so that $dx = dX$ and $dy = dY$

$\therefore$ From (1), $$\frac{dY}{dX} = \frac{2(X+h)+9(Y+k)-20}{6(X+h)+2(Y+k)-10} = \frac{(2X+9Y)+(2h+9k-20)}{(6X+2Y)+(6h+2k-10)} \quad ...(2)$$

Choose $h$ and $k$ such that

$$2h + 9k - 20 = 0 \quad \text{and} \quad 6h + 2k - 10 = 0$$

$\Rightarrow$ $h = 1, k = 2$

$\therefore$ From (2), $$\frac{dY}{dX} = \frac{2X+9Y}{6X+2Y} \quad ...(3)$$

Put $Y = vX$ so that $\dfrac{dY}{dX} = v + X\dfrac{dv}{dX}$

$\therefore$ From (3), $v + X\dfrac{dv}{dX} = \dfrac{2X+9vX}{6X+2vX}$

or $$X\frac{dv}{dX} = \frac{2+9v}{6+2v} - v = \frac{2+3v-2v^2}{2(3+v)}$$

or $$\frac{3+v}{(1+2v)(2-v)}\,dv = \frac{1}{2}\cdot\frac{dX}{X} \quad \text{or} \quad \left(\frac{1}{1+2v}+\frac{1}{2-v}\right)dv = \frac{1}{2}\cdot\frac{dX}{X}$$

Integrating $\frac{1}{2}\log(1+2v) - \log(2-v) = \frac{1}{2}\log X + c'$

or $\log(1+2v) - 2\log(2-v) = \log X + 2c' = \log X + \log c$, where $\log c = 2c'$

or $$\log\frac{1+2v}{(2-v)^2} = \log cX \quad \text{or} \quad \frac{1+2v}{(2-v)^2} = cX$$

or $$1 + \frac{2Y}{X} = cX\left(2 - \frac{Y}{X}\right)^2 \quad \text{or} \quad X + 2Y = c\,(2X - Y)^2$$

or $$(x-h) + 2(y-k) = c\,[2(x-h) - (y-k)]^2$$

or $$x + 2y - 5 = c\,(2x - y)^2.$$

**Case of Failure :** $\dfrac{a}{a'} = \dfrac{b}{b'}$ .

**Example 1.** *Solve* :

(*i*) $(4x + 6y + 3)\,dx = (6x + 9y + 2)\,dy$ (*ii*) $\dfrac{dy}{dx} = \dfrac{x+y+1}{2x+2y+3}$ (Lucknow, 1998)

(*iii*) $(2x + 3y + 4)dx - (4x + 6y + 5)\,dy = 0.$

**Sol.** (*i*) Given equation is $\dfrac{dy}{dx} = \dfrac{4x+6y+3}{6x+9y+2}$ or $\dfrac{dy}{dx} = \dfrac{2(2x+3y)+3}{3(2x+3y)+2}$ ...(*i*)

Put $2x + 3y = z$, then $2 + 3\dfrac{dy}{dx} = \dfrac{dz}{dx}$ or $\dfrac{dy}{dx} = \dfrac{1}{3}\left[\dfrac{dz}{dx} - 2\right]$

$\therefore$ From (*i*), $\dfrac{1}{3}\left[\dfrac{dz}{dx} - 2\right] = \dfrac{2z+3}{3z+2}$

$$\frac{dz}{dx} - 2 = \frac{6z+9}{3z+2} \quad \text{or} \quad \frac{dz}{dx} = \frac{6z+9}{3z+2} + 2 = \frac{12z+13}{3z+2}$$

Separating the variables, $\dfrac{3z+2}{12z+13}\,dz = dx$

or $$\left[\frac{1}{4} - \frac{5}{4}\cdot\frac{1}{12z+13}\right]dz = dx$$

$$\begin{array}{r|l} 12z+13 & 3z+2\ \left(\tfrac{1}{4}\right. \\ & 3z+\tfrac{13}{4} \\ \hline & -\tfrac{5}{4} \end{array}$$

Integrating both sides,

$$\frac{z}{4} - \frac{5}{4}\cdot\frac{1}{12}\log(12z+13) = x + c_1$$

or $$12z - 5\log(12z+13) = 48x + 48c_1$$

or $12(2x + 3y) - 5\log(24x + 36y + 13) = 48x + c$

or $12(3y - 2x) - 5\log(24x + 36y + 13) = c$ is the required solution.

(*ii*) Given equation is $\frac{dy}{dx} = \frac{x+y+1}{2x+2y+3} = \frac{(x+y)+1}{2(x+y)+3}$ ...(*i*)

Put $x + y = z$, then $1 + \frac{dy}{dx} = \frac{dz}{dx}$ or $\frac{dy}{dx} = \frac{dz}{dx} - 1$

$\therefore$ From (*i*), $\frac{dz}{dx} - 1 = \frac{z+1}{2z+3}$ or $\frac{dz}{dx} = \frac{3z+4}{2z+3}$

Separating the variables, $\frac{2z+3}{3z+4}\,dz = dx$

$$3z+4\,\overline{)\,2z+3\,}\left(\tfrac{2}{3}\right.$$
$$\frac{2z+\frac{8}{3}}{\frac{1}{3}}$$

or $\left[\frac{2}{3} + \frac{1}{3}\cdot\frac{1}{3z+4}\right] dz = dx$

Integrating both sides, $\frac{2}{3}z + \frac{1}{3}\cdot\frac{1}{3}\log(3z+4) = x + c_1$

or $6z + \log(3z+4) = 9x + 9c_1$

or $6(x+y) + \log(3x+3y+4) = 9x + c$

or $3(2y - x) + \log(3x+3y+4) = c$

is the required solution.

(*iii*) Given equation is $\frac{dy}{dx} = \frac{2x+3y+4}{4x+6y+5} = \frac{2x+3y+4}{2(2x+3y)+5}$ ...(*i*)

Put $2x + 3y = z$, then

$$2 + 3\frac{dy}{dx} = \frac{dz}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{3}\left[\frac{dz}{dx} - 2\right]$$

$\therefore$ From (*i*), $\frac{1}{3}\left[\frac{dz}{dx} - 2\right] = \frac{z+4}{2z+5}$

or $\frac{dz}{dx} - 2 = \frac{3z+12}{2z+5}$ or $\frac{dz}{dx} = \frac{7z+22}{2z+5}$

Separating the variables, $\frac{2z+5}{7z+22}\,dz = dx$

$$7z+22\,\overline{)\,2z+5\,}\left(\tfrac{2}{7}\right.$$
$$\frac{2z+\frac{44}{7}}{-\frac{9}{7}}$$

or $\left[\frac{2}{7} - \frac{9}{7}\cdot\frac{1}{7z+22}\right] dz = dx$

Integrating both sides, $\frac{2}{7}z - \frac{9}{7}\cdot\frac{1}{7}\log(7z+22) = x + c_1$

or $14z - 9\log(7z+22) = 49x + 49c_1$

or $14(2x+3y) - 9\log(14x+21y+22) = 49x + c$

or $21(2y - x) - 9\log(14x+21y+22) = c$

or $7(2y - x) - 3\log(14x+21y+22) = \frac{1}{3}c = A$

which is the required solution.

**Example 2.** *Solve* :

(*i*) $\dfrac{dy}{dx} = \dfrac{x + 2y + 1}{2x + 4y + 3}$ (*ii*) $(2x - 2y + 5)\dfrac{dy}{dx} = x - y + 3$

(*iii*) $(2x - 4y + 5)\,dy = (x - 2y + 3)\,dx$ (*iv*) $(2x + y + 1)\,dx + (4x + 2y - 1)\,dy = 0.$

**Sol.** (*i*) Given equation is $\dfrac{dy}{dx} = \dfrac{(x + 2y) + 1}{2(x + 2y) + 3}$ ...(*i*)

Put $x + 2y = z$, then

$$1 + 2\frac{dy}{dx} = \frac{dz}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{2}\left[\frac{dz}{dx} - 1\right]$$

$\therefore$ From (*i*),
$$\frac{1}{2}\left[\frac{dz}{dx} - 1\right] = \frac{z + 1}{2z + 3} \quad \text{or} \quad \frac{dz}{dx} - 1 = \frac{2z + 2}{2z + 3}$$

$$\therefore \quad \frac{dz}{dx} = \frac{2z + 2}{2z + 3} + 1 = \frac{4z + 5}{2z + 3}$$

Separating the variables $\dfrac{2z + 3}{4z + 5}\,dz = dx$

$4z + 5 \,)\, 2z + 3 \,(\, \frac{1}{2}$
$2z + \frac{5}{2}$
$\frac{1}{2}$

or
$$\left[\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{4z + 5}\right] dz = dx$$

Integrating both sides, $\dfrac{1}{2}z + \dfrac{1}{2}\cdot\dfrac{1}{4}\log(4z + 5) = x + c_1$

or
$$4z + \log(4z + 5) = 8x + 8c_1$$
or
$$4(x + 2y) + \log(4x + 8y + 5) = 8x + c$$
or
$$4(2y - x) + \log(4x + 8y + 5) = c$$
is the required solution.

(*ii*) Given equation is $\dfrac{dy}{dx} = \dfrac{x - y + 3}{2(x - y) + 5}$ ...(*i*)

Put $x - y = z$, then

$$1 - \frac{dy}{dx} = \frac{dz}{dx} \quad \text{or} \quad \frac{dy}{dx} = 1 - \frac{dz}{dx}$$

$\therefore$ From (*i*),
$$1 - \frac{dz}{dx} = \frac{z + 3}{2z + 5} \quad \text{or} \quad \frac{dz}{dx} = 1 - \frac{z + 3}{2z + 5} = \frac{z + 2}{2z + 5}$$

Separating the variables,

$$\frac{2z + 5}{z + 2}\,dz = dx \quad \text{or} \quad \left(2 + \frac{1}{z + 2}\right) dz = dx$$

Integrating both sides

$$2z + \log(z + 2) = x + c$$
or
$$2(x - y) + \log(x - y + 2) = x + c$$
or
$$x - 2y + \log(x - y + 2) = c$$ is the required solution.

(*iii*) Given equation is

$$\frac{dy}{dx} = \frac{x - 2y + 3}{2x - 4y + 5} = \frac{x - 2y + 3}{2(x - 2y) + 5} \qquad ...(i)$$

Put $x - 2y = z$, then $1 - 2\frac{dy}{dx} = \frac{dz}{dx}$

or $$\frac{dy}{dx} = \frac{1}{2}\left(1 - \frac{dz}{dx}\right)$$

$\therefore$ From (*i*), $\frac{1}{2}\left(1 - \frac{dz}{dx}\right) = \frac{z+3}{2z+5}$ or $1 - \frac{dz}{dx} = \frac{2z+6}{2z+5}$

or $$\frac{dz}{dx} = 1 - \frac{2z+6}{2z+5} = -\frac{1}{2z+5}$$

Separating the variables,

$$(2z + 5)\, dz = -\, dx$$

Integrating, $z^2 + 5z = -x + c$

or $(x - 2y)^2 + 5(x - 2y) = -x + c$

or $x^2 - 4xy + 4y^2 + 6x - 10y = c$

which is the required solution.

(*iv*) Please try yourself. [**Ans.** $x + 2y + \log(2x + y - 1) = c$]

**Example 3.** *Solve* :

$$\left(\frac{x+y-1}{x+y-2}\right)\frac{dy}{dx} = \frac{x+y+1}{x+y+2}.$$

**Sol.** Put $x + y = z$

then $1 + \frac{dy}{dx} = \frac{dz}{dx} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{dz}{dx} - 1$

The given equation becomes

$$\left(\frac{z-1}{z-2}\right)\left(\frac{dz}{dx} - 1\right) = \frac{z+1}{z+2}$$

$\Rightarrow$ $$\frac{dz}{dx} - 1 = \frac{(z+1)(z-2)}{(z+2)(z-1)} = \frac{z^2 - z - 2}{z^2 + z - 2}$$

$\Rightarrow$ $$\frac{dz}{dx} = \frac{z^2 - z - 2}{z^2 + z - 2} + 1 = \frac{2z^2 - 4}{z^2 + z - 2}$$

$\Rightarrow$ $\frac{z^2 + z - 2}{z^2 - 2}\, dz = 2dx \quad \Rightarrow \quad \left(1 + \frac{z}{z^2 - 2}\right) dz = 2dx$

Integrating $z + \frac{1}{2}\int \frac{2z}{z^2 - 2}\, dz = 2x + c \quad \Rightarrow \quad z + \frac{1}{2}\log(z^2 - 2) = 2x + c$

$\Rightarrow \quad x + y + \frac{1}{2}\log[(x + y)^2 - 2] = 2x + c \quad \Rightarrow \quad \frac{1}{2}\log[(x + y)^2 - 2] = x - y + c.$

**Example 4.** *Solve the following differential equations* :

(*i*) $\frac{dy}{dx} = \frac{x+y+7}{2x+2y+3}$ (*ii*) $(x + y)(dx - dy) = dx + dy.$

**Sol.** (*i*) The given equation is

$$\frac{dy}{dx} = \frac{(x+y)+7}{2(x+y)+3} \qquad ...(1)$$

Put $x + y = z$ so that $\quad 1 + \frac{dy}{dx} = \frac{dz}{dx} \quad$ or $\quad \frac{dy}{dx} = \frac{dz}{dx} - 1$

$\therefore$ From (1), $\quad \frac{dz}{dx} - 1 = \frac{z+7}{2z+3}$

or $\quad \frac{dz}{dx} = \frac{z+7}{2z+3} + 1 = \frac{3z+10}{2z+3}$

or $\quad \frac{2z+3}{3z+10}\, dz = dx \quad$ or $\quad \left[\frac{2}{3} - \frac{11}{3(3z+10)}\right] dz = dx$

Integrating $\quad \frac{2}{3} z - \frac{11}{9} \log (3z + 10) = x + c'$

or $\quad 6(x + y) - 11 \log [3(x + y) + 10] = 9x + 9c'$

or $\quad 3(2y - x) - 11 \log (3x + 3y + 10) = c \quad$ where $c = 9c'$.

(*ii*) The given equation is

$$(x + y)(dx - dy) = dx + dy$$

or $\quad (x + y - 1)\, dx = (x + y + 1)\, dy \quad$ or $\quad \frac{dy}{dx} = \frac{x+y-1}{x+y+1}$

Put $x + y = z$ and proceed further yourself. [**Ans.** $(y - x) + \log (x + y) + c$]

## TYPE IV. LINEAR DIFFERENTIAL EQUATION

**Def.** A linear differential equation is that in which the dependent variable and its differential co-efficient occur only in the first degree and are not multiplied together.

Thus, the standard form of a linear differential equation of the first order is $\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$, where P and Q are functions of $x$ or constants (*i.e.*, independent of $y$).

### Art. To solve the equation $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only

The given equation is $\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$

Multiplying throughout by $e^{\int \mathrm{P}\,dx}$, we get

$$\frac{dy}{dx} \cdot e^{\int \mathrm{P}\,dx} + \mathrm{P}y \cdot e^{\int \mathrm{P}\,dx} = \mathrm{Q} \cdot e^{\int \mathrm{P}\,dx} \qquad ...(i)$$

Now $\quad \frac{d}{dx}\left[ye^{\int \mathrm{P}\,dx}\right] = \frac{dy}{dx} \cdot e^{\int \mathrm{P}\,dx} + y \cdot \frac{d}{dx}\left[e^{\int \mathrm{P}\,dx}\right]$

$$= \frac{dy}{dx} \cdot e^{\int \mathrm{P}\,dx} + y \cdot e^{\int \mathrm{P}\,dx} \cdot \frac{d}{dx}\left[\int \mathrm{P}\,dx\right]$$

$$\left[\because \ \frac{d}{dx}\left\{e^{f(x)}\right\} = e^{f(x)} \cdot \frac{d}{dx}\{f(x)\}\right]$$

$$= \frac{dy}{dx} \cdot e^{\int P\,dx} + y \cdot e^{\int P\,dx} \cdot P = \frac{dy}{dx} \cdot e^{\int P\,dx} + Py \cdot e^{\int P\,dx}$$

$\therefore$ From (*i*), $\frac{d}{dx}\,[y \cdot e^{\int P\,dx}] = Q \cdot e^{\int P\,dx}$

Integrating both sides w.r.t. $x$, we have

$$y \cdot e^{\int P\,dx} = \int Q \cdot e^{\int P\,dx}\,dx + c$$

which is the required solution of the given linear differential equation.

**Note 1.** The factor $e^{\int P\,dx}$, on multiplying by which the L.H.S. of the differential equation becomes the differential co-efficient of some function of $x$ and $y$, is called an integrating factor of the differential equation and is shortly written as I.F.

**Note 2. The solution of the linear equation $\frac{dy}{dx} + Py = Q$, where P and Q are functions of x only, is**

$$\mathbf{y(I.F.) = \int Q(I.F.)\,dx + c} \qquad \text{| Remember}$$

**Note 3.** Sometimes a differential equation becomes linear if we take $y$ as the independent variable and $x$ as dependent variable. In that case, the equation can be put in the form **$\frac{dx}{dy} + Px = Q$, where P and Q are functions of y (and not of x) or constants.**

**I.F. (in this case) = $e^{\int P\,dy}$, and the solution is**

$$\mathbf{x(I.F.) = \int Q.\,(I.F.)\,dy + c.} \qquad \text{| Remember}$$

**Note 4. While evaluating the I.F., it is very useful to remember that $e^{\log f(x)} = f(x)$.**

**Thus** $\qquad e^{\log x^2} = x^2.$

**Note 5. The co-efficient of $\frac{dy}{dx}$, if not unity, must be made unity by dividing throughout by it.**

**Example 1.** *Solve the following* :

(*i*) $(1 + x^2)\,\frac{dy}{dx} + 2xy = 4x^2$ $\qquad$ (*ii*) $\frac{dy}{dx} + y \sec x = \tan x$

(*iii*) $\frac{dy}{dx} + \frac{y}{x} = x^2$ $\qquad$ (*iv*) $\frac{dy}{dx} = y \tan x - 2 \sin x.$

**Sol.** (*i*) Given equation is $(1 + x^2)\,\frac{dy}{dx} + 2xy = 4x^2$

Dividing throughout by $1 + x^2$, $\left(\text{to make the co-efficient of } \frac{dy}{dx} \text{ unity.}\right)$

$$\frac{dy}{dx} + \frac{2x}{1+x^2} \cdot y = \frac{4x^2}{1+x^2} \qquad \text{...(i)}$$

It is of the form $\qquad \frac{dy}{dx} + Py = Q$

Here $\qquad P = \frac{2x}{1+x^2},\ Q = \frac{4x^2}{1+x^2}$

$\therefore$ $\quad$ I.F. $= e^{\int P\,dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$

Hence the solution is

$$y\,.\,(\text{I.F.}) = \int Q\,.\,(\text{I.F.})\,dx + c$$

or $$y(1+x^2) = \int \frac{4x^2}{1+x^2}\,.\,(1+x^2)\,dx + c$$

or $$y(1+x^2) = \int 4x^2 dx + c$$

or $$y(1+x^2) = \frac{4x^3}{3} + c.$$

(*ii*) Given equation is $\dfrac{dy}{dx} + (\sec x)\,.\,y = \tan x$

It is of the form $\dfrac{dy}{dx} + \text{P}y = \text{Q}$

Here $\quad$ P $= \sec x$, $\quad$ Q $= \tan x$

$\therefore$ $\quad$ I.F. $= e^{\int P\,dx} = e^{\int \sec x\,dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$

Hence the solution is

$$y\,.\,(\text{I.F.}) = \int \text{Q}.\,(\text{I.F.})\,dx + c$$

or $$y\,(\sec x + \tan x) = \int \tan x\,(\sec x + \tan x)\,dx + c$$

$$= \int \sec x \tan x\,dx + \int \tan^2 x\,dx + c$$

$$= \sec x + \int (\sec^2 x - 1)\,dx + c = \sec x + \tan x - x + c.$$

**Another Form of Above Question**

*Solve* $\quad \cos x\,\dfrac{dy}{dx} + y = \sin x$

[**Hint.** Divide throughout by $\cos x$]

(*iii*) Given equation is $\dfrac{dy}{dx} + \dfrac{1}{x}\,.\,y = x^2$

It is of the form $\dfrac{dy}{dx} + \text{P}y = \text{Q}$

Here $\quad$ P $= \dfrac{1}{x}$, $\quad$ Q $= x^2$

$\therefore$ $\quad$ I.F. $= e^{\int P\,dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

Hence the solution is

$$y(\text{I.F.}) = \int \text{Q}\,(\text{I.F.})\,dx + c$$

or $$yx = \int x^2\,.\,x dx + c \quad \text{or} \quad xy = \frac{x^4}{4} + c.$$

(*iv*) Given equation is $\dfrac{dy}{dx} - (\tan x) \cdot y = -2 \sin x$

It is of the form $\dfrac{dy}{dx} + \text{P}y = \text{Q}$

Here $\text{P} = -\tan x, \text{Q} = -2 \sin x$

$$\therefore \quad \text{I.F.} = e^{\int \text{P}\, dx} = e^{-\int \tan x\, dx} = e^{-(-\log \cos x)}$$
$$= e^{\log \cos x} = \cos x$$

Hence the solution is

$$y\,(\text{I.F.}) = \int \text{Q} \cdot (\text{I.F.})\, dx + c$$

or
$$y \cos x = \int -2 \sin x \cos x\, dx + c$$
$$= -\int \sin 2x\, dx + c = -\frac{-\cos 2x}{2} + c$$

or
$$y \cos x = \tfrac{1}{2} \cos 2x + c.$$

**Example 2.** *Solve the following* :

(*i*) $\sec x \dfrac{dy}{dx} = y + \sin x$ (*ii*) $\dfrac{dy}{dx} + y \tan x = \sec x$

(*iii*) $(1 + x^2) \dfrac{dy}{dx} + 2xy = \cos x$ (Meerut, 1998) (*iv*) $x \log x \dfrac{dy}{dx} + y = 2 \log x$ (Delhi, 1996)

(*v*) $\dfrac{dy}{dx} = \dfrac{x + y + 1}{x + 1}$.

**Sol.** (*i*) Given equation is $\sec x \cdot \dfrac{dy}{dx} - y = \sin x$

Dividing throughout by $\sec x$, to make the co-efficient of $\dfrac{dy}{dx}$ unity,

$$\frac{dy}{dx} - (\cos x) \cdot y = \sin x \cos x$$

It is of the form $\dfrac{dy}{dx} + \text{P}y = \text{Q}$

Here $\text{P} = -\cos x, \quad \text{Q} = \sin x \cos x$

$$\therefore \quad \text{I.F.} = e^{\int \text{P}\, dx} = e^{\int -\cos x\, dx} = e^{-\sin x}$$

Hence the solution is

$$y \cdot (\text{I.F.}) = \int \text{Q} \cdot (\text{I.F.})\, dx + c$$

or
$$y \cdot e^{-\sin x} = \int \sin x \cos x \cdot e^{-\sin x}\, dx + c = \int te^{-t}\, dt + c, \text{ where } t = \sin x$$
$$= t \cdot \frac{e^{-t}}{-1} - \int 1 \cdot \frac{e^{-t}}{-1}\, dt + c = -te^{-t} - e^{-t} + c$$
$$= -e^{-t}(t + 1) + c = -e^{-\sin x}(\sin x + 1) + c$$

or
$$y = -(\sin x + 1) + c\, e^{\sin x}.$$

(*ii*) Given equation is $\frac{dy}{dx} + (\tan x) \cdot y = \sec x$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $P = \tan x$ ; $Q = \sec x$

$\therefore$ $\quad$ $\text{I.F.} = e^{\int P\,dx} = e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x$

Hence the solution is

$$y \cdot (\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c$$

or $$y \sec x = \int \sec x \cdot \sec x\,dx + c = \int \sec^2 x\,dx + c$$

or $$y \sec x = \tan x + c \qquad \text{or} \qquad y = \sin x + c \cos x.$$

(*iii*) Given equation is $(1 + x^2)\frac{dy}{dx} + 2xy = \cos x$

Dividing throughout by $(1 + x^2)$ to make the co-efficient of $\frac{dy}{dx}$ unity,

$$\frac{dy}{dx} + \frac{2x}{1+x^2} \cdot y = \frac{1}{1+x^2}\cos x$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $P = \frac{2x}{1+x^2}$, $Q = \frac{1}{1+x^2}\cos x$

$\therefore$ $\quad$ $\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{2x}{1+x^2}dx} = e^{\log(1+x^2)} = 1 + x^2$

Hence the solution is

$$y \cdot (\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c$$

or $$y(1 + x^2) = \int \frac{1}{1+x^2}\cos x \cdot (1 + x^2)\,dx + c$$

$$= \int \cos x\,dx + c \quad \text{or} \quad y(1 + x^2) = \sin x + c.$$

(*iv*) Given equation is $x \log x \frac{dy}{dx} + y = 2 \log x$

Dividing throughout by $x \log x$ to make the co-efficient of $\frac{dy}{dx}$ unity,

$$\frac{dy}{dx} + \frac{1}{x \log x} \cdot y = \frac{2}{x}.$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $P = \frac{1}{x \log x}$, $Q = \frac{2}{x}$

$\therefore$ $\quad$ $\text{I.F.} = e^{\int P\,dx} = e^{\int \frac{1}{x \log x}dx} = e^{\int \frac{1/x}{\log x}dx} = e^{\log \log x} = \log x$

Hence the solution is

$$y \cdot (\text{I.F.}) = \int Q.\,(\text{I.F.})\,dx + c$$

or $$y \log x = \int \frac{2}{x} \log x \, dx + c$$

or $$y \log x = 2 \int \frac{1}{x} . \log x \, dx + c = 2 . \frac{(\log x)^2}{2} + c$$ $\because \int [f(x)]^n f'(x) \, dx = \frac{[f(x)]^{n+1}}{n+1}, n \neq -1$

or $$y \log x = (\log x)^2 + c.$$

(*v*) Given equation is $\frac{dy}{dx} = \frac{x+1+y}{x+1} = 1 + \frac{y}{x+1}$ | **Note**

or $$\frac{dy}{dx} - \frac{1}{x+1} . y = 1$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $P = -\frac{1}{x+1}, \quad Q = 1$

$\therefore$ $$\text{I.F.} = e^{\int P\,dx} = e^{-\int \frac{1}{x+1} dx} = e^{-\log(x+1)} = e^{\log(x+1)^{-1}}$$

$$= (x+1)^{-1} = \frac{1}{x+1}$$

Hence the solution is

$$y . (\text{I.F.}) = \int Q . (\text{I.F.}) \, dx + c$$

or $$y . \frac{1}{x+1} = \int \frac{1}{x+1} dx + c$$

or $$\frac{y}{x+1} = \log (x+1) + c.$$

**Example 3.** *Solve the following* :

(*i*) $(x+1) \frac{dy}{dx} - ny = e^x (x+1)^{n+1}$ (*ii*) $\cos^2 x \frac{dy}{dx} + y = \tan x.$

**Sol.** (*i*) Given equation is $(x+1) \frac{dy}{dx} - ny = e^x (x+1)^{n+1}$

Dividing throughout by $(x + 1)$ to make of the co-eff. of $\frac{dy}{dx}$ unity,

$$\frac{dy}{dx} - \frac{n}{x+1} . y = e^x (x+1)^n$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $P = -\frac{n}{n+1}, \quad Q = e^x (x+1)^n$

$\therefore$ $$\text{I.F.} = e^{\int P\,dx} = e^{-n\int \frac{1}{x+1} dx} = e^{-n\log(x+1)} = e^{\log(x+1)^{-n}}$$

$$= (x+1)^{-n} = \frac{1}{(x+1)^n}$$

Hence the solution is

$$y \cdot \frac{1}{(x+1)^n} = \int e^x (x+1)^n \cdot \frac{1}{(x+1)^n}\, dx + c$$

$$= \int e^x\, dx + c \qquad \text{or} \qquad y = (x+1)^n (e^x + c)$$

(*ii*) Given equation is $\cos^2 x \dfrac{dy}{dx} + y = \tan x$

Dividing throughout by $\cos^2 x$ to make the co-eff. of $\dfrac{dy}{dx}$ unity,

$$\frac{dy}{dx} + (\sec^2 x) \cdot y = \sec^2 x \tan x$$

It is of the form $\qquad \dfrac{dy}{dx} + \text{P}y = \text{Q}$

Here $\qquad \text{P} = \sec^2 x, \quad \text{Q} = \sec^2 x \tan x$

$\therefore \qquad \text{I.F.} = e^{\int \text{P}\, dx} = e^{\int \sec^2 x\, dx} = e^{\tan x}$

Hence the solution is

$$y \cdot e^{\tan x} = \int \sec^2 x \tan x \cdot e^{\tan x}\, dx + c$$

$$= \int t e^t\, dt + c, \text{ where } t = \tan x$$

$$= te^t - e^t + c = e^t (t-1) + c$$

$$= e^{\tan x} (\tan x - 1) + c$$

or $$y = \tan x - 1 + ce^{-\tan x}.$$

**Another Form of Above Question**

$$\cos^3 x \cdot \frac{dy}{dx} + y \cos x = \sin x$$

**Example 4.** *Solve the following :*

(*i*) $\dfrac{dy}{dx} + \dfrac{2x}{x^2+1} \cdot y = \dfrac{1}{(x^2+1)^2}$ *given that* $y = 0$, *when* $x = 1$

(*ii*) $(1 + x^2) \dfrac{dy}{dx} + y = \tan^{-1} x$

(*iii*) $\dfrac{dy}{dx} + 2y \tan x = \sin x$ *given that* $y = 0$, *when* $x = \dfrac{\pi}{3}$

(*iv*) $\dfrac{dy}{dx} + \dfrac{4x}{x^2+1} y = \dfrac{1}{(x^2+1)^3}$ $\qquad$ (*v*) $x \dfrac{dy}{dx} + 2y - x^2 \log x = 0.$

**Sol.** (*i*) Given equation is $\dfrac{dy}{dx} + \dfrac{2x}{x^2+1} \cdot y = \dfrac{1}{(x^2+1)^2}$

It is of the form $\qquad \dfrac{dy}{dx} + \text{P}y = \text{Q}$

Here $\qquad \text{P} = \dfrac{2x}{x^2+1}, \quad \text{Q} = \dfrac{1}{(x^2+1)^2}$

$\therefore \qquad \text{I.F.} = e^{\int \text{P}\, dx} = e^{\int \frac{2x}{x^2+1} dx} = e^{\log (x^2+1)} = x^2 + 1$

Hence the solution is

$$y(x^2+1) = \int \frac{1}{(x^2+1)^2} \cdot (x^2+1)\, dx + c = \int \frac{1}{x^2+1}\, dx + c$$

or $\quad y(x^2+1) = \tan^{-1} x + c$

when $\quad x = 1, y = 0$

$\therefore \quad 0 = \tan^{-1} 1 + c \quad$ or $\quad 0 = \dfrac{\pi}{4} + c \quad \therefore \quad c = -\dfrac{\pi}{4}$

$\therefore$ Required solution is $y(x^2+1) = \tan^{-1} x - \dfrac{\pi}{4}$.

(*ii*) Given equation is $(1+x^2)\dfrac{dy}{dx} + y = \tan^{-1} x$.

Dividing both sides by $(1+x^2)$ to make co-eff. of $\dfrac{dy}{dx}$ unity

$$\frac{dx}{dy} + \frac{1}{1+x^2} \cdot y = \frac{\tan^{-1} x}{1+x^2}$$

It is of the form $\quad \dfrac{dy}{dx} + Py = Q$

Here $\quad P = \dfrac{1}{1+x^2},\ Q = \dfrac{\tan^{-1} x}{1+x^2}$

$\therefore \quad$ I.F. $= e^{\int P\,dx} = e^{\int \frac{1}{1+x^2} dx} = e^{\tan^{-1} x}$

Hence the solution is

$$y \cdot e^{\tan^{-1} x} = \int \frac{\tan^{-1} x}{1+x^2} \cdot e^{\tan^{-1} x}\, dx + c$$

$$= \int t e^t\, dt + c, \text{ where } t = \tan^{-1} x$$

$$= t.e^t - 1\, e^t + c = e^t(t-1) + c = e^{\tan^{-1} x}(\tan^{-1} x - 1) + c$$

or $\quad y = \tan^{-1} x - 1 + c e^{-\tan^{-1} x}$.

(*iii*) Given equation is $\dfrac{dy}{dx} + (2\tan x) \cdot y = \sin x$

It is of the form $\dfrac{dy}{dx} + Py = Q$

Here $\quad P = 2\tan x,\ Q = \sin x$

$\therefore \quad$ I.F. $= e^{\int P\,dx} = e^{\int 2\tan x\,dx} = e^{2\log\sec x} = e^{\log\sec^2 x} = \sec^2 x$

Hence the solution is

$$y \cdot \sec^2 x = \int \sin x \sec^2 x\, dx + c$$

$$= \int \sec x \tan x\, dx + c = \sec x + c$$

or $\quad y = \cos x + c\cos^2 x \quad$ ...(*i*)

Now when $x = \dfrac{\pi}{3}, y = 0$

$\therefore \quad 0 = \cos\dfrac{\pi}{3} + c\cos^2\dfrac{\pi}{3} \quad$ or $\quad 0 = \dfrac{1}{2} + c.\dfrac{1}{4}$

$\therefore \qquad c = -2$

Hence the required solution is $y = \cos x - 2\cos^8 x$.

(*iv*) Given equation is $\dfrac{dy}{dx} + \dfrac{4x}{x^2+1} \cdot y = \dfrac{1}{(x^2+1)^3}$

It is of the form $\dfrac{dy}{dx} + \text{P}y = \text{Q}$

Here $\qquad \text{P} = \dfrac{4x}{x^2+1}, \text{Q} = \dfrac{1}{(x^2+1)^2}$

$$\therefore \qquad \text{I.F.} = e^{\int \text{P}\,dx} = e^{2\int \frac{2x}{x^2+1}dx} = e^{2\log(x^2+1)}$$

$$= e^{\log(x^2+1)^2} = (x^2+1)^2$$

Hence the solution is

$$y(x^2+1)^2 = \int \frac{1}{(x^2+1)^3}(x^2+1)^2\,dx + c = \int \frac{dx}{x^2+1} + c$$

or $\qquad y(x^2+1)^2 = \tan^{-1} x + c.$

(*v*) Given equation is $x\dfrac{dy}{dx} + 2y = x^2 \log x$

Dividing throughout by $x$ to make the co-eff. of $\dfrac{dy}{dx}$ unity

$$\frac{dy}{dx} + \frac{2}{x} \cdot y = x \log x$$

It is of the form $\dfrac{dy}{dx} + \text{P}y = \text{Q}$

Here $\qquad \text{P} = \dfrac{2}{x}, \ \text{Q} = x \log x$

$$\therefore \qquad \text{I.F.} = e^{\int \text{P}\,dx} = e^{\int \frac{2}{x}dx} = e^{2\log x} = e^{\log x^2} = x^2$$

Hence the solution is

$$yx^2 = \int x \log x . x^2\,dx + c = \int x^3 \log x\,dx + c$$

$$= \log x . \frac{x^4}{4} - \int \frac{1}{x} . \frac{x^4}{4}\,dx + c = \frac{x^4}{4}\log x - \frac{x^4}{16} + c$$

or $\qquad x^2 y = \dfrac{x^4}{4}\log x - \dfrac{x^4}{16} + c.$

**Example 5.** *Solve the following :*

(*i*) $x(x-1)\dfrac{dy}{dx} - (x-2)y = x^3(2x-1)$ $\qquad$ (*ii*) $\dfrac{dy}{dx} + y\tan x = x^m \cos x$

(*iii*) $\dfrac{dy}{dx} = x(x^2-2y)$ $\qquad$ (*iv*) $\dfrac{dy}{dx} + y\cos x = \sin 2x.$

**Sol.** (*i*) Given equation is

$$x(x-1)\frac{dy}{dx} - (x-2)y = x^3(2x-1)$$

or $$\frac{dy}{dx} - \frac{x-2}{x(x-1)}\,y = \frac{x^2(2x-1)}{x-1}$$

It is of the form $\dfrac{dy}{dx} + Py = Q$

Here $P = -\dfrac{x-2}{x(x-1)}$, $Q = \dfrac{x^2(2x-1)}{x-1}$

$\therefore$ $$\text{I.F.} = e^{\int P\,dx} = e^{-\int \frac{x-2}{x(x-1)}dx} = e^{-\int\left(\frac{2}{x}-\frac{1}{x-1}\right)dx}$$

$$= e^{-[2\log x - \log(x-1)]} = e^{-[\log x^2 - \log(x-1)]}$$

$$= e^{-\log\frac{x^2}{x-1}} = e^{\log\left(\frac{x^2}{x-1}\right)^{-1}} = \left(\frac{x^2}{x-1}\right)^{-1} = \frac{x-1}{x^2}$$

$\therefore$ The solution is

$$y\,.\,\frac{x-1}{x^2} = \int \frac{x^2(2x-1)}{x-1}\,.\,\frac{x-1}{x^2}\,dx + c = \int(2x-1)\,dx + c = x^2 - x + c$$

or $$y(x-1) = x^2(x^2 - x + c).$$

(*ii*) Given equation is $\dfrac{dy}{dx} + (\tan x)\,.\,y = x^m \cos x$

It is of the form $\dfrac{dy}{dx} + Py = Q$

Here $P = \tan x,\ Q = x^m \cos x$

$\therefore$ $$\text{I.F.} = e^{\int P\,dx}\ e^{\int \tan x\,dx} = e^{\log \sec x} = \sec x$$

The solution is,

$$y \sec x = \int x^m \cos x.\ \sec x\,dx + c = \int x^m\,dx + c$$

or $$y \sec x = \frac{x^{m+1}}{m+1} + c.$$

(*iii*) The given equation is $\dfrac{dy}{dx} = x^3 - 2xy$

or $$\frac{dy}{dx} + 2x.\,y = x^3$$

It is of the form $\dfrac{dy}{dx} + Py = Q$

Here $P = 2x,\quad Q = x^3$

$\therefore$ $$\text{I.F.} = e^{\int P\,dx} = e^{\int 2x\,dx} = e^{x^2}$$

The solution is

$$y\,.\,e^{x^2} = \int x^3.\,e^{x^2}\,dx + c = \frac{1}{2}\int x^2\,.\,e^{x^2}\,.\,2x\,dx + c$$ | **Note**

$$= \tfrac{1}{2}\int te^t\,dt + c, \text{ where } t = x^2$$

$$= \tfrac{1}{2} e^t(t-1) + c$$

or $$y \cdot e^{x^2} = \frac{1}{2} e^{x^2} (x^2 - 1) + c \quad \text{or} \quad y = \frac{1}{2}(x^2-1) + ce^{-x^2}.$$

(*iv*) The given equation is $\dfrac{dy}{dx} + \cos x \cdot y = \sin 2x$

It is of the form $\dfrac{dy}{dx} + \text{P}y = \text{Q}$

Here $\text{P} = \cos x, \quad \text{Q} = \sin 2x$

$\therefore$ $\text{I.F.} = e^{\int \cos x\, dx} = e^{\sin x}$

$\therefore$ The solution is

$$y \cdot e^{\sin x} = \int \sin 2x \cdot e^{\sin x}\, dx = c$$
$$= \int 2 \sin x \cos x \cdot e^{\sin x}\, dx + c = 2 \int te^t\, dt + c, \text{ where } t = \sin x$$
$$= 2e^t(t-1) + c$$

or $$y \cdot e^{\sin x} = 2 e^{\sin x}(\sin x - 1) + c$$

or $$y = 2(\sin x - 1) + ce^{-\sin x}.$$

**Example 6.** *Solve the following* :

(*i*) $\sin x \dfrac{dy}{dx} + y \cos x = 2 \sin^2 x \cos x$ (*ii*) $(x^2 - 1)\dfrac{dy}{dx} + 2(x+2)y = 2(x+1)$

(*iii*) $(1 - x^2)\dfrac{dy}{dx} + xy = ax$ (*iv*) $x(x-1)\dfrac{dy}{dx} - y = x^2(x-1)^2$

(*v*) $x(1-x^2)\dfrac{dy}{dx} + (2x^2 - 1)y = x^3$.

**Sol.** (*i*) The given equation is

$$\sin x \frac{dy}{dx} + y \cos x = 2 \sin^2 x \cdot \cos x$$

or $$\frac{dy}{dx} + \cot x \cdot y = 2 \sin x \cos x$$ | **Note**

It is of the form $\dfrac{dy}{dx} + \text{P}y = \text{Q}$

Here $\text{P} = \cot x, \quad \text{Q} = 2 \sin x \cos x$

$\therefore$ $\text{I.F.} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$

$\therefore$ The solution is

$$y \cdot \sin x = \int 2 \sin x \cos x \cdot \sin x\, dx + c = 2 \int \sin^2 x \cdot \cos x\, dx + c$$

or $$y \sin x = \tfrac{2}{3} \sin^3 x + c.$$

(*ii*) The given equation is

$$(x^2 - 1)\frac{dy}{dx} + 2(x+2)y = 2(x+1)$$

or $$\frac{dy}{dx} + \frac{2(x+2)}{(x^2-1)} \cdot y = \frac{2}{x-1}$$ | **Note**

It is of the form $\dfrac{dy}{dx} + \text{P}y = \text{Q}$

Here $\text{P} = \dfrac{2(x+2)}{x^2-1}, \quad \text{Q} = \dfrac{2}{x-1}$

$$\therefore \qquad \text{I.F.} = e^{\int P\,dx} = e^{2\int \frac{x+2}{(x+1)(x-1)}\,dx} = e^{2\int\left[\frac{3}{2(x-1)} - \frac{1}{2(x+1)}\right]dx}$$

$$= e^{\int\left[\frac{3}{x-1} - \frac{1}{x+1}\right]dx} = e^{3\log(x-1) - \log(x+1)}$$

$$= e^{\log(x-1)^3 - \log(x+1)} = e^{\log\frac{(x-1)^3}{x+1}} = \frac{(x-1)^3}{(x+1)}$$

$\therefore$ The solution is

$$y\,\frac{(x-1)^3}{(x+1)} = \int \frac{2}{x-1}\cdot\frac{(x-1)^3}{x+1}\,dx + c$$

$$= 2\int \frac{(x-1)^2}{x+1}\,dx + c = 2\int \frac{x^2-2x+1}{x+1}\,dx + c$$

$$= 2\int\left(x - 3 + \frac{4}{x+1}\right)dx + c$$

or
$$y\,.\,\frac{(x-1)^3}{x+1} = 2\left[\frac{x^2}{2} - 3x + 4\log(x+1)\right] + c$$

or
$$y(x-1)^3 = (x+1)[x^2 - 6x + 8\log(x+1) + c].$$

(*iii*) The given equation is $(1-x^2)\,.\,\dfrac{dy}{dx} + xy = ax$

or
$$\frac{dy}{dx} + \frac{x}{1-x^2}\,.\,y = \frac{ax}{1-x^2}$$

It is of the form $\dfrac{dy}{dx} + \text{P}y = \text{Q}$

Here $\qquad \text{P} = \dfrac{x}{1-x^2},\ \text{Q} = \dfrac{ax}{1-x^2}$

$$\therefore \qquad \text{I.F.} = e^{\int \frac{x}{1-x^2}\,dx} = e^{-\frac{1}{2}\int \frac{-2x}{1-x^2}\,dx} \qquad | \textbf{ Note}$$

$$= e^{-\frac{1}{2}\log(1-x^2)} = e^{\log(1-x^2)^{-1/2}} = (1-x^2)^{-1/2} = \frac{1}{\sqrt{1-x^2}}$$

$\therefore$ The solution is

$$y\,.\,\frac{1}{\sqrt{1-x^2}} = \int \frac{ax}{1-x^2}\cdot\frac{1}{\sqrt{1-x^2}}\,dx + c$$

$$= a\int \frac{x}{(1-x^2)^{3/2}}\,dx + c = -\frac{a}{2}\int -2x\,(1-x^2)^{-3/2}\,dx + c$$

$$= -\frac{a}{2}\cdot\frac{(1-x^2)^{-1/2}}{-\frac{1}{2}} + c$$

or
$$\frac{y}{\sqrt{1-x^2}} = \frac{a}{\sqrt{1-x^2}} + c \qquad \text{or} \qquad y = a + c\sqrt{1-x^2}.$$

(*iv*) The given equation is

$$x(x-1)\frac{dy}{dx}-y=x^2(x-1)^2$$

or
$$\frac{dy}{dx}-\frac{1}{x(x-1)}\cdot y=x(x-1)$$

It is of the form $\quad \dfrac{dy}{dx}+\text{P}y=\text{Q}$

Here $\quad \text{P}=-\dfrac{1}{x(x-1)}\ ;\quad \text{Q}=x(x-1)$

$\therefore$
$$\text{I.F.}=e^{\int -\frac{1}{x(x-1)}dx}=e^{-\int\left(\frac{1}{x-1}-\frac{1}{x}\right)dx}$$

$$=e^{-[\log(x-1)-\log x]}=e^{\log x-\log(x-1)}=e^{\log\frac{x}{x-1}}=\frac{x}{x-1}$$

Hence the solution is

$$y.\frac{x}{x-1}=\int x(x-1).\frac{x}{x-1}\,dx+c$$

or
$$y\cdot\frac{x}{x-1}=\int x^2\,dx+c=\frac{x^3}{3}+c.$$

(*v*) Dividing by $x(1-x^2)$ to make the co-efficient of $\dfrac{dy}{dx}$ unity, the given equation becomes

$$\frac{dy}{dx}+\frac{2x^2-1}{x(1-x^2)}y=\frac{x^2}{1-x^2}$$

It is of the form $\quad \dfrac{dy}{dx}+\text{P}y=\text{Q}$

Here $\quad \text{P}=\dfrac{2x^2-1}{x(1-x^2)},\ \text{Q}=\dfrac{x^2}{1-x^2}$

Now $\quad \text{P}=\dfrac{2x^2-1}{x(1-x)(1+x)}=-\dfrac{1}{x}+\dfrac{1}{2(1-x)}-\dfrac{1}{2(1+x)}$ [Partial fractions]

$\therefore$
$$\int \text{P}dx=-\log x-\frac{1}{2}\log(1-x)-\frac{1}{2}\log(1+x)=-\log[x(1-x)^{1/2}(1+x)^{1/2}]$$

$$=-\log[x\sqrt{1-x^2}]=\log(x\sqrt{1-x^2})^{-1}$$

$\therefore$
$$\text{I.F.}=e^{\int \text{P}\,dx}=e^{\log(x\sqrt{1-x^2})^{-1}}=\frac{1}{x\sqrt{1-x^2}}$$

The solution is

$$y\cdot\frac{1}{x\sqrt{1-x^2}}=\int\frac{x^2}{1-x^2}\cdot\frac{1}{x\sqrt{1-x^2}}\,dx+c$$

$$= \int \frac{x}{(1-x^2)^{3/2}}\, dx + c = -\frac{1}{2} \int (1-x^2)^{-3/2} . (-2x)\, dx + c$$

$$= -\frac{1}{2} . \frac{(1-x^2)^{-1/2}}{-\frac{1}{2}} + c$$

$$\Rightarrow \quad \frac{y}{x\sqrt{1-x^2}} = \frac{1}{\sqrt{1-x^2}} + c \quad \Rightarrow \quad y = x + cx\sqrt{1-x^2}$$

which is the required solution.

**Example 7.** *Integrate* $(1 + x^2) . \frac{dy}{dx} + 2xy - 4x^2 = 0$ *and obtain the equation of the cubic curve satisfying this equation and passing through the origin.*

**Sol.** Proceeding as in Example 1 (*i*), the solution is

$$y(1 + x^2) = \frac{4x^3}{3} + c$$

As it passes through the origin (0, 0) $\therefore$ $c = 0$

Hence, the equation of the curve is

$$y(1 + x^2) = \frac{4x^3}{3} \quad \text{or} \quad 3y(1 + x^2) = 4x^3.$$

**Example 8.** *Solve the following* :

(*i*) $x(x^2 + 1) . \frac{dy}{dx} = y(1 - x^2) + x^3 \log x$ (*ii*) $\cos x \frac{dy}{dx} + y \sin x = 1$

(*iii*) $\frac{dy}{dx} + \frac{2}{x} y = \sin x$.

**Sol.** (*i*) The given equation is

$$x(x^2 + 1) . \frac{dy}{dx} = y(1 - x^2) + x^3 \log x$$

or
$$\frac{dy}{dx} + \frac{x^2 - 1}{x(x^2 + 1)} y = \frac{x^2 \log x}{x^2 + 1}$$

It is of the form
$$\frac{dy}{dx} + \text{P}y = \text{Q}$$

Here
$$\text{P} = \frac{x^2 - 1}{x(x^2 + 1)}, \quad \text{Q} = \frac{x^2 \log x}{x^2 + 1}$$

$\therefore$
$$\text{I.F.} = e^{\int \frac{x^2 - 1}{x(x^2 + 1)} dx} = e^{\frac{1}{2}\int \frac{x^2 - 1}{x^2 (x^2 + 1)} 2x\, dx}$$ | **Note**

$$= e^{\frac{1}{2}\int \frac{t - 1}{t(t + 1)} dt} \quad \text{where } x^2 = t$$

$$= e^{\frac{1}{2}\int \left(\frac{2}{t+1} - \frac{1}{t}\right) dt} = e^{\log (t + 1) - \frac{1}{2} \log t}$$

$$= e^{\log \frac{t+1}{\sqrt{t}}} = \frac{t + 1}{\sqrt{t}} = \frac{x^2 + 1}{x}$$

$\therefore$ The solution is

$$y \cdot \frac{x^2+1}{x} = \int \frac{x^2 \log x}{x^2+1} \cdot \frac{x^2+1}{x}\, dx + c$$

or $$y. \frac{x^2+1}{x} = \int x \log x\, dx + c = \log x. \frac{x^2}{2} - \int \frac{1}{x} . \frac{x^2}{2}\, dx + c$$

or $$y. \frac{x^2+1}{x} = \frac{x^2}{2} \log x - \frac{x^2}{4} + c$$

or $$y(x^2+1) = \frac{x^3}{2} \log x - \frac{x^3}{4} + cx.$$

(*ii*) The given equation is

$$\cos x \frac{dy}{dx} + y \sin x = 1 \quad \text{or} \quad \frac{dy}{dx} + \tan x \cdot y = \sec x$$

Here $$\text{I.F.} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$$

$\therefore$ The solution is $y \cdot \sec x = \int \sec^2 x\, dx + c$

or $$y \sec x = \tan x + c.$$

(*iii*) Given equation is

$$\frac{dy}{dx} + \frac{2}{x} \cdot y = \sin x$$

$$\text{I.F.} = e^{\int \frac{2}{x} dx} = e^{2 \log x} = e^{\log x^2} = x^2$$

$\therefore$ The solution is

$$y \cdot x^2 = \int \sin x . x^2\, dx + c$$
$$= x^2(-\cos x) - \int 2x(-\cos x)\, dx + c$$
$$= -x^2 \cos x + 2[x \sin x - \int 1. \sin x\, dx] + c$$

or $$x^2 y = -x^2 \cos x + 2x \sin x + 2 \cos x + c.$$

**Example 9.** *Solve the following* :

(*i*) $\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right) \frac{dx}{dy} = 1$ (*ii*) $\frac{dy}{dx} + \frac{1-2x}{x^2} y = 1$

(*iii*) $y\, dx - x\, dy + \log x\, dx = 0$ (*iv*) $\frac{dy}{dx} + 2y \cot x = 3x^2 \operatorname{cosec}^2 x$

(*v*) $\sin 2x \frac{dy}{dx} = y + \tan x$ (*vi*) $\frac{dy}{dx} + \frac{y}{(1-x)\sqrt{x}} = 1 - \sqrt{x}$

(*vii*) $dt + 2st\, ds = se^{-s^2}\, ds.$

**Sol.** (*i*) The given equation is

$$\left(\frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}}\right) \frac{dx}{dy} = 1$$

or $$\frac{dy}{dx} = \frac{e^{-2\sqrt{x}}}{\sqrt{x}} - \frac{y}{\sqrt{x}} \quad \text{or} \quad \frac{dy}{dx} + \frac{1}{\sqrt{x}} y = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $$P = \frac{1}{\sqrt{x}}, \quad Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$$

$$\therefore \quad \text{I.F.} = e^{\int \frac{1}{\sqrt{x}} dx} = e^{\int x^{-1/2} dx} = e^{2\sqrt{x}}$$

$\therefore$ Hence the solution is

$$y \,.\, e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \,.\, e^{2\sqrt{x}}\, dx + c$$

or $$y \,.\, e^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}}\, dx + c = 2\sqrt{x} + c.$$

(*ii*) The given equation is

$$\frac{dy}{dx} + \frac{1-2x}{x^2} \,.\, y = 1$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $$P = \frac{1-2x}{x^2}, \; Q = 1$$

$$\therefore \quad \text{I.F.} = e^{\int \frac{1-2x}{x^2} dx} = e^{\int \left(\frac{1}{x^2} - \frac{2}{x}\right) dx}$$

$$= e^{-\frac{1}{x} - 2\log x} = e^{-\frac{1}{x}} \,.\, e^{-2\log x} \qquad \text{| Note}$$

$$= e^{-1/x} \,.\, e^{\log x^{-2}} = e^{-1/x} \,.\, x^{-2} = \frac{1}{x^2} \,.\, e^{-1/x}$$

$\therefore$ Hence the solution is

$$y \,.\, \frac{1}{x^2} e^{-1/x} = \int 1 \,.\, \frac{1}{x^2} \,.\, e^{-1/x}\, dx + c = \int e^t\, dt + c$$

$$= e^t + 3 \qquad \left| \text{ where } -\frac{1}{x} = t \;\therefore\; \frac{1}{x^2} dx = dt \right.$$

or $$y \,.\, \frac{1}{x^2} e^{-1/x} = e^{-1/x} + c \quad \text{or} \quad \frac{y}{x^2} = 1 + ce^{1/x}$$

or $$y = x^2(1 + ce^{1/x}).$$

(*iii*) The given equation is $y dx - x dy + \log x \, dx = 0$

or $$y - x \,.\, \frac{dy}{dx} + \log x = 0 \quad \text{or} \quad \frac{dy}{dx} - \frac{1}{x} \,.\, y = \frac{\log x}{x}$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $$P = -\frac{1}{x}, \qquad Q = \frac{\log x}{x}$$

$$\therefore \quad \text{I.F.} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}.$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $P = \frac{1}{\sqrt{x}}, \quad Q = \frac{e^{-2\sqrt{x}}}{\sqrt{x}}$

$\therefore$ $\text{I.F.} = e^{\int \frac{1}{\sqrt{x}} dx} = e^{\int x^{-1/2} dx} = e^{2\sqrt{x}}$

$\therefore$ Hence the solution is

$$y \,.\, e^{2\sqrt{x}} = \int \frac{e^{-2\sqrt{x}}}{\sqrt{x}} \,.\, e^{2\sqrt{x}}\, dx + c$$

or $$y \,.\, e^{2\sqrt{x}} = \int \frac{1}{\sqrt{x}}\, dx + c = 2\sqrt{x} + c.$$

(*ii*) The given equation is

$$\frac{dy}{dx} + \frac{1-2x}{x^2} \,.\, y = 1$$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $P = \frac{1-2x}{x^2}, \; Q = 1$

$\therefore$ $$\text{I.F.} = e^{\int \frac{1-2x}{x^2} dx} = e^{\int \left(\frac{1}{x^2} - \frac{2}{x}\right) dx}$$

$$= e^{-\frac{1}{x} - 2\log x} = e^{-\frac{1}{x}} \,.\, e^{-2\log x}$$ | Note

$$= e^{-1/x} \,.\, e^{\log x^{-2}} = e^{-1/x} \,.\, x^{-2} = \frac{1}{x^2} \,.\, e^{-1/x}$$

$\therefore$ Hence the solution is

$$y \,.\, \frac{1}{x^2} e^{-1/x} = \int 1 \,.\, \frac{1}{x^2} \,.\, e^{-1/x}\, dx + c = \int e^t\, dt + c$$

$$= e^t + 3$$ | where $-\frac{1}{x} = t \quad \therefore \quad \frac{1}{x^2} dx = dt$

or $y \,.\, \frac{1}{x^2} e^{-1/x} = e^{-1/x} + c$ or $\frac{y}{x^2} = 1 + ce^{1/x}$

or $y = x^2(1 + ce^{1/x})$.

(*iii*) The given equation is $ydx - xdy + \log x\, dx = 0$

or $y - x \,.\, \frac{dy}{dx} + \log x = 0$ or $\frac{dy}{dx} - \frac{1}{x} \,.\, y = \frac{\log x}{x}$

It is of the form $\frac{dy}{dx} + Py = Q$

Here $P = -\frac{1}{x}, \qquad Q = \frac{\log x}{x}$

$\therefore$ $\text{I.F.} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$.

where $$\mathrm{P} = \frac{1}{(1-x)\sqrt{x}}, \quad \mathrm{Q} = 1 - \sqrt{x}$$

$$\therefore \quad \text{I.F.} = e^{\int \frac{1}{(1-x)\sqrt{x}}dx} = e^{\int \frac{2dt}{1-t^2}} \quad \text{where } t = \sqrt{x}$$

$$= e^{2 \cdot \frac{1}{2} \log \frac{1+t}{1-t}} = \frac{1+t}{1-t} = \frac{1+\sqrt{x}}{1-\sqrt{x}}$$

$\therefore$ The solution is $$y \cdot \frac{1+\sqrt{x}}{1-\sqrt{x}} = \int (1-\sqrt{x}) \cdot \frac{1+\sqrt{x}}{1-\sqrt{x}}\, dx + c$$

$$= \int (1+\sqrt{x})\, dx + c = x + \frac{2}{3} x^{\frac{3}{2}} + c.$$

(*vii*) The given equation is $\dfrac{dt}{ds} + 2s.\ t = se^{-s^2}$

It is of the form $\dfrac{dt}{ds} + \mathrm{P}t = \mathrm{Q}$

where P, Q are functions of $s$ only.

Here $$\mathrm{P} = 2s, \mathrm{Q} = se^{-s^2}$$

$$\therefore \quad \text{I.F.} = e^{\int 2s\, ds} = e^{s^2}$$

$\therefore$ The solution is $$t.\ e^{s^2} = \int se^{-s^2} \cdot e^{s^2}\, ds + c$$

or $$t \cdot e^{s^2} = \int s ds + c \quad \text{or} \quad t\, e^{s^2} = \frac{s^2}{2} + c.$$

**Example 10.** (*i*) *Solve :* $\dfrac{dy}{dx} + \dfrac{y}{(1-x^2)^{3/2}} = \dfrac{x+\sqrt{1-x^2}}{(1-x^2)^2}$

(*ii*) *For the equation* $x^2 \dfrac{dy}{dx} + 2xy = 1$, *show that every solution tends to zero as x tends to* $\infty$.

**Sol.** (*i*) The given equation is of the form

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$$

where P, Q are functions of $x$ only.

Here $$\mathrm{P} = \frac{1}{(1-x^2)^{3/2}}, \mathrm{Q} = \frac{x+\sqrt{1-x^2}}{(1-x^2)^2}$$

$$\therefore \quad \text{I.F.} = e^{\int \frac{1}{(1-x^2)^{3/2}} dx}$$

Put $x = \sin\theta$, so that $dx = \cos\theta\, d\theta$

$$\int \frac{1}{(1-x^2)^{3/2}}\, dx = \int \frac{\cos\theta\, d\theta}{\cos^3\theta} = \int \sec^2\theta\, d\theta = \tan\theta = \frac{x}{\sqrt{1-x^2}}$$

$$\therefore \quad \text{I.F.} = e^{\frac{x}{\sqrt{1-x^2}}}$$

$\therefore$ The solution is

$$y \cdot e^{\frac{x}{\sqrt{1-x^2}}} = \int \frac{x+\sqrt{1-x^2}}{(1-x^2)^2} \cdot e^{\frac{x}{\sqrt{1-x^2}}}\, dx + c$$

Put $x = \sin\theta$ so that $dx = \cos\theta\, d\theta$

$$\therefore \quad \int \frac{x+\sqrt{1-x^2}}{(1-x^2)^2} \cdot e^{\frac{x}{\sqrt{1-x^2}}}\, dx = \int \frac{\sin\theta+\cos\theta}{\cos^4\theta} e^{\frac{\sin\theta}{\cos\theta}} \cdot \cos\theta\, d\theta$$

$$= \int \frac{\sin\theta+\cos\theta}{\cos\theta} e^{\tan\theta} \frac{1}{\cos^2\theta}\, d\theta$$

$$= \int (\tan\theta+1)\, e^{\tan\theta} \sec^2\theta\, d\theta$$

$$= \int (z+1) e^z\, dz \qquad \text{where } z = \tan\theta$$

$$= (z+1)\, e^z - \int 1.\, e^z\, dz = (z+1)\, e^z - e^z = ze^z$$

$$= \tan\theta.\, e^{\tan\theta} = \frac{x}{\sqrt{1-x^2}} e^{\frac{x}{\sqrt{1-x^2}}}$$

$\therefore$ The solution is

$$y.\, e^{\frac{x}{\sqrt{1-x^2}}} = \frac{x}{\sqrt{1-x^2}} e^{\frac{x}{\sqrt{1-x^2}}} + c \qquad \text{or} \qquad y = \frac{x}{\sqrt{1-x^2}} + c\, e^{\frac{-x}{\sqrt{1-x^2}}}.$$

(*ii*) The given equation is

$$x^2 \frac{dy}{dx} + 2xy = 1$$

or

$$\frac{dy}{dx} + \frac{2}{x} y = \frac{1}{x^2} \qquad \text{Form } \frac{dy}{dx} + \text{P}y = \text{Q}$$

Here

$$\text{P} = \frac{2}{x}, \text{Q} = \frac{1}{x^2}$$

$\therefore$

$$\text{I.F.} = e^{\int \frac{2}{x} dx} = e^{2\log x} = e^{\log x^2} = x^2$$

The solution is

$$y \cdot x^2 = \int \frac{1}{x^2} x^2\, dx + c$$

or

$$yx^2 = x + c \qquad \text{or} \qquad y = \frac{1}{x} + \frac{c}{x^2}$$

As $x \to \infty, y \to 0$

Hence every solution tends to zero as $x \to \infty$.

**Equations of the Form $\frac{dx}{dy} + Px = Q$ where P and Q are functions of y only.**

**Example 11.** *Solve the following :*

(*i*) $(x + 2y^3) \frac{dy}{dx} = y$

(*ii*) $(1 + y^2)dx = (\tan^{-1} y - x)dy$ (Delhi, 1999 ; Calcutta, 1996)

(*iii*) $(1 + y^2) + (x - e^{\tan^{-1} y}) \frac{dy}{dx} = 0$ (*iv*) $(2x - 10y^3) \frac{dy}{dx} + y = 0.$

**Sol.** (*i*) The given equation is $(x + 2y^3) \frac{dy}{dx} = y$

or $$y \cdot \frac{dx}{dy} = x \cdot 2y^3 \quad \text{or} \quad \frac{dx}{dy} - \frac{1}{y} \cdot x = 2y^2$$

It is of the form $\frac{dx}{dy} + Px = Q$

Here $P = -\frac{1}{y}$, $Q = 2y^2$

$\therefore$ $$\text{I.F.} = e^{\int -\frac{1}{y} dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1} = \frac{1}{y}$$

$\therefore$ The solution is

$$x \cdot \frac{1}{y} = \int 2y^2 \cdot \frac{1}{y} \, dy + c = \int 2y \, dy + c$$

or $$\frac{x}{y} = y^2 + c \quad \text{or} \quad x = y^3 + cy.$$

(*ii*) The given equation is $(1 + y^2) \, dx = (\tan^{-1} y - x) \, dy$

or $$(1 + y^2) \frac{dx}{dy} = \tan^{-1} y - x$$

or $$\frac{dx}{dy} + \frac{1}{1+y^2} \cdot x = \frac{\tan^{-1} y}{1+y^2}$$

It is of the form $\frac{dx}{dy} + Px = Q$

$$\text{I.F.} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1} y}$$

$\therefore$ The solution is

$$x \cdot e^{\tan^{-1} y} = \int \frac{\tan^{-1} y}{1+y^2} \cdot e^{\tan^{-1} y} \, dy + c$$

$$= \int te^t \, dt + c, \text{ where } t = \tan^{-1} y$$

$$= e^t(t - 1) + c$$

$$= e^{\tan^{-1} y} (\tan^{-1} y - 1)c$$

or $$x = \tan^{-1} y - 1 + ce^{-\tan^{-1} y}.$$

(*iii*) The given equation is

$$(1+y^2)+(x-e^{\tan^{-1}y})\frac{dy}{dx}=0$$

or $$(1+y^2)\frac{dx}{dy}+x-e^{\tan^{-1}y}=0$$

or $$\frac{dx}{dy}+\frac{1}{1+y^2}\cdot x=\frac{e^{\tan^{-1}y}}{1+y^2}$$

It is of the form $\frac{dx}{dy}+\text{P}x=\text{Q}$

$$\text{I.F.}=e^{\int\frac{1}{1+y^2}dy}=e^{\tan^{-1}y}$$

$\therefore$ The solution is

$$x.\,e^{\tan^{-1}y}=\int\frac{e^{\tan^{-1}y}}{1+y^2}\cdot e^{\tan^{-1}y}\,dy+c=\int e^t\cdot e^t\,dt+c \quad \text{where } t=\tan^{-1}y$$

$$=\int e^{2t}\,dt+c=\tfrac{1}{2}e^{2t}+c$$

or $$x\,.\,e^{\tan^{-1}y}=\tfrac{1}{2}e^{2\tan^{-1}y}+c.$$

(*iv*) The given equation is $(2x-10y^3)\frac{dy}{dx}+y=0$

or $$y.\frac{dx}{dy}+2x-10y^3=0$$

or $$\frac{dx}{dy}+\frac{2}{y}\cdot x=10y^2$$

It is of the form $\frac{dx}{dy}+\text{P}x=\text{Q}$

$$\text{I.F.}=e^{\int \text{P}\,dx}=e^{\int\frac{2}{y}dy}=e^{2\log y}=e^{\log y^2}=y^2$$

$\therefore$ The solution is

$$xy^2=\int 10y^2\cdot y^2dy+c=10\int y^4\,dy+c$$

or $$xy^2=\frac{10y^5}{5}+c=2y^5+c.$$

**Example 12.** *Solve the following differential equations :*

(*i*) $e^y\,dx+(1+x\,e^y)\,dy=0$ (*ii*) $\cosh x\,dy+(y+\cosh x)\sinh x\,dx=0$

(*iii*) $\frac{dy}{dx}+y\cot x=2\cos x$ (*iv*) $\sin x\frac{dy}{dx}+3y=\cos x$

(*v*) $(x+\log y)\,dy+y\,dx=0.$

**Sol.** (*i*) The given equation can be written as

$$e^y\frac{dx}{dy}+1+x\,e^y=0 \qquad \text{or} \qquad \frac{dx}{dy}+x=-e^{-y}$$

It is of the form $\frac{dx}{dy} + Px = Q$

where $P = 1, \quad Q = -e^{-y}$

$$\text{I.F.} = e^{\int P\,dy} = e^{y}$$

$\therefore$ The solution is

$$x \, . \, e^{y} = \int -e^{-y} \, . \, e^{y}\, dy + c$$

or $$x\, e^{y} = -\int dy + c \qquad \text{or} \qquad x\, e^{y} + y = c.$$

(*ii*) The given equation can be written as

$$\cosh x \frac{dy}{dx} + y \sinh x + \cosh x \sinh x = 0$$

or $$\frac{dy}{dx} + y \tanh x = -\sinh x$$

It is of the form $\frac{dy}{dx} + Py = Q$

where $P = \tanh x, \quad Q = -\sinh x$

$$\text{I.F.} = e^{\int P\,dy} = e^{\int \tanh x\, dx} = e^{\log \cosh x} = \cosh x$$

$\therefore$ The solution is

$$y \cosh x = \int -\sinh x \cosh x\, dx + c$$

$$= -\frac{1}{2}\int \sinh 2x\, dx + c$$

or $$y \cosh x = -\frac{1}{4} \cosh 2x + c.$$

(*iii*) Please try yourself. **[Ans.** $y \sin x = -\frac{1}{2} \cos 2x + c$]

(*iv*) The given equation can be written as

$$\frac{dy}{dx} + 3y \operatorname{cosec} x = \cot x$$

It is of the form $\frac{dy}{dx} + Py = Q$

where $P = 3 \operatorname{cosec} x$, $Q = \cot x$

$$\text{I.F.} = e^{\int P\,dx} = e^{\int 3 \operatorname{cosec} x\, dx} = e^{3 \log \tan \frac{x}{2}} = e^{\log \left(\tan \frac{x}{2}\right)^3} = \tan^3 \frac{x}{2}$$

$\therefore$ The solution is

$$y \tan^3 \frac{x}{2} = \int \cot x \, . \, \tan^3 \frac{x}{2}\, dx + c = \int \frac{\tan^3 \frac{x}{2}}{\tan x}\, dx + c$$

$$= \int \frac{\tan^3 \frac{x}{2}\left(1 - \tan^2 \frac{x}{2}\right)}{2 \tan \frac{x}{2}}\, dx + c \qquad \left[\because \ \tan x = \frac{2 \tan \frac{x}{2}}{1 - \tan^2 \frac{x}{2}}\right]$$

$$= \frac{1}{2} \int \tan^2 \frac{x}{2} \left(1 - \tan^2 \frac{x}{2}\right) dx + c \qquad ...(1)$$

Putting $\tan \frac{x}{2} = t$ so that $\frac{1}{2} \sec^2 \frac{x}{2}\, dx = dt$

or $\left(1 + \tan^2 \frac{x}{2}\right) dx = 2dt$ or $dx = \frac{2\,dt}{1+t^2}$

$\therefore$ From (1), $y \tan^3 \frac{x}{2} = \frac{1}{2} \int t^2 (1 - t^2) \cdot \frac{2\,dt}{1+t^2} + c$

$$= \int \frac{-t^4 + t^2}{t^2 + 1}\, dt + c = \int \left(-t^2 + 2 - \frac{2}{t^2 + 1}\right) dt + c$$

$$= -\frac{t^3}{3} + 2t - 2 \tan^{-1} t + c$$

or $y \tan^3 \frac{x}{2} = -\frac{1}{3} \tan^3 \frac{x}{2} + 2 \tan \frac{x}{2} - x + c.$

($v$) The given equation can be written as

$$x + \log y + y \frac{dx}{dy} = 0 \qquad \text{or} \qquad \frac{dx}{dy} + \frac{x}{y} = -\frac{\log y}{y}$$

It is of the form $\frac{dx}{dy} + Px = Q$

where $P = \frac{1}{y}, \quad Q = -\frac{\log y}{y}$

$$\text{I.F.} = e^{\int P\,dy} = e^{\int \frac{1}{y} dy} = e^{\log y} = y$$

$\therefore$ The solution is

$$xy = \int -\frac{\log y}{y} \cdot y\, dy + c$$

$$= -\int (\log y) . 1\, dy + c = -\left[(\log y) . y - \int \frac{1}{y} y\, dy\right] + c$$

or $xy = -y \log y + y + c$ or $x = 1 - \log y + \frac{c}{y}$.

**Example 13.** *Solve the following differential equations :*

($i$) $x \frac{dy}{dx} - y = 2x^2 \operatorname{cosec} 2x$ (Kanpur, 1996)

($ii$) $(1 + x^2) \frac{dy}{dx} + y = e^{\tan^{-1} x}$ ($iii$) $(1 + y + x^2 y)\, dx + (x + x^3)\, dy = 0$

($iv$) $\sec x\, dy = (y + \sin x)\, dx$.

**Sol.** ($i$) The given equation is

$$x \frac{dy}{dx} - y = 2x^2 \operatorname{cosec} 2x \qquad \text{or} \qquad \frac{dy}{dx} - \frac{y}{x} = 2x \operatorname{cosec} 2x$$

It is of the form $\frac{dy}{dx} + \text{P}y = \text{Q}$

where $\quad \text{P} = -\frac{1}{x}, \quad \text{Q} = 2x \operatorname{cosec} 2x$

$$\text{I.F.} = e^{\int \text{P}\,dx} = e^{\int -\frac{1}{x}\,dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

$\therefore$ The solution is

$$y \cdot \frac{1}{x} = \int (2x \operatorname{cosec} 2x)\, \frac{1}{x}\, dx + c$$

$$= 2\int \operatorname{cosec} 2x\, dx + c = 2 \times \frac{\log \tan x}{2} + c$$

or $\quad y = x(\log \tan x + c).$

(*ii*) The given equation is

$$(1 + x^2)\frac{dy}{dx} + y = e^{\tan^{-1} x} \qquad \text{or} \qquad \frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e^{\tan^{-1} x}}{1+x^2}$$

It is of the form $\frac{dy}{dx} + \text{P}y = \text{Q}$

where $\quad \text{P} = \frac{1}{1+x^2}, \quad \text{Q} = \frac{e^{\tan^{-1} x}}{1+x^2}$

$$\text{I.F.} = e^{\int \text{P}\,dx} = e^{\int \frac{1}{1+x^2}\,dx} = e^{\tan^{-1} x}$$

$\therefore$ The solution is

$$y \cdot e^{\tan^{-1} x} = \int \frac{e^{\tan^{-1} x}}{1+x^2} \cdot e^{\tan^{-1} x}\, dx + c$$

$$= \int t\, dt + c = \tfrac{1}{2} t^2 + c \quad \text{where } t = e^{\tan^{-1} x}$$

or $\quad y\, e^{\tan^{-1} x} = \tfrac{1}{2} e^{2\tan^{-1} x} + c \qquad \text{or} \qquad y = \tfrac{1}{2} e^{\tan^{-1} x} + c e^{-\tan^{-1} x}.$

(*iii*) The given equation is

$$(1 + y + x^2 y)\, dx + (x + x^3)\, dy = 0$$

or
$$x(1 + x^2)\frac{dy}{dx} + y(1 + x^2) = -1$$

or
$$\frac{dy}{dx} + \frac{y}{x} = \frac{-1}{x(1+x^2)}$$

It is of the form $\frac{dy}{dx} + \text{P}y = \text{Q}$

where $\quad \text{P} = \frac{1}{x}, \quad \text{Q} = \frac{-1}{x(1+x^2)}$

$$\text{I.F.} = e^{\int \text{P}\,dx} = e^{\int \frac{1}{x}\,dx} = e^{\log x} = x$$

$\therefore$ The solution is

$$y \,.\, x = \int \frac{-1}{x(1+x^2)} \,.\, x\,dx + c = -\int \frac{dx}{1+x^2} + c$$

or $$xy + \tan^{-1} x = c.$$

(*iv*) The given equation is

$$\sec x\,dy = (y + \sin x)\,dx$$

or $$\frac{dy}{dx} = \cos x\,(y + \sin x)$$

or $$\frac{dy}{dx} - y\cos x = \sin x \cos x$$

It is of the form $\dfrac{dy}{dx} + \text{P}y = \text{Q}$

where $$\text{P} = -\cos x,\ \text{Q} = \sin x \cos x$$

$$\text{I.F.} = e^{\int \text{P}\,dx} = e^{\int -\cos x\,dx} = e^{-\sin x}$$

$\therefore$ The solution is

$$y \,.\, e^{-\sin x} = \int \sin x \cos x \,.\, e^{-\sin x}\,dx + c$$

$$= \int t\,e^{-t}\,dt + c, \quad \text{where } t = \sin x$$

$$= t\,(-e^{-t}) - \int 1 \,.\, (-e^{-t})\,dt + c$$

$$= -te^{-t} - e^{-t} + c = -(t+1)\,e^{-t} + c$$

or $$y \,.\, e^{-\sin x} = -(\sin x + 1)\,e^{-\sin x} + c$$

or $$y = -(\sin x + 1) + ce^{\sin x}.$$

**Example 14.** *Obtain the equation of the curve whose slope at any point is equal to $y + 2x$ and which passes through the origin.*

**Sol.** Since the slope of a curve at any point is $\dfrac{dy}{dx}$.

$\therefore$ The differential equation of the curve is

$$\frac{dy}{dx} = y + 2x \qquad \text{or} \qquad \frac{dy}{dx} - y = 2x$$

It is of the form $\dfrac{dy}{dx} + \text{P}y = \text{Q}$

where $$\text{P} = -1, \quad \text{Q} = 2x$$

$$\text{I.F.} = e^{\int \text{P}\,dx} = e^{\int -1\,dx} = e^{-x}$$

$\therefore$ The solution is

$$ye^{-x} = \int 2x\,e^{-x}\,dx + c$$

$$= 2\left[x\,(-e^{-x}) - \int 1.(-e^{-x})\,dx\right] + c = 2\,[-x\,e^{-x} - e^{-x}] + c$$

or $$y = -2x - 2 + ce^x \quad \text{...(1)}$$
which is the equation of the family of curves.

If it passes through the origin (0, 0), then
$$0 = -2 + c \quad \text{or} \quad c = 2$$
$\therefore$ The required particular member of the family is
$$2x + y + 2 = 2e^x.$$

**Example 15.** *Solve the following differential equations :*

(*i*) $\dfrac{dy}{dx} + 2y = 6e^x$ (*ii*) $2y' + 4y = x^2 - x$

(*iii*) $y' - 2y = \cos 3x$ (*iv*) $y' + y = \sin x + \cos x$.

**Sol.** (*i*) **Hint.** I.F. $= e^{2x}$

The solution is $y \cdot e^{2x} = \int 6e^x \cdot e^{2x}\, dx + c = 6 \int e^{3x}\, dx + c$

or $$ye^{2x} = 6 \cdot \frac{e^{3x}}{3} + c \quad \text{or} \quad y = 2e^x + ce^{-2x}.$$

(*ii*) **Hint.** I.F. $= e^{2x}$

The solution is
$$y\, e^{2x} = \int \frac{1}{2} (x^2 - x)\, e^{2x}\, dx + c$$
$$= \frac{1}{2}\left[(x^2 - x) \cdot \frac{e^{2x}}{2} - \int (2x - 1) \cdot \frac{e^{2x}}{2}\, dx\right] + c$$
$$= \frac{1}{4} (x^2 - x)e^{2x} - \frac{1}{4} \int (2x - 1)\, e^{2x}\, dx + c$$
$$= \frac{1}{4} (x^2 - x)\, e^{2x} - \frac{1}{4}\left[(2x - 1) \cdot \frac{e^{2x}}{2} - \int 2 \cdot \frac{e^{2x}}{2}\, dx\right] + c$$
$$= \frac{1}{4} (x^2 - x)\, e^{2x} - \frac{1}{8} (2x - 1)\, e^{2x} + \frac{1}{4} \cdot \frac{e^{2x}}{2} + c$$
or $$y = \tfrac{1}{8} (2x^2 - 2x - 2x + 1 + 1) + ce^{-2x}$$
or $$y = \tfrac{1}{4} (x - 1)^2 + ce^{-2x}.$$

(*iii*) **Hint.** I.F. $= e^{-2x}$

The solution is $ye^{-2x} = \int e^{-2x} \cos 3x\, dx + c$
$$= \frac{e^{-2x}}{(-2)^2 + 3^2} [-2 \cos 3x + 3 \sin 3x] + c$$
$$\left[\because \int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \cos bx + b \sin bx)\right]$$
or $$y = \frac{1}{13} (3 \sin 3x - 2 \cos 3x) + ce^{2x}.$$

(*iv*) **Hint.** I.F. $= e^x$

The solution is $ye^x = \int (\sin x + \cos x)\, e^x\, dx + c$

$$= \int \sin x \,.\, e^x\, dx + \int \cos x\, e^x\, dx + c$$

$$= \sin x \,.\, e^x - \int \cos x\, e^x\, dx + \int \cos x\, e^x\, dx + c$$

$$= e^x \sin x + c \quad \text{or} \quad y = \sin x + c\, e^{-x}.$$

**Example 16.** *Solve the following differential equations :*

(*i*) $y' + y = \dfrac{x}{(x+1)^2}$ (*ii*) $y' + y = \dfrac{1 + x \log x}{x}$

(*iii*) $y' + y = \dfrac{1 + \sin x}{1 + \cos x}$ (*iv*) $xy' - y = (x - 1)\, e^x$.

**Sol.** (*i*) **Hint.** I.F. $= e^x$

The solution is $y\, e^x = \int \dfrac{x}{(x-1)^2}\, e^x\, dx + c$

or $$ye^x = \int \frac{(x+1) - 1}{(x+1)^2}\, e^x\, dx + c$$

$$= \int \left[\frac{1}{x+1} + \frac{-1}{(x+1)^2}\right] e^x\, dx + c \qquad \left[\text{Form} : \int [f(x) + f'(x)]\, e^x\, dx\right]$$

$$= \int \frac{1}{x+1}\, e^x\, dx + \int \frac{-1}{(x+1)^2}\, e^x\, dx + c$$

$$= \frac{1}{x+1}\, e^x - \int \frac{-1}{(x+1)^2}\, e^x\, dx + \int \frac{-1}{(x+1)^2}\, e^x\, dx + c$$

$$= \frac{e^x}{x+1} + c \quad \text{or} \quad y = \frac{1}{x+1} + ce^{-x}.$$

(*ii*) **Hint.** I.F. $= e^x$

The solution is $$ye^x = \int \left(\frac{1 + x \log x}{x}\right) e^x\, dx + c$$

$$= \int \left(\log x + \frac{1}{x}\right) e^x\, dx + c \qquad \left[\text{Form} : \int \{f(x) + f'(x)\}\, e^x\, dx\right]$$

$$= \int \log x \,.\, e^x\, dx + \int \frac{1}{x}\, e^x\, dx + c$$

$$= \log x\, e^x - \int \frac{1}{x}\, e^x\, dx + \int \frac{1}{x}\, e^x\, dx + c$$

$$= e^x \log x + c \quad \text{or} \quad y = \log x + ce^{-x}.$$

(*iii*) **Hint.** I.F. $= e^x$

The solution is $$ye^x = \int \frac{1 + \sin x}{1 + \cos x} \,.\, e^x\, dx + c$$

$$= \int \frac{1 + 2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}} . e^x \, dx + c$$

$$= \int \left(\tan\frac{x}{2} + \frac{1}{2}\sec^2\frac{x}{2}\right) e^x \, dx + c \qquad \left[\text{Form} : \int \{f(x) + f'(x)\} e^x \, dx\right]$$

$$= \int \tan\frac{x}{2} . e^x \, dx + \int \frac{1}{2}\sec^2\frac{x}{2} . e^x \, dx + c$$

$$= \tan\frac{x}{2} . e^x - \int \frac{1}{2}\sec^2\frac{x}{2} . e^x dx + \int \frac{1}{2}\sec^2\frac{x}{2} . e^x \, dx + c$$

$$= e^x \tan\frac{x}{2} + c \qquad \text{or} \qquad y = \tan\frac{x}{2} + c\, e^{-x}.$$

(*iv*) **Hint.** I.F. $= e^{\int -\frac{1}{x} dx} = \frac{1}{x}$

The solution is

$$y . \frac{1}{x} = \int \left(\frac{x-1}{x}\right) e^x . \frac{1}{x} \, dx + c$$

$$= \int \left(\frac{1}{x} - \frac{1}{x^2}\right) e^x \, dx + c \qquad \left[\text{Form} : \int \{f(x) + f'(x)\} e^x \, dx\right]$$

$$= \int \frac{1}{x} e^x \, dx + \int \frac{-1}{x^2} e^x \, dx + c$$

$$= \frac{1}{x} e^x - \int \frac{-1}{x^2} e^x \, dx + \int \frac{-1}{x^2} e^x \, dx + c = \frac{1}{x} + e^x + c \quad \text{or} \quad y = e^x + cx.$$

**Example 17.** *Solve the following differential equations :*

(*i*) $\frac{dy}{dx} + y \tan x = 2x + x^2 \tan x$ (*ii*) $\frac{dy}{dx} + \frac{y}{x} = \cos x + \frac{\sin x}{x}$

(*iii*) $ydx + (x - y^3)\, dy = 0.$

**Sol.** (*i*) **Hint.** I.F. $= e^{\log \sec x} = \sec x$

The solution is

$$y \sec x = \int (2x + x^2 \tan x) \sec x \, dx + c$$

$$= \int 2x \sec x \, dx + \int x^2 (\sec x \tan x) \, dx + c$$

$$= \int 2x \sec x \, dx + x^2 \sec x - \int 2x \sec x \, dx + c$$

$$= x^2 \sec x + c \quad \text{or} \quad y = x^2 + c \cos x.$$

(*ii*) Please try yourself. $\left[\textbf{Ans. } y = \sin x \frac{c}{x}\right]$

(*iii*) The given equation is $y\frac{dx}{dy} + x - y^3 = 0$

or
$$\frac{dx}{dy} + \frac{x}{y} = y^2$$

It is of the form $\frac{dx}{dy} + Px = Q$

where $P = \frac{1}{y}, \quad Q = y^2$

$$\text{I.F.} = e^{\int P\,dy} = e^{\log y} = y$$

$\therefore$ The solution is

$$xy = \int y^2 \, . \, y dy + c \quad \text{or} \quad xy = \frac{y^4}{4} + c.$$

**Example 18.** *Solve the following differential equations :*

(*i*) $dr + (2r \cot\theta + \sin 2\theta)\, d\theta = 0$ (*ii*) $e^{-y} \sec^2 y\, dy = dx + x dy$

(*iii*) $y\, e^y\, dx = (y^3 + 2xe^y) dy$ (*iv*) $\frac{dy}{dx} = \frac{y}{2y \log y + y - x}$

(*v*) $\sqrt{1 - y^2}\; dx = (\sin^{-1} y - x)\, dy$ (*vi*) $x^2 \frac{dy}{dx} = 3x^2 - 2xy + 1.$

**Sol.** (*i*) The given equation can be written as

$$\frac{dr}{d\theta} + (2 \cot\theta)\, r = -\sin 2\theta$$

It is of the form $\frac{dr}{d\theta} + Pr = Q$, where P and Q are functions of $\theta$ only.

$$\text{I.F.} = e^{\int 2 \cot\theta\, d\theta} = e^{2 \log \sin\theta} = e^{\log \sin^2\theta} = \sin^2\theta$$

$\therefore$ The solution is

$$r \, . \, \sin^2\theta = \int -\sin 2\theta \, . \, \sin^2\theta\, d\theta + c$$

$$= -2 \int \sin^3\theta \cos\theta\, d\theta + c = -2 \, . \, \frac{\sin^4\theta}{4} + c$$

or $\quad 2r \sin^2\theta + \sin^4\theta = C \quad$ where $C = 2c$.

(*ii*) The given equation can be written as

$$\frac{dx}{dy} + x = e^{-y} \sec^2 y$$

$$\text{I.F.} = e^{\int 1\, dy} = e^y$$

$\therefore$ The solution is

$$x \, . \, e^y = \int e^{-y} \sec^2 y \, . \, e^y\, dy + c$$

$$= \int \sec^2 y\, dy + c = \tan y + c$$

or $\quad x = e^{-y} (\tan y + c).$

(*iii*) The given equation can be written as

$$ye^y \frac{dx}{dy} = y^3 + 2xe^y$$

Dividing by $ye^y$ to make the co-efficient of $\frac{dx}{dy}$ unity, we have

$$\frac{dx}{dy} - \frac{2}{y}x = y^2 e^{-y}$$

It is of the form $\frac{dx}{dy} + Px = Q$, where P and Q are functions of $y$ only.

$$\text{I.F.} = e^{\int -\frac{2}{y}dy} = e^{-2\log y} = e^{\log y^{-2}} = \frac{1}{y^2}$$

$\therefore$ The solution is

$$x \cdot \frac{1}{y^2} = \int y^2 e^{-y} \cdot \frac{1}{y^2}\, dy + c$$

or $$\frac{x}{y^2} = \int e^{-y}\, dy + c = -e^{-y} + c \qquad \text{or} \qquad x = y^2(c - e^{-y})$$

(*iv*) The given equation can be written as

$$\frac{dx}{dy} = \frac{2y\log y + y - x}{y} = 2\log y + 1 - \frac{x}{y}$$

or $$\frac{dx}{dy} + \frac{x}{y} = 2\log y + 1$$

$$\text{I.F.} = e^{\int \frac{1}{y}dy} = e^{\log y} = y$$

$\therefore$ The solution is

$$xy = \int (2\log y + 1)\, y\, dy + c$$

$$= 2\int (\log y) \cdot y\, dy + \int y\, dy + c$$

$$= 2\left[(\log y)\frac{y^2}{2} - \int \frac{1}{y} \cdot \frac{y^2}{2}\, dy\right] + \int y\, dy + c$$

$$= y^2 \log y - \int y\, dy + \int y\, dy + c = y^2\log y + c$$

or $$x = y\log y + \frac{c}{y}$$

(*v*) The given equation can be written as

$$\sqrt{1-y^2} \cdot \frac{dx}{dy} = \sin^{-1} y - x$$

or $$\frac{dx}{dy} + \frac{x}{\sqrt{1-y^2}} = \frac{\sin^{-1} y}{\sqrt{1-y^2}}$$

$$\text{I.F.} = e^{\int \frac{dy}{\sqrt{1-y^2}}} = e^{\sin^{-1} y}$$

The solution is $x \, . \, e^{\sin^{-1} y} = \int \frac{\sin^{-1} y}{\sqrt{1-y^2}} . e^{\sin^{-1} y} \, dy + c = \int \; t \, e^t \, dt + c$ where $t = \sin^{-1} y$

$$= (t-1) \, e^t + c = (\sin^{-1} y - 1) \, e^{\sin^{-1} y} + c$$

or $$x = \sin^{-1} y - 1 + ce^{-\sin^{-1} y}.$$

(*iv*) The given equation can be written as

$$\frac{dy}{dx} + \frac{2}{x} y = 3 + \frac{1}{x^2}$$

Proceed further yourself. [**Ans.** $y = x + x^{-1} + cx^{-2}$]

## TYPE V. BERNOULLI'S EQUATION

**To solve the equation $\frac{dy}{dx} + Py = Qy^n$, where P, Q are functions of x only.**

The given equation is $\frac{dy}{dx} + Py = Qy^n$ ...(*i*)

Dividing both sides of (*i*) by $y^n$, **to make the R.H.S. a functions of x only.**

$$y^{-n} \frac{dy}{dx} + Py^{1-n} = Q \qquad ...(ii)$$

Put $y^{1-n} = z$, then

$$(1-n) \, . \, y^{-n} \frac{dy}{dx} = \frac{dz}{dx} \qquad \text{or} \qquad y^{-n} \frac{dy}{dx} = \frac{1}{1-n} \cdot \frac{dz}{dx}$$

$\therefore$ (*ii*) becomes $\frac{1}{1-n} \cdot \frac{dz}{dx} + Pz = Q$

or $$\frac{dz}{dx} + (1-n) \, . \, Pz = (1-n) \, Q.$$

which is a linear equation in $z$ and can be solved.

In the solution, putting $z = y^{1-n}$, we get the reqd. solution.

**Example 1.** *Solve the following* :

(*i*) $\frac{dy}{dx} + \frac{y}{x} = y^2$ (*ii*) $2 \frac{dy}{dx} = \frac{y}{x} + \frac{y^2}{x^2}$

(*iii*) $\frac{dy}{dx} = x^3y^3 - xy$ (*iv*) $(y \log x - 1) \, y \, dx = xdy$.

**Sol.** (*i*) The given equation is $\frac{dy}{dx} + \frac{y}{x} = y^2$

Dividing both sides by $y^2$, we have

$$y^{-2} . \frac{dy}{dx} + y^{-1} . \frac{1}{x} = 1 \qquad ...(i)$$

Put $y^{-1} = z$, then $-y^{-2} \frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ (*i*) becomes $-\dfrac{dz}{dx} + z \cdot \dfrac{1}{x} = 1$ or $\dfrac{dz}{dx} - \dfrac{1}{x} \cdot z = -1$

which is linear in $z$.

$$P = -\frac{1}{x}, Q = -1$$

$$\text{I.F.} = e^{\int P\,dx} = e^{-\int \frac{1}{x}dx} = e^{-\log x} = e^{\log\frac{1}{x}} = \frac{1}{x}$$

$\therefore$ The solution is $z \cdot \dfrac{1}{x} = \int -\dfrac{1}{x}\,dx + c$

or $$y^{-1} \cdot \frac{1}{x} = -\log x + c \qquad | \because z = y^{-1}$$

or $$\frac{1}{xy} + \log x = c.$$

(*ii*) The given equation is $2 \cdot \dfrac{dy}{dx} - \dfrac{y}{x} = \dfrac{y^2}{x^2}$.

Dividing throughout by $y^2$

$$2y^{-2}\frac{dy}{dx} - \frac{1}{x} \cdot y^{-1} = \frac{1}{x^2} \qquad ...(i)$$

Put $y^{-1} = z$, then $-y^{-2}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ (*i*) becomes

$$-2\frac{dz}{dx} - \frac{1}{x}z = \frac{1}{x^2} \quad \text{or} \quad \frac{dz}{dx} + \frac{1}{2x}z = -\frac{1}{2x^2}$$

which is linear in $z$.

$$P = \frac{1}{2x}, Q = -\frac{1}{2x^2}$$

$$\text{I.F.} = e^{\int \frac{1}{2x}dx} = e^{\frac{1}{2}\log x} = e^{\log\sqrt{x}} = \sqrt{x}$$

$\therefore$ The solution is $z \cdot \sqrt{x} = \int -\dfrac{1}{2x^2}\sqrt{x}\,dx + c$

or $$y^{-1}\sqrt{x} = -\frac{1}{2}\int x^{-3/2}\,dx + c \quad \text{or} \quad \frac{\sqrt{x}}{y} = \frac{1}{\sqrt{x}} + c$$

or $$x = y(1 + c\sqrt{x}).$$

(*iii*) The given equation is

$$\frac{dy}{dx} = x^3y^3 - xy = x^3y^3$$

Dividing throughout by $y^3$

$$y^{-3}\frac{dy}{dx} + xy^{-2} = x^3 \qquad ...(i)$$

Put $y^{-2} = z$, then $-2y^{-3}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ (*i*) becomes

$$-\frac{1}{2}\frac{dz}{dx}+xz=x^3 \quad \text{or} \quad \frac{dz}{dx}-2xz=-2x^3$$

which is linear in $z$. $\quad P=-2x,\ Q=-2x^3$

$$\text{I.F.}=e^{\int -2x\,dx}=e^{-x^2}$$

$\therefore$ The solution is

$$z\,.\,e^{-x^2}=\int -2x^3\,e^{-x^2}\,dx+c$$

$$=\int -2x\,.\,x^2\,.\,e^{-x^2}\,dx+c=\int -te^t\,dt+c \quad \text{where } t=-x^2$$

$$=-e^t(t-1)+c$$

or $\quad y^{-2}\,e^{-x^2}=-e^{-x^2}(-x^2-1)+c \quad$ or $\quad y^{-2}=x^2+1+ce^{x^2}$.

(*iv*) The given equation is $(y\log x-1)\,y\,dx=x\,dy$

or $\quad x\,.\,\dfrac{dy}{dx}=y^2\log x-y \quad$ or $\quad \dfrac{dy}{dx}+\dfrac{1}{x}\,.\,y=y^2\,.\,\dfrac{\log x}{x}$

Dividing throughout by $y^2$,

$$y^{-2}\frac{dy}{dx}+\frac{1}{x}y^{-1}=\frac{\log x}{x} \qquad \text{...}(i)$$

Put $y^{-1}=z$, then $-y^{-2}\dfrac{dy}{dx}=\dfrac{dz}{dx}$

$\therefore$ (*i*) becomes

$$-\frac{dz}{dx}+\frac{1}{x}z=\frac{\log x}{x} \quad \text{or} \quad \frac{dz}{dx}-\frac{1}{x}\,.\,z=-\frac{\log x}{x}$$

which is linear in $\quad z\,.\,P=-\dfrac{1}{x},\ Q=-\dfrac{\log x}{x}$

$$\text{I.F.}=e^{\int -\frac{1}{x}dx}=e^{-\log x}=e^{\log\frac{1}{x}}=\frac{1}{x}$$

$\therefore$ The solution is $\quad z\,.\,\dfrac{1}{x}=\displaystyle\int -\frac{\log x}{x}\,.\,\frac{1}{x}\,dx+c$

or $\quad y^{-1}\,.\,\dfrac{1}{x}=-\displaystyle\int \log x\,.\,\frac{1}{x^2}\,dx+c$

or $\quad \dfrac{1}{xy}=-\left[\log x\,.\left(-\dfrac{1}{x}\right)-\displaystyle\int \frac{1}{x}\left(-\frac{1}{x}\right)dx\right]+c$

or $\quad \dfrac{1}{xy}=-\left[-\dfrac{1}{x}\log x+\displaystyle\int \frac{1}{x^2}\,dx\right]+c$

or $\quad \dfrac{1}{xy}=-\left[-\dfrac{1}{x}\log x-\dfrac{1}{x}\right]+c \quad$ or $\quad \dfrac{1}{y}=\log x+1+cx.$

**Example 2.** *Solve the following :*

(i) $\frac{dy}{dx} - 2y \tan x = y^2 \tan^2 x$ (ii) $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$ (Kerala, 2001)

(iii) $3\frac{dy}{dx} + \frac{2}{x+1}y = \frac{x^3}{y^2}$ (iv) $\frac{dy}{dx} + \frac{x}{1-x^2}y = x\sqrt{y}$.

**Sol.** (i) The given equation is

$$\frac{dy}{dx} - 2y \tan x = y^2 \tan^2 x$$

Dividing both sides by $y^2$

$$y^{-2} \cdot \frac{dy}{dx} - 2y^{-1} \tan x = \tan^2 x \quad \text{...(i)}$$

Put $y^{-1} = z$, then $-y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ From (i), $-\frac{dz}{dx} - 2z \tan x = \tan^2 x$

or $$\frac{dz}{dx} + 2 \tan x \cdot z = -\tan^2 x$$

which is linear in $z$. $P = 2 \tan x$, $Q = -\tan^2 x$.

$$\text{I.F.} = e^{\int 2 \tan x\, dx} = e^{2 \log \sec x} = e^{\log \sec^2 x} = \sec^2 x$$

$\therefore$ The solution is

$$z \cdot \sec^2 x = \int -\tan^2 x \cdot \sec^2 x\, dx + c$$

or $$y^{-1} \cdot \sec^2 x = -\frac{\tan^3 x}{3} + c.$$

(ii) The given equation is $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$

Dividing throughout by $e^y$

$$e^{-y}\frac{dy}{dx} + e^{-y}\frac{dy}{dx} + \frac{1}{x} = \frac{1}{x^2} \quad \text{...(i)}$$

Put $e^{-y} = z$, then $-e^{-y}\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ (i) becomes

$$-\frac{dz}{dx} + z \cdot \frac{1}{x} = \frac{1}{x^2} \quad \text{or} \quad \frac{dz}{dx} - \frac{1}{x} \cdot z = -\frac{1}{x^2}$$

which is linear in $z$. $P = -\frac{1}{x}$, $Q = -\frac{1}{x^2}$

$$\text{I.F.} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log \frac{1}{x}} = \frac{1}{x}$$

$\therefore$ The solution is $z \cdot \frac{1}{x} = \int -\frac{1}{x^2} \cdot \frac{1}{x}\, dx + c$

or $$e^{-y} \cdot \frac{1}{x} = -\int \frac{1}{x^3}\, dx + c \quad \text{or} \quad e^{-y} \cdot \frac{1}{x} = \frac{1}{2x^2} + c$$

or $$2x = e^y + 2cx^2e^y.$$

(*iii*) The given equation is $3\,\dfrac{dy}{dx}+\dfrac{2}{x+y}\,y=\dfrac{x^3}{y^2}$

Multiplying throughout by $y^2$

$$3y^2\,\frac{dy}{dx}+2y^3\,.\,\frac{1}{x+1}=x^3 \qquad \text{...}(i)$$

Put $y^3=z$, then $\qquad 3y^2\,.\,\dfrac{dy}{dx}=\dfrac{dz}{dx}$

$\therefore$ (*i*) becomes

$$\frac{dz}{dx}+2z\,.\,\frac{1}{x+1}=x^3 \qquad \text{or} \qquad \frac{dz}{dx}+\frac{2}{x+1}\,.\,z=x^3$$

which is linear in $z$. $\qquad \text{P}=\dfrac{2}{x+1}\,,\ \text{Q}=x^3$

$$\text{I.F.}=e^{\int \frac{2}{x+1}dx}=e^{2\log(x+1)}=e^{\log(x+1)^2}=(x+1)^2$$

$\therefore$ The solution is

$$z\,.\,(x+1)^2=\int x^3(x+1)^2\,dx+c$$
$$=\int x^3(x^2+2x+1)\,dx+c=\int(x^5+2x^4+x^3)\,dx+c$$

or $\qquad y^2\,(x+1)^2=\dfrac{x^6}{6}+\dfrac{2x^5}{5}+\dfrac{x^4}{4}+c.$

(*iv*) The given equation is $\dfrac{dy}{dx}+\dfrac{x}{1-x^2}\,y=x\,\sqrt{y}$

Dividing throughout by $\sqrt{y}$,

$$y^{1/2}\,.\,\frac{dy}{dx}+\frac{x}{1-x^2}\,y^{1/2}=x \qquad \text{...}(i)$$

Put $y^{1/2}=z$ ; then $\qquad \dfrac{1}{2}\,y^{-1/2}\,.\,\dfrac{dy}{dx}=\dfrac{dz}{dx}$

$\therefore$ (*i*) becomes

$$2\,.\,\frac{dz}{dx}+\frac{x}{1-x^2}\,.\,z=x \qquad \text{or} \qquad \frac{dz}{dx}+\frac{x}{2(1-x^2)}\,.\,z=\frac{x}{2}$$

which is linear in $z$. $\qquad \text{P}=\dfrac{x}{2(1-x^2)}\,,\ \text{Q}=\dfrac{x}{2}$

$\therefore \qquad \text{I.F.}=e^{\int \frac{x}{2(1-x^2)}dx}=e^{-\frac{1}{4}\int \frac{-2x}{1-x^2}dx}$ | **Note**

$$=e^{-\frac{1}{4}\log(1-x^2)}=e^{\log(1-x^2)^{-1/4}}=(1-x^2)^{-1/4}$$

$\therefore$ The solution is $\quad z\,.\,(1-x^2)^{-1/4}=\displaystyle\int \frac{x}{2}\,(1-x^2)^{-1/4}\,dx+c$

or $\qquad \sqrt{y}\,.\,(1-x^2)^{-1/4}=-\dfrac{1}{4}\displaystyle\int -2x(1-x^2)^{-1/4}\,dx+c$

or $$\sqrt{y}\,.\,(1-x^2)^{-1/4} = -\frac{1}{4}\cdot\frac{(1-x^2)^{3/4}}{\frac{3}{4}} + c$$

or $$\sqrt{y} = -\tfrac{1}{3}\,(1-x^2) + c(1-x^2)^{1/4}.$$

**Another Form of Above Question**

*Solve* $$\frac{1}{y}\cdot\frac{dy}{dx} + \frac{x}{1-x^2} = xy^{-1/2}.$$

**Example 3.** *Solve the following differential equations :*

(*i*) $\dfrac{dy}{dx} + y\cos x = y^n \sin 2x$ (*ii*) $(x^2y^3 + xy)\,dy = dx$ (Ranchi, 1996)

(*iii*) $\dfrac{dy}{dx} + x\sin 2y = x^3\cos^2 y.$ (Kanpur, 1997 ; Lucknow, 1996 ; Madras, 1996)

**Sol.** (*i*) The given equation is $\dfrac{dy}{dx} + y\cos x = y^n \sin 2x$

Dividing throughout by $y^n$

$$y^{-n}\cdot\frac{dy}{dx} + y^{1-n}\cos x = \sin 2x \qquad ...(i)$$

Put $y^{1-n} = z$, then $(1-n)y^{-n}\cdot\dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ (*i*) becomes $\dfrac{1}{1-n}\cdot\dfrac{dz}{dx} + z\cos x = \sin 2x$

or $$\frac{dz}{dx} + (1-n)\cos x\,.\,z = (1-n)\sin 2x$$

which is linear in $z$. $P = (1-n)\cos x,\ Q = (1-n)\sin 2x$

$$\text{I.F.} = e^{\int (1-n)\cos x\,dx} = e^{(1-n)\sin x}$$

$\therefore$ The solution is

$$z\,.\,e^{(1-n)\sin x} = \int (1-n)\sin 2x\,.\,e^{(1-n)\sin x}\,dx + c$$

or $$y^{1-n}\,.\,e^{(1-n)\sin x} = 2\int \sin x\,.\,(1-n)\cos x\,.\,e^{(1-n)\sin x}\,dx + c$$

$$= 2\int \frac{t}{1-n}\,.\,e^t\,dt + c, \quad \text{where } t = (1-n)\sin x$$

$$= \frac{2}{1-n}\,e^t\,(t-1) + c = \frac{2}{1-n}\,.\,e^{(1-n)\sin x}\,[(1-n)\sin x - 1] + c$$

or $$y^{1-n} = \frac{2}{1-n}\,[(1-n)\sin x - 1] + c\,e^{(n-1)\sin x}.$$

(*ii*) The given equation is $(x^2y^3 + xy)dy = dx$

or $$\frac{dx}{dy} = x^2y^3 + xy$$

or $$\frac{dx}{dy} - xy = x^2y^3$$

**Note**

Form $\dfrac{dx}{dy} + Px = Qx^n$

Dividing throughout by $x^2$

$$x^{-2}\frac{dx}{dy} - x^{-1}y = y^3 \qquad ...(i)$$

Put $x^{-1} = z$, then $\qquad -x^{-2}\frac{dx}{dy} = \frac{dz}{dy}$

$\therefore$ $(i)$ becomes

$$-\frac{dz}{dy} - zy = y^3 \qquad \text{or} \qquad \frac{dz}{dy} + yz = -y^3$$

which is linear in $z$. $\qquad P = y, Q = -y^3$

$$\text{I.F.} = e^{\int y\,dy} = e^{y^2/2}$$

$\therefore$ The solution is $\quad z\,.\,e^{1/2y^2} = \int -y^3\,.\,e^{1/2y^2}\,dy + c$

or $\qquad x^{-1}\,.\,e^{1/2y^2} = -\int y^2\,.\,y\,.\,e^{1/2y^2}\,dy + c$

$$= -\int 2t\,e^t\,dt + c, \quad \text{where } t = \tfrac{1}{2}y^2$$

or $\qquad x^{-1}\,.\,e^{1/2y^2} = -2e^t(t-1) + c$

or $\qquad x^{-1}\,.\,e^{1/2y^2} = -2e^{1/2y^2}\left(\tfrac{1}{2}y^2 - 1\right) + c \qquad \text{or} \qquad x^{-1} = -y^2 + 2 + ce^{-1/2y^2}.$

$(iii)$ The given equation is $\frac{dy}{dx} + x\sin 2y\,x^3\cos^2 y$

Dividing throughout by $\cos^2 y$

$$\sec^2 y\,.\,\frac{dy}{dx} + 2x\tan y = x^3 \qquad ...(i) \qquad \Big|\because \frac{\sin 2y}{\cos^2 y} = \frac{2\sin y\cos y}{\cos^2 y} = 2\tan y$$

Put $\tan y = z$, then $\sec^2 y\,.\,\frac{dy}{dx} = \frac{dz}{dx}$

$\therefore$ $(i)$ becomes $\quad \frac{dz}{dx} + 2xz = x^3$

which is linear in $z$. $\qquad P = 2x, Q = x^2$

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

$\therefore$ The solution is $\qquad z\,.\,e^{x^3} = \int x^3\,.\,e^{x^2}\,dx + c$

or $\qquad \tan y\,.\,e^{x^2} = \tfrac{1}{2}\int 2x.\,x^2\,e^{x^3}\,dx + c$

$$= \tfrac{1}{2}\int te^t\,dt + c, \quad \text{where } t = x^2$$

$$= \tfrac{1}{2}e^t(t-1) + c$$

or $\qquad \tan y\,.\,e^{x^3} = \tfrac{1}{2}e^{x^2}(x^2 - 1) + c$

or $\qquad \tan y = \tfrac{1}{2}(x^2 - 1) + ce^{-x^3}.$

**Example 4.** *Solve the following :*

(*i*) $(x+1)\dfrac{dy}{dx}+1=2e^{-y}$ (*ii*) $\dfrac{dy}{dx}=e^{x-y}(e^x-e^y)$

(Delhi, 1997 ; Kanpur, 1997)

(*iii*) $\dfrac{dy}{dx}-\dfrac{\tan y}{1+x}=(1+x)\,e^x \sec y$ (Kanpur, 1998)

(*iv*) $(1-x^2)\dfrac{dy}{dx}+xy=xy^2.$

**Sol.** (*i*) The given equation is $(x+1)\dfrac{dy}{dx}+1=2e^{-y}$

or $$\frac{dy}{dx}+\frac{1}{x+1}=\frac{2e^{-y}}{x+1}$$

or $$e^y\cdot\frac{dy}{dx}+\frac{1}{x+1}\cdot e^y=\frac{2}{x+1} \qquad \ldots(i)$$

Put $e^y=z$, then $e^y\cdot\dfrac{dy}{dx}=\dfrac{dz}{dx}$

$\therefore$ From (*i*), $\dfrac{dz}{dx}+\dfrac{1}{x+1}\cdot z=\dfrac{2}{x+1}$

which is linear in $z$. $P=\dfrac{1}{x+1}$, $Q=\dfrac{2}{x+1}$

$$\text{I.F.}=e^{\int\frac{1}{x+1}dx}=e^{\log(x+1)}=x+1$$

$\therefore$ The solution is $z(x+1)=\displaystyle\int\frac{2}{x+1}\cdot(x+1)\,dx+c$

or $$c^y\cdot(x+1)=2x+c.$$

(*ii*) The given equation is

$$\frac{dy}{dx}=e^{x-y}(e^x-e^y) \quad\text{or}\quad \frac{dy}{dx}=e^{2x}\cdot e^{-y}-e^x$$

or $$\frac{dy}{dx}+e^x=e^{2x}\cdot e^{-y} \quad\text{or}\quad e^y\cdot\frac{dy}{dx}+e^x\cdot e^y=e^{2x} \qquad \ldots(i)$$

Put $e^y=z$, then $e^y\cdot\dfrac{dy}{dx}=\dfrac{dz}{dx}$

$\therefore$ (*i*) becomes $\dfrac{dz}{dx}+e^x\cdot z=e^{2x}$

which is linear in $z$. $P=e^x$, $Q=e^{2x}$

$$\text{I.F.}=e^{\int e^x dx}=e^{e^x}$$

$\therefore$ The solution is $z\cdot e^{e^x}=\displaystyle\int e^{2x}\cdot e^{e^x}\,dx+c$

or
$$\begin{aligned} e^y\cdot e^{e^x} &= \int e^x\cdot e^x\cdot e^{e^x}\,dx+c \\ &= \int t\,e^t\,dt+c, \quad \text{where } t=e^x \\ &= e^t(t-1)+c \end{aligned}$$

or $$e^y \cdot e^{e^x} = e^{e^x}(e^x - 1) + c \quad \text{or} \quad e^y = e^x - 1 + c\, e^{-e^x}.$$

(*iii*) The given equation is

$$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)\, e^x \cdot \sec y$$

Dividing throughout by sec $y$

$$\cos y \cdot \frac{dy}{dx} - \frac{\sin y}{1+x} = (1+x)\, e^y \qquad \text{...}(i)$$

Put $\sin y = z$, then $\cos y \cdot \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ (*i*) becomes $$\frac{dz}{dx} - \frac{1}{1+x} \cdot z = (1+x)\, e^x$$

which is linear in $z$, $$P = -\frac{1}{1+x},\ Q = (1+x)\, e^x$$

$$\text{I.F.} = e^{\int -\frac{1}{1+x}dx} = e^{-\log(1+x)} = e^{\log\frac{1}{1+x}} = \frac{1}{1+x}$$

$\therefore$ The solution is

$$z \cdot \frac{1}{1+x} = \int (1+x)\, e^x \frac{1}{1+x}\, dx + c$$

or $$z \cdot \frac{1}{1+x} = e^x + c \quad \text{or} \quad \sin y = (1+x)(e^x + c).$$

(*iv*) The given equation is $(1-x^2)\dfrac{dy}{dx} + xy = xy^2$

or $$\frac{dy}{dx} + \frac{x}{1-x^2} \cdot y = \frac{xy^2}{1-x^2}$$

or $$y^{-2}\frac{dy}{dx} + \frac{x}{1-x^2} \cdot y^{-1} = \frac{x}{1-x^2} \qquad \text{...}(i)$$

Put $y^{-1} = z$, then $$-y^{-2}\frac{dy}{dx} = \frac{dz}{dx}$$

$\therefore$ (*i*) becomes $$-\frac{dz}{dx} + \frac{x}{1-x^2} \cdot z = \frac{x}{1-x^2}$$

or $$\frac{dz}{dx} - \frac{x}{1-x^2} \cdot z = -\frac{x}{1-x^2}$$

which is linear in $z$. $$P = -\frac{x}{1-x^2}, \qquad Q = \frac{-x}{1-x^2}$$

$$\text{I.F.} = e^{\int -\frac{x}{1-x^2}dx} = e^{\frac{1}{2}\int \frac{-2x}{1-x^2}dx} = e^{\frac{1}{2}\log(1-x^2)}$$

$$= e^{\log(1-x^2)^{1/2}} = (1-x^2)^{1/2}$$

$\therefore$ The solution is

$$z(1-x^2)^{1/2} = \int \frac{-x}{1-x^2} .(1-x^2)^{1/2}\, dx + c$$

or
$$y^{-1}(1-x^2)^{1/2} = \int -x\,(1-x^2)^{-1/2}\, dx + c$$

$$= \frac{1}{2}\int -2x(1-x^2)^{-1/2}\, dx + c$$

$$= \frac{1}{2}\cdot\frac{(1-x^2)^{1/2}}{\frac{1}{2}} + c = (1-x^2)^{1/2} + c$$

or
$$y^{-1} = 1 + c(1-x^2)^{-1/2}.$$

**Example 5.** *Solve*

*(i)* $(x-y^2)dx + 2xydy = 0$ *(ii)* $x^2y\dfrac{dy}{dx} = xy^2 - e^{-1/x^2}$

*(iii)* $\cos x\, dy = y(\sin x - y)\, dx$ *(iv)* $xy - \dfrac{dy}{dx} = y^3 e^{-x^2}$.

**Sol.** (*i*) The given equation is $(x-y^2)dx + 2xy\, dy = 0$

or
$$2xy\frac{dy}{dx} + x - y^2 = 0 \quad \text{or} \quad 2y\cdot\frac{dy}{dx} - \frac{1}{x}\cdot y^2 = -1 \qquad ...(i)$$

Put $y^2 = z$, then $2y\dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ (*i*) becomes $\dfrac{dz}{dx} - \dfrac{1}{x}\cdot z = -1$

which is linear in $z$. $P = -\dfrac{1}{x}$, $Q = -1$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log\frac{1}{x}} = \frac{1}{x}$$

$\therefore$ The solution is $z\cdot\dfrac{1}{x} = \displaystyle\int -1.\frac{1}{x}\,dx + c$

or
$$y^2\cdot\frac{1}{x} = -\log x + c \quad \text{or} \quad y^2 = x(-\log x + c).$$

(*ii*) The given equation is $x^2y\dfrac{dy}{dx} = xy^2 - e^{-1/x^3}$

or
$$y\cdot\frac{dy}{dx} - \frac{1}{x}\cdot y^2 = -\frac{1}{x^2}\cdot e^{-1/x^3} \qquad ...(i)$$

Put $y^2 = z$, then $2y\dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ (*i*) becomes $\dfrac{1}{2}\dfrac{dz}{dx} - \dfrac{1}{x}z = -\dfrac{1}{x^2}\cdot e^{-1/x^3}$

or
$$\frac{dz}{dx} - \frac{2}{x}\cdot z = -\frac{2}{x^2}\cdot e^{-1/x^3}$$

which is linear in $z$. $P = -\dfrac{2}{x}$, $Q = -\dfrac{2}{x^2}\cdot e^{\frac{-1}{x^3}}$

$$\text{I.F.} = e^{\int -\frac{2}{x}dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2} = \frac{1}{x^2}$$

$\therefore$ The solution is $\quad z \cdot \dfrac{1}{x^2} = \displaystyle\int -\frac{2}{x^2} \cdot e^{-1/x^3} \frac{1}{x^2}\, dx + c$

or $$y^2 \cdot \frac{1}{x^2} = -2\int \frac{1}{x^4} \cdot e^{-1/x^3}\, dx + c \qquad \ldots(ii)$$

To integrate $\displaystyle\int \frac{1}{x^4} \cdot e^{-1/x^3}\, dx$. Put $-\dfrac{1}{x^3} = t$

*i.e.,* $$-x^{-3} = t, \text{ then } \frac{3}{x^4}\, dx = dt$$

$$\therefore \quad \int \frac{1}{x^4} \cdot e^{-1/x^3}\, dx = \frac{1}{3}\int e^t\, dt = \frac{1}{3}e^t = \frac{1}{3} \cdot e^{-1/x^3}$$

$\therefore$ From $(ii)$, the solution is

$$\frac{y^2}{x^2} = -\frac{2}{3} \cdot e^{-1/x^3} + c \quad \text{or} \quad y^2 = x^2\left(-\frac{2}{3} \cdot e^{-1/x^3} + c\right).$$

$(iii)$ The given equation is $\cos x\, dy = y(\sin x - y)\, dx$

or $$\cos x \frac{dy}{dx} = y \sin x - y^2$$

or $$\frac{dy}{dx} - y \tan x = -y^2 \sec x$$

Dividing throughout by $y^2$

$$y^{-2}\frac{dy}{dx} - y^{-1}\tan x = -\sec x \qquad \ldots(i)$$

Put $y^{-1} = z$, then $\quad -y^{-2}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ $(i)$ becomes $\quad -\dfrac{dz}{dx} - z \cdot \tan x = -\sec x$

or $$\frac{dz}{dx} + \tan x \cdot z = \sec x$$

which is linear in $z$. $\quad P = \tan x, \quad Q = \sec x$

$$\text{I.F.} = e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$$

$\therefore$ The solution is

$$z \cdot \sec x = \int \sec x \cdot \sec x\, dx + c$$

or $$y^{-1}\sec x = \int \sec^2 x\, dx + c = \tan x + c$$

$(iv)$ The given equation is $xy - \dfrac{dy}{dx} = y^3 \cdot e^{-x^2}$

or $$\frac{dy}{dx} - xy = -y^3 e^{-x^2}$$

Dividing throughout by $y^3$,

$$y^{-3}\frac{dy}{dx} - xy^{-2} = -e^{-x^2} \qquad \ldots(i)$$

Put $y^{-2} = z$, then $\qquad -2y^{-3}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ (*i*) becomes

$$-\frac{1}{2}\frac{dz}{dx} - xz = -e^{-x^2} \qquad \text{or} \qquad \frac{dz}{dx} + 2xz = 2e^{-x^2}$$

which is linear in $z$. $\qquad P = 2x,\ Q = 2e^{-x^2}$

$$\therefore \qquad \text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

$\therefore$ The solution is

$$z \cdot e^{x^2} = \int 2e^{-x^2} \cdot e^{x^2}\,dx + c$$

or $$y^{-2} \cdot e^{x^2} = 2\int 1\,dx + c = 2x + c.$$

**Example 6.** *Solve the following* :

(*i*) $\left\{xy^2 - e^{\frac{1}{x^3}}\right\} dx - x^2y\,dy = 0$ $\qquad$ (*ii*) $\dfrac{dy}{dx} + (2x\tan^{-1} y - x^3)(1 + y^2) = 0.$

**Sol.** (*i*) The given equation is

$$\left\{xy^2 - e^{\frac{1}{x^3}}\right\} dx - x^2y\,dy = 0 \qquad \text{or} \qquad x^2y\frac{dy}{dx} = xy^2 - e^{\frac{1}{x^3}}$$

Now proceeding as in Ex. 5. (*ii*) the solution is

$$y^2 = x^2\left(\frac{2}{3}e^{\frac{1}{x^3}} + c\right).$$

(*ii*) The given equation is

$$\frac{dy}{dx} + (2x\tan^{-1} y - x^3)(1 + y^2) = 0$$

or $$\frac{1}{1+y^2} \cdot \frac{dy}{dx} + 2x\tan^{-1} y - x^3 = 0$$

or $$\frac{1}{1+y^2} \cdot \frac{dy}{dx} + 2x\tan^{-1} y = x^3 \qquad \text{...}(i)$$

Put $\tan^{-1} y = z$, then $\dfrac{1}{1+y^2} \cdot \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ From (*i*), $\dfrac{dz}{dx} + 2xz = x^3$

which is linear in $z$. $\qquad P = 2x,\ Q = x^3$

$$\text{I.F.} = e^{\int 2x\,dx} = e^{x^2}$$

$\therefore$ The solution is $z \cdot e^{x^2} = \int x^3 \cdot e^{x^2}\,dx + c$

or $$\tan^{-1} y \cdot e^{x^2} = \tfrac{1}{2}\int 2x \cdot x^2 e^{x^2}\,dx + c$$

$$= \tfrac{1}{2}\int t\,e^t\,dt + c, \quad \text{where } t = x^2$$

$$= \tfrac{1}{2}e^t(t-1) + c$$

or $$\tan^{-1} y \,.\, e^{x^2} = \tfrac{1}{2}\, e^{x^2}\,(x^2 - 1) + c$$

or $$\tan^{-1} y = \tfrac{1}{2}\,(x^2 - 1) + ce^{-x^2}.$$

**Example 7.** *Solve the following differential equations :*

(*i*) $(x^3y^2 + xy)\, dx = dy$ (Delhi, 1996) (*ii*) $\dfrac{dy}{dx} + \dfrac{y}{x} \log y = \dfrac{y}{x^2} (\log y)^2$

(*iii*) $\dfrac{dy}{dx} + \dfrac{y}{x} = y^2 \log x$ (*iv*) $e^y \left(\dfrac{dy}{dx} + 1\right) = e^x$

(*v*) $\dfrac{dy}{dx} + 2y \tan x = y^2$.

**Sol.** (*i*) The given equation is $(x^3y^2 + xy)\, dx = dy$

or $$\frac{dy}{dx} = x^3y^2 + xy \quad \text{or} \quad \frac{dy}{dx} - xy = x^3y^2$$

Dividing both sides by $y^2$, $\quad y^{-2} \dfrac{dy}{dx} - xy^{-1} = x^3 \qquad ...(i)$

Put $y^{-1} = z$, then $\quad -y^{-2} \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ (*i*) becomes $\quad \dfrac{dz}{dx} - xz = x^3 \quad$ or $\quad \dfrac{dz}{dx} + xz = -x^3$

which is linear in $z$. $\quad P = x, \quad Q = -x^3$

$$\text{I.F.} = e^{\int x\, dx} = e^{\frac{x^2}{2}}$$

$\therefore$ The solution is
$$z \,.\, e^{\frac{x^2}{2}} = \int -x^3 \,.\, e^{\frac{x^2}{2}}\, dx + c = -\int x^2 \,.\, xe^{\frac{x^2}{2}}\, dx + c$$
$$= -\int 2te^t\, dt + c, \quad \text{where } t = \frac{x^2}{2}$$
$$= -\int 2te^t dt + c = -2e^t\,(t - 1) + c$$
$$y^{-1} \,.\, e^{\frac{x^2}{2}} = -2e^{\frac{x^2}{2}} \left(\frac{x^2}{2} - 1\right) + c$$

or $$y^{-1} = -x^2 + 2 + ce^{-\frac{x^2}{2}}.$$

(*ii*) The given equation is $\dfrac{dy}{dx} + \dfrac{y}{x} \log y = \dfrac{y}{x^2} (\log y)^2$

Dividing both sides by $y\,(\log y)^2$, we get

$$\frac{1}{y(\log y)^2} \cdot \frac{dy}{dx} + \frac{1}{\log y} \cdot \frac{1}{x} = \frac{1}{x^2} \qquad ...(i)$$

Put $\dfrac{1}{\log y} = (\log y)^{-1} = z$, then $-(\log y)^{-2} \,.\, \dfrac{1}{y}\, \dfrac{dy}{dx} = \dfrac{dz}{dx}$

or $$\frac{1}{y(\log y)^2} \cdot \frac{dy}{dx} = -\frac{dz}{dx}$$

$\therefore$ From (*i*), $-\dfrac{dz}{dx} + z \cdot \dfrac{1}{x} = \dfrac{1}{x^2}$ or $\dfrac{dz}{dx} - \dfrac{1}{x} z = -\dfrac{1}{x^2}$

which is linear in $z$. $\quad \text{P} = -\dfrac{1}{x}, \text{Q} = -\dfrac{1}{x^2}$.

$$\text{I.F.} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

$\therefore$ The solution is $z \cdot \dfrac{1}{x} = \displaystyle\int -\frac{1}{x^2} \cdot \frac{1}{x}\, dx + c$

or $$z \cdot \frac{1}{x} = -\int x^{-3}\, dx + c = -\frac{x^{-2}}{-2} + c$$

or $$\frac{1}{\log y} \frac{1}{x} = \frac{1}{2x^2} + c \quad \text{or} \quad \frac{1}{\log y} = \frac{1}{2x} + cx.$$

(*iii*) The given equation is $\dfrac{dy}{dx} + \dfrac{y}{x} = y^2 \log x$

Dividing both sides by $y^2$, we get

$$y^{-2} \cdot \frac{dy}{dx} + \frac{1}{x} \cdot y^{-1} = \log x \qquad \text{...}(i)$$

Put $y^{-1} = z$, then $\quad -y^{-2} \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ From (*i*), $\quad -\dfrac{dz}{dx} + \dfrac{1}{x} \cdot z = \log x$

or $$\frac{dz}{dx} - \frac{1}{x} \cdot z = -\log x$$

which is linear in $z$. $\quad \text{P} = -\dfrac{1}{x}, \quad \text{Q} = -\log x$

$$\text{I.F.} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

$\therefore$ The solution is $z \dfrac{1}{x} = \displaystyle\int -\log x \cdot \frac{1}{x}\, dx + c$

or $$y^{-1} \cdot \frac{1}{x} = -\int \log x \cdot \frac{1}{x}\, dx + c$$

$$= -\frac{(\log x)^2}{2} + c \qquad \left| \because \int [f(x)]^n f'(x)\, dx\, (n \neq -1) = \frac{[f(x)]^{n+1}}{n+1} \right.$$

or $$y^{-1} = -\tfrac{1}{2} (\log x)^2 + cx.$$

(*iv*) The given equation is $e^y \left(\dfrac{dy}{dx} + 1\right) = e^x$

or $$e^y \cdot \frac{dy}{dx} + e^y = e^x \qquad \text{...}(i)$$

Put $e^y = z$, then $\quad e^y \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ From (*i*), $\quad \dfrac{dz}{dx} + z = e^x$

which is linear in $z$. $\quad P = 1, \quad Q = e^x$

$$\text{I.F.} = e^{\int 1\,dx} = e^x$$

$\therefore$ The solution is

$$z \,.\, e^x = \int e^x \,.\, e^x \,dx + c$$

or $$e^x \,.\, e^x = \int e^{2x}\,dx + c = \tfrac{1}{2}\,e^{2x} + c$$

or $$e^{x+y} = \frac{1}{2}\,e^{2x} + c.$$

(*v*) The given equation is $\dfrac{dy}{dx} + 2y \tan x = y^2$

Dividing both sides by $y^2$,

$$y^{-2} \,.\, \frac{dy}{dx} + 2y^{-1} \tan x = 1 \qquad \text{...}(i)$$

Put $y^{-1} = z$, then $\quad -y^{-2}\,\dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ From (*i*), $\quad -\dfrac{dz}{dx} + 2z \tan x = 1$

or $$\frac{dz}{dx} - 2 \tan x \,.\, z = -1$$

which is linear in $z$. $\quad P = -2\tan x, \quad Q = -1$

$$\text{I.F.} = e^{\int -2\tan x\,dx} = e^{-2(-\log\cos x)} = e^{\log\cos^2 x} = \cos^2 x$$

$\therefore$ The solution is

$$z \,.\, \cos^2 x = \int -1 \,.\, \cos^2 x \,dx + c$$

or $$y^{-1}\cos^2 x = -\int \frac{1+\cos 2x}{2}\,dx + c = -\frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) + c$$

or $$\cos^2 x + \frac{1}{2}\left(x + \frac{\sin 2x}{2}\right) y = cy.$$

**Example 8.** *Solve the following* :

(*i*) $\dfrac{dy}{dx} + \dfrac{1}{x}\,y = x^2y^6$ $\qquad$ (*ii*) $y(2xy + e^x)\,dx - e^x\,dy = 0$

(*iii*) $x\,\dfrac{dy}{dx} + y = x^3y^4$ $\qquad$ (*iv*) $\dfrac{dy}{dx} + \dfrac{y}{x} = x^2y^4$

(*v*) $x\,\dfrac{dy}{dx} + y = y^2x^3 \cos x$ $\qquad$ (*vi*) $2\,xydy - (x^2 + y^2 + 1)\,dx = 0.$ $\quad$ (Delhi 1999)

**Sol.** (*i*) The given equation is

$$\frac{dy}{dx} + \frac{1}{x}\,y = x^2y^6$$

Dividing both sides by $y^6$,

$$y^{-6}\frac{dy}{dx}+\frac{1}{x}y^{-5}=x^2 \qquad ...(i)$$

Put $y^{-5}=z$, then $\quad -5y^{-6}\dfrac{dy}{dx}=\dfrac{dz}{dx}$

$\therefore$ From (*i*), $\quad -\dfrac{1}{5}\dfrac{dz}{dx}+\dfrac{1}{x}\,.\,z=x^2$

or $$\frac{dz}{dx}-\frac{5}{x}\,.\,z=-5x^2$$

which is linear in $z$. $\quad P=-\dfrac{5}{x}, \qquad Q=-5x^2$

$$\text{I.F.}=e^{\int -\frac{5}{x}dx}=e^{-5\log x}=e^{\log x^{-5}}=x^{-5}=\frac{1}{x^5}$$

$\therefore$ The solution is

$$z\,.\,\frac{1}{x^5}=\int -5x^2\,.\,\frac{1}{x^5}\,dx+c$$

or $$y^{-5}\,.\,\frac{1}{x^5}=-5\int x^{-3}\,dx+c=-5\,.\,\frac{x^{-2}}{-2}+c$$

or $$y^{-5}=\frac{5}{2}x^3+cx^5.$$

(*ii*) The given equation is

$$e^x\frac{dy}{dx}=y\,(2xy+e^x)=2xy^2+ye^x \quad \text{or} \quad \frac{dy}{dx}-y=2xe^{-x}\,.\,y^2$$

Dividing both sides by $y^2$,

$$y^{-2}\frac{dy}{dx}-y^{-1}=2xe^{-x} \qquad ...(i)$$

Putting $y^{-1}=z$ etc.

$$\frac{dz}{dx}+z=-2xe^{-x}$$

which is linear in $z$. $\quad P=1,\ Q=-2x\,e^{-x}$

$$\text{I.F.}=e^{\int 1\,dx}=e^x$$

$\therefore$ The solution is $\quad z\,.\,e^x=\int -2x\,e^{-x}\,.\,e^x\,dx+e=-\int 2x\,dx+c$

or $$y^{-1}\,.\,e^x=-x^2+c.$$

(*iii*) The given equation is

$$x\frac{dy}{dx}+y=x^3y^4 \qquad \text{or} \qquad \frac{dy}{dx}+\frac{1}{x}y=x^2y^4$$

Dividing both sides by $y^4$

$$y^{-4}\frac{dy}{dx}+\frac{1}{x}y^{-3}=x^2 \qquad ...(i)$$

Putting $y^{-3}=z$ etc.

$$\frac{dz}{dx} - \frac{3}{x} z = -3x^2$$

which is linear in $z$.

$$P = -\frac{3}{x}, \; Q = -3x^2$$

$$\text{I.F.} = e^{\int -\frac{3}{x} dx} = e^{-3\log x} = e^{\log x^{-3}} = x^{-3} = \frac{1}{x^3}$$

$\therefore$ The solution is

$$z \cdot \frac{1}{x^3} = \int -3x^2 \cdot \frac{1}{x^3}\, dx + c = -3\log x + c$$

or $$y^{-3} = -3x^3 \log x + cx^3.$$

(*iv*) Please try yourself. **[Ans.** $y^{-3} = -3x^4 + cx^3$**]**

(*v*) The given equation is

$$x\frac{dy}{dx} + y = y^2x^3 \cos x \qquad \text{or} \qquad \frac{dy}{dx} + \frac{1}{x} y = y^2x^2 \cos x$$

Dividing both sides $y^2$,

$$y^{-2}\frac{dy}{dx} + \frac{1}{x} y^{-1} = x^2 \cdot \cos x \qquad \text{...}(i)$$

Putting $y^{-1} = z$ etc.

(*i*) becomes $\dfrac{dz}{dx} - \dfrac{1}{x} z = -x^2 \cos x$

which is linear in $z$.

$$P = -\frac{1}{x}, \; Q = -x^2 \cos x$$

$$\text{I.F.} = e^{\int -\frac{1}{x} dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

$\therefore$ The solution is

$$z \cdot \frac{1}{x} = \int -x^2 \cos x \cdot \frac{1}{x}\, dx + c = -\int x \cos x\, dx + c$$

or $$y^{-1} \cdot \frac{1}{x} = -\left[x \sin x - \int 1 . \sin x\, dx\right] + c$$

or $$\frac{1}{xy} = -x \sin x - \cos x + c.$$

(*vi*) The given equation is

$$2xy\frac{dy}{dx} = x^2 + y^2 + 1 \qquad \text{or} \qquad 2y\frac{dy}{dx} - \frac{y^2}{x} = \frac{x^2+1}{x} \qquad \text{...}(i)$$

Putting $y^2 = z$ etc. (*i*) becomes

$$\frac{dz}{dx} - \frac{1}{x} z = \frac{x^2+1}{x}$$

which is linear in $z$. $\quad P = -\dfrac{1}{x}, \; Q = \dfrac{x^2+1}{x}$

$$\text{I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = \frac{1}{x}$$

$\therefore$ The solution is

$$z \cdot \frac{1}{x} = \int \frac{x^2+1}{x} \cdot \frac{1}{x}\, dx + c$$

or $$y^2 \cdot \frac{1}{x} = \int (1 + x^{-2})\, dx + c = x - \frac{1}{x} + c$$

or $$y^2 = x^2 - 1 + cx.$$

**Example 9.** *Solve the following* :

(*i*) $(\sin y + e^{\sin x})\, dx + \tan x \cos y\, dy = 0$ (*ii*) $x \dfrac{dy}{dx} + y \log y = xye^x$.

**Sol.** (*i*) The given equation is

$$\tan x \cos y \frac{dy}{dx} + \sin y + e^{\sin x} = 0$$

or $$\cos y \frac{dy}{dx} + \cot x \sin y = -\cot x \cdot e^{\sin x} \quad \ldots(i)$$

Putting $\sin y = z$, $\cos y \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ From (*i*), $\dfrac{dz}{dx} = z \cot x = -\cot x \cdot e^{\sin x}$

which is linear in $z$.

$$\text{P} = \cot x,\ \text{Q} = -\cot x \cdot e^{\sin x}$$

$$\text{I.F.} = e^{\int \cot x\, dx} = e^{\log \sin x} = \sin x$$

$\therefore$ The solution is $z \cdot \sin x = \int -\cot x \cdot e^{\sin x} \cdot \sin x\, dx + c$

or $$\sin y \sin x = -\int e^{\sin x} \cos x\, dx + c$$
$$= -\int e^t\, dt + c, \quad \text{where } t = \sin x$$
$$= -e^t + c = -e^{\sin x} + c$$

or $$\sin y = \text{cosec}\, x(c - e^{\sin x}).$$

(*ii*) The given equation is

$$x \frac{dy}{dx} + y \log y = xye^x \quad \text{or} \quad \frac{1}{y}\frac{dy}{dx} + \log y \cdot \frac{1}{x} = e^x \quad \ldots(1)$$

Putting $\log y = z$, $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ From (1), $\dfrac{dz}{dx} + \dfrac{1}{x} z = e^x$

which is linear in $z$.

$$\text{P} = \frac{1}{x},\ \text{Q} = e^x$$

$$\text{I.F.} = e^{\int \frac{1}{x} dx} = e^{\log x} = x$$

$\therefore$ The solution is $\quad z \,.\, x = \int xe^x \, dx + c = xe^x - \int e^x \, dx + c = xe^x - e^x + c$

or $\quad x \log y = (x - 1)\, e^x + c.$

**Example 10.** *Show how to solve an equation of the form*

$$f'(y) \frac{dy}{dx} + Pf(y) = Q$$

*where P, Q are functions of x only. Hence or otherwise solve the differential equation*

$$3x(1 - x^2)y^2 \frac{dy}{dx} + (2x^2 - 1)y^3 = ax^3.$$

**Sol.** (*a*) The given equation is

$$f'(y) \frac{dy}{dx} + Pf(y) = Q \qquad ...(i)$$

where P, Q are functions of $x$ only.

Put $f(y) = z$, then $\quad f'(y) \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ (*i*) becomes $\qquad \dfrac{dz}{dx} + Pz = Q$

which is linear in $z$ and can be solved.

[I.F. $= e^{\int P\, dx}$ and the solution is

$$z(\text{I.F.}) = \int Q \,.\, (\text{I.F.})\, dx + c$$

or $\quad f(y) \,.\, (\text{I.F.}) = \int Q \,.\, (\text{I.F.})\, dx + c]$

(*b*) The given equation is

$$3x\,(1 - x^2)\, y^2 \frac{dy}{dx} + (2x^2 - 1)y^3 = ax^3$$

or
$$3y^2 \frac{dy}{dx} + \frac{2x^2 - 1}{x(1 - x^2)}\, y^3 = \frac{ax^2}{1 - x^2} \qquad ...(i)$$

It is of the form

$$f'(y) \frac{dy}{dx} + Pf(y) = Q \text{ of part } (a)$$

Here $\quad f(y) = y^3$ Put $y^3 = z$

then $\quad 3y^2 \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ (*i*) becomes $\dfrac{dz}{dx} + \dfrac{2x^2 - 1}{x(1 - x^2)}\, z = \dfrac{ax^2}{1 - x^2}$

which is linear in $z$.

$$P = \frac{2x^2 - 1}{x(1 - x^2)} = \frac{2x^2 - 1}{x(1 + x)(1 - x)} = -\frac{1}{x} - \frac{1}{2(1 + x)} + \frac{1}{2(1 - x)} \qquad \text{[Partial fractions]}$$

$$Q = \frac{ax^2}{1 - x^2}$$

$$\text{I.F.} = e^{\int \left[-\frac{1}{x} - \frac{1}{2(1+x)} + \frac{1}{2(1-x)}\right] dx}$$

$$= e^{-\log x - \frac{1}{2}\log(1+x) - \frac{1}{2}\log(1-x)}$$

$$= e^{-\log x(1+x)^{1/2}(1-x)^{1/2}} = e^{-\log x\sqrt{1-x^2}} = e^{\log(x\sqrt{1-x^2})^{-1}}$$

$$= (x\sqrt{1-x^2})^{-1} = \frac{1}{x\sqrt{1-x^2}}$$

$\therefore$ The solution is

$$z \cdot \frac{1}{x\sqrt{1-x^2}} = \int \frac{ax^2}{1-x^2} \cdot \frac{1}{x\sqrt{1-x^2}}\, dx + c$$

or
$$y^3 \cdot \frac{1}{x\sqrt{1-x^2}} = -\frac{a}{2}\int (1-x^2)^{-3/2}(-2x)\, dx + c$$

$$= -\frac{a}{2} \cdot \frac{(1-x^2)^{-1/2}}{-\frac{1}{2}} + c = \frac{a}{\sqrt{1-x^2}} + c$$

or
$$y^2 = ax + cx\sqrt{1-x^2}.$$

**Example 11.** *Solve the following differential equations :*

*(i)* $(x+1)\dfrac{dy}{dx} + 1 = e^{x-y}$ *(ii)* $\dfrac{dy}{dx} = y\tan x - y^2 \sec x$

*(iii)* $\dfrac{dy}{dx} + \dfrac{1}{x}\sin 2y = x^3\cos^2 y$ *(iv)* $x^3\dfrac{dy}{dx} - x^2y + y^4\cos x = 0.$

**Sol.** (*i*) The given equation is

$$(x+1)\frac{dy}{dx} + 1 = \frac{e^x}{e^y} \quad \text{or} \quad e^y\frac{dy}{dx} + \frac{e^y}{x+1} = \frac{e^x}{x+1} \qquad ...(i)$$

Putting $e^y = z$ so that $e^y\dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ (*i*) becomes $\dfrac{dz}{dx} + \dfrac{z}{x+1} = \dfrac{e^x}{x+1}$

which is linear in $z$ with $\text{P} = \dfrac{1}{x+1}$, $\text{Q} = \dfrac{e^x}{x+1}$

$$\text{I.F.} = e^{\int \text{P}\, dx} = e^{\int \frac{1}{x+1} dx} = e^{\log(x+1)} = x + 1$$

$\therefore$ The solution is

$$z(x+1) = \int \frac{e^x}{x+1} \cdot (x+1)\, dx + c \text{ or } e^y(x+1) = e^x + c.$$

(*ii*) The given equation is

$$\frac{dy}{dx} - y\tan x = -y^2\sec x$$

or $$-\frac{1}{y^2}\cdot\frac{dy}{dx}+\frac{1}{y}\tan x=\sec x \qquad ...(1)$$

Putting $\frac{1}{y}=z$ so that $-\frac{1}{y^2}\frac{dy}{dx}=\frac{dz}{dx}$

$\therefore$ Equation (1) becomes $\frac{dz}{dx}+z\tan x=\sec x$

which is linear in $z$ with $\quad P=\tan x,\quad Q=\sec x$

$$\text{I.F.}=e^{\int P\,dx}=e^{\int \tan x\,dx}=e^{\log\sec x}=\sec x$$

$\therefore$ The solution is

$$z\,.\sec x=\int \sec x\,.\sec x\,dx+c$$

or $$\frac{1}{y}\sec x=\tan x+c \text{ or } \frac{1}{y}=\sin x+c\cos x.$$

(*iii*) The given equation is

$$\frac{dy}{dx}+\frac{1}{x}\,.\,2\sin y\cos y=x^3\cos^2 y$$

Dividing by $\cos^2 y$

$$\sec^2 y\,\frac{dy}{dx}+\frac{2\tan y}{x}=x^3 \qquad ...(1)$$

Putting $\tan y=z$ so that $\sec^2 y\,\frac{dy}{dx}=\frac{dz}{dx}$

$\therefore$ Equation (1) becomes $\frac{dz}{dx}+\frac{2}{x}z=x^3$

which is linear in $z$ with $\quad P=\frac{2}{x},\quad Q=x^3$

$$\text{I.F.}=e^{\int P\,dx}=e^{\int \frac{2}{x}dx}=e^{2\log x}=e^{\log x^2}=x^2$$

$\therefore$ The solution is

$$z\,.\,x^2=\int x^3\,.\,x^2\,dx+c \text{ or } x^2\tan y=\frac{x^6}{6}+c.$$

(*iv*) The given equation is

$$x^3\frac{dy}{dx}-x^2y=-y^4\cos x \qquad \text{or} \qquad \frac{dy}{dx}-\frac{y}{x}=-\frac{y^4\cos x}{x^3}$$

Dividing by $y^4$

$$y^{-4}\frac{dy}{dx}-\frac{1}{x}y^{-3}=-\frac{\cos x}{x^3} \qquad ...(1)$$

Putting $y^{-3}=z$ so that $-3y^{-4}\frac{dy}{dx}=\frac{dz}{dx}$

or $$y^{-4}\frac{dy}{dx}=-\frac{1}{3}\cdot\frac{dz}{dx}$$

$\therefore$ Equation (1) becomes $$-\frac{1}{3}\frac{dz}{dx}-\frac{1}{x}z=-\frac{\cos x}{x^3}$$

or $$\frac{dz}{dx}+\frac{3}{x}z=\frac{3\cos x}{x^3}$$

which is linear in $z$ with $P=\frac{3}{x}$, $Q=\frac{3\cos x}{x^3}$

$$\text{I.F.}=e^{\int P\,dx}=e^{\int \frac{3}{x}dx}=e^{3\log x}=e^{\log x^3}=x^3$$

$\therefore$ The solution is $$z\,.\,x^3=\int \frac{3\cos x}{x^3}\,.\,x^3\,dx+c$$

or $$y^{-3}x^3=3\sin x+c \qquad \text{or} \qquad \frac{x^3}{y^3}=3\sin x+c.$$

**Example 12.** *Solve the following differential equations :*

(*i*) $y^2\frac{dy}{dx}=x+y^3$ (*ii*) $\frac{dy}{dx}+\frac{2y}{x}=\frac{y^3}{x^3}$

(*iii*) $\frac{dy}{dx}\sin x-y\cos x+y^2=0$ (*iv*) $\sec^2 y\frac{dy}{dx}+2x\tan y=x^3.$

**Sol.** (*i*) The given equation is

$$y^2\frac{dy}{dx}-y^3=x \qquad ...(1)$$

Putting $y^3=z$ so that $$3y^2\frac{dy}{dx}=\frac{dz}{dx} \qquad \text{or} \qquad y^2\frac{dy}{dx}=\frac{1}{3}\frac{dz}{dx}$$

$\therefore$ Equation (1) becomes $$\frac{1}{3}\frac{dz}{dx}-z=x \qquad \text{or} \qquad \frac{dz}{dx}-3z=3x$$

which is linear in $z$ with $P=-3$, $Q=3x$

$$\text{I.F.}=e^{\int P\,dx}=e^{-3x}$$

$\therefore$ The solution is $$z\,.\,e^{-3x}=\int 3xe^{-3x}\,dx+c$$

or $$y^3e^{-3x}=3x\,.\,\frac{e^{-3x}}{-3}-\int 3\,.\,\frac{e^{-3x}}{-3}\,dx+c$$

$$=-x\,e^{-3x}+\frac{e^{-3x}}{-3}+c \quad \text{or} \quad y^3=-x-\frac{1}{3}+ce^{3x}.$$

(*ii*) Please try yourself. $\left[\textbf{Ans. } \frac{1}{y^2}=cx^4+\frac{1}{3x^2}\right]$

(*iii*) The given equation is

$$\frac{dy}{dx}-y\cot x=-y^2\operatorname{cosec} x \qquad \text{or} \qquad \frac{1}{y^2}\,.\,\frac{dy}{dx}-\frac{1}{y}\cot x=-\operatorname{cosec} x$$

Put $-\frac{1}{y}=z$ and proceed further yourself. $\left[\textbf{Ans. } x-\frac{\sin x}{y}=c\right]$

(*iv*) The given equation is

$$\sec^2 y \frac{dy}{dx} + 2x \tan y = x^3 \qquad ...(1)$$

Putting $\tan y = z$ so that $\sec^2 y \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ Equation (1) becomes $\dfrac{dz}{dx} + 2xz = x^3$

which is linear in $z$ with $P = 2x$, $Q = x^3$

$$\text{I.F.} = e^{\int P\,dx} = e^{x^2}$$

$\therefore$ The solution is $z \cdot e^{x^2} = \int x^3 e^{x^2}\, dx + c$

or

$$e^{x^2} \tan y = \int x \cdot x^2 e^{x^2}\, dx + c$$

$$= \tfrac{1}{2} \int te^t\, dt + c \quad \text{where } t = x^2$$

$$= \frac{1}{2}\left[te^t - \int 1 \cdot e^t\, dt\right] + c = \frac{1}{2}(te^t - e^t) + c$$

$$= \frac{1}{2}(x^2 - 1)e^{x^2} + c$$

or

$$\tan y = \frac{1}{2}(x^2 - 1) + c\, e^{-x^2}.$$

**Example 13.** *Solve the following differential equations :*

(*i*) $\sin y \dfrac{dy}{dx} = \cos y (1 - x \cos y)$ (*ii*) $\dfrac{dy}{dx} + xy = x^3y^4$

(*iii*) $2 \dfrac{dy}{dx} - y \sec x = y^3 \tan x$.

**Sol.** (*i*) The given equation is

$$\sin y \frac{dy}{dx} - \cos y = -x \cos^2 y$$

Dividing by $\cos^2 y$

$$\sec y \tan y \frac{dy}{dx} - \sec y = -x \qquad ...(1)$$

Putting $\sec y = z$ so that $\sec y \tan y \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ Equation (1) becomes $\dfrac{dz}{dx} - z = -x$

which is linear in $z$ with $P = -1$, $Q = -x$

$$\text{I.F.} = e^{\int P\,dx} = e^{-x}$$

$\therefore$ The solution is $z \cdot e^{-x} = \int -x e^{-x}\, dx + c$

or $$\sec y \,.\, e^{-x} = -x \,.\, \frac{e^{-x}}{-1} - \int -1 \,.\, \frac{e^{-x}}{-1}\, dx + c$$

$$= x\, e^{-x} + e^{-x} + c \qquad \text{or} \qquad \sec y = x + 1 + ce^{x}.$$

(*ii*) The given equation is $\dfrac{dy}{dx} + xy = x^3y^4$

Dividing by $y^4$, $$y^{-4}\frac{dy}{dx} + xy^{-3} = x^3 \qquad ...(1)$$

Putting $y^{-3} = z$, so that $$-3y^{-4}\frac{dy}{dx} = \frac{dz}{dx} \qquad \text{or} \qquad y^{-4} \,.\, \frac{dy}{dx} = -\frac{1}{3}\frac{dz}{dx}$$

$\therefore$ Equation (1) becomes $-\dfrac{1}{3} \,.\, \dfrac{dz}{dx} + x\,z = x^3$ or $\dfrac{dz}{dx} - 3xz = -3x^3$

which is linear in $z$ with $P = -3x$, $Q = -3x^3$

$$\text{I.F.} = e^{\int P\,dx} = e^{-\frac{3}{2}x^2}$$

$\therefore$ The solution is

$$z \,.\, e^{-\frac{3}{2}x^2} = \int -3x^3 \,.\, e^{-\frac{3}{2}x^2}\, dx + c = \int x^2\,(-3x)\, e^{-\frac{3}{2}x^2}\, dx + c$$

$$= \int -\frac{2}{3}\, t\, e^t\, dt + c, \text{ where } t = -\frac{3}{2}x^2$$

$$= -\frac{2}{3}\left[t\,e^t - \int 1 . e^t\, dt\right] + c = -\frac{2}{3}\,[t\,e^t - e^t] + c$$

or $$y^{-3} \,.\, e^{-\frac{3}{2}x^2} = -\frac{2}{3}\left(-\frac{3}{2}x^2 - 1\right) e^{-\frac{3}{2}x^2} + c \qquad \text{or} \qquad \frac{1}{y^3} = x^2 + \frac{2}{3} + c\, e^{\frac{3}{2}x^2}$$

(*iii*) The given equation is

$$2\frac{dy}{dx} - y\sec x = y^3\tan x \qquad \text{or} \qquad \frac{2}{y^3}\frac{dy}{dx} - \frac{1}{y^2}\sec x = \tan x \qquad ...(1)$$

Putting $-\dfrac{1}{y^2} = z$ so that $\dfrac{2}{y^3} \,.\, \dfrac{dy}{dx} = \dfrac{dz}{dx}$

$\therefore$ Equation (1) becomes $\dfrac{dz}{dx} + z\sec x = \tan x$

which is linear in $z$ with $P = \sec x$, $Q = \tan x$

$$\text{I.F.} = e^{\int P\,dx} = e^{\log(\sec x + \tan x)} = \sec x + \tan x$$

$\therefore$ The solution is

$$z\,(\sec x + \tan x) = \int \tan x\,(\sec x + \tan x)\, dx + c$$

or $$-\frac{1}{y^2}\,(\sec x + \tan x) = \int (\sec x \tan x + \sec^2 x - 1)\, dx + c$$

or $$-\frac{1}{y^2}\,(\sec x + \tan x) = \sec x + \tan x - x + c.$$

**Example 14.** *Solve the following differential equations :*

(*i*) $r \sin \theta \, d\theta + (r^3 - 2r^2 \cos \theta + \cos \theta) \, dr = 0$

(*ii*) $\dfrac{dy}{dx} = \dfrac{y}{x + \sqrt{xy}}$.

**Sol.** (*i*) The given equation can be written as

$$r \sin \theta \frac{d\theta}{dr} + (1 - 2r^2) \cos \theta = -r^3$$

Dividing by $r$, we have

$$\sin \theta \frac{d\theta}{dr} + \left(\frac{1}{r} - 2r\right) \cos \theta = -r^2 \qquad ...(i)$$

Putting $\cos \theta = z$ so that $-\sin \theta \dfrac{d\theta}{dr} = \dfrac{dz}{dr}$

$\therefore$ Equation (1) becomes $-\dfrac{dz}{dr} + \left(\dfrac{1}{r} - 2r\right) z = -r^2$

or $$\frac{dz}{dr} + \left(2r - \frac{1}{r}\right) z = r^2$$

It is of the form $\dfrac{dz}{dr} + Pz = Q$ where P and Q are functions of $r$ only.

$$\text{I.F.} = e^{\int\left(2r - \frac{1}{r}\right) dr} = e^{r^2 - \log r} = e^{r^2 + \log \frac{1}{r}} = e^{r^2} \cdot e^{\log \frac{1}{r}} = \frac{1}{r} e^{r^2}$$

$\therefore$ The solution is $z \cdot \dfrac{1}{r} e^{r^2} = \displaystyle\int r^2 \cdot \frac{1}{r} e^{r^2} \, dr + c = \int e^{r^2} \cdot r \, dr + c$

$$= \frac{1}{2} \int e^t \, dt + c \quad \text{where } t = r^2$$

$$= \frac{1}{2} e^t + c = \frac{1}{2} e^{r^2} + c$$

or $$2z = r \, e^{-r^2} (e^{r^2} + 2c)$$

or $$2 \cos \theta = r (1 + Ce^{-r^2}) \qquad \text{where } C = 2c.$$

(*ii*) The given equation can be written as

$$y \frac{dx}{dy} = x + \sqrt{xy} \qquad \text{or} \qquad \frac{dx}{dy} - \frac{x}{y} = \sqrt{\frac{x}{y}}$$

Dividing by $\sqrt{x}$, we have $\dfrac{1}{\sqrt{x}} \cdot \dfrac{dx}{dy} - \dfrac{\sqrt{x}}{y} = \dfrac{1}{\sqrt{y}}$ ...(1)

Putting $\sqrt{x} = z$ so that $\dfrac{1}{2\sqrt{x}} \cdot \dfrac{dx}{dy} = \dfrac{dz}{dy}$

$\therefore$ Equation (1) becomes $2 \dfrac{dz}{dy} - \dfrac{z}{y} = \dfrac{1}{\sqrt{y}}$ or $\dfrac{dz}{dy} - \dfrac{1}{2y} z = \dfrac{1}{2\sqrt{y}}$

It is of the form $\dfrac{dz}{dy} + Pz = Q$, where P and Q are functions of $y$ only.

$$\text{I.F.} = e^{\int -\frac{1}{2y}\,dy} = e^{-\frac{1}{2}\log y} = e^{\log y^{-1/2}} = y^{-1/2} = \frac{1}{\sqrt{y}}$$

$\therefore$ The solution is $z \,.\, \dfrac{1}{\sqrt{y}} = \displaystyle\int \frac{1}{2\sqrt{y}} \cdot \frac{1}{\sqrt{y}}\, dy + c$ or $\dfrac{\sqrt{x}}{\sqrt{y}} = \dfrac{1}{2}\displaystyle\int \frac{dy}{y} + c$

or $\sqrt{x} = \sqrt{y}\left(\dfrac{1}{2}\log y + c\right)$ or $\sqrt{x} = \sqrt{y}\left(\log\sqrt{y} + c\right)$.

## TYPE VI. EXACT DIFFERENTIAL EQUATIONS

**Definition.** The equation $Mdx + Ndy = 0$ [where M, N are functions of $x$ and $y$] is said to be exact if $Mdx + Ndy$ is the exact differential of a function of $x$ and $y$, *i.e.*, if $Mdx + Ndy = du$, where $u$ is a function of $x$ and $y$.

**Remember.** If $u = f(x, y)$ be a function of two variables $x$ and $y$, then the **total differential** of $u$ or **the exact differential** of $u$ is given by

$$du = \frac{\partial u}{\partial x}\,.\,dx + \frac{\partial u}{\partial y}\,.\,dy$$

*i.e.*, $du$ is the sum of the two partial differentials

$$\frac{\partial u}{\partial x}\,dx \quad \text{and} \quad \frac{\partial u}{\partial y}\,dy$$

**Art. To find the necessary and sufficient condition that the equation Mdx + Ndy = 0 may be exact**

**To find the necessary condition.**

Let the equation $Mdx + Ndy = 0$ be exact.

Then by def. $Mdx + Ndy = du$, where $u$ is a function of $x$ and $y$

$$= \frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy$$

Equating co-efficients of $dx$, $\quad M = \dfrac{\partial u}{\partial x}$

Equating co-efficients of $dy$, $\quad N = \dfrac{\partial u}{\partial y}$

so that
$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x}, \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}$$

**Hence**
$$\frac{\partial \mathbf{M}}{\partial y} = \frac{\partial \mathbf{N}}{\partial \mathbf{x}} \qquad \left[\because \frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}\right]$$

**is the required necessary condition for exactness.**

**To prove that the condition is sufficient**

$$\left[\begin{array}{c} i.e., \text{ Given } \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}, \text{ to prove that} \\ Mdx + Ndy = 0 \text{ is exact} \end{array}\right]$$

Let $\int M dx = u$ ...(*i*)

**where integration has been performed treating y as constant**

so that $\dfrac{\partial}{\partial x}\left(\int M\,dx\right) = \dfrac{\partial u}{\partial x} \quad \Rightarrow \quad M = \dfrac{\partial u}{\partial x}$ ...(*ii*)

$\therefore$ $\dfrac{\partial M}{\partial y} = \dfrac{\partial^2 u}{\partial y \partial x}$ ...(*iii*)

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ (*given*) and $\dfrac{\partial^2 u}{\partial y \partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$

$\therefore$ (*iii*) becomes $\dfrac{\partial N}{\partial x} = \dfrac{\partial^2 u}{\partial x \partial y}$

Integrating both sides w.r.t. $x$, **treating y as constant.**

$$N = \frac{\partial u}{\partial y} + \text{a function of } y = \frac{\partial u}{\partial y} + f(y) \ (say) \qquad ...(iv)$$

From (*ii*) and (*iv*)

$$Mdx + Ndy = \frac{\partial u}{\partial x}\,dx + \left(\frac{\partial u}{\partial y} + f(y)\right) dy$$

$$= \left(\frac{\partial u}{\partial x}\,dx + \frac{\partial u}{\partial y}\,dy\right) + f(y)dy$$

$$= du + f(y)\,dy = d[u + \textstyle\int f(y)dy] \qquad ...(v)$$

which is an exact differential.

Hence $Mdx + Ndy = 0$ is an exact differential equation.

**Cor. Solution of an exact differential equation.**

If the equation $Mdx + Ndy = 0$ is exact

then $Mdx + Ndy = d[u + \int f(y)\,dy]$

$\therefore$ $Mdx + Ndy = 0 \quad \Rightarrow \quad d[u + \int f(y)\,dy] = 0$ | using (*v*)

Integrating both sides, we get

$$u + \int f(y)\,dy = c$$

where $c$ is an arbitrary constant.

From (*i*) $u = \displaystyle\int_{y\,-\,\text{constant}} Mdx$

From (*iv*) $f(y)$ = terms in N not containing $x$.

$\therefore$ From (*vi*) $\displaystyle\int_{y\,-\,\text{constant}} Mdx + \int$ (terms in N not containing $x$) $dy = c$

which is the reqd. solution.

**Working Rule**

**1. If for an equation of the form Mdx + Ndy = 0.**

**[where M, N are functions of x and y]** $\dfrac{\partial \mathbf{M}}{\partial \mathbf{y}} = \dfrac{\partial \mathbf{N}}{\partial \mathbf{x}}$**, it is exact.**

**2. Then the solution is**

$$\int_{y\text{ - constant}} M dx + \int (\text{terms in N } \textbf{not containing x})\, dy = c.$$

**Example 1.** *Solve the following* :

(*i*) $(e^y + 1) \cos x\, dx + e^y \sin x\, dy = 0$ (Rohilkhand, 1997)   (*ii*) $(a^2 - 2xy - y^2)dx - (x + y)^2 dy = 0$

(*iii*) $(3x^2 + 6xy^2)dx + (6x^2y + 4y^3)dy = 0$   (*iv*) $(x^2 - ay)dx = (ax - y^2)dy$. (Delhi, 1996)

**Sol.** (*i*) The given equation is $(e^y + 1) \cos x\, dx + e^y \sin x\, dy = 0$ ...(*i*)

Comparing it with $M dx + N dy = 0$

Here $M = (e^y + 1) \cos x$ ; $N = e^y \sin x$

$$\frac{\partial M}{\partial y} = e^y \cos x, \quad \frac{\partial N}{\partial x} = e^y \cos x$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ the equation (*i*) is exact.

Hence the solution of (*i*) is

$$\int_{y\text{ - constant}} M dx + \int (\text{terms in N not containing } x)\, dy = c$$

*i.e.*,
$$\int_{y\text{ - constant}} (e^y + 1) \cos x\, dx = c$$

or
$$(e^y + 1) \int \cos x dx = c \quad \text{or} \quad (e^y + 1) \sin x = c.$$

(*ii*) The given equation is $(a^2 - 2xy - y^2)\, dx - (x + y)^2\, dy = 0$ ...(*i*)

Comparing it with $M dx + N dy = 0$

Here $M = a^2 - 2xy - y^2$ ; $N = -(x + y)^2$

$$\frac{\partial M}{\partial y} = 0 - 2x - 2y = -2(x + y) \; ; \; \frac{\partial N}{\partial x} = -2(x + y)$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation (*i*) is exact.

Hence the solution of (*i*) is

$$\int_{y\text{ - constant}} M dx + \int (\text{terms in N not containing } x)\, dy = c$$

*i.e.*,
$$\int_{y\text{ - constant}} (a^2 - 2xy - y^2)\, dx + \int - y^2\, dy = c$$

or
$$a^2x - 2y \cdot \frac{x^2}{2} - y^2 \cdot x - \frac{y^3}{3} = c \quad \text{or} \quad a^2x - x^2y - xy^2 - \frac{y^3}{3} = c.$$

(*iii*) The given equation is $(3x^2 + 6xy^2)\, dx + (6x^2y + 4y^3)dy = 0$

Comparing it with $M dx + N dy = 0$

Here $M = 3x^2 + 6xy^2$ ; $N = 6x^2y + 4y^3$

$$\frac{\partial M}{\partial y} = 6x \cdot 2y = 12xy \; ; \; \frac{\partial N}{\partial x} = 6y \cdot 2x = 12xy$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ equation (*i*) is exact

Hence the solution of (*i*) is

$$\int_{y\text{ - constant}} M dx + \int (\text{terms in N not containing } x)\, dy = c$$

*i.e.,*
$$\int_{y\text{ - constant}} (3x^2 + 6xy^2)\, dx + \int 4y^3\, dy = c$$

or
$$3 \cdot \frac{x^3}{3} + 6y^2 \frac{x^2}{2} + 4 \cdot \frac{x^4}{4} = c \quad \text{or} \quad x^3 + 3x^2y^2 + y^4 = c.$$

(*iv*) The given equation is $(x^2 - ay)dx + (y^2 - ax)dy = 0$ ...(*i*)

Comparing it with $Mdx + Ndx = 0$

Here $M = x^2 - ay\,; \quad N = y^2 - ax$

$$\frac{\partial M}{\partial y} = -a\,; \qquad \frac{\partial N}{\partial x} = -a$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ equation (*i*) is exact.

Hence the solution is

$$\int_{y\text{ - constant}} M dx + \int (\text{terms in N not containing } x)\, dy = c$$

*i.e.,*
$$\int_{y\text{ - constant}} (x^2 - ay)\, dx + \int y^2 dy = c$$

or
$$\frac{x^3}{3} - ayx + \frac{y^3}{3} = c \qquad \text{or} \qquad x^3 - 3axy + y^3 = 3c$$

or
$$x^3 - 3axy + y^3 = A.$$

**Example 2.** *Solve the following :*

(*i*) $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$ (*ii*) $(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0$

(*iii*) $\left[y\left(1 + \frac{1}{x}\right) + \cos y\right] dx + [x + \log x - x \sin y] dy = 0$

(*iv*) $\cos x(\cos x - \sin a \sin y)\, dx + \cos y(\cos y - \sin a \sin x)\, dy = 0.$

**Sol.** (*i*) The given equation is $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0$ ...(*i*)

Comparing it with $Mdx + Ndy = 0$

Here $M = \sec^2 x \tan y\,; N = \sec^2 y \tan x$

$$\frac{\partial M}{\partial y} = \sec^2 x \sec^2 y\,; \frac{\partial N}{\partial x} = \sec^2 y \sec^2 x$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ equation (*i*) is exact.

Hence the solution is

$$\int_{y\text{ - constant}} M dx + \int (\text{ terms in N not containing } x)\, dy = c$$

i.e.,
$$\int_{y\text{ - constant}} \sec^2 x \tan y \, dx = c$$

or
$$\tan y \int \sec^2 x \, dx = c \quad \text{or} \quad \tan y \tan x = c.$$

(ii) The given equation is $$(1 + e^{x/y})\, dx + e^{x/y}\left(1 - \frac{x}{y}\right) dy = 0 \quad ...(i)$$

Comparing it with $M dx + N dy = 0$

Here $$M = 1 + e^{x/y},\ N = e^{x/y}\left(1 - \frac{x}{y}\right)$$

$$\frac{\partial M}{\partial y} = e^{x/y}\left(-\frac{x}{y^2}\right) = -\frac{x}{y^2} e^{x/y}$$

$$\frac{\partial N}{\partial x} = e^{x/y} \cdot \frac{1}{y}\left(1 - \frac{x}{y}\right) + e^{x/y}\left(-\frac{1}{y}\right) = e^{x/y}\left(\frac{1}{y} - \frac{x}{y^2} - \frac{1}{y}\right) = -\frac{x}{y^2} e^{x/y}$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, $\therefore$ equation (i) is exact.

Hence the solution is

$$\int_{y\text{ - constant}} M dx + \int (\text{terms in N not containing } x)\, dy = c$$

i.e.,
$$\int_{y\text{ - constant}} (1 + e^{x/y})\, dx = c \quad \text{or} \quad x + \frac{e^{x/y}}{\frac{1}{y}} = c \quad \text{or} \quad x + ye^{x/y} = c.$$

(iii) The given equation is

$$\left[y\left(1 + \frac{1}{x}\right) + \cos y\right] dx + [x + \log x - x \sin y]\, dy = 0 \quad ...(i)$$

Comparing it with $M dx + N dy = 0$

Here $$M = y\left(1 + \frac{1}{x}\right) + \cos y,\ N = x + \log x - x \sin y$$

$$\frac{\partial M}{\partial y} = 1 + \frac{1}{x} - \sin y,\ \frac{\partial N}{\partial x} = 1 + \frac{1}{x} - \sin y$$

Since, $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ $\therefore$ equation (i) is exact.

Hence the solution is

$$\int_{y\text{ - constant}} M dx + \int (\text{terms in N not containing } x)\, dy = c$$

i.e., $$\int_{y-\text{constant}} \left[y\left(1+\frac{1}{x}\right)+\cos y\right] dx = c$$

or $$y\int\left(1+\frac{1}{x}\right) dx + \cos y \int 1\, dx = c$$

or $$y(x+\log x) + x\cos y = c.$$

(*iv*) The given equation is

$$\cos x\,(\cos x - \sin a \sin y)\, dx + \cos y\,(\cos y - \sin a \sin x)\, dy = 0 \qquad \text{...}(i)$$

Comparing it with $Mdx + Ndy = 0$

Here $M = \cos x(\cos x - \sin a \sin y) = \cos^2 x - \cos x \sin a \sin y$

and $N = \cos y\,(\cos y - \sin a \sin x) = \cos^2 y - \cos y \sin a \sin x$

$$\frac{\partial M}{\partial y} = -\cos x \sin a \cos y \;;\; \frac{\partial N}{\partial x} = -\cos y \sin a \cos x$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ $\therefore$ equation (*i*) is exact.

Hence the solution is

$$\int_{y-\text{constant}} Mdx + \int (\text{terms in N not containing } x)\, dy = c$$

i.e., $$\int_{y-\text{constant}} (\cos^2 x - \sin a \sin y \cos x)\, dx + \int \cos^2 y\, dy = c$$

or $$\int \frac{1+\cos 2x}{2}\, dx - \sin a \sin y \sin x + \int \frac{1+\sin 2y}{2}\, dy = c$$

or $$\frac{1}{2}\left(x+\frac{\sin 2x}{2}\right) - \sin a \sin x \sin y + \frac{1}{2}\left(y+\frac{\sin 2y}{2}\right) = c.$$

**Example 3.** *Solve the following* :.

(*i*) $xdx + ydy = \dfrac{a^2(xdy - ydx)}{x^2+y^2}$ (Kanpur, 1998)

(*ii*) $(y^2 e^{xy^2} + 4x^3)\, dx + (2xy\, e^{xy^2} - 3y^2) dy = 0$

(*iii*) $(2xy + y - \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)\, dy = 0$

(*iv*) $[\cos x \tan y + \cos(x+y)]dx + [\sin x \sec^2 y + \cos(x+y)]dy = 0.$

**Sol.** (*i*) The given equation is

$$xdx + ydy = \frac{a^2 x}{x^2+y^2}\, dy - \frac{a^2 y}{x^2+y^2}\, dx$$

or $$\left[x+\frac{a^2 y}{x^2+y^2}\right] dx + \left[y - \frac{a^2 x}{x^2+y^2}\right] dy = 0 \qquad \text{...}(i)$$

Comparing it with $Mdx + Ndy = 0$

Here $$M = x + \frac{a^2 y}{x^2+y^2}\ ;\ N = y - \frac{a^2 x}{x^2+y^2}$$

$$\frac{\partial M}{\partial y} = 0 + a^2 \frac{(x^2+y^2) - y(2y)}{(x^2+y^2)^2} = \frac{a^2(x^2-y^2)}{(x^2+y^2)^2}$$

$$\frac{\partial N}{\partial x} = 0 - a^2 \frac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \frac{a^2(x^2-y^2)}{(x^2+y^2)^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ equation (*i*) is exact.

Hence the solution is

$$\int_{y\text{ - constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c$$

*i.e.,* $$\int_{y\text{ - constant}} \left(x + \frac{a^2 y}{x^2+y^2}\right) dx + \int y\,dy = c$$

or $$\frac{x^2}{2} + a^2 y \int_{y\text{ - constant}} \frac{dx}{x^2+y^2} + \frac{y^2}{2} = c$$

| From $\int \frac{dx}{x^2+a^2}$
| $\because$ here $y$ is constant

or $$\frac{x^2}{2} + a^2 y \,.\, \frac{1}{y}\tan^{-1}\frac{x}{y} + \frac{y^2}{2} = c$$

or $$x^2 + y^2 + 2a^2 \tan^{-1}\frac{x}{y} = 2c$$

or $$x^2 + y^2 + 2a^2 \tan^{-1}\frac{x}{y} = A$$

(*ii*) The given equation is

$$(y^2 e^{xy^2} + 4x^3)\,dx + (2xy \,.\, e^{xy^2} - 3y^2)\,dy = 0 \qquad ...(i)$$

Comparing it with $M dx + N dy = 0$

Here $$M = y^2 e^{xy^2} + 4x^3\ ;\ N = 2xy\, e^{xy^2} - 3y^2$$

$$\frac{\partial M}{\partial y} = 2y \,.\, e^{xy^2} + y^2 \,.\, e^{xy^2} \,.\, 2xy\ ;\ \frac{\partial N}{\partial x} = 2y\, e^{xy^2} + 2xy \,.\, e^{xy^2} \,.\, y^2$$

$$= 2y \,.\, e^{xy^2} + 2xy^3 \,.\, e^{xy^2} = 2y \,.\, e^{xy^2} + 2xy^3 \,.\, e^{xy^2}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ equation (*i*) is exact.

Hence the solution is

$$\int_{y\text{ - constant}} M\,dx + \int (\text{terms in N not containing } x)\,dy = c$$

*i.e.,* $$\int_{y\text{ - constant}} (y^2 e^{xy^2} + 4x^3)\,dx + \int (-3y^2)\,dy = c$$

or $$y^2 \int_{y-\text{constant}} e^{y^2 x}\, dx + \frac{4x^4}{4} - 3\frac{y^3}{3} = c$$

or $$y^2 \cdot \frac{e^{y^2 x}}{y^2} + x^4 - y^3 = c \text{ or } e^{xy^2} + x^4 - y^3 = c$$

$$\left[ \textbf{Note. } \int ae^{ax}\, dx = a \cdot \frac{e^{ax}}{a} = e^{ax} \qquad \therefore \int_{y-\text{constant}} ye^{xy}\, dx = y \int e^{xy}\, dx = y \cdot \frac{e^{xy}}{y} = e^{xy} \right].$$

(*iii*) The given equation is

$$(2xy + y - \tan y)dx + (x^2 - x\tan^2 y + \sec^2 y)dy = 0 \qquad ...(i)$$

Comparing it with $Mdx + Ndy = 0$

Here $$M = 2xy + y - \tan y\ ;\ N = x^2 - x\tan^2 y + \sec^2 y$$

$$\frac{\partial M}{\partial y} = 2x + 1 - \sec^2 y\ ;\ \frac{\partial N}{\partial x} = 2x - \tan^2 y$$

$$= 2x - (\sec^2 y - 1) = 2x - \tan^2 y$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, $\quad\therefore$ equation (*i*) is exact.

Hence the solution is

$$\int_{y-\text{constant}} Mdx + \int (\text{terms in N not containing } x)\, dy = c$$

*i.e.*, $$\int_{y-\text{constant}} (2xy + y - \tan y)dx + \int \sec^2 y dy = c$$

or $$2y \cdot \frac{x^2}{2} + yx - x\tan y + \tan y = c$$

or $$x^2 y + xy - x\tan y + \tan y = c.$$

(*iv*) The given equation is

$$[\cos x \tan y + \cos(x + y)]dx + [\sin x \sec^2 y + \cos(x + y)dy = 0 \qquad ...(i)$$

Comparing it with $Mdx + Ndy = 0$

Here, $$M = \cos x \tan y + \cos(x + y)\ ;$$
$$N = \sin x \sec^2 y + \cos(x + y)$$

$$\frac{\partial M}{\partial y} = \cos x \sec^2 y - \sin(x + y)$$

$$\frac{\partial N}{\partial x} = \cos x \sec^2 y - \sin(x + y)$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ $\therefore$ equation (*i*) is exact.

Hence the solution is

$$\int_{y-\text{constant}} Mdx + \int (\text{terms in N not containing } x)\, dy = c$$

*i.e.*, $$\int_{y\text{-constant}} [\cos x \tan y + \cos(x+y)]\, dx = c$$

or $$\tan y \sin x + \sin(x+y) = c.$$

**Example 4.** *Solve the following :*

(*i*) $y \sin 2x\, dx - (y^2 + \cos^2 x)\, dy = 0$

(*ii*) $(x^4 - 2xy^2 + y^4)\, dx - (2x^2y - 4xy^3 + \sin y) dy = 0$

(*iii*) $(x^2 - 2xy + 3y^2) dx + (4y^3 + 6xy - x^2) dy = 0$

(*iv*) $(x^2 + y^2 + e^x) dx + 2xy\, dy = 0$

(*v*) $(2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x}) dy + (12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y) dx = 0$

(*vi*) $(2x^2y^3 + xy^2 + 3y) dx + (x^3y^2 + x^2y + 3x) dy = 0.$

**Sol.** (*i*) The given equation is

$$y \sin 2x\, dx - (y^2 + \cos^2 x)\, dy = 0 \qquad ...(i)$$

Comparing it with $Mdx + Ndy = 0$

Here $M = y \sin 2x,\ N = (y^2 + \cos^2 x)$

$$\frac{\partial M}{\partial y} = \sin 2x,\ \frac{\partial N}{\partial x} = -(-2\cos x \sin x) = \sin 2x$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, $\therefore$ equation (*i*) is exact.

Hence the solution is

$$\int_{y\text{-constant}} Mdx + \int (\text{terms in N not containing } x)\, dy = c$$

*i.e.*, $$\int_{y\text{-constant}} y \sin 2x\, dx + \int -y^2 dy = c$$

or $$y \cdot \frac{-\cos 2x}{2} - \frac{y^3}{3} = c \qquad \text{or} \qquad \frac{y\cos 2x}{2} + \frac{y^3}{3} + c = 0.$$

(*ii*) The given equation is

$$(x^4 - 2xy^2 + y^4) dx - (2x^2y - 4xy^3 + \sin y) dy = 0 \qquad ...(i)$$

Comparing it with $Mdx + Ndy = 0$

Here $M = x^4 - 2xy^2 + y^4$

$N = -(2x^2y - 4xy^3 + \sin y)$

$$\frac{\partial M}{\partial y} = -4xy + 4y^3$$

$$\frac{\partial N}{\partial x} = -4xy + 4y^3$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, equation (*i*) is exact.

Hence the solution is

$$\int_{y\text{-constant}} Mdx + \int (\text{terms in N not containing } x)\, dy = c$$

*i.e.,* $$\int_{y\text{ - constant}} (x^4 - 2xy^2 + y^4)dx + \int -\sin y\, dy = c$$

or $$\frac{y^5}{5} - 2y^2 \cdot \frac{x^2}{2} + y^4 \cdot x + \cos y = c \quad \text{or} \quad \frac{y^5}{5} - x^2y^2 + xy^4 + \cos y = c.$$

(*iii*) Please try yourself. $\left[\textbf{Ans. } \frac{x^3}{3} - x^4y + 3xy^2 + y^4 = c\right]$

(*iv*) Please try yourself. $\left[\textbf{Ans. } \frac{x^3}{3} + xy^2 + e^x = c\right]$

(*v*) The given equation is

$$(2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x})\, dy + (12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{3x} - e^y)\, dx = 0 \quad ...(i)$$

Comparing it with $Mdx + Ndy = 0$

Here $M = 12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y$

$N = 2x^2y + 4x^3 - 12xy^2 + 3y^2 - xe^y + e^{2x}$

$$\frac{\partial M}{\partial y} = 12x^2 + 4xy - 12y^2 + 2e^{2x} - e^y$$

$$\frac{\partial N}{\partial x} = 4xy + 12x^2 - 12y^2 - e^y + 2e^{2x}$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, equation (*i*) is exact.

Hence the solution is

$$\int_{y\text{ - constant}} Mdx + \int (\text{terms in N not containing } x)\, dy = c$$

*i.e.,* $$\int_{y\text{ - constant}} (12x^2y + 2xy^2 + 4x^3 - 4y^3 + 2ye^{2x} - e^y)\, dx + \int 3y^2\, dy = c$$

or $$4x^3y + x^2y^2 + x^4 - 4xy^3 + ye^{2x} - xe^y + y^3 = c.$$

(*vi*) Please try yourself. [**Ans.** $4x^3y^3 + 3x^2y^2 + 18xy = c$]

**Example 5.** *Solve* $\frac{dy}{dx} + \frac{ax + hy + g}{hx + by + f} = 0.$ (Delhi, 1997)

*Show that this differential equation represents a family of conics.*

**Sol.** The given equation is $(ax + hy + g)dx + (hx + by + f)dy = 0$ ...(*i*)

Comparing it with $Mdx + Ndy = 0$

Here $M = ax + hy + g,\ N = hx + by + f$

$$\frac{\partial M}{\partial y} = h\ ;\ \frac{\partial N}{\partial x} = h$$

Since $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ equation (*i*) is exact.

Hence the solution is $\int_{y\text{ - constant}} Mdx + \int (\text{terms in N not containing } x)\, dy = c$

*i.e.,* $$\int_{y-\text{constant}} (ax + hy + g)dx + \int (by + f)\, dy = c_1$$

or $$a \cdot \frac{x^2}{2} + hy \cdot x + gx + b \cdot \frac{y^2}{2} + fy = c_1$$

or $$ax^2 + 2hxy + by^2 + 2gx + 2fy - 2c_1 = 0$$

Replacing $-2c_1$ by $c$

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

which evidently represents a family of conics.

**Another Form of Above Equation**

*Show that the equation $(ax + hy + g)\, dx + (hx + by + f)dy = 0$ is exact and hence solve it.*

**Example 6.** *Solve the following differential equations :*

*(i)* $y(x^2 + y^2 + a^2) \dfrac{dy}{dx} + x(x^2 + y^2 - a^2) = 0$

*(ii)* $\left[y\left(1+\dfrac{1}{x}\right)\cos y\right] dx + (x + \log x)(\cos y - y \sin y)\, dy = 0.$

**Sol.** (*i*) The given equation can be written as

$$x(x^2 + y^2 - a^2)dx + y(x^2 + y^2 + a^2)dy = 0 \qquad ...(1)$$

Comparing it with $Mdx + Ndy = 0$

Here $\quad M = x(x^2 + y^2 - a^2)\,;\, N = y(x^2 + y^2 + a^2)$

$$\frac{\partial M}{\partial y} = 2xy\,,\ \frac{\partial N}{\partial x} = 2xy$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, equation (1) is exact.

$\therefore$ The solution is

$$\int_{y-\text{constant}} Mdx + \int (\text{terms in N not containing } x)\, dy = c$$

*i.e.,* $$\int_{y-\text{constant}} x(x^2 + y^2 - a^2)\, dx + \int y(y^2 + a^2)\, dy = c$$

or $$\int_{y-\text{constant}} [x^3 + (y^2 - a^2)\, x]\, dx + \int (y^3 + a^2 y)\, dy = c$$

or $$\frac{x^4}{4} + (y^2 - a^2) \cdot \frac{x^2}{2} + \frac{y^4}{4} + a^2 \cdot \frac{y^2}{2} = c$$

or $$(x^4 + y^4 + 2x^2y^2) + 2a^2(y^2 - x^2) = 4c$$

or $$(x^2 + y^2)^2 + 2a^2(y^2 - x^2) = C, \quad \text{where } C = 4c.$$

(*ii*) The given equation is

$$\left[y\left(1+\frac{1}{x}\right)\cos y\right] dx + (x + \log x)(\cos y - y \sin y)\, dy = 0$$

Comparing it with M$dx$ + N$dy$ = 0

Here $$M = y\left(1+\frac{1}{x}\right)\cos y,\ N = (x + \log x)(\cos y - y \sin y)$$

$$\frac{\partial M}{\partial y} = \left(1+\frac{1}{x}\right)(\cos y - y\sin y),\ \frac{\partial N}{\partial x} = \left(1+\frac{1}{x}\right)(\cos y - y \sin y)$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$ , the given equation is exact.

∴ The solution is

$$\int_{y\text{ - constant}} M dx + \int (\text{terms in N not containing } x)\, dy = c$$

*i.e.,*
$$\int_{y\text{ - constant}} y\left(1+\frac{1}{x}\right)\cos y\, dx = c$$

or
$$y\cos y \int \left(1+\frac{1}{x}\right) dx = c \qquad \text{or} \qquad y \cos y\,(x + \log x) = c.$$

**Example 7.** *Solve the following differential equations :*

(*i*) $\dfrac{dy}{dx} + \dfrac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$ (*ii*) $(y \cos x + 1)\, dx + \sin x\, dy = 0$

(*iii*) $ye^{xy}\, dx + (xe^{xy} + 2y)\, dy = 0$

(*iv*) $(5x^4 + 3x^2y^2 - 2xy^3)\, dx + (2x^3y - 3x^2y^2 - 5y^4)\, dy = 0.$

**Sol.** (*i*) The given equation can be written as

$$(y\cos x + \sin y + y)dx + (\sin x + x\cos y + x)\, dy = 0 \qquad ...(1)$$

Comparing it with M$dx$ + N$dy$ = 0

Here $M = y\cos x + \sin y + y, \quad N = \sin x + x \cos y + x$

$$\frac{\partial M}{\partial y} = \cos x + \cos y + 1, \quad \frac{\partial N}{\partial x} = \cos x + \cos y + 1$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation (1) is exact.

Hence the solution of (1) is

$$\int_{y\text{ - constant}} M dx + \int (\text{terms in N not containing } x)\, dy = c$$

*i.e.*
$$\int_{y\text{ - constant}} (y\cos x + \sin y + y)\, dx = c$$

or
$$y \sin x + (\sin y + y)\, x = c.$$

(*ii*) Please try yourself. **[Ans.** $y \sin x + x = c$]

(*iii*) The given equation is $ye^{xy}\, dx + (xe^{xy} + 2y)dy = 0$ ...(1)

Comparing it with M$dx$ + N$dy$ = 0

Here $M = ye^{xy}, \quad N = xe^{xy} + 2y$

$$\frac{\partial M}{\partial y} = y\,(e^{xy}\,.\,x) + e^{xy} = e^{xy}\,(xy+1)$$

$$\frac{\partial N}{\partial x} = x(e^{xy}\,.\,y) + e^{xy} = e^{xy}\,(xy+1)$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation (1) is exact.

Hence the solution of (1) is

$$\int_{y\,-\,\text{constant}} M dx + \int \text{(terms in N not containing } x)\, dy = c$$

*i.e.,*
$$\int_{y\,-\,\text{constant}} ye^{xy}\, dx + \int 2y\, dy = c$$

or
$$y\,.\,\frac{e^{xy}}{y} + 2\,.\,\frac{y^2}{2} = c \qquad \text{or} \qquad e^{xy} + y^2 = c.$$

(*iv*) Please try yourself. **[Ans.** $x^5 + x^3y^2 - x^2y^3 - y^5 = c$]

**Example 8.** *Solve the following differential equations :*

(*i*) $(\sec x \tan x \tan y - e^x)\, dx + \sec x \sec^2 y\, dy = 0$

(*ii*) $(2\, xy \cos x^2 - 2xy + 1)\, dx + (\sin x^2 - x^2)\, dy = 0$

(*iii*) $(\sin x \cos y + e^{2x})\, dx + (\cos x \sin y + \tan y)\, dy = 0$

(*iv*) $\left[y\left(1+\dfrac{1}{x}\right) + \cos y\right] dx + (x + \log x - x \sin y)\, dy = 0.$

**Sol.** (*i*) The given equation is

$$(\sec x \tan x \tan y - e^x)\, dx + \sec x \sec^2 y\, dy = 0 \qquad ...(1)$$

Comparing it with $Mdx + Ndy = 0$

Here $M = \sec x \tan x \tan y - e^x$, $N = \sec x \sec^2 y$

$$\frac{\partial M}{\partial y} = \sec x \tan x \sec^2 y, \quad \frac{\partial N}{\partial x} = \sec x \tan x \sec^2 y$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation (1) is exact.

Hence the solution of (1) is

$$\int_{y\,-\,\text{constant}} M dx + \int \text{(terms in N not containing } x)\, dy = c$$

*i.e.,*
$$\int_{y\,-\,\text{constant}} (\sec x \tan x \tan y - e^x)\, dx = c \qquad \text{or} \qquad \tan y \sec x - e^x = c.$$

(*ii*) The given equation is

$$(2xy \cos x^2 - 2xy + 1)\, dx + (\sin x^2 - x^2) dy = 0 \qquad ...(1)$$

Comparing it with $Mdx + Ndy = 0$

Here $M = 2xy \cos x^2 - 2xy + 1$, $N = \sin x^2 - x^2$

$$\frac{\partial M}{\partial y} = 2x \cos x^2 - 2x \quad , \quad \frac{\partial N}{\partial x} = 2x \cos x^2 - 2x$$

Since $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$, the equation (1) is exact.

Hence the solution of (1) is

$$\int_{y\text{ - constant}} M dx + \int (\text{terms in N not containing } x)\, dy = c$$

*i.e.,*
$$\int_{y\text{ - constant}} (2xy \cos x^2 - 2xy + 1) dx = c$$

or
$$y \int \cos t\, dt - y \int 2x\, dx + x = c \qquad \text{where } t = x^2$$

or
$$y \sin t - y\, x^2 + x = c$$

or
$$y \sin x^2 - x^2 y + x = c$$

(*iii*) Please try yourself. [**Ans.** $-\cos x \cos y + \dfrac{1}{2} e^{2x} + \log \sec y = c$]

(*iv*) Please try yourself. [**Ans.** $y\,(x + \log x) + x \cos y = c$]

**Integrating Factors.** Some of the equations which are not exact can sometimes be made exact after multiplying them by some suitable function of $x$ and $y$. Such a function is called an integrating factor.

Thus, an **integrating factor** of a differential equation is a factor such that if the equation is multiplied by it, the resulting equation is exact.

For example, consider the equation $y\, dx - x\, dy = 0$ ...(1)

Here $M = y$ and $N = -x$

$\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, therefore the equation is not exact.

(*i*) Multiplying the equation by $\dfrac{1}{y^2}$, it becomes $\dfrac{ydx - xdy}{y^2} = 0$ or $d\left(\dfrac{x}{y}\right) = 0$

which is exact.

(*ii*) Multiplying the equation by $\dfrac{1}{x^2}$, it becomes $\dfrac{ydx - xdy}{x^2} = 0$ or $d\left(\dfrac{y}{x}\right) = 0$

which is exact.

(*iii*) Multiplying the equation by $\dfrac{1}{xy}$, it becomes $\dfrac{dx}{x} - \dfrac{dy}{y} = 0$ or $d(\log x - \log y) = 0$

which is exact.

$\therefore$ $\dfrac{1}{y^2}, \dfrac{1}{x^2}$ and $\dfrac{1}{xy}$ are integrating factors of (1).

*If a differential equation has one integrating factor, it has an infinite number of integrating factors.*

**I.F. found by inspection.** In a number of problems, a little analysis helps to find the integrating factor. The following differentials are useful in selecting a suitable integrating factor.

(i) $ydx + xdy = d(xy)$ (ii) $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$

(iii) $\dfrac{ydx - xdy}{y^2} = d\left(\dfrac{x}{y}\right)$ (iv) $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

(v) $\dfrac{xdy - ydx}{xy}\, d\left[\log\left(\dfrac{y}{x}\right)\right]$ (vi) $\dfrac{ydx + xdy}{xy} = d\,[\log(xy)]$

(vii) $\dfrac{xdx + ydy}{x^2 + y^2} = d\left[\dfrac{1}{2}\log(x^2 + y^2)\right]$ (viii) $\dfrac{xdy - ydx}{x^2 - y^2} = d\left(\dfrac{1}{2}\log\dfrac{x+y}{x-y}\right)$

**Example 1.** *Solve* $ydx - xdy + 3x^2y^2e^{x^3}\,dx = 0$.

**Sol.** Since $3x^2e^{x^3} = d(e^{x^3})$, the term $3x^2y^2e^{x^3}\,dx$ should not involve $y^2$.

This suggests that $\dfrac{1}{y^2}$ may be an I.F.

Multiplying throughout by $\dfrac{1}{y^2}$, we have $\dfrac{ydx - xdy}{y^2} + 3x^2e^{x^3}\,dx = 0$

or $$d\left(\frac{x}{y}\right) + d\,(e^{x^3}) = 0, \text{ which is exact.}$$

Integrating, we get $\dfrac{x}{y} + e^{x^3} = c$ which is the required solution.

**Example 2.** *Solve* $xdy - ydx = x\sqrt{x^2 - y^2}\,dx$.

**Sol.** The given equation is $xdy - ydx = x^2\sqrt{1 - \left(\dfrac{y}{x}\right)^2}\,dx$ or $\dfrac{\dfrac{xdy - ydx}{x^2}}{\sqrt{1 - \left(\dfrac{y}{x}\right)^2}} = dx$

or $d\left(\sin^{-1}\dfrac{y}{x}\right) = dx$, which is exact.

Integrating, we get $\sin^{-1}\dfrac{y}{x} = x + c$ or $y = x\sin(x + c)$

which is the required solution.

**Example 3.** *Solve* $xdx + ydy = \dfrac{a^2\,(x\,dy - y\,dx)}{x^2 + y^2}$.

**Sol.** The given equation is $xdx + ydy - a^2d\left(\tan^{-1}\dfrac{y}{x}\right) = 0$

Integrating, we get $\dfrac{x^2}{2} + \dfrac{y^2}{2} - a^2\tan^{-1}\dfrac{y}{x} = c$

or $$x^2 + y^2 - 2a^2\tan^{-1}\frac{y}{x} = C \quad \text{where} \quad C = 2c.$$

**Example 4.** *Solve* $y\,(2xy + e^x)\,dx = e^x\,dy$ (Lucknow, 1998)

**Sol.** Re-writing the given equation, we have

$$2xy^2\,dx + y\,e^x\,dx - e^x\,dy = 0$$

Dividing by $y^2$, $$2x dx + \frac{ye^x dx - e^x dy}{y^2} = 0 \quad \text{or} \quad 2x\,dx + d\left(\frac{e^x}{y}\right) = 0$$

Integrating $$x^2 + \frac{e^x}{y} = c.$$

**Example 5.** *Solve* $$x^2\,\frac{dy}{dx} + xy = \sqrt{1-x^2y^2}$$

**Sol.** Re-writing the given equation, we have

$$x^2dy + xy\,dx = \sqrt{1-x^2y^2}\ dx$$

Dividing by $x$, $$x\,dy + y\,dx = \sqrt{1-x^2y^2}\ .\ \frac{dx}{x}$$

or $$\frac{xdy + ydx}{\sqrt{1-x^2y^2}} = \frac{dx}{x} \quad \text{or} \quad d\,(\sin^{-1}(xy)) - \frac{dx}{x} = c$$

Integrating $$\sin^{-1}(xy) - \log x = c.$$

## RULES FOR FINDING INTEGRATING FACTORS

**Rule 1. If the equation Mdx + Ndy = 0 is homogeneous in x and y, i.e., if M, N are homogeneous functions of the same degree in x and y, then $\frac{1}{Mx + Ny}$ is an integrating factor.** [*Provided* $Mx + Ny \neq 0$]

**Proof.** The given equation is $Mdx + Ndy = 0$ ...(*i*)

where M, N are homogeneous functions of the same degree, say $n$, in $x$ and $y$.

∴ By Euler's Theorem on Partial Differentiation,

$$\left.\begin{aligned} x\,\frac{\partial M}{\partial x} + y\,\frac{\partial M}{\partial y} &= nM \\ \text{and} \quad x\,\frac{\partial N}{\partial x} + y\,\frac{\partial N}{\partial y} &= nN \end{aligned}\right] \quad \text{...}(ii)$$

Multiplying (*i*) by $\frac{1}{Mx + Ny}$, $(Mx + Ny \neq 0)$, we get

$$\frac{M}{Mx + Ny}\,dx + \frac{N}{Mx + Ny}\,dy = 0$$

It will be exact if

$$\frac{\partial}{\partial y}\left[\frac{M}{Mx + Ny}\right] = \frac{\partial}{\partial x}\left[\frac{N}{Mx + Ny}\right]$$

*i.e.,* if $$\frac{(Mx + Ny)\,.\,\frac{\partial M}{\partial y} - M\left(x\,\frac{\partial M}{\partial y} + N + y + \frac{\partial N}{\partial y}\right)}{(Mx + Ny)^2} = \frac{(Mx + Ny)\,.\,\frac{\partial N}{\partial y} - N\left(M + x\,\frac{\partial M}{\partial x} + y\,\frac{\partial N}{\partial x}\right)}{(Mx + Ny)^2}$$

or if $$\mathrm{N}y \frac{\partial \mathrm{M}}{\partial y} - \mathrm{MN} - \mathrm{M}y \frac{\partial \mathrm{N}}{\partial y} = \mathrm{M}x \frac{\partial \mathrm{N}}{\partial x} - \mathrm{MN} - \mathrm{N}x \frac{\partial \mathrm{M}}{\partial y}$$

or if $$\mathrm{N}\left(x \frac{\partial \mathrm{M}}{\partial x} + y \frac{\partial \mathrm{M}}{\partial y}\right) = \mathrm{M}\left(x \frac{\partial \mathrm{N}}{\partial x} + y \frac{\partial \mathrm{N}}{\partial y}\right)$$

or if $$\mathrm{N} \,.\, n\mathrm{M} = \mathrm{M} \,.\, n\mathrm{N} \qquad \text{[using } (ii)]$$

which is true. Hence the result.

**Second Method.** The given equation is $\mathrm{M}dx + \mathrm{N}dy = 0$ ...(*i*)

where M, N are homogeneous functions of the same degree in $x$ and $y$.

Now $$\mathrm{M}dx + \mathrm{N}dy \equiv \frac{1}{2}\left[(\mathrm{M}x + \mathrm{N}y)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (\mathrm{M}x - \mathrm{N}y)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$$

$$\Rightarrow \quad \mathrm{M}dx + \mathrm{N}dy \equiv \frac{1}{2}\left[(\mathrm{M}x + \mathrm{N}y)\, d(\log xy) + (\mathrm{M}x - \mathrm{N}y)\, d\left(\log \frac{x}{y}\right)\right]$$

Dividing by $\mathrm{M}x + \mathrm{N}y$ (which is $\neq 0$)

$$\frac{\mathrm{N}dx + \mathrm{N}dy}{\mathrm{M}x + \mathrm{N}y} = \frac{1}{2}\left[d\,(\log xy) + \frac{\mathrm{M}x - \mathrm{N}y}{\mathrm{M}x + \mathrm{N}y}\, d\left(\log \frac{x}{y}\right)\right]$$

Since M and N are homogeneous functions of the same degree in $x$ and $y$, the expression $\frac{\mathrm{M}x - \mathrm{N}y}{\mathrm{M}x + \mathrm{N}y}$ is homogeneous and equal to a function of $\frac{x}{y}$, say $f\left(\frac{x}{y}\right)$.

$$\therefore \quad \frac{\mathrm{M}dx + \mathrm{N}dy}{\mathrm{M}x + \mathrm{N}y} = \tfrac{1}{2}\, d(\log xy) + \tfrac{1}{2}\, f\left(\frac{x}{y}\right) d\left(\log \frac{x}{y}\right)$$

Since $$\frac{x}{y} = e^{\log \frac{x}{y}},\ f\left(\frac{x}{y}\right) = f\left(e^{\log \frac{x}{y}}\right) = \mathrm{F}\left(\log \frac{x}{y}\right)$$

$$\therefore \quad \frac{\mathrm{M}dx + \mathrm{N}dy}{\mathrm{M}x + \mathrm{N}y} = \tfrac{1}{2}\, d\,(\log xy) + \tfrac{1}{2}\, \mathrm{F}\left(\log \frac{x}{y}\right) d\left(\log \frac{x}{y}\right)$$

which is an exact differential.

$$\Rightarrow \quad \frac{\mathrm{M}dx + \mathrm{N}dy}{\mathrm{M}x + \mathrm{N}y} = 0 \quad \text{or} \quad \frac{\mathrm{M}}{\mathrm{M}x + \mathrm{N}y}\, dx + \frac{\mathrm{N}}{\mathrm{M}x + \mathrm{N}y}\, dy = 0$$

is an exact differential equation.

**Note.** If the equation $\mathrm{M}dx + \mathrm{N}dy = 0$ is homogeneous in $x$ and $y$ and if $\mathrm{M}x + \mathrm{N}y$ consists of only one term, use the method of integrating factor. Otherwise, proceed as in Type II by putting $y = vx$.

**Example.** *Solve :*

(*i*) $x^2y dx - (x^3 + y^3)\, dy = 0$ (*ii*) $(x^2y - 2xy^2)\, dx - (x^3 - 3x^2y)\, dy = 0$

(*iii*) $(3xy^2 - y^3)\, dx - (2x^2y - xy^2)\, dy = 0$.

**Sol.** (*i*) The given equation is $x^2ydx - (x^3 + y^3)dy = 0$ ...(*i*)

Comparing it with $\mathrm{M}dx + \mathrm{N}dy = 0$

$$\mathrm{M} = x^2y \,;\ \mathrm{N} = -x^3 - y^3$$

$$\frac{\partial \mathrm{M}}{\partial y} = x^2 \,;\ \frac{\partial \mathrm{N}}{\partial x} = -3x^2$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$, equation (*i*) is not exact.

But (*i*) is homogeneous in $x$ and $y$

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{x^3y - x^3y - y^4} = -\frac{1}{y^4}$$

Multiplying (*i*) by $-\dfrac{1}{y^4}$ it becomes

$$-\frac{x^2}{y^3}\,dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0 \qquad ...(ii)$$

which is exact. $\left[\because \ \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = \dfrac{3x^2}{y^4}\right]$

Hence the solution is

$$\int_{y\,-\,\text{constant}} -\frac{x^2}{y^3}\,dx + \int \frac{1}{y}\,dy = c \qquad \text{or} \qquad -\frac{x^3}{3y^3} + \log y = c.$$

(*ii*) The given equation is $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0$ ...(*i*)

Comparing with $M dx + N dy = 0$

Here $M = x^2y - 2xy^2$ ; $N = -x^3 + 3x^2y$

$$\frac{\partial M}{\partial y} = x^2 - 4xy, \ \frac{\partial N}{\partial x} = -3x^2 + 6xy$$

Since $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$ equation (*i*) is not exact.

But (*i*) is homogeneous in $x$ and $y$

$$\therefore \qquad \text{I.F.} = \frac{1}{Mx + Ny} = \frac{1}{x^3y - 2x^2y^2 - x^3y + 3x^2y^2} = \frac{1}{x^2y^2}$$

Multiplying (*i*) by $\dfrac{1}{x^2y^2}$ it becomes

$$\left(\frac{1}{y} - \frac{2}{x}\right)dx - \left(\frac{x}{y^2} - \frac{3}{y}\right)dy = 0 \qquad ...(ii)$$

which is exact. $\left[\because \ \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = -\dfrac{1}{y^2}\right]$

Hence the solution is

$$\int_{y\,-\,\text{constant}} \left(\frac{1}{y} - \frac{2}{x}\right)dx - \int -\frac{3}{y}\,dy = c$$

or

$$\frac{x}{y} - 2\log x + 3\log y = c.$$

(*iii*) Please try yourself. $\left[\textbf{Ans.} \ \ 3\log x + \dfrac{y}{x} - 2\log y = c\right]$

**Rule 2. If the equation Mdx + Ndy = 0 is of the form**

$$\mathbf{f_1(xy)ydx + f_2(xy)xdy = 0, \text{ then } \frac{1}{Mx - Ny} \text{ is an integrating factor.}}$$

**[Provided Mx − Ny ≠ 0]**

**Proof.** The given equation is $Mdx + Ndy = 0$ ...(*i*)

where $M = f_1(xy)y \qquad N = f_2(xy)x$

Now $\qquad Mdx + Ndy \equiv \frac{1}{2}\left[(Mx + Ny)\left(\frac{dx}{x} + \frac{dy}{y}\right) + (Mx - Ny)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right]$

$\Rightarrow \qquad Mdx + Ndy \equiv \frac{1}{2}\left[(Mx + Ny)\, d(\log xy) + (Mx - Ny)\, d\left(\log \frac{x}{y}\right)\right]$

Dividing by $Mx - Ny$ (which is $\neq 0$)

$$\frac{Mdx + Ndy}{Mx - Ny} = \frac{1}{2}\left[\frac{Mx + Ny}{Mx - Ny}\, d\,(\log xy) + d\left(\log \frac{x}{y}\right)\right]$$

$$= \frac{1}{2}\left[\frac{f_1(xy)xy + f_2(xy)xy}{f_1(xy)xy - f_2(xy)\,xy}\, d\,(\log xy) + d\left(\log \frac{x}{y}\right)\right]$$

$$= \frac{1}{2}\left[f(xy)\, d(\log xy) + d\left(\log \frac{x}{y}\right)\right]$$

Since $xy = e^{\log xy}$, $f(xy) = f(e^{\log xy}) = F(\log xy)$

$\therefore \qquad \frac{Mdx + Ndy}{Mx - Ny} = \frac{1}{2} F(\log xy)\, d\,(\log xy) + \frac{1}{2}\, d\left(\log \frac{x}{y}\right)$

which is an exact differential.

$\Rightarrow \qquad \frac{Mdx + Ndy}{Mx - Ny} = 0 \qquad \text{or} \qquad \frac{M}{Mx - Ny} dx + \frac{N}{Mx - Ny}\, dy = 0$

is an exact differential equation.

$\Rightarrow \quad \frac{1}{Mx - Ny}$ is an integrating factor of (*i*).

**Example.** *Solve (i)* $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$

(*ii*) $(xy \sin xy + \cos xy)ydx + (xy \sin xy - \cos xy)\, xdy = 0$ (Lucknow, 1997)

(*iii*) $(x^2y^2 + xy + 1)ydx + (x^2y^2 - xy + 1)xdy = 0$

(*iv*) $(x^4y^4 + x^2y^2 + xy)ydx + (x^4y^4 - x^2y^2 + xy)xdy = 0$

(*v*) $(1 + xy)ydx + (1 - xy)xdy = 0.$

**Sol.** (*i*) The given equation is $\quad (xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$ ...(*i*)

Comparing it with $Mdx + Ndy = 0$

Here $\qquad M = xy^2 + 2x^2y^3,\ N = x^2y - x^3y^2$

$$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2,\ \frac{\partial N}{\partial x} = 2xy - 3x^2y^2$$

Since $\qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, equation (*i*) is not exact.

But $(i)$ can be written as

$$(xy + 2x^2y^2)ydx + (xy - x^2y^2)xdy = 0$$

$\therefore$ $$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{x^2y^2 + 2x^3y^3 - x^2y^2 + x^3y^3} = \frac{1}{3x^3y^3}$$

Multiplying $(i)$ by $\dfrac{1}{3x^3y^3}$, it becomes

$$\left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \left(\frac{1}{3xy^2} - \frac{1}{3y}\right)dy = 0$$

which is exact. $\left[\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = -\dfrac{1}{3x^2y^2}\right]$

Hence the solution is

$$\int_{y\text{ - constant}} \left(\frac{1}{3x^2y} + \frac{2}{3x}\right)dx + \int -\frac{1}{3y}\,dy = c_1$$

or $$\frac{1}{3y}\int \frac{1}{x^2}\,dx + \frac{2}{3}\log x - \frac{1}{3}\log y = c_1$$

or $$-\frac{1}{3xy} + \frac{2}{3}\log x - \frac{1}{3}\log y = c_1$$

or $$-\frac{1}{xy} + 2\log x - \log y = 3c_1 \quad \text{or} \quad -\frac{1}{xy} + 2\log x - \log y = c.$$

$(ii)$ The given equation is

$$(xy \sin xy + \cos xy)y\,dx + (xy \sin xy - \cos xy)\,x\,dy = 0 \qquad ...(i)$$

It is of the form $f_1(xy)\,ydx + f_2(xy)\,x\,dy = 0$

Here $M = (xy \sin xy + \cos xy)y$

$N = (xy \sin xy - \cos xy)\,x$

$$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{xy(xy \sin xy + \cos xy) - xy(xy \sin xy - \cos xy)} = \frac{1}{2xy \cos xy}$$

Multiplying $(i)$ by $\dfrac{1}{2xy\cos xy}$, it becomes

$$\left[\frac{y \sin xy}{2\cos xy} + \frac{1}{2x}\right]dx + \left[\frac{x \sin xy}{2\cos xy} - \frac{1}{2y}\right]dy = 0$$

or $$\left(y \tan xy + \frac{1}{x}\right)dx + \left(x \tan xy - \frac{1}{y}\right)dy = 0$$

which is exact. $\left[\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = (\tan xy + xy\sec^2 xy)\right]$

Hence the solution is

$$\int_{y\text{ - constant}} \left(y \tan xy + \frac{1}{x}\right)dx + \int -\frac{1}{y}\,dy = c_1$$

or $$y \cdot \frac{\log \sec xy}{y} + \log x - \log y = c_1$$

or $$\log \sec xy + \log x - \log y = c_1$$

or $$\log \sec xy + \log \frac{x}{y} = \log c \qquad \text{or} \qquad \log \frac{x}{y} \sec xy = \log c$$

or $$\frac{x}{y} \sec xy = c \qquad \text{or} \qquad x \sec xy = cy.$$

(*iii*) The given equation is $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)\,x\,dy = 0$ ...(*i*)

It is of the form $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$

Here $M = (x^2y^2 + xy + 1)y$

$N = (x^2y^2 - xy + 1)x$

$$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{(x^2y^2 + xy + 1)xy - (x^2y^2 - xy + 1)xy} = \frac{1}{2x^2y^2}$$

Multiplying (*i*) by $\frac{1}{2x^2y^2}$, it becomes

$$\left(\frac{1}{2} + \frac{1}{2xy} + \frac{1}{2x^2y^2}\right) y\,dx + \left(\frac{1}{2} - \frac{1}{2xy} + \frac{1}{2x^2y^2}\right) x\,dy = 0$$

or $$\left(y + \frac{1}{x} + \frac{1}{x^2y}\right) dx + \left(x - \frac{1}{y} + \frac{1}{xy^2}\right) dy = 0$$

which is exact. $\left[\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 1 - \frac{1}{x^2y^2}\right]$

Hence the solution is

$$\int_{y\text{ - constant}} \left(y + \frac{1}{y} + \frac{1}{x^2y}\right) dx + \int \left(-\frac{1}{y}\right) dy = c$$

or $$yx + \log x + \frac{1}{y}\left(-\frac{1}{x}\right) - \log y = c$$

or $$yx + \log x - \frac{1}{xy} - \log y = c \qquad \text{or} \qquad xy - \frac{1}{xy} + \log \frac{x}{y} = c.$$

(*iv*) The given equation is $(x^4y^4 + x^2y^2 + xy)y\,dx + (x^4y^4 - x^2y^2 + xy)x\,dy = 0$ ...(*i*)

It is of the form $f_1(xy)y\,dx + f_2(xy)x\,dy = 0$

Here $M = (x^4y^4 + x^2y^2 + xy)y$, $N = (x^4y^4 - x^2y^2 + xy)x$

$$\text{I.F.} = \frac{1}{Mx - Ny} = \frac{1}{(x^4y^4 + x^2y^2 + xy)xy - (x^4y^4 - x^2y^2 + xy)xy} = \frac{1}{2x^3y^3}$$

Multiplying (*i*) by $\frac{1}{2x^3y^3}$, it becomes

$$\frac{1}{2}\left(xy + \frac{1}{xy} + \frac{1}{x^2y^2}\right) y\,dx + \frac{1}{2}\left(xy - \frac{1}{xy} + \frac{1}{x^2y^2}\right) x\,dy = 0$$

or $$\left(xy^2+\frac{1}{x}+\frac{1}{x^2y}\right)dx+\left(x^2y-\frac{1}{y}+\frac{1}{xy^2}\right)dy=0$$

which is exact. $\left[\because\ \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=2xy-\frac{1}{x^2y^2}\right]$

Hence the solution is

$$\int_{y-\text{constant}}\left(xy^2+\frac{1}{x}+\frac{1}{x^2y}\right)dx+\int\left(-\frac{1}{y}\right)dy=c$$

or $$y^2\,\frac{x^2}{2}+\log x+\frac{1}{y}\left(-\frac{1}{x}\right)-\log y=c$$

or $$\frac{1}{2}\,x^2y^2-\frac{1}{xy}+\log\frac{x}{y}=c.$$

(*v*) Please try yourself. $\left[\textbf{Ans.}\quad -\frac{1}{xy}+\log\frac{x}{y}=c\right]$

**Rule 3. If in the equation Mdx + Ndy = 0**

$$\frac{\dfrac{\partial \mathbf{M}}{\partial \mathbf{y}}-\dfrac{\partial \mathbf{N}}{\partial \mathbf{x}}}{\mathbf{N}}\ \textbf{is a function of x only} = \mathbf{f(x)}\quad \textbf{(say)}$$

**then $\mathbf{e^{\int f(x)\,dx}}$ is an integrating factor.**

**Proof.** The given equation is $Mdx + Ndy = 0$ ...(*i*)

where $\dfrac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=f(x)$ ...(*ii*)

Multiplying (*i*) by $e^{\int f(x)\,dx}$, we get

$$M\,e^{\int f(x)\,dx}\,dx+N\,e^{\int f(x)\,dx}\,dy=0$$

It will be exact if

$$\frac{\partial}{\partial y}\left[M\,e^{\int f(x)\,dx}\right]=\frac{\partial}{\partial x}\left[N\,e^{\int f(x)\,dx}\right]$$

or if $$e^{\int f(x)\,dx}\cdot\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}\cdot e^{\int f(x)\,dx}+N\cdot e^{\int f(x)\,dx}f(x)$$

[cancelling $e^{\int f(x)\,dx}$ throughout]

or if $$\frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}+N\,f(x)$$

or if $$\frac{\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}}{N}=f(x)$$

which is true from (*ii*).

$\Rightarrow$ $e^{\int f(x)\,dx}$ is an integrating factor of (*i*) provided (*ii*) is true.

**Example.** *Solve*

*(i)* $(x^2 + y^2 + 2x)\, dx + 2y dy = 0$ *(ii)* $(xy^2 - e^{1/x^3})\, dx - x^2 y dy = 0$

*(iii)* $\left(y + \dfrac{y^3}{3} + \dfrac{x^2}{2}\right) dx + \dfrac{1}{4}(x + xy^2)\, dy = 0$

*(iv)* $(xy^3 + y)dx + 2(x^2y^2 + x + y^4)\, dy = 0.$

**Sol.** *(i)* The given equation is $(x^2 + y^2 + 2x)\, dx + 2y dy = 0$ ...(*i*)

Here $M = x^2 + y^2 + 2x,\ N = 2y$

$$\frac{\partial M}{\partial y} = 2y,\ \frac{\partial N}{\partial x} = 0$$

$$\therefore \quad \frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2y - 0}{2y} = 1 \quad \therefore \quad \text{I.F.} = e^{\int 1\, dx} = e^x$$

Multiplying (*i*) by $e^x$, it becomes

$$(x^2e^x + y^2e^x + 2xe^x)\, dx + 2ye^x dy = 0 \qquad \left[\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = 2ye^x\right]$$

which is exact.

Hence the solution is $\displaystyle\int_{y\,-\,\text{constant}} (x^2e^x + y^2e^x + 2xe^x)\, dx = c$

or $e^x(x^2 - 2x + 2) + y^2e^x + 2e^x(x - 1) = c$

or $e^x(x^2 - 2x + 2 + y^2 + 2x - 2) = c$

or $e^x(x^2 + y^2) = c.$

*(ii)* The given equation is

$$\left(xy^2 - e^{\frac{1}{x^3}}\right) dx - x^2 y dy = 0 \qquad ...(i)$$

Here $M = xy^2 - e^{\frac{1}{x^3}},\ N = -x^2y$

$$\frac{\partial M}{\partial y} = 2xy,\ \frac{\partial N}{\partial x} = -2xy$$

$$\frac{\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}}{N} = \frac{2xy - (-2xy)}{-x^2y} = \frac{4xy}{-x^2y} = -\frac{4}{x} = f(x)$$

$$\therefore \quad \text{I.F.} = e^{\int f(x)\, dx} = e^{\int -\frac{4}{x} dx} = e^{-4\log x} = e^{\log x^{-4}} = x^{-4} = \frac{1}{x^4}$$

Multiplying (*i*) by $\dfrac{1}{x^4}$, it becomes

$$\left(\frac{y^2}{x^3} - \frac{1}{x^4} e^{\frac{1}{x^3}}\right) dx - \frac{y}{x^2}\, dy = 0$$

which is exact. $\left[\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = \dfrac{2y}{x^3}\right]$

Hence the solution is

$$\int \left( \frac{y^2}{x^3} - \frac{1}{x^4} e^{\frac{1}{x^3}} \right) dx = c \quad \text{or} \quad y^2 \int \frac{1}{x^3} dx - \int \frac{1}{x^4} e^{\frac{1}{x^3}} dx = c$$

or
$$y^2 \cdot \frac{x^{-2}}{-2} - \int \frac{1}{x^4} e^{\frac{1}{x^3}} dx + c \qquad ...(ii)$$

To integrate $\int \frac{1}{x^4} e^{\frac{1}{x^3}} dx$, put $\frac{1}{x^3} = t$

$$\therefore \qquad -\frac{3}{x^4} dx = dt \quad \text{or} \quad \frac{1}{x^4} dx = -\frac{1}{3} dt$$

$$\therefore \qquad \int \frac{1}{x^4} e^{\frac{1}{x^3}} dx = \int e^t \left(-\frac{1}{3}\right) dt = -\frac{1}{3} e^t = -\frac{1}{3} e^{\frac{1}{x^3}}$$

Hence, from (*ii*), the solution is

$$-\frac{y^2}{2x^2} + \frac{1}{3} e^{\frac{1}{x^3}} = c.$$

(*iii*) The given equation is

$$(y + \tfrac{1}{3} y^3 + \tfrac{1}{2} x^2)\, dx + \tfrac{1}{4} (x + xy^2)\, dy = 0 \qquad ...(i)$$

$$M = y + \tfrac{1}{3} y^3 + \tfrac{1}{2} x^2,\ N = \tfrac{1}{4} (x + xy^2)$$

$$\frac{\partial M}{\partial y} = 1 + y^2,\ \frac{\partial N}{\partial x} = \frac{1}{4} (1 + y^2)$$

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{(1+y^2) - \frac{1}{4}(1+y^2)}{\frac{1}{4}(x + xy^2)} = \frac{\frac{3}{4}(1+y^2)}{\frac{1}{4}x(1+y^2)} = \frac{3}{x} = f(x)$$

$$\text{I.F.} = e^{\int f(x)\, dx} = e^{\int \frac{3}{x} dx} = e^{3 \log x} = e^{\log x^3} = x^3$$

Multiplying (*i*) by $x^3$, it becomes

$$(x^3y + \tfrac{1}{3} x^3y^3 + \tfrac{1}{2} x^5)\, dx + \tfrac{1}{4} (x^4 + x^4y^2) dy = 0$$

which is exact. $\left[\because \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} = x^3 + x^3y^2\right]$

Hence the solution is

$$\int_{y\text{ - constant}} (x^3y + \tfrac{1}{3} x^3y^3 + \tfrac{1}{2} x^5)\, dx = c_1$$

or
$$y \cdot \frac{x^4}{4} + \frac{1}{3} y^3 \cdot \frac{x^4}{4} + \frac{1}{2} \cdot \frac{x^6}{6} = c_1$$

or
$$3x^4y + x^4y^3 + x^6 = 12c_1 \quad \text{or} \quad 3x^4y + x^4y^3 + x^6 = c.$$

(*iv*) Please try yourself. $\left[\textbf{Ans. } \frac{x^2y^4}{2} + xy^2 + \frac{y^6}{3} = c\right]$

**Rule 4. If in the equation Mdx + Ndy = 0**

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} \text{ is a function of y only} = f(y) \text{ (say)}$$

**then $e^{\int f(x)\,dx}$ is an integrating factor.**

**Proof.** The proof of Rule 4 is similar to the proof of Rule 3 and is, therefore, left as an exercise for the student.

**Example.** *Solve the following :*

(*i*) $(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)dy = 0$ (*ii*) $(3x^2y^4 + 2xy)\,dx + (2x^3y^3 - x^2)dy = 0$

(*iii*) $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0.$

**Sol.** (*i*) The given equation is $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)dy = 0$ ...(*i*)

Here $M = y^4 + 2y$ , $N = xy^3 + 2y^4 - 4x$

$$\frac{\partial M}{\partial y} = 4y^3 + 2\,, \quad \frac{\partial N}{\partial x} = y^3 - 4$$

$$\frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \frac{(y^3-4)-(4y^3+2)}{y^4+2y} = \frac{-3(y^3+2)}{y(y^3+2)} = -\frac{3}{y} = f(y)$$

$$\therefore \qquad \text{I.F.} = e^{\int f(y)\,dy} = e^{\int -\frac{3}{y}\,dy} = e^{-3\log y} = e^{\log y^{-3}} = y^{-3} = \frac{1}{y^3}$$

Multiplying (*i*) by $\dfrac{1}{y^3}$ , it becomes

$$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0$$

which is exact. $\left[\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = 1 - \dfrac{4}{y^3}\right]$

Hence the solution is

$$\int_{y\,-\,\text{constant}} \left(y + \frac{2}{y^2}\right)dx + \int 2y\,dy = c \qquad \text{or} \qquad \left(y + \frac{2}{y^2}\right)x + y^2 = c.$$

(*ii*) Please try yourself. **[Ans.** $x^3y^3 + x^2 = cy$]

(*iii*) The given equation is $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$ ...(1)

Here $M = 2xy^4e^y + 2xy^3 + y$, $N = x^2y^4e^y - x^2y^2 - 3x$

$$\frac{\partial M}{\partial y} = 2x\,(y^4e^y + 4y^3e^y) + 6xy^2 + 1 = 2xy^4e^y + 8xy^3e^y + 6xy^2 + 1$$

$$\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -8xy^3e^y - 8xy^2 - 4 = -4\,(2xy^3e^y + 2xy^2 + 1) = -4\left(\frac{M}{y}\right)$$

$$\Rightarrow \qquad \frac{\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}}{M} = \frac{-4}{y} = f(y)$$

$$\therefore \qquad \text{I.F.} = e^{\int f(y)\,dy} = e^{\int -\frac{4}{y}dy} = e^{-4\log y} = e^{\log y^{-4}} = y^{-4} = \frac{1}{y^4}$$

Multiplying (1) by $\frac{1}{y^4}$, we get

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \left(x^2e^y - \frac{x^2}{y^2} - \frac{3x}{y^4}\right)dy = 0$$

which is exact.

Hence the solution is

$$\int_{y\text{ - constant}} \left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx = c \qquad \text{or} \qquad x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c.$$

**Rule 5. An equation of the form**

$$\mathbf{x^a y^b\,(mydx + nxdy) + x^c y^d\,(pydx + qxdy) = 0}$$

**where a, b, c, d, m, n, p, q are all constants, has an integrating factor $x^\alpha y^\beta$, where $\alpha$, $\beta$ are so chosen that after multiplying by $x^\alpha y^\beta$, the equation becomes exact.**

**Note. $\alpha$ and $\beta$ can also be determined by solving the equations**

$$\frac{a+\alpha+1}{m} = \frac{b+\beta+1}{n} \quad \text{and} \quad \frac{c+\alpha+1}{p} = \frac{d+\beta+1}{q}.$$

The following example will illustrate the procedure.

**Example.** *Solve the following* :

(*i*) $(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0$ (*ii*) $(2x^2y^2 + y)dx - (x^3y - 3x)dy = 0$

(*iii*) $(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$ (*iv*) $(2x^2y - 3y^4)dx + (3x^3 + 2xy^3)dy = 0$.

**Sol.** (*i*) The given equation is $(y^3 - 2yx^2)dx + (2xy^2 - x^3)dy = 0$ ...(*i*)

Here $M = y^3 - 2yx^2,\ N = 2xy^2 - x^3$

$$\frac{\partial M}{\partial y} = 3y^2 - 2x^2,\ \frac{\partial N}{\partial x} = 2y^2 - 3x^2$$

$$\because \qquad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \qquad \therefore (i) \text{ is not exact.}$$

Now (*i*) can be written as

$$x^0y^2(ydx + 2xdy) + x^2y^0(-2ydx - xdy) = 0$$

which is of the form $x^ay^b\,(mydx + nxdy) + x^cy^d\,(pydx + qxdy) = 0$

Let $x^\alpha y^\beta$ be an intergrating factor of (*i*)

Multiplying (*i*) by $x^\alpha y^\beta$, it becomes

$$(x^\alpha y^{\beta+3} - 2x^{\alpha+2}y^{\beta+1})\,dx + (2x^{\alpha+1}y^{\beta+2} - x^{\alpha+3}y^\beta)\,dy = 0$$

It will be exact if

$$\frac{\partial}{\partial y}(x^\alpha y^{\beta+3} - 2x^{\alpha+2}y^{\beta+1}) = \frac{\partial}{\partial x}(2x^{\alpha+1}y^{\beta+2} - x^{\alpha+3}y^\beta)$$

*i.e.,* if $(\beta+3)x^{\alpha}y^{\beta+2} - 2(\beta+1)x^{\alpha+2}y^{\beta} = 2(\alpha+1)x^{\alpha}y^{\beta+2} - (\alpha+3)x^{\alpha+2}y^{\beta}$

or if $\begin{cases} \beta+3 = 2(\alpha+1) & [\textit{equating co-effi. of } x^{\alpha}y^{\beta+\theta}] \\ \text{and } -2(\beta+1) = -(\alpha+3) & [\textit{equating co-effi. of } x^{\alpha+2}y^{\beta}] \end{cases}$

or if $\begin{cases} 2\alpha-\beta-1=0 \\ \text{and } \alpha-2\beta+1=0 \end{cases}$

or if $\alpha = 1, \beta = 1$. Hence $xy$ is an integrating factor.

Multiplying (*i*) by $xy$, we have

$$(xy^4 - 2x^3y^2)dx + (2x^2y^3 - x^4y)dy = 0$$

which is exact. $\left[\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = 4xy^3 - 4x^3y\right]$

Hence the solution is

$$\int\limits_{y-\text{constant}} (xy^4 - 2x^3y^2)dx = c_1 \quad \text{or} \quad y^4 \cdot \frac{x^2}{2} - 2y^2 \cdot \frac{x^4}{4} = c_1$$

or $$x^2y^2(y^2 - x^2) = 2c_1 = c.$$

[**Note.** The given equation can be written as

$$x^0y^2(ydx + 2xdy) + x^2y^0(-2ydx - xdy) = 0$$

Here $a = 0, b = 2, m = 1, n = 2$

$c = 2, d = 0, p = -2, q = -1$

$\therefore \dfrac{a+\alpha+1}{m} = \dfrac{b+\beta+1}{n} \Rightarrow \dfrac{\alpha+1}{1} = \dfrac{\beta+3}{2}$ or $2\alpha - \beta - 1 = 0$ ...(1)

Also $\dfrac{c+\alpha+1}{p} = \dfrac{d+\beta+1}{q} \Rightarrow \dfrac{\alpha+3}{-2} = \dfrac{\beta+1}{-1}$ or $\alpha - 2\beta + 1 = 0$ ...(2)

Solving (1) and (2), $\dfrac{\alpha}{-1-2} = \dfrac{\beta}{-1-2} = \dfrac{1}{-4+1} \Rightarrow \alpha = \beta = 1.$]

(*ii*) The given equation is $(2x^2y^2 + y)dx - (x^3y - 3x)dy = 0$ ...(*i*)

Here $M = 2x^2y^2 + y, N = -(x^3y - 3x)$

$$\frac{\partial M}{\partial y} = 4x^2y + 1, \quad \frac{\partial N}{\partial x} = -3x^2y + 3$$

$\because \dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x} \therefore$ (*i*) is not exact.

Now (*i*) can be written as

$$x^2y(2ydx - xdy) + x^0y^0(ydx + 3xdy) = 0$$

which is of the form

$$x^ay^b(mydx + nxdy) + x^cy^d(pydx + qxdy) = 0$$

Let $x^{\alpha}y^{\beta}$ be an integrating factor of (*i*).

Multiplying (*i*) by $x^{\alpha}y^{\beta}$, it becomes

$$(2x^{\alpha+2}y^{\beta+2} + x^{\alpha}y^{\beta+1})dx - (x^{\alpha+3}y^{\beta+1} - 3x^{\alpha+1}y^{\beta})dy = 0$$

It will be exact if

$$\frac{\partial}{\partial y}(2x^{\alpha+2}y^{\beta+2}+x^{\alpha}y^{\beta+1})=\frac{\partial}{\partial x}(-x^{\alpha+3}y^{\beta+1}+3x^{\alpha+1}y^{\beta})$$

*i.e.*, if $2(\beta+2)x^{\alpha+2}y^{\beta+1}+(\beta+1)x^{\alpha}y^{\beta}=-(\alpha+3)x^{\alpha+2}y^{\beta+1}+3(\alpha+1)x^{\alpha}y^{\beta}$

or if $\begin{cases}2(\beta+2)=-(\alpha+3) & [\textit{equating co-effi. of } x^{\alpha+2}y^{\beta+1}]\\(\beta+1)=3(\alpha+1) & [\textit{equating co-effi. of } x^{\alpha}y^{\beta}]\end{cases}$

or if $\begin{cases}\alpha+2\beta+7=0\\3\alpha-\beta+2=0\end{cases}$

or if $\alpha=-\frac{11}{7},\beta=-\frac{19}{7}$.

Hence $x^{-\frac{11}{7}}y^{-\frac{19}{7}}$ is an integrating factor.

Multiplying (*i*) by $x^{-\frac{11}{7}}y^{-\frac{19}{7}}$, we have

$$\left(2x^{\frac{3}{7}}y^{-\frac{5}{7}}+x^{-\frac{11}{7}}y^{-\frac{12}{7}}\right)dx-\left(x^{\frac{10}{7}}y^{-\frac{12}{7}}-3x^{-\frac{4}{7}}y^{-\frac{19}{7}}\right)dy=0$$

which is exact. $\left[\because \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}=-\frac{10}{7}x^{\frac{3}{7}}y^{-\frac{12}{7}}-\frac{12}{7}x^{-\frac{11}{7}}y^{-\frac{19}{7}}\right]$

Hence the solution is

$$\int_{y\text{ - constant}}\left(2x^{\frac{3}{7}}y^{-\frac{5}{7}}+x^{-\frac{11}{7}}y^{-\frac{12}{7}}\right)dx=c$$

or $$2y^{-\frac{5}{7}}\cdot\frac{x^{\frac{10}{7}}}{\frac{10}{7}}+y^{-\frac{12}{7}}\cdot\frac{x^{-\frac{4}{7}}}{-\frac{4}{7}}=c \quad\text{or}\quad \frac{5}{7}x^{\frac{10}{7}}y^{-\frac{5}{7}}-\frac{7}{4}x^{-\frac{4}{7}}y^{-\frac{12}{7}}=c.$$

(*iii*) The given equation is $(y^2+2x^2y)\,dx+(2x^3-xy)\,dy=0$ ...(1)

Here $M=y^2+2x^2y$, $N=2x^3-xy$

$$\frac{\partial M}{\partial y}=2y+2x^2,\ \frac{\partial N}{\partial x}=6x^2-y$$

$\because$ $\frac{\partial M}{\partial y}\neq\frac{\partial N}{\partial x}$, equation (1) is not exact.

Now (1) can be written as

$$y(ydx-xdy)+x^2(2ydx+2xdy)=0$$

or $$x^0y^1(ydx-xdy)+x^2y^0(2ydx+2xdy)=0$$

which is of the form

$$x^ay^b(mydx+nxdy)+x^cy^d(pydx+qxdy)=0$$

Let $x^{\alpha}y^{\beta}$ be an integrating factor of (1).

Multiplying (1) by $x^\alpha y^\beta$, it becomes

$$(x^\alpha y^{\beta+2} + 2x^{\alpha+2}y^{\beta+1})\,dx + (2x^{\alpha+3}y^\beta - x^{\alpha+1}y^{\beta+1})\,dy = 0$$

It will be exact if

$$\frac{\partial}{\partial y}(x^\alpha y^{\beta+2} + 2x^{\alpha+2}y^{\beta+1}) = \frac{\partial}{\partial x}(2x^{\alpha+3}y^\beta - x^{\alpha+1}y^{\beta+1})$$

*i.e.*, if $(\beta+2)x^\alpha y^{\beta+1} + 2(\beta+1)x^{\alpha+2}y^\beta = 2(\alpha+3)x^{\alpha+2}y^\beta - (\alpha+1)x^\alpha y^{\beta+1}$

or if $\begin{cases} \beta + 2 = -(\alpha+1) & [\textit{equating co-effi. of } x^\alpha y^{\beta+1}] \\ 2(\beta+1) = 2(\alpha+3) & [\textit{equating co-effi. of } x^{\alpha+2} y^\beta] \end{cases}$

or if $\begin{cases} \alpha + \beta + 3 = 0 \\ \alpha - \beta + 2 = 0 \end{cases}$

or if $\alpha = -\dfrac{5}{2}, \beta = -\dfrac{1}{2}$

$\therefore$ $x^{-\frac{5}{2}} y^{-\frac{1}{2}}$ is an integrating factor.

Multiplying (1) by $x^{-\frac{5}{2}} y^{-\frac{1}{2}}$, we have

$$\left(x^{-\frac{5}{2}} y^{\frac{3}{2}} + 2x^{-\frac{1}{2}} y^{\frac{1}{2}}\right)dx + \left(2x^{\frac{1}{2}} y^{-\frac{1}{2}} + x^{-\frac{3}{2}} y^{\frac{1}{2}}\right)dy = 0$$

which is exact. $\left[\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = \dfrac{3}{2} x^{-\frac{5}{2}} y^{\frac{1}{2}} + x^{-\frac{1}{2}} y^{-\frac{1}{2}}\right]$

Hence the solution is

$$\int_{y\text{ - constant}} \left(x^{-\frac{5}{2}} y^{\frac{3}{2}} + 2x^{-\frac{1}{2}} y^{\frac{1}{2}}\right) dx = c \qquad \text{or} \qquad y^{\frac{3}{2}} \cdot \frac{x^{-\frac{3}{2}}}{-\frac{3}{2}} + 2y^{\frac{1}{2}} \cdot \frac{x^{\frac{1}{2}}}{\frac{1}{2}} = c$$

or $$4\sqrt{xy} - \frac{2}{3}\left(\frac{y}{x}\right)^{3/2} = c \Rightarrow 6\sqrt{xy} - \left(\frac{y}{x}\right)^{3/2} = C, \qquad \text{where } C = \frac{3}{2}c.$$

(*iv*) Please try yourself. $\left[\textbf{Ans. } 5x^{-\frac{36}{13}} y^{\frac{24}{13}} - 12x^{-\frac{10}{13}} y^{-\frac{15}{13}} = c\right]$

$\left[\textbf{Hint. } \alpha = -\dfrac{49}{13}, \beta = -\dfrac{23}{13}\right]$

# 3

# Linear Equations with Constant Co-efficients

## 1. Linear Equation with Constant Co-efficients

**Definition.** A linear differential equation with constant co-efficients is that in which the dependent variable and its differential co-efficients occur only in the first degree and are not multiplied together, and the co-efficients are all constants.

Thus, the equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + a_n y = X$$

where $a_0, a_1, a_2, \ldots\ldots, a_n$ are constants and X is any function of $x$ is a **linear differential equation of nth order.**

**2. Operators.** The part $\frac{d}{dx}$ of the symbol $\frac{dy}{dx}$ may be regarded as an **operator,** such that when it operates on $y$, the result is the derivative of $y$.

Similarly, $\frac{d^2}{dx^2}, \frac{d^3}{dx^3}, \ldots\ldots, \frac{d^n}{dx^n}$ may be regarded as operators.

In symbolic form, we have

$$\frac{d}{dx} = D,\ \frac{d^2}{dx^2} = D^2, \ldots\ldots, \frac{d^n}{dx^n} = D^n$$

Thus, the equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + a_n y = X$$

when written in the symbolic form, becomes

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \ldots\ldots + a_n y = X$$

or

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n) y = X$$

**3.** $a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n = f(D)$ is regarded as a single operator, operating on $y$.

**4. Theorem. If $y = y_1, y = y_2, \ldots\ldots, y = y_n$ are n linearly independent solutions of**

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n) y = 0 \qquad \ldots(i)$$

**then $y = c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n$ is the general or complete solution of (i) where $c_1, c_2, \ldots\ldots, c_n$ are n arbitrary constants.**

**Proof.** The given equation is

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \ldots\ldots + a_n y = 0 \qquad \ldots(i)$$

$\because$ $y = y_1, y = y_2, \ldots\ldots, y = y_n$ are its solutions.

$$\therefore \qquad \left.\begin{aligned} a_0 D^n y_1 + a_1 D^{n-1} y_1 + a_2 D^{n-2} y_1 + \ldots\ldots + a_n y_1 = 0 \\ a_0 D^n y_2 + a_1 D^{n-1} y_2 + a_2 D^{n-2} y_2 + \ldots\ldots + a_n y_2 = 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_0 D^n y_n + a_1 D^{n-1} y_n + a_2 D^{n-2} y_n + \ldots\ldots + a_n y_n = 0 \end{aligned}\right] \qquad \ldots(ii)$$

If $\qquad y = c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n$, then

$$\begin{aligned} \text{L.H.S. of } (i) &= a_0 D^n (c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n) \\ &\quad + a_1 D^{n-1} (c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n) \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + a_n (c_1 y_1 + c_2 y_2 + \ldots\ldots + c_n y_n) \\ &= c_1 (a_0 D^n y_1 + a_1 D^{n-1} y_1 + \ldots\ldots + a_n y_1) \\ &\quad + c_2 (a_0 D^n y_2 + a_1 D^{n-1} y_2 + \ldots\ldots + a_n y_2) \\ &\quad + \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ &\quad + c_n (a_0 D^n y_n + a_1 D^{n-1} y_n + \ldots\ldots + a_n y_n) \\ &= c_1(0) + c_2(0) + \ldots\ldots + c_n(0) \qquad \mid \because \text{ of } (ii) \\ &= 0 \end{aligned}$$

Thus the equation $(i)$ is satisfied when $y = c_1 y_1 + c_2 y_2 + \ldots\ldots\ldots + c_n y_n$, which is therefore, its solution. Since it contains $n$ arbitrary constants, it is the general or complete solution.

### 5. Auxiliary Equation (A.E.)

Consider the differential equation

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n) y = 0 \qquad \ldots(i)$$

or

$$a_0 D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \ldots\ldots + a_n y = 0 \qquad \ldots(ii)$$

Let $y = e^{mx}$ be a solution of $(i)$. Then

$$Dy = me^{mx},\ D^2 y = m^2 e^{mx},\ \ldots\ldots,\ D^n y = m^n e^{mx}$$

Substituting the values of $y$, $Dy$, $D^2 y$, ......, $D^n y$ in $(ii)$, we get

$$a_0 m^n e^{mx} + a_1 m^{n-1} e^{mx} + \ldots\ldots + a_n e^{mx} = 0$$

or

$$(a_0 m^n + a_1 m^{n-1} + \ldots\ldots + a_n)\, e^{mx} = 0$$

Cancelling $e^{mx}$ $\qquad (\because e^{mx} \neq 0$ for any $m)$

$$a_0 m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots\ldots + a_n = 0 \qquad \ldots(iii)$$

Thus, $e^{mx}$ will be a solution of $(i)$ if $m$ satisfies $(iii)$ *i.e.*, if $m$ is a root of $(iii)$.

Equation $(iii)$ is called the auxiliary equation for the differential equation $(i)$.

Replacing $m$ by D in $(iii)$, we get

$$a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n = 0 \qquad \ldots(iv)$$

Equation $(iii)$ gives the same values of $m$ as equation $(iv)$ gives of D. In practice, we take equation $(iv)$ as the auxiliary equation which is obtained by equating to zero the symbolic co-efficient of $y$ in $(i)$. [D is considered as an algebraic quantity]

**Definition.** The equation obtained by equating to zero the symbolic co-efficient of $y$ is called the auxiliary equation, provided D is taken as an algebraic quantity.

If the differential equation is $f(D)\, y = 0$, its auxiliary equation is $f(D) = 0$.

### ☞ 6. To Find the Complete Solution of the Equation

$$(a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n) y = 0 \qquad \ldots(i)$$

The A.E. for $(i)$ is $\quad a_0 D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots\ldots + a_n = 0 \qquad \ldots(ii)$

**Case I. When all the roots of the A.E. (ii) are real and different.**

Let $m_1, m_2, m_3, \ldots\ldots, m_n$ be the $n$ real and different roots of (*ii*).

Then $y = e^{m_1 x}$, $y = e^{m_2 x}$, $y = e^{m_3 x}$, ......., $y = e^{m_n x}$ are $n$ independent solutions of (*i*).

Hence the complete solution of (*i*) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + c_n e^{m_n x} \qquad \ldots(iii)$$

**Case II. When two roots of the A.E. (ii) are equal and all others different.**

Let $m_1$, $m_1$ (*i.e.*, $m_2 = m_1$), $m_3$, ......, $m_n$ be the $n$ real roots of (*ii*). Then solution (*iii*) becomes $y = (c_1 + c_2)e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$ which is a solution of (*i*). It contains $(n - 1)$ arbitrary constants (since $c_1 + c_2$ is equivalent to only one arbitrary constant) and not $n$.

$\therefore$ it is not a complete solution of (*i*).

Consider the equation $(D - m_1)^2 y = 0$, a differential equation of 2nd order whose A.E. $(D - m_1)^2 = 0$ has both the roots equal. It may be written as $(D - m_1)(D - m_1)y = 0$.

Put $(D - m_1)y = V$, it becomes

$$(D - m_1)V = 0 \quad \text{or} \quad \frac{dV}{dx} = m_1 V$$

Separating the variables, $\dfrac{dV}{V} = m_1 dx$

Integrating, $\qquad \log V = m_1 x + \log c$

or $\qquad \log V - \log c = m_1 x \quad \text{or} \quad \log \dfrac{V}{c} = m_1 x$

$\therefore \qquad \dfrac{V}{c} = e^{m_1 x} \quad \text{or} \quad V = ce^{m_1 x}$

or $\qquad (D - m_1)y = ce^{m_1 x} \qquad\qquad | \because \quad V = (D - m_1)y$

or $\qquad \dfrac{dy}{dx} - m_1 y = ce^{m_1 x}$

which is a linear equation of the first order.

$$\text{I.F.} = e^{\int -m_1 dx} = e^{-m_1 x}$$

$\therefore$ its solution is $\quad y \cdot e^{-m_1 x} = \int ce^{m_1 x} \cdot e^{-m_1 x}\, dx + c'$

$$= \int c\,dx + c' = cx + c'$$

or $\qquad y = (c' + cx)\, e^{m_1 x} \quad \text{or} \quad y = (c_1 + c_2 x)e^{m_1 x}$

Hence the complete solution of (*i*) is

$$y = (c_1 + c_2 x)\, e^{m_1 x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}.$$

**Cor.** In case three roots of A.E. are equal *i.e.*, $m_1 = m_2 = m_3$ the complete solution is

$$y = (c_1 + c_2 x + c_3 x^2)\, e^{m_1 x} + c_4 e^{m_4 x} + \ldots\ldots + c_n\, e^{m_n x}.$$

**Case III. When two roots of the A.E. (ii) are imaginary and the rest are all real and different.**

Let $\alpha + i\beta$, $\alpha - i\beta$, $m_3$, $m_4$, ......, $m_n$ be the $n$ roots (*ii*). Then the complete solution is

$$y = c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}[c_1(\cos \beta x + i \sin \beta x) + c_2(\cos \beta x - i \sin \beta x)] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}.$$

[$\because \quad e^{i\theta} = \cos\theta + i \sin\theta$]

$$= e^{\alpha x}[(c_1 + c_2)\cos \beta x + i(c_1 - c_2)\sin \beta x] + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

$$= e^{\alpha x}(c' \cos \beta x + c'' \sin \beta x) + c_3 e^{m_3 x} + \ldots\ldots + c_n e^{m_n x}$$

[Taking $c_1 + c_2 = c'$, $i(c_1 - c_2) = c''$]

**Note.** After suitably adjusting the constants, $e^{\alpha x}(c' \cos \beta x + c'' \sin \beta x)$ may also be written as $e^{\alpha x}.\, c' \cos(\beta x + c'')$ or $e^{\alpha x}.\, c' \sin(\beta x + c'')$.

**Cor.** In case the A.E. has two equal pairs of imaginary roots *i.e.,* $\alpha \pm i\beta, \pm i\beta$, then the complete solution is

$$y = e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x] + c_5\, e^{m_5 x} + \ldots\ldots + c_n\, e^{m_n x}.$$

## REVISION THROUGH UNIVERSITY PAPERS

**Q. 1. If $\lambda_1, \lambda_2$ are real and distinct roots of the auxiliary equation $\lambda^2 + a_1\lambda + a_2 = 0$, and $y_1 = e^{\lambda_1 x}$, $y_2 = e^{\lambda_1 x}$, then $y = c_1 y_1 + c_2 y_2$ is a solution of**

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0.$$

**Is this solution general ?**

**Sol.** The auxiliary equation is $\lambda^2 + a_1\lambda + a_2 = 0$

or $$D^2 + a_1 D + a_2 = 0 \qquad \ldots(i)$$

[Replacing $\lambda$ by D]

$\because \quad \lambda_1, \lambda_2$ are its roots ($\lambda_1 \neq \lambda_2$)

$\therefore \quad y_1 = e^{\lambda_1 x}$ and $y_2 = e^{\lambda_2 x}$ are two independent solutions of

$$(D^2 + a_1 D + a_2)y = 0$$

or $$D^2 y + a_1 Dy + a_2 y = 0 \qquad \ldots(ii)$$

$$\Rightarrow \quad \left.\begin{aligned} D^2 y_1 + a_1 Dy_1 + a_2 y_1 = 0 \\ D^2 y_2 + a_1 Dy_2 + a_2 y_2 = 0 \end{aligned}\right\} \qquad \ldots(iii)$$

Putting $y = c_1 y + c_2 y_2$ in (*ii*), we have

$$D^2(c_1 y_1 + c_2 y_2) + a_1 D(c_1 y_1 + c_2 y_2) + a_2(c_1 y_1 + c_2 y_2) = 0$$

or $$c_1(D^2 y_1 + a_1 Dy_1 + a_2 y_1) + c_2(D^2 y_2 + a_1 Dy_2 + a_2 y_2) = 0$$

or $$c_1(0) + c_2(0) = 0 \qquad \text{| Using } (iii)]$$

which is true.

$\Rightarrow \quad y = c_1 y_1 + c_2 y_2$ is a solution of (*ii*)

*i.e.,* $\quad y = c_1 y_1 + c_2 y_2$ is a solution of

$$\frac{d^2y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$$

Since the solution involves two arbitrary constants and the order of the given equation is also two, this solution is general.

**Q. 2. Show that the solution of $(D - m)^2 y = 0$ is $(c_1 + c_2x)e^{mx}$, where $D = \frac{d}{dx}$ and $c_1$, $c_2$ are arbitrary constants.**

**Sol.** See Case II of the article.

**Q. 3. Obtain the solution of the differential equation f(D)y = 0 when the auxiliary equation has all roots real, two of them are equal and the remaining distinct.**

**Sol.** Reproduce Case II of the article.

**Q. 4. Explain the method to solve the differential equation f(D)y = 0 when the auxiliary equation has two equal roots.**

**Sol.** See Case II of the article.

**Q. 5. If the two roots of the auxiliary equation**

$$\lambda^2 + a_1\lambda + a_2 = 0$$

**are real and equal, then prove that**

**$y = (c_1 + c_2x)\, e^{\lambda x}$ is a solution of the equation**

**$y_2 + a_1y_1 + a_2y = 0$, where $c_1$, $c_2$ are arbitrary constants.**

**Sol.** The given equation is

$$y_2 + a_1y_1 + a_2 = 0 \quad \text{or} \quad D^2y + a_1Dy + a_2y = 0$$

or $$(D^2 + a_1D_1 + a_2)y = 0 \quad ...(i)$$

Its A.E. is $\lambda^2 + a_1\lambda + a_2 = 0$ or $D^2 + a_1D + a_2 = 0$ ...(*ii*)

[Replacing $\lambda$ by D]

The roots of (*ii*) are real and equal. (given)

Let each root be $\lambda$. Then (*ii*) can be written as

$$D^2 + a_1D + a_2 \equiv (D - \lambda)^2 = 0$$

$\therefore$ (*i*) becomes $(D - \lambda)^2y = 0$ or $(D - \lambda)(D - \lambda)y = 0$ ...(*iii*)

Put $(D - \lambda)y = V$, then (*iii*), becomes $(D - \lambda)V = 0$

or $$\frac{dV}{dx} = \lambda V \quad \text{or} \quad \frac{dV}{V} = \lambda\, dx$$

Integrating $\log V = \lambda x + \log c$ or $\log V - \log c = \lambda x$

or $$\log \frac{V}{c} = \lambda x \quad \therefore \quad \frac{V}{c} = e^{\lambda x} \quad \text{or} \quad V = ce^{\lambda x}$$

or $$(D - \lambda)y = ce^{\lambda x} \qquad [\because \quad V = (D - \lambda)y]$$

or $$\frac{dy}{dx} - \lambda y = ce^{\lambda x}$$, which is a linear equation of the first order.

$$\text{I.F.} = e^{\int -\lambda\, dx} = e^{-\lambda x}$$

$\therefore$ its solution is $y \,.\, e^{-\lambda x} = \int ce^{\lambda x} \,.\, e^{-\lambda x}\, dx + c' = \int c dx + c' = cx + c'$

or $$y = (cx + c')\, e^{\lambda x} \text{ or } y = (c_1 + c_2x)\, e^{\lambda x}.$$

**7. Rule to Solve the Equation**

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + a_ny = 0 \qquad | \text{ R.H.S.} = 0$$

(*i*) Write the equation in the symbolic form

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots + a_n)y = 0$$

(*ii*) Write the auxiliary equation (A.E.) $a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots + a_n = 0$

Solve it for D as if D were an ordinary algebraic quantity.

(*iii*) From the roots of A.E. write down the corresponding part of the complete solution as follows :

| *Roots of A.E.* | *Corresponding part of C.S.* |
|---|---|
| 1. One real root $m_1$. | $c_1e^{m_1x}$ |
| 2. Two real and different roots $m_1, m_2$. | $c_1 e^{m_1x} + c_2 e^{m_2x}$ |
| 3. Two real and equal roots $m_1, m_1$. | $(c_1 + c_2x) e^{m_1x}$ |
| 4. Three real and equal roots $m_1, m_1, m_1$. | $(c_1 + c_2x + c_3x^2) e^{m_1x}$ |
| 5. One pair of complex roots $\alpha \pm i\beta$. | $e^{\alpha x} (c_1 \cos \beta x + c_2 \sin \beta x)$ |
| 6. Two pairs of complex and equal roots $\alpha \pm i\beta, \alpha \pm i\beta$. | $e^{\alpha x} [(c_1 + c_2x) \cos \beta x + (c_3 + c_4x) \sin \beta x]$ |

**Example 1.** *Solve the following :*

(*i*) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} - 4y = 0$ (*ii*) $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = 0$

(*iii*) $\dfrac{d^3y}{dx^3} - 9\dfrac{d^2y}{dx^2} + 23\dfrac{dy}{dx} - 15y = 0$ (*iv*) $\dfrac{d^2y}{dx^2} + (a + b)\dfrac{dy}{dx} + aby = 0$

(*v*) $(D^3 - 2D^2 + 4D - 8)y = 0$.

**Sol.** (*i*) Given equation in the symbolic form is

$$(D^2 - 3D - 4)y = 0$$

Auxiliary equation is

$$D^2 - 3D - 4 = 0 \quad \text{or} \quad (D - 4)(D + 1) = 0$$

$$\Rightarrow \quad D = 4, -1 \qquad | \text{ Roots are real and distinct}$$

$\therefore$ The complete solution is $y = c_1e^{4x} + c_2e^{-x}$.

(*ii*) Given equation in the symbolic form is

$$(D^3 - 7D - 6)y = 0$$

Auxiliary equation is $D^3 - 7D - 6 = 0$

Putting $D = -1$ L.H.S. of A.E. $= -1 + 7 - 6 = 0$

$\therefore$ $D + 1$ is a factor of L.H.S. of A.E.

By synthetic division

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$\therefore$ A.E. may be written as $(D + 1)(D^2 - D - 6) = 0$

or $\quad (D + 1)(D - 3)(D + 2) = 0 \quad \therefore \quad D = -1, -2, 3$

$\therefore$ The complete solution is $y = c_1 e^{-x} + c_2e^{-2x} + c_3e^{3x}$.

(*iii*) Given equation in the symbolic form is $(D^3 - 9D^2 + 23D - 15)y = 0$

Auxiliary equation is $\quad D^3 - 9D^2 + 23D - 15 = 0$

Putting $D = 1$ L.H.S. of A.E. $= 1 - 9 + 23 - 15 = 0$

$\therefore$ D – 1 is a factor of L.H.S. of A.E.

By synthetic division

$$\begin{array}{r|rrrr} 1 & 1 & -9 & 23 & -15 \\ & & 1 & -8 & 15 \\ \hline & 1 & -8 & 15 & \boxed{0} \end{array}$$

$\therefore$ A.E. may be written as

$$(D-1)(D^2-8D+15)=0 \quad \text{or} \quad (D-1)(D-3)(D-5)=0$$

$\therefore$ $$D = 1, 3, 5$$

$\therefore$ The complete solution is $y = c_1e^x + c_2e^{3x} + c_3e^{5x}$.

(*iv*) Given equation in the symbolic form is

$$[D^2 + (a+b)D + ab]y = 0$$

Auxiliary equation is $D^2 + (a+b)D + ab = 0$

or $$(D+a)(D+b) = 0 \therefore D = -a, -b$$

$\therefore$ The complete solution is $y = c_1e^{-ax} + c_2e^{-bx}$

(*v*) The given equation in symbolic form is

$$(D^3 - 2D^2 + 4D - 8)y = 0$$

Auxiliary equation is

$$D^3 - 2D^2 + 4D - 8 = 0$$

or $$D^2(D-2) + 4(D-2) = 0$$

or $$(D-2)(D^2+4) = 0 \quad \therefore \quad D = 2, \pm 2i$$

The complete solution is

$$y = c_1 e^{2x} + e^{ax}(c_2 \cos 2x + c_3 \sin 2x)$$

or $$y = c_1e^{2x} + c_2 \cos 2x + c_3 \sin 2x.$$

**Example 2.** *Solve the following :*

(*i*) $\dfrac{d^4y}{dx^4} - 5\dfrac{d^2y}{dx^2} + 4y = 0$ (*ii*) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = 0$

(*iii*) $\dfrac{d^4y}{dx^4} + m^4y = 0$ (*iv*) $(D^4 + 2D^3 + 3D^2 + 2D + 1)y = 0$

(*v*) $(D^3 - D^2 - D - 2)y = 0$.

**Sol.** (*i*) Given equation in the symbolic form is

$$(D^4 - 5D^2 + 4)y = 0$$

Auxiliary equation is $$D^4 - 5D^2 + 4 = 0$$

or $$(D^2-4)(D^2-1) = 0 \quad \text{or} \quad D^2 = 4, 1$$

$\therefore$ $$D = \pm 2, \pm 1 \quad \textit{i.e.,} \quad D = 1, -1, 2, -2$$

$\therefore$ The complete solution is $y = c_1e^x + c_2e^{-x} + c_3e^{2x} + c_4e^{-2x}$.

(*ii*) Given equation in the symbolic form is $(D^2 - 4D + 1)y = 0$

Auxiliary equation is $D^2 - 4D + 1 = 0$

$$D = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

*i.e.,* $$D = 2+\sqrt{3},\ 2-\sqrt{3}$$ | **Note.** The roots are real (irrational) and not complex.

∴ The complete solution is

$$y = c_1 e^{(2+\sqrt{3})x} + c_2 e^{(2-\sqrt{3})x}.$$

(*iii*) Given equation in the symbolic form is

$$(D^4 + m^4)y = 0$$

Auxiliary equation is $\quad D^4 + m^4 = 0$

or $\quad D^4 + 2m^2D^2 + m^4 - 2m^2D^2 = 0$ | **Note carefully**

or $\quad (D^2 + m^2)^2 - 2m^2D^2 = 0 \quad$ or $\quad (D^2 + m^2)^2 - (\sqrt{2}\,mD)^2 = 0$

or $\quad (D^2 + \sqrt{2}\,mD + m^2)(D^2 - \sqrt{m}\ D + m^2) = 0$

$$\therefore \quad D = \frac{-\sqrt{2}m \pm \sqrt{2m^2 - 4m^2}}{2};\ \frac{\sqrt{2}m \pm \sqrt{m^2 - 4m^2}}{2}$$

$$= \frac{-\sqrt{2}m \pm i\,.\sqrt{2}m}{2};\ \frac{\sqrt{2}m \pm i\,.\sqrt{2}m}{2}$$

$$= -\frac{m}{\sqrt{2}} \pm i\,.\frac{m}{\sqrt{2}}\ ;\ \frac{m}{\sqrt{2}} \pm i\,.\frac{m}{\sqrt{2}}$$ | Form $\alpha \pm i\beta$

∴ The complete solution is

$$y = e^{-\frac{m}{\sqrt{2}}x}\left[c_1 \cos\frac{m}{\sqrt{2}}x + c_2 \sin\frac{m}{\sqrt{2}}x\right] + e^{\frac{m}{\sqrt{2}}x}\left[c_3 \cos\frac{m}{\sqrt{2}}x + c_4 \sin\frac{m}{\sqrt{2}}x\right]$$

(*iv*) The given equation in the symbolic form is

$$(D^4 + 2D^3 + 3D^2 + 2D + 1)y = 0$$

Auxiliary equation is $\quad D^4 + 2D^3 + 3D^2 + 2D + 1 = 0$

or $\quad (D^4 + 2D^3 + D^2) + 2D^2 + 2D + 1 = 0$

or $\quad (D^2 + D)^2 + 2(D^2 + D) + 1 = 0$ | **Note carefully**

or $\quad (D^2 + D + 1)^2 = 0$

$$\therefore \quad D = \frac{-1 \pm \sqrt{1-4}}{2},\ \frac{-1 \pm \sqrt{1-4}}{2}$$

$$= \frac{-1 \pm i\sqrt{3}}{2};\ \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\,.\frac{\sqrt{3}}{2};\ -\frac{1}{2} \pm i\,.\frac{\sqrt{3}}{2}$$

**[Two pairs of complex and equal roots]**

∴ The complete solution is

$$y = e^{-\frac{1}{2}x}\left[(c_1 + c_2x)\cos\frac{\sqrt{3}}{2}x + (c_3 + c_4x)\sin\frac{\sqrt{3}}{2}x\right]$$

(*v*) The given equation is $(D^3 - D^2 - D - 2)y = 0$

Auxiliary equation is $\quad D^3 - D^2 - D - 2 = 0$

D = 2 satisfies it ∴ D − 2 is a factor of L.H.S.

By synthetic division

$$\begin{array}{c|cccc} 2 & 1 & -1 & -1 & -2 \\ & & 2 & 2 & 2 \\ \hline & 1 & 1 & 1 & \boxed{0} \end{array}$$

$\therefore$ A.E. may be written as $(D-2)(D^2+D+1)=0$

$\Rightarrow$ $$D = 2, \frac{-1 \pm i\sqrt{3}}{2} \Rightarrow D = 2, -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

The complete solution is

$$y = c_1 e^{2x} + e^{-\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right).$$

**Example 3.** *Solve the following :*

(*i*) $\dfrac{d^4y}{dx^4} + (m^2+n^2)\dfrac{d^2y}{dx^2} + m^2n^2y = 0$ (*ii*) $\dfrac{d^4y}{dx^4} + 13\dfrac{d^2y}{dx^2} + 36y = 0$

(*iii*) $\dfrac{d^4y}{dx^4} + 8\dfrac{d^2y}{dx^2} + 16y = 0$ (*iv*) $\dfrac{d^4y}{dx^4} + 4y = 0$

(*v*) $\dfrac{d^6y}{dx^6} - 64y = 0.$

**Sol.** (*i*) Given equation in symbolic form is

$$[D^4 + (m^2+n^2)D^2 + m^2n^2]y = 0$$

Auxiliary equation is $D^4 + (m^2+n^2)D^2 + m^2n^2 = 0$

or $$(D^2+m^2)(D^2+n^2) = 0 \Rightarrow D = \pm i\,m, \pm i\,n$$

$\therefore$ The complete solution is

$$y = c_1 \cos mx + c_2 \sin mx + c_3 \cos nx + c_4 \sin nx.$$

(*ii*) Given equation in symbolic form is $(D^4 + 13D^2 + 36)y = 0$

Auxiliary equation is

$$D^4 + 13D^2 + 36 = 0 \quad \text{or} \quad (D^2+4)(D^2+9) = 0$$

$\therefore$ $$D^2 = -4, -9 \quad \text{or} \quad D = \pm 2i, \pm 3i$$

or $$D = 0 \pm 2i, 0 \pm 3i$$ | Form $\alpha \pm i\beta$, $\alpha = 0$

$\therefore$ the complete solution is

$$y = e^{0x}(c_1 \cos 2x + c_2 \sin 2x) + e^{0x}(c_3 \cos 3x + c_4 \sin 3x)$$

$$y = c_1 \cos 2x + c_2 \sin 2x + c_3 \cos 3x + c_4 \sin 3x.$$

(*iii*) Given equation in symbolic form is $(D^4 + 8D^2 + 16)y = 0$

Auxiliary equation is

$$D^4 + 8D^2 + 16 = 0 \quad \text{or} \quad (D^2+4)^2 = 0 \therefore D^2 = -4\,; -4$$

$$D = \pm 2i, \pm 2i$$ | Form $\alpha \pm i\beta$, $\alpha = 0$

$\therefore$ The complete solution is

$$y = (c_1 + c_2x)\cos 2x + (c_3 + c_4x)\sin 2x$$ | $\because$ $e^{\alpha x} = e^0 = 1$

(*iv*) Given equation in symbolic form is $(D^4 + 4)y = 0$

Auxiliary equation is $D^4 + 4 = 0$

or $$D^4 + 4D^2 + 4 - 4D^2 = 0$$

or $$(D^2+2)^2 - (2D)^2 = 0$$ **Note carefully**

or $$(D^2 + 2D + 2)(D^2 - 2D + 2) = 0$$

$$\therefore \quad D = \frac{-2 \pm \sqrt{4-8}}{2}, \frac{2 \pm \sqrt{4-8}}{2} = \frac{-2 \pm 2i}{2}, \frac{2 \pm 2i}{2} \qquad \Big| -1 \pm i, \alpha = -1, \beta = 1$$

$$= -1 \pm i, 1 \pm i \qquad \Big| \; 1 \pm i, \alpha = 1, \beta = 1$$

$\therefore$ The complete solution is

$$y = e^{-x}(c_1 \cos x + c_2 \sin x) + e^x (c_3 \cos x + c_4 \sin x).$$

(*v*) Given equation in symbolic form is $(D^6 - 64)y = 0$

Auxiliary equation is $D^6 - 64 = 0$ or $(D^3 - 8)(D^3 + 8) = 0$

or $$(D-2)(D^2+2D+4)(D+2)(D^2-2D+4) = 0$$

$$\therefore \quad D = 2, \frac{-2 \pm \sqrt{4-16}}{2}, -2, \frac{2 \pm \sqrt{4-16}}{2} = 2, \frac{-2 \pm i2\sqrt{3}}{2}, -2, \frac{2 \pm i \cdot 2\sqrt{3}}{2}$$

$$= 2, -2, -1 \pm i\sqrt{3}, 1 \pm i\sqrt{3}$$

$\therefore$ The complete solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} + e^{-x}(c_3 \cos \sqrt{3}x + c_4 \sin \sqrt{3}\, x)$$
$$+ e^x (c_5 \cos \sqrt{3}x + c_6 \sin \sqrt{3}\, x).$$

**Example 4.** *Solve the following* :

(*i*) $\dfrac{d^4y}{dx^4} + 5\dfrac{d^3y}{dx^3} + 6\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} - 8y = 0$ (*ii*) $\dfrac{d^6y}{dx^6} + 6\dfrac{d^4y}{dx^4} + 12\dfrac{d^2y}{dx^2} + 8y = 0$

(*iii*) $(D^2+1)^3(D^2+D+1)^2 y = 0$ (*iv*) $(D^2 - 2aD + a^2 + b^2)y = 0$

(*v*) $(D^2+1)^2(D^2+D+1)y = 0.$

**Sol.** (*i*) Given equation in symbolic form is

$$(D^4 + 5D^3 + 6D^2 - 4D - 8)y = 0$$

Auxiliary equation is $D^4 + 5D^3 + 6D^2 - 4D - 8 = 0$

Putting $D = 1$ L.H.S. of A.E. $= 0$

$\therefore$ $D - 1$ is a factor of L.H.S. of A.E.

By synthetic division

| | | | | | |
|---|---|---|---|---|---|
| 1 | 1 | 5 | 6 | −4 | −8 |
| | | 1 | 6 | 12 | 8 |
| | 1 | 6 | 12 | 8 | 0 |

$\therefore$ Auxiliary equation may be written as

$$(D-1)(D^3 + 6D^2 + 12D + 8) = 0 \quad \text{or} \quad (D-1)(D+2)^3 = 0$$

$\Rightarrow$ $$D = 1, -2, -2, -2 \qquad | -2 \text{ is repeated thrice}$$

$\therefore$ The complete solution is $y = c_1 e^x + (c_2 + c_3 x + c_4 x^2)e^{-2x}.$

(*ii*) Given equation in symbolic form is

$$(D^6 + 6D^4 + 12D^2 + 8)y = 0$$

Auxiliary equation is $D^6 + 6D^4 + 12D^2 + 8 = 0$

or $$(D^2+2)^3 = 0 \quad \therefore \quad D^2 = -2, -2, -2$$

$$D = \pm\sqrt{2}\, i, \pm\sqrt{2}\, i, \pm\sqrt{2}\, i \qquad | \text{ Form } \alpha \pm i\beta, \alpha = 0, \beta = \sqrt{2}$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x + c_3 x^2)\cos\sqrt{2}x + (c_4 + c_5 x + c_6 x^2)\sin\sqrt{2}x. \qquad | \because \; e^{0x} = e^0 = 1$$

(*iii*) Given equation in symbolic form is

$$(D^2+1)^3(D^2+D+1)^2y=0$$

Auxiliary equation is $(D^2+1)^3(D^2+D+1)^2=0$

$$\therefore \qquad D=\pm i,\pm i,\pm i,\frac{-1\pm\sqrt{1-4}}{2},\frac{-1\pm\sqrt{1-4}}{2}$$

$$=\pm i,\pm i,\pm i,-\frac{1}{2}\pm i\frac{\sqrt{3}}{2},-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$$

$\therefore$ The complete solution is

$$y=(c_1+c_2x+c_3x^2)\cos x+(c_4+c_5x+c_6x^2)\sin x$$

$$+e^{-\frac{1}{2}x}\left[(c_7+c_8x)\cos\frac{\sqrt{3}}{2}x+(c_9+c_{10}x)\sin\frac{\sqrt{3}}{2}x\right].$$

(*iv*) Given equation in symbolic form is

$$(D^2-2aD+a^2+b^2)y=0$$

Auxiliary equation is $D^2-2aD+a^2+b^2=0$

or $$(D-a)^2=-b^2 \quad \text{or} \quad D-a=\pm ib$$

$$\therefore \qquad D=a\pm ib$$

$\therefore$ The complete solution is

$$y=e^{ax}(c_1\cos bx+c_2\sin bx).$$

(*v*) Please try yourself.

$$\left[\textbf{Ans.}\quad y=(c_1+c_2x)\cos x+(c_3+c_4x)\sin x+e^{-\frac{1}{2}x}\left(c_5\cos\frac{\sqrt{3}}{2}x+c_6\sin\frac{\sqrt{3}}{2}x\right)\right]$$

**Example 5.** (*a*) *Solve* $\frac{d^2x}{dt^2}-3\frac{dx}{dt}+2x=0$, *given that when* $t=0$, $x=0$ *and* $\frac{dx}{dt}=0$.

(*b*) *Solve* $y''-2y'+10y=0$, given $y(0)=4$, $y'(0)=1$.

**Sol.** (*a*) Given equation in the symbolic form is

$$(D^2-3D+2)x=0 \qquad \Big|\text{ Here } D=\frac{d}{dt}$$

Auxiliary equation is $D^2-3D+2=0$

or $$(D-1)(D-2)=0 \quad \therefore \quad D=1,2$$

$\therefore$ The complete solution is

$$x=c_1e^t+c_2e^{2t} \qquad \ldots(i)$$ | **Note carefully**

Now, when $t=0, x=0$ (given)

$\therefore$ From (*i*), $0=c_1e^0+c_2e^0=c_1+c_2$

i.e., $$c_1+c_2=0 \qquad \ldots(ii)$$

Differentiating (*i*) w.r.t. '*t*', | To find $\frac{dx}{dt}$

$$\frac{dx}{dt}=c_1e^t+2c_2e^{2t}$$

when $t = 0, \frac{dx}{dt} = 0$ (given)

$\therefore \quad 0 = c_1 e^0 + 2c_2 e^0 \quad \Rightarrow \quad c_1 + 2c_2 = 0$ ...(iii)

From (ii) and (iii), $c_1 = c_2 = 0$

$\therefore$ From (i), the solution is $x = 0$.

(b) Given equation in symbolic form is $(D^2 - 2D + 10)y = 0$

Its auxiliary equation is $D^2 - 2D + 10 = 0 \quad \therefore \quad D = \frac{2 \pm 6i}{2} = 1 \pm 3i$

$\therefore$ The complete solution is, $y = e^x (c_1 \cos 3x + c_2 \sin 3x)$ ...(i)

Now $y(0) = 4 \quad \Rightarrow \quad y = 4$ when $x = 0 \quad \therefore \quad 4 = c_1$

Equation (i) becomes $y = e^x (4 \cos 3x + c_2 \sin 3x)$ ...(ii)

so that $y' = e^x (4 \cos 3x + c_2 \sin 3x) + e^x (-12 \sin 3x + 3c_2 \cos 3x)$

Since $y'(0) = 1$ *i.e.*, $y' = 1$ when $x = 0$

we have $1 = 4 + 3c_2 \quad \Rightarrow \quad c_2 = -1$

Equation (ii) becomes $y = e^x (4 \cos 3x - \sin 3x)$

which is the required particular solution.

**Example 6.** *Solve* $\frac{d^2x}{dt^2} + 4\frac{dx}{dt} + 3x = 0$, *given that for* $t = 0$, $x = 0$ *and* $\frac{dx}{dt} = 12$.

**Sol.** Please try yourself. [**Ans.** $x = -6e^{-3t} + 6e^{-t}$]

**Example 7.** *Solve* $\frac{d^2y}{dx^2} + y = 0$, *given* $y = 2$ *for* $x = 0$, $y = -2$ *for* $x = \frac{\pi}{2}$.

**Sol.** Given equation in symbolic form is $(D^2 + 1)y = 0$

Auxiliary equation is $D^2 + 1 = 0 \quad \therefore \quad D = \pm i$

$\therefore$ The complete solution is $y = c_1 \cos x + c_2 \sin x$ ...(i)

when $x = 0, y = 2 \quad \therefore$ from (i), $2 = c_1$

when $x = \frac{\pi}{2}, y = -2 \quad \therefore$ from (i), $-2 = c_2$

Substituting the values of $c_1$ and $c_2$ in (i), the solution is

$$y = 2 \cos x - 2 \sin x.$$

**Example 8.** *If for* $t = \frac{\pi}{\sqrt{\mu}}$, $x = a$ *and* $\frac{dx}{dt} = 0$, *solve the equation* $\frac{d^2x}{dt^2} + \mu x = 0$ $(\mu > 0)$.

**Sol.** Please try yourself. [**Ans.** $x = -a \cos \sqrt{\mu}t$.]

**Example 9.** *Show that the solution of* $\ddot{x} + k\dot{x} + \mu x = 0$ *is*

$$x = e^{-\frac{1}{2}kt} (A \cos nt + B \sin nt)$$

*where* $n^2 = \mu - \frac{1}{4}k^2$ *and n is real, dots denoting differentiation w.r.t. 't'.*

**Sol.** Given equation is $\frac{d^2x}{dt^2} + k\frac{dx}{dt} + \mu x = 0$

Writing it in symbolic form $(D^2 + kD + \mu)x = 0 \quad \Big| \; D = \frac{d}{dt}$

Auxiliary equation is $D^2 + kD + \mu = 0$

$$\therefore \quad D = \frac{-k \pm \sqrt{k^2 - 4\mu}}{2} = \frac{-k \pm \sqrt{k^2 - 4(n^2 + \frac{1}{4}k^2)}}{2} \qquad \left| \begin{array}{l} \because \; n^2 = \mu - \frac{1}{4}k^2 \\ \therefore \; \mu = n^2 + \frac{1}{4}k^2 \end{array} \right.$$

$$= \frac{-k \pm \sqrt{-4n^2}}{2} = \frac{-k \pm i.2n}{2} \qquad \left| \begin{array}{r} \because \; n \text{ is real} \\ n^2 \text{ is} + \text{ve} \\ \therefore \; -4n^2 \text{ is} - \text{ve} \end{array} \right.$$

$$= -\frac{k}{2} \pm i\,n$$

Hence the complete solution is

$$x = e^{-\frac{1}{2}kt} (A \cos nt + B \sin nt).$$

**Example 10.** *If* $\dfrac{d^4y}{dx^4} - a^4y = 0$, *show that*

$$y = c_1 \cos ax + c_2 \sin ax + c_3 \cosh ax + c_4 \sinh ax.$$

**Sol.** Given equation in symbolic form is $(D^4 - a^4)y = 0$

A.E. is $\qquad D^4 - a^4 = 0 \quad \text{or} \quad (D^2 + a^2)(D^2 - a^2) = 0$

$\therefore \qquad D = \pm ia, \pm a$

$\therefore$ The complete solution is

$$y = c_1 \cos ax + c_2 \sin ax + Ae^{ax} + Be^{-ax}$$

$$= c_1 \cos ax + c_2 \sin ax + A(\cosh ax + \sinh ax) + B(\cosh ax - \sinh ax)$$

$$\left[\because \; \cosh ax = \frac{e^{ax} + e^{-ax}}{2};\; \sinh ax = \frac{e^{ax} - e^{-ax}}{2}\right.$$

$$\left.\begin{array}{r} \therefore \; e^{ax} = \cosh ax + \sinh ax, \\ e^{-ax} = \cosh ax - \sinh ax \end{array}\right]$$

$$= c_1 \cos ax + c_2 \sin ax + (A + B) \cosh ax + (A - B) \sinh ax$$

$$= c_1 \cos ax + c_2 \sin ax + c_3 \cosh ax + c_4 \sinh ax$$

[on replacing A + B and A − B by $c_3$ and $c_4$]

**Theorem 1. If y = Y be the complete solution of**

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots + a_n)y = 0 \qquad \ldots(i)$$

**and y = u be a particular solution [containing no arbitrary constants] of**

$$(a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots + a_n)y = X \qquad \ldots(ii)$$

**where X is a function of x, then the complete solution of (ii) is y = Y + u.**

**Proof.** $\because$ $y = Y$ is the complete solution of (*i*)

$$\therefore \qquad (a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots + a_n)Y = 0 \qquad \ldots(iii)$$

Again $\because$ $y = u$ is a solution of (*ii*)

$$\therefore \qquad (a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots + a_n)\,u = X \qquad \ldots(iv)$$

If $y = Y + u$, then

L.H.S. of (*ii*) $\qquad = (a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots + a_n)(Y + u)$

$$= (a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots + a_n)Y$$
$$+ (a_0D^n + a_1D^{n-1} + a_2D^{n-2} + \ldots\ldots + a_n)\, u$$
$$= 0 + X \qquad \mid \because \text{ of } (iii) \text{ and } (iv)$$
$$= X = \text{R.H.S. of } (ii)$$

$\therefore$ $y = Y + u$ is a solution of $(ii)$.

It contains $n$ arbitrary constants

Also $(ii)$ is of the $n$th order.

Hence $y = Y + u$ is the complete solution of $(ii)$.

**Note 1.** $f(D) = 0$ is called the **auxiliary equation (A.E.)** ;

Y the **complementary function (C.F.)**

$u$ the **particular integral (P.I.)**

**Note 2. The complete solution (C.S.) of $(ii)$ is**

$$\mathbf{y = C.F. + P.I.}$$

**The inverse operator $\dfrac{1}{f(D)}$ .**

**Definition.** $\dfrac{1}{f(D)}$ X is the function of $x$, **free from arbitrary constants,** which when operated upon by $f(D)$ gives X.

*i.e.,*
$$f(D) \cdot \frac{1}{f(D)} X = X$$

Thus $f(D)$ and $\dfrac{1}{f(D)}$ are inverse operators (since they cancel each other's effect when both operate on any function, leaving the function intact.)

**Theorem 2. Prove that $\dfrac{1}{D}$ X = $\int$ Xdx, no arbitrary constants being added.**

**Proof.** Let
$$\frac{1}{D} X = Z$$

Operating on both sides by D

$$D \cdot \frac{1}{D} X = DZ \qquad \text{or} \qquad X = DZ \qquad \text{or} \qquad \frac{dZ}{dx} = X$$

Integrating both sides w.r.t. $x$

$Z = \int X dx$, **no arbitrary constant being added.**

$$\left[\because\ Z = \frac{1}{D} X \text{ constants no arbtrary constant}\right]$$

$\therefore$
$$\frac{1}{D} X = \int X dx, \text{ no arbitrary constant being added.}$$

**Note. As proved above the symbol $\dfrac{1}{D}$ stands for integration.**

**Theorem 3. Prove that $\dfrac{1}{f(D)}$ X is the particular integral of the equation f(D)y = X.**

**Proof.** The equation is $f(D)y = X$ ...(i)

Putting $$y = \frac{1}{f(D)} X$$

L.H.S. of $(i)$ $$= f(D) \cdot \frac{1}{f(D)} X = X = \text{R.H.S. of } (i)$$

$\therefore$ $$y = \frac{1}{f(D)} X \text{ is a solution of } (i)$$

$\because$ It contains no arbitrary constants.

$\therefore$ It is the particular integral.

**Theorem 4. Prove that $\frac{1}{D-\alpha} X = e^{ax}\int Xe^{-ax}\, dx$, no arbitrary constant being added.**

**Proof.** Let $\frac{1}{D-a} X = y$

Operating on both sides by $(D - \alpha)$, we get

$$(D-\alpha) \cdot \frac{1}{D-\alpha} X = (D-\alpha)y$$

*i.e.,* $$X = Dy - \alpha y = \frac{dy}{dx} - \alpha y \quad \text{or} \quad \frac{dy}{dx} - ay = X$$

which is linear in $y$.

$$\text{I.F.} = e^{\int -\alpha dx} = e^{-\alpha x}$$

$\therefore$ its solution is $y \cdot e^{-\alpha x} = \int Xe^{-\alpha x}\, dx$, no arbitrary constant being added.

$$\left[\because \; y\left(= \frac{1}{D-\alpha} X\right) \text{contains no arbitrary constant}\right]$$

or $$y = e^{\alpha x}\int Xe^{-\alpha x}\, dx \quad \text{or} \quad \frac{1}{D-\alpha} X = e^{\alpha x}\int Xe^{-\alpha x}\, dx.$$

**Example 1.** *Show that $\frac{1}{D-a} Q = e^{ax}\int e^{-ax} Q dx$, where Q is a function of x.*

*Hence prove that $\frac{1}{(D-a)^2} e^{ax} = \frac{x^2}{2} e^{ax}$.*

**Sol.** For the first part, see Theorem 4 above.

**Second part**

$\therefore$ $$\frac{1}{D-a} e^{ax} = e^{ax}\int e^{-ax} \cdot e^{ax}\, dx \qquad | \because \; Q = e^{ax}$$

$$= e^{ax}\int dx = xe^{ax}$$

$$\frac{1}{(D-a)^2} e^{ax} = \frac{1}{D-a}\left[\frac{1}{D-a} e^{ax}\right] = \frac{1}{D-a}\,(xe^{ax})$$

$$= e^{ax}\int e^{-ax} \cdot xe^{ax}\, dx \qquad | \because \; Q = xe^{ax}$$

$$= e^{ax}\int x dx = \frac{x^2}{2} e^{ax}.$$

**Example 2.** *Show that* $\dfrac{1}{(D-a)^n} e^{ax} = \dfrac{x^n}{n!} e^{ax}$.

**Sol.** $\dfrac{1}{(D-a)^n} e^{ax} = \dfrac{1}{(D-a)^{n-1}} \cdot \dfrac{1}{D-a} e^{ax}$

$$= \frac{1}{(D-a)^{n-1}} e^{ax} . \int e^{-ax} . e^{ax}\, dx = \frac{1}{(D-a)^{n-1}} . e^{ax} \int 1\, dx$$

$$= \frac{1}{(D-a)^{n-1}} \cdot \frac{x}{1!} e^{ax} \qquad ...(i)$$

$$= \frac{1}{(D-a)^{n-2}} \cdot \frac{1}{D-a} xe^{ax} = \frac{1}{(D-a)^{n-2}} . e^{ax} \int e^{-ax} x\, e^{ax}\, dx$$

$$= \frac{1}{(D-a)^{n-2}} . e^{ax} \int x\, dx = \frac{1}{(D-a)^{n-2}} \frac{x^2}{2!} e^{ax} \qquad ...(ii)$$

$$= \frac{1}{(D-a)^{n-3}} \cdot \frac{1}{D-a} \cdot \frac{x^2}{2!} . e^{ax}$$

$$= \frac{1}{(D-a)^{n-3}} . e^{ax} \int e^{-ax} \frac{x^2}{2!} e^{ax} dx = \frac{1}{(D-a)^{n-3}} . e^{ax} . \frac{x^3}{3!} \qquad ...(iii)$$

Generalising from (*i*), (*ii*) and (*iii*),

$$\frac{1}{(D-a)^n} . e^{ax} = \frac{1}{(D-a)^{n-n}} . \frac{x^n}{n!} e^{ax} = \frac{x^n e^{ax}}{n!} .$$

**Standard Cases of Particular Integrals**

**Case I.** $\quad \mathbf{\dfrac{1}{f(D)} e^{ax} = \dfrac{1}{f(a)} e^{ax} \text{ if } f(a) \neq 0.}$

**Proof.** By successive differentiation, we have

$$De^{ax} = ae^{ax}$$
$$D^2e^{ax} = a^2e^{ax}$$
$$\dots\dots\dots\dots$$
$$\dots\dots\dots\dots$$
$$D^ne^{ax} = a^ne^{ax}$$

[We observe that the R.H.S. is obtained by replacing D by $a$]

$\therefore \qquad f(D)e^{ax} = f(a)e^{ax}$

Operating open both sides by $\dfrac{1}{f(D)}$, we get

$$\frac{1}{f(D)} f(D)\, e^{ax} = \frac{1}{f(D)} f(a)e^{ax}$$

or $\qquad e^{ax} = \dfrac{1}{f(D)} f(a)e^{ax} \qquad$ or $\qquad e^{ax} = f(a) . \dfrac{1}{f(D)} e^{ax} \qquad \mid \because\ f(a)$ is constant

Dividing both sides by $f(a)$ which is not zero

$$\frac{1}{f(a)} e^{ax} = \frac{1}{f(D)} e^{ax}$$

Hence $$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \qquad [f(a) \neq 0]$$

**Remember . To evaluate $\frac{1}{f(D)} e^{ax}$, put D = a [f(a) ≠ 0].**

**Example 1.** *Solve the following :*

(*i*) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = e^{-x}$ (*ii*) $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = e^{5x}$

(*iii*) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = e^{4x}$ (*iv*) $\frac{d^3y}{dx^3} + y = 3 + 5e^x$

(*v*) $4\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 3y = e^{2x}$.

**Sol.** (*i*) Given equation in symbolic form is

$$(D^2 + D + 1)y = e^{-x}$$

Auxiliary equation is $D^2 + D + 1 = 0$

$$\therefore \quad D = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1 \pm i\sqrt{3}}{2} = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\therefore \quad \text{C.F.} = e^{-\frac{1}{2}x}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I.} = \frac{1}{D^2 + D + 1} e^{-x} \qquad \left| \frac{1}{f(D)} X \right.$$

$$= \frac{1}{(-1)^2 + (-1) + 1} e^{-x}$$

$$= \frac{1}{1} e^{-x} = e^{-x}$$

Comparing with $\frac{1}{f(D)} e^{ax}$

here $a = -1$ ∴ Put D = – 1

∴ the complete solution is y = C.F. + P.I.

*i.e.,* $$y = e^{-\frac{1}{2}x}\left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x\right] + e^{-x}.$$

(*ii*) Given equation in symbolic form is

$$(D^2 - 3D + 2)y = e^{5x}$$

Auxiliary equation is

$$D^2 - 3D + 2 = 0 \quad i.e., \quad (D-1)(D-2) = 0 \quad \therefore \quad D = 1, 2$$

$$\text{C.F.} = c_1e^x + c_2e^{2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} e^{5x} = \frac{1}{(5)^2 - 3(5) + 2} e^{5x} = \frac{1}{12} e^{5x}$$

∴ the complete solution is y = C.F. + P.I.

*i.e.,* $$y = c_1e^x + c_2e^{2x} + \frac{1}{12} e^{5x}.$$

(*iii*) Given equation in symbolic form is

$$(D^2 - 5D + 6)y = e^{4x}$$

Auxiliary equation is

$$D^2 - 5D + 6 = 0 \quad i.e., \quad (D-2)(D-3) = 0 \quad \therefore \quad D = 2, 3$$

$\therefore$
$$\text{C.F.} = c_1e^{2x} + c_2e^{3x}$$

$$\text{P.I.} = \frac{1}{D^2 - 5D + 6} e^{4x} = \frac{1}{(4)^2 - 5(4) + 6} \cdot e^{4x} = \frac{1}{2} e^{4x}$$

$\therefore$ The complete solution is $y = \text{C.F.} + \text{P.I.}$

*i.e.*,
$$y = c_1e^{2x} + c_2e^{3x} + \frac{1}{2} e^{4x}.$$

(*iv*) Given equation in symbolic form is $(D^3 + 1)y = 3 + 5e^x$

Auxiliary equation is $D^3 + 1 = 0$ *i.e.*, $(D + 1)(D^2 - D + 1) = 0$

$$D = -1, \frac{1 + \sqrt{1-4}}{2} \quad i.e., \quad D = -1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\text{C.F.} = c_1e^{-x} + e^{\frac{1}{2}x}\left[c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I.} = \frac{1}{D^3 + 1}(3 + 5e^x) = \frac{1}{D^3+1}(3e^{0x} + 5e^x) \quad | \textbf{ Note}$$

$$= 3 \cdot \frac{1}{D^3+1} e^{0x} + 5 \cdot \frac{1}{D^3+1} e^x$$

$$= 3 \cdot \frac{1}{0+1} e^{0x} + 5 \cdot \frac{1}{(1)^3+1} e^x = 3 + \frac{5}{2} e^x$$

$\therefore$ The complete solution is $y = \text{C.F.} + \text{P.I.}$

*i.e.*,
$$y = c_1e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right) + 3 + \frac{5}{2} e^x.$$

(*v*) Please try yourself.
$$\left[\textbf{Ans. } y = c_1e^{\frac{1}{2}x} = c_2e^{\frac{3}{2}x} + \frac{1}{21}e^{2x}\right]$$

**Example 2.** *Solve the following :*

(*i*) $\dfrac{d^2y}{dx^2} - 2k\dfrac{dy}{dx} + k^2y = e^x$
(*ii*) $(D^3 + 2D^2 + D)y = e^{2x}$

(*iii*) $(D^3 - 3D^2 + 4)y = e^{3x}$
(*iv*) $\dfrac{d^2y}{dx^2} + 2p\dfrac{dy}{dx} + (p^2 + q^2)\, y = e^{ax}$

(*v*) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 2e^{2x}$
(*vi*) $(D^2 - 1)y = a^x$.

**Sol.** (*i*) Given equation in symbolic form is

$$(D^2 - 2kD + k^2)y = e^x$$

Auxiliary equation is $D^2 - 2kD + k^2 = 0$

$\therefore$
$$(D - k)^2 = 0 \Rightarrow D = k, k \quad \therefore \quad \text{C.F.} = (c_1 + c_2x)e^{kx}$$

$$\text{P.I.} = \frac{1}{D^2 - 2kD + k^2} e^x = \frac{1}{(1)^2 - 2k.1 + k^2} e^x = \frac{1}{(1-k)^2} e^x$$

$\therefore$ The complete solution is

$$y = \text{C.F.} + \text{P.I.} \qquad i.e., \qquad y = (c_1 + c_2 x)e^{kx} + \frac{1}{(1-k)^2}\, e^x$$

(*ii*) Given equation in symbolic form is

$$(D^3 + 2D^2 + D)y = e^{2x}$$

Auxiliary equation is $D^3 + 2D^2 + D = 0$

*i.e.,* $D(D^2 + 2D + 1) = 0$ or $D(D + 1)^2 = 0$ $\therefore$ $D = 0, -1, -1$

$$\text{C.F.} = c_1 e^{0x} + (c_2 + c_3 x)e^{-x} = c_1 + (c_2 + c_3 x)e^{-x}$$

$$\text{P.I.} = \frac{1}{D^3 + 2D^2 + D}\, e^{2x} = \frac{1}{(2)^3 + 2(2)^2 + 2}\, e^{2x} = \frac{1}{18}\, e^{2x}$$

$\therefore$ The complete solution is $y = \text{C.F.} + \text{P.I.}$

*i.e.,*
$$y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{1}{18}\, e^{2x}.$$

(*iii*) Given equation in symbolic form is

$$(D^3 - 3D^2 + 4)y = e^{3x}$$

Auxiliary equation is $D^3 - 3D^2 + 4 = 0$

Putting $D = -1$, L.H.S. of A.E. $= 0$

$\therefore$ D + 1 is a factor of L.H.S. of A.E.

By synthetic division

$$\begin{array}{r|rrrr} -1 & 1 & -3 & 0 & 4 \\ & & -1 & 4 & -4 \\ \hline & 1 & -4 & 4 & \boxed{0} \end{array}$$

$\therefore$ A.E. may be written as $(D + 1)(D^2 - 4D + 4) = 0$

or $(D + 1)(D - 2)^2 = 0$ $\therefore$ $D = -1, 2, 2$

$$\text{C.F.} = c_1 e^{-x} + (c_2 + c_3 x)e^{2x}$$

$$\text{P.I.} = \frac{1}{D^3 - 3D^2 + 4}\, e^{3x} = \frac{1}{(3)^3 - 3(3)^2 + 4}\, e^{3x} = \frac{1}{4}\, e^{3x}$$

$\therefore$ The complete solution is $y = \text{C.F.} + \text{P.I.}$

*i.e.,*
$$y = c_1 e^{-x} + (c_2 + c_3 x)e^{2x} + \frac{1}{4}\, e^{3x}.$$

(*iv*) Given equation in symbolic form is

$$(D^2 + 2pD + p^2 + q^2)y = e^{ax}$$

Auxiliary equation is $D^2 + 2pD + p^2 + q^2 = 0$ or $(D + p)^2 = -q^2$

$\therefore$ $D + p = \pm iq \Rightarrow D = -p \pm iq$

$\therefore$ $\text{C.F.} = e^{-px}(c_1 \cos qx + c_2 \sin qx)$

$$\text{P.I.} = \frac{1}{D^2 + 2pD + p^2 + q^2}\, e^{ax} = \frac{1}{a^2 + 2pa + p^2 + q^2}\, e^{ax} = \frac{1}{(a+p)^2 + q^2}\, e^{ax}$$

$\therefore$ The complete solution is $y = \text{C.F.} + \text{P.I.}$

*i.e.,*
$$y = e^{-px}(c_1 \cos qx + c_2 \sin qx) + \frac{1}{(a+p)^2 + q^2}\, e^{ax}.$$

($v$) Given equation in symbolic form is

$$(D^2 + 2D + 1)y = 2e^{2x}$$

Auxiliary equation is $D^2 + 2D + 1 = 0$

*i.e.,* $(D + 1)^2 = 0 \quad \therefore \quad D = -1, -1$

$$\text{C.F.} = (c_1 + c_2x)e^{-x}$$

$$\text{P.I.} = \frac{1}{D^2 + 2D + 1}(2e^{2x}) = 2 \cdot \frac{1}{D^2 + 2D + 1}\, e^{2x}$$

$$= 2 \cdot \frac{1}{(2)^2 + 2(2) + 1}\, e^{2x} = \frac{2}{9}\, e^{2x}$$

$\therefore$ The complete solution is $y = \text{C.F.} + \text{P.I.}$

*i.e.,* $$y = (c_1 + c_2x)e^{-x} + \frac{2}{9}\, e^{2x}.$$

($vi$) Given equation in symbolic form is

$$(D^2 - 1)y = a^x$$

A.E. is $D^2 - 1 = 0 \therefore D = \pm 1$

$$\text{C.F.} = c_1 e^x + c_2e^{-x}$$

$$\text{P.I.} = \frac{1}{D^2 - 1}\, a^x = \frac{1}{D^2 - 1}\, e^{x \log a}$$

Since $e^{\log f(x)} = f(x)$

$\therefore \quad a^x = e^{\log a^x} = e^{x \log a}$

$$= \frac{1}{(\log a)^2 - 1}\, e^{x \log a} = \frac{a^x}{(\log a)^2 - 1}$$

$\therefore$ The complete solution is $y = c_1e^x + c_2e^{-x} + \dfrac{a^x}{(\log a)^2 - 1}$.

**Example 3.** *Solve* : $\dfrac{d^2y}{dx^2} + 3\,\dfrac{dy}{dx} + 2y = e^{e^x}$

**Sol.** Given equation in symbolic form is $(D^2 + 3D + 2)y = e^{e^x}$

Its A.E. is $D^2 + 3D + 2 = 0$ or $(D + 1)(D + 2) = 0 \Rightarrow D = -1, -2$

$\therefore$ $\text{C.F.} = c_1e^{-x} + c_2e^{-2x}$

$$\text{P.I.} = \frac{1}{(D + 1)(D + 2)}\, e^{e^x} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^x}$$

$$= \frac{1}{D+1}\, e^{e^x} - \frac{1}{D+2}\, e^{e^x} \qquad \text{...}(i)$$

Now $$\frac{1}{D+1}\, e^{e^x} = \frac{1}{D-(-1)}\, e^{e^x} = e^{-x}\int e^{e^x} \cdot e^x\, dx \qquad \left[\because \frac{1}{D-a}X = e^{ax}\int X e^{-ax}\, dx\right]$$

$$= e^{-x}\int e^t\, dt \quad \text{where} \quad t = e^x$$

$$= e^{-x} \cdot e^t = e^{-x} \cdot e^{e^x}$$

and $$\frac{1}{D+2}\, e^{e^x} = \frac{1}{D-(-2)}\, e^{e^x} = e^{-2x}\int e^{e^x} \cdot e^{2x}\, dx$$

$$= e^{-2x} \int e^{e^x} \cdot e^x \cdot e^x\, dx = e^{-2x} \int e^t \cdot t\, dt \quad \text{where } t = e^x$$

$$= e^{-2x} \left[ te^t - \int 1 \cdot e^t\, dt \right] = e^{-2x} [te^t - e^t] = e^{-2x} [e^x - 1]\, e^{e^x}$$

$$= (e^{-x} - e^{-2x})\, e^{e^x}$$

$\therefore$ From (1), $\qquad$ P.I. $= e^{-x} \cdot e^{e^x} - (e^{-x} - e^{-2x}) e^{e^x} = e^{-2x} \cdot e^{e^x}$

The complete solution is $\qquad y =$ C.F. + P.I.

*i.e.,*
$$y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} \cdot e^{e^x}.$$

**Example 4.** *Solve* $\dfrac{d^2y}{dx^2} - 7\dfrac{dy}{dx} + 6y = e^{2x}$, *given that* $y = 0$ *when* $x = 0$.

**Sol.** Given equation in symbolic form is

$$(D^2 - 7D + 6)y = e^{2x}$$

Auxiliary equation is

$$D^2 - 7D + 6 = 0 \quad \text{or} \quad (D - 1)(D - 6) = 0 \quad \therefore \quad D = 1, 6$$

$$\text{C.F.} = c_1 e^x + c_2 e^{6x}$$

$$\text{P.I.} = \frac{1}{D^2 - 7D + 6}\, e^{2x} = \frac{1}{(2)^2 - 7(2) + 6}\, e^{2x} = -\frac{1}{4}\, e^{2x}$$

$\therefore$ The complete solution is $\quad y =$ C.F. + P.I.

*i.e.,*
$$y = c_1 e^x + c_2 e^{6x} - \frac{1}{4}\, e^{2x} \qquad \ldots(i)$$

When $x = 0$, $\qquad y = 0$

$\therefore$ From $(i)$,
$$0 = c_1 e^0 + c_2 e^0 - \frac{1}{4}\, e^0 = c_1 + c_2 - \frac{1}{4}$$

$\therefore$
$$c_2 = \frac{1}{4} - c_1$$

$\therefore$ From $(i)$, required solution is

$$y = c_1 e^x + \left( \frac{1}{4} - c_1 \right) e^{6x} - \frac{1}{4}\, e^{2x}$$

$$y = c_1 (e^x - e^{6x}) + \frac{1}{4}\, (e^{6x} - e^{2x}).$$

**Case II.**
$$\left.\begin{aligned} &\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax \\ \text{and} \quad &\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax \end{aligned}\right\} f(a - a^2) \neq 0$$

**Proof.** By successive differentiation, we have

$$D \sin ax = a \cos ax$$

$$D^2 \sin ax = -a^2 \sin ax$$

$$\Rightarrow \qquad \mathbf{(D^2)^1 \sin ax = (-a^2)^1 \sin ax}$$

$$D^3 \sin ax = -a^3 \cos ax$$

$$D^4 \sin ax = a^4 \cos ax$$

$$\Rightarrow \qquad (\mathbf{D^2})^2 \sin \mathbf{ax} = (-\mathbf{a}^2)^2 \sin \mathbf{ax}$$

..............................................

..............................................

$$(\mathbf{D^2})^n \sin \mathbf{ax} = (-\mathbf{a}^2)^n \sin \mathbf{ax}$$

From these results, it is evident that $f(D^2) \sin ax = f(-a^2) \sin ax$

Operating upon both sides by $\frac{1}{f(D^2)}$ , we have

$$\frac{1}{f(D^2)} f(D^2) \sin ax = \frac{1}{f(D^2)} \cdot f(-a^2) \sin ax$$

or
$$\sin ax = f(-a^2) \cdot \frac{1}{f(D^2)} \sin ax$$

Dividing both sides by $f(-a^2)$ which is not zero

$$\frac{1}{f(-a^2)} \sin ax = \frac{1}{f(D^2)} \sin ax$$

Hence
$$\frac{1}{f(D^2)} \sin ax = \frac{1}{f(-a^2)} \sin ax, \text{ where } f(-a^2) \neq 0$$

Similarly
$$\frac{1}{f(D^2)} \cos ax = \frac{1}{f(-a^2)} \cos ax, \text{ where } f(-a^2) \neq 0$$
(Prove it yourself.)

**Remember. To evaluate $\frac{1}{f(D^2)}$ sin ax, or $\frac{1}{f(D^2)}$ cos ax, put $D^2 = -a^2$ [$(f(-a^2) \neq 0$].**

**Important.** When the particular integral is of the form

$$\frac{1}{f(D)} \sin ax \quad \text{or} \quad \frac{1}{f(D)} \cos ax$$

for $D^2$ put $-a^2$

for $D^3 = D^2 . D$ put $-a^2D$

for $D^4 = D^2 . D^2$, put $-a^2(-a^2) = a^4$ and so on

Ultimately $f(D)$ becomes a linear in D, say of the form D + A. Now proceed as under :

$$\frac{1}{D+A} \sin ax = \frac{D-A}{(D+A)(D-A)} \sin ax$$
| **Note**

$$= \frac{D-A}{D^2-A^2} \sin ax = \frac{D-A}{-a^2-A^2} \sin ax$$

[Putting $-a^2$ for $D^2$ in the denominator]

$$= -\frac{1}{a^2+A^2} (D-A) \sin ax$$

$$= -\frac{1}{a^2+A^2} \left[\frac{d}{dx}(\sin ax) - A \sin ax\right]$$

$$= -\frac{1}{a^2+A^2} [a \cos ax - A \sin ax]$$

In this way, we can evaluate the particular integral completely.

**Example 1.** *Evaluate the particular integral*

$$\frac{1}{f(D^2)} \sin(ax+b) \qquad [f(-a^2) \neq 0]$$

**Sol.** By successive differentiation, we have

$$D \sin(ax+b) = a \cos(ax+b)$$

$$D^2 \sin(ax+b) = -a^2 \sin(ax+b)$$

$$\Rightarrow \qquad (D^2)^1 \sin(ax+b) = (-a^2)^1 \sin(ax+b)$$

$$D^3 \sin(ax+b) = -a^3 \cos(ax+b)$$

$$D^4 \sin(ax+b) = a^4 \sin(ax+b)$$

$$\Rightarrow \qquad (D^2)^2 \sin(ax+b) = (-a^2)^2 \sin(ax+b)$$

..........................................................................

..........................................................................

$$(D^2)^n \sin(ax+b) = (-a^2)^n \sin(ax+b)$$

From these results, it is evident that

$$f(D^2) \sin(ax+b) = f(-a^2) \sin(ax+b)$$

Operating upon both sides by $\frac{1}{f(D^2)}$, we have

$$\frac{1}{f(D^2)} \cdot f(D^2) \sin(ax+b) = \frac{1}{f(D^2)} f(-a^2) \sin(ax+b)$$

or
$$\sin(ax+b) = f(-a^2) \cdot \frac{1}{f(D^2)} \sin(ax+b)$$

Dividing both sides by $f(-a^2)$ which is not zero

$$\frac{1}{f(-a^2)} \sin(ax+b) = \frac{1}{f(D^2)} \sin(ax+b)$$

Hence
$$\frac{1}{f(D^2)} \sin(ax+b) = \frac{1}{f(-a^2)} \sin(ax+b).$$

**Exercise.** *Prove that*

$$\frac{1}{f(D^2)} \cos(mx+n) = \frac{1}{f(-m^2)} \cos(mx+n) \text{ where } f(-m^2) \neq 0.$$

Please try yourself.

**Example 2.** *Solve the following :*

(*i*) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$ (*ii*) $\frac{d^4y}{dx^4} + y = \cos x$

(*iii*) $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = \sin 3x.$ (*iv*) $(D^4 - 2D^2 + 1)y = \cos x$

(*v*) $(D^3 + 1)y = \sin(2x+3).$

**Sol.** (*i*) Given equation in symbolic form is $(D^2 + D + 1)y = \sin 2x$

Auxiliary equation is $D^2 + D + 1 = 0$

$$\Rightarrow \qquad D = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm i \cdot \frac{\sqrt{3}}{2}$$

$\therefore$ $$\text{C.F.} = e^{-1/2x}\left[c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right]$$

$$\text{P.I.} = \frac{1}{D^2 + D + 1}\sin 2x$$

$$= \frac{1}{-2^2 + D + 1}\sin 2x = \frac{1}{D-3}\sin 2x$$

| Form $\frac{1}{f(D)}\sin ax$
| Here $a = 2$
| $\therefore$ put $D^2 = -2^2$

$$= \frac{D+3}{(D-3)(D+3)}\sin 2x = \frac{D+3}{D^2-9}\sin 2x$$

$$= \frac{D+3}{-2^2-9}\sin 2x = -\frac{1}{13}\left[\frac{d}{dx}(\sin 2x) + 3\sin 2x\right]$$

$$= -\frac{1}{13}(2\cos 2x + 3\sin 2x)$$

$\therefore$ The complete solution is

$$y = e^{-\frac{1}{2}x}\left(c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x\right) - \frac{1}{13}(2\cos 2x + 3\sin 2x)$$

(*ii*) Given equation in symbolic form is $(D^4 + 1)y = \cos x$

Auxiliary equation is

$$D^4 + 1 = 0 \quad i.e., \quad D^4 + 2D^2 + 1 - 2D^2 = 0$$

$$(D^2 + 1)^2 - (\sqrt{2}D)^2 = 0$$ | **Note**

$\therefore$ $$(D^2 + \sqrt{2}D + 1)(D^2 - \sqrt{2}D + 1) = 0$$

$$D = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2}, \frac{\sqrt{2} \pm \sqrt{2-4}}{2}$$

$$= \frac{-\sqrt{2} \pm i.\sqrt{2}}{2}, \frac{\sqrt{2} \pm i.\sqrt{2}}{2} = -\frac{1}{\sqrt{2}} \pm i.\frac{1}{\sqrt{2}} \,;\, \frac{1}{\sqrt{2}} \pm i.\frac{1}{\sqrt{2}}$$

$\therefore$ $$\text{C.F.} = e^{-\frac{1}{\sqrt{2}}x}\left(c_1 \cos \frac{x}{\sqrt{2}} + c_2 \sin \frac{x}{\sqrt{2}}\right) + e^{\frac{1}{\sqrt{2}}x}\left(c_3 \cos \frac{x}{\sqrt{2}} + c_4 \sin \frac{x}{\sqrt{2}}\right)$$

$$\text{P.I.} = \frac{1}{D^4 + 1}\cos x$$

$$= \frac{1}{(-1)^2 + 1}\cos x = \frac{1}{2}\cos x$$

| Form $\frac{1}{f(D)}\cos ax$
| Here $a = 1$
| $\therefore$ put $D^2 = -1^2 = -1$.

$\therefore$ The complete solution is

$$y = e^{-\frac{1}{\sqrt{2}}x}\left(c_1 \cos \frac{x}{\sqrt{2}} + c_2 \sin \frac{x}{\sqrt{2}}\right) + e^{\frac{1}{\sqrt{2}}x}\left(c_3 \cos \frac{x}{\sqrt{2}} + c_4 \sin \frac{x}{\sqrt{2}}\right) + \frac{1}{2}\cos x.$$

(*iii*) Given equation in symbolic form is $(D^2 - 5D + 6)y = \sin 3x$

Auxiliary equation is

$$D^2 - 5D + 6 = 0 \quad \text{or} \quad (D-2)(D-3) = 0 \therefore D = 2, 3$$

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{3x}$$

$$\text{P.I.} = \frac{1}{D^2 - 5D + 6} \sin 3x \qquad \text{| Put } D^2 = -3^2 = -9$$

$$= \frac{1}{-9 - 5D + 6} \sin 3x = -\frac{1}{5D+3} \sin 3x$$

$$= -\frac{5D-3}{(5D+3)(5D-3)} \sin 3x = -\frac{5D-3}{25D^2 - 9} \sin 3x$$

$$= -\frac{5D-3}{25(-9)-9} \sin 3x = \frac{1}{234}\left[5\frac{d}{dx}(\sin 3x) - 3\sin 3x\right]$$

$$= \frac{1}{234}[15\cos 3x - 3\sin 3x] = \frac{1}{78}[5\cos 3x - \sin 3x]$$

$\therefore$ The complete solution is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{78}[5\cos 3x - \sin 3x].$$

(*iv*) Given equation in symbolic form is

$$(D^4 - 2D^2 + 1)\, y = \cos x \quad \text{or} \quad (D^2 - 1)^2 y = \cos x$$

A.E. is $(D^2 - 1)^2 = 0 \quad \therefore \quad D = \pm 1, \pm 1 \quad$ *i.e.*, $\quad D = 1, 1, -1, -1$

$$\therefore \qquad \text{C.F.} = (c_1 + c_2 x)\, e^x + (c_3 + c_4 x)\, e^{-x}$$

$$\text{P.I.} = \frac{1}{(D^2-1)^2} \cos x = \frac{1}{(-1^2-1)^2} \cos x \qquad [\text{Replacing } D^2 \text{ by } -1^2]$$

$$= \frac{1}{4}\cos x$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x)\, e^x + (c_3 + c_4 x)\, e^{-x} + \frac{1}{4}\cos x.$$

(*v*) The given equation is $(D^3 + 1)y = \sin(2x + 3)$

Its A.E. is $\quad D^3 + 1 = 0 \quad \text{or} \quad (D+1)(D^2 - D + 1) = 0$

so that $\quad D = -1, \dfrac{1 \pm i\sqrt{3}}{2} \quad \therefore \quad D = -1, \dfrac{1}{2} \pm \dfrac{1}{2}\sqrt{3}\, i$

$$\text{C.F.} = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$$

$$\text{P.I.} = \frac{1}{D^3+1}\sin(2x+3) = \frac{1}{DD^2+1}\sin(2x+3)$$

$$= \frac{1}{D(-2^2)+1}\sin(2x+3) = \frac{1}{1-4D}\sin(2x+3)$$

$$= \frac{1+4D}{(1+4D)(1-4D)} \sin(2x+3) = \frac{1+4D}{1-16D^2} \sin(2x+3)$$

$$= \frac{1+4D}{1-16(-2^2)} \sin(2x+3) = \frac{1}{65} [\sin(2x+3) + 4D \sin(2x+3)]$$

$$= \frac{1}{65} [\sin(2x+3) + 8\cos(2x+3)] \qquad \left[\because \ D \equiv \frac{d}{dx}\right]$$

Hence the complete solution is

$$y = c_1 e^{-x} + e^{\frac{1}{2}x} \left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right) + \frac{1}{65} [\sin(2x+3) + 8\cos(2x+3)].$$

**Example 3.** *Solve the following :*

*(i)* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 5y = \sin 3x$ *(ii)* $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} + \frac{dy}{dx} + y = \sin 2x$

*(iii)* $(D^3 + 6D^2 + 11D + 6)\, y = 2 \sin x.$

**Sol.** (*i*) The given equation in symbolic form is

$$(D^2 - 2D + 5)y = \sin 3x$$

A.E. is $\quad D^2 - 2D + 5 = 0$

$\therefore \quad D = \frac{2 \pm \sqrt{4-20}}{2} = 1 \pm 2i$

$\therefore \quad$ C.F. $= e^x(c_1 \cos 2x + c_2 \sin 2x)$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 5} \sin 3x = \frac{1}{-9-2D+5} \sin 3x$$

$$= -\frac{1}{2} \cdot \frac{1}{D+2} \sin 3x = -\frac{1}{2} \cdot \frac{D-2}{D^2-4} \sin 3x$$

$$= -\frac{1}{2} \cdot \frac{D-2}{-9-4} \sin 3x = \frac{1}{26} [D(\sin 3x) - 2 \sin 3x]$$

$$= \frac{1}{26} (3\cos 3x - 2\sin 3x)$$

$\therefore$ The complete solution is

$$y = e^x (c_1 \cos 2x + c_2 \sin 2x) + \frac{1}{26} (3\cos 3x - 2\sin 3x)$$

(*ii*) The given equation in symbolic form is

$$(D^3 + D^2 + D + 1)y = \sin 2x$$

A.E. is $\quad D^3 + D^2 + D + 1 = 0 \quad$ or $\quad D^2(D+1) + (D+1) = 0 \quad$ or $\quad (D+1)(D^2+1) = 0$

$\therefore \quad D = -1, \pm i$

C.F. $= c_1 e^{-x} + c_2 \cos x + c_3 \sin x$

$$\text{P.I.} = \frac{1}{D^3 + D^2 + D + 1} \sin 2x$$

$$= \frac{1}{(D+1)(D^2+1)} \sin 2x = \frac{1}{(D+1)} \cdot \frac{1}{(-4+1)} \sin 2x$$

$$= -\frac{1}{3}\cdot\frac{1}{D+1}\sin 2x = -\frac{1}{3}\cdot\frac{D-1}{D^2-1}\sin 2x$$

$$= -\frac{1}{3}\frac{D-1}{-4-1}\sin 2x = \frac{1}{15}\,[D(\sin 2x) - \sin 2x]$$

$$= \frac{1}{15}\,(2\cos 2x - \sin 2x)$$

$\therefore$ The complete solution is

$$y = c_1e^{-x} + c_2\cos x + c_3\sin x + \frac{1}{15}\,(2\cos 2x - \sin 2x).$$

(*iii*) The given equation is $(D^3 + 6D^2 + 11D + 6)y = 2\sin x$

A.E. is $D^3 + 6D^2 + 11D + 6 = 0$

By trial, $-1$ is a root.

$$\begin{array}{r|rrrr} -1 & 1 & 6 & 11 & 6 \\ & & -1 & -5 & -6 \\ \hline & 1 & 5 & 6 & \boxed{0} \end{array}$$

$\therefore$ $(D + 1)(D^2 + 5D + 6) = 0$ or $(D + 1)(D + 2)(D + 3) = 0$

$\Rightarrow$ $D = -1, -2, -3$

$\therefore$ $\text{C.F.} = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x}$

$$\text{P.I.} = \frac{1}{D^3 + 6D^2 + 11D + 6}\,(2\sin x) = 2\,.\,\frac{1}{D^3 + 6D^2 + 11D + 6}\sin x$$

$$= 2\,.\,\frac{1}{(-1)D + 6(-1) + 11D + 6}\sin x \qquad [\text{Putting } D^2 = -1]$$

$$= 2\,.\,\frac{1}{10D}\sin x = \frac{1}{5}\cdot\frac{1}{D}\sin x = \frac{1}{5}\int \sin x\,dx = -\frac{1}{5}\cos x$$

$\therefore$ The complete solution is

$$y = c_1e^{-x} + c_2e^{-2x} + c_3e^{-3x} - \frac{1}{5}\cos x.$$

**Example 4.** *Solve the following* :

(*i*) $(D^3 + 1)y = \cos 2x$ (*ii*) $(D^3 + 1)y = \sin 2x$

(*iii*) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + y = a\cos 2x$ (*iv*) $\dfrac{d^2y}{dx^2} - 4y = 2\sin\dfrac{x}{2}$

(*v*) $\dfrac{d^3y}{dx^3} + y = \sin 3x - \cos^2\dfrac{x}{2}$.

**Sol.** (*i*) Given the equation in symbolic form is

$$(D^3 + 1)y = \cos 2x$$

Auxiliary equation is $D^3 + 1 = 0$ or $(D + 1)(D^2 - D + 1) = 0$

$$\therefore \qquad D = -1, \frac{1 \pm \sqrt{1-4}}{2} = -1, \frac{1}{2} \pm i\,.\,\frac{\sqrt{3}}{2}$$

$$\therefore \qquad \text{C.F.} = c_1e^{-x} + e^{\frac{1}{2}x}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right)$$

$$\text{P.I.} = \frac{1}{D^3+1}\cos 2x = \frac{1}{D^2 . D+1}\cos 2x$$

$$= \frac{1}{-4D+1}\cos 2x \qquad | \text{ By putting } D^2 = -2^2 = -4$$

$$= \frac{1+4D}{(1-4D)(1+4D)}\cos 2x = \frac{1+4D}{1-16D^2}\cos 2x$$

$$= \frac{1+4D}{1-16(-4)}\cos 2x = \frac{1}{65}\left[\cos 2x + 4\frac{d}{dx}(\cos 2x)\right]$$

$$= \frac{1}{65}[\cos 2x - 8\sin 2x]$$

$\therefore$ The complete solution is

$$y = c_1e^{-x} + e^{\frac{1}{2}x}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{65}(\cos 2x - 8\sin 2x).$$

(*ii*) Please try yourself.

$$\left[\textbf{Ans. } y = c_1e^{-x} + e^{\frac{1}{2}x}\left(c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{65}(\sin 2x + 8\cos 2x)\right]$$

(*iii*) Given equation in symbolic form is

$$(D^2 - 4D + 1)y = a\cos 2x$$

Auxiliary equation is $D^2 - 4D + 1 = 0$

$$D = \frac{4 \pm \sqrt{16-4}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$$

$\therefore$ $D = 2 + \sqrt{3}, 2 - \sqrt{3}$ | **Note.** The roots are irrational, not imaginary

$$\text{C.F.} = c_1e^{(2+\sqrt{3})x} + c_2e^{(2-\sqrt{3})x}$$

$$= e^{2x}[c_1e^{\sqrt{3}x} + c_2e^{-\sqrt{3}x}]$$

$$\text{P.I.} = \frac{1}{D^2-4D+1}(a\cos 2x) = a.\frac{1}{D^2-4D+1}\cos 2x$$

$$= a . \frac{1}{-2^2-4D+1}\cos 2x = -a\frac{1}{4D+3}\cos 2x$$

$$= -a . \frac{4D-3}{(4D+3)(4D-3)}\cos 2x$$

$$= -a . \frac{4D-3}{16D^2-9}\cos 2x = -a . \frac{4D-3}{16(-2^2)-9}\cos 2x$$

$$= -a . \frac{4D-3}{-73}\cos 2x = \frac{a}{73}\left[4\frac{d}{dx}(\cos 2x) - 3\cos 2x\right]$$

$$= \frac{a}{73}[-8\sin 2x - 3\cos 2x] = -\frac{a}{73}[8\sin 2x + 3\cos 2x]$$

$\therefore$ The complete solution is

$$y = e^{2x}[c_1 e^{\sqrt{3}x} + c_2 e^{-\sqrt{3}x}] - \frac{a}{73}[8 \sin 2x + 3 \cos 2x].$$

(*iv*) Given equation in symbolic form is

$$(D^2 - 4)y = 2 \sin \frac{x}{2}$$

Auxiliary equation is

$$D^2 - 4 = 2 \quad \Rightarrow \quad D = \pm 2$$

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 4}\left(2 \sin \frac{x}{2}\right) = 2 \cdot \frac{1}{D^2 - 4} \sin \frac{x}{2}$$

$$= 2 \cdot \frac{1}{-\left(\frac{1}{2}\right)^2 - 4} \sin \frac{x}{2} = -\frac{8}{17} \sin \frac{x}{2}$$

$\therefore$ The complete solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{8}{17} \sin \frac{x}{2}.$$

(*v*) Given equation in symbolic form is

$$(D^3 + 1)y = \sin 3x - \cos^2 \frac{x}{2}$$

Auxiliary equation is $D^3 + 1 = 0$

or $(D + 1)(D^2 - D + 1) = 0$

$$\therefore \qquad D = -1, \frac{1 \pm \sqrt{1-4}}{2} = -1, \frac{1}{2} \pm i \cdot \frac{\sqrt{3}}{2}$$

$$\text{C.F.} = c_1 \cdot e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right)$$

$$\text{P.I.} = \frac{1}{D^3 + 1}\left[\sin 3x - \cos^2 \frac{x}{2}\right] = \frac{1}{D^3 + 1}\left[\sin 3x - \frac{1 + \cos x}{2}\right]$$

$$= \frac{1}{D^3 + 1}\left[\sin 3x - \frac{1}{2} - \frac{1}{2} \cos x\right]$$

$$= \frac{1}{D^3 + 1} \sin 3x - \frac{1}{2} \cdot \frac{1}{D^3 + 1} e^{0x} - \frac{1}{2} \cdot \frac{1}{D^3 + 1} \cos x$$

$$= \frac{1}{D^2 . D + 1} \sin 3x - \frac{1}{2} \cdot \frac{1}{0 + 1} e^{0x} - \frac{1}{2} \cdot \frac{1}{D^2 . D + 1} \cos x$$

$$= \frac{1}{-9D + 1} \sin 3x - \frac{1}{2} - \frac{1}{2} \cdot \frac{1}{-D + 1} \cos x$$

$$= \frac{1 + 9D}{(1 - 9D)(1 + 9D)} \sin 3x - \frac{1}{2} - \frac{1}{2} \frac{1 + D}{(1 - D)(1 + D)} \cos x$$

$$= \frac{1+9D}{1-81D^2} \sin 3x - \frac{1}{2} - \frac{1}{2} \cdot \frac{1+D}{1-D^2} \cos x$$

$$= \frac{1+9D}{1-81(-9)} \sin 3x - \frac{1}{2} - \frac{1}{2} \cdot \frac{1+D}{1-(-1)} \cos x$$

$$= \frac{1}{730}\left[\sin 3x + 9 \cdot \frac{d}{dx}(\sin 3x)\right] - \frac{1}{2} - \frac{1}{4}\left[\cos x + \frac{d}{dx}(\cos x)\right]$$

$$= \frac{1}{730} [\sin 3x + 27 \cos 3x] - \frac{1}{2} - \frac{1}{4} [\cos x - \sin x]$$

Hence, the complete solution is

$$y = c_1 e^{-x} + e^{\frac{1}{2}x}\left[c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right]$$

$$+ \frac{1}{730} [\sin 3x + 27 \cos 3x] - \frac{1}{2} - \frac{1}{4} (\cos x - \sin x)].$$

**Example 5.** *Solve the following :*

(*i*) $(D^3 - 2D^2 + 3)y = \cos x$ (*ii*) $(D^2 + D - 2)y = \sin x$

(*iii*) $(D^3 + D^2 - D - 1)y = \cos 2x$ (*iv*) $\frac{d^4y}{dx^4} + 2n^2 \frac{d^2y}{dx^2} n^4 y = \cos mx$

(*v*) $\frac{d^2y}{dx^2} + 2k \frac{dy}{dx} + k^2 y = A \cos px.$

**Sol.** (*i*) Given equation in symbolic form is

$$(D^3 - 2D^2 + 3)y = \cos x$$

Auxiliary equation is $D^3 - 2D^2 + 3 = 0$

Putting $D = -1$, L.H.S. of A.E. = 0

$\therefore$ $D + 1$ is a factor of L.H.S. of A.E.

By synthetic division

$$\begin{array}{r|rrrr} -1 & 1 & -2 & 0 & 3 \\ & & -1 & 3 & -3 \\ \hline & 1 & -3 & 3 & \boxed{0} \end{array}$$

$\therefore$ A.E. may be written as $(D + 1)(D^2 - 3D + 3) = 0$

$$D = -1, \frac{3 \pm \sqrt{9-12}}{2} = -1, \frac{3 \pm i\sqrt{3}}{2} = -1, \frac{3}{2} \pm i \frac{\sqrt{3}}{2}$$

$$\text{C.F.} = c_1 e^{-x} + e^{\frac{3}{2}x}\left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right)$$

$$\text{P.I.} = \frac{1}{D^3 - 2D^2 + 3} \cos x = \frac{1}{D^2 \cdot D - 2D^2 + 3} \cos x$$

$$= \frac{1}{-1^2 \cdot D - 2(-1^2) + 3} \cos x = \frac{1}{5-D} \cos x$$

$$= \frac{5+D}{(5-D)(5+D)} \cos x = \frac{5+D}{25-D^2} \cos x$$

$$= \frac{5 + D}{25 - (-1^2)} \cos x = \frac{1}{26}\left[5 \cos x + \frac{d}{dx}(\cos x)\right]$$

$$= \frac{1}{26} [5 \cos x - \sin x]$$

$\therefore$ The complete solution is

$$= c_1 e^{-x} + e^{\frac{3}{2}x}\left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right) + \frac{1}{26}(5 \cos x - \sin x)$$

(*ii*) Please try yourself. $\left[\textbf{Ans.} \quad y = c_1 e^{x} + c_2 e^{-2x} - \frac{1}{10}(\cos x + 3 \sin x)\right]$

(*iii*) Given equation in symbolic form is

$$(D^3 + D^2 - D - 1)\, y = \cos 2x$$

Auxiliary equation is $D^3 + D^2 - D - 1 = 0$

or $$D^2(D + 1) - (D + 1) = 0$$

or $$(D + 1)(D^2 - 1) = 0 \quad \text{or} \quad (D + 1)^2(D - 1) = 0$$

$\therefore$ $$D = 1, -1, -1$$

$$\text{C.F.} = c_1 e^{x} (c_2 + c_3 x) e^{-x}$$

$$\text{P.I.} = \frac{1}{D^3 + D^2 - D - 1} \cos 2x = \frac{1}{D\,.\,D^2 + D^2 - D - 1} \cos 2x$$

$$= \frac{1}{D(-4) - 4 - D - 1} \cos 2x = -\frac{1}{5(D + 1)} \cos 2x$$

$$= -\frac{1}{5} \cdot \frac{D - 1}{(D + 1)(D - 1)} \cos 2x$$

$$= -\frac{1}{5} \cdot \frac{D - 1}{D^2 - 1} \cos 2x = -\frac{1}{5} \cdot \frac{D - 1}{-4 - 1} \cos 2x$$

$$= \frac{1}{25}\left[\frac{d}{dx}(\cos 2x) - \cos 2x\right] = \frac{1}{25} [-2 \sin 2x - \cos 2x]$$

$\therefore$ The complete solution is

$$y = c_1 e^{x} + (c_2 + c_3 x)\, e^{-x} - \frac{1}{25}(2 \sin 2x + \cos 2x).$$

(*iv*) Given equation in symbolic form is

$$(D^4 + 2n^2D^2 + n^4)y = \cos mx$$

Auxiliary equation is $D^4 + 2n^2D^2 + n^4 = 0$

or $$(D^2 + n^2)^2 = 0 \quad \therefore \quad D^2 = -n^2\,;\, -n^2$$

$$D = \pm\, in, \pm\, in \text{ i.e., } 0 \pm in, 0 \pm in$$

$\therefore$ $$\text{C.F.} = e^{0x} [(c_1 + c_2 x) \cos nx + (c_3 + c_4 x) \sin nx]$$

$$= (c_1 + c_2 x) \cos nx + (c_3 + c_4 x) \sin nx$$

$$\text{P.I.} = \frac{1}{D^4 + 2n^2D^2 + n^4} \cos mx = \frac{1}{(-m^2)^2 + 2n^2(-m^2) + n^4} \cos mx$$

$$= \frac{1}{m^4 - 2m^2n^2 + n^4} \cos mx = \frac{1}{(m^2 - n^2)^2} \cos mx$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2x)\cos nx + (c_3 + c_4x)\sin nx + \frac{1}{(m^2 - n^2)^2}\cos mx.$$

(*v*) Given equation in symbolic form is

$$(D^2 + 2kD + k^2)y = A\cos px$$

Auxiliary equation is $D^2 + 2kD + k^2 = 0$

or $\qquad (D + k)^2 = 0 \quad \therefore \quad D = -k, -k$

$$\text{C.F.} = (c_1 + c_2x)e^{-kx}$$

$$\text{P.I.} = \frac{1}{D^2 + 2kD + k^2}\,.\,A\cos px = A\,.\,\frac{1}{D^2 + 2kD + k^2}\cos px$$

$$= A\,.\,\frac{1}{-p^2 + 2kD + k^2}\cos px = A.\,\frac{1}{2kD + (k^2 - p^2)}\cos px$$

$$= A\,.\,\frac{2kD - (k^2 - p^2)}{[2kD + (k^2 - p^2)][2kD - (k^2 - p^2)]}\cos px$$

$$= A\,.\,\frac{2kD - (k^2 - p^2)}{4k^2D^2 - (k^2 - p^2)^2}\cos px = A\,.\,\frac{2kD - (k^2 - p^2)}{4k^2(-p^2) - (k^2 - p^2)^2}\cos px$$

$$= A\,.\,\frac{2kD - (k^2 - p^2)}{-[(k^2 - p^2) + 4k^2p^2]}\cos px$$

$$= -\frac{A}{(k^2 + p^2)^2}\left[2k\frac{d}{dx}(\cos px) - (k^2 - p^2)\cos px\right]$$

$$= -\frac{A}{(k^2 - p^2)^2}\,[-2kp\sin px - (k^2 - p^2)\cos px]$$

$$= \frac{A}{(k^2 + p^2)^2}\,[2kp\sin px + (k^2 - p^2)\cos px]$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2x)e^{-kx} + \frac{A}{(k^2 - p^2)^2}\,[2k\,p\sin px + (k^2 - p^2)\cos px].$$

**Example 6.** *Solve the following :*

(*i*) $(D^2 - 4D + 4)y = e^{-4x} + 5\cos 3x$ $\qquad$ (*ii*) $\dfrac{d^2y}{dx^2} - 4y = e^x + \sin 2x$

(*iii*) $(D^2 - 3D + 2)y = 6\,e^{-3x} + \sin 2x.$

**Sol.** (*i*) Given equation in symbolic form is

$$(D^2 - 4D + 4)y = e^{-4x} + 5\cos 3x$$

Auxiliary equation is $D^2 - 4D + 4 = 0$

or $\qquad (D - 2)^2 = 0 \quad \therefore \quad D = 2, 2$

$$\text{C.F.} = (c_1 + c_2x)\,e^{2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 4}\,(e^{-4x} + 5\cos 3x)$$

$$= \frac{1}{D^2 - 4D + 4} e^{-4x} + 5 \cdot \frac{1}{D^2 - 4D + 4} \cos 3x$$

$$= \frac{1}{(-4)^2 - 4(-4) + 4} e^{-4x} + 5 \cdot \frac{1}{-9 - 4D + 4} \cos 3x$$

$$= \frac{1}{36} e^{-4x} - 5 \cdot \frac{1}{4D + 5} \cos 3x$$

$$= \frac{1}{36} e^{-4x} - 5 \cdot \frac{4D - 5}{(4D + 5)(4D - 5)} \cos 3x$$

$$= \frac{1}{36} e^{-4x} - 5 \cdot \frac{4D - 5}{16D^2 - 25} \cos 3x$$

$$= \frac{1}{36} e^{-4x} - 5 \cdot \frac{4D - 5}{16(-9) - 25} \cos 3x$$

$$= \frac{1}{36} e^{-4x} + \frac{5}{169}\left[4 \frac{d}{dx} (\cos 3x) - 5 \cos 3x\right]$$

$$= \frac{1}{36} e^{-4x} + \frac{5}{169}[-12 \sin 3x - 5 \cos 3x]$$

$$= \frac{1}{36} e^{-4x} - \frac{5}{169}(12 \sin 3x + 5 \cos 3x)$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x) e^{2x} + \frac{1}{36} e^{-4x} - \frac{5}{169}(12 \sin 3x + 5 \cos 3x).$$

(*ii*) The given equation in symbolic form is

$$(D^2 - 4)y = e^x + \sin 2x$$

Auxiliary equation is $D^2 - 4 = 0, D = \pm 2$

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 4} (e^x + \sin 2x) = \frac{1}{D^2 + 4} e^x + \frac{1}{D^2 - 4} \sin 2x$$

$$= \frac{1}{1^3 - 4} e^x + \frac{1}{-4 - 4} \sin 2x = -\frac{1}{3} e^x - \frac{1}{8} \sin 2x$$

$\therefore$ The complete solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{3} e^x - \frac{1}{8} \sin 2x.$$

(*iii*) Given equation in symbolic form is

$$(D^2 - 3D + 2)y = 6 e^{-3x} + \sin 2x$$

Auxiliary equation is

$$D^2 - 3D + 2 = 0 \quad \text{or} \quad (D - 1)(D - 2) = 0 \quad \therefore \quad D = 1, 2$$

$\therefore$ $\text{C.F.} = c_1 e^x + c_2 e^{2x}.$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2}[6e^{-3x} + \sin 2x]$$

$$= 6 \cdot \frac{1}{D^2 - 3D + 2} e^{-3x} + \frac{1}{D^2 - 3D + 2} \sin 2x$$

$$= 6. \frac{1}{(-3)^2 - 3(-3) + 2} e^{-3x} + \frac{1}{-4 - 3D + 2} \sin 2x$$

$$= \frac{6}{20} e^{-3x} - \frac{1}{3D + 2} \sin 2x = \frac{3}{10} e^{-3x} - \frac{3D - 2}{(3D + 2)(3D - 2)} \sin 2x$$

$$= \frac{3}{10} e^{-3x} - \frac{3D - 2}{9D^2 - 4} \sin 2x = \frac{3}{10} e^{-3x} - \frac{3D - 2}{9(-4) - 4} \sin 2x$$

$$= \frac{3}{10} e^{-3x} + \frac{1}{40} \left[ 3 \frac{d}{dx} (\sin 2x) - 2 \sin 2x \right]$$

$$= \frac{3}{10} e^{-3x} + \frac{1}{40} [6 \cos 2x - 2 \sin 2x]$$

$$= \frac{3}{10} e^{-3x} + \frac{1}{20} [3 \cos 2x - \sin 2x]$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{2x} + \frac{3}{10} e^{-3x} + \frac{1}{20} [3 \cos 2x - \sin 2x].$$

**Example 7.** *Solve the following* :

*(i)* $(D^2 - D - 2)y = \sin 2x + e^x$ *(ii)* $(D^4 + 4)y = \sin 3x + e^x$

*(iii)* $\frac{d^2y}{dx^2} + 9y = e^x - \cos 2x$ *(iv)* $(D^2 - 3D + 2)y = \cos 3x \cos 2x$

*(v)* $(D^2 + 1)y = 3 \cos^2 x + 2 \sin^3 2x$ *(vi)* $\frac{d^2y}{dx^2} - 4y = \cos^2 x$.

**Sol.** (*i*) Please try yourself. $\left[ \textbf{Ans.} \;\; y = c_1 e^{2x} = c_2 e^{-x} + \frac{1}{20} (\cos 2x - 3 \sin 2x) - \frac{1}{2} e^x \right]$

(*ii*) The given equation in symbolic form is

$$(D^4 + 4)y = \sin 3x + e^x$$

A.E. is $D^4 + 4 = 0$ or $(D^4 + 4D^2 + 4) - 4D^2 = 0$

or $(D^2 + 2)^2 - (2D)^2 = 0$ or $(D^2 + 2D + 2)(D^2 - 2D + 2) = 0$

$$\therefore \quad D = \frac{-2 \pm 2i}{2} \cdot \frac{2 \pm 2i}{2} = -1 \pm i, 1 \pm i$$

$$\text{C.F.} = e^{-x} (c_1 \cos x + c_2 \sin x) + e^x (c_3 \cos x + c_4 \sin x)$$

$$\text{P.I.} = \frac{1}{D^4 + 4} (\sin 3x + e^x) = \frac{1}{(D^2)^2 + 4} \sin 3x + \frac{1}{D^4 + 4} e^x$$

$$= \frac{1}{(-3^2)^2 + 4} \sin 3x + \frac{1}{1^4 + 4} e^x = \frac{1}{85} \sin 3x + \frac{1}{5} e^x$$

$\therefore$ The complete solution is

$$y = e^{-x} (c_1 \cos x + c_2 \sin x) + e^x (c_3 \cos x + c_4 \sin x) + \frac{1}{85} \sin 3x + \frac{1}{5} e^x.$$

(*iii*) Please try yourself. $\left[\textbf{Ans.}\quad y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{10} e^x - \frac{1}{5} \cos 2x\right]$

(*iv*) The given equation in symbolic form is

$$(D^2 - 3D + 2)y = \cos 3x \cos 2x$$

A.E. is $\quad D^2 - 3D + 2 = 0$

or $\quad (D-1)(D-2) = 0 \quad \therefore \quad D = 1, 2$

$$\text{C.F.} = c_1 e^x + c_2 e^{2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} \cos 3x \cos 2x \qquad \left[\because \cos 3x \cos 2x = \frac{1}{2}(2\cos 3x \cos 2x) = \frac{1}{2}(\cos 5x + \cos x)\right]$$

$$= \frac{1}{2} \cdot \frac{1}{D^2 - 3D + 2} (\cos 5x + \cos x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 3D + 2} \cos 5x + \frac{1}{D^2 - 3D + 2} \cos x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-5^2 - 3D + 2} \cos 5x + \frac{1}{-1^2 - 3D + 2} \cos x\right]$$

$$= \frac{1}{2}\left[-\frac{1}{3D + 23} \cos 5x + \frac{1}{1 - 3D} \cos x\right]$$

$$= \frac{1}{2}\left[-\frac{3D - 23}{9D^2 - 529} \cos 5x + \frac{1 + 3D}{1 - 9D^2} \cos x\right]$$

$$= \frac{1}{2}\left[-\frac{3D - 23}{9(-5^2) - 529} \cos 5x + \frac{1 + 3D}{1 - 9(-1^2)} \cos x\right]$$

$$= \frac{1}{2}\left[\frac{1}{754}(-15 \sin 5x - 23 \cos 5x) + \frac{1}{10}(\cos x - 3 \sin x)\right]$$

$$= -\frac{1}{1508}(15 \sin 5x + 23 \cos 5x) + \frac{1}{20}(\cos x - 3 \sin x)$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{2x} - \frac{1}{1508}(15 \sin 5x + 23 \cos 5x) + \frac{1}{20}(\cos x - 3 \sin x).$$

(*v*) The given equation in symbolic form is

$$(D^2 + 1)y = 3 \cos^2 x + 2 \sin^3 2x$$

A.E. is $\quad D^2 + 1 = 0 \quad$ or $\quad D = \pm i$

$\therefore \quad \text{C.F.} = e^{0x}(c_1 \cos x + c_2 \sin x) = c_1 \cos x + c_2 \sin x$

$$\text{P.I.} = \frac{1}{D^2 + 1}(3 \cos^2 x + 2 \sin^3 2x)$$

Since $\quad \cos^2 A = \dfrac{1 + \cos 2A}{2} \quad$ and $\quad \sin 3A = 3 \sin A - 4 \sin^3 A$

$$\Rightarrow \qquad \sin^3 A = \frac{3}{4}\sin A - \frac{1}{4}\sin^3 A$$

$$\therefore \quad 3\cos^2 x + 2\sin^3 2x = 3\left(\frac{1-\cos 2x}{2}\right) + 2\left(\frac{3}{4}\sin 2x - \frac{1}{4}\sin 6x\right)$$

$$= \frac{3}{2} + \frac{3}{2}\cos 2x + \frac{3}{2}\sin 2x - \frac{1}{2}\sin 6x$$

$$\text{P.I.} = \frac{3}{2}\frac{1}{D^2+1}e^{0x} + \frac{3}{2}\frac{1}{D^2+1}\cos 2x + \frac{3}{2}\frac{1}{D^2+1}\sin 2x - \frac{1}{2}\cdot\frac{1}{D^2+1}\sin 6x$$

$$= \frac{3}{2}\cdot\frac{1}{0+1}e^{0x} + \frac{3}{2}\cdot\frac{1}{-2^2+1}\cos 2x + \frac{3}{2}\cdot\frac{1}{-2^2+1}\sin 2x - \frac{1}{2}\cdot\frac{1}{-6^2+1}\sin 6x$$

$$= \frac{3}{2} - \frac{1}{2}(\cos 2x + \sin 2x) + \frac{1}{70}\sin 6x$$

$\therefore$ The complete solution is

$$y = c_1\cos x + c_2\sin x + \frac{3}{2} - \frac{1}{2}(\cos 2x + \sin 2x) + \frac{1}{70}\sin 6x.$$

(*vi*) Please try yourself. $\left[\textbf{Ans.}\ \ y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{16}(2+\cos x)\right]$

**Example 8.** *Solve the following* :

(*i*) $(D^4 + D^3)y = \cos 4x$ (*ii*) $(D^2 + 2D + 10)y + 37\sin 3x = 0$

(*iii*) $(D^2 - 4D + 3)y = \sin 3x\cos 2x$.

**Sol.** (*i*) Given equation is $(D^4 + D^3)y = \cos 4x$

Auxiliary equation is $D^4 + D^3 = 0$ or $D^3(D+1) = 0$

$\therefore \qquad D = 0, 0, 0, -1$ | **Note**

$$\therefore \qquad \text{C.F.} = (c_1 + c_2x + c_3x^2)\,e^{0x} + c_4e^{-x} = c_1 + c_2x + c_3x^2 + c_4e^{-x}$$

$$\text{P.I.} = \frac{1}{D^4+D^3}\cos 4x = \frac{1}{(D^2)^2 + D^2D}\cos 4x$$

$$= \frac{1}{(-16)^2 + (-16)D}\cos 4x = \frac{1}{256-16D}\cos 4x$$

$$= \frac{1}{16}\cdot\frac{1}{16-D}\cos 4x = \frac{1}{16}\cdot\frac{16+D}{256-D^2}\cos 4x$$

$$= \frac{1}{16}\cdot\frac{16+D}{256-(-16)}\cos 4x = \frac{1}{16\times 272}\left[16\cos 4x + \frac{d}{dx}(\cos 4x)\right]$$

$$= \frac{1}{4352}[16\cos 4x - 4\sin 4x] = \frac{1}{1088}[4\cos 4x - \sin 4x]$$

$\therefore$ The complete solution is

$$y = c_1 + c_2x + c_3x^2 + c_4e^{-x} + \frac{1}{1088}[4\cos 4x - \sin 4x].$$

(*ii*) Given equation is $(D^2 + 2D + 10)y = -37\sin 3x$

Auxiliary equation is $D^2 + 2D + 10 = 0$

$$D = \frac{-2 \pm \sqrt{4-40}}{2} = \frac{-2 \pm 6i}{2} = -1 \pm 3i$$

$\therefore$ $\quad$ C.F. $= e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$

$$\text{P.I.} = \frac{1}{D^2 + 2D + 10}[-37 \sin 3x]$$

$$= -37 \cdot \frac{1}{D^2 + 2D + 10} \sin 3x = -37 \cdot \frac{1}{-9 + 2D + 10} \sin 3x$$

$$= -37 \cdot \frac{1}{1 + 2D} \sin 3x = -37 \frac{1 - 2D}{1 - 4D^2} \sin 3x$$

$$= -37 \cdot \frac{1 - 2D}{1 - 4(-9)} \sin 3x = -(1 - 2D) \sin 3x$$

$$= -\left[\sin 3x - 2 \cdot \frac{d}{dx}(\sin 3x)\right] = -[\sin 3x - 6 \cos 3x]$$

$\therefore$ The complete solution is

$$y = e^{-x}(c_1 \cos 3x + c_2 \sin 3x) + (6 \cos 3x - \sin 3x).$$

(*iii*) Given equation is

$$(D^2 - 4D + 3)y = \sin 3x \cos 2x$$

Auxiliary equation is $\quad D^2 - 4D + 3 = 0$

or $\quad (D - 1)(D - 3) = 0 \quad \therefore \quad D = 1, 3$

$\therefore$ $\quad$ C.F. $= c_1 e^x + c_2 e^{3x}$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 3}[\sin 3x \cos 2x] = \frac{1}{D^2 - 4D + 3}\left[\frac{1}{2} \cdot 2 \sin 3x \cos 2x\right]$$

$$= \frac{1}{D^2 - 4D + 3}\left[\frac{1}{2}(\sin 5x + \sin x)\right]$$

$$= \frac{1}{2} \cdot \left[\frac{1}{D^2 - 4D + 3} \sin 5x + \frac{1}{D^2 - 4D + 3} \sin x\right]$$

$$= \frac{1}{2} \cdot \left[\frac{1}{-25 - 4D + 3} \sin 5x + \frac{1}{-1 - 4D + 3} \sin x\right]$$

$$= \frac{1}{2} \cdot \left[-\frac{1}{2} \cdot \frac{1}{11 + 2D} \sin 5x + \frac{1}{2} \cdot \frac{1}{1 - 2D} \sin x\right]$$

$$= -\frac{1}{4} \frac{11 - 2D}{121 - 4D^2} \sin 5x + \frac{1}{4} \cdot \frac{1 + 2D}{1 - 4D^2} \sin x$$

$$= -\frac{1}{4} \cdot \frac{11 - 2D}{121 - 4(-25)} \sin 5x + \frac{1}{4} + \frac{1 + 2D}{1 - 4(-1)} \sin x$$

$$= -\frac{1}{884}\left[11 \sin 5x - 2 \frac{d}{dx}(\sin 5x)\right] + \frac{1}{20}\left[\sin x + 2 \frac{d}{dx}(\sin x)\right]$$

$$= -\frac{1}{884}[11 \sin 5x - 10 \cos 5x] + \frac{1}{20}[\sin x + 2 \cos x]$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{3x} + \frac{1}{884}[10 \cos 5x - 11 \sin 5x] + \frac{1}{20}[\sin x + 2 \cos x].$$

**Example 9.** *Solve the following :*

*(i)* $(D^4 + 1)y = \cos^2 x$ *(ii)* $(D^4 + 1)y = \sin^2 x.$

**Sol.** *(i)* Given equation is $(D^4 + 1)y = \cos^2 x$

Auxiliary equation is $D^4 + 1 = 0$

$$\Rightarrow \quad (D^4 + 2D^2 + 1) - 2D^2 = 0 \quad \Rightarrow \quad (D^2 + 1)^2 - (\sqrt{2}\,D)^2 = 0$$

$$\Rightarrow \quad (D^2 + \sqrt{2}\,D + 1)(D^2 - \sqrt{2}\,D + 1) = 0$$

$$\Rightarrow \quad D = \frac{-\sqrt{2} \pm \sqrt{2-4}}{2}, \frac{\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{-\sqrt{2} \pm i\sqrt{2}}{2}, \frac{\sqrt{2} \pm i\sqrt{2}}{2}$$

$$= -\frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$$

$$\therefore \quad \text{C.F.} = e^{-\frac{1}{\sqrt{2}}x}\left(c_1 \cos\frac{x}{\sqrt{2}} + c_2 \sin\frac{x}{\sqrt{2}}\right) + e^{\frac{1}{\sqrt{2}}x}\left(c_3 \cos\frac{x}{\sqrt{2}} + c_4 \sin\frac{x}{\sqrt{2}}\right)$$

$$\text{P.I.} = \frac{1}{D^4+1}\cos^2 x = \frac{1}{D^4+1}\left(\frac{1+\cos 2x}{2}\right) = \frac{1}{2}\cdot\frac{1}{D^4+1}e^{0x} + \frac{1}{2}\frac{1}{D^4+1}\cos 2x$$

$$= \frac{1}{2}\cdot\frac{1}{0+1}\cdot\frac{1}{2}\cdot\frac{1}{(-4)^2+1}\cos 2x \qquad \left|\ \begin{array}{l}\text{Replacing } D^2 \text{ by} \\ -(2^2) = -4\end{array}\right.$$

$$= \frac{1}{2} + \frac{1}{34}\cos 2x$$

$\therefore$ The complete solution is

$y$ = C.F. + P.I.

$$= e^{-\frac{1}{\sqrt{2}}x}\left(c_1 \cos\frac{x}{\sqrt{2}} + c_2 \sin\frac{x}{\sqrt{2}}\right) + e^{\frac{x}{\sqrt{2}}}\left(c_3 \cos\frac{x}{\sqrt{2}} + c_4 \sin\frac{x}{\sqrt{2}}\right) + \frac{1}{2} + \frac{1}{34}\cos 2x.$$

*(ii)* Please try yourself. $\left(\sin^2 x = \dfrac{1-\cos 2x}{2}\right)$

$$\left[\textbf{Ans. } y = e^{-\frac{x}{\sqrt{2}}}\left(c_1 \cos\frac{x}{\sqrt{2}} + c_2 \sin\frac{x}{\sqrt{2}}\right)\right.$$

$$\left. + e^{-\frac{x}{\sqrt{2}}}\left(c_3 \cos\frac{x}{\sqrt{2}} + c_4 \sin\frac{x}{\sqrt{2}}\right) - \frac{1}{2} - \frac{1}{34}\cos 2x.\right]$$

**Example 10.** *Show that the solution of the differential equation* $\dfrac{d^2y}{dt^2} + 4y = A \sin pt$ *which is such that* $y = 0$ *and* $\dfrac{dy}{dt} = 0$ *when* $t = 0$ *is .*

$$y = \frac{A\,(\sin pt - \frac{1}{2}\,p \sin 2t)}{4 - p^2} \quad [\text{if } p \neq 2].$$

**Sol.** Given equation in symbolic form is

$$(D^2 + 4)y = A \sin pt$$

Auxiliary equation is $\quad D^2 + 4 = 0 \quad$ or $\quad D^2 = -4 \quad \therefore \quad D = \pm 2i$

$$\text{C.F.} = c_1 \cos 2t + c_2 \sin 2t$$

$$\text{P.I.} = \frac{1}{D^2 + 4} A \sin pt = A \cdot \frac{1}{-p^2 + 4} \sin pt \qquad [\text{if } p \neq 2]$$

$$= \frac{A \sin pt}{4 - p^2}$$

$\therefore$ The complete solution is

$$y = c_1 \cos 2t + c_2 \sin 2t + \frac{A \sin pt}{4 - p^2} \qquad \ldots(i)$$

when $t = 0,\quad y = 0$ (given)

$\therefore \quad 0 = c_1$

Differentiating $(i)$ w.r.t. '$t$',

$$\frac{dy}{dt} = -2c_1 \sin 2t + 2c_2 \cos 2t + \frac{Ap \cos pt}{4 - p^2}$$

when $t = 0,\quad \dfrac{dy}{dt} = 0$ (given)

$$\therefore \quad 0 = 2c_2 + \frac{Ap}{4 - p^2} \quad \text{or} \quad c_2 = -\frac{1}{2} \cdot \frac{Ap}{4 - p^2}$$

Substituting the values of $c_1$ and $c_3$ in $(i)$, the required solution is

$$y = \frac{A \sin pt}{4 - p^2} - \frac{1}{2} \cdot \frac{Ap \sin 2t}{4 - p^2} = \frac{A\,(\sin pt - \frac{1}{2} p \sin 2t)}{4 - p^2}.$$

**Example 11.** *Solve* $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 10y + 37 \sin 3x = 0$ *and find the value of* $y$ *when* $x = \dfrac{\pi}{2}$, *if it is given that* $y = 3$ *and* $\dfrac{dy}{dx} = 0$ *when* $x = 0$.

**Sol.** The complete solution is

$$y = e^{-x}(c_1 \cos 3x + c_2 \sin 3x) + 6 \cos 3x - \sin 3x \qquad \ldots(i)$$

when $\quad x = 0,\quad y = 3$ [given]

$\Rightarrow \quad 3 = c_1 + 6 \quad \Rightarrow \quad c_1 = -3$

Differentiating $(i)$ w.r.t. $x$,

$$\frac{dy}{dx} = e^{-x}(-3c_1 \sin 3x + 3c_2 \cos 3x)$$
$$+ (-e^{-x})(c_1 \cos 3x + c_2 \sin 3x) - 18 \sin 3x - 3 \cos 3x$$

when $x = 0,\quad \dfrac{dy}{dx} = 0$ (given)

$\Rightarrow \quad 0 = 3c_2 - c_1 - 3 = 3c_2 + 3 - 3 \quad \Rightarrow \quad c_2 = 0$

$\therefore$ From $(i)$, $\quad y = -3 \cos 3x e^{-x} + 6 \cos 3x - \sin 3x$

When $x = \dfrac{\pi}{2}$,

$$y = -\sin\frac{3\pi}{2} = -(-1) \qquad \left|\because\ \cos\frac{3\pi}{2} = 0\right.$$

*i.e.*, $y = 1.$

**Case III. To evaluate $\dfrac{1}{f(D)}\,x^m$, where m is a positive integer.**

1. Take out the lowest degree term from $f(D)$ **to make the first term unity** [so that Binomial Theorem for a negative index is applicable].

The remaining factor will be of the form

$$[1 + \phi(D)] \qquad \text{or} \qquad [1 - \phi(D)].$$

2. Take this factor in the numerator. It takes the form

$$[1 + \phi(D)]^{-1} \qquad \text{or} \qquad [1 - \phi(D)]^{-1}$$

3. Expand it by Binomial Theorem, remembering that

$$(1 + x)^{-1} = 1 - x + x^2 - x^3 + \ldots\ldots$$
$$(1 - x)^{-1} = 1 + x + x^2 + x^3 + \ldots\ldots$$

The expansion is to be carried upto the term $D^m$, since $D^{m+1}(x^m) = 0$, $D^{m+2}(x^m) = 0$, and all the higher differential co-efficients of $x^m$ vanish.

Thus $$\frac{1}{D-2}\,x^2 = \frac{1}{-2\left(1-\dfrac{D}{2}\right)}\,x^2 = -\frac{1}{2}\left(1-\frac{D}{2}\right)^{-1} x^2$$

$$= -\frac{1}{2}\left[1+\frac{D}{2}+\frac{D^2}{4}+\frac{D^3}{8}+\ldots\ldots\right]x^3 \qquad \left|\begin{array}{l}\text{Now the expansion is to be}\\ \text{carried upto the term } D^3\end{array}\right.$$

$$= -\frac{1}{2}\left[x^3+\frac{1}{2}D(x^3)+\frac{1}{4}D^2(x^3)+\frac{1}{8}D^3(x^3)\right]$$

$$= -\frac{1}{2}\left[x^3+\frac{1}{2}.3x^2+\frac{1}{4}.6x+\frac{1}{8}.6\right] = -\frac{1}{2}\left[x^3+\frac{3}{2}x^2+\frac{3}{2}x+\frac{3}{4}\right].$$

**Example 1.** *Solve the following :*

(*i*) $(D^2 + D - 6)y = x$ (*ii*) $(D^3 - 3D - 2)y = x^2$

(*iii*) $\dfrac{d^2y}{dx^2} - 4y = x^2$ (*iv*) $\dfrac{d^3y}{dx^3} - 13\dfrac{dy}{dx} + 12y = x.$

**Sol.** (*i*) Given equation in symbolic form is

$$(D^2 + D - 6)y = x$$

Auxiliary equation is $D^2 + D - 6 = 0$

or $(D + 3)(D - 2) = 0 \quad \therefore \quad D = -3, 2$

$$\text{C.F.} = c_1e^{-3x} + c_2e^{2x}$$

$$\text{P.I.} = \frac{1}{D^2+D-6}\,x = \frac{1}{-6\left(1-\dfrac{D}{6}-\dfrac{D^2}{6}\right)}\,x = -\frac{1}{6}\left[1-\left(\frac{D}{6}+\frac{D^2}{6}\right)\right]^{-1} x$$

$$= -\frac{1}{6}\left[1+\left(\frac{D}{6}+\frac{D^2}{6}\right)+\ldots\ldots\right]x = -\frac{1}{6}\left[x+\frac{1}{6}D(x)\right] = -\frac{1}{6}\left[x+\frac{1}{6}\right]$$

$\therefore$ The complete solution is

$$y = c_1 e^{-3x} + c_1 e^{2x} - \frac{1}{36}(6x+1).$$

(*ii*) Given equation in symbolic form is

$$(D^3 - 3D - 2)y = x^2$$

Auxiliary equation is $\quad D^3 - 3D - 2 = 0$

Putting $D = -1$ $\quad$ L.H.S. of A.E. $= 0$

$\therefore$ $D + 1$ is a factor of L.H.S. of A.E.

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -3 & -2 \\ & & -1 & 1 & 2 \\ \hline & 1 & -1 & -2 & \boxed{0} \end{array}$$

$\therefore$ A.E. may be written as $(D + 1)(D^2 - D - 2) = 0$

or $\quad (D + 1)(D - 2)(D + 1) = 0 \quad \therefore \quad D = -1, -1, 2$

$$\text{C.F.} = (c_1 + c_2 x)e^{-x} + c_3 e^{2x}$$

$$\text{P.I.} = \frac{1}{D^3 - 3D - 2}x^2 = \frac{1}{-2\left(1 + \frac{3D}{2} - \frac{D^3}{2}\right)}x^2 = -\frac{1}{2}\left[1 + \left(\frac{3D}{2} - \frac{D^3}{2}\right)\right]^{-1} x^2$$

$$= -\frac{1}{2}\left[1 - \left(\frac{3D}{2} - \frac{D^3}{2}\right) + \left(\frac{3D}{2} - \frac{D^3}{2}\right)^2 \ldots\ldots\right] x^2$$

[$\because$ expansion is to be carried upto the term containing $D^2$ only]

$$= -\frac{1}{2}\left[1 - \frac{3D}{2} + \frac{9D^2}{4} \ldots\ldots\right] x^2 = -\frac{1}{2}\left[x^2 - \frac{3}{2}D(x^2) + \frac{9}{4}D^2(x^2)\right]$$

$$= -\frac{1}{2}\left[x^2 - \frac{3}{2}.2x + \frac{9}{4}.2\right] = -\frac{1}{2}\left[x^2 - 3x + \frac{9}{2}\right]$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x)e^{-x} + c_3 e^{2x} - \frac{1}{2}\left(x^2 - 3x + \frac{9}{2}\right).$$

(*iii*) Given equation in symbolic form is $(D^2 - 4)y = x^2$

Auxiliary equation is $\quad D^2 - 4 = 0 \quad$ or $\quad D = \pm 2$

$\therefore \quad$ C.F. $= c_1 e^{2x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2 - 4}x^2 = \frac{1}{-4\left(1 - \frac{D^2}{4}\right)}x^2$$

$$= -\frac{1}{4}\left(1 - \frac{D^2}{4}\right)^{-1} x^2 = -\frac{1}{4}\left[1 + \frac{D^2}{4} + \ldots\ldots\right] x^2$$

$$= -\frac{1}{4}\left[x^2 + \frac{1}{4}D^2(x^2)\right] = -\frac{1}{4}\left(x^2 + \frac{1}{4}.2\right) = -\frac{1}{4}\left[x^2 + \frac{1}{2}\right]$$

$\therefore$ The complete solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{4}\left(x^2 + \frac{1}{2}\right).$$

(*iv*) Given equation in symbolic form is $(D^3 - 13D + 12)y = x$

Auxiliary equation is $D^3 - 13D + 12 = 0$ or $D^3 - D - 12D + 12 = 0$

or $D(D^2 - 1) - 12(D - 1) = 0$ or $(D - 1)[D(D + 1) - 12] = 0$

or $(D - 1)(D^2 + D - 12) = 0$

or $(D - 1)(D + 4)(D - 3) = 0$

$\therefore$ $D = 1, -4, 3$

$$\text{C.F.} = c_1 e^{x} + c_2 e^{-4x} + c_3 e^{3x}$$

$$\text{P.I.} = \frac{1}{D^3 - 13D + 12}\, x = \frac{1}{12\left[1 - \frac{13}{12}D + \frac{D^3}{12}\right]}\, x = \frac{1}{12}\left[1 - \left(\frac{13}{12}D - \frac{D^3}{12}\right)\right]^{-1} x$$

$$= \frac{1}{12}\left[1 + \left(\frac{13}{12}D - \frac{D^3}{12}\right) + \ldots\ldots\right] x = \frac{1}{12}\left[x + \frac{13}{12}D(x)\right] = \frac{1}{12}\left[x + \frac{13}{12}\right]$$

$\therefore$ The complete solution is

$$y = c_1 e^{x} + c_2 e^{-4x} + c_3 e^{3x} + \frac{1}{144}(12x + 13).$$

**Example 2.** *Solve* :

(*i*) $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} = x^2$ (*ii*) $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 8y = x$

(*iii*) $(D^3 - D^2 - 6D)y = x^2$ (*iv*) $(D^4 - 2D^3 + D^2)y = x^3$.

**Sol.** (*i*) Given equation in symbolic form is

$$(D^3 + 3D^2 + 2D)y = x^2$$

Auxiliary equation is $D(D^2 + 3D + 2) = 0$

or $D(D + 1)(D + 2) = 0$ $\therefore$ $D = 0, -1, -2$

$$\text{C.F.} = c_1 e^{0x} + c_2 e^{-x} + c_3 e^{-2x} = c_1 + c_2 e^{-x} + c_3 e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^3 + 3D^2 + 2D}\, x^2 = \frac{1}{2D\left[1 + \frac{3D}{2} + \frac{D^2}{2}\right]}\, x^2$$

$$= \frac{1}{2D}\left[1 + \left(\frac{3D}{2} + \frac{D^2}{2}\right)\right]^{-1} x^2 = \frac{1}{2D}\left[1 - \left(\frac{3D}{2} + \frac{D^2}{2}\right) + \left(\frac{3D}{2} + \frac{D^2}{2}\right)^2 \ldots\ldots\right] x^2$$

$$= \frac{1}{2D}\left[1 - \frac{3D}{2} - \frac{D^2}{2} + \frac{9D^2}{4} \ldots\ldots\right] x^2 = \frac{1}{2D}\left[1 - \frac{3D}{2} + \frac{7D^2}{4} \ldots\ldots\right] x^2$$

$$= \frac{1}{2D}\left[x^2 - \frac{3}{2}D(x^2) + \frac{7}{4}D^2(x^2)\right] = \frac{1}{2D}\left[x^2 - \frac{3}{2}.2x + \frac{7}{4}.2\right] = \frac{1}{2D}\left[x^2 - 3x + \frac{7}{2}\right]$$

$$= \frac{1}{2}\int \left(x^2 - 3x + \frac{7}{2}\right) dx \qquad \because \frac{1}{D} \equiv \text{integration}$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{3x^2}{2} + \frac{7}{2}x\right]$$

$\therefore$ The complete solution is

$$y = c_1 + c_2e^{-x} + c_3e^{-2x} + \frac{1}{2}\left[\frac{x^3}{3} - \frac{3x^2}{2} + \frac{7}{2}x\right].$$

(*ii*) Given equation in symbolic form is

$$(D^3 - 3D^2 - 6D + 8)y = x$$

Auxiliary equation is $\quad D^3 - 3D^2 - 6D + 8 = 0$

Putting $D = 1$, $\quad$ L.H.S. of A.E. $= 0$

$\therefore$ $D - 1$ is a factor of L.H.S. of A.E.

By synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -3 & -6 & 8 \\ & & 1 & -2 & -8 \\ \hline & 1 & -2 & -8 & \boxed{0} \end{array}$$

$\therefore$ A.E. may be written as

$$(D - 1)(D^2 - 2D - 8) = 0 \quad \text{or} \quad (D - 1)(D - 4)(D + 2) = 0$$

$\therefore$ $\quad D = 1, -2, 4$

$$\text{C.F.} = c_1e^{x} + c_2e^{-2x} + c_3e^{4x}$$

$$\text{P.I.} = \frac{1}{D^3 - 3D^2 - 6D + 8}x = -\frac{1}{8\left(1 - \frac{3D}{4} - \frac{3D^2}{8} + \frac{D^3}{8}\right)}x$$

$$= \frac{1}{8}\left[1 - \left(\frac{3D}{4} + \frac{3D^2}{8} - \frac{D^3}{8}\right)\right]^{-1} x = \frac{1}{8}\left[1 + \left(\frac{3D}{4} + \frac{3D^2}{2} - \frac{D^3}{8}\right) + \ldots\ldots\right] x$$

$$= \frac{1}{8}\left[x + \frac{3}{4}D(x)\right] = \frac{1}{8}\left[x + \frac{3}{4}\right]$$

$\therefore$ The complete solution is

$$y = c_1e^{x} + c_2e^{-2x} + c_3e^{4x} + \frac{1}{8}\left[x + \frac{3}{4}\right].$$

(*iii*) Given equation in symbolic form is

$$(D^3 - D^2 - 6D)y = x^2$$

Auxiliary equation is $\quad D^3 - D^2 - 6D = 0$

or $\quad D(D^2 - D - 6) = 0 \quad$ or $\quad D(D - 3)(D + 2) = 0$

$\therefore$ $\quad D = 0, 3, -2$

$$\text{C.F.} = c_1 + c_2e^{3x} + c_3e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^3 - D^2 - 6D} x^2 = \frac{1}{-6D\left[1 + \frac{D}{6} - \frac{D^2}{6}\right]} x^2$$

$$= -\frac{1}{6D}\left[1 + \left(\frac{D}{6} - \frac{D^2}{6}\right)\right]^{-1} x^2 = -\frac{1}{6D}\left[1 - \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2 \ldots\ldots\right] x^2$$

$$= -\frac{1}{6D}\left[1 - \frac{D}{6} + \frac{D^2}{6} + \frac{D^2}{36} \ldots\ldots\right] x^2 = -\frac{1}{6D}\left[1 - \frac{D}{6} + \frac{7D^2}{36} \ldots\ldots\right] x^2$$

$$= -\frac{1}{6D}\left[x^2 - \frac{1}{6} D(x^2) + \frac{7}{36} D^2(x^2)\right]$$

$$= -\frac{1}{6D}\left[x^2 - \frac{1}{6} . 2x + \frac{7}{36} . 2\right] = -\frac{1}{6D}\left[x^2 - \frac{x}{3} + \frac{7}{18}\right]$$

$$= -\frac{1}{6}\left[\int \left(x^2 - \frac{x}{3} + \frac{7}{18}\right) dx\right] \qquad \left|\because \frac{1}{D} \equiv \text{integration}\right.$$

$$= -\frac{1}{6}\left[\frac{x^3}{3} - \frac{1}{3} . \frac{x^2}{2} + \frac{7}{18} x\right] = -\frac{1}{18}\left[x^3 - \frac{1}{2} x^2 + \frac{7}{6} x\right]$$

$\therefore$ The complete solution is

$$y = c_1 + c_1 e^{3x} + c_3 e^{-2x} - \frac{1}{18}\left(x^3 - \frac{1}{2} x^2 + \frac{7}{6} x\right).$$

(*iv*) Given equation in symbolic form is

$$(D^4 - 2D^3 + D^2)y = x^3$$

Auxiliary equation is $\quad D^4 - 2D^3 + D^2 = 0$

or $\quad D^2(D^2 - 2D + 1) = 0 \quad$ or $\quad D^2(D - 1)^2 = 0$

$\therefore \quad D = 0, 0, 1, 1$ | **Note**

$\therefore \quad \text{C.F.} = (c_1 + c_2 x)e^{0x} + (c_3 + c_4 x) e^x = c_1 + c_2 x + (c_3 + c_4 x)e^x$

$$\text{P.I.} = \frac{1}{D^4 - 2D^3 + D^2} x^3 = \frac{1}{D^2(1 - 2D + D^2)} x^3$$

$$= \frac{1}{D^2} (1 - D)^{-2} x^3 = \frac{1}{D^2} (1 + 2D + 3D^2 + 4D^3 + \ldots) x^3$$

$$= \frac{1}{D^2} [x^2 + 2D(x^3) + 3D^2(x^3) + 4D^3(x^3)] = \frac{1}{D^2} [x^3 + 2.3x^2 + 3.6x + 4.6]$$

$$= \frac{1}{D} \int (x^3 + 6x^2 + 18x + 24)\, dx \qquad \left|\because \frac{1}{D} \equiv \text{integration}\right.$$

$$= \frac{1}{D}\left[\frac{x^4}{4} + 2x^3 + 9x^2 + 24x\right]$$

$$= \int \left(\frac{x^4}{4} + 2x^3 + 9x^2 + 24x\right) dx = \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2$$

$\therefore$ The complete solution is

$$y = c_1 + c_2x + (c_3 + c_4x)e^x + \frac{x^5}{20} + \frac{x^4}{2} + 3x^3 + 12x^2.$$

**Example 3.** *Solve the following :*

(*i*) $2\frac{d^2y}{dx^2} + 5\frac{dy}{dx} + 2y = 5 + 2x$

(*ii*) $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + x^2 + x$ (Lucknow 1998)

(*iii*) $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = x + \sin x$ (Meerut 1998 ; Calcutta 1996)

(*iv*) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = x^2 + e^x + \cos 2x.$

**Sol.** (*i*) Given equation in symbolic form is

$$(2D^2 + 5D + 2)y = 5 + 2x$$

Auxiliary equation is $2D^2 + 5D + 2 = 0$

or $(2D + 1)(D + 2) = 0 \quad \therefore \quad D = -\frac{1}{2}, -2$

$$\therefore \quad \text{C.F.} = c_1e^{-\frac{1}{2}x} + c_2e^{-2x}$$

$$\text{P.I.} = \frac{1}{2D^2 + 5D + 2}(5 + 2x) = \frac{1}{2\left(1 + \frac{5D}{2} + D^2\right)}(5 + 2x)$$

$$= \frac{1}{2}\left[1 + \left(\frac{5D}{2} + D^2\right)\right]^{-1}(5 + 2x) = \frac{1}{2}\left[1 - \left(\frac{5D}{2} + D^2\right) + \ldots\ldots\right](5 + 2x)$$

$$= \frac{1}{2}\left[(5 + 2x) - \frac{5}{2}D(5 + 2x)\right] = \frac{1}{2}\left[5 + 2x - \frac{5}{2}.2\right] = x$$

$\therefore$ The complete solution is

$$y = c_1e^{-\frac{1}{2}x} + c_2e^{-2x+x}.$$

(*ii*) Given equation in symbolic form is

$$(D^3 + 2D^2 + D)y = e^2 + x^2 + x$$

Auxiliary equation is $D^3 + 2D^2 + D = 0$

or $D(D^2 + 2D + 1) = 0$ or $D(D + 1)^2 = 0$

$\therefore$ $D = 0, -1, -1$

$$\text{C.F.} = c_1 + (c_2 + c_3x)e^{-x}$$

$$\text{P.I.} = \frac{1}{D^3 + 2D^2 + D}(e^{2x} + x^2 + x)$$

$$= \frac{1}{D^3 + 2D^2 + D}e^{2x} + \frac{1}{D^3 + 2D^2 + D}(x^2 + x)$$

$$= \frac{1}{2^3 + 2.2^2 + 2} e^{2x} + \frac{1}{D(1 + 2D + D^2)} (x^2 + x)$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D(1+D)^2} (x^2 + x) = \frac{1}{18} e^{2x} + \frac{1}{D} (1 + D)^{-2}(x^2 + x)$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D} (1 - 2D + 3D^2 \ldots\ldots)(x^2 + x)$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D} [x^2 + x - 2D(x^2 + x) + 3D^2(x^2 + x)]$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D} [x^2 + x - 2(2x + 1) + 3(2)]$$

$$= \frac{1}{18} e^{2x} + \frac{1}{D} (x^2 - 3x + 4) = \frac{1}{18} e^{2x} + \int (x^2 - 3x + 4)\, dx$$

$$= \frac{1}{18} e^{2x} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x$$

$\therefore$ The complete solution is

$$y = c_1 + (c_2 + c_3 x)e^{-x} + \frac{1}{18} e^{2x} + \frac{x^3}{3} - \frac{3x^2}{2} + 4x.$$

(*iii*) Given equation in symbolic form is

$$(D^2 + D - 2)y = x + \sin x$$

Auxiliary equation is $\quad D^2 + D - 2 = 1$

or $\quad (D + 2)(D - 1) = 0 \quad \therefore \quad D = -2, 1$

$$\text{C.F.} = c_1 e^{-2x} + c_2 e^x$$

$$\text{P.I.} = \frac{1}{D^2 + D - 2} (x + \sin x) = \frac{1}{D^2 + D - 2} x + \frac{1}{D^2 + D - 2} \sin x$$

$$= \frac{1}{-2\left(1 - \dfrac{D}{2} - \dfrac{D^2}{2}\right)} x + \frac{1}{-1^2 + D - 2} \sin x$$

$$= -\frac{1}{2}\left[1 - \left(\frac{D}{2} + \frac{D^2}{2}\right)\right]^{-1} x + \frac{1}{D - 3} \sin x$$

$$= -\frac{1}{2}\left[1 + \left(\frac{D}{2} + \frac{D^2}{2}\right) + \ldots\ldots\right] x + \frac{D + 3}{(D - 3)(D + 3)} \sin x$$

$$= -\frac{1}{2}\left[x + \frac{1}{2} D(x)\right] + \frac{D + 3}{D^2 - 9} \sin x = -\frac{1}{2}\left(x + \frac{1}{2}\right) + \frac{D + 3}{-1 - 9} \sin x$$

$$= -\frac{1}{2}\left(x + \frac{1}{2}\right) - \frac{1}{10} [D(\sin x) + 3 \sin x]$$

$$= -\frac{1}{2}\left(x + \frac{1}{2}\right) - \frac{1}{10} (\cos x + 3 \sin x)$$

$\therefore$ The complete solution is

$$y = c_1 e^{-2x} + c_2 e^x - \frac{1}{2}\left(x + \frac{1}{2}\right) - \frac{1}{10}(\cos x + 3\sin x).$$

(*iv*) Given equation in the symbolic form is

$$(D^2 - 4D + 4)y = x^2 + e^x + \cos 2x$$

Auxiliary equation is $D^2 - 4D + 4 = 0$

or $$(D-2)^2 = 0 \quad \therefore \quad D = 2, 2$$

$\therefore$ $$\text{C.F.} = (c_1 + c_2 x)e^{2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 4}(x^2 + e^x + \cos 2x)$$

$$= \frac{1}{D^2 - 4D + 4}x^2 + \frac{1}{D^2 - 4D + 4}e^x + \frac{1}{D^2 - 4D + 4}\cos 2x$$

$$= \frac{1}{4\left(1 - D + \frac{D^2}{4}\right)}x^2 + \frac{1}{1^2 - 4.1 + 4}e^x + \frac{1}{-2^2 - 4D + 4}\cos 2x$$

$$= \frac{1}{4}\left[1 - \left(D - \frac{D^2}{4}\right)\right]^{-1} x^2 + e^x - \frac{1}{4D}\cos 2x$$

$$= \frac{1}{4}\left[1 + \left(D - \frac{D^2}{4}\right) + \left(D - \frac{D^2}{4}\right)^2 + \ldots\ldots\right]x^2 + e^x - \frac{1}{4}\int \cos 2x\, dx$$

$$= \frac{1}{4}\left[1 + D - \frac{D^2}{2} + D^2 + \ldots\ldots\right]x^2 + e^x - \frac{1}{4}.\frac{\sin 2x}{2}$$

$$= \frac{1}{4}\left[1 + D + \frac{3}{4}D^2 + \ldots\ldots\right]x^2 + e^x - \frac{1}{8}\sin 2x$$

$$= \frac{1}{4}\left[x^2 + D(x^2) + \frac{3}{4}D^2(x^2)\right] + e^x - \frac{1}{8}\sin 2x$$

$$= \frac{1}{4}\left[x^2 + 2x + \frac{3}{2}\right] + e^x - \frac{1}{8}\sin 2x$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x)e^{2x} + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right) + e^x - \frac{1}{8}\sin 2x.$$

**Example 4.** *Solve the following* :

(*i*) $(D^2 + 5D + 4)y = x^2 + 7x + 9$ (*ii*) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 3y = \cos x + x^2$

(*iii*) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 2y = e^x + \cos x.$

**Sol.** (*i*) Given equation in the symbolic form is

$$(D^2 + 5D + 4)y = x^2 + 7x + 9$$

Auxiliary equation is

$$D^2 + 5D + 4 = 0 \quad \text{or} \quad (D + 1)(D + 4) = 0$$

$$\therefore \quad D = -1, -4$$

$$\text{C.F.} = c_1 e^{-x} + c_2 e^{-4x}$$

$$\text{P.I.} = \frac{1}{D^2 + 5D + 4}(x^2 + 7x + 9) = \frac{1}{4\left(1 + \frac{5D}{4} + \frac{D^2}{4}\right)}(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 + \left(\frac{5D}{4} + \frac{D^2}{4}\right)\right]^{-1}(x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 - \left(\frac{5D}{4} + \frac{D^2}{4}\right) + \left(\frac{5D}{4} + \frac{D^2}{4}\right)^2 \ldots\ldots\right](x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 - \frac{5D}{4} - \frac{D^2}{4} + \frac{25D^2}{16} + \ldots\ldots\right](x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[1 - \frac{5D}{4} + \frac{21D^2}{16} \ldots\ldots\right](x^2 + 7x + 9)$$

$$= \frac{1}{4}\left[(x^2 + 7x + 9) - \frac{5}{4}D(x^2 + 7x + 9) + \frac{21}{16}D^2(x^2 + 7x + 9)\right]$$

$$= \frac{1}{4}\left[(x^2 + 7x + 9) - \frac{5}{4}(2x + 7) + \frac{21}{16}(2)\right]$$

$$= \frac{1}{4}\left[x^2 + 7x - \frac{5}{2}x + 9 - \frac{35}{4} + \frac{21}{8}\right] = \frac{1}{4}\left[x^2 + \frac{9}{2}x + \frac{23}{8}\right]$$

$\therefore$ The complete solution is

$$y = c_1 e^{-x} + c_2 e^{-4x} + \frac{1}{4}\left[x^2 + \frac{9}{2}x + \frac{23}{8}\right].$$

(*ii*) Given equation in symbolic form is

$$(D^2 - 2D + 3)y = \cos x + x^2$$

Auxiliary equation is $\quad D^2 - 2D + 3 = 0$

or

$$D = \frac{2 \pm \sqrt{4 - 12}}{2} = \frac{2 \pm i2\sqrt{2}}{2} = 1 \pm i\sqrt{2}$$

$$\therefore \quad \text{C.F.} = (c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x)\, e^x$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 3}(\cos x + x^2) = \frac{1}{D^2 - 2D + 3}\cos x + \frac{1}{D^2 - 2D + 3}x^2$$

$$= \frac{1}{-1^2 - 2D + 3}\cos x + \frac{1}{3\left(1 - \frac{2D}{3} + \frac{D^2}{3}\right)}x^2$$

$$= \frac{1}{2-2D}\cos x + \frac{1}{3}\left[1-\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right]^{-1} x^2$$

$$= \frac{1+D}{2(1-D)(1+D)}\cos x + \frac{1}{3}\left[1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)+\left(\frac{2D}{3}-\frac{D^2}{3}\right)^2+\ldots\ldots\right]x^2$$

$$= \frac{1+D}{2(1-D^2)}\cos x + \frac{1}{3}\left[1+\frac{2D}{3}-\frac{D^2}{3}+\frac{4D^2}{9}+\ldots\ldots\right]x^2$$

$$= \frac{1+D}{2[1-(-1^2)]}\cos x + \frac{1}{3}\left[1+\frac{2D}{3}+\frac{D^2}{9}+\ldots\ldots\right]x^2$$

$$= \frac{1}{4}\,[\cos x + D(\cos x)] + \frac{1}{3}\left[x^2+\frac{2}{3}D(x^2)+\frac{1}{9}D^2(x^2)\right]$$

$$= \frac{1}{4}\,[\cos x - \sin x] + \frac{1}{3}\left[x^2+\frac{4x}{3}+\frac{2}{9}\right]$$

$\therefore$ The complete solution is

$$y = e^x(c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x) + \frac{1}{4}(\cos x - \sin x) + \frac{1}{3}\left[x^2+\frac{4x}{3}+\frac{2}{9}\right].$$

*(iii)* Given equation in symbolic form is

$$(D^2 - 2D + 2)y = e^x + \cos x$$

Auxiliary equation is

$$D^2 - 2D + 2 = 0$$

$$\therefore \qquad D = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm 2i}{2} = 1 \pm i$$

$$\text{C.F.} = e^x(c_1 \cos x + c_2 \sin x)$$

$$\text{P.I.} = \frac{1}{D^2-2D+2}(e^x + \cos x) = \frac{1}{D^2-2D+2}e^x + \frac{1}{D^2-2D+2}\cos x$$

$$= \frac{1}{1^2-2.1+2}e^x + \frac{1}{-1^2-2D+2}\cos x$$

$$= e^x + \frac{1}{1-2D}\cos x = e^x + \frac{1+2D}{(1-2D)(1+2D)}\cos x$$

$$= e^x + \frac{1+2D}{1-4D^2}\cos x = e^x + \frac{1+2D}{1-4(-1^2)}\cos x$$

$$= e^x + \frac{1}{5}[\cos x + 2D(\cos x)] = e^x + \frac{1}{5}[\cos x - 2\sin x]$$

$\therefore$ The complete solution is

$$y = e^x(c_1 \cos x + c_2 \sin x) + e^x + \frac{1}{2}(\cos x - 2\sin x).$$

**Example 5.** *Solve the following :*

(i) $\frac{d^2y}{dx^2} + 4y = e^x + \sin 3x + x^2$ (ii) $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = e^{-4x} + 5\cos 3x$

(iii) $(D^5 - D)\, y = 12e^{2x} + 8\sin 2x$ (iv) $(D^4 - a^4)y = x^4 + \sin bx$

(v) $\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 6\frac{dy}{dx} = 1 + x^2$ (vi) $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \sin 2x + x^2$.

**Sol.** (*i*) Given equation in symbolic form is

$$(D^2 + 4)\, y = e^x + \sin 3x + x^2$$

Auxiliary equation is $D^2 + 4 = 0 \quad \therefore \quad D = \pm 2i$

$$\text{C.F.} = c_1 \cos 2x + c_2 \sin 2x$$

$$\text{P.I.} = \frac{1}{D^2+4}(e^x + \sin 3x + x^2) = \frac{1}{D^2+4}e^x + \frac{1}{D^2+4}\sin 3x + \frac{1}{D^2+4}x^2$$

$$= \frac{1}{1^2+4}e^x + \frac{1}{-3^2+4}\sin 3x + \frac{1}{4\left(1+\frac{D^2}{4}\right)}x^2$$

$$= \frac{1}{5}e^x - \frac{1}{5}\sin 3x + \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x^2 = \frac{1}{5}e^x - \frac{1}{5}\sin 3x + \frac{1}{4}\left(1-\frac{D^2}{4}+\ldots\ldots\right)x^2$$

$$= \frac{1}{5}e^x - \frac{1}{5}\sin 3x + \frac{1}{4}\left(x^2 - \frac{1}{4}D^2(x^2)\right) = \frac{1}{5}e^x - \frac{1}{5}\sin 3x + \frac{1}{4}\left(x^2 - \frac{1}{2}\right)$$

$\therefore$ The complete solution is

$$y = c_1\cos 2x + c_2 \sin 2x + \frac{1}{5}e^x - \frac{1}{5}\sin 3x + \frac{1}{4}\left(x^2 - \frac{1}{2}\right).$$

(*ii*) Given equation in symbolic form is

$$(D^2 - 4D + 4)y = e^{-4x} + 5\cos 3x$$

Auxiliary equation is $D^2 - 4D + 4 = 0$

or $(D-2)^2 = 0 \quad \therefore \quad D = 2, 2$

$$\text{C.F.} = (c_1 + c_2 x)\, e^{2x}$$

$$\text{P.I.} = \frac{1}{D^2-4D+4}(e^{-4x} + 5\cos 3x) = \frac{1}{D^2-4D+4}e^{-4x} + 5\frac{1}{D^2-4D+4}\cos 3x$$

$$= \frac{1}{(-4)^2 - 4(-4)+4}e^{-4x} + 5\frac{1}{-3^2-4D+4}\cos 3x$$

$$= \frac{1}{36}e^{-4x} - 5\,.\frac{1}{4D+5}\cos 3x = \frac{1}{36}e^{-4x} - 5\,.\frac{4D-5}{(4D+5)(4D-5)}\cos 3x$$

$$= \frac{1}{36}e^{-4x} - 5\,.\frac{4D-5}{16D^2-25}\cos 3x = \frac{1}{36}e^{-4x} - 5\,.\frac{4D-5}{16(-3^2)-25}\cos 3x$$

$$= \frac{1}{36}e^{-4x} + \frac{5}{169}\left[4D(\cos 3x) - 5\cos 3x\right]$$

$$= \frac{1}{36} e^{-4x} + \frac{5}{169} [-12 \sin 3x - 5 \cos 3x]$$

$$= \frac{1}{36} e^{-4x} - \frac{5}{169} (12 \sin 3x + 5 \cos 3x)$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x)e^{2x} + \frac{1}{36} e^{-4x} - \frac{5}{169} (12 \sin 3x + 5 \cos 3x).$$

(*iii*) Given equation in symbolic form is

$$(D^5 - D)y = 12 e^{2x} + 8 \sin 2x$$

Auxiliary equation is $D^5 - D = 0$

or $D(D^4 - 1) = 0$ or $D(D^2 - 1)(D^2 + 1) = 0$

$\therefore$ $D = 0, \pm 1, \pm i$

$$\text{C.F.} = c_1 + c_2 e^x + c_3 e^{-x} + c_4 \cos x + c_5 \sin x$$

$$\text{P.I.} = \frac{1}{D^5 - D} (12e^{2x} + 8 \sin 2x) = 12 \cdot \frac{1}{D^5 - D} e^{2x} + 8 \cdot \frac{1}{D^5 - D} \sin 2x$$

$$= 12 \cdot \frac{1}{2^5 - 2} e^{2x} + 8 \cdot \frac{1}{D(D^4 - 1)} \sin 2x$$

$$= \frac{12}{30} e^{2x} + 8 \cdot \frac{1}{D[(-4)^2 - 1]} \sin 2x \qquad | \text{ Putting } D^2 = -2^2 = -4$$

$$= \frac{2}{5} e^{2x} + \frac{8}{15D} \sin 2x = \frac{2}{5} e^{2x} + \frac{8}{15} \int \sin 2x \, dx$$

$$= \frac{2}{5} e^{2x} + \frac{8}{15} \cdot \frac{-\cos 2x}{2} = \frac{2}{5} e^{2x} - \frac{4}{15} \cos 2x$$

$\therefore$ The complete solution is

$$y = c_1 + c_2 e^x + c_3 e^{-x} + c_4 \cos x + c_5 \sin x + \frac{2}{5} e^{2x} - \frac{4}{15} \cos 2x.$$

(*iv*) Given equation in symbolic form is

$$(D^4 - a^4)y = x^4 + \sin bx$$

Auxiliary equation is $D^4 - a^4 = 0$

or $(D^2 - a^2)(D^2 + a^2) = 0 \quad \therefore \quad D = \pm a, \pm ia$

$$\text{C.F.} = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax$$

$$\text{P.I.} = \frac{1}{D^4 - a^4} (x^4 + \sin bx) = \frac{1}{D^4 - a^4} x^4 + \frac{1}{D^4 - a^4} \sin bx$$

$$= \frac{1}{-a^4 \left(1 - \frac{D^4}{a^4}\right)} x^4 + \frac{1}{(-b^2)^2 - a^4} \sin bx$$

$$= -\frac{1}{a^4} \left(1 - \frac{D^4}{a^4}\right)^{-1} x^4 + \frac{1}{b^4 - a^4} \sin bx$$

$$= -\frac{1}{a^4} \left[1 + \frac{D^4}{a^4} + \ldots\ldots\right] x^4 + \frac{1}{b^4 - a^4} \sin bx$$

$$= -\frac{1}{a^4}\left[x^4 + \frac{1}{a^4}D^4(x^4)\right] + \frac{1}{b^4 - a^4}\sin bx$$

$$= -\frac{1}{a^4}\left[x^4 + \frac{24}{a^4}\right] + \frac{1}{b^4 - a^4}\sin bx$$

$[\because\ D^4(x^4) = D^2\,D(x^4) = D^3(4x^3) = D^2\,D(4x^3) = D^2(12x^2)$
$= D.D(12x^2) = D(24x) = 24]$

$\therefore$ The complete solution is

$$y = c_1e^{ax} + c_2e^{-ax} + c_3\cos ax + c_4\sin ax - \frac{1}{a^4}\left(x^4 + \frac{24}{a^4}\right) + \frac{\sin bx}{b^4 - a^4}.$$

(*v*) Given equation in symbolic form is

$$(D^3 - D^2 - 6D)y = 1 + x^2$$

Auxiliary equation is $(D^3 - D^2 - 6D) = 0$

or $D(D^2 - D - 6) = 0$ or $D(D - 3)(D + 2) = 0$

$\therefore$ $D = 0, 3, -2$

$$\text{C.F.} = c_1 + c_2e^{3x} + c_3e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^3 - D^2 - 6D}(1 + x^2) = \frac{1}{-6D\left(1 + \frac{D}{6} - \frac{D^2}{6}\right)}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1 + \left(\frac{D}{6} - \frac{D^2}{6}\right)\right]^{-1}(1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \left(\frac{D}{6} - \frac{D^2}{6}\right) + \left(\frac{D}{6} - \frac{D^2}{6}\right)^2 \ldots\ldots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[1 - \frac{D}{6} + \frac{D^2}{6} + \frac{D^2}{36} + \ldots\ldots\right](1 + x^2) = -\frac{1}{6D}\left[1 - \frac{D}{6} + \frac{7D^2}{36} + \ldots\ldots\right](1 + x^2)$$

$$= -\frac{1}{6D}\left[1 + x^2 - \frac{1}{6}D(1 + x^2) + \frac{7}{36}D^2(1 + x^2)\right]$$

$$= -\frac{1}{6D}\left[1 + x^2 - \frac{x}{3} + \frac{7}{18}\right] = -\frac{1}{6}\int\left(x^2 - \frac{x}{3} + \frac{25}{18}\right)dx \qquad \left|\because\ \frac{1}{D} \equiv \text{integration}\right.$$

$$= -\frac{1}{6}\left[\frac{x^3}{3} - \frac{x^2}{6} + \frac{25}{18}x\right] = -\frac{1}{18}\left[x^3 - \frac{x^2}{2} + \frac{25}{6}x\right]$$

$\therefore$ The complete solution is

$$y = c_1 + c_2e^{3x} + c_3e^{-2x} - \frac{1}{18}\left[x^2 - \frac{x^2}{2} + \frac{25}{6}x\right].$$

(*vi*) Please try yourself.

$$\left[\textbf{Ans.}\quad y = c_1e^x + (c_2 + c_3x)e^{-x} + \frac{1}{25}(2\cos 2x - \sin 2x) - x^2 + 2x - 4\right]$$

**Case IV. To evaluate $\frac{1}{f(D)}(e^{ax}V)$, where V is a function of x**

$$\frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V.$$

Let X be a function of $x$. By successive differentiation, we have

$$D(e^{ax}X) = e^{ax}DX + ae^{ax}X = e^{ax}(D+a)X$$

$$D^2(e^{ax}X) = e^{ax}D^2X + ae^{ax}DX + ae^{ax}DX + a^2e^{ax}X$$

$$= e^{ax}(D^2 + 2aD + a^2)X = e^{ax}(D+a)^2X$$

Similarly, $\quad D^3(e^{ax}X) = e^{ax}(D+a)^3X$

..........................................

..........................................

$$D^n(e^{ax}X) = e^{ax}(D+a)^n X$$

$\therefore \quad f(D)e^{ax}X = e^{ax}f(D+a)X \quad$ ...(1)

Now let $\quad f(D+a)X = V \quad$ ...(2)

so that $\quad \frac{1}{f(D+a)}V = X \quad$ ...(3)

$\therefore$ From (1), $f(D)\left[e^{ax}\frac{1}{f(D+a)}V\right] = e^{ax}V \quad$ [Using (2) and (3)]

$\Rightarrow \quad e^{ax}\frac{1}{f(D+a)}V = \frac{1}{f(D)}(e^{ax}V)$

Hence $\quad \frac{1}{f(D)}(e^{ax}V) = e^{ax}\frac{1}{f(D+a)}V$

Hence the theorem.

☞**Note. The above theorem proves that $e^{ax}$ which is on the right of $\frac{1}{f(D)}$ may be taken out to the left provided D is replaced by D + a.**

For example,

$$\frac{1}{D^2+3D+2}e^{2x}\sin x = e^{2x}\cdot\frac{1}{(D+2)^2+3(D+2)+2}\sin x = e^{2x}\cdot\frac{1}{D^2+7D+12}\sin x$$

Now proceeding as in Case II, we can find P.I. completely.

**Example 1.** *Solve the following :*

*(i)* $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + 4y = e^x\cos x$

*(ii)* $\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^{2x}\cos x + k$, *where k is a constant.*

*(iii)* $(D^2 - 5D + 6)y = xe^{4x}$ *(iv)* $\frac{d^2y}{dx^2} + y = xe^{2x}$.

**Sol.** (*i*) Given equation in symbolic form is $(D^2 - 2D + 4)y = e^x\cos x$

Auxiliary equation is $D^2 - 2D + 4 = 0$

$$\therefore \quad D = \frac{2 \pm \sqrt{4-16}}{2} = \frac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3}$$

$$\text{C.F.} = e^x(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 4}(e^x \cos x) = e^x \cdot \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$= e^x \cdot \frac{1}{D^2 + 3} \cos x = e^x \cdot \frac{1}{-1+3} \cos x = \frac{1}{2} e^x \cos x$$

$\therefore$ The complete solution is

$$y = e^x(c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x) + \frac{1}{2} e^x \cos x.$$

(*ii*) Given equation in the symbolic form is

$$(D^3 - 2D^2 + D - 2)y = e^{2x} \cos x + k$$

Auxiliary equation is $D^3 - 2D^2 + D - 2 = 0$

or $D^2(D-2) + (D-2) = 0$ or $(D-2)(D^2+1) = 0$

$\Rightarrow$ $D = 2, \pm i$

$\therefore$ $\text{C.F.} = c_1 e^{2x} + c_2 \cos x + c_3 \sin x$

$$\text{P.I.} = \frac{1}{D^3 - 2D^2 + D - 2}(e^{2x} \cos x + k)$$

$$= \frac{1}{D^3 - 2D^2 + D - 2}(e^{2x} \cos x) + k \frac{1}{D^3 - 2D^2 + D - 2} e^{0x}$$

$$= e^{2x} \cdot \frac{1}{(D+2)^3 - 2(D+2)^2 + (D+2) - 2} \cos x + k \cdot \frac{1}{0 - 0 + 0 - 2} e^{0x}$$

$$= e^{2x} \cdot \frac{1}{D^3 + 4D^2 + 5D} \cos x - \frac{k}{2} = e^{2x} \frac{1}{D.D^2 + 4D^2 + 5D} \cos x - \frac{k}{2}$$

$$= e^{2x} \cdot \frac{1}{D(-1) + 4(-1) + 5D} \cos x - \frac{k}{2}$$

$$= e^{2x} \cdot \frac{1}{4(D-1)} \cos x - \frac{k}{2} = \frac{1}{4} e^{2x} \frac{D+1}{D^2 - 1} \cos x - \frac{k}{2}$$

$$= \frac{1}{4} e^{2x} \cdot \frac{D+1}{-1-1} \cos x - \frac{k}{2} = -\frac{1}{8} e^{2x} [D(\cos x) + \cos x] - \frac{k}{2}$$

$$= -\frac{1}{8} e^{2x}(-\sin x + \cos x) - \frac{k}{2} = \frac{1}{8} e^{2x}(\sin x - \cos x) - \frac{k}{2}$$

$\therefore$ The complete solution is

$$y = c_1 e^{2x} + c_2 \cos x + c_3 \sin x + \frac{1}{8} e^{2x}(\sin x - \cos x) - \frac{k}{2}.$$

(*iii*) Given equation in symbolic form is

$$(D^2 - 5D + 6)y = xe^{4x}$$

Auxiliary equation is $(D^2 - 5D + 6) = 0$

or $(D-2)(D-3) = 0$ $\therefore$ $D = 2, 3$

$\text{C.F.} = c_1 e^{2x} + c_2 e^{3x}$

$$\text{P.I.} = \frac{1}{D^2 - 5D + 6}(xe^{4x}) = e^{4x} \frac{1}{(D+4)^2 - 5(D+4) + 6} x$$

$$= e^{4x} \frac{1}{D^2 + 3D + 2} x = e^{4x} \frac{1}{2\left(1 + \frac{3D}{2} + \frac{D^2}{2}\right)} x = \frac{1}{2} e^{4x} \left[1 + \left(\frac{3D}{2} + \frac{D^2}{2}\right)\right]^{-1} x$$

$$= \frac{1}{2} e^{4x} \left[1 - \left(\frac{3D}{2} + \frac{D^2}{2}\right) + \ldots\ldots\right] x = \frac{1}{2} e^{4x} \left[1 - \frac{3D}{2} \ldots\ldots\right] x$$

$$= \frac{1}{2} e^{4x} \left[x - \frac{3}{2} D(x)\right] = \frac{1}{2} e^{4x} \left(x - \frac{3}{2}\right)$$

$\therefore$ The complete solution is

$$y = c_1 e^{2x} + c_2 e^{3x} + \frac{1}{2} e^{4x} \left(x - \frac{3}{2}\right).$$

(*iv*) Given equation in symbolic form is

$$(D^2 + 1)y = xe^{2x}$$

Auxiliary equation is $\quad D^2 + 1 = 0 \quad \therefore \quad D = \pm i$

$$\text{C.F.} = c_1 \cos x + c_2 \sin x$$

$$\text{P.I.} = \frac{1}{D^2 + 1} (xe^{2x}) = e^{2x} \cdot \frac{1}{(D+2)^2 + 1} x$$

$$= e^{2x} \frac{1}{D^2 + 4D + 5} x = e^{2x} \frac{1}{5\left(1 + \frac{4D}{5} + \frac{D^3}{5}\right)} x$$

$$= \frac{1}{5} e^{2x} \left[1 + \left(\frac{4D}{5} + \frac{D^2}{5}\right)\right]^{-1} x = \frac{1}{5} e^{2x} \left[1 - \left(\frac{4D}{5} + \frac{D^2}{5}\right) + \ldots\ldots\right] x$$

$$= \frac{1}{5} e^{2x} \left[x - \frac{4}{5} D(x)\right] = \frac{1}{5} e^{2x} \left(x - \frac{4}{5}\right)$$

$\therefore$ The complete solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{5} e^{2x} \left(x - \frac{4}{5}\right).$$

**Example 2.** *Solve the following :*

(*i*) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + 4y = e^x \cos x + x$ (Kanpur 1996) $\qquad$ (*ii*) $(D^3 + 1)y = xe^x$

(*iii*) $\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} + 4y = x^2 e^{-2x}$ $\qquad$ (*iv*) $(D^2 + D - 2)y = e^x \sin x$.

**Sol.** (*i*) The given equation in symbolic form is

$$(D^2 - 2D + 4)y = e^x \cos x + x$$

A.E. is $\quad D^2 - 2D + 4 = 0 \quad \therefore \quad D = \dfrac{2 \pm 2i\sqrt{3}}{2} = 1 \pm i\sqrt{3}$

$$\text{C.F.} = e^x (c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 4}(e^x \cos x + x) = \frac{1}{D^2 - 2D + 4} e^x \cos x + \frac{1}{D^2 - 2D + 4} x$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x + \frac{1}{4\left(1 - \frac{D}{2} + \frac{D^2}{4}\right)} x$$

$$= e^x \frac{1}{D^2 + 3} \cos x + \frac{1}{4}\left[1 - \left(\frac{D}{2} - \frac{D^2}{4}\right)\right]^{-1} x$$

$$= e^x \cdot \frac{1}{-1^2 + 3} \cos x + \frac{1}{4}\left[1 + \left(\frac{D}{2} - \frac{D^2}{4}\right) + \ldots\ldots\right] x = \frac{1}{2} e^x \cos x + \frac{1}{4}\left(x + \frac{1}{2}\right)$$

$\therefore$ The complete solution is

$$y = e^x (c_1 \cos \sqrt{3}\, x + c_2 \sin \sqrt{3}\, x) + \frac{1}{2} e^x \cos x + \frac{1}{4}\left(x + \frac{1}{2}\right).$$

(*ii*) The given equation in symbolic form is $(D^3 + 1)\, y = x\, e^x$

A.E. is $D^3 + 1 = 0$ or $(D + 1)(D^2 - D + 1) = 0$

$$\therefore \qquad D = -1, \frac{1 \pm i\sqrt{3}}{2} = -1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\text{C.F.} = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right)$$

$$\text{P.I.} = \frac{1}{D^3 + 1} x\, e^x = e^x \frac{1}{(D+1)^3 + 1} x$$

$$= e^x \frac{1}{2 + 3D + 3D^2 + D^3} x = e^x \frac{1}{2\left(1 + \frac{3}{2}D + \frac{3}{2}D^2 + \frac{1}{2}D^3\right)} x$$

$$= \frac{1}{2} e^x \left[1 + \left(\frac{3}{2}D + \frac{3}{2}D^2 + \frac{1}{2}D^3\right)\right]^{-1} x$$

$$= \frac{1}{2} e^x \left[1 - \left(\frac{3}{2}D + \ldots\ldots\right) + \ldots\ldots\right] x = \frac{1}{2} e^x \left(x - \frac{3}{2}\right)$$

$\therefore$ The complete solution is

$$y = c_1 e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right) + \frac{1}{2} e^x \left(x - \frac{3}{2}\right).$$

(*iii*) The given equation in symbolic form is

$$(D^2 + 4D + 4)y = x^2 e^{-2x} \quad \text{or} \quad (D + 2)^2 y = x^2 e^{-2x}$$

A.E. is $(D + 2)^2 = 0 \quad \therefore \quad D = -2, -2$

$$\text{C.F.} = (c_1 + c_2 x)\, e^{-2x}$$

$$\text{P.I.} = \frac{1}{(D+2)^2} x^2 e^{-2x} = e^{-2x} \frac{1}{(D - 2 + 2)^2} x^2 = e^{-2x} \frac{1}{D^2} x^2$$

$$= e^{-2x}\frac{1}{D}\int x^2dx = e^{-2x}\frac{1}{D}\frac{x^3}{3} = e^{-2x}\int \frac{x^3}{3}dx = \frac{x^4}{12}e^{-2x}$$

$\therefore$ The complete solution is $y = (c_1 + c_2x)e^{-2x} + \dfrac{x^4}{12}e^{-2x}$.

(*iv*) The given equation in symbolic form is $(D^2 + D - 2)y = e^x \sin x$

A.E. is $D^2 + D - 2 = 0$ or $(D + 2)(D - 1) = 0$

$\therefore$ $D = 1, -2$

$$\text{C.F.} = c_1e^x + c_2e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^2+D-2}e^x \sin x = e^x\frac{1}{(D+1)^2+(D+1)-2}\sin x$$

$$= e^x\frac{1}{D^2+3D}\sin x = e^x\frac{1}{-1^2+3D}\sin x$$

$$= e^x\frac{3D+1}{9D^2-1}\sin x = e^x\frac{3D+1}{9(-1^2)-1}\sin x$$

$$= -\frac{1}{10}e^x(3\cos x + \sin x)$$

$\therefore$ The complete solution is

$$y = c_1e^x + ce^{-2x} - \frac{1}{10}e^x(3\cos x + \sin x).$$

**Example 3.** *Solve the following* :

(*i*) $(D^2 + 3D + 2)y = e^{2x}\sin x$ (Delhi 1996)

(*ii*) $(D^2 - 2D + 1)y = x^2e^{3x}$ (Rohilkhand 1997)

(*iii*) $(D^3 + 3D^2 - 4)y = xe^{-2x}$

(*iv*) $\dfrac{d^4y}{dx^4} - y = e^x \cos x$.

**Sol.** (*i*) Given equation is

$$(D^2 + 3D + 2)y = e^{2x}\sin x$$

Auxiliary equation is $D^2 + 3D + 2 = 0$

or $(D + 1)(D + 2) = 0$ $\therefore$ $D = -1, -2$

$$\text{C.F.} = c_1e^{-x} + c_2e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^2+3D+2}(e^{2x}\sin x) = e^{2x}\cdot\frac{1}{(D+2)^2+3(D+2)+2}\sin x$$

$$= e^{2x}\cdot\frac{1}{D^2+7D+12}\sin x = e^{2x}\frac{1}{-1+7D+12}\sin x$$

$$= e^{2x}\frac{1}{7D+11}\sin x = e^{2x}\frac{7D-11}{(7D+11)(7D-11)}\sin x$$

$$= e^{2x}\frac{7D-11}{49D^2-121}\sin x = e^{2x}\frac{7D-11}{49(-1)-121}\sin x$$

$$= -\frac{1}{170}e^{2x}[7D(\sin x) - 11\sin x] = -\frac{1}{170}e^{2x}[7\cos x - 11\sin x]$$

$\therefore$ The complete solution is

$$y = c_1e^{-x} + c_2e^{-2x} - \frac{1}{170}e^{2x}\,(7\cos x - 11\sin x).$$

(*ii*) Given equation is $(D^2 - 2D + 1)y = x^2e^{3x}$

Auxiliary equation is $D^2 - 2D + 1 = 0$

or $(D-1)^2 = 0 \;\therefore\; D = 1, 1$

$$\text{C.F.} = (c_1 + c_2x)e^x$$

$$\text{P.I.} = \frac{1}{(D-1)^2}(x^2e^{3x}) = e^{3x}\frac{1}{(D+3-1)^2}x^2$$

$$= e^{3x}\cdot\frac{1}{(D+2)}x^2 = e^{3x}\cdot\frac{1}{4\left(1+\frac{D}{2}\right)^2}x^2$$

$$= \frac{1}{4}e^{3x}\left(1+\frac{D}{2}\right)^{-2}x^2 = \frac{1}{4}e^{3x}\left(1-2\frac{D}{2}+3\,.\,\frac{D^2}{4}+\ldots\ldots\right)x^2$$

$$= \frac{1}{4}e^{3x}\left[x^2 - D(x^2) + \frac{3}{4}D^2(x^2)\right] = \frac{1}{4}e^{3x}\left(x^2 - 2x + \frac{3}{2}\right)$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2x)\,e^x + \frac{1}{4}e^{3x}\left(x^2 - 2x + \frac{3}{2}\right).$$

(*iii*) Given equation is $(D^3 + 3D^2 - 4)y = xe^{-2x}$

Auxiliary equation is $D^3 + 3D^2 - 4 = 0$

Putting $D = 1$, L.H.S. of A.E. $= 0$

$\therefore$ $D - 1$ is a factor of L.H.S. of A.E.

By synthetic division

$$\begin{array}{c|cccc} 1 & 1 & 3 & 0 & -4 \\ & & 1 & 4 & 4 \\ \hline & 1 & 4 & 4 & \boxed{0} \end{array}$$

$\therefore$ A.E. may be written as $(D-1)(D^2 + 4D + 4) = 0$

or $(D-1)(D+2)^2 = 0 \quad \therefore \quad D = 1, -2, -2$

$$\text{C.F.} = c_1e^x + (c_2 + c_3x)^{-2x}$$

$$\text{P.I.} = \frac{1}{D^3 + 3D^2 - 4}(x.e^{-2x}) = e^{-2x}\frac{1}{(D-2)^3 + 3(D-2)^2 - 4}x$$

$$= e^{-2x}\cdot\frac{1}{D^3 - 3D^2}x = e^{-2x}\cdot\frac{1}{-3D^2\left(1-\frac{D}{3}\right)}x$$

$$= -\frac{1}{3}e^{-2x}\cdot\frac{1}{D^2}\left(1-\frac{D}{3}\right)^{-1}x = -\frac{1}{3}e^{-2x}\frac{1}{D^2}\left(1+\frac{D}{3}+\ldots\ldots\right)x$$

$$= -\frac{1}{3}e^{-2x}\frac{1}{D^2}\left(x+\frac{1}{3}\right) = -\frac{1}{3}e^{-2x}\frac{1}{D}\left(\frac{x^2}{2}+\frac{x}{3}\right)$$

$$= -\frac{1}{3}e^{-2x}\left(\frac{x^3}{6}+\frac{x^2}{6}\right) = -\frac{1}{18}x^2e^{-2x}\,(x+1)$$

$\therefore$ The complete solution is

$$y = c_1e^x + (c_2 + c_3x)\,e^{-2x} - \left(\frac{1}{18}\right)x^2e^{-2x}(x+1).$$

(*iv*) Given equation in symbolic form is

$$(D^4 - 1)y = e^x \cos x$$

Auxiliary equation is $D^4 - 1 = 0$

or $$(D^2-1)(D^2+1) = 0 \quad \therefore \quad D = 1, -1, \pm i$$

$$\text{C.F.} = c_1e^x + c_2e^{-x} + c_3\cos x + c_4 \sin x$$

$$\text{P.I.} = \frac{1}{D^4-1}(e^x\cos x) = e^x\,.\,\frac{1}{(D+1)^4-1}\cos x = e^x\,.\,\frac{1}{D^4+4D^3+6D^2+4D}\cos x$$

$$= e^x\,.\,\frac{1}{(-1)^2+4D(-1)+6(-1)+4D}\cos x = e^x\,.\,\frac{1}{-5}\cos x = -\frac{1}{5}e^x\cos x$$

$\therefore$ The complete solution is

$$y = c_1e^x + c_2e^{-x} + c_3\cos x + c_4\sin x - \frac{1}{3}e^x\cos x.$$

**Example 4.** *Solve the following* :

(*i*) $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = x(x+e^x)$ (*ii*) $(D^2 - 4D + 3)y = e^x\cos 2x + \cos 3x$

(*iii*) $\dfrac{d^3y}{dx^3} - 7\dfrac{dy}{dx} - 6y = e^{2x}(1+x).$

**Sol.** (*i*) Given equation in symbolic form is

$$(D^2 - 5D + 6)y = x(x+e^x)$$

Auxiliary equation is $D^2 - 5D + 6 = 0$

or $$(D-2)(D-3) = 0 \quad \therefore \quad D = 2, 3$$

$$\text{C.F.} = c_1e^{2x} + c_2e^{3x}$$

$$\text{P.I.} = \frac{1}{D^2-5D+6}[x(x+e^x)] = \frac{1}{D^2-5D+6}x^2 + \frac{1}{D^2-5D+6}xe^x$$

$$= \frac{1}{6\left(1-\frac{5D}{6}+\frac{D^2}{6}\right)}x^2 + e^x\,.\,\frac{1}{(D+1)^2-5(D+1)+6}x$$

$$= \frac{1}{6}\left[1-\left(\frac{5D}{6}-\frac{D^2}{6}\right)\right]^{-1}x^2 + e^x\,.\,\frac{1}{D^2-3D+2}x$$

$$= \frac{1}{6}\left[1+\left(\frac{5D}{6}-\frac{D^2}{6}\right)+\left(\frac{5D}{6}-\frac{D^2}{6}\right)^2+\ldots\ldots\right]x^2 + e^x\frac{1}{2\left(1-\frac{3D}{2}+\frac{D^2}{2}\right)}x$$

$$= \frac{1}{6}\left[1+\frac{5D}{6}-\frac{D^2}{6}+\frac{25D^2}{36}+\ldots\ldots\right]x^2+\frac{1}{2}e^x\left[1-\left(\frac{3D}{2}-\frac{D^2}{2}\right)\right]^{-1}x$$

$$= \frac{1}{6}\left[x^2+\frac{5}{6}D(x^2)+\frac{19}{36}D^2(x^2)+\frac{1}{2}e^x\right]\left[1+\left(\frac{3D}{2}-\frac{D^2}{2}\right)+\ldots\ldots\right]x$$

$$= \frac{1}{6}\left[x^2+\frac{5}{3}x+\frac{19}{18}\right]+\frac{1}{2}e^x\left[x+\frac{3}{2}D(x)\right]=\frac{1}{6}\left(x^2+\frac{5}{3}x+\frac{19}{18}\right)+\frac{1}{2}e^x\left(x+\frac{3}{2}\right)$$

$\therefore$ The complete solution is

$$y = c_1e^{2x}+c_2e^{3x}+\frac{1}{6}\left(x^2+\frac{5}{3}x+\frac{19}{18}\right)+\frac{1}{2}e^x\left(x+\frac{3}{2}\right).$$

(*ii*) Given equation is $(D^2-4D+3)\,y = e^x\cos 2x+\cos 3x$

Auxiliary equation is $D^2-4D+3=0$

or $(D-1)(D-3)=0 \quad \therefore \quad D=1, 3$

$$\text{C.F.} = c_1e^x+c_2e^{3x}$$

$$\text{P.I.} = \frac{1}{D^2-4D+3}(e^x\cos 2x+\cos 3x)$$

$$= \frac{1}{D^2-4D+3}e^x\cos 2x+\frac{1}{D^2-4D+3}\cos 3x$$

$$= e^x\,.\,\frac{1}{(D+1)^2-4(D+1)+3}\cos 2x+\frac{1}{-9-4D+3}\cos 3x$$

$$= e^x\,.\,\frac{1}{D^2-2D}\cos 2x-\frac{1}{2(2D+3)}\cos 3x$$

$$= e^x\,.\,\frac{1}{-4-2D}\cos 2x-\frac{1}{2}\,.\,\frac{2D-3}{(2D+3)(2D-3)}\cos 3x$$

$$= e^x\,.\,\frac{D-2}{-2(D+2)(D-2)}\cos 2x-\frac{1}{2}\,.\,\frac{2D-3}{4D^2-9}\cos 3x$$

$$= e^x\,.\,\frac{D-2}{-2(D^2-4)}\cos 2x-\frac{1}{2}\,.\,\frac{2D-3}{4(-9)-9}\cos 3x$$

$$= e^x\,.\,\frac{D-2}{-2(-4-4)}\cos 2x+\frac{1}{90}\,[2D\,(\cos 3x)-3\cos 3x]$$

$$= \frac{1}{16}e^x\,[D\,(\cos 2x)-2\cos 2x]+\frac{1}{90}\,[-6\sin 3x-3\cos 3x]$$

$$= \frac{1}{16}e^x\,[-2\sin 2x-2\cos 2x]-\frac{1}{30}\,[2\sin 3x+\cos 3x]$$

$$= -\frac{1}{8}e^x\,[\sin 2x+\cos 2x]-\frac{1}{30}\,[2\sin 3x+\cos 3x]$$

$\therefore$ The complete solution is

$$y = c_1e^x+c_2e^{3x}-\frac{1}{8}e^x(\sin 2x+\cos 2x)-\frac{1}{30}(2\sin 3x+\cos 3x).$$

(*iii*) Given equation in symbolic form is

$$(D^3 - 7D - 6)y = e^{2x}(1 + x)$$

Auxiliary equation is $D^3 - 7D - 6 = 0$

Putting $D = -1$. L.H.S. of A.E. $= 0$

$\therefore$ D + 1 is a factor of L.H.S. of A.E.

By synthetic division

$$\begin{array}{r|rrrr} -1 & 1 & 0 & -7 & -6 \\ & & -1 & 1 & 6 \\ \hline & 1 & -1 & -6 & \boxed{0} \end{array}$$

$\therefore$ A.E. may be written as $(D + 1)(D^2 - D - 6) = 0$

or $(D + 1)(D - 3)(D + 2) = 0 \quad \therefore \quad D = -1, 3, -2$

$\therefore \quad$ C.F. $= c_1e^{-x} + c_2e^{3x} + c_3e^{-2x}$

$$\text{P.I.} = \frac{1}{D^3 - 7D - 6} e^{2x}(1 + x) = e^{2x} \cdot \frac{1}{(D+2)^3 - 7(D+2) - 6}(1 + x)$$

$$= e^{2x} \cdot \frac{1}{D^3 + 6D^2 + 5D - 12}(1 + x) = e^{2x} \cdot \frac{1}{-12\left(1 - \dfrac{5D}{12} - \dfrac{D^2}{2} - \dfrac{D^3}{12}\right)}(1 + x)$$

$$= -\frac{1}{12}e^{2x}\left[1 - \left(\frac{5D}{12} + \frac{D^2}{2} + \frac{D^3}{12}\right)\right]^{-1}(1 + x)$$

$$= -\frac{1}{12}e^{2x}\left[1 + \left(\frac{5D}{12} + \ldots\ldots\right) + \ldots\ldots\right](1 + x) = -\frac{1}{12}e^{2x}\left[(1 + x) + \frac{5}{12}D(1 + x)\right]$$

$$= -\frac{1}{12}e^{2x}\left[1 + x + \frac{5}{12}\right] = -\frac{1}{12}e^{2x}\left[x + \frac{17}{12}\right]$$

$\therefore$ The complete solution is

$$y = c_1e^{-x} + c_2e^{3x} + c_3e^{-2x} - \frac{1}{12}e^{2x}\left(x + \frac{17}{12}\right).$$

**Example 5.** *Solve the following :*

(*i*) $(D^2 - 4D + 4)y = e^{2x}\cos^2 x$ (*ii*) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^x \sin 2x$

(*iii*) $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} - y = xe^x + e^x$ (*iv*) $\dfrac{d^3y}{dx^3} - 3\dfrac{dy}{dx} + 2y = x^2e^x$

(*v*) $\dfrac{d^2y}{dx^2} + 2y = x^2e^{3x} + e^x\cos 2x$ (*vi*) $(D^3 + 1)y = e^x\cos x$

(*vii*) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 3y = 5 + x^2e^x$.

**Sol.** (*i*) Given equation is $(D^2 - 4D + 4)y = e^{2x}\cos^2 x$

A.E. is $D^2 - 4D + 4 = 0$ or $(D - 2)^2 = 0 \Rightarrow D = 2, 2$

$\therefore \quad$ C.F. $= (c_1 + c_2x)e^{2x}$

$$\text{P.I.} = \frac{1}{(D-2)^2}\,(e^{2x}\cos^2 x) = e^{2x}\cdot\frac{1}{(D+2-2)^2}\cos^2 x = e^{2x}\cdot\frac{1}{D^2}\left(\frac{1+\cos 2x}{2}\right)$$

$$= \frac{1}{2}e^{2x}\cdot\frac{1}{D}\left[\int(1+\cos 2x)\,dx\right]$$

$$= \frac{1}{2}e^{2x}\cdot\frac{1}{D}\left(x+\frac{\sin 2x}{2}\right) = \frac{1}{2}e^{2x}\int\left(x+\frac{\sin 2x}{2}\right)dx$$

$$= \frac{1}{2}e^{2x}\left(\frac{x^2}{2}-\frac{\cos 2x}{4}\right) = \frac{1}{8}\,e^{2x}\,(2x^2-\cos 2x)$$

$\therefore$ **The complete solution is**

$$y = (c_1+c_2x)e^{2x} + \frac{1}{8}\,e^{2x}\,(2x^2-\cos 2x).$$

(*ii*) **Given equation in symbolic form is** $(D^2+2D+1)y = e^x \sin 2x$

**A.E. is** $\quad D^2+2D+1=0 \quad$ or $\quad (D+1)^2=0 \quad \Rightarrow \quad D=-1,-1$

$$\text{C.F.} = (c_1+c_2x)\,e^{-x}$$

$$\text{P.I.} = \frac{1}{(D+1)^2}\,e^x\sin 2x = e^x\cdot\frac{1}{(D+1+1)^2}\sin 2x$$

$$= e^x\cdot\frac{1}{D^2+4D+4}\sin 2x = e^x\cdot\frac{1}{-4+4D+4}\sin 2x$$

$$= e^x\cdot\frac{1}{4D}\sin 2x = \frac{1}{4}e^x\int\sin 2x\,dx = \frac{1}{4}e^x\cdot\frac{-\cos 2x}{2} = -\frac{1}{8}\,e^x\cos 2x$$

$\therefore$ **The complete solution is**

$$y = (c_1+c_2x)e^{-x} - \frac{1}{8}\,e^x\cos 2x.$$

(*iii*) **Given equation in symbolic form is** $(D^3-3D^2+3D-1)y = e^x\,(x+1)$

**A.E. is** $\quad D^3-3D^2+3D-1=0 \quad$ or $\quad (D-1)^3=0 \quad \Rightarrow \quad D=1,1,1$

$\therefore$ $\quad \text{C.F.} = (c_1+c_2x+c_3x^2)e^x$

$$\text{P.I.} = \frac{1}{(D-1)^3}\,e^x(x+1) = e^x\cdot\frac{1}{(D+1-1)^3}\,(x+1)$$

$$= e^x\cdot\frac{1}{D^3}\,(x+1) = e^x\cdot\frac{1}{D^2}\left[\int(x+1)\,dx\right]$$

$$= e^x\cdot\frac{1}{D^2}\frac{(x+1)^2}{2} = \frac{1}{2}e^x\cdot\frac{1}{D}\left[\int(x+1)^2\,dx\right]$$

$$= \frac{1}{2}e^x\cdot\frac{1}{D}\frac{(x+1)^3}{3} = \frac{1}{6}e^x\cdot\frac{1}{D}\,(x+1)^3$$

$$= \frac{1}{6}\int(x+1)^3\,dx = \frac{1}{6}e^x\cdot\frac{(x+1)^4}{4} = \frac{1}{24}\,e^x(x+1)^4$$

$\therefore$ **The complete solution is**

$$y = (c_1+c_2x+c_3x^2)\,e^x + \frac{1}{24}\,e^x(x+1)^4.$$

(*iv*) Given equation in symbolic form is $(D^3 - 3D + 2)y = x^2e^x$

A.E. is $\quad D^3 - 3D + 2 = 0 \quad$ or $\quad (D-1)(D^2 + D - 2) = 0$

or $\quad (D-1)(D+2)(D-1) = 0 \quad \Rightarrow \quad D = 1, 1, -2$

$\therefore \quad$ C.F. $= (c_1 + c_2 x)e^x + c_3e^{-2x}$

$$\text{P.I.} = \frac{1}{D^3 - 3D + 2} x^2e^x = e^x \cdot \frac{1}{(D+1)^3 - 3(D+1) + 2} x^2$$

$$= e^x \cdot \frac{1}{D^3 + 3D^2} x^2 = e^x \cdot \frac{1}{3D^2\left(1+\frac{D}{3}\right)} x^2 = \frac{1}{3} e^x \cdot \frac{1}{D^2}\left(1+\frac{D}{3}\right)^{-1} x^2$$

$$= \frac{1}{3} e^x \cdot \frac{1}{D^2}\left[1 - \frac{D}{3} + \frac{D^2}{9} \ldots\ldots\right] x^2 = \frac{1}{3} e^x \cdot \frac{1}{D^2}\left[x^2 - \frac{2x}{3} + \frac{2}{9}\right]$$

$$= \frac{1}{3} e^x \cdot \frac{1}{D}\left[\int \left(x^2 - \frac{2x}{3} + \frac{2}{9}\right) dx\right] = \frac{1}{3} e^x \cdot \frac{1}{D}\left(\frac{x^3}{3} - \frac{x^2}{3} + \frac{2x}{9}\right)$$

$$= \frac{1}{3} e^x \cdot \int \left(\frac{x^3}{3} - \frac{x^2}{3} + \frac{2x}{9}\right) dx = \frac{1}{3} e^x \left[\frac{x^4}{12} - \frac{x^3}{9} + \frac{x^2}{9}\right] = \frac{1}{108} x^2e^x(3x^2 - 4x + 4)$$

$\therefore \quad$ The complete solution is

$$y = (c_1 + c_2 x)e^x + c_3e^{-2x} + \frac{1}{108} x^2e^x(3x^2 - 4x + 4).$$

(*v*) Given equation in symbolic form is

$$(D^2 + 2)y = x^2e^{3x} + e^x \cos 2x$$

A.E. is $\quad D^2 + 2 = 0 \;\Rightarrow\; D = \pm i\sqrt{2}$

$\therefore \quad$ C.F. $= c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x$

$$\text{P.I.} = \frac{1}{D^2+2}(x^2e^{3x} + e^x\cos 2x) = \frac{1}{D^2+2}(x^2e^{3x}) + \frac{1}{D^2+2}(e^x\cos 2x)$$

$$= e^{3x} \cdot \frac{1}{(D+3)^2+2} x^2 + e^x \cdot \frac{1}{(D+1)^2+2} \cos 2x$$

$$= e^{3x} \cdot \frac{1}{D^2+6D+11} x^2 + e^x \cdot \frac{1}{D^2+2D+3} \cos 2x$$

$$= e^{3x} \cdot \frac{1}{11\left(1+\frac{6D}{11}+\frac{D^2}{11}\right)} x^2 + e^x \cdot \frac{1}{-4+2D+3} \cos 2x$$

$$= \frac{1}{11} e^{3x}\left[1+\left(\frac{6D}{11}+\frac{D^2}{11}\right)\right]^{-1} x^2 + e^x \cdot \frac{1}{2D-1} \cos 2x$$

$$= \frac{1}{11} e^{3x}\left[1-\left(\frac{6D}{11}+\frac{D^2}{11}\right)+\left(\frac{6D}{11}+\frac{D^2}{11}\right)^2 \ldots\ldots\right] x^2 + e^x \frac{2D+1}{4D^2-1} \cos 2x$$

$$= \frac{1}{11} e^{3x} \left[1 - \frac{6D}{11} - \frac{D^2}{11} + \frac{36D^2}{121} \ldots\ldots\right] x^2 + e^x \cdot \frac{2D+1}{4(-4)-1} \cos 2x$$

$$= \frac{1}{11} e^{3x} \left[1 - \frac{6D}{11} + \frac{25}{121} D^2 \ldots\ldots\right] x^2 - \frac{1}{17} e^x (2D + 1) \cos 2x$$

$$= \frac{1}{11} e^{3x} \left[x^2 - \frac{12}{11} x + \frac{50}{121}\right] - \frac{1}{17} e^x [2(-2 \sin 2x) + \cos 2x]$$

$$= \frac{1}{11} e^{3x} \left(x^2 - \frac{12}{11} x + \frac{50}{121}\right) + \frac{1}{17} e^x (4 \sin 2x - \cos 2x)$$

$\therefore$ The complete solution is

$$y = c_1 \cos \sqrt{2} x + c_2 \sin \sqrt{2} x + \frac{1}{11} e^{3x} \left(x^2 - \frac{12}{11} x + \frac{50}{121}\right) + \frac{1}{17} e^x (4 \sin 2x - \cos 2x).$$

(*vi*) Given equation is $(D^3 + 1)y = e^x \cos x$

A.E. is $D^3 + 1 = 0$ or $(D + 1)(D^2 - D + 1) = 0$

$$D = -1, \frac{1 \pm i\sqrt{3}}{2}$$

$$\therefore \quad \text{C.F.} = c_1 e^{-x} + e^{\frac{1}{2}x} \left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right)$$

$$\text{P.I.} = \frac{1}{D^3 + 1} (e^x \cos x) = e^x \cdot \frac{1}{(D+1)^3 + 1} \cos x$$

$$= e^x \cdot \frac{1}{D^3 + 3D^2 + 3D + 2} \cos x = e^x \cdot \frac{1}{-D - 3 + 3D + 2} \cos x$$

$$= e^x \cdot \frac{1}{2D - 1} \cos x = e^x \cdot \frac{2D + 1}{4D^2 - 1} \cos x$$

$$= e^x \cdot \frac{4D + 1}{4(-1) - 1} \cos x = -\frac{1}{5} e^x [2(-\sin x) + \cos x]$$

$$= \frac{1}{5} e^x (2 \sin x - \cos x)$$

$\therefore$ The complete solution is

$$y = c_1 e^{-x} + e^{\frac{1}{2}x} \left(c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x\right) + \frac{1}{5} e^x (2 \sin x - \cos x).$$

(*vii*) Please try yourself. $\left[\textbf{Ans.}\quad y = c_1 e^x + c_2 e^{3x} + \frac{5}{3} - \frac{1}{2} e^x \left(\frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{2}\right)\right]$

**Example 6.** *Solve the following :*

(*i*) $(D^2 - 1)y = \cosh x \cos x + a^x$ (*ii*) $(D^2 - 1)\, y = \cos x \sinh x$

(*iii*) $\dfrac{d^2y}{dx^2} - 4y = x \sinh x.$

**Sol.** (*i*) Given equation in symbolic form is

$$(D^2 - 1)y = \cosh x \cos x + a^x$$

A.E. is $\quad D^2 - 1 = 0 \quad \therefore \quad D = \pm 1$

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}$$

$$\text{P.I.} = \frac{1}{D^2-1}(\cosh x \cos x + a^x) = \frac{1}{D^2-1}\left(\frac{e^x + e^{-x}}{2}\cos x\right) + \frac{1}{D^2-1}a^x$$

$$= \frac{1}{2}\frac{1}{D^2-1}e^x \cos x + \frac{1}{2}\frac{1}{D^2-1}e^{-x}\cos x + \frac{1}{D^2-1}e^{\log a^x} \qquad [\because \; e^{\log f(x)} = f(x)]$$

$$= \frac{1}{2}e^x \frac{1}{(D+1)^2-1}\cos x + \frac{1}{2}e^{-x}\frac{1}{(D-1)^2-1}\cos x + \frac{1}{D^2-1}e^{x\log a}$$

$$= \frac{1}{2}e^x \frac{1}{D^2+2D}\cos x + \frac{1}{2}e^{-x}\frac{1}{D^2-2D}\cos x + \frac{1}{(\log a)^2-1}e^{x\log a}$$

$$= \frac{1}{2}e^x \frac{1}{-1^2+2D}\cos x + \frac{1}{2}e^{-x}\frac{1}{-1^2-2D}\cos x + \frac{a^x}{(\log a)^2-1}$$

$$= \frac{1}{2}e^x \frac{2D+1}{4D^2-1}\cos x - \frac{1}{2}e^{-x}\frac{2D-1}{4D^2-1}\cos x + \frac{a^x}{(\log a)^2-1}$$

$$= \frac{1}{2}e^x \frac{2D+1}{4(-1^2)-1}\cos x - \frac{1}{2}e^{-x}\frac{2D-1}{4(-1^2)-1}\cos x + \frac{a^x}{(\log a)^2-1}$$

$$= -\frac{1}{10}e^x(-2\sin x + \cos x) + \frac{1}{10}e^{-x}(-2\sin x - \cos x) + \frac{a^x}{(\log a)^2-1}$$

$$= \frac{1}{5}(e^x - e^{-x})\sin x - \frac{1}{10}(e^x + e^{-x})\cos x + \frac{a^x}{(\log a)^2-1}$$

$$= \frac{2}{5}\left(\frac{e^x - e^{-x}}{2}\right)\sin x - \frac{1}{5}\left(\frac{e^x + e^{-x}}{2}\right)\cos x + \frac{a^x}{(\log a)^2-1}$$

$$= \frac{2}{5}\sinh x \sin x - \frac{1}{5}\cosh x \cos x + \frac{a^x}{(\log a)^2-1}$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{-x} + \frac{2}{5}\sinh x \sin x - \frac{1}{5}\cosh x \cos x + \frac{a^x}{(\log a)^2-1}.$$

(*ii*) Please try yourself.

(*iii*) Given equation in symbolic form is $(D^2 - 4)y = x \sinh x$

Its A.E. is $\quad D^2 - 4 = 0 \quad$ so that $D = \pm 2$

$\therefore \qquad \text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2-4}x \sinh x = \frac{1}{D^2-4}x\left(\frac{e^x - e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4}\,xe^{x} - \frac{1}{D^2-4}\,xe^{-x}\right] = \frac{1}{2}\left[e^{x}\,\frac{1}{(D+1)^2-4}\,x - e^{-x}\,\frac{1}{(D-1)^2-4}\,x\right]$$

$$= \frac{1}{2}\left[e^{x}\,\frac{1}{D^2+2D-3}\,x - e^{-x}\,\frac{1}{D^2-2D-3}\,x\right]$$

$$= \frac{1}{2}\left[e^{x}\,.\,\frac{1}{-3\left(1-\frac{2D}{3}-\frac{D^2}{3}\right)}\,x - e^{-x}\,.\,\frac{1}{-3\left(1+\frac{2D}{3}-\frac{D^2}{3}\right)}\,x\right]$$

$$= -\frac{1}{6}\left[e^{x}\left\{1-\left(\frac{2D}{3}+\frac{D^2}{3}\right)\right\}^{-1} x - e^{-x}\left\{1+\left(\frac{2D}{3}-\frac{D^2}{3}\right)\right\}^{-1} x\right]$$

$$= -\frac{1}{6}\left[e^{x}\left(1+\frac{2D}{3}-\ldots\right)x - e^{-x}\left(1-\frac{2D}{3}\ldots\right)x\right]$$

$$= -\frac{1}{6}\left[e^{x}\left(x+\frac{2}{3}\right) - e^{-x}\left(x-\frac{2}{3}\right)\right] = -\frac{1}{6}\left[x\,(e^{x}-e^{-x}) + \frac{2}{3}(e^{x}+e^{-x})\right]$$

$$= -\frac{x}{3}\left(\frac{e^{x}-e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^{x}+e^{-x}}{2}\right) = -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$$

Hence the complete solution is $y = c_1e^{2x} + c_2e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x.$

**Case V. To evaluate $\frac{1}{f(D)}$ (xV), where V is a function of x.**

$$\mathbf{\frac{1}{f(D)}\,(xV) = x\,.\,\frac{1}{f(D)}\,V + \left(\frac{d}{dD}\,\frac{1}{f(D)}\right)V.}$$

**Proof.** Let X be a function of $x$.

By Leibnitz's Theorem on successive differentiation, we have

$$= D^n(xX) = D^n(Xx) = D^nX\,.\,x + {}^nC_1\,D^{n-1}X\,.\,1 = xD^nX + nD^{n-1}X$$

$$= xD^nX + \frac{d}{dD}\,(D^n)X \qquad \left|\because\ \frac{d}{dD}(D^n) = nD^{n-1}\right.$$

$$\Rightarrow \qquad f(D)(xX) = xf(D)X + \left(\frac{d}{dD}\,f(D)\right)X \qquad \ldots(i)$$

Putting $f(D)X = V$, we have

$$\frac{1}{f(D)} = (f(D)X) = \frac{1}{f(D)}\,V \qquad \Rightarrow \qquad X = \frac{1}{f(D)}\,V$$

Also, since X is a function of $x$, so is V.

$\therefore$ From (*i*), $$f(D)\left(x\frac{1}{f(D)}V\right)=xV+\left(\frac{d}{dD}f(D)\right)\frac{1}{f(D)}V$$

Operating on both sides by $\frac{1}{f(D)}$, we have

$$\left(x\frac{1}{f(D)}V\right)=\frac{1}{f(D)}(xV)+\frac{f'(D)}{[f(D)]^2}V$$

$$\Rightarrow \quad \frac{1}{f(D)}(xV)=x\,.\,\frac{1}{f(D)}V-\frac{f'(D)}{[f(D)]^2}V=x\,.\,\frac{1}{f(x)}V+\left(\frac{d}{dD}\frac{1}{f(D)}\right)V$$

which proves the theorem.

**Exercise.** *Prove that* $\frac{1}{f(D)}[x\,\phi(x)]=\left[x-\frac{1}{f(D)}f'(D)\right]\frac{1}{f(D)}\phi(x)$.

**Sol.** Let $\phi(x) = V$, then we have to prove that

$$\frac{1}{f(D)}(xV)=\left[x-\frac{f'(D)}{f(D)}\right]\frac{1}{f(D)}V=x\,.\,\frac{1}{f(D)}V-\frac{f'(D)}{[f(D)]^2}V$$

Now reproduce the above article.

**Example 1.** *Solve the following :*

(*i*) $\frac{d^2y}{dx^2}+4y=x\sin x$ (*ii*) $\frac{d^2y}{dx^2}+4y=x\cos x$

(*iii*) $\frac{d^2y}{dx^2}-4y=x\cos 2x$ (*iv*) $(D^2+2)y=x\sin x$

(*v*) $(D^2-1)y=x\sin x$.

**Sol.** (*i*) Given equation in symbolic form is

$$(D^2+4)y=x\sin x$$

A.E. is $D^2+4=0 \Rightarrow D=\pm 2i$

$\therefore$ C.F. $= c_1\cos 2x+c_2\sin 2x$

$$\text{P.I.}=\frac{1}{D^2+4}(x\sin x)=x\,.\,\frac{1}{D^2+4}\sin x-\frac{2D}{(D^2+4)^2}\sin x$$

$$\left[\text{using } \frac{1}{f(D)}(xV)=x\,.\,\frac{1}{f(D)}V-\frac{f'(D)}{[f(D)]^2}V\right]$$

$$=x\,.\,\frac{1}{-1+4}\sin x-\frac{2D}{(-1+4)^2}\sin x$$

$$=\frac{1}{3}x\sin x-\frac{2}{9}D(\sin x)=\frac{1}{3}x\sin x-\frac{2}{9}\cos x$$

$\therefore$ The complete solution is

$$y=c_1\cos 2x+c_2\sin 2x+\frac{1}{3}x\sin x-\frac{2}{9}\cos x$$

(*ii*) Please try yourself. $\left[\textbf{Ans.}\ \ y=c_1\cos 2x+c_2\sin 2x+\frac{1}{3}x\cos x+\frac{2}{9}\sin x.\right]$

(*iii*) Given equation in symbolic form is

$$(D^2 - 4)y = x \cos 2x$$

A.E. is $D^2 - 4 = 0 \Rightarrow D = \pm 2$

$\therefore$ C.F. $= c_1 e^{2x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2 - 4}(x \cos 2x) = x \cdot \frac{1}{D^2 - 4}\cos 2x - \frac{2D}{(D^2 - 4)^2}\cos 2x$$

$$\left[\text{using } \frac{1}{f(D)}(xV) = x \cdot \frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V\right]$$

$$= x \cdot \frac{1}{-4-4}\cos 2x - \frac{2D}{(-4-4)^2}\cos 2x = -\frac{1}{8}x\cos 2x - \frac{1}{32}D(\cos 2x)$$

$$= -\frac{1}{8}x\cos 2x - \frac{1}{32}(-2\sin 2x) = -\frac{1}{8}x\cos 2x + \frac{1}{16}\sin 2x$$

$\therefore$ The complete solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{1}{8}x\cos 2x + \frac{1}{16}\sin 2x.$$

(*iv*) Please try yourself. [**Ans.** $y = c_1 \cos\sqrt{2}x + c_2 \sin\sqrt{2}x + x\sin x - 2\cos x$]

(*v*) Please try yourself. $\left[\textbf{Ans.}\ \ y = c_1 e^x + c_2 e^{-x} - \frac{1}{2}(x\sin x + \cos x)\right]$

**Example 2.** *Solve the following :*

(*i*) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = x \sin x$ (Allahabad 1998 ; Agra 1996)

(*ii*) $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \sin x$ (Lucknow 1996)

(*iii*) $\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 2y = xe^x \sin x$ (*iv*) $\dfrac{d^2y}{dx^2} - y = x \sin x + (1 + x^2)e^x$

(*v*) $\dfrac{d^2y}{dx^2} - y = x \sin x + x^2e^x$.

**Sol.** (*i*) Given equation in symbolic form is

$$(D^2 - 2D + 1)y = x \sin x$$

A.E. is $D^2 - 2D + 1 = 0$ or $(D - 1)^2 = 0 \Rightarrow D = 1, 1$

$\therefore$ C.F. $= (c_1 + c_2 x)e^x$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1}(x \sin x)$$

$$= x \cdot \frac{1}{D^2 - 2D + 1}\sin x + \frac{d}{dD}\left(\frac{1}{D^2 - 2D + 1}\right)\sin x$$

$$= x \cdot \frac{1}{D^4 - 2D + 1}\sin x - \frac{2D - 2}{(D^2 - 2D + 1)^2}\sin x$$

$$= x \cdot \frac{1}{-1-2D+1} \sin x - \frac{2(D-1)}{(-1-2D+1)^2} \sin x$$

$$= -\frac{x}{2} \cdot \frac{1}{D} \sin x - \frac{D-1}{2D^2} \sin x = -\frac{x}{2} \int \sin x \, dx - \frac{D-1}{2(-1)} \sin x$$

$$= \frac{x}{2} \cos x + \frac{1}{2} [D(\sin x) - \sin x] = \frac{x}{2} \cos x + \frac{1}{2} (\cos x - \sin x)$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x) e^x + \frac{x}{2} \cos x + \frac{1}{2} (\cos x - \sin x).$$

(*ii*) Here C.F. $= (c_1 + c_2 x) e^x$

$$\text{P.I.} = \frac{1}{(D-1)^2} (xe^x \sin x) = e^x \cdot \frac{1}{(D+1-1)^2} (x \sin x) = e^x \cdot \frac{1}{D^2} (x \sin x)$$

$$= e^x \left[ x \cdot \frac{1}{D^2} \sin x + \frac{d}{dD} \left( \frac{1}{D^2} \right) \sin x \right] = e^x \left[ x \cdot \frac{1}{-1} \sin x - \frac{2}{D^3} \sin x \right]$$

$$= e^x \left[ -x \sin x - \frac{2}{D^2} (-\cos x) \right] \qquad \left[ \because \frac{1}{D} \equiv \text{integration} \right]$$

$$= e^x \left[ -x \sin x + \frac{2}{D} \sin x \right] = e^x [-x \sin x - 2 \cos x]$$

$\therefore$ The complete solution is $\quad y = (c_1 + c_2 x) e^x - e^x (x \sin x + 2 \cos x)$.

**Note. Second Method for Particular Integral**

$$\text{P.I.} = e^x \cdot \frac{1}{D^2} (x \sin x) = e^x \frac{1}{D} \int x \sin x \, dx \quad \text{Integrating by parts}$$

$$= e^x \cdot \frac{1}{D} \left[ x(-\cos x) - \int 1(-\cos x) \, dx \right] = e^x \cdot \frac{1}{D} (-x \cos x + \sin x)$$

$$= e^x \int (-x \cos x + \sin x) \, dx = e^x \left[ -\left\{ x \sin x - \int 1 . \sin x \, dx \right\} - \cos x \right]$$

$$= e^x [-x \sin x - \cos x - \cos x] = -e^x (x \sin x + 2 \cos x).$$

(*iii*) Here AE is $\quad D^2 + 3D + 2 = 0 \quad \Rightarrow \quad D = -1, -2$

$\therefore \quad$ C.F. $= c_1 e^{-x} + c_2 e^{-2x}$

$$\text{P.I.} = \frac{1}{D^2 + 3D + 2} (xe^x \sin x)$$

$$= e^x \cdot \frac{1}{(D+1)^2 + 3(D+1) + 2} (x \sin x) = e^x \cdot \frac{1}{D^2 + 5D + 6} (x \sin x)$$

$$= e^x \left[ x \cdot \frac{1}{D^2 + 5D + 6} \sin x + \left( \frac{d}{dD} \frac{1}{D^2 + 5D + 6} \right) \sin x \right]$$

$$= e^x \left[ x \cdot \frac{1}{-1 + 5D + 6} \sin x - \frac{2D + 5}{(D^2 + 5D + 6)^2} \sin x \right]$$

$$= e^x \left[ x \cdot \frac{1}{5(D+1)} \sin x - \frac{2D + 5}{(-1 + 5D + 6)^2} \sin x \right]$$

$$= e^x\left[\frac{x}{5}\cdot\frac{D-1}{D^2-1}\sin x - \frac{2D+5}{25(D+1)^2}\sin x\right]$$

$$= e^x\left[\frac{x}{5}\cdot\frac{D-1}{-1-1}\sin x - \frac{1}{25}\cdot\frac{2D+5}{(D^2+2D+1)^2}\sin x\right]$$

$$= e^x\left[-\frac{x}{10}(D\sin x - \sin x) - \frac{1}{25}\cdot\frac{2D+5}{(-1+2D+1)^2}\sin x\right]$$

$$= e^x\left[-\frac{x}{10}(\cos x - \sin x) - \frac{1}{25}\cdot\frac{2D+5}{4D^2}\sin x\right]$$

$$= e^x\left[\frac{x}{10}(\sin x - \cos x) - \frac{1}{100}\left(\frac{2D+5}{D^2}\right)\sin x\right]$$

$$= e^x\left[\frac{x}{10}(\sin x - \cos x) - \frac{1}{100}\left(\frac{2}{D}+\frac{5}{D^2}\right)\sin x\right]$$

$$= e^x\left[\frac{x}{10}(\sin x - \cos x) - \frac{1}{50}\cdot\frac{1}{D}\sin x - \frac{1}{20}\cdot\frac{1}{D^2}\sin x\right]$$

$$= e^x\left[\frac{x}{10}(\sin x - \cos x) - \frac{1}{50}(-\cos x) - \frac{1}{20}\cdot\frac{1}{-1}\sin x\right]$$

$$= e^x\left[\frac{x}{10}(\sin x - \cos x) + \frac{1}{50}\cos x + \frac{1}{20}\sin x\right]$$

$\therefore$ The complete solution is

$$y = c_1e^{-x} + c_2e^{-2x} + e^x\left[\frac{x}{10}(\sin x - \cos x) + \frac{1}{50}\cos x + \frac{1}{20}\sin x\right]$$

(*iv*) Here C.F. $= c_1e^x + c_2e^{-x}$

$$\text{P.I.} = \frac{1}{D^2-1}[x\sin x + (1+x^2)e^x] = \frac{1}{D^2-1}(x\sin x) + \frac{1}{D^2-1}[e^x(1+x^2)]$$

$$= x\cdot\frac{1}{D^2-1}\sin x + \left(\frac{d}{dD}\frac{1}{D^2-1}\right)\sin x + e^x\cdot\frac{1}{(D+1)^2-1}(1+x^2)$$

$$= x\cdot\frac{1}{-1-1}\sin x - \frac{2D}{(D^2-1)^2}\sin x + e^x\cdot\frac{1}{D^2+2D}(1+x^2)$$

$$= -\frac{x}{2}\sin x - \frac{2D}{(-1-1)^2}\sin x + e^x\cdot\frac{1}{2D\left(1+\frac{D}{2}\right)}(1+x^2)$$

$$= -\frac{x}{2}\sin x - \frac{1}{2}D\sin x + \frac{1}{2}e^x\cdot\frac{1}{D}\left(1+\frac{D}{2}\right)^{-1}(1+x^2)$$

$$= -\frac{x}{2}\sin x - \frac{1}{2}\cos x + \frac{1}{2}e^x\,\frac{1}{D}\left(1-\frac{D}{2}+\frac{D^2}{4}\,\ldots\ldots\right)(1+x^2)$$

$$= -\frac{x}{2}\sin x - \frac{1}{2}\cos x + \frac{1}{2}e^x \cdot \frac{1}{D}\left[(1+x^2) - \frac{1}{2}(2x) + \frac{1}{4}(2)\right]$$

$$= -\frac{x}{2}\sin x - \frac{1}{2}\cos x + \frac{1}{2}e^x \cdot \int \left(x^2 - x + \frac{3}{2}\right) dx$$

$$= -\frac{x}{2}\sin x - \frac{1}{2}\cos x + \frac{1}{2}e^x \left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{3}{2}x\right)$$

$$= -\frac{1}{2}(x\sin x + \cos x) + \frac{1}{12}xe^x(2x^2 - 3x + 9)$$

$\therefore$ The complete solution is

$$y = c_1e^x + c_2e^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{1}{12}xe^x(2x^2 - 3x + 9).$$

(*v*) Please try yourself.

$$\left[\textbf{Ans.}\quad y = c_1e^x + c_2e^{-x} - \frac{1}{2}(x\sin x + \cos x) + \frac{1}{12}xe^x(2x^2 - 3x + 3)\right]$$

## CASES OF FAILURE

We have proved that

(1) $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$ provided $f(a) \neq 0$

(2) $\frac{1}{f(D)^2}\sin ax = \frac{1}{f(-a^2)}\sin ax$ provided $f(-a^2) \neq 0$

(3) $\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax$ provided $f(-a^2) \neq 0$

If $f(a) = 0$ in (1) or $f(-a^2) = 0$ in (2) and (3), the above results fail and therefore, we have the following theorems.

**Theorem I.** $\mathbf{\frac{1}{f(D)}e^{ax} = x \cdot \frac{1}{\frac{d}{dD}[(fD)]}e^{ax}}$ **if f(a) = 0.**

**Proof.** If $f(a) = 0$, $(D - a)$ is a factor of $f(D)$ [Factor Theorem]

Let $f(D) = (D - a)\,\phi(D)$ ...(*i*)

$$\therefore \quad \frac{1}{f(D)}e^{ax} = \frac{1}{(D-a)\,\phi(D)}e^{ax} = \frac{1}{D-a}\left(\frac{1}{\phi(D)}e^{ax}\right)$$

$$= \frac{1}{D-a}\left(\frac{1}{\phi(a)}e^{ax}\right) \qquad [\text{Assuming } \phi(a) \neq 0]$$

$$= \frac{1}{\phi(a)} \cdot \frac{1}{D-a}e^{ax} = \frac{1}{\phi(a)} \cdot e^{ax}\int e^{-ax} \cdot e^{ax}\,dx$$

$$= \frac{1}{\phi(a)} \cdot e^x \int 1dx = \frac{xe^{ax}}{\phi(a)} \qquad \text{...}(ii)$$

Differentiating (*i*) w.r.t. 'D', we get

$$f'(\mathrm{D}) = \phi(\mathrm{D}) + (\mathrm{D} - a)\,\phi'(\mathrm{D})$$

Putting $\quad \mathrm{D} = a, f'(a) = \phi(a)$

$\therefore$ From (*ii*), $\dfrac{1}{f(\mathrm{D})} e^{ax} = \dfrac{xe^{ax}}{f'(a)} = x \cdot \dfrac{1}{f'(\mathrm{D})} e^{ax} = x \cdot \dfrac{1}{\dfrac{d}{d\mathrm{D}}[f(\mathrm{D})]} e^{ax}.$

**Euler's Theorem.** $e^{ix} = \cos x + i \sin x$ [From Trigonometry]

$\Rightarrow \quad \cos x = \text{Real Part (R.P.) of } e^{ix}$

$\quad \sin x = \text{Imaginary Part (I.P. ) of } e^{ix}$

**Theorem II.**

(*i*) $\dfrac{1}{\mathbf{f(D^2)}} \mathbf{\sin ax = x \cdot \dfrac{1}{\dfrac{d}{dD}[f(D^2)]} \sin ax \text{ if } f(-a^2) = 0}$

(*ii*) $\dfrac{1}{\mathbf{f(D^2)}} \mathbf{\cos ax = x \cdot \dfrac{1}{\dfrac{d}{dD}[f(D^2)]} \cos ax \text{ if } f(-a^2) = 0}.$

**Proof.** $\dfrac{1}{f(\mathrm{D}^2)} (\cos ax + i \sin ax) = \dfrac{1}{f(\mathrm{D}^2)} e^{iax}$ [Euler's Theorem]

[If we replace D by $ia$, $f(\mathrm{D}^2) = f[(ia)^2] = f(-a^2) = 0$]

$$= x \cdot \frac{1}{\dfrac{d}{d\mathrm{D}}[f(\mathrm{D}^2)]} e^{iax} \qquad \text{[Using Th. I]}$$

$$= x \cdot \frac{1}{\dfrac{d}{d\mathrm{D}}[f(\mathrm{D}^2)]} (\cos ax + i \sin ax).$$

Equating imaginary parts, $\dfrac{1}{f(\mathrm{D}^2)} \sin ax = x \cdot \dfrac{1}{\dfrac{d}{d\mathrm{D}}[f(\mathrm{D}^2)]} \sin ax$

Equating real parts, $\dfrac{1}{f(\mathrm{D}^2)} \cos ax = x \cdot \dfrac{1}{\dfrac{d}{d\mathrm{D}}[f(\mathrm{D}^2)]} \cos ax.$

☞**Note. Thus, in the case of failure, we have**

$$\mathbf{x \cdot \frac{1}{\text{diff. co-eff. of denom. w.r.t. D}} e^{ax}} \textbf{ (or sin ax or cos ax) as the case may be.}$$

☞**If by using this rule, denominator is again zero, repeat the rule.**

**Example 1.** *Solve the following* :

(*i*) $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = e^x$ (*ii*) $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + y = e^x$

(*iii*) $\dfrac{d^2y}{dx^2} - 4y - (1 + e^x)^2 = 0$ (*iv*) $\dfrac{d^3y}{dx^3} - 3\dfrac{d^2y}{dx^2} + 4\dfrac{dy}{dx} - 2y = e^x.$

**Sol.** (*i*) Given equation in symbolic form is

$$(D^2 - 3D + 2)y = e^x$$

Auxiliary equation is $D^2 - 3D + 2 = 0$

or $(D-1)(D-2) = 0 \quad \therefore \quad D = 1, 2$

$$\text{C.F.} = c_1e^x + c_2e^{2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2}\, e^x \qquad = \frac{1}{1-3+2}\, e^x = \frac{1}{0}\, e^x$$

Case of failure

$$= x \,.\, \frac{1}{2D-3}\, e^x$$

$$= x \,.\, \frac{1}{2.1-3}\, e^x = -\,xe^x \qquad \therefore \quad \frac{1}{f(D)}\, e^{ax} = x \,.\, \frac{1}{\frac{d}{dD}[f(D)]}\, e^{ax}$$

$\therefore$ The complete solution is $y = c_1e^x + c_2e^{2x} - xe^x$.

(*ii*) Given equation in symbolic form is

$$(D^3 - D^2 - D + 1)y = e^x$$

Auxiliary equation is $D^3 - D^2 - D + 1 = 0$

or $D^2(D-1) - (D-1) = 0$ or $(D-1)(D^2-1) = 0$

or $(D-1)^2(D+1) = 0 \quad \therefore \quad D = 1, 1, -1$

$\therefore$ $\text{C.F.} = (c_1 + c_2x)e^x + c_3e^{-x}$

$$\text{P.I.} = \frac{1}{D^3 - D^2 - D + 1}\, e^x \qquad | \text{Case of failure}$$

$$= x\, \frac{1}{3D^2 - 2D - 1}\, e^x \qquad | \text{Case of failure}$$

$$= x^2 \,.\, \frac{1}{6D-2}\, e^x = x^2 \,.\, \frac{1}{6.1-2}\, e^x = \frac{1}{4}\, x^2e^x$$

$\therefore$ The complete solution is $y = (c_1 + c_2x)e^x + c_3e^{-x} + \frac{1}{4}\, x^2e^x$.

(*iii*) Given equation in symbolic form is $(D^2 - 4)y = (1 + e^x)^2$

Auxiliary equation is $D^2 - 4 = 0 \quad \therefore \quad D = \pm 2$

$$\text{C.F.} = c_1e^{2x} + c_2e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^2-4}\,(1+e^x)^2 = \frac{1}{D^2-4}\,(1 + 2e^x + e^{2x})$$

$$= \frac{1}{D^2-4}\, e^{0x} + 2 \,.\, \frac{1}{D^2-4}\, e^x + \frac{1}{D^2-4}\, e^{2x}$$

$$= \frac{1}{0-4}\, e^{0x} + 2\, \frac{1}{1-4}\, e^x + x \,.\, \frac{1}{2D}\, e^{2x}$$

$$= -\frac{1}{4} - \frac{2}{3}\, e^x + \frac{1}{2}\, x \int e^{2x}\, dx \qquad \Big|\ \because\ \frac{1}{D} \equiv \text{integration}$$

$$= -\frac{1}{4} - \frac{2}{3}\, e^x + \frac{1}{4}\, xe^{2x}$$

$\therefore$ The complete solution is

$$y = c_1e^{2x} + c_2e^{-2x} - \frac{1}{4} - \frac{2}{3}e^x + \frac{1}{4}xe^{2x}.$$

(*iv*) Given equation in symbolic form is

$$(D^3 - 3D^2 + 4D - 2)y = e^x$$

Auxiliary equation is $D^3 - 3D^2 + 4D - 2 = 0$

Putting $D = 1$, L.H.S. of A.E. = 0

$\therefore$ $D - 1$ is a factor of L.H.S. of A.E.

$\therefore$ By synthetic division

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 4 & -2 \\ & & 1 & -2 & 3 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

$\therefore$ A.E. may be written as $(D - 1)(D^2 - 2D + 2) = 0$

$$\therefore \qquad D = 1, \frac{2 \pm \sqrt{4-8}}{2} = 1, 1 \pm i$$

$$\text{C.F.} = c_1e^x + e^x(c_2 \cos x + c_3 \sin x)$$

$$\text{P.I.} = \frac{1}{D^3 - 3D^2 + 4D - 2}e^x \qquad \text{| Case of failure}$$

$$= x \cdot \frac{1}{3D^2 - 6D + 4}e^x = x \cdot \frac{1}{3 \cdot 1^2 - 6 \cdot 1 + 4}e^x = xe^x$$

$\therefore$ The complete solution is $y = c_1e^x + e^x(c_2 \cos x + c_3 \sin x) + xe^x$.

**Example 2.** *Solve the following :*

(*i*) $\frac{d^3y}{dx^3} + y = 3 + e^{-x} + 5e^{2x}$ (*ii*) $(D^2 - a^2)y = e^{ax} + e^{nx}$

(*iii*) $\frac{d^2y}{dx^2} + a^2y = \sin ax$ (Delhi 1997) (*iv*) $\frac{d^2y}{dx^2} + 4y = e^x + \sin 2x$

(*v*) $(D^2 - 3D + 2)y = \cosh x$ (*vi*) $(D^2 + 4D + 4)y = e^{2x} + e^{-2x}$.

**Sol.** (*i*) Given equation in symbolic form is $(D^3 + 1)y = 3 + e^{-x} + 5e^{2x}$

Auxiliary equation is $D^3 + 1 = 0$ or $(D + 1)(D^2 - D + 1) = 0$

$$\Rightarrow \qquad D = -1, \frac{1 \pm \sqrt{1-4}}{2} = -1, \frac{1}{2} \pm i \cdot \frac{\sqrt{3}}{2}$$

$$\text{C.F.} = c_1e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right)$$

$$\text{P.I.} = \frac{1}{D^3 + 1}(3 + e^{-x} + 5e^{2x}) = 3 \cdot \frac{1}{D^3 + 1}e^{0x} + \frac{1}{D^3 + 1}e^{-x} + 5 \cdot \frac{1}{D^3 + 1}e^{2x}$$

$$= 3 \cdot \frac{1}{0 + 1}e^{0x} + x \cdot \frac{1}{3D^2}e^{-x} + 5 \cdot \frac{1}{2^3 + 1}e^{2x}$$

$$= 3 + \frac{x}{3} \cdot \frac{1}{(-1)^2}e^{-x} + \frac{5}{9}e^{2x} = 3 + \frac{xe^{-x}}{3} + \frac{5}{9} - e^{2x}$$

$\therefore$ The complete solution is

$$y = c_1e^{-x} + e^{\frac{1}{2}x}\left(c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x\right) + 3 + \frac{xe^{-x}}{3} + \frac{5}{9}e^{2x}.$$

(*ii*) Given equation is $(D^2 - a^2)y = e^{ax} + e^{nx}$

Auxiliary equation is $D^2 - a^2 = 0 \Rightarrow D = \pm a$

$$\therefore \quad \text{C.F.} = c_1e^{ax} + c_2e^{-ax}$$

$$\text{P.I.} = \frac{1}{D^2 - a^2}(e^{ax} + e^{nx}) = \frac{1}{D^2 - a^2}e^{ax} + \frac{1}{D^2 - a^2}e^{nx}$$

$$= x \,.\, \frac{1}{2D}e^{ax} \div \frac{1}{n^2 - a^2}e^{nx} = \frac{x}{2}\int e^{ax}\,dx + \frac{1}{n^2 - a^2}e^{nx}$$

$$= \frac{xe^{ax}}{2a} + \frac{1}{n^2 - a^2}e^{nx}$$

$\therefore$ the complete solution is

$$y = c_1e^{ax} + c_2e^{-ax} + \frac{xe^{ax}}{2a} + \frac{1}{n^2 - a^2}e^{nx}.$$

(*iii*) Given equation in symbolic form is $(D^2 + a^2)y = \sin ax$

A.E. is $D^2 + a^2 = 0 \Rightarrow D = \pm ia$

$$\therefore \quad \text{C.F.} = c_1 \cos ax + c_2 \sin ax$$

$$\text{P.I.} = \frac{1}{D^2 + a^2}\sin ax$$

Case of failure

$$= x \,.\, \frac{1}{2D}\sin ax = \frac{x}{2}\int \sin ax\,dx = -\frac{x\cos ax}{2a}$$

$\therefore$ The complete solution is

$$y = c_1 \cos ax + c_2 \sin ax - \frac{x\cos ax}{2a}.$$

(*iv*) Given equation in symbolic form is $(D^2 + 4)y = e^x + \sin 2x$

Auxiliary equation is $D^2 + 4 = 0 \quad \therefore \quad D = \pm 2i$

$$\text{C.F.} = c_1 \cos 2x + c_2 \sin 2x$$

$$\text{P.I.} = \frac{1}{D^2 + 4}(e^x + \sin 2x) = \frac{1}{D^2 + 4}e^x + \frac{1}{D^2 + 4}\sin 2x$$

$$= \frac{1}{1 + 4}e^x + x \,.\, \frac{11}{2D}\sin 2x = \frac{1}{5}e^x + \frac{x}{2}\int \sin 2x\,dx = \frac{1}{5}e^x - \frac{x}{4}\cos 2x$$

$\therefore$ The complete solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{5}e^x - \frac{x}{4}\cos 2x.$$

(*v*) Given equation is

$$(D^2 - 3D + 2)y = \cosh x$$

Auxiliary equation is $D^2 - 3D + 2 = 0$

or $\quad (D - 1)(D - 2) = 0 \quad \therefore \quad D = 1, 2$

$$\text{C.F.} = c_1e^x + c_2e^{2x}$$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2} \cosh x = \frac{1}{D^2 - 3D + 2}\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{1}{D^2 - 3D + 2} e^x + \frac{1}{2} \cdot \frac{1}{D^2 - 3D + 2} e^{-x}$$

$$= \frac{1}{2} \cdot x \cdot \frac{1}{2D - 3} e^x + \frac{1}{2} \cdot \frac{1}{(-1)^2 - 3(-1) + 2} e^{-x}$$

$$= \frac{x}{2} \cdot \frac{1}{2-3} e^x + \frac{1}{12} e^{-x} = -\frac{x}{2} e^x + \frac{1}{12} e^{-x}$$

$\therefore$ The complete solution is

$$y = c_1e^x + c_2e^{2x} - \frac{x}{2} e^x + \frac{1}{12} e^{-x}.$$

(*vi*) Please try yourself. $\left[\textbf{Ans.}\quad y = (c_1 + c_2 x) e^{-2x} + \frac{e^{2x}}{16} + \frac{x^2}{2} e^{-2x}.\right]$

**Example 3.** *Solve the following* :

(*i*) $\dfrac{d^3y}{dx^3} + 3\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + y = e^{-x}$ (*ii*) $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = e^x + x^2$

(*iii*) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 8(x^2 + e^{2x} + \sin 2x)$ (*iv*) $(D^2 + 1)y = \sin x \sin 2x$

(*v*) $\dfrac{d^2y}{dx^2} + y = e^{-x} + \cos x + x^3 + e^x \sin x.$

**Sol.** (*i*) Given equation in symbolic form is

$$(D^3 + 3D^2 + 3D + 1)y = e^{-x}$$

Auxiliary equation is $D^3 + 3D^2 + 3D + 1 = 0$

or $$(D + 1)^3 = 0 \quad \therefore \quad D = -1, -1, -1$$

$$\text{C.F.} = (c_1 + c_2 x + c_3x^2)e^{-x}$$

$$\text{P.I.} = \frac{1}{(D+1)^3} e^{-x} \qquad \Big|\ \text{Case of failure}$$

$$= x \cdot \frac{1}{3(D+1)^2} e^{-x} \qquad \Big|\ \text{Case of failure}$$

$$= x^2 \cdot \frac{1}{6(D+1)} e^{-x} \qquad \Big|\ \text{Case of failure}$$

$$= x^3 \cdot \frac{1}{6} e^{-x}$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2x + c_3x^2)e^{-x} + \frac{x^3}{6} e^{-x}.$$

(*ii*) Given equation in symbolic form is $(D^2 + 2D + 1)y = e^x + x^2$

Auxiliary equation is $D^2 + 2D + 1 = 0$

or $$(D + 1)^2 = 0 \Rightarrow D = -1, -1$$

$\therefore$ $$\text{C.F.} = (c_1 + c_2 x)e^{-x}$$

$$\text{P.I.} = \frac{1}{(D+1)^2} e^x + \frac{1}{(D+1)^2} x^2 = \frac{1}{(1+1)^2} e^x + (1 + D)^{-2} x^2$$

$$= \frac{1}{4} e^x + (1 - 2D + 3D^2 + \ldots\ldots) x^2$$

$$= \frac{1}{4} e^x + [x^2 - 2D(x^2) + 3D^2(x^2)] = \frac{1}{4} e^x + [x^2 - 4x + 6]$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x)e^{-x} + \frac{1}{4} e^x + (x^2 - 4x + 6).$$

(*iii*) Given equation in symbolic form is

$$(D^2 - 4D + 4)y = 8(x^2 + e^{2x} + \sin 2x)$$

Auxiliary equation is $D^2 - 4D + 4 = 0$

or $(D - 2)^2 = 0 \Rightarrow D = 2, 2$

$$\text{C.F.} = (c_1 + c_2 x)e^{2x}$$

$$\text{P.I.} = \frac{1}{(D-2)^2} [8(x^2 + e^{2x} + \sin 2x)]$$

$$= \frac{8}{(D-2)^2} x^2 + \frac{8}{(D-2)^2} e^{2x} + \frac{8}{(D-2)^2} \sin 2x$$

$$= \frac{8}{4\left(1 - \frac{D}{2}\right)^2} x^2 + 8x \cdot \frac{1}{2(D-2)} e^{2x} + \frac{8}{D^2 - 4D + 4} \sin 2x$$

$$= 2\left(1 - \frac{D}{2}\right)^{-2} x^2 + 4x \cdot \frac{1}{D-2} e^{2x} + \frac{8}{-4 - 4D + 4} \sin 2x$$

$$= 2\left(1 + 2 \cdot \frac{D}{2} + 3 \cdot \frac{D^2}{4} + \ldots\ldots\right) x^2 + 4x^2 \cdot \frac{1}{1} e^{2x} - \frac{2}{D} \sin 2x$$

$$= 2\left[x^2 + D(x^2) + \frac{3}{3} D^2(x^2)\right] + 4x^2 e^{2x} - 2 \int \sin 2x \, dx$$

$$= 2\left[x^2 + 2x + \frac{3}{2}\right] + 4x^2 e^{2x} + \cos 2x$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x)e^{2x} + 2x^2 + 4x + 3 + 4x^2 e^{2x} + \cos 2x.$$

**Solve.** $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = x^2 + e^{2x}$

Please try yourself. $\left[\textbf{Ans.} \quad y = (c_1 + c_2 x)e^{2x} + \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right) + \frac{1}{2} x^2 e^{2x}\right]$

(*iv*) Given equation is $(D^2 + 1)y = \sin x \sin 2x$

Auxiliary equation is $D^2 + 1 = 0 \quad \therefore \quad D = \pm i$

$$\text{C.F.} = c_1 \cos x + c_2 \sin x$$

$$\text{P.I.} = \frac{1}{D^2+1} \sin x \sin 2x = \frac{1}{2}\cdot\frac{1}{D^2+1} (2 \sin 2x \sin x)$$

$$= \frac{1}{2}\cdot\frac{1}{D^2+1} (\cos x - \cos 3x) = \frac{1}{2}\cdot\frac{1}{D^2+1} \cos x - \frac{1}{2}\cdot\frac{1}{D^2+1} \cos 3x$$

$$= \frac{1}{2}\cdot x\cdot\frac{1}{2D} \cos x - \frac{1}{2}\cdot\frac{1}{-9+1} \cos 3x$$

$$= \frac{x}{4}\int \cos x\, dx + \frac{1}{16} \cos 3x = \frac{x}{4} \sin x + \frac{1}{16} \cos 3x$$

$\therefore$ The complete solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{x}{4} \sin x + \frac{1}{16} \cos 3x.$$

(*v*) Given equation in symbolic form is

$$(D^2 + 1)y = e^{-x} + \cos x + x^3 + e^x \sin x$$

Auxiliary equation is $D^2 + 1 = 0 \quad \therefore \quad D = \pm i$

$$\text{C.F.} = c_1 \cos x + c_2 \sin x$$

$$\text{P.I.} = \frac{1}{D^2+1} (e^{-x} + \cos x + x^3 + e^x \sin x)$$

$$= \frac{1}{D^2+1} e^{-x} + \frac{1}{D^2+1} \cos x + \frac{1}{D^2+1} x^3 + \frac{1}{D^2+1} e^x \sin x$$

$$= \frac{1}{(-1)^2+1} e^{-x} + x\cdot\frac{1}{2D} \cos x + (1 + D^2)^{-1}x^3 + e^x \frac{1}{(D+1)^2+1} \sin x$$

$$= \frac{1}{2} e^{-x} + \frac{x}{2} \sin x + (1 - D^2 + D^4 \,......)x^3 + e^x \cdot \frac{1}{D^2+2D+2} \sin x$$

$$= \frac{1}{2} e^{-x} + \frac{x}{2} \sin x + x^3 - D^2(x^3) + e^x \cdot \frac{1}{-1+2D+2} \sin x$$

$$= \frac{1}{2} e^{-x} + \frac{x}{2} \sin x + x^3 - 6x + e^x \cdot \frac{2D-1}{(2D+1)(2D-1)} \sin x$$

$$= \frac{1}{2} e^{-x} + \frac{x}{2} \sin x + x^3 - 6x + e^x \cdot \frac{2D-1}{4D^2-1} \sin x$$

$$= \frac{1}{2} e^{-x} + \frac{x}{2} \sin x + x^3 - 6x + e^x \cdot \frac{2D-1}{4(-1)-1} \sin x$$

$$= \frac{1}{2} e^{-x} + \frac{x}{2} \sin x + x^3 - 6x - \frac{1}{5} e^x [(2D(\sin x) - \sin x]$$

$$= \frac{1}{2} e^{-x} + \frac{x}{2} \sin x + x^3 - 6x - \frac{1}{5} e^x (2 \cos x - \sin x)$$

$\therefore$ The complete solution is

$$y = c_1 \cos x + c_2 \sin x + \frac{1}{2} e^{-x} + \frac{x}{2} \sin x + x^3 - 6x - \frac{1}{5} e^x (2 \cos x - \sin x).$$

**Example 4.** *Solve :*

*(i)* $(D-1)^2(D+1)^2 y = \sin^2 \frac{x}{2} + e^x + x$ *(ii)* $\frac{d^2y}{dx^2} - 4y = \cosh (2x-1) + 3^x$

*(iii)* $(D+2)(D-1)^2 y = e^{-2x} + 2 \sinh x.$

**Sol.** (*i*) Given equation is $(D-1)^2(D+1)^2 y = \sin^2 \frac{x}{2} + e^x + x.$

Auxiliary equation is $(D-1)^2(D+1)^2 = 0$

$\therefore$ $D = 1, 1, -1, -1$

$$\text{C.F.} = (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x}$$

$$\text{P.I.} = \frac{1}{(D-1)^2(D+1)^2}\left(\sin^2 \frac{x}{2} + e^x + x\right) = \frac{1}{(D^2-1)^2} \cdot \left[\frac{1-\cos x}{2} + e^x + x\right]$$

$$= \frac{1}{2} \cdot \frac{1}{(D^2-1)^2} e^{0x} - \frac{1}{2} \cdot \frac{1}{(D^2-1)^2} \cos x + \frac{1}{(D^2-1)^2} e^x + \frac{1}{(D^2-1)^2} x$$

$$= \frac{1}{2} \cdot \frac{1}{(0-1)^2} - \frac{1}{2} \cdot \frac{1}{(-1-1)^2} \cos x + x \cdot \frac{1}{2(D^2-1)\,2D} e^x + \frac{1}{(1-D^2)^2} x$$

$$= \frac{1}{2} - \frac{1}{8} \cos x + \frac{x}{4} \cdot \frac{1}{D^3 - D} e^x + (1-D^2)^{-2} x$$

$$= \frac{1}{2} - \frac{1}{8} \cos x + \frac{x^2}{4} \cdot \frac{1}{3D^3 - 1} e^x + (1 + 2D^2 + \ldots\ldots)x$$

$$= \frac{1}{2} - \frac{1}{8} \cos x + \frac{x^2}{4} \cdot \frac{1}{3(1)^2 - 1} e^x + x = \frac{1}{2} - \frac{1}{8} \cos x + \frac{x^2}{8} e^x + x$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2x)e^x + (c_3 + c_4x)\, e^{-x} + \frac{1}{2} - \frac{1}{8} \cos x + \frac{x^2}{8} e^x + x.$$

(*ii*) Given equation in symbolic form is $(D^2 - 4)y = \cosh (2x-1) + 3^x$

Its A.E. is $D^2 - 4 = 0 \quad \Rightarrow \quad D = \pm 2$

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{-2x}$$

$$\text{P.I.} = \frac{1}{D^2-4} [\cosh (2x-1) + 3^x]$$

$$= \frac{1}{D^2-4} \left[\frac{e^{2x-1} + e^{-(2x-1)}}{2} + e^{\log 3^x}\right] \qquad \left(\because \cosh t = \frac{e^t + e^{-t}}{2} \text{ and } u = e^{\log u}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^2-4} e^{2x-1} + \frac{1}{D^2-4} e^{-(2x-1)}\right] + \frac{1}{D^2-4} e^{x \log 3}$$

$$= \frac{1}{2}\left[x \cdot \frac{1}{2D} e^{2x-1} + x \frac{1}{2D} e^{-(2x-1)}\right] + \frac{1}{(\log 3)^2 - 4} e^{x \log 3} \qquad \text{(Case of failure)}$$

$$= \frac{x}{4}\left[\int e^{2x-1}\,dx + \int e^{-(2x-1)}\,dx\right] + \frac{3^x}{(\log 3)^2 - 4}$$

$$= \frac{x}{4}\left[\frac{e^{2x-1}}{2} + \frac{e^{-(2x-1)}}{-2}\right] + \frac{3^x}{(\log 3)^2 - 4} = \frac{x}{4}\left[\frac{e^{2x-1} - e^{-(2x-1)}}{2}\right] + \frac{3^x}{(\log 3)^2 - 4}$$

$$= \frac{x}{4}\sinh(2x-1) + \frac{3^x}{(\log 3)^2 - 4}$$

Hence the complete solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} + \frac{x}{4}\sinh(2x-1) + \frac{3^x}{(\log 3)^2 - 4}.$$

(*iii*) Given equation is $(D+2)(D-1)^2 y = e^{-2x} + 2\sinh x$

Its A.E. is $(D+2)(D-1)^2 = 0 \Rightarrow D = -2, 1, 1$

$$\text{C.F.} = c_1 e^{-2x} + (c_2 + c_3 x)e^x$$

$$\text{P.I.} = \frac{1}{(D+2)(D-1)^2}(e^{-2x} + 2\sinh x)$$

$$= \frac{1}{(D+2)(D-1)^2}(e^{-2x} + e^x - e^{-x}) \qquad \left[\because \sinh x = \frac{e^x - e^{-x}}{2}\right]$$

Now $$\frac{1}{(D+2)(D-1)^2}e^{-2x} = \frac{1}{D+2}\left[\frac{1}{(D-1)^2}e^{-2x}\right]$$

$$= \frac{1}{D+2}\left[\frac{1}{(-2-1)^2}e^{-2x}\right] = \frac{1}{9}\cdot\frac{1}{D+2}e^{-2x} \qquad \text{[Case of failure]}$$

$$= \frac{1}{9}\cdot x\cdot\frac{1}{1}e^{-2x} = \frac{x}{9}e^{-2x}$$

$$\frac{1}{(D+2)(D-1)^2}e^x = \frac{1}{(D-1)^2}\left[\frac{1}{D+2}e^x\right] = \frac{1}{(D-1)^2}\left[\frac{1}{1+2}e^x\right]$$

$$= \frac{1}{3}\cdot\frac{1}{(D-1)^2}e^x \qquad \text{[Case of failure]}$$

$$= \frac{1}{3}\cdot x\cdot\frac{1}{2(D-1)}e^x \qquad \text{[Case of failure]}$$

$$= \frac{x}{6}\cdot x\cdot\frac{1}{1}e^x = \frac{x^2}{6}e^x$$

$$\frac{1}{(D+2)(D-1)^2}e^{-x} = \frac{1}{(-1+2)(-1-1)^2}e^{-x} = \frac{1}{4}e^{-x}$$

$\therefore$ $$\text{P.I.} = \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^x + \frac{1}{4}e^{-x}$$

Hence the complete solution is

$$y = c_1 e^{-2x} + (c_2 + c_3 x)e^x + \frac{x}{9}e^{-2x} + \frac{x^2}{6}e^x + \frac{1}{4}e^{-x}.$$

**Example 5.** *Solve the following :*

(i) $\dfrac{d^2y}{dx^2} + 4y = x^2 + \cos 2x$ (ii) $\dfrac{d^2y}{dx^2} - y = (1 + x^2)e^x$

(iii) $\dfrac{d^3y}{dx^3} + \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + y = 2e^{-x} + x^2$ (iv) $(D^3 - D^2 + 3D + 5)y = e^x \cos 2x.$

**Sol.** (*i*) Given equation in symbolic form is

$$(D^2 + 4)y = x^2 + \cos 2x$$

A.E. is $D^2 + 4 = 0 \Rightarrow D = \pm 2i$

$\therefore$ C.F. $= c_1 \cos 2x + c_2 \sin 2x$

$$\text{P.I.} = \frac{1}{D^2+4}(x^2 + \cos 2x) = \frac{1}{D^2+4}x^2 + \frac{1}{D^2+4}\cos 2x$$

$$= \frac{1}{4\left(1+\frac{D^2}{4}\right)}x^2 + x\,.\frac{1}{2D}\cos 2x = \frac{1}{4}\left(1+\frac{D^2}{4}\right)^{-1}x^2 + \frac{x}{2}\int \cos 2x\,dx$$

$$= \frac{1}{4}\left[1-\frac{D^2}{4}......\right]x^2 + \frac{x}{2}.\frac{\sin 2x}{2} = \frac{1}{4}\left[x^2 - \frac{1}{4}D^2(x^2)\right] + \frac{x}{4}\sin 2x$$

$$= \frac{1}{4}\left(x^2 - \frac{1}{2}\right) + \frac{x}{4}\sin 2x$$

$\therefore$ The complete solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + \frac{1}{4}\left(x^2 - \frac{1}{2}\right) + \frac{x}{4}\sin 2x.$$

(*ii*) Given equation in symbolic form is $(D^2 - 1)y = (1 + x^2)e^x$

A.E. is $D^2 - 1 = 0 \Rightarrow D = \pm 1$

$\therefore$ C.F. $= c_1e^x + c^2e^{-x}$

$$\text{P.I.} = \frac{1}{D^2-1}[(1+x^2)e^x] = e^x\,.\frac{1}{(D+1)^2-1}(1+x^2)$$

$$= e^x\,.\frac{1}{D^2+2D}(1+x^2) = e^x\,.\frac{1}{2D\left(1+\frac{D}{2}\right)}(1+x^2)$$

$$= \frac{1}{2}e^x\,.\frac{1}{D}\left(1+\frac{D}{2}\right)^{-1}(1+x^2) = \frac{1}{2}e^x\,.\frac{1}{D}\left(1-\frac{D}{2}+\frac{D^2}{4}......\right)(1+x^2)$$

$$= \frac{1}{2}e^x\,.\frac{1}{D}\left[(1+x^2) - \frac{1}{2}D(1+x^2) + \frac{1}{4}D^2(1+x^2)\right]$$

$$= \frac{1}{2}e^x\,.\frac{1}{D}\left[(1+x^2) - \frac{1}{2}(2x) + \frac{1}{4}(2)\right]$$

$$= \frac{1}{2}e^x\,.\int\left(x^2 - x + \frac{3}{2}\right)dx = \frac{1}{2}e^x\left(\frac{x^3}{3} - \frac{x^2}{2} + \frac{3}{2}x\right) = \frac{1}{12}xe^x(2x^2 - 3x + 9)$$

$\therefore$ The complete solution is

$$y = c_1e^x + c_2e^{-x} + \frac{1}{12}\, x\, e^x(2x^2 - 3x + 9).$$

(*iii*) Given equation in symbolic form is

$$(D^3 + D^2 + D + 1)y = 2e^{-x} + x^2$$

A.E. is $D^3 + D^2 + D + 1 = 0$ or $(D + 1)(D^2 + 1) = 0 \Rightarrow D = -1, \pm i$

$\therefore$ C.F. $= c_1e^{-x} + c_2 \cos x + c_3 \sin x$

$$\text{P.I.} = \frac{1}{D^3 + D^2 + D + 1}(2e^{-x} + x^2) = 2 \,.\, \frac{1}{D^3 + D^2 + D + 1}\, e^{-x} + \frac{1}{D^3 + D^2 + D + 1}\, x^2$$

$$= 2x \,.\, \frac{1}{3D^2 + 2D + 1}\, e^{-x} + [1 + (D + D^2 + D^3)]^{-1}\, x^2$$

$$= 2x \,.\, \frac{1}{3(-1)^2 + 2(-1) + 1}\, e^{-x} + [1 - (D + D^2 + D^3) + (D + D^2 + D^3)^2 \,......]x^2$$

$$= x\, e^{-x} + [1 - (D + D^2) + D^2 \,......]x^2 = x\, e^{-x} + [1 - D]x^2 = x\, e^{-x} + (x^2 - 2x)$$

$\therefore$ the complete solution is

$$y = c_1e^{-x} + c_2 \cos x + c_3 \sin x + x\, e^{-x} + (x^2 - 2x)$$

(*iv*) Given equation is $(D^3 - D^2 + 3D + 5)y = e^x \cos 2x$

A.E. is $D^3 - D^2 + 3D + 5 = 0$ or $(D + 1)(D^2 - 2D + 5) = 0$

$$\therefore \quad D = -1, \frac{2 \pm \sqrt{4 - 20}}{2} = -1, 1 \pm 2i$$

$\therefore$ C.F. $= c_1\, e^{-x} + e^x(c_2 \cos 2x + c_3 \sin 2x)$

$$\text{P.I.} = \frac{1}{D^3 - D^2 + 3D + 5}(e^x \cos 2x)$$

$$= e^x \,.\, \frac{1}{(D + 1)^3 + (D + 1)^2 + 3(D + 1) + 5} \cos 2x$$

$$= e^x \,.\, \frac{1}{D^3 + 2D^2 + 4D + 8} \cos 2x \qquad | \text{ Case of failure}$$

$$= e^x \,.\, x\, \frac{1}{3D^2 + 4D + 4} \cos 2x$$

$$= x\, e^x \,.\, \frac{1}{3(-4) + 4D + 4} \cos 2x = xe^x \,.\, \frac{1}{4(D - 2)} \cos 2x$$

$$= \frac{1}{4}\, xe^x \,.\, \frac{D + 2}{D^2 - 4} \cos 2x = \frac{1}{4}\, xe^x \,.\, \frac{D + 2}{-4 - 4} \cos 2x$$

$$= -\frac{1}{32}\, x\, e^x[D(\cos 2x) + 2 \cos 2x] = -\frac{1}{32}\, x\, e^x(-2 \sin 2x + 2 \cos 2x)$$

$$= \frac{1}{16}\, x\, e^x(\sin 2x - \cos 2x)$$

$\therefore$ The complete solution is

$$y = c_1e^{-x} + e^x(c_2 \cos 2x + c_3 \sin 2x) + \frac{1}{16}\, x\, e^x(\sin 2x - \cos 2x).$$

**Example 6.** *Solve the following :*

(*i*) $\dfrac{d^3y}{dx^3}+\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}-y=e^x+x^3$ (*ii*) $\dfrac{d^3y}{dx^3}-\dfrac{d^2y}{dx^2}-\dfrac{dy}{dx}+y=e^x+\sin 2x$

(*iii*) $\dfrac{d^4y}{dx^4}-y=2\sinh x+\sin x$ (*iv*) $\dfrac{d^2y}{dx^2}-2\dfrac{dy}{dx}+y=e^x(1+xe^x\sin x)$.

**Sol.** (*i*) Given equation in symbolic form is

$$(D^3+D^2-D-1)y=e^x+x^3$$

Auxiliary equation is $D^3+D^2-D-1=0$

$\Rightarrow$ $D^2(D+1)-(D+1)=0 \Rightarrow (D+1)(D^2-1)=0$

$\Rightarrow$ $(D+1)^2(D-1)=0 \quad \therefore \quad D=1,-1,-1$

$$\text{C.F.}=c_1e^x+(c_2+c_3x)e^{-x}$$

$$\text{P.I.}=\frac{1}{D^3+D^2-D-1}(e^x+x^3)=\frac{1}{D^3+D^2-D-1}e^x+\frac{1}{D^3+D^2-D-1}x^3$$

$$=x\,.\,\frac{1}{3D^2+2D-1}e^x-\frac{1}{1+D-D^2-D^3}x^3$$

$$=x\,.\,\frac{1}{3(1)^2+2(1)-1}e^x-[1+(D-D^2-D^3)]^{-1}x^3$$

$$=\frac{1}{4}xe^x+[1-(D-D^2-D^3)+(D-D^2-D^3)^2-(D-D^2-D^3)^3\ \ldots\ldots]x^3$$

$$=\frac{1}{4}xe^x+[1-D+D^2+D^2+D^3-2D^3-D^3\ \ldots\ldots]x^3$$

$$=\frac{1}{4}xe^x+(1-D+2D^2-2D^3\ \ldots\ldots)x^3=\frac{1}{4}x\,e^x+x^3-3x^2+2.6x-2.6$$

$$=\frac{1}{4}x\,e^x+x^3-3x^2+12x-12$$

$\therefore$ The complete solution is

$$y=c_1e^x+(c_2+c_3x)e^{-x}+\frac{1}{4}xe^x+x^3-3x^2+12x-12.$$

(*ii*) Given equation in symbolic form is

$$(D^3-D^2-D+1)y=e^x\sin 2x$$

Auxiliary equation is $D^3-D^2-D+1=0$

$\Rightarrow$ $D^2(D-1)-(D-1)=0 \Rightarrow (D-1)(D^2-1)=0$

$\Rightarrow$ $(D-1)^2(D+1)=0 \Rightarrow D=1,1,-1$

$$\text{C.F.}=(c_1+c_2x)e^x+c_3e^{-x}$$

$$\text{P.I.}=\frac{1}{D^3-D^2-D+1}(e^x+\sin 2x)$$

$$=\frac{1}{D^3-D^2-D+1}e^x+\frac{1}{D^3-D^2-D+1}\sin 2x$$

$$=x\,.\,\frac{1}{3D^2-2D-1}e^x+\frac{1}{(-4)D-(-4)-D+1}\sin 2x$$

$$= x^2 \cdot \frac{1}{6D-2} e^x + \frac{1}{5(1-D)} \sin 2x = x^2 \cdot \frac{1}{6(1)-2} e^x + \frac{1}{5} \cdot \frac{1+D}{1-D^2} \sin 2x$$

$$= \frac{1}{4} x^2 e^x + \frac{1}{5} \cdot \frac{1+D}{1-(-4)} \sin 2x = \frac{1}{4} x^2 e^x + \frac{1}{25} (\sin 2x + 2\cos 2x)$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x)e^x + c_3 e^{-x} + \frac{1}{4} x^2 e^x + \frac{1}{25} (\sin 2x + 2\cos 2x).$$

(*iii*) Given equation in symbolic form is $(D^4 - 1)y = 2 \sinh x + \sin x$

Auxiliary equation is $D^4 - 1 = 0$

$$\Rightarrow \quad (D^2 - 1)(D^2 + 1) = 0 \quad \Rightarrow \quad D = \pm 1, \pm i$$

$$\text{C.F.} = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$$

$$\text{P.I.} = \frac{1}{D^4 - 1} (2 \sinh x + \sin x) = \frac{1}{D^4 - 1} \left( 2 \cdot \frac{e^x - e^{-x}}{2} \right) + \frac{1}{D^4 - 1} \sin x$$

$$= \frac{1}{D^4 - 1} e^x - \frac{1}{D^4 - 1} e^{-x} + \frac{1}{D^4 - 1} \sin x$$

$$= x \cdot \frac{1}{4D^3} e^x - x \cdot \frac{1}{4D^3} e^{-x} + x \cdot \frac{1}{4D^3} \sin x$$

$$= \frac{x}{4} \cdot \frac{1}{(1)^3} e^x - \frac{x}{4} \cdot \frac{1}{(-1)^3} e^{-x} + \frac{x}{4} \cdot \frac{1}{D^2} (-\cos x)$$

$$= \frac{x}{4} e^x + \frac{x}{4} e^{-x} + \frac{x}{4} \frac{1}{D} (-\sin x)$$

$$= \frac{x}{4} (e^x + e^{-x}) + \frac{x}{4} \cdot \cos x = \frac{x}{4} (e^x + e^{-x} + \cos x)$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + \frac{x}{4} (e^x + e^{-x} + \cos x).$$

(*iv*) Given equation in symbolic form is

$$(D^2 - 2D + 1)y = e^x (1 + x e^x \sin x)$$

Auxiliary equation is $D^2 - 2D + 1 = 0 \quad \Rightarrow \quad (D-1)^2 = 0 \quad \Rightarrow \quad D = 1, 1$

$$\text{C.F.} = (c_1 + c_2 x)e^x$$

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} (e^x + x e^{2x} \sin x) = \frac{1}{D^2 - 2D + 1} e^x + \frac{1}{D^2 - 2D + 1} x e^{2x} \sin x$$

$$= x \cdot \frac{1}{2D-2} e^x + e^{2x} \cdot \frac{1}{(D+2)^2 - 2(D+2) + 1} x \sin x$$

$$= x^2 \cdot \frac{1}{2} e^x + e^{2x} \cdot \frac{1}{D^2 + 2D + 1} x \sin x$$

$$= \frac{1}{2} x^2 e^x + e^{2x} \left[ x \cdot \frac{1}{D^2 + 2D + 1} \sin x - \frac{2D + 2}{(D^2 + 2D + 1)} \sin x \right]$$

$$\left[ \because \frac{1}{f(D)} (x \sin x) = x \frac{1}{f(D)} \sin x - \frac{f'(D)}{(f(D))^2} \sin x \right]$$

$$= \frac{1}{2} x^2e^x + e^{2x}\left[x \cdot \frac{1}{-1+2D+1}\sin x - \frac{2(D+1)}{(-1+2D+1)^2}\sin x\right]$$

$$= \frac{1}{2} x^2e^x + e^{2x}\left[\frac{x}{2} \cdot \frac{1}{D}\sin x - \frac{2(D+1)}{4D^2}\sin x\right]$$

$$= \frac{1}{2} x^2e^x + e^{2x}\left[\frac{x}{2}(-\cos x) - \frac{D+1}{2(-1)}\sin x\right]$$

$$= \frac{1}{2} x^2e^x + e^{2x}\left[-\frac{x}{2}\cos x + \frac{1}{2}(\cos x + \sin x)\right]$$

$\therefore$ The complete solution is $y = (c_1 + c_2 x)e^x + \frac{1}{2} x^2 e^x + \frac{1}{2} e^{2x}[(1-x)\cos x + \sin x]$.

**Example 7.** *Solve the following :*

*(i)* $\frac{d^2y}{dx^2} + 4y = 3e^x + \sin 2x + x^2$ *(ii)* $(2D+1)^2 y = 4e^{-\frac{x}{2}}$

*(iii)* $\frac{d^2y}{dx^2} + 2y = \sin \sqrt{2}x$ *(iv)* $(D^2+4)y = \cos 2x$

*(v)* $9\frac{d^2y}{dx^2} + 12\frac{dy}{dx} + 4y = e^{-\frac{2x}{3}}$.

**Sol.** (*i*) Please try yourself. [See example 5 (*i*)]

$$\left[\textbf{Ans. } y = c_1 \cos 2x + c_2 \sin 2x + \frac{3}{5}e^x - \frac{x}{4}\cos 2x + \frac{1}{4}\left(x^2 - \frac{1}{2}\right)\right]$$

(*ii*) Please try yourself. $\left[\textbf{Ans. } y = (c_1 + c_2 x)e^{-\frac{1}{2}x} + \frac{x^2}{2}e^{-\frac{x}{2}}\right]$

(*iii*) Please try yourself. $\left[\textbf{Ans. } y = c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x - \frac{x}{2\sqrt{2}}\cos \sqrt{2}x\right]$

(*iv*) Please try yourself. $\left[\textbf{Ans. } c_1 \cos 2x + c_2 \sin 2x + \frac{x}{4}\sin 2x\right]$

(*v*) The given equation in symbolic form is

$$(9D^2 + 12D + 4)y = e^{-\frac{2x}{3}} \quad \text{or} \quad (3D+2)^2 y = e^{-\frac{2x}{3}}$$

A.E. is $(3D+2)^2 = 0 \quad \therefore \quad D = -\frac{2}{3}, -\frac{2}{3}$

$$\text{C.F.} = (c_1 + c_2 x)\, e^{-\frac{2}{3}x}$$

$$\text{P.I.} = \frac{1}{(3D+2)^2} e^{-\frac{2}{3}x} \qquad | \text{ Case of failure}$$

$$= x.\ \frac{1}{2(3D+2)(3)} e^{-\frac{2}{3}x} = \frac{x}{6} \cdot \frac{1}{3D+2} e^{-\frac{2}{3}x} \qquad | \text{ Case of failure}$$

$$= \frac{x}{6}\left(x \cdot \frac{1}{3} e^{-\frac{2}{3}x}\right) = \frac{x^2}{18} e^{-\frac{2}{3}x}$$

$\therefore$ The complete solution is $y = (c_1 + c_2 x)\, e^{-\frac{2}{3}x} + \dfrac{x^2}{18} e^{-\frac{2}{3}x}$.

**Example 8.** *Solve the following :*

(i) $\dfrac{d^3y}{dx^3} - y = (e^x + 1)^2$ (ii) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 8y = (e^{2x} - 1)^2$

(iii) $\dfrac{d^4y}{dx^4} + \dfrac{d^3y}{dx^3} - 6\dfrac{d^2y}{dx^2} = 3e^{2x}$ (iv) $4\dfrac{d^2y}{dx^2} + 16\dfrac{dy}{dx} - 9y = e^{\frac{x}{2}} + 3\sin\dfrac{x}{4}$

(v) $(D^2 - 2D + 1)y = (e^x + 1)^2$.

**Sol.** (i) Given equation in symbolic form is $(D^3 - 1)y = (e^x + 1)^2$

A.E. is $D^3 - 1 = 0$ or $(D - 1)(D^2 + D + 1) = 0$

$$\therefore \quad D = 1, \frac{-1 \pm i\sqrt{3}}{2}$$

$$\text{C.F.} = c_1 e^x + e^{-\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right)$$

$$\text{P.I.} = \frac{1}{D^3 - 1}(e^x + 1)^2 = \frac{1}{D^3 - 1}(e^{2x} + 2e^x + 1)$$

$$= \frac{1}{D^3 - 1}e^{2x} + 2\frac{1}{D^3 - 1}e^x + \frac{1}{D^3 - 1}e^{0x} = \frac{1}{2^3 - 1}e^{2x} + 2x \cdot \frac{1}{3D^2}e^x + \frac{1}{0 - 1}e^{0x}$$

$$= \frac{1}{7}e^{2x} + \frac{2}{3}x \cdot \frac{1}{1^2}e^x - 1 = \frac{1}{7}e^{2x} + \frac{2}{3}xe^x - 1$$

$\therefore$ The complete solution is

$$y = c_1 e^x + e^{-\frac{1}{2}x}\left(c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x\right) + \frac{1}{7}e^{2x} + \frac{2}{3}xe^x - 1.$$

(ii) Given equation in symbolic form is

$$(D^2 - 6D + 8)y = (e^{2x} - 1)^2$$

A.E. is $D^2 - 6D + 8 = 0$ or $(D - 2)(D - 4) = 0$

$$\therefore \quad D = 2, 4$$

$$\text{C.F.} = c_1 e^{2x} + c_2 e^{4x}$$

$$\text{P.I.} = \frac{1}{D^2 - 6D + 8}(e^{2x} - 1)^2 = \frac{1}{D^2 - 6D + 8}(e^{4x} - 2e^{2x} + 1)$$

$$= \frac{1}{D^2 - 6D + 8}e^{4x} - 2\frac{1}{D^2 - 6D + 8}e^{2x} + \frac{1}{D^2 - 6D + 8}e^{0x}$$

$$= x\frac{1}{2D - 6}e^{4x} - 2x\frac{1}{2D - 6}e^{2x} + \frac{1}{0 - 0 + 8}e^{0x}$$

$$= x\frac{1}{2(4) - 6}e^{4x} - 2x\frac{1}{2(2) - 6}e^{2x} + \frac{1}{8} = \frac{x}{2}e^{4x} + xe^{2x} + \frac{1}{8}$$

$\therefore$ The complete solution is $y = c_1 e^{2x} + c_2 e^{4x} + \dfrac{x}{2}e^{4x} + xe^{2x} + \dfrac{1}{8}$.

(*iii*) Please try yourself. $\left[\textbf{Ans.}\quad y=(c_1+c_2x)+c_3\,e^{2x}+c_4\,e^{-3x}+\dfrac{3x}{20}\,e^{2x}\right]$

(*iv*) Given equation in symbolic form is

$$(4D^2 + 16D - 9)y = e^{x/2} + 3\sin\frac{x}{4}$$

A.E. is $4D^2 + 16D - 9 = 0$

$$\therefore \qquad D = \frac{-16 \pm 20}{8} = \frac{1}{2}, -\frac{9}{2}$$

$$\text{C.F.} = c_1 e^{\frac{1}{2}x} + c_2\, e^{-\frac{9}{2}x}$$

$$\text{P.I.} = \frac{1}{4D^2+16D-9}\left(e^{\frac{x}{2}} + 3\sin\frac{x}{4}\right) = \frac{1}{4D^2+16D-9}\,e^{\frac{x}{2}} + 3\,\frac{1}{4D^2+16D-9}\sin\frac{x}{4}$$

$$= x\,\frac{1}{8D+16}\,e^{\frac{x}{2}} + 3\,\frac{1}{4\left(-\frac{1}{4^2}\right)+16D-9}\sin\frac{x}{4}$$

$$= x\,\frac{1}{8\left(\frac{1}{2}\right)+16}\,e^{\frac{x}{2}} + 3\,.\,\frac{4}{64D-37}\sin\frac{x}{4}$$

$$= \frac{x}{20}\,e^{\frac{x}{2}} + 12\,.\,\frac{64D+37}{(64D)^2-(37)^2}\sin\frac{x}{4} = \frac{x}{20}\,e^{\frac{x}{2}} + 12\,.\,\frac{64D+37}{(64)^2\left(-\frac{1}{4^2}\right)-(37)^2}\sin\frac{x}{4}$$

$$= \frac{x}{20}\,e^{\frac{x}{2}} - \frac{12}{1625}\,(64D+37)\sin\frac{x}{4} = \frac{x}{20}\,e^{\frac{x}{2}} - \frac{12}{1625}\left(64\times\frac{1}{4}\cos\frac{x}{4} + 37\sin\frac{x}{4}\right)$$

$$= \frac{x}{20}\,e^{\frac{x}{2}} - \frac{12}{1625}\left(16\cos\frac{x}{4} + 37\sin\frac{x}{4}\right)$$

$\therefore$ The complete solution is

$$y = c_1 e^{\frac{1}{2}x} + c_2\, e^{-\frac{9}{2}x} + \frac{x}{20}\,e^{\frac{x}{2}} - \frac{12}{1625}\left(16\cos\frac{x}{4} + 37\sin\frac{x}{4}\right).$$

(*v*) Please try yourself. [**Ans.** $y = (c_1 + c_2 x)\,e^x + e^{2x} + x^2 e^x + 1$]

**Example 9.** *Solve the following :*

(*i*) $(D^2 - 3D + 2)y = e^{2x} + x^2$

(*ii*) $\dfrac{d^3y}{dx^3} - \dfrac{d^2y}{dx^2} - 4\,\dfrac{dy}{dx} + 4y = e^{2x}$

(*iii*) $(D^2 - 3D + 2)y = e^{2x} + \sin x$

(*iv*) $(D^2 - 4D + 4)y = e^x\,(1 + e^x)$

(*v*) $\dfrac{d^2y}{dx^2} - 4\,\dfrac{dy}{dx} + 4y = x + e^{2x}.$

**Sol.** (*i*) Please try yourself. $\left[\textbf{Ans.}\quad y = c_1\,e^x + c_2\,e^{2x} + x\,e^{2x} + \dfrac{1}{2}\left(x^2 + 3x + \dfrac{7}{2}\right)\right]$

(*ii*) Please try yourself. $\left[\textbf{Ans.}\quad y = c_1\,e^x + c_2\,e^{2x} + c_3\,e^{-2x} + \dfrac{x}{4}\,e^{2x}\right]$

(*iii*) Please try yourself. $\left[\textbf{Ans. } y = c_1 e^x + c_2 e^{2x} + x e^{2x} + \frac{1}{10}(\sin x + 3\cos x)\right]$

(*iv*) Given equation in symbolic form is $(D-2)^2 y = e^x + e^{2x}$

A.E. is $(D-2)^2 = 0 \quad \therefore \quad D = 2, 2$

$$\text{C.F.} = (c_1 + c_2 x) e^{2x}$$

$$\text{P.I.} = \frac{1}{(D-2)^2} e^x + \frac{1}{(D-2)^2} e^{2x}$$

$$= \frac{1}{(1-2)^2} e^x + x \cdot \frac{1}{2(D-2)} e^{2x} = e^x + \frac{x}{2}\left(\frac{1}{D-2} e^{2x}\right)$$

$$= e^x + \frac{x}{2} \cdot x \frac{1}{1} e^{2x} = e^x + \frac{x^2}{2} e^{2x}$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x) e^{2x} + e^x + \frac{x}{2} e^{2x}.$$

(*v*) Please try yourself. $\left[\textbf{Ans. } y = (c_1 + c_2 x) e^{2x} + \frac{1}{4}(x+1) + \frac{x^2}{2} e^{2x}\right]$

**Example 10.** *Solve the following :*

(*i*) $(D^3 - 5D^2 + 7D - 3)y = e^{2x} \cosh x$ (*ii*) $(D^2 - 1)y = x \sin x + (1 + x^2) e^x$.

**Sol.** (*i*) Given equation in symbolic form is

$$(D^3 - 5D^2 + 7D - 3)y = e^{2x} \cosh x$$

A.E. is $D^3 - 5D^2 + 7D - 3 = 0$ or $(D-1)(D^2 - 4D + 3) = 0$ [$\because$ D = 1 satisfies it]

or $(D-1)^2 (D-3) = 0 \quad \therefore \quad D = 1, 1, 3$

$$\text{C.F.} = (c_1 + c_2 x) e^x + c_3 e^{3x}$$

$$\text{P.I.} = \frac{1}{D^3 - 5D^2 + 7D - 3} e^{2x} \cosh x = \frac{1}{D^3 - 5D^2 + 7D - 3} e^{2x} \left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{1}{2}\left[\frac{1}{D^3 - 5D^2 + 7D - 3} e^{3x} + \frac{1}{D^3 - 5D^2 + 7D - 3} e^x\right]$$

$$= \frac{1}{2}\left[x \frac{1}{3D^2 - 10D + 7} e^{3x} + x \frac{1}{3D^2 - 10D + 7} e^x\right]$$

$$= \frac{x}{2}\left[\frac{1}{3 \times 3^2 - 10 \times 3 + 7} e^{3x} + x^2 \frac{1}{6D - 10} e^x\right]$$

$$= \frac{x}{2}\left[\frac{1}{4} e^{3x} + x^2 \frac{1}{6 \times 1 - 10} e^x\right] = \frac{x}{2}\left(\frac{1}{4} e^{3x} - \frac{x^2}{4} e^x\right) = \frac{x}{8}\left(e^{3x} - x^2 e^x\right)$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x) e^x + \frac{x}{8}\left(e^{3x} - x^2 e^x\right).$$

(*ii*) Given equation in symbolic form is $(D^2 - 1)y = x \sin x + (1 + x^2) e^x$

A.E. is $D^2 - 1 = 0 \quad \therefore \quad D = \pm 1$

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}$$

$$\text{P.I.} = \frac{1}{D^2-1} x \sin x + \frac{1}{D^2-1}(1+x^2)\, e^x$$

$$= \left[x \frac{1}{D^2-1} \sin x + \frac{d}{dD}\left(\frac{1}{D^2-1}\right) \sin x\right] + e^x \frac{1}{(D+1)^2-1}(1+x^2)$$

$$= x \frac{1}{-1^2-1} \sin x - \frac{2D}{(D^2-1)^2} \sin x + e^x \frac{1}{D^2+2D}(1+x^2)$$

$$= -\frac{x}{2} \sin x - \frac{2D}{(-1^2-1)^2} \sin x + e^x \frac{1}{2D\left(1+\frac{D}{2}\right)}(1+x^2)$$

$$= -\frac{x}{2} \sin x - \frac{1}{2} \cos x + \frac{1}{2} e^x \cdot \frac{1}{D}\left(1+\frac{D}{2}\right)^{-1} (1+x^2)$$

$$= -\frac{x}{2} \sin x - \frac{1}{2} \cos x + \frac{1}{2} e^x \cdot \frac{1}{D}\left(1-\frac{D}{2}+\frac{D^2}{4} \ldots\ldots\right)(1+x^2)$$

$$= -\frac{1}{2}(x \sin x + \cos x) + \frac{1}{2} e^x \frac{1}{D}\left[(1+x^2) - \frac{1}{2}(2x) + \frac{1}{4}(2)\right]$$

$$= -\frac{1}{2}(x \sin x + \cos x) + \frac{1}{2} e^x \frac{1}{D}\left(\frac{3}{2} - x + x^2\right)$$

$$= -\frac{1}{2}(x \sin x + \cos x) + \frac{1}{2} e^x \left(\frac{3}{2}x - \frac{x^2}{2} + \frac{x^3}{3}\right)$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{-x} - \frac{1}{2}(x \sin x + \cos x) + \frac{1}{2} e^x \left(\frac{3}{2}x - \frac{x^2}{2} + \frac{x^3}{3}\right).$$

**Example 11.** *(i) Solve the equation* $\frac{d^2y}{dx^2} = a + bx + cx^2$*, given that* $\frac{dy}{dx} = 0$ *when* $x = 0$ *and* $y = d$ *when* $x = 0$

*(ii) If* $\frac{d^2x}{dt^2} + \frac{g}{b}(x-a) = 0$ *(a, b and g being positive numbers) and* $x = a'$ *and* $\frac{dx}{dt} = 0$ *when* $t = 0$*, show that* $x = a + (a'-a)\cos\left(\sqrt{g/b - t}\right)$.

*(iii) Find the solution of the equation* $\frac{d^2y}{dx^2} - y = 1$*, which vanishes when* $x = 0$ *and tends to a finite limit as* $x \to -\infty$.

*(iv) Solve* $\frac{d^2x}{dt^2} + b^2x = k \cos bt$*, given that* $x = 0$ *and* $\frac{dx}{dt} = 0$ *when* $t = 0$.

**Sol.** (*i*) Given equation in symbolic form is $D^2y = a + bx + cx^2$

Auxiliary equation is $D^2 = 0 \quad \therefore \quad D = 0, 0$

$$\text{C.F.} = (c_1 + c_2 x)\, e^{0x} = c_1 + c_2 x$$

$$\text{P.I.} = \frac{1}{D^2}\,(a + bx + cx^2) = \frac{1}{D}\left(\int (a + bx + cx^2)\, dx\right)$$

$$= \frac{1}{D}\left(ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right) = \int \left(ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right) dx = \frac{ax^2}{2} + \frac{bx^3}{6} + \frac{cx^4}{12}$$

$\therefore$ The complete solution is

$$y = c_1 + c_2 x + \frac{ax^2}{2} + \frac{bx^3}{6} + \frac{cx^4}{12} \qquad ...(i)$$

when $x = 0, \quad y = d$ (given)

$\therefore$ From $(i)$, $\quad d = c_1$

Differentiating $(i)$ w.r.t. '$x$',

$$\frac{dy}{dx} = c_2 + ax + \frac{bx^2}{2} + \frac{cx^3}{3} \qquad ...(ii)$$

when $x = 0, \quad \frac{dy}{dx} = 0$ (given)

$\therefore$ From $(ii)$, $\quad 0 = c_2$

Substituting the values of $c_1$ and $c_2$ in $(i)$, the required solution is

$$y = d + \frac{ax^2}{2} + \frac{bx^3}{6} + \frac{cx^4}{12}\,.$$

$(ii)$ Given equation in symbolic form is $\left(D^2 + \frac{g}{b}\right) x = \frac{ag}{b}$ | **Note**

Auxiliary equation is

$$D^2 + \frac{g}{b} = 0 \quad \therefore \quad D = \pm i \sqrt{\frac{g}{b}}$$

$$\therefore \qquad \text{C.F.} = c_1 \cos \sqrt{\frac{g}{b}}\, t + c_2 \sin \sqrt{\frac{g}{b}}\, t$$

$$\text{P.I.} = \frac{1}{D^2 + \frac{g}{b}} \cdot \frac{ag}{b} = \frac{ag}{b} \cdot \frac{1}{D^2 + \frac{g}{b}}\, e^{0t} = \frac{ag}{b} \cdot \frac{1}{0 + \frac{g}{b}}\, e^{0t} = a$$

$\therefore$ The complete solution is

$$x = c_1 \cos \sqrt{\frac{g}{b}}\; t + c_2 \sin \sqrt{\frac{g}{b}}\; t + a \qquad ...(i)$$

when $t = 0, \quad x = a'$ (given)

$\therefore$ From $(i)$, $\quad a' = c_1 \cos 0 + a = c_1 + a \quad \therefore \quad c_1 = a' - a$

Differentiating $(i)$ w.r.t. '$t$',

$$\frac{dx}{dt} = -c_1 \sqrt{\frac{g}{b}} \sin \sqrt{\frac{g}{b}}\, t + c_2 \sqrt{\frac{g}{b}} \cos \sqrt{\frac{g}{b}}\, t \qquad ...(ii)$$

when $\quad t = 0, \quad \frac{dx}{dt} = 0$ (given)

$\therefore$ From (*ii*), $0 = c_2 \sqrt{\frac{g}{b}} \,.\, 1 \quad \therefore \quad c_2 = 0$

Substituting the values of $c_1$ and $c_2$ in (*i*), the required solution is

$$x = a + (a' - a) \cos \left( \sqrt{\frac{g}{b}}\, t \right).$$

(*iii*) Given equation in symbolic form is $(D^2 - 1)y = 1$

Auxiliary equation is $D^2 - 1 = 0 \quad \therefore \quad D = \pm 1$

$$\text{C.F.} = c_1 e^x + c_2 e^{-x}$$

$$\text{P.I.} = \frac{1}{D^2 - 1} \,.\, 1 = \frac{1}{D^2 - 1} e^{0x} = \frac{1}{0 - 1} e^{0x} = -1$$

$\therefore$ The complete solution is $y = c_1 e^x + c_2 x^{-x} - 1$ ...(*i*)

when $x = 0, \quad y = 0$ (given)

$\therefore$ From (*i*) $\quad 0 = c_1 + c_2 - 1$ ...(*ii*)

when $x \to \infty$, $y \to a$ finite limit

$\therefore$ from (*i*) a finite quantity $= c_1 e^{-\infty} + c_2 e^{\infty} - 1$

or a finite quantity $= c_1 \times 0 + c_2 e^{\infty} - 1$ or a finite quantity $+ 1 = c_2 e^{\infty}$

or a finite quantity $= c_2 e^{\infty}$

$$\therefore \quad c_2 = \frac{\text{a finite quantity}}{e^{\infty}} = \frac{\text{a finite quantity}}{\infty}$$

$\therefore \quad c_2 = 0 \quad \therefore$ from (*ii*) $c_1 = 1$

Substituting the values of $c_1$ and $c_2$ in (i), the required solution is $y = e^x - 1$.

(*iv*) Given equation in symbolic form is

$$(D^2 + b^2)x = k \cos bt$$

Auxiliary equation is $\quad D^2 + b^2 = 0 \Rightarrow D = \pm ib$

$$\therefore \quad \text{C.F.} = c_1 \cos bt + c_2 \sin bt$$

$$\text{P.I.} = \frac{1}{D^2 + b^2} (k \cos bt) = k \,.\, \frac{1}{D^2 + b^2} \cos bt$$

$$= kt \,.\, \frac{1}{2D} \cos bt \qquad | \textbf{ Note}$$

$$= \frac{kt}{2} \,.\, \frac{\sin bt}{b} = \frac{kt}{2b} \sin bt$$

$\therefore$ The complete solution is

$$x = c_1 \cos bt + c_2 \sin bt + \frac{kt}{2b} \sin bt \qquad ...(i)$$

when $t = 0, \quad x = 0$ (given)

$\therefore$ From (*i*), $\quad 0 = c_1 \cos 0 = c_1$

Differentiating (*i*) w.r.t. '*t*',

$$\frac{dx}{dt} = -bc_1 \sin bt + bc_2 \cos bt + \frac{bt}{2} \cos bt \qquad ...(ii)$$

when $t = 0, \quad \frac{dx}{dt} = 0$ (given)

$\therefore$ From (*ii*), $0 = bc_2 \cos 0 = bc_2 \quad \therefore \quad c_2 = 0$

Substituting the values of $c_1$ and $c_2$ in (*i*), the required solution is $x = \dfrac{kt}{2b} \sin bt$.

**Note.** If P.I. of an equation does not fall in any of the five cases, then P.I. is determined by using the formula

$$\frac{1}{D-a} X = e^{ax} \int X \, . \, e^{-ax} \, dx$$

Also, $e^{ix} = \cos x + i \sin x$ [Euler's Theorem]

$\Rightarrow$ $\cos x = \text{R.P. of } e^{ix}$ ; $\sin x = \text{I.P. of } e^{ix}$.

**Example 1.** *Solve the following :*

(*i*) $\dfrac{d^2y}{dx^2} + a^2y = \sec ax$ (*ii*) $\dfrac{d^2y}{dx^2} + y = \text{cosec } x.$

**Sol.** (*i*) Given equation in symbolic form is $(D^2 + a^2)y = \sec ax$

A.E. is $D^2 + a^2 = 0 \Rightarrow D = \pm ia$

$\therefore$ $\text{C.F.} = c_1 \cos ax + c_2 \sin ax$

$$\text{P.I.} = \frac{1}{D^2 + a^2} \sec ax = \frac{1}{D^2 - (-a^2)} \sec ax$$

$$= \frac{1}{D^2 - i^2 a^2} \sec ax = \frac{1}{(D + ia)(D - ia)} \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right] \sec ax \qquad \text{[Partial fractions]}$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} \sec ax - \frac{1}{D + ia} \sec ax\right] \qquad ...(i)$$

Now $\dfrac{1}{D - ia} \sec ax = e^{iax} \displaystyle\int \sec ax \, . \, e^{-iax} \, dx$

$$= e^{iax} \int \sec ax \, (\cos ax - i \sin ax) \, dx \qquad \text{[Euler's Theorem]}$$

$$= e^{iax} \int (1 - i \tan ax) \, dx = e^{iax} \left[x - i\left(-\frac{\log \cos ax}{a}\right)\right]$$

$$= e^{iax}\left[x + i \frac{\log \cos ax}{a}\right]$$

Changing $i$ to $-i$, we get

$$\frac{1}{D + ia} \sec ax = e^{-iax}\left[x - i \frac{\log \cos ax}{a}\right]$$

$\therefore$ From (*i*),

$$\text{P.I.} = \frac{1}{2ia}\left[e^{iax}\left(x + i \frac{\log \cos ax}{a}\right) - e^{-ia}\left(x - i \frac{\log \cos ax}{a}\right)\right]$$

$$= \frac{1}{2ia}\left[x(e^{iax} - e^{-iax}) + i \frac{\log \cos ax}{a}(e^{iax} + e^{-iax})\right]$$

$$= \frac{1}{a}\left[x\left(\frac{e^{iax}-e^{-iax}}{2i}\right)+\frac{\log\cos ax}{a}.\left(\frac{e^{iax}+e^{-iax}}{2}\right)\right]$$

$$= \frac{1}{a}\left[x\sin ax+\frac{\log\cos ax}{a}.\cos ax\right]$$

$$\left[\because\ \cos\theta=\frac{e^{i\theta}+e^{-i\theta}}{2};\sin\ \theta=\frac{e^{i\theta}-e^{-i\theta}}{2i}\right]$$

$\therefore$ The complete solution is

$$y=c_1\cos ax+c_2\sin ax+\frac{1}{a}\left[x\sin ax+\cos ax.\frac{\log\cos ax}{a}\right].$$

(*ii*) Given equation in symbolic form is

$$(D^2+1)y=\text{cosec } x$$

A.E. is $\quad D^2+1=0 \Rightarrow D=\pm i$

$\therefore \quad$ C.F. $=c_1\cos x+c_2\sin x$

$$\text{P.I.}=\frac{1}{D^2+1}\text{cosec } x=\frac{1}{D^2-i^2}\text{cosec } x$$

$$=\frac{1}{(D+i)(D-i)}\text{cosec } x=\frac{1}{2i}\left[\frac{1}{D-i}-\frac{1}{D+i}\right]\text{cosec } x$$

[Partial Fractions]

$$=\frac{1}{2i}\left[\frac{1}{D-i}\text{cosec } x-\frac{1}{D+i}\text{cosec } x\right] \quad ...(i)$$

Now $\quad \dfrac{1}{D-i}\text{cosec } x=e^{ix}\displaystyle\int\text{cosec } x.e^{ix}\,dx=e^{ix}\int\text{cosec } x(\cos x-i\sin x)\,dx$

$$=e^{ix}\int(\cot x-i)\,dx=e^{ix}[\log\sin x-ix]$$

Changing $i$ to $-i$, we get

$$\frac{1}{D+i}\text{cosec } x=e^{-ix}[\log\sin x+ix]$$

$\therefore$ From (*i*), P.I. $=\dfrac{1}{2i}[e^{ix}(\log\sin x-ix)-e^{-ix}(\log\sin x+ix)]$

$$=\frac{1}{2i}[\log\sin x(e^{ix}-e^{-ix})-ix(e^{ix}+e^{-ix})$$

$$=\log\sin x\left(\frac{e^{ix}-e^{-ix}}{2i}\right)-x\left(\frac{e^{ix}+e^{-ix}}{2}\right)=\sin x\log\sin x-x\cos x$$

$\therefore$ The complete solution is

$$y=c_1\cos x+c_2\sin x+\sin x\log\sin x-x\cos x.$$

**Example 2.** *Solve the following :*

(*i*) $\dfrac{d^2y}{dx^2}+y=\sec x$ $\qquad$ (*ii*) $\dfrac{d^2y}{dx^2}+9y=\sec 3x.$

**Sol.** Please try yourself. [See example (*i*)]

(*i*) Here $a = 1$ [**Ans.** $y = c_1 \cos x + c_2 \sin x + x \sin x + \cos x \log \cos x$]

(*ii*) Here $a = 3$ $\left[\textbf{Ans. } y = c_1 \cos 3x + c_2 \sin 3x + \frac{1}{3}\left[x \sin 3x + \cos 3x \,.\, \frac{\log \cos 3x}{3}\right]\right]$

**Example 3.** *Solve the following :*

(*i*) $\dfrac{d^2y}{dx^2} + a^2y = x \cos ax$ (*ii*) $\dfrac{d^2y}{dx^2} - y = x^2 \cos x$

(*iii*) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 3x^2 e^{2x} \sin 2x$ (Kanpur 1997) (*iv*) $(D^4 + 2D^2 + 1)y = x^2 \cos x$

(*v*) $\dfrac{d^4y}{dx^4} - y = x \sin x$ (*vi*) $(D^2 + 1)y = x^2 \sin 2x.$

**Sol.** (*i*) Here C.F. $= c_1 \cos ax + c_2 \sin ax$

$$\text{P.I.} = \frac{1}{D^2 + a^2}\, x \cos ax$$

$$= \text{Real Part of } \frac{1}{D^2 + a^2}\, x\, e^{iax} = \text{R.P. of } e^{iax} \,.\, \frac{1}{(D + ia)^2 + a^2}\, x$$

$$= \text{R.P. of } e^{iax} \,.\, \frac{1}{D^2 + 2iaD}\, x = \text{R.P. of } e^{iax} \,.\, \frac{1}{2iaD\left(1 + \frac{1}{2ia} D\right)}\, x$$

$$= \text{R.P. of } e^{iax} \,.\, \frac{1}{2iaD\left(1 - \frac{1}{2a} iD\right)}\, x \qquad \left[\because \ \frac{1}{i} = \frac{i}{i^2} = -i\right]$$

$$= \text{R.P. of } e^{iax} \,.\, \frac{1}{2iaD}\left(1 - \frac{1}{2a} - iD\right)^{-1} x$$

$$= \text{R.P. of } e^{iax} \,.\, \frac{1}{2iaD}\left(1 + \frac{1}{2a} iD \ldots\ldots\right) x$$

$$= \text{R.P. of } e^{iax} \,.\, \frac{1}{2iaD}\left(x + \frac{i}{2a}\right) = \text{R.P. of } e^{iax} \,.\, \frac{1}{2ia} \int \left(x + \frac{i}{2a}\right) dx$$

$$= \text{R.P. of} - \frac{i}{2a}\,(\cos ax + i \sin ax)\left(\frac{x^2}{2} + \frac{i}{2a} x\right)$$

$$= \frac{1}{4a^2}\, x \cos ax + \frac{1}{4a}\, x^2 \sin ax = \frac{x}{4a^2}\,[\cos ax + ax \sin ax]$$

$\therefore$ The complete solution is

$$y = c_1 \cos ax + c_2 \sin ax + \frac{1}{4a^2}\,(\cos ax + ax \sin ax).$$

(*ii*) Here C.F. $= c_1 e^x + c_2 e^{-x}$

$$\text{P.I.} = \frac{1}{D^2 - 1}\,(x^2 \cos x) = \text{Real Part of } \frac{1}{D^2 - 1}\,(x^2 e^{ix})$$

$$= \text{R.P. of } e^{ix} \cdot \frac{1}{(D+i)^2 - 1} x^2 = \text{R.P. of } e^{ix} \cdot \frac{1}{D^2 + 2iD - 2} x^2$$

$$= \text{R.P. of} e^{ix} \cdot \frac{1}{-2\left(1 - iD - \frac{D^2}{2}\right)} x^2 = \text{R.P. of} -\frac{1}{2} e^{ix} \left[1 - \left(iD + \frac{D^2}{2}\right)\right]^{-1} x^2$$

$$= \text{R.P. of} -\frac{1}{2} e^{ix} \left[1 + \left(iD + \frac{D^2}{2}\right) + \left(iD + \frac{D^2}{2}\right)^2 \ldots\ldots\right] x^4$$

$$= \text{R.P. of} -\frac{1}{2} e^{ix} \left[1 + iD + \frac{D^2}{2} + i^2 D^2 \ldots\ldots\right] x^2$$

$$= \text{R.P. of} -\frac{1}{2} e^{ix} \left[1 + iD - \frac{D^2}{2} \ldots\ldots\right] x^2$$

$$= \text{R.P. of} -\frac{1}{2} (\cos x + i \sin x)(x^2 + 2ix - 1)$$

$$= -\frac{1}{2} x^2 \cos x + \frac{1}{2} \cos x + x \sin x = -\frac{1}{2} (x^2 - 1) \cos x + \sin x$$

$\therefore$ The complete solution is

$$y = c_1 e^x + c_2 e^{-x} - \frac{1}{2} (x^2 - 1) \cos x + x \sin x.$$

(*iii*) Here $\quad \text{C.F.} = (c_1 + c_2 x) e^{2x}$

$$\text{P.I.} = \frac{1}{(D-2)^2} (8x^2 e^{2x} \sin 2x)$$

$$= 8e^{2x} \cdot \frac{1}{(D+2-2)^2} (x^2 \sin 2x) = 8e^{2x} \cdot \frac{1}{D^2} x^2 \sin 2x$$

$$= \text{Imaginary Part of } 8e^{2x} \cdot \frac{1}{D^2} x^2 e^{2ix} = \text{I.P. of } 8\, e^{2x} \cdot e^{2ix} \cdot \frac{1}{(D+2i)^2} x^2$$

$$= \text{I.P. of } 8\, e^{2x} \cdot e^{2ix} \cdot \frac{1}{\left[2i\left(1 + \frac{D}{2i}\right)\right]^2} x^2$$

$$= \text{I.P. of } 8\, e^{2x} \cdot e^{2ix} \cdot \frac{-1}{4} \left(1 - \frac{iD}{2}\right)^{-1} x^2$$

$$= \text{I.P. of} -2\, e^{2x} \cdot e^{2ix} \cdot \left[1 + 2 \cdot \frac{iD}{2} + 3\left(\frac{iD}{2}\right)^2 \ldots\ldots\right] x^2$$

$$= \text{I.P. of} -2\, e^{2x} \cdot e^{2ix} \cdot \left[1 + iD - \frac{3}{4} D^2 \ldots\ldots\right] x^2$$

$$= \text{I.P. of} -2\, e^{2x} \cdot e^{2ix} \cdot \left(x^2 + 2ix - \frac{3}{2}\right)$$

$$= \text{I.P. of } -2\,e^{2x}(\cos 2x + i\sin 2x)\left(x^2 + 2ix - \frac{3}{2}\right)$$

$$= -2\,e^{2x}\left[2x\cos 2x + x^2\sin 2x - \frac{3}{2}\sin 2x\right]$$

$$= -e^{2x}\,[((2x^2-3)\sin 2x + 4x\cos 2x]$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 x)\,e^{2x} - \frac{3}{8}\,e^{2x}\,[((2x^2-3)\sin 2x + 4x\cos 2x].$$

(*iv*) Here A.E. is $D^4 + 2D^2 + 1 = 0$ or $(D^2+1)^2 = 0 \quad \Rightarrow \quad D = \pm i, \pm i$

$\therefore$ $\quad$ C.F. $= (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$

$$\text{P.I.} = \frac{1}{(D^2+1)^2}\,x^2\cos x$$

$$= \text{Real Part of } \frac{1}{(D^2+1)}\,x^2 e^{ix} = \text{R.P. of } e^{ix}\,.\,\frac{1}{[(D+i)^2+1]^2}\,x^2$$

$$= \text{R.P. of } e^{ix}\,.\,\frac{1}{(D^2+2iD)^2}\,x^2 = \text{R.P. of } e^{ix}\,.\,\frac{1}{4i^2D^2\left(1+\frac{D}{2i}\right)^2}\,x^2$$

$$= \text{R.P. of } -\frac{1}{4}e^{ix}\,.\,\frac{1}{D^2}\left(1-\frac{iD}{2}\right)^{-2}x^2$$

$$= \text{R.P. of } -\frac{1}{4}e^{ix}\,.\,\frac{1}{D^2}\left[1+2\left(\frac{iD}{2}\right)+3\left(\frac{iD}{2}\right)^2 \ldots\ldots\right]x^2$$

$$= \text{R.P. of } -\frac{1}{4}e^{ix}\,.\,\frac{1}{D^2}\left[1+iD-\frac{3}{4}D^2 \ldots\ldots\right]x^2$$

$$= \text{R.P. of } -\frac{1}{4}e^{ix}\,.\,\frac{1}{D^2}\left(x^2+2ix-\frac{3}{2}\right)$$

$$= \text{R.P. of } -\frac{1}{4}e^{ix}\,.\,\frac{1}{D}\int\left(x^2+2ix-\frac{3}{2}\right)dx$$

$$= \text{R.P. of } -\frac{1}{4}e^{ix}\,.\,\frac{1}{D}\left(\frac{x^3}{3}+ix^2-\frac{3}{2}x\right)$$

$$= \text{R.P. of } -\frac{1}{4}e^{ix}\int\left(\frac{x^3}{3}+ix^2-\frac{3}{2}x\right)dx$$

$$= \text{R.P. of } -\frac{1}{4}(\cos x + i\sin x)\left(\frac{x^4}{12}+\frac{ix^3}{3}-\frac{3}{4}x^2\right)$$

$$= -\frac{1}{4}\left[\left(\frac{x^4}{12}-\frac{3}{4}x^2\right)\cos x - \frac{1}{3}x^3\sin x\right]$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x - \frac{1}{4}\left[\left(\frac{x^4}{12} - \frac{3}{4}x^2\right)\cos x - \frac{1}{3}x^3 \sin x\right]$$

(*v*) Here A.E. is $D^4 - 1 = 0$ or $(D^2 + 1)(D^2 - 1) = 0$

$\Rightarrow$ $F = \pm i, \pm 1$

$\therefore$ $\text{C.F.} = c_1 \cos x + c_2 \sin x + c_3 e^x + c_4 e^{-x}$

$$\text{P.I.} = \frac{1}{D^4 - 1}(x \sin x) = \text{Imaginary Part of } \frac{1}{D^4 - 1} x e^{ix}$$

$$= \text{I.P. of } e^{ix} \cdot \frac{1}{(D + i)^4 - 1} x = \text{I.P. of } e^{ix} \cdot \frac{1}{D^4 + 4i D^3 + 6i^2 D^2 + 4i^3 D + i^4 - 1} x$$

$$= \text{I.P. of } e^{ix} \cdot \frac{1}{D^4 + 4i D^3 - 6D^2 - 4iD} x$$

$$= \text{I.P. of } e^{ix} \cdot \frac{1}{-4i D\left[1 + \frac{3}{2i}D - D^2 - \frac{1}{4i}D^3\right]} x$$

$$= \text{I.P. of} - \frac{1}{4i} e^{ix} \cdot \frac{1}{D}\left[1 - \frac{3}{2}iD - D^2 + \frac{i}{4}D^3\right]^{-1} x$$

$$= \text{I.P. of } \frac{i}{4} e^{ix} \cdot \frac{1}{D}\left[1 - \left(\frac{3}{2}iD + D^2 - \frac{i}{4}D^3\right)\right]^{-1} x$$

$$= \text{I.P. of } \frac{i}{4} e^{ix} \cdot \frac{1}{D}\left[1 + \frac{3}{2}iD \ldots\ldots\right] x = \text{I.P. of } \frac{i}{4} e^{ix} \cdot \int\left(x + \frac{3}{2}i\right) dx$$

$$= \text{I.P. of } \frac{i}{4}(\cos x + i \sin x)\left(\frac{x^2}{2} + \frac{3}{2}ix\right)$$

$$= \frac{1}{8}x^2 \cos x - \frac{3}{8}x \sin x = \frac{x}{8}(x \cos x - 3 \sin x)$$

$\therefore$ The complete solution is

$$y = c_1 \cos x + c_2 \sin x + c_3 e^x + c_4 e^{-x} \frac{x}{8}(x \cos x - 3 \sin x).$$

(*vi*) A.E. is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$\therefore$ $\text{C.F.} = c_1 \cos x + c_2 \sin x$

$$\text{P.I.} = \frac{1}{D^2 + 1} x^2 \sin 2x = \text{I.P. of } \frac{1}{D^2 + 1} x^2 e^{2ix}$$

$$= \text{I.P. of } e^{2ix} \frac{1}{(D + 2i)^2 + 1} x^2 = \text{I.P. of } e^{2ix} \frac{1}{D^2 + 4iD - 3} x^2$$

$$= \text{I.P. of } e^{2ix} \frac{1}{-3\left(1 - \frac{4}{3}iD - \frac{D^2}{3}\right)} x^2 = \text{I.P. of } \frac{e^{2ix}}{-3}\left[1 - \left(\frac{4iD + D^2}{3}\right)\right]^{-1} x^2$$

$$= \text{I.P. of } -\frac{1}{3}\, e^{2ix}\left[1+\left(\frac{4iD+D^2}{3}\right)+\left(\frac{4iD+D^2}{3}\right)^2+\ldots\ldots\right]x^2$$

$$= \text{I.P. of } -\frac{1}{3}\, e^{2ix}\left[1+\frac{4iD}{3}+\left(\frac{1}{3}-\frac{16}{9}\right)D^2+\ldots\ldots\right]x^2$$

$$= \text{I.P. of } -\frac{1}{3}\, e^{2ix}\left[x^2+\frac{4i}{3}(2x)-\frac{13}{9}(2)\right]$$

$$= \text{I.P. of } -\frac{1}{3}(\cos 2x+i\sin 2x)\left[\left(x^2-\frac{26}{9}\right)+\left(\frac{8x}{3}\right)i\right]$$

$$= -\frac{1}{3}\left[\frac{8x}{3}\cos 2x+\left(x^2-\frac{26}{9}\right)\sin 2x\right]$$

$$= -\frac{1}{27}\,[24x\cos 2x+(9x^2-26)\sin 2x]$$

Hence the C.S. is $y = c_1 \cos x + c_2 \sin x - \frac{1}{27}\,[24x \cos 2x + (9x^2 - 26) \sin 2x]$.

**Example 4.** *Solve the following :*

*(i)* $(D^2 + a^2)y = \tan ax$ *(ii)* $(D^2 + 4)y = \tan 2x$.

**Sol.** *(i)* The given equation in symbolic form is $(D^2 + a^2)y = \tan ax$

A.E. is $D^2 + a^2 = 0 \quad \therefore \quad D = \pm ia$

$$\text{C.F.} = e^{0x}(c_1 \cos ax + c_2 \sin ax) = c_1 \cos ax + c_2 \sin ax$$

$$\text{P.I.} = \frac{1}{D^2+a^2}\tan ax = \frac{1}{(D+ia)(D-ia)}\tan ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ia}-\frac{1}{D+ia}\right]\tan x \qquad ...(1)$$

Now $\frac{1}{D-ia}\tan ax = e^{iax}\int e^{-iax}\tan ax\,dx = e^{iax}\int(\cos ax - i\sin ax)\tan ax\,dx$

$$= e^{iax}\int\left(\sin ax - i\,\frac{\sin^2 ax}{\cos ax}\right)dx = e^{iax}\int\left(\sin ax - i\,\frac{1-\cos^2 ax}{\cos ax}\right)dx$$

$$= e^{iax}\int(\sin ax - i\sec ax + i\cos ax)\,dx$$

$$= e^{iax}\left[\frac{-\cos ax}{a}-\frac{i}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)+\frac{i\sin ax}{a}\right]$$

$$= -\frac{1}{a}\,e^{iax}(\cos ax - i\sin ax)-\frac{ie^{iax}}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

$$= -\frac{1}{a}\,e^{iax}(e^{-iax})-\frac{ie^{iax}}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

$$= -\frac{1}{a}-\frac{ie^{iax}}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)$$

# 4

# Application to Geometry and Mechanics

**Remember.** (*a*) **Facts from Cartesian curves f(x, y) = 0**

(*i*) $\dfrac{dy}{dx}$ = *slope of the tangent at any point (x, y).*

(*ii*) Equation of tangent at $(x, y)$ is $Y - y = \dfrac{dy}{dx}(X - x)$

(*iii*) Equation of normal at $(x, y)$ is $Y - y = -\dfrac{dx}{dy}(X - x)$

(*iv*) Length of tangent at $(x, y) = y\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}$

(*v*) Length of normal at $(x, y) = y\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}$

(*vi*) Length of sub-tangent at $(x, y) = y \cdot \dfrac{dx}{dy}$

(*vii*) Length of sub-normal at $(x, y) = y \cdot \dfrac{dy}{dx}$

(*viii*) If 's' denotes the length of arc of a curve from a fixed point on the curve, then

$$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}, \quad \frac{ds}{dx} = \sqrt{1+\left(\frac{dx}{dy}\right)^2}$$

(*b*) **Facts from Polar curves f(r, θ) = 0**

(*i*) At any point $(r, \theta)$ of the curve, the angle $\phi$ between the radius vector and the tangent is given by $\tan\phi = r\dfrac{d\theta}{dr}$.

(*ii*) Length of polar sub-tangent at $(r, \theta) = r^2\dfrac{d\theta}{dr}$

(*iii*) Length of polar sub-normal at $(r, \theta) = \dfrac{dr}{d\theta}$

(*iv*) Length of polar tangent at $(r, \theta) = r\sqrt{1+\left(r\dfrac{d\theta}{dr}\right)^2}$

(*v*) Length of polar normal at $(r, \theta) = r\sqrt{1+\left(\frac{1}{r}\cdot\frac{dr}{d\theta}\right)^2}$

(*vi*) '*p*', the perpendicular from the pole on the tangent at the point $(r, \theta)$ is given by

$$p = r\sin\phi, \text{ also } \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2$$

(*vii*) If $\psi$ is the angle which the tangent at $(r, \theta)$ makes with the initial line, then $\psi = \theta + \phi$.

(*viii*) If '*s*' denotes the length of arc of a curve from a fixed point on the curve, then

$$\frac{ds}{dr} = \sqrt{1+\left(r\frac{d\theta}{dr}\right)^2}, \quad \frac{ds}{d\theta} = \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}.$$

**Example 1.** (*a*) *Find the equation of the curve for which the cartesian sub-tangent is constant.*

(*b*) *Find the equation of the curve for which the cartesian sub-tangent at any point varies as the reciprocal of the square of the abscissa of that point.*

**Sol.** (*a*) Cartesian sub-tangent $= y\,\dfrac{dx}{dy}$

$\therefore$ $$y\,\frac{dx}{dy} = a, \text{ where } a \text{ is constant}$$

Separating the variables $\dfrac{dy}{y} = \dfrac{1}{a}\,dx$

Integrating $\log y = \dfrac{1}{a}x + c = \log e^{\left(\frac{x}{a}+c\right)}$ $\quad \because f(x) = f(x)\log e = \log e^{f(x)}$

or $$y = e^{\frac{x}{a}+c} = e^c \cdot e^{\frac{x}{a}} = Ae^{\frac{x}{a}}$$ [Replacing $e^c$ by A]

$\therefore$ $y = \mathrm{A}e^{\frac{x}{a}}$ is the required equation of the curve.

(*b*) Cartesian sub-tangent at any point $(x, y)$ is $y\,\dfrac{dx}{dy}$

$\therefore$ $$y\,\frac{dx}{dy} \propto \frac{1}{x^2} \quad \text{or} \quad y\,\frac{dx}{dy} = \frac{k}{x^2} \quad \text{or} \quad x^2\,dx = k\,\frac{dy}{y}$$

Integrating $\dfrac{x^3}{3} = k\log y + c$ is the required equation of the curve.

**Example 2.** *Find the equation of the curve for which the cartesian sub-normal is constant.*

**Sol.** Cartesian sub-normal $= y\,\dfrac{dy}{dx}$

$$y\,\frac{dy}{dx} = a, \text{ where a is constant}$$

$\Rightarrow$ $$y\,dy = a\,dx$$

Integrating $$\frac{y^2}{2} = ax + c$$

or $$y^2 = 2ax + 2c \quad \text{or} \quad y^2 = 2ax + A \qquad \text{[Replacing } 2c \text{ by A]}$$
which is the required equation of the curve.

**Example 3.** *Determine the curve whose sub-tangent is n times the abscissa of the point of contact and find the particular curve which passes through the point (2, 3). What is the curve when*

(*i*) $n = 1$ (*ii*) $n = 2$.

**Sol.** If $(x, y)$ is any point on the curve, the sub-tangent at $(x, y) = y \dfrac{dx}{dy}$

$$\therefore \quad y\frac{dx}{dy} = nx \qquad \text{(given)}$$

or $$n\frac{dy}{y} = \frac{dx}{x}$$

Integrating $\quad n \log y = \log x + \log c$

or $$\log y^n = \log cx \quad \text{or} \quad y^n = cx \qquad ...(i)$$

which is the required equation of the family of curves.

Putting $x = 2, y = 3$ in (*i*), we have $3^n = 2c$ or $c = \dfrac{3^n}{2}$

Putting this value of $c$ in (*i*)

$$y^n = \frac{3^n}{2}x \quad \text{or} \quad 2y^n = 3^n x \qquad ...(ii)$$

which is the particular curve passing through the point (2, 3)

Putting $n = 1$ in (*ii*), we have $2y = 3x$

which is a straight line.

Putting $n = 2$ in (*ii*), we have $2y^2 = 9x$

which is a parabola.

**Example 4.** *Find the curve whose sub-tangent is twice the abscissa of the point of contact and which passes through the point (3, 4).*

**Sol.** Please try yourself.

**Example 5.** *Determine the curve in which the length of the sub-normal is proportional to the square of the ordinate.*

**Sol.** If $(x, y)$ is any point on the curve, the sub-normal $= y\dfrac{dy}{dx}$

$$\therefore \quad y\frac{dy}{dx} \propto y^2 \qquad \text{(given)}$$

Let $$y\frac{dy}{dx} = ky^2 \quad \text{or} \quad \frac{dy}{dx} = ky \quad \text{or} \quad \frac{dy}{y} = k\,dx$$

Integrating $\quad \log y = kx + c = \log e^{(kx+c)}$

$$\therefore \quad y = e^{kx+c} = e^c \,.\, e^{kx} = Ae^{kx}$$

which is the required equation of the curve.

**Example 6.** (*a*) *Determine the curve in which the length of the sub-normal is proportional to the square of the abscissa.*

(*b*) *Find the family of curves whose tangents form an angle $\dfrac{\pi}{4}$ with the hyperbolas $xy = c$.*

**Sol.** (*a*) If $(x, y)$ is any point on the curve, then

$$\text{sub-normal} = y\,\frac{dy}{dx}$$

$\therefore$ $$y\,\frac{dy}{dx} \propto x^2 \qquad \text{(given)}$$

Let $$y\,\frac{dy}{dx} = kx^2 \quad \text{or} \quad y\,dy = kx^2\,dx$$

Integrating $$\frac{y^2}{2} = k\,\frac{x^3}{3} + c \quad \text{or} \quad 3y^2 = 2kx^3 + 6c \quad \text{or} \quad 3y^2 = 2kx^3 + A$$

which is the required equation of the curve.

(*b*) Slope of tangent to the curve at any point $(x, y) = \dfrac{dy}{dx} = m_1$

Equation of hyperbola is $xy = c$ or $y = \dfrac{c}{x}$

Slope of tangent to hyperbola $= \dfrac{dy}{dx} = -\dfrac{c}{x^2} = m_2$

Angle between the curve and hyperbola is $\dfrac{\pi}{4}$

$\therefore$ $$\tan\frac{\pi}{4} = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$\Rightarrow$ $$1 = \frac{\dfrac{dy}{dx} + \dfrac{c}{x^2}}{1 + \dfrac{dy}{dx} \times \dfrac{-c}{x^2}} \quad \text{or} \quad 1 - \frac{c}{x^2}\cdot\frac{dy}{dx} = \frac{dy}{dx} + \frac{c}{x^2}$$

or $$\left(1 + \frac{c}{x^2}\right)\frac{dy}{dx} = 1 - \frac{c}{x^2} \quad \text{or} \quad dy = \frac{x^2 - c}{x^2 + c}\,dx$$

or $$dy = \frac{(x^2 + c) - 2c}{x^2 + c}\,dx \quad \text{or} \quad dy = \left(1 - \frac{2c}{x^2 + c}\right)dx$$

Integrating $$y = x - 2c\,.\,\frac{1}{\sqrt{c}}\tan^{-1}\frac{x}{\sqrt{c}} + c'$$

or $$y = x - 2\sqrt{c}\,\tan^{-1}\frac{x}{\sqrt{c}} + c'$$

which is the required equation of the curve.

**Example 7.** *Determine the curve in which the sub-tangent is n times the sub-normal.*

**Sol.** By the given condition $y\,\dfrac{dx}{dy} = ny\,\dfrac{dy}{dx}$

or $$\left(\frac{dx}{dy}\right)^2 = n \quad \text{or} \quad \frac{dx}{dy} = \sqrt{n} \quad \text{or} \quad \sqrt{n}\,dy = dx$$

Integrating $\sqrt{n}\,y = x + c$ which is the required equation of the curve.

**Example 8.** *Find the curve for which the length of subnormal at any point is* $\frac{a^2}{x^3}$. *Find the particular curve which passes through the point (1, 0).*

**Sol.** Length of subnormal $= y\frac{dy}{dx} = \frac{a^2}{x^3}$ (given)

$\Rightarrow \quad y dy = a^2 x^{-3}\, dx$

Integrating $\quad \frac{y^2}{2} = \frac{a^2 x^{-2}}{-2} + c_1 \quad$ or $\quad y^2 = -\frac{a^2}{x^2} + 2c_1$

or $\quad x^2y^2 = -a^2 + cx^2 \quad$ [Replacing $2c_1$ by $c$]

or $\quad x^2y^2 = cx^2 - a^2 \quad$ ...(*i*)

which is the required equation of the family of curves.

If it passes through (1, 0), then

$$0 = c - a^2 \quad \text{or} \quad c = a^2$$

$\therefore$ The required particular curve is $x^2y^2 = a^2(x^2 - 1)$.

**Example 9.** *Find the curve for which the product of subtangent at any point and the abscissa of that point is constant.*

**Sol.** Let $(x, y)$ be any point on the curve. Then

$$y\frac{dx}{dy} \times x = a, \text{ where } a \text{ is constant.}$$

$\Rightarrow \quad a\frac{dy}{y} = x dx$

Integrating $\quad a \log y = \frac{x^2}{2} + c$

$\Rightarrow \quad \log y^a = \log e^{\frac{x^2}{2}+c} \quad \Rightarrow \quad y^a = e^{\frac{x^2}{2}+c}$

or $\quad y^a = e^c \,.\, e^{\frac{x^2}{2}} \quad$ or $\quad y^a = \text{A} \,.\, e^{\frac{x^2}{2}}$

which is the required equation of the curve.

**Example 10.** *Find the curve for which length of the normal at any point is proportional to the square of the ordinate of the point.*

**Sol.** Let $(x, y)$ be any point on the curve. Then

$$y\sqrt{1+\left(\frac{dy}{dx}\right)^2} \propto y^2 \quad \text{(given)}$$

Let $\quad y\sqrt{1+\left(\frac{dy}{dx}\right)^2} = ky^2$

or $\quad 1 + \left(\frac{dy}{dx}\right)^2 = k^2y^2 \quad \Rightarrow \quad \frac{dy}{dx} = \sqrt{k^2y^2 - 1}$

or $\quad \frac{dy}{\sqrt{k^2y^2-1}} = dx \quad$ or $\quad \frac{dy}{k\sqrt{y^2 - \frac{1}{k^2}}} = dx \quad$ or $\quad \frac{dy}{\sqrt{y^2-\frac{1}{k^2}}} = k dx$

Integrating $\cosh^{-1}\left(\dfrac{y}{\frac{1}{k}}\right) = kx + c$ or $ky = \cosh(kx + c)$.

**Example 11.** *Find the curve in which the length of the arc measured from a fixed point A to any point P is proportional to the square root of the abscissa of P.*

**Sol.** Let P be the point $(x, y)$ and arc AP $= s$. Then $s \propto \sqrt{x}$

Let $$s = k\sqrt{x}$$

Differentiating $$\frac{ds}{dx} = \frac{k}{2\sqrt{x}}$$

But $$\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2} \qquad \therefore \qquad \sqrt{1+\left(\frac{dy}{dx}\right)^2} = \frac{k}{2\sqrt{x}}$$

Squaring $$1 + \left(\frac{dy}{dx}\right)^2 = \frac{k^2}{4x} = \frac{a}{x} \qquad \left[\text{Replacing } \frac{k^2}{4} \text{ by } a\right]$$

$$\Rightarrow \qquad \left(\frac{dy}{dx}\right)^2 = \frac{a}{x} - 1 = \frac{a-x}{x}$$

$$\therefore \qquad \frac{dy}{dx} = \sqrt{\frac{a-x}{x}} \quad \text{or} \quad dy = \sqrt{\frac{a-x}{x}}\, dx$$

Integrating $$y + c = \int \sqrt{\frac{a-x}{x}}\, dx$$

Put $x = a\sin^2\theta$ so that
$$dx = 2a\sin\theta\cos\theta\, d\theta$$

$$\therefore \qquad y + c = \int \sqrt{\frac{a - a\sin^2\theta}{a\sin^2\theta}}\, .\, 2a\sin\theta\cos\theta\, d\theta$$
$$= \int \frac{\cos\theta}{\sin\theta}\, .\, 2a\sin\theta\cos\theta\, d\theta = 2a\int \cos^2\theta\, d\theta$$
$$= a\int (1+\cos 2\theta)\, d\theta = a\left[\theta + \frac{\sin 2\theta}{2}\right]$$
$$= a\theta + a\sin\theta\cos\theta$$
$$= a\sin^{-1}\sqrt{\frac{x}{a}} + a\, .\sqrt{\frac{x}{a}}\sqrt{1-\frac{x}{a}} = a\sin^{-1}\frac{x}{a} + \sqrt{ax - x^2}$$

which is the required equation of the curve.

**Example 12.** *Find the curve in which the perpendicular upon the tangent from the foot of the ordinate of the point of contact is constant and equal to a.*

**Sol.** Let P$(x, y)$ be any point on the curve.

Equation of tangent at P is $\quad Y - y = \dfrac{dy}{dx}(X - x)$

or $$X\frac{dy}{dx} - Y + y - x\frac{dy}{dx} = 0 \qquad \ldots(i)$$

$$\therefore \qquad \frac{r}{c} = e^{\theta \cot \alpha} \qquad \text{or} \qquad r = ce^{\theta \cot \alpha}$$

which is the required equation of the curve.

**Example 19.** *Find the curve in which the angle between the radius vector and the tangent is n times the vectorial angle. What is the curve when n = 1 ? when n =* $\frac{1}{2}$ *?*

**Sol.** $\phi$, the angle between the radius vector and tangent at $(r, \theta)$ is given by

$$\tan \phi = r \frac{d\theta}{dr}$$

But $\qquad \phi = n\theta \qquad$ (given)

$$\therefore \qquad \tan n\theta = r \frac{d\theta}{dr} \qquad \text{or} \qquad \frac{dr}{r} = \cot n\theta \, d\theta$$

Integrating $\qquad \log r = \frac{1}{n} \log \sin n\theta + k$

or $\qquad n \log r = \log \sin n\theta + nk$

or $\qquad \log r^n = \log \sin n\theta + \log c \qquad$ where $\log c = nk$

or $\qquad \log r^n = \log e \sin n\theta$

or $\qquad r^n = c \sin n\theta$ which is the required equation of the curve

when $n = 1$, equation of curve is $r = c \sin \theta$ which is a circle

when $n = \frac{1}{2}$, equation of curve is $\sqrt{r} = c \sin \theta/2$

$$\text{or} \qquad r = c^2 \sin^2 \frac{\theta}{2} = c^2 \cdot \frac{1 - \cos \theta}{2} \qquad \text{or} \qquad r = \frac{c^2}{2} (1 - \cos \theta)$$

or $\qquad r = a(1 - \cos \theta) \qquad \left[\text{where } a = \frac{c^2}{2}\right]$

which is a cardioid.

**Example 20.** *Find the equation of the curve in which the angle between the radius vector and the tangent is half the vectorial angle.*

**Sol.** $\phi$, the angle between the radius vector and the tangent at $(r, \theta)$ is given by

$$\tan \phi = r \frac{d\theta}{dr}$$

But $\qquad \phi = \frac{\theta}{2} \qquad$ (given)

$$\therefore \qquad \tan \frac{\theta}{2} = r \frac{d\theta}{dr} \qquad \text{or} \qquad \frac{dr}{r} = \cot \frac{\theta}{2} \, d\theta$$

Integrating $\qquad \log r = 2 \log \sin \frac{\theta}{2} + \log c$

or $\qquad \log r = \log \sin^2 \frac{\theta}{2} + \log c \quad$ or $\quad \log r = \log c \sin^2 \frac{\theta}{2}$

or $\qquad r = c \sin^2 \frac{\theta}{2} = \frac{c}{2} (1 - \cos \theta)$

or $\qquad r = a(1 - \cos \theta) \qquad \left[\text{where } a = \frac{c}{2}\right]$

which is a cardioid.

**Example 21.** (*a*) *Find the equation of the curve in which the angle between the radius vector and tangent is supplementary of half the vectorial angle.*

(*b*) *Find the curve for which the normal makes equal angles with the radius vector and the initial line.*

**Sol.** Here $\phi = \pi - \dfrac{\theta}{2}$

$\therefore$ $\tan\phi = r\dfrac{d\theta}{dr}$ becomes

$\therefore$ $\tan\left(\pi - \dfrac{\theta}{2}\right) = r\dfrac{d\theta}{dr}$ or $-\tan\dfrac{\theta}{2} = r\dfrac{d\theta}{dr}$ or $\dfrac{dr}{r} + \cot\dfrac{\theta}{2}\,d\theta = 0$

Integrating $\log r + 2\log\sin\dfrac{\theta}{2} = \log c$

or $\log r + \log\sin^2\dfrac{\theta}{2} = \log c$

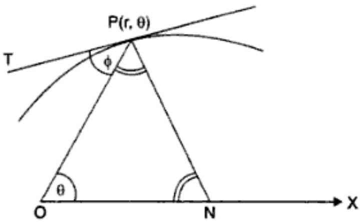
or $\log r\sin^2\dfrac{\theta}{2} = \log c$ or $r\sin^2\dfrac{\theta}{2} = c$

or $\dfrac{r}{2}(1 - \cos\theta) = c$ or $\dfrac{2c}{r} = 1 - \cos\theta$

which is a parabola.

(*b*) Let PT and PN be the tangent and normal to the curve at any point $(r, \theta)$. Then

$$\tan\phi = r\frac{d\theta}{dr} \qquad \text{...(1)}$$




By the condition of the problem

$\angle ONP = \angle OPN = 90° - \phi$

$\therefore$ $\theta = 180° - (90° - \phi + 90° - \phi)$

or $\theta = 2\phi$ or $\phi = \dfrac{\theta}{2}$

$\therefore$ From (1), $\tan\dfrac{\theta}{2} = r\dfrac{d\theta}{dr}$

$\Rightarrow$ $\dfrac{dr}{r} = \cot\dfrac{\theta}{2}\,d\theta$

Integrating $\log r = \dfrac{\log\sin\dfrac{\theta}{2}}{\dfrac{1}{2}} + \log c$

or $\log r = 2\log\sin\dfrac{\theta}{2} + \log c = \log c\sin^2\dfrac{\theta}{2}$

or $r = c\sin^2\dfrac{\theta}{2}$ or $r = \dfrac{c}{2}(1 - \cos\theta)$

or $r = a(1 - \cos\theta)$ where $a = \dfrac{c}{2}$.

Hence the required curve is the cardioid $r = a(1 - \cos\theta)$.

**Example 22.** *Show that all curves for which the square of the normal is equal to the square of the radius vector are either circles or rectangular hyperbolas.*

**Sol.** At any point P$(x, y)$ of a curve ; length of the normal $= y\sqrt{1+\left(\frac{dy}{dx}\right)^2}$

Length of radius vector OP $= \sqrt{x^2+y^2}$

By the given condition, we have

$$y^2\left[1+\left(\frac{dy}{dx}\right)^2\right] = x^2+y^2 \quad \text{or} \quad y^2\left(\frac{dy}{dx}\right)^2 = x^2 \quad \text{or} \quad y\frac{dy}{dx} = \pm x$$

when $\quad y\frac{dy}{dx} = +x$

we have $\quad xdx = ydy$

Integrating $\quad \frac{x^2}{2} = \frac{y^2}{2} + \frac{c}{2}$

or $\quad x^2 - y^2 = c$ which is a rectangular hyperbola

when $\quad y\frac{dy}{dx} = -x$

we have $\quad xdx + ydy = 0$

Integrating $\quad \frac{x^2}{2} + \frac{y^2}{2} = \frac{c}{2} \quad$ or $\quad x^2 + y^2 = c \quad$ which is a circle.

**Example 23.** *Find the equation of the curve in which the perpendicular from the pole upon the tangent at any point is k times the radius vector of the point.*

**Sol.** Let P$(r, \theta)$ be any point on the curve. Then $\quad p = kr \quad$ (given) ...(*i*)

Using $\quad \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$, we have

$$\frac{1}{k^2r^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 \quad \text{or} \quad r^2 = r^2k^2 + k^2\left(\frac{dr}{d\theta}\right)^2$$

or $$\frac{r^2(1-k^2)}{k^2} = \left(\frac{dr}{d\theta}\right)^2$$

$\therefore$ $$\frac{dr}{d\theta} = \frac{r\sqrt{1-k^2}}{k} \quad \text{or} \quad \frac{dr}{r} = \frac{\sqrt{1-k^2}}{k}\,d\theta$$

Integrating $$\log r = \frac{\sqrt{1-k^2}}{k}\,\theta + c$$

$\Rightarrow$ $$r = e^{\frac{\sqrt{1-k^2}}{k}\theta + c} = e^{\frac{\sqrt{1-k^2}}{k}\theta}\,.\,e^c$$

$\therefore$ $$r = \mathrm{A}e^{\frac{\sqrt{1-k^2}}{k}\theta}$$

which is the required equation of the curve.

☞**Remember** 1. Expression for velocity $v$ at any time $t$ is

$$v = \frac{ds}{dt}$$

2. Expressions for acceleration $f$ are

$$f = \frac{d^2s}{dt^2} \text{ or } \frac{dv}{dt} \text{ or } v\,\frac{dv}{ds}$$

which $s$ denotes the displacement at any time $t$.

**Example 1.** *The magnitude of the velocity of a particle moving along the x-axis is given by the equation* $V = \frac{x}{4}$, *where V is metres per second and x is in metres. When t = 0, the particle is 2 metres to the right of the origin. Determine the position of the particle when t = 3 seconds.*

**Sol.** The velocity is given by $V = \frac{x}{4}$

$$\therefore \quad \frac{dx}{dt} = \frac{x}{4} \qquad \left| \because \; V = \frac{dx}{dt} \right.$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{1}{4}\,dt$$

Integrating $\quad \log x = \frac{1}{4}\,t + c$

$$\Rightarrow \quad x = e^{\frac{1}{4}t + c} = e^c \,.\, e^{\frac{1}{4}t} \quad \text{or} \quad x = Ae^{\frac{1}{4}t} \qquad \ldots(i)$$

Initially when $\quad t = 0, x = 2$ m (given)

$\therefore \quad 2 = Ae^0 = A.$

$\therefore$ From $(i)$, $\quad x = 2e^{\frac{1}{4}t}$

when $t = 3$ secs., $\quad x = 2e^{3/4}$ m.

which gives the position at $t = 3$ seconds.

**Example 2.** *A particle initially at rest, moves from a fixed point in a straight line so that at the end of t seconds, its acceleration is* $\sin t + \left(\frac{1}{t+1}\right)^2$. *Show that at the end of π seconds from start, it is at a distance of* $2\pi - \log(\pi + 1)$ *from the fixed point.*

**Sol.** If $f$ is the acceleration at any time $t$, then

$$f = \sin t + \frac{1}{(t+1)^2}$$

or
$$\frac{dv}{dt} = \sin t + \left(\frac{1}{t+1}\right)^2 \quad \text{or} \quad dv = \left[\sin t + \frac{1}{(t+1)^2}\right] dt$$

Integrating $\quad v = -\cos t - \frac{1}{t+1} + c_1 \qquad \ldots(i)$

when $\quad t = 0, v = 0 \qquad$ [∵ Particle is initially at rest]

Integrating $\dfrac{v^3}{3} + \dfrac{v^2}{2} = 10x + c$ ...(i)

when $x = 0$, $v = 0$

$\therefore$ From (i), $c = 0$ $\therefore$ $\dfrac{v^3}{3} + \dfrac{v^2}{2} = 10x$ ...(ii)

when $v = 44$ m/sec.

From (ii), $\dfrac{(44)^3}{3} + \dfrac{(44)^2}{2} = 10x$ or $60x = (44)^2 [2 \times 44 + 3] = 1936 \times 91$

$$\therefore \quad y = \frac{1936 \times 91}{60} = 2936.27 \text{ m.}$$

**Example 5.** *A particle moving in a straight line is subject to a resistance which produces retardation $kv^3$. Show that v and t are given in terms of s by the equations*

$$v = \frac{u}{1 + ksu}, \; t = \frac{1}{2} ks^2 + \frac{s}{u}$$

*where u is the initial velocity.*

**Sol.** Retardation $= kv^3$ $\therefore$ $f = -kv^3$

or $v \dfrac{dv}{ds} = -kv^3$ or $\dfrac{dv}{v^2} = -kds$

Integrating $-\dfrac{1}{v} = -ks + c_1$ ...(i)

when $s = 0$, $v = u$

$\therefore$ From (i), $-\dfrac{1}{u} = c_1$

putting $c_1 = -\dfrac{1}{u}$ in (i)

$$-\frac{1}{v} = -ks - \frac{1}{u} \quad \text{or} \quad \frac{1}{v} = ks + \frac{1}{u} = \frac{ksu + 1}{u}$$

$\therefore$ $v = \dfrac{u}{1 + ksu}$ ...(I)

Again $v = \dfrac{ds}{dt} = \dfrac{u}{1 + ksu}$

$\therefore$ $(1 + ksu)\, dx = udt$

Integrating $s + ku \,.\, \dfrac{s^2}{2} = ut + c_2$ ...(ii)

when $t = 0$, $s = 0$ $\therefore$ $c_2 = 0$

$\therefore$ From (ii), $s + ku \,.\, \dfrac{s^2}{2} = ut$ or $t = \dfrac{1}{2} ks^2 + \dfrac{s}{u}$. ...(II)

**Example 6.** *A particle of unit mass being to more from a distance 'a' towards a fixed centre which repels according to the law* $\mu x$, *x being the distance from the fixed centre. Its initial velocity is* $\sqrt{\mu}\ a$, *show that it will continually approach the fixed centre, but will never reach it.*

**Sol.** Mass $m = 1$

$\therefore$ Equation of motion of the particle is

$$mf = \mu x \quad \text{or} \quad f = \mu x \quad \text{or} \quad v\frac{dv}{dx} = \mu x$$

Separating the variables $v dv = \mu\, x dx$

Integrating $$\frac{v^2}{2} = \mu \cdot \frac{x^2}{2} + \text{A} \qquad ...(i)$$

when $x = a$, $v = -\sqrt{\mu}\, a$ (negative sign has been taken because the particle is moving towards the centre)

$\therefore$ From $(i)$, $$\frac{\mu a^2}{2} = \frac{\mu a^2}{2} + \text{A} \quad \text{or} \quad \text{A} = 0$$

$\therefore$ $$\frac{v^2}{2} = \mu\frac{x^2}{2} \quad \text{or} \quad v^2 = \mu x^2 \quad \text{or} \quad v = -\sqrt{\mu}\, x \qquad | \textbf{ Note}$$

or $$\frac{dx}{dt} = -\sqrt{\mu}\, x \quad \text{or} \quad \frac{dx}{x} = -\sqrt{\mu}\ dt$$

Integrating $$\log x = -\sqrt{\mu}\, t + c \qquad ...(ii)$$

when $t = 0$, $x = a$ $\therefore$ from $(ii)$ $\log a = c$

Putting this value of $c$ in $(ii)$,

$$\log x = -\sqrt{\mu}\, t + \log a \quad \text{or} \quad \log x - \log a = -\sqrt{\mu}\, t$$

or $$\log\frac{x}{a} = -\sqrt{\mu}\, t \quad \text{or} \quad \frac{x}{a} = e^{-\sqrt{\mu}t} \quad \therefore \quad x = ae^{-\sqrt{\mu}t}$$

Now as $t$ increases, $x$ decreases. Therefore the particle is continually approaching the centre. Since $x \to 0$ only when $t \to \infty$, the particle does not reach the centre in any finite time.

**Example 7.** *A particle moves from a point O with an acceleration k times the distance from O with a starting velocity* $\sqrt{k}$. *Find the velocity of the particle at a distance 2 units from O.*

**Sol.** Let $f$ be the acceleration and $s$, the distance of the particle from O at any instant. Then

$$f = ks \qquad \text{(given)}$$

$$\Rightarrow \quad v\frac{dv}{ds} = ks \quad \text{or} \quad v dv = ks\, ds$$

Integrating $$\frac{v^2}{2} = \frac{ks^2}{2} + c \qquad ...(i)$$

Initially, when $s = 0$, $v = \sqrt{k}$

$\therefore$ $$\frac{k}{2} = c \quad \therefore \quad \text{from } (i)\ \frac{v^2}{2} = \frac{ks^2}{2} + \frac{k}{2} \quad \text{or} \quad v = \sqrt{k(s^2+1)}$$

When $s = 2$, $v = \sqrt{k(4+1)} = \sqrt{5k}$.

When $t = 0$, $\qquad v = v_0$ (given)

$$\therefore \qquad -\frac{1}{2v_0^2} = c_1$$

$$\therefore \text{ From } (i), \qquad -\frac{1}{2v^2} = -kt - \frac{1}{2v_0^2}$$

$$\frac{1}{2v^2} = \frac{2kv_0^2t + 1}{2v_0^2} \qquad \Rightarrow \qquad v^2 = \frac{v_0^2}{2kv_0^2t + 1}$$

$$\therefore \qquad v = \frac{v_0}{\sqrt{2kv_0^2t + 1}} \qquad \text{or} \qquad \frac{ds}{dt} = \frac{v_0}{\sqrt{2kv_0^2t + 1}}$$

or
$$ds = v_0(2kv_0{}^2 + 1)^{-\frac{1}{2}}\, dt$$

Integrating
$$s = v_0 \cdot \frac{(2kv_0^2t + 1)^{\frac{1}{2}}}{\frac{1}{2} \cdot 2kv_0^2} + c_2$$

or
$$s = \frac{\sqrt{2kv_0^2t + 1}}{kv_0} + c_2 \qquad \ldots(ii)$$

When $t = 0$, $\qquad s = 0$ (given)

$$\therefore \qquad 0 = \frac{1}{kv_0} + c_2 \qquad \text{or} \qquad c_2 = -\frac{1}{kv_0}$$

$$\therefore \text{ From } (ii), \qquad s = \frac{\sqrt{2kv_0^2t + 1}}{kv_0} - \frac{1}{kv_0} \qquad \Rightarrow \qquad s = \frac{\sqrt{2kv_0^2t + 1} - 1}{kv_0}$$

which gives the distance passed over by the particle in time $t$.

# 5

# Homogeneous Linear Equations

## 1. Homogeneous Linear Equation

**Definition.** An equation of the form

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + a_{n-1} x \frac{dy}{dx} + a_n y = \mathrm{X}$$

where $a_0, a_1, a_2, \ldots\ldots, a_n$ are constants and X is a function of $x$ is called a homogeneous linear equation of the $n$th order.

## 2. Method of Solution

**Reduce the homogeneous linear equation**

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_{n-1} x \frac{dy}{dx} + a_n y = \mathrm{X}$$

**into linear equation with constant co-efficients.**

The given homogeneous linear equation is

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2} y}{dx^{n-2}} + \ldots\ldots + a_{n-1} x \frac{dy}{dx} + a_n y = \mathrm{X} \quad \ldots(1)$$

Put $x = e^z \Rightarrow z = \log x \quad \therefore \quad \frac{dz}{dx} = \frac{1}{x}$

$$\therefore \quad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x} \frac{dy}{dz} \Rightarrow x \frac{dy}{dx} = \frac{dy}{dz} \quad \ldots(2)$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{1}{x} \cdot \frac{dy}{dz}\right) = \frac{1}{x} \cdot \frac{d^2 y}{dz^2} \cdot \frac{dz}{dx} - \frac{1}{x^2} \cdot \frac{dy}{dz}$$

$$= \frac{1}{x^2}\left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right) \qquad \left[\because \frac{dz}{dx} = \frac{1}{x}\right]$$

$$\Rightarrow \quad x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz}$$

Put $\quad x \frac{d}{dx} = \frac{d}{dz} = \mathrm{D}$

From (2), $\quad x \frac{dy}{dx} = \mathrm{D}y$

From (3), $\quad x^2 \frac{d^2 y}{dx^2} = \mathrm{D}^2 y - \mathrm{D}y = \mathrm{D}(\mathrm{D} - 1)y$

Similarly $\quad x^3 \dfrac{d^3y}{dx^3} = D(D-1)(D-2)y$

..........................................................................................

..........................................................................................

$$x^n \frac{d^ny}{dx^n} = D(D-1)(D-2) \ldots\ldots (D-n+1)y$$

$\therefore$ From (1), we have

$$[a_0D(D-1) \ldots\ldots (D-n+1) + a_1D(D-1) \ldots\ldots (D-n+2) + \ldots\ldots$$
$$+ a_{n-2}D(D-1) + a_{n-1}D + a_n]y = f(e^z), \text{ where } X = f(x)$$

This is a linear equation with constant co-efficients and is, therefore, solvable for $y$ in terms of $z$ (by methods done in Chapter 3).

If $y = F(z)$ is its solution, then putting $z = \log x$, the required solution is $y = F(\log x)$

**Remember.** Put $\quad x = e^z$ *(so that $z = \log x$)*

and
$$D \equiv x\frac{d}{dx} = \frac{d}{dz}$$

Then
$$x^2 \frac{d^2}{dx^2} \equiv D(D-1)$$

$$x^3 \frac{d^3}{dx^3} \equiv D(D-1)(D-2) \quad \text{and so on.}$$

Given equation becomes $f(D)y = Z$ (*a function of z*)

Solve it and replace $z$ by $\log x$.

**Example 1.** *Solve the following :*

(*i*) $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 9y = 0$ (*ii*) $x^2 \dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = 0$

(*iii*) $(x^2D^2 + 2xD - 2)y = 0.$

**Sol.** (*i*) Put $x = e^x$ so that $z \log x$

Let $\quad D = \dfrac{d}{dz} = x\dfrac{d}{dx}$ the given equation becomes

$$[D(D-1) + D - 9]y = 0 \quad \text{or} \quad (D^2 - 9)y = 0$$

A.E. is $\quad D^2 - 9 = 0 \quad \therefore \quad D = \pm 3$

Complete solution is $\quad y = c_1e^{3z} + c_2e^{-3x} \quad$ or $\quad y = c_1x^3 + c_2x^{-3}.$

(*ii*) Put $x = e^z$, so that $\quad z = \log x$

Let $\quad D = \dfrac{d}{dz} = x\dfrac{d}{dx}$ , the given equation becomes

$$[D(D-1) + 3D + 1]y = 0 \quad \text{or} \quad (D^2 + 2D + 1)y = 0 \quad \text{or} \quad (D+1)^2y = 0$$

A.E. is $\quad (D+1)^2 = 0 \quad \therefore \quad D = -1, -1$

Complete solution is $\quad y = (c_1 + c_2z)e^{-z} \quad$ or $\quad y = (c_1 + c_2 \log x) \,.\, x^{-1}$

or
$$y = \frac{1}{x}(c_1 + c_2 \log x).$$

(*iii*) Put $x = e^z$ so that $z = \log x$

Let $D' = \dfrac{d}{dz} = xD = x\dfrac{d}{dx}$ the given equation becomes,

$[D'(D'-1) + 2D' - 2]y = 0$ or $(D'^2 + D' - 2)y = 0$

A.E. is $D'^2 + D' - 2 = 0$ or $(D'+2)(D'-1) = 0$

$\therefore$ $D' = -2, 1$

Complete solution is $y = c_1 e^{-2z} + c_2 e^z$ or $y = c_1 x^{-2} + c_2 x$.

**Example 2.** *Solve the following :*

(*i*) $x^3 \dfrac{d^3y}{dx^3} + 6x^2 \dfrac{d^2y}{dx^2} + 4x\dfrac{dy}{dx} - 4y = 0$ (*ii*) $\dfrac{d^3y}{dx^3} = \dfrac{6y}{x^3}$

(*iii*) $x^3 \dfrac{d^2y}{dx^2} + 9x\dfrac{dy}{dx} + 25y = 50$ (*iv*) $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 4y = 0.$

**Sol.** (*i*) Put $x = e^z$ so that $z = \log x$

Let $D = \dfrac{d}{dz} = x\dfrac{d}{dx}$, the given equation becomes

$[D(D-1)(D-2) + 6D(D-1) + 4D - 4]y = 0$ or $(D^3 + 3D^2 - 4)y = 0$

A.E. is $D^3 + 3D^2 - 4 = 0$ or $(D-1)(D^2 + 4D + 4) = 0$

or $(D-1)(D+2)^2 = 0 \Rightarrow D = 1, -2, -2$

$\therefore$ the complete solution is

$$y = c_1 e^z (c_2 + c_3 z)e^{-2z} \quad \text{or} \quad y = c_1 x + (c_2 + c_3 \log x)x^{-2}.$$

(*ii*) The given equation is $x^3 \dfrac{d^3y}{dx^3} - 6y = 0$

Put $x = e^z$ so that $z = \log x$

Let $D = \dfrac{d}{dz} = x\dfrac{d}{dx}$, the given equation becomes

$[D(D-1)(D-2) - 6]y = 0$ or $(D^3 - 3D^2 + 2D - 6)y = 0$

A.E. is $D^3 - 3D^2 + 2D - 6 = 0$ or $D^2(D-3) + 2(D-3) = 0$

or $(D-3)(D^2+2) = 0 \Rightarrow D = 3, \pm\sqrt{2}\,i$

$\therefore$ the complete solution is $y = c_1 e^{3z} + e^{0z}(c_2 \cos \sqrt{2}z + c_3 \sin \sqrt{2}z)$

or $$y = c_1 x^3 + c_2 \cos(\sqrt{2} \log x) + c_3 \sin(\sqrt{2} \log x).$$

(*iii*) Put $x = e^z$ so that $z = \log x$

Let $D = \dfrac{d}{dx} = x\dfrac{d}{dx}$, the given equation becomes

$[D(D-1) + 9(D + 25]y = 50$ or $(D^2 + 8D + 25)y = 50$

A.E. is $D^2 + 8D + 25 = 0$

$$\therefore \quad D = \frac{-8 \pm \sqrt{64 - 100}}{2} = \frac{-8 \pm 6i}{2} = -4 \pm 3i$$

$\therefore$ C.F. $= e^{-4z}(c_1 \cos 3z + c_2 \sin 3z)$

$$\text{P.I.} = \frac{1}{D^2 + 8D + 25}(50) = 50\frac{1}{D^2 + 8D + 25}e^{0z} = 50 \cdot \frac{1}{0+0+25} = 2$$

$\therefore$ The complete solution is

$$y = e^{-4z}(c_1 \cos 3z + c_2 \sin 3z) + 2$$

or
$$y = x^{-4}[c_1 \cos(3 \log x) + c_2 \sin(3 \log x)] + 2.$$

(*iv*) Please try yourself. $\left[\textbf{Ans.}\quad y = c_1x^2 + \dfrac{c^2}{x^2}\right]$

**Example 3.** *Solve*

(*i*) $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} - 4y = x^4$ (*ii*) $x^2 \dfrac{d^2y}{dx^2} - 2y = x^2 + \dfrac{1}{x}$

(*iii*) $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = 2x^2$ (*iv*) $(x^2D^2 + 5xD + 4)y = x^4$

(*v*) $x^2 \dfrac{d^2y}{dx^2} + 2x\dfrac{dy}{dx} - 20y = (x+1)^2.$ (Delhi, 2000)

**Sol.** (*i*) Put $x = e^z$ so that $z = \log x$

Let $D = \dfrac{d}{dz} = x\dfrac{d}{dx}$, the given equation becomes

$$[D(D-1) - 2D - 4]y = e^{4z} \quad \text{or} \quad (D^2 - 3D - 4)y = e^{4z}$$

A.E. is $D^2 - 3D - 4 = 0$ or $(D-4)(D+1) = 0 \therefore D = 4, -1$

$$\text{C.F.} = c_1e^{4z} + c_2e^{-z}$$

$$\text{P.I.} = \frac{1}{D^2 - 3D - 4}e^{4z} \qquad | \text{ Case of failure}$$

$$= z \cdot \frac{1}{2D-3}e^{4z} = z \cdot \frac{1}{2(4)-3}e^{4z} = \frac{1}{5}ze^{4z}$$

$\therefore$ Complete solution is

$$y = c_1e^{4z} + c_2e^{-z} + \frac{1}{5}ze^{4z} = c_1x^4 + c_2x^{-1} + \frac{1}{5}x^4 \log x.$$

(*ii*) Put $x = e^z$ so that $z = \log x$

Let $D = \dfrac{d}{dz} = x\dfrac{d}{dx}$, the given equation becomes

$$[D(D-1) - 2]y = e^{2z} + e^{-z} \quad \text{or} \quad (D^2 - D - 2)y = e^{2z} + e^{-z}$$

A.E. is $D^2 - D - 2 = 0$ or $(D-2)(D+1) = 0$

$\therefore$ $D = 2, -1$

$$\text{C.F.} = c_1e^{2z} + c_2e^{-z}$$

$$\text{P.I.} = \frac{1}{D^2 - D - 2}(e^{2z} + e^{-z}) = \frac{1}{D^2 - D - 2}e^{2z} + \frac{1}{D^2 - D - 2}e^{-z}$$

$$= z \cdot \frac{1}{2D-1}e^{2z} + z\frac{1}{2D-1}e^{-z} = z \cdot \frac{1}{2(2)-1}e^{2z} + z \cdot \frac{1}{2(-1)-1}e^{-z}$$

$$= \frac{1}{3}ze^{2z} - \frac{1}{3}ze^{-z} = \frac{1}{3}(e^{2z} - e^{-z})z$$

$\therefore$ Complete solution is

$$y = c_1e^{2z} + c_2e^{-z} + \frac{1}{3}(e^{2z} - e^{-z})z = c_1x^3 + \frac{c_2}{x} + \frac{1}{3}\left(x^2 - \frac{1}{x}\right)\log x.$$

(*iii*) Please try yourself. [**Ans.** $y = (c_1 + c_2 \log x) \,.\, x^3 + x^2(\log x)^2$]

(*iv*) Please try yourself. $\left[\textbf{Ans.} \quad y = (c_1 + c_2 \log x)\, x^{-2} + \dfrac{1}{36} x^4\right]$

(*v*) Put $x = e^z$ so that $z = \log x$

Let $\quad D = \dfrac{d}{dz} = x\dfrac{d}{dx}$, the given equation becomes

$$[D(D-1) + 2D - 20]y = (e^z + 1)^2 \quad \text{or} \quad (D^2 + D - 20)y = e^{2z} + 2e^z + 1$$

A.E. is $\qquad D^2 + D - 20 = 0 \quad$ or $\quad (D+5)(D-4) = 0$

$$D = -5, 4$$

$$\text{C.F.} = c_1 e^{-5z} + c_2 e^{4z}$$

$$\text{P.I.} = \frac{1}{D^2 + D - 20}(e^{2z} + 2e^z + 1)$$

$$= \frac{1}{D^2 + D - 20} e^{2z} + 2\frac{1}{D^2 + D - 20} e^z + \frac{1}{D^2 + D - 20} e^{0z}$$

$$= \frac{1}{2^2 + 2 - 20} e^{2z} + 2\frac{1}{1^2 + 1 - 20} e^z + \frac{1}{0^2 + 0 - 20} e^{0z}$$

$$= -\frac{1}{14} e^{2z} - \frac{1}{9} e^z - \frac{1}{20}$$

$\therefore$ Complete solution is

$$y = c_1 e^{-5z} + c_2 e^{4z} - \frac{1}{14} e^{2z} - \frac{1}{9} e^z - \frac{1}{20}$$

$$= c_1 x^{-5} + c_2 x^4 - \frac{1}{14} x^2 - \frac{1}{9} x - \frac{1}{20}.$$

**Example 4.** *Solve the following* :

(*i*) $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 4x^3$ $\qquad$ (*ii*) $x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - 4y = x^2$

(*iii*) $x^2 \dfrac{d^2y}{dx^2} + 7x\dfrac{dy}{dx} + 5y = x^5$ $\qquad$ (*iv*) $x^2 \dfrac{d^3y}{dx^3} - 4x\dfrac{d^2y}{dx^2} + 6\dfrac{dy}{dx} = 4.$

**Sol.** (*i*) Put $x = e^z$ so that $z = \log x$

Let $\qquad D = \dfrac{d}{dz} = x\dfrac{d}{dx}$, the given equation becomes

$$[D(D-1) - 2D + 2]y = 4e^{3z} \quad \text{or} \quad (D^2 - 3D + 2)y = 4e^{3z}$$

A.E. is $\qquad D^2 - 3D + 2 = 0 \quad$ or $\quad (D-1)(D-2) = 0$

$\Rightarrow \qquad D = 1, 2$

$\therefore \qquad \text{C.F.} = c_1 e^z + c_2 e^{2z}$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2}(4e^{3z}) = 4 \,.\, \frac{1}{D^2 - 3D + 2} e^{3z}$$

$$= 4 \,.\, \frac{1}{(3)^2 - 3(3) + 2} e^{3z} = 2e^{3z}$$

or $\quad (D-2)(D^2-5D+1)=0$

$$\Rightarrow \quad D = 2, \frac{5 \pm \sqrt{25-4}}{2} = 2, \frac{1}{2}(5+\sqrt{21}), \frac{1}{2}(5-\sqrt{21})$$

$$\therefore \quad \text{C.F.} = c_1 e^{2z} + c_2 e^{\frac{1}{2}(5+\sqrt{21})z} + c_3 e^{\frac{1}{2}(5-\sqrt{21})z}$$

$$\text{P.I.} = \frac{1}{D^3 - 7D^2 + 11D - 2} e^{3z}$$

$$= \frac{1}{(3)^3 - 7(3)^2 + 11(3) - 2} e^{3z} = -\frac{1}{5} e^{3z}$$

$\therefore$ The complete solution is

$$y = c_1 e^{2z} + c_2 e^{\frac{1}{2}(5+\sqrt{21})z} + c_3 e^{\frac{1}{2}(5-\sqrt{21})z} - \frac{1}{5} e^{3z}$$

$$= c_1 x^2 + c_2 x^{\frac{1}{2}(5+\sqrt{21})} + c_3 x^{\frac{1}{2}(5-\sqrt{21})} - \frac{x^3}{5}$$

or
$$y = c_1 x^2 + x^{5/2}\left[c_2 x^{\frac{1}{2}\sqrt{21}} + c_3 x^{-\frac{1}{2}\sqrt{21}}\right] - \frac{x^3}{5}.$$

**Example 6.** *Solve* $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = x$ *given that* $y = 0$ *where* $x = 1$ *and* $y = e^2$ *when* $x = e$.

**Sol.** Put $x = e^z$ so that $z = \log x$

Let $\quad D = \dfrac{d}{dz} = x\dfrac{d}{dx}$, the given equation becomes

$$[D(D-1) - 3D + 4]y = e^z \quad \text{or} \quad (D^2 - 4D + 4)y = e^z \quad \text{or} \quad (D-2)^2 y = e^z$$

A.E. is $\quad (D-2)^2 = 0 \quad \therefore \quad D = 2, 2$

$$\text{C.F.} = (c_1 + c_2 z)e^{2z}$$

$$\text{P.I.} = \frac{1}{(D-2)^2} e^z = \frac{1}{(1-2)} e^z = e^z$$

$\therefore$ Complete solution is

$$y = (c_1 + c_2 z)e^{2z} + e^z \quad \text{or} \quad y = (c_1 + c_2 \log x) \,.\, x^2 + x \qquad \ldots(i)$$

when $\quad x = 1, y = 0 \qquad$ (given)

$\therefore$ From (*i*), $\quad 0 = c_1 + 1 \qquad$ | $\because \log 1 = 0$

$\therefore \quad c_1 = -1$

When $\quad x = e, y = e^2 \qquad$ (given)

$\therefore$ From (*i*), $\quad e^2 = (-1 + c_2)e^2 + e \qquad$ | $\because \log e = 1$

or
$$\frac{e^2 - e}{e^2} = -1 + c_2 \quad \text{or} \quad c_2 = 1 + 1 - \frac{1}{e} = 2 - \frac{1}{e}$$

$\therefore$ From (*i*), the required particular solution is

$$y = \left[-1 + \left(2 - \frac{1}{e}\right)\log x\right] x^2 + x.$$

**Example 7.** *Solve*

(*i*) $x^3 \dfrac{d^3y}{dx^3} - x^2 \dfrac{d^2y}{dx^2} + 2x \dfrac{dy}{dx} - 2y = x^3 + 3x$

(*ii*) $x^4 \dfrac{d^3y}{dx^3} + 2x^3 \dfrac{d^2y}{dx^2} - x^2 \dfrac{dy}{dx} + xy = 1$

(*iii*) $x^3 \dfrac{d^3y}{dx^3} + 2x^2 \dfrac{d^2y}{dx^2} + 2y = 10\left(x + \dfrac{1}{x}\right)$. (Delhi, 1997 ; I.A.S., 1999)

**Sol.** (*i*) Put $x = e^z$ so that $z = \log x$

Let $D = \dfrac{d}{dz} = x\dfrac{d}{dx}$, the given equation becomes

$$[D(D-1)(D-2) - D(D-1) + 2D - 2]y = e^{3z} + 3e^z$$

or $$[D^3 - 3D^2 + 2D - D^2 + D + 2D - 2]y = e^{3z} + 3e^z$$

or $$(D^3 - 4D^2 + 5D - 2)y = e^{3z} + 3e^z$$

A.E. is $D^3 - 4D^2 + 5D - 2 = 0$ D = 1 satisfies it.

By synthetic division

$$\begin{array}{c|cccc} 1 & 1 & -4 & 5 & -2 \\ & & 1 & -3 & 2 \\ \hline & 1 & -3 & 2 & \boxed{0} \end{array}$$

The other two roots are the roots of

$$D^2 - 3D + 2 = 0 \quad \Rightarrow \quad D = 1, 2$$

$\therefore$ Roots of A.E. are 1, 1, 2

$$\text{C.F.} = (c_1 + c_2 z)e^z + c_2 e^{2z}$$

$$\text{P.I.} = \frac{1}{D^3 - 4D^2 + 5D - 2} e^{3z} + 3\frac{1}{D^3 - 4D^2 + 5D - 2} e^z$$

$$= \frac{1}{3^3 - 4.3^2 + 5.3 - 2} e^{3z} + 3z \frac{1}{3D^2 - 8D + 5} e^z$$

$$= \frac{1}{4} e^{3z} + 3z^2 . \frac{1}{6D - 8} e^z = \frac{1}{4} e^{3z} + 3z^2 \frac{1}{6(1) - 8} e^z = \frac{1}{4} e^{3z} - \frac{3}{2} z^2 e^z$$

$\therefore$ Complete solution is

$$y = (c_1 + c_2 z)e^z + c_3 e^{2z} \frac{1}{4} e^{3z} - \frac{3}{2} z^2 e^z$$

$$= (c_1 + c_2 \log x) . x + c_3 x^2 + \frac{1}{4} x^3 - \frac{3}{2} x (\log x)^2.$$

(*ii*) Please try yourself. $\left[\textbf{Ans.} \quad y = (c_1 + c_2 \log x)\, x + c_3 x^{-1} + \dfrac{1}{4x} \log x\right]$

[**Hint.** First divide throughout by $x$]

(*iii*) Put $x = e^z$ so that $z = \log x$

Let $D = \dfrac{d}{dz} = x\dfrac{d}{dx}$, the given equation becomes

$$[D(D-1)(D-2) + 2D(D-1) + 2]y = 10(e^z + e^{-z})$$

or $$(D^3 - D^2 + 2)y = 10(e^z + e^{-z})$$

A.E. is $D^3 - D^2 + 2 = 0$ $D = -1$ satisfies it.

By synthetic division

$$\begin{array}{r|rrrr} -1 & 1 & -1 & 0 & 2 \\ & & -1 & & -2 \\ \hline & 1 & -2 & 2 & \boxed{0} \end{array}$$

The other two roots are the roots of $D^2 - 2D + 2 = 0$ whence

$$D = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

Roots of A.E. are $-1, 1 \pm i$

$$\text{C.F.} = c_1 e^{-z} + e^z (c_2 \cos z + c_3 \sin z)$$

$$\text{P.I.} = 10 \frac{1}{D^3 - D^2 + 2} e^z + 10 \frac{1}{D^3 - D^2 + 2} e^{-z}$$

$$= 10 \frac{1}{1-1+2} e^z + 10z \cdot \frac{1}{3D^2 - 2D} e^{-z}$$

$$= 5e^z + 10z \cdot \frac{1}{3(-1)^2 - 2(-1)} e^{-z} = 5e^z + 2ze^{-z}$$

$\therefore$ Complete solution is

$$y = c_1 e^{-z} + e^z (c_2 \cos z + c_3 \sin z) + 5e^z + 2ze^{-z}$$

$$= c_1 x^{-1} + x\,[c_2 \cos(\log x) + c_3 \sin(\log x)] + 5x + \frac{2}{x} \log x.$$

**Example 8.** *Solve the following :*

(*i*) $(x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)y = 0$ (*ii*) $(x^2D^2 + 7xD + 5)y = 2x^4$

(*iii*) $(x^2D^2 + 4xD + 2)y = x$ (*iv*) $(x^2D^2 + xD - 1)y = x^3$

(*v*) $x^2 \dfrac{d^2y}{dx^2} - 4x \dfrac{dy}{dx} + 6y = x.$

**Sol.** (*i*) Put $x = e^z$ so that $z = \log x$

Let $D' = \dfrac{d}{dz} = xD = x\dfrac{d}{dx}$

The given equation becomes

$$[D'(D'-1)(D'-2)(D'-3) + 6D'(D'-1)(D'-2) + 9D'(D'-1) + 3D' + 1]y = 0$$

or $(D'^4 - 6D'^3 + 11D'^2 - 6D' + 6D'^3 - 18D'^2 + 12D' + 9D'^2 - 9D' + 3D' + 1)y = 0$

or $(D'^4 + 2D'^2 + 1)y = 0$ or $(D'^2 + 1)^2 y = 0$

A.E. is $(D'^2 + 1)^2 = 0$ $\therefore$ $D' = \pm i, \pm i$

The complete solution is

$$y = e^{0z}\,[(c_1 + c_2 z) \cos z + (c_3 + c_4 z) \sin z]$$

or $$y = (c_1 + c_2 \log x) \cos(\log x) + (c_3 + c_4 \log x) \sin(\log x).$$

(*ii*) Put $x = e^z$ so that $z = \log x$

Let $D' = \dfrac{d}{dz} = xD = x\dfrac{d}{dx}$

The given equation becomes

$$[D'(D'-1) + 7D' + 5]y = 2e^{4z} \quad \text{or} \quad (D'^2 + 6D' + 5)y = 2e^{4z}$$

**Example 10.** *Solve the following :*

(*i*) $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} - 3y = x^2 \log x$ (Lucknow, 1998)

(*ii*) $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = 2 \log x$

(*iii*) $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\cdot\dfrac{dy}{dx} = \dfrac{12 \log x}{x^2}$ (Delhi, 1997)

(*iv*) $x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = x \log x$ (Kanpur, 1998 ; Lucknow, 1997)

(*v*) $x^2 \dfrac{d^2y}{dx^2} + 8x\dfrac{dy}{dx} + 13y = \log x.$

**Sol.** (*i*) Put $x = e^z$ so that $z = \log x$

Let $\quad D = \dfrac{d}{dz} = x\dfrac{d}{dx}$, the given equation becomes

$$[D(D-1) - D - 3]y = ze^{2z} \quad \text{or} \quad (D^2 - 2D - 3)y = ze^{2z}$$

A.E. is $\quad D^2 - 2D - 3 = 0 \quad$ or $\quad (D-3)(D+1) = 0$

$$D = 3, -1$$

$$\text{C.F.} = c_1e^{3z} + c_2e^{-z}$$

$$\text{P.I.} = \frac{1}{D^2 - 2D - 3}\, ze^{2z}$$

$$= e^{2z}\frac{1}{(D+2)^2 - 2(D+2) - 3}\, z \qquad \left|\ \because\ \frac{1}{f(D)}e^{ax}X = e^{ax}\frac{1}{f(D+a)}X\right.$$

$$= e^{2z}\frac{1}{D^2 + 2D - 3}\, z = e^{2z}\frac{1}{-3\left(1 - \dfrac{2D}{3} - \dfrac{D^2}{3}\right)}\, z$$

$$= -\frac{1}{3}e^{2z}\left[1 - \left(\frac{2D}{3} + \frac{D^2}{3}\right)\right]^{-1} z$$

$$= -\frac{1}{3}e^{2z}\left[1 + \frac{2D}{3}\right] z = -\frac{1}{3}e^{2z}\left(z + \frac{2}{3}\right)$$

$\therefore$ Complete solution is

$$y = c_1e^{3z} + c_1e^{-z} - \frac{1}{3}e^{2z}\left(z + \frac{2}{3}\right) = c_1x^3 + \frac{c_2}{x} - \frac{x^2}{3}\left(\log x + \frac{2}{3}\right).$$

(*ii*) Please try yourself. **[Ans.** $y = (c_1 + c_2 \log x)x + 2 \log x + 4$]

(*iii*) Multiplying throughout by $x^2$, the given equation becomes

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12 \log x \qquad ...(1)$$

Put $\quad x = e^z$ so that $z = \log x$

Let $\quad D = \dfrac{d}{dz} = x\dfrac{d}{dx}$, then (1) becomes

$$[D(D-1) + D]y = 12z \quad \text{or} \quad D^2y = 12z$$

A.E. is $D^2 = 0$ whence $D = 0, 0$

$$\text{C.F.} = (c_1 + c_2 z)e^{0z} = c_1 + c_2 z$$

$$\text{P.I.} = \frac{1}{D^2}(12z) = \frac{1}{D}\left(\int 12\,z dz\right) = \frac{1}{D}(6z^2) = \int 6z^2\,dz = 2z^3$$

$\therefore$ Complete solution is

$$y = c_1 + c_2 z + 2z^3 = c_1 + c_2 \log x + 2(\log x)^3.$$

(*iv*) Please try yourself. **[Ans.** $y = x[c_1 \cos(\log x) + c_2 \sin(\log x)] + x \log x$]

(*v*) Please try yourself.

$$\left[\textbf{Ans.}\quad y = x^{-\frac{7}{2}}\left[c_1 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_2 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right] + \frac{1}{169}[13 \log x - 7]\right]$$

**Example 11.** *Solve the following :*

(*i*) $x^2 \dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = x^2 \log x$ (*ii*) $x^2 \dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + 4y = x^2 \log x$

(*iii*) $x^2D^2y + 2xDy = \log x$ (*iv*) $x^2 \dfrac{d^2y}{dx^2} + 5x\dfrac{dy}{dx} + 4y = x \log x$

(*v*) $x^3 \dfrac{d^3y}{dx^3} + 3x^2\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + y = x \log x$ (I.A.S. 1996)

(*vi*) $(x^2D^3 + 3xD^2 + D)y = x^2 \log x$.

**Sol.** (*i*) Put $x = e^z$ so that $z = \log x$

Let $D = \dfrac{d}{dz} = x\dfrac{d}{dx}$

The given equation becomes

$$[D(D-1) - 2D + 2]y = ze^{2z} \quad \text{or} \quad (D^2 - 3D + 2)y = ze^{2z}$$

A.E. is $D^2 - 3D + 2 = 0$ or $(D-1)(D-2) = 0$

$\therefore$ $D = 1, 2$

$$\text{C.F.} = c_1e^z + c_2e^{2z} = c_1x + c_2x^2$$

$$\text{P.I.} = \frac{1}{D^2 - 3D + 2}ze^{2z} = e^{2z}\frac{1}{(D+2)^2 - 3(D+2) + 2}z$$

$$= e^{2z}\frac{1}{D^2 + D}z = e^{2z}\frac{1}{D(1+D)}z$$

$$= e^{2z}\frac{1}{D}(1+D)^{-1}z = e^{2z}\frac{1}{D}(1 - D + \ldots\ldots)z$$

$$= e^{2z}\frac{1}{D}(z-1) = e^{2z}\left(\frac{z^2}{2} - z\right) = x^2\left[\frac{1}{2}(\log x)^2 - \log x\right]$$

$\therefore$ The complete solution is

$$y = c_1x + c_2x^2 + x^2\left[\frac{1}{2}(\log x)^2 - \log x\right].$$

(*ii*) Put $x = e^z$ so that $z = \log x$

Let $D = \dfrac{d}{dz} = x\dfrac{d}{dx}$

The given equation becomes

$$[D(D-1)-3D+4]y = ze^{2z}$$

or $\quad (D^2-4D+4)y = ze^{2z}$ or $(D-2)^2y = ze^{2z}$

A.E. is $\quad (D-2)^2 = 0 \quad \therefore \quad D = 2, 2$

$$\text{C.F.} = (c_1 + c_2 z)\, e^{2z} = (c_1 + c_2 \log x)x^2$$

$$\text{P.I.} = \frac{1}{(D-2)^2} ze^{2z} = e^{2z} \frac{1}{(D+2-2)^2} z$$

$$= e^{2z} \frac{1}{D^2} z = e^{2z} \frac{1}{D}\left(\frac{z^2}{2}\right) = e^{2z} \cdot \frac{z^3}{6} = \frac{1}{6} x^2 (\log x)^3$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 \log x)x^2 + \frac{1}{6} x (\log x)^3.$$

(*iii*) Please try yourself. $\left[\textbf{Ans.} \quad y = c_1 + c_2 x^{-1} + \frac{1}{2}(\log x)^2 - \log x\right]$

(*iv*) Please try yourself. $\left[\textbf{Ans.} \quad y = (c_1 + c_2 \log x)x^{-2} + \frac{1}{9} x\left(\log x - \frac{2}{3}\right)\right]$

(*v*) Put $x = e^z$ so that $z = \log x$

Let $\quad D = \dfrac{d}{dz} = x\dfrac{d}{dx}$

The given equation becomes

$$[D(D-1)(D-2) + 3D(D-1) + D + 1]y = ze^z$$

or

$$(D^3 + 1)y = ze^z$$

A.E. is $\quad D^3 + 1 = 0 \quad$ or $\quad (D+1)(D^2 - D + 1) = 0$

$$\therefore \qquad D = -1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$$

$$\text{C.F.} = c_1 e^{-z} + e^{\frac{1}{2}z}\left(c_2 \cos\frac{\sqrt{3}}{2} z + c_3 \sin\frac{\sqrt{3}}{2} z\right)$$

$$= c_1 x^{-1} + x^{1/2}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right]$$

$$\text{P.I.} = \frac{1}{D^3+1} ze^z = e^z \frac{1}{(D+1)^3+1} z = e^z \frac{1}{2 + 3D + 3D^2 + D^3} z$$

$$= e^z \frac{1}{2\left(1 + \frac{3}{2}D + \frac{3}{2}D^2 + \frac{1}{2}D^3\right)} z = \frac{1}{2} e^z \left[1 + \left(\frac{3}{2}D + \frac{3}{2}D^2 + \frac{1}{2}D^3\right)\right]^{-1} z$$

$$= \frac{1}{2} e^z \left[1 - \left(\frac{3}{2}D \ldots\ldots\right) + \ldots\ldots\right] z = \frac{1}{2} e^z \left[z - \frac{3}{2}\right] = \frac{1}{2} x\left(\log x - \frac{3}{2}\right)$$

$\therefore$ The complete solution is

$$y = \frac{c_1}{x} + \sqrt{x}\left[c_2 \cos\left(\frac{\sqrt{3}}{2}\log x\right) + c_3 \sin\left(\frac{\sqrt{3}}{2}\log x\right)\right] + \frac{x}{2}\left(\log x - \frac{3}{2}\right).$$

(*vi*) The given equation is not homogeneous.

Multiplying throughout by $x$, it becomes

$$(x^3D^3 + 3x^2D^2 + xD)y = x^3 \log x \qquad ...(1)$$

Put $x = e^z$ so that $z = \log x$

Let $D' = \dfrac{d}{dz} = x\dfrac{d}{dx} = xD$

Equation (1) becomes

$$[D'(D'-1)(D'-2) + 3D'(D'-1) + D']y = ze^{3z} \quad \text{or} \quad D'^3 y = ze^{3z}$$

A.E. is $D'^3 = 0 \quad \therefore \quad D' = 0, 0, 0$

$$\text{C.F.} = (c_1 + c_2 z + c_3 z^2)\, e^{0z} = c_1 + c_2 \log x + c_3 (\log x)^2$$

$$\text{P.I.} = \frac{1}{D'^3} ze^{3z} = e^{3z} \frac{1}{(D'+3)^3} z$$

$$= e^{3z} \frac{1}{\left[3\left(1+\frac{D'}{3}\right)\right]^3} z = \frac{1}{27} e^{3z}\left(1+\frac{D'}{3}\right)^{-3} z$$

$$= \frac{1}{27} e^{3z}(1 - D' + \ldots\ldots)\, z = \frac{1}{27} e^{3z}(z-1) = \frac{1}{27} x^3 (\log x - 1)$$

$\therefore$ The complete solution is

$$y = c_1 + c_2 \log x + c_3 (\log x)^2 + \frac{x^3}{27}(\log x - 1).$$

**Example 12.** *Solve the following :*

(*i*) $(x^2D^2 - 3xD + 5)y = \sin(\log x)$

(*ii*) $x^3 \dfrac{d^3y}{dx^3} + 3x^2 \dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 8y = 65 \cos(\log x)$

(*iii*) $x^4 \dfrac{d^4y}{dx^4} + 2x^3 \dfrac{d^3y}{dx^3} + x^2 \dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + y = x + \log x$

(*iv*) $(x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)y = (1 + \log x)^2$.

**Sol.** (*i*) Put $x = e^z$ so that $z = \log x$

Let $D' = \dfrac{d}{dz} = xD = x\dfrac{d}{dx}$, the given equation becomes

$$[D'(D'-1) - 3D' + 5]y = \sin z \quad \text{or} \quad (D'^2 + 4D' + 5)y = \sin z$$

A.E. is $\quad D'^2 - 4D' + 5 = 0 \Rightarrow D' = \dfrac{4 \pm \sqrt{16-20}}{2} = 2 \pm i$

$\therefore \quad \text{C.F.} = e^{2z}(c_1 \cos z + c_2 \sin z)$

$$\text{P.I.} = \frac{1}{D'^2 - 4D' + 5} \sin z = \frac{1}{-1 - 4D' + 5} \sin z$$

$$= \frac{1}{4} \cdot \frac{1}{1-D'} \sin z = \frac{1}{4} \cdot \frac{1+D'}{1-D'^2} \sin z$$

$$= \frac{1}{4} \cdot \frac{1+D'}{1-(-1)} \sin z = \frac{1}{8} [\sin z + D' (\sin z)] = \frac{1}{8} (\sin z + \cos z)$$

$\therefore$ The complete solution is

$$y = e^z (c_1 \cos z + c_2 \sin z) + \frac{1}{8} (\sin z + \cos z)$$

$$= x^2 [c_1 \cos (\log x) + c_2 \sin (\log x)] + \frac{1}{8} [\sin (\log x) + \cos (\log x)].$$

(*ii*) Put $x = e^z$ so that $z = \log x$

Let $D = \frac{d}{dz} = x \frac{d}{dx}$, the given equation becomes

$$[D(D-1)(D-2) + 3D(D-1) + D + 8]y = 65 \cos z$$

or $$(D^3 + 8)y = 65 \cos z$$

A.E. is $D^3 + 8 = 0$ or $(D+2)(D^2 - 2D + 4) = 0$

$\therefore$ $$D = -2, \frac{2 \pm \sqrt{4-16}}{2} = -2, \frac{2 \pm 2i\sqrt{3}}{2} = -2, 1 \pm i\sqrt{3}$$

$\therefore$ $$\text{C.F.} = c_1 e^{-2z} + e^z (c_2 \cos \sqrt{3} z + c_3 \sin \sqrt{3} z)$$

$$\text{P.I.} = \frac{1}{D^3+8} (65 \cos z) = 65 \cdot \frac{1}{D \cdot D^2 + 8} \cos z$$

$$= 65 \cdot \frac{1}{D(-1)+8} \cos z = 65 \cdot \frac{1}{8-D} \cos z$$

$$= 65 \cdot \frac{8+D}{64-D^2} \cos z = 65 \cdot \frac{8+D}{64-(-1)} \cos z$$

$$= 8 \cos z + D (\cos z) = 8 \cos z - \sin z$$

$\therefore$ the complete solution is

$$y = c_1 e^{-2z} + e^z (c_2 \cos \sqrt{3} z + c_3 \sin \sqrt{3} z) + 8 \cos z - \sin z$$

$$= c_1 x^{-2} + x[c_2 \cos (\sqrt{3} \log x) + c_3 \sin (\sqrt{3} \log x)] + 8 \cos (\log x) - \sin (\log x)$$

(*iii*) Put $x = e^z$ so that $z = \log x$

Let $D = \frac{d}{dz} = x \frac{d}{dx}$, the given equation becomes

$$[D(D-1)(D-2)(D-3) + 2D(D-1)(D-2) + D(D-1) - D + 1]y = e^z + z$$

or $$(D^4 - 4D^3 + 6D^2 - 4D + 1)y = e^z + z$$

$$(D-1)^4 y = e^z + z$$

A.E. is $(D-1)^4 = 0 \Rightarrow D = 1, 1, 1, 1$

$\therefore$ $$\text{C.F.} = (c_1 + c_2 z + c_3 z^2 + c_4 z^3) e^z$$

$$\text{P.I.} = \frac{1}{(D-1)^4} e^z + \frac{1}{(D-1)^4} z = z \cdot \frac{1}{4(D-1)^3} e^z + \frac{1}{(1-D)^4} z$$

$$= \frac{z}{4} \cdot z \frac{1}{3(D-1)^2} e^z + (1-D)^{-4} z = \frac{z^2}{12} \cdot z \frac{1}{3(D-1)} e^z + (1 + 4D \ldots\ldots) z$$

$$= \frac{z^3}{24} \cdot z \frac{1}{1} e^z + z + 4 = \frac{1}{24} z^4 e^z + z + 4$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 z + c_3 z^2 + c_4 z^3) e^z + \frac{1}{24} z^4 e^z + z + 4$$

$$= [c_1 + c_2 \log x + c_3 (\log x)^2 + c_4 (\log x)^3] x + \frac{x(\log x)^4}{24} + \log x + 4.$$

(*iv*) Put $x = e^z$ so that $z = \log x$

Let $D' \equiv xD = x \dfrac{d}{dx}$, the given equation becomes

$$[D'(D'-1)(D'-2)(D'-3) + 6D'(D'-1)(D'-2) + 9D'(D'-1) + 3D' + 1]y = (1+z)^2$$

or $$(D'^4 + 2D'^2 + 1)y = (1+z)^2 \quad \text{or} \quad (D'^2+1)^2 y = (1+z)^2$$

A.E. is $\quad (D'^2 + 1)^2 = 0 \quad \Rightarrow \quad D' = \pm i, \pm i$

$\therefore$ C.F. $= (c_1 + c_2 z) \cos z + (c_3 + c_4 z) \sin z$

$$\text{P.I.} = \frac{1}{(D'^2+1)^2} (1+z)^2 = (1 + D'^2)^{-2} (1+z)^2$$

$$= (1 + 2D'^2 \ldots\ldots)(1+z)^2 = (1+z)^2 - 2.\,2 = (1+z)^2 - 4$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 z) \cos z + (c_3 + c_4 z) \sin z + (1+z)^2 - 4$$

$$= (c_1 + c_2 \log x) \cos(\log x) + (c_3 + c_4 \log x) \sin(\log x) + (1 + \log x)^2 - 4$$

or $$y = (c_1 + c_2 \log x) \cos(\log x) + (c_3 + c_4 \log x) \sin(\log x) + (\log x)^2 + 2 \log x - 3.$$

**Example 13.** *Solve the following :*

(*i*) $(x^2D^2 + xD + 1)y = \log x \sin(\log x)$

(*ii*) $(x^2D^2 - xD + 4)y = \cos(\log x) + x \sin(\log x)$

**Sol.** (*i*) Put $x = e^z$ so that $z = \log x$

Let $\quad D' = \dfrac{d}{dz} = xD = x \dfrac{d}{dx}$, the given equation becomes

$$[D'(D'-1) + D' + 1]y = z \sin z \quad \text{or} \quad (D'^2+1)y = z \sin z$$

A.E. is $\quad D'^2 + 1 = 0 \quad$ whence $\quad D' = \pm i$

$\therefore$ C.F. $= c_1 \cos z + c_2 \sin z$

$$\text{P.I.} = \frac{1}{D'^2+1} z \sin z = \text{Imaginary part of } \frac{1}{D'^2+1} z e^{iz}$$

$$= \text{I.P. of } e^{iz} \frac{1}{(D+i)^2+1} z = \text{I.P. of } e^{iz} \frac{1}{D'^2 + 2iD'} z$$

$$= \text{I.P. of } e^{iz} \frac{1}{2iD'\left(1 + \frac{D'}{2i}\right)} z = \text{I.P. of } \frac{1}{2i} e^{iz} \cdot \frac{1}{D'} \left(1 - \frac{D'}{2}\right)^{-1} z$$

$$= \text{I.P. of } \frac{1}{2i} e^{iz} \cdot \frac{1}{D'}\left(1 + \frac{D'}{2}\right) z$$

$$= \text{I.P. of } \frac{1}{2i} e^{iz} \left(\frac{1}{D'} + \frac{i}{2}\right) z = \text{I.P. of } \frac{1}{2i} e^{iz} \left(\int z dz + \frac{iz}{2}\right)$$

$$= \text{I.P. of} - \frac{i}{2} e^{iz} \left(\frac{z^2}{2} + \frac{iz}{2}\right) = \text{I.P. of } e^{iz} \left(-\frac{iz^2}{4} - \frac{i^2 z}{4}\right)$$

$$= \text{I.P. of } (\cos z + i \sin z) \left(-\frac{iz^2}{4} + \frac{z}{4}\right) = -\frac{z^2}{4} \cos z + \frac{z}{4} \sin z$$

$\therefore$ Complete solution is

$$y = c_1 \cos z + c_2 \sin z - \frac{z^2}{4} \cos z + \frac{z}{4} \sin z$$

$$= c_1 \cos(\log x) + c_2 \sin(\log x) - \frac{1}{4} (\log x)^2 \cos(\log x) + \frac{1}{4} \log x \sin(\log x).$$

(*ii*) Put $x = e^z$ so that $z = \log x$

Let $D' = \frac{d}{dz} = xD = x \frac{d}{dx}$, the given equation becomes

$$[D'(D' - 1) - D' + 4]y = \cos z + e^z \sin z$$

or
$$(D'^2 - 2D' + 4)y = \cos z + e^z \sin z$$

A.E. is $D'^2 - 2D' + 4 = 0$ whence $D' = \frac{2 \pm \sqrt{4 - 16}}{2} = 1 \pm i\sqrt{3}$

$$\text{C.F.} = e^z (c_1 \cos \sqrt{3}z + c_2 \sin \sqrt{3}z)$$

$$\text{P.I.} = \frac{1}{D'^2 - 2D + 4} \cos z + \frac{1}{D'^2 - 2D' + 4} e^z \sin z$$

$$= \frac{1}{(-1) - 2D' + 4} \cos z + e^z \frac{1}{(D' + 1)^2 - 2(D' + 1) + 4} \sin z$$

$$= \frac{1}{3 - 2D'} \cos z + e^z \frac{1}{D'^2 + 3} \sin z = \frac{3 + 2D'}{9 - 4D'^2} \cos z + e^z - \frac{1}{-1 + 3} \sin z$$

$$= \frac{3 + 2D'}{9 - 4(-1)} \cos z + \frac{1}{2} e^z \sin z = \frac{1}{13} (3 + 2D') \cos z + \frac{1}{2} e^z \sin z$$

$$= \frac{1}{13} (3 \cos z - 2 \sin z) + \frac{1}{2} e^z \sin z$$

$\therefore$ Complete solution is

$$y = e^z(c_1 \cos \sqrt{3}z + c_3 \sin \sqrt{3}z) + \frac{1}{13} (3 \cos z - 2 \sin z) + \frac{1}{2} e^z \sin z$$

$$= x\text{[illegible]} \cos(\sqrt{3} \log x) + c_2 \sin(\sqrt{3} \log x)]$$

$$+ \frac{1}{13} [3 \cos(\log x) - 2 \sin(\log x)] + \frac{1}{2} x \sin(\log x).$$

**Example 14.** *Solve the following :*

(i) $x^2 \dfrac{d^2y}{dx^2} - 2x \dfrac{dy}{dx} + 2y = x + x^2 \log x + x^3$

(ii) $x^2 \dfrac{d^2y}{dx^2} - 3 \dfrac{dy}{dx} + 5y = x^2 \sin (\log x)$

(iii) $(x^2D^2 + 4xD + 2)y = x + \log x$ (iv) $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} - 5y = \sin (\log x)$.

**Sol.** (i) Put $x = e^z$ so that $z = \log x$

Let $\mathrm{D} = \dfrac{d}{dz} = x \dfrac{d}{dx}$

The given equation becomes

$$[\mathrm{D}(\mathrm{D}-1) - 2\mathrm{D} + 2]y = e^z + ze^{2z} + e^{3z}$$

or
$$(\mathrm{D}^2 - 3\mathrm{D} + 2)y = e^z + ze^{2z} + e^{3z}$$

A.E. is $\mathrm{D}^2 - 3\mathrm{D} + 2 = 0$ or $(\mathrm{D}-1)(\mathrm{D}-2) = 0$ $\therefore$ $\mathrm{D} = 1, 2$

$$\text{C.F.} = c_1e^z + c_2e^{2z} = c_1x + c_2x^2$$

$$\text{P.I.} = \frac{1}{\mathrm{D}^2 - 3\mathrm{D} + 2}(e^z + ze^{2z} + e^{3z})$$

$$= \frac{1}{\mathrm{D}^2 - 3\mathrm{D}+2} e^z + e^{2z} \frac{1}{(\mathrm{D}+2)^2 - 3(\mathrm{D}+2)+2} z + \frac{1}{\mathrm{D}^2 - 3\mathrm{D}+2} e^{3z}$$

$$= z \frac{1}{2\mathrm{D}-3} e^z + e^{2z} \frac{1}{\mathrm{D}+\mathrm{D}^2} z + \frac{1}{3^2 - 3\times 3 + 2} e^{3z}$$

$$= z \frac{1}{2\times 1 - 3} e^z + e^{2z} \frac{1}{\mathrm{D}(1+\mathrm{D})} z + \frac{1}{2} e^{3z} = -ze^z + e^{2z} \cdot \frac{1}{\mathrm{D}} (1+\mathrm{D})^{-1} z + \frac{1}{2} e^{3z}$$

$$= -ze^z + e^{2z} \cdot \frac{1}{\mathrm{D}} (1 - \mathrm{D} + \ldots\ldots) z + \frac{1}{2} e^{3z} = -ze^z + e^{2z} \frac{1}{\mathrm{D}} (z-1) + \frac{1}{2} e^{3z}$$

$$= -ze^z + e^{2z} \left(\frac{z^2}{2} - z\right) + \frac{1}{2} e^{3z} = -x \log x + x^2 \left[\frac{1}{2} (\log x)^2 - \log x\right] + \frac{1}{2} x^3$$

$\therefore$ The complete solution is

$$y = c_1x + c_2x^2 - x \log x + x^2 \left[\frac{1}{2}(\log x)^2 - \log x\right] + \frac{x^3}{2}.$$

(ii) Put $x = e^z$ so that $z = \log x$

Let $\mathrm{D} = \dfrac{d}{dz} = x \dfrac{d}{dx}$

The given equation becomes

$[\mathrm{D}(\mathrm{D}-1) - 3\mathrm{D} + 5]y = e^{2z} \sin z$ or $(\mathrm{D}^2 - 4\mathrm{D} + 5)y = e^{2z} \sin z$

A.E. is $\mathrm{D}^2 - 4\mathrm{D} + 5 = 0$

$\therefore$
$$\mathrm{D} = \frac{4 \pm 2i}{2} = 2 \pm i$$

$$\text{C.F.} = e^{2z}(c_1 \cos x + c_2 \sin x) = x^2 (c_1 \cos(\log x) + c_2 \sin(\log x))]$$

$$\text{P.I.} = \frac{1}{D^2 - 4D + 5} e^{2z} \sin z = e^{2z} \frac{1}{(D+2)^2 - 4(D+2) + 5} \sin z$$

$$= e^{2z} \frac{1}{D^2 + 1} \sin z \qquad \text{(Case of failure)}$$

$$= e^{2z}\left(z \cdot \frac{1}{2D} \sin z\right) = e^{2z}\left(\frac{z}{2}(-\cos z)\right) = -\frac{1}{2} x^2 \log x \cos(\log x)$$

∴ The complete solution is

$$y = x^2 [c_1 \cos(\log x) + c_2 \sin(\log x)] - \frac{x^2}{2} \log x \cos(\log x).$$

(*iii*) Please try yourself. $\left[\textbf{Ans.}\quad y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{1}{6}x + \frac{1}{2}\left(\log x - \frac{3}{2}\right)\right]$

(*iv*) Please try yourself. $\left[\textbf{Ans.}\quad y = \frac{c_1}{x} + c_2 x^5 + \frac{1}{26}[2\cos(\log x) - 3\sin(\log x)]\right]$

**Example 15.** *Solve :*

(*i*) $x^2 \dfrac{d^2y}{dx^2} + 4x \dfrac{dy}{dx} + 2y = e^x$ (*ii*) $x^2 \dfrac{d^2y}{dx^2} + x \dfrac{dy}{dx} - y = x^2 e^x$

(*iii*) $x^2 \dfrac{d^2y}{dx^2} + 4x \dfrac{dy}{dx} + 2y = x + \sin x$ (*iv*) $x^2 \dfrac{d^2y}{dx^2} + 3x \dfrac{dy}{dx} + y = \dfrac{1}{(1-x)^2}$.

**Sol.** (*i*) Put $x = e^z$ so that $z = \log x$

Let $D = \dfrac{d}{dz} = x\dfrac{d}{dx}$, the given equation becomes

$$[D(D-1) + 4D + 2]y = e^{e^z} \quad \text{or} \quad [D^2 + 3D + 2]y = e^{e^z}$$

A.E. is $D^2 + 3D + 2 = 0$ or $(D+1)(D+2) = 0$

∴ $D = -1, -2$

$$\text{C.F.} = c_1 e^{-x} + ce^{-2x}$$

$$\text{P.I.} = \frac{1}{(D+1)(D+2)} e^{e^z} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^z} \qquad \text{[Partial fractions]}$$

$$= \frac{1}{D+1} e^{e^z} - \frac{1}{D+2} e^{e^z} = \frac{1}{D-(-1)} e^{e^z} - \frac{1}{D-(-2)} e^{e^z}$$

$$= e^{-z} \int e^{e^z} . e^z dz - e^{-2z} \int e^{e^z} . e^{2z}\, dz \qquad \left[\because \frac{1}{D-a} X = e^{ax} \int X e^{-ax}\, dx\right]$$

$$= e^{-z} \int e^{e^z} . e^z dz - e^{-2z} \int e^z . e^{e^z} . e^z\, dz$$

$$= e^{-z} \int e^t dt - e^{-2z} \int te^t dt, \quad \text{where } t = e^z$$

$$= e^{-z} . e^t - e^{-2z} . e^t (t-1) = e^{-z} . e^{e^z} - e^{-2z} . e^{e^z} (e^z - 1)$$

$$= e^{e^z}(e^{-z} - e^{-z} + e^{-2z}) = e^{-2z} . e^{e^z}$$

$\therefore$ The complete solution is

$$y = c_1e^{-z} + c_2e^{-2z} + e^{-2z}.\ e^{e^z} = c_1x^{-1} + c_2x^{-2} + x^{-2}e^x$$

or
$$y = \frac{c_1}{x} + \frac{c_2}{x^2} + \frac{e^x}{x^2}\ .$$

(*ii*) Put $x = e^z$ so that $z = \log x$

Let $D = \dfrac{d}{dz} = x\dfrac{d}{dx}$, the given equation becomes

$$[D(D-1) + D - 1]y = e^{2z}\ .\ e^{e^z} \quad \text{or} \quad (D^2-1)y = e^{2z}\ .\ e^{e^z}$$

A.E. is $\qquad D^2 - 1 = 0 \ \Rightarrow\ D = \pm 1$

$\therefore \qquad$ C.F. $= c_1e^z + c_2e^{-z}$

$$\text{P.I.} = \frac{1}{D^2-1}(e^{2z}\ e^{e^z}) = \frac{1}{(D+1)(D-1)}(e^{2z}\ e^{e^z})$$

$$= \frac{1}{2}\left[\frac{1}{D-1} - \frac{1}{D+1}\right](e^{2z}\ .\ e^{e^z}) \qquad \text{[Partial fractions]}$$

$$= \frac{1}{2}\cdot\frac{1}{D-1}(e^{2z}\ e^{e^z}) - \frac{1}{2}\cdot\frac{1}{D+1}(e^{2z}\ e^{e^z})$$

$$= \frac{1}{2}e^z\int e^{2z}\ e^{e^z}\ .\ e^{-z}dz - \frac{1}{2}e^{-z}\int e^{2z}\ .\ e^{e^z}\ .\ e^z\ dz$$

$$\left[\because\ \frac{1}{D-a}X = e^{ax}\int Xe^{-ax}\,dx\right]$$

$$= \frac{1}{2}e^z\int e^{e^z}\ e^z dz - \frac{1}{2}e^{-z}\int e^{2z}\ .\ e^{e^z}\ .\ e^z\ dz$$

$$= \frac{1}{2}e^z\int e^t dt - \frac{1}{2}e^{-z}\int t^2e^t dt, \quad \text{where } t = e^z$$

$$= \frac{1}{2}\ e^z.\ e^t - \frac{1}{2}\ e^{-z}\,(t^2 - 2t + 2)\ e^t$$

$$= \frac{1}{2}\ e^z\ .\ e^{e^z} - \frac{1}{2}\ e^{-z}\,(e^{2z} + 2e^z + 2)\ e^{e^z}$$

$$= \frac{1}{2}\ e^{e^z}\ (e^z - e^z + 2 - 2e^{-z}) = e^{e^z} - e^{-z}\ .e^{e^z}$$

$\therefore$ The complete solution is

$$y = c_1e^z + c_2e^{-z} + e^{e^z} - e^{-z}.\ e^{e^z}$$

$$= c_1x + c_2x^{-1} + e^x - x^{-1}\ .\ e^x \quad \text{or} \quad y = c_1x + \frac{c_2}{x} + e^x\left(1 - \frac{1}{x}\right).$$

(*iii*) Put $x = e^z$ so that $z = \log x$

Let $D = \dfrac{d}{dz} = x\dfrac{d}{dx}$ , the given equation becomes

$$[D(D-1) + 4D + 2]y = e^z + \sin(e^z) \qquad \text{or} \qquad (D^2 + 3D + 2)y = e^z + \sin(e^z)$$

$$\text{P.I.} = \frac{1}{(D+1)^2} \cdot \frac{1}{(1-e^z)^2} = \frac{1}{D+1} \cdot \frac{1}{D+1} \frac{1}{(1-e^z)^2}$$

$$= \frac{1}{D+1} e^{-z} \int \frac{1}{(1-e^z)^2} \cdot e^z \, dz$$

$$= \frac{1}{D+1} e^{-z} \int (1-t)^{-2} \, dt, \quad \text{where } t = e^z$$

$$= \frac{1}{D+1} e^{-z} \frac{(1-t)^{-1}}{(-1)(-1)} = \frac{1}{D+1} \frac{e^{-z}}{(1-e^z)}$$

$$= e^{-z} \int \frac{e^{-z}}{1-e^z} e^z dz = e^{-z} \int \frac{dz}{1-e^z}$$

$$= e^{-z} \int \frac{e^{-z}}{e^{-z}-1} \, dz = -e^{-z} \int \frac{-e^{-z}}{e^{-z}-1} \, dz$$

$$= -e^{-z} \log (e^{-z} - 1) \qquad \Bigg| \because \int \frac{f'(x)}{f(x)} \, dx = \log f(x)$$

$\therefore$ The complete solution is

$$y = (c_1 + c_2 z)e^{-z} - e^{-z} \log (e^{-z} - 1) = (c_1 + c_2 \log x)\, x^{-1} - x^{-1} \log (x^{-1} - 1)$$

$$= (c_1 + c_2 \log x) \frac{1}{x} - \frac{1}{x} \log \left( \frac{1}{x} - 1 \right)$$

or
$$y = (c_1 + c_2 \log x) \frac{1}{x} - \frac{1}{x} \log \frac{1-x}{1}.$$

## LINEAR EQUATIONS REDUCIBLE TO HOMOGENEOUS LINEAR FORM

**Method of solution of linear differential equations reducible to homogeneous linear form.**

Consider the equation

$$a_0 (a+bx)^n \frac{d^n y}{dx^n} + a_1 (a+bx)^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_{n-1}(a+bx)\frac{dy}{dx} + a_n y = f(x) \quad \ldots(1)$$

where $a_0, a_1, a_2, \ldots\ldots, a_n$ are constants.

Put $\quad a + bx = e^z \quad$ so $\quad$ that $z = \log (a + bx)$

$$\frac{dz}{dx} = \frac{b}{a+bx} \qquad \therefore \qquad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{b}{a+bx} \cdot \frac{dy}{dz}$$

$$\Rightarrow \qquad (a+bx) \frac{dy}{dx} = b \frac{dy}{dz} \quad \ldots(2)$$

$$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b}{a+bx} \cdot \frac{dy}{dz}\right) = -\frac{b^2}{(a+bx)^2} \frac{dy}{dz} + \frac{b}{a+bx} \cdot \frac{d^2 y}{dz^2} \cdot \frac{dz}{dx}$$

$$= -\frac{b^2}{(a+bx)^2} \frac{dy}{dz} + \frac{b}{a+bx} \cdot \frac{d^2 y}{dz^2} \cdot \frac{b}{a+bx}$$

$$[2^2.\ D(D-1) - 6.\ 2D + 8]y = 0 \qquad \mid \text{ Here } b = 2$$

$$4(D^2 - 4D + 2)y = 0 \quad \text{or} \quad (D^2 - 4D + 2)y = 0$$

A.E. is $\quad D^2 - 4D + 2 = 0$ whence

$$D = \frac{4 \pm \sqrt{16-8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} = 2 \pm \sqrt{2}$$

$\therefore$ Complete solution is

$$y = c_1\, e^{(2+\sqrt{2})z} + c_2 e^{(2-\sqrt{2})z} = c_1(5+2x)^{2+\sqrt{2}} + c_2(5+2x)^{2-\sqrt{2}}.$$

(*ii*) Put $2x - 1 = e^z$ so that $z = \log(2x-1)$

Let $\frac{d}{dx} \equiv D$, the given equation becomes

$$[2^3.\ D(D-1)(D-2) + 2D - 2]y = 0 \qquad \mid \text{ Here } b = 2$$

or

$$2(4D^3 - 12D^2 + 8D + D - 1)y = 0$$

$$(4D^3 - 12D^2 + 9D - 1)y = 0$$

A.E. is $\quad 4D^3 - 12D^2 + 9D - 1 = 0$, $D = 1$ satisfies it

By synthetic division

$$\begin{array}{c|cccc} 1 & 4 & -12 & 9 & -1 \\ & & 4 & -8 & 1 \\ \hline & 4 & -8 & 1 & \boxed{0} \end{array}$$

The other two roots are the roots of $4D^2 - 8D + 1 = 0$

$$D = \frac{8 \pm \sqrt{64-16}}{8} = \frac{8 \pm 4\sqrt{3}}{8} = 1 \pm \frac{\sqrt{3}}{2}$$

$\therefore$ Roots of A.E. are $1,\ 1 + \frac{\sqrt{3}}{2},\ 1 - \frac{\sqrt{3}}{2}$

Complete solution is $y = c_1\, e^z + c_2 e^{\left(1+\frac{\sqrt{3}}{2}\right)z} + c_3 e^{\left(1-\frac{\sqrt{3}}{2}\right)z}$

$$= c_1(2x-1) + c_2(2x-1)^{1+\frac{\sqrt{3}}{2}} + c_3(2x-1)^{1-\frac{\sqrt{3}}{2}}.$$

(*iii*) Put $x + a = e^z$ so that $z = \log(x+a)$

Let $\frac{d}{dz} \equiv D$, the given equation becomes

$$[D(D-1) - 4D + 6]y = e^z - a \quad \text{or} \quad (D^2 - 5D + 6)y = e^z - a$$

A.E. is $\quad D^2 - 5D + 6 = 0 \quad$ or $\quad (D-2)(D-3) = 0 \quad \Rightarrow \quad D = 2, 3$

$\therefore \quad$ C.F. $= c_1 e^{2z} + c_2 e^{3z}$

$$\text{P.I.} = \frac{1}{D^2 - 5D + 6}(e^z - a) = \frac{1}{D^2 - 5D + 6}\, e^z - a \cdot \frac{1}{D^2 - 5D + 6}\, e^{0z}$$

$$= \frac{1}{1^2 - 5(1) + 6}\, e^z - a \cdot \frac{1}{6} = \frac{1}{2} e^2 - \frac{1}{6}\, a$$

$\therefore$ The complete solution is

$$y = c_1 e^{2z} + c_2 e^{3z} + \frac{1}{2} e^z - \frac{1}{6}\, a$$

$$= c_1(x+a)^2 + c_2(x+a)^3 + \frac{1}{2}(x+a) - \frac{1}{6}a$$

or $$y = c_1(x+a)^2 + c_2(x+a)^3 + \frac{1}{6}(3x+2a).$$

**Example 2.** *Solve*

(*i*) $(1+x)^2 \dfrac{d^2y}{dx^2} + (1+x)\dfrac{dy}{dx} + y = 4\cos\log(1+x)$

(*ii*) $(3x+2)^2 \dfrac{d^2y}{dx^2} + 3(3x+2)\dfrac{dy}{dx} - 36y = 3x^2 + 4x + 1$ (Allahabad, 1996)

(*iii*) $(x+1)^2 \dfrac{d^2y}{dx^2} + (x+1)\dfrac{dy}{dx} = (2x+3)(2x+4)$

(*iv*) $(1+2x)^3 \dfrac{d^2y}{dx^2} - 6(1+2x)\dfrac{dy}{dx} + 16y = 8(1+2x)^2.$

**Sol.** (*i*) Put $1 + x = e^z$ so that $z = \log(1+x)$

Let $\dfrac{d}{dz} \equiv D$, the given equation becomes

$$[D(D-1) + D + 1]y = 4\cos z \qquad | \text{ Here } b = 1$$

or $$(D^2+1)y = 4\cos z$$

A.E. is $$D^2 + 1 = 0 \quad \Rightarrow \quad D = \pm i$$

$$\text{C.F.} = c_1\cos z + c_2 \sin z$$

$$\text{P.I.} = 4\,\frac{1}{D^2+1}\cos z \qquad | \text{ Case of failure}$$

$$= 4z\,\frac{1}{2D}\cos z = 2z\int \cos z\, dz = 2z\sin z$$

$\therefore$ Complete solution is

$$y = c_1\cos z + c_2\sin z + 2z\sin z$$
$$= c_1\cos[\log(1+x)] + c_2\sin[\log(1+x)] + 2\log(1+x)\sin[\log(1+x)].$$

(*ii*) Put $3x + 2 = e^z$ so that $z = \log(3x+2)$

Let $\dfrac{d}{dz} \equiv D$, the given equation becomes

$$[3^2D(D-1) + 3.\,3D - 36]y = 3\left(\frac{e^z-2}{3}\right)^2 + 4\left(\frac{e^z-2}{3}\right) + 1 \qquad | \text{ Here } b = 3$$

or $$9(D^2 - D + D - 4)y = \frac{1}{3}(e^{2z} - 4e^z + 4) + \frac{4}{3}(e^z - 2) + 1$$

or $$9(D^2-4)y = \frac{1}{3}e^{2z} - \frac{1}{3}$$

A.E. is $9(D^2-4) = 0$ whence $D = \pm 2$

$$\text{C.F.} = c_1 e^{2z} + c_2 e^{-2z}$$

$$\text{P.I.} = \frac{1}{9(D^2-4)}\left(\frac{1}{3}e^{2z} - \frac{1}{3}\right) = \frac{1}{27}\cdot\frac{1}{D^2-4}e^{2z} - \frac{1}{27}\cdot\frac{1}{D^2-4}\,e^{0z}$$

$$= \frac{1}{27}z\cdot\frac{1}{2D}e^{2z} - \frac{1}{27}\cdot\frac{1}{0-4} = \frac{z}{54}\int e^{2z}\,dz + \frac{1}{108}$$

$$= \frac{x}{54}\cdot\frac{e^{2z}}{2} + \frac{1}{108} = \frac{1}{108}\,(ze^{2z}+1)$$

$\therefore$ Complete solution is

$$y = c_1 e^{2z} + c_2 e^{-2z} + \frac{1}{108}\,(ze^{2z}+1)$$

$$= c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}\,[(3x+2)^2\log(3x+2)+1)]$$

(*iii*) Please try yourself. **[Ans.** $y = c_1 + c_2\log(1+x) + x^2 + 8x + [\log(x+1)]^2$

(*iv*) Please try yourself. **[Ans.** $y = (1+2x)^2[c_1 + c_2\log(1+2x) + \{\log(1+2x)\}^2]$

**Example 3.** *Solve* $x^2\dfrac{d^2y}{dx^2} - 3x\dfrac{dy}{dx} + y = \log x\,\dfrac{\sin(\log x)+1}{x}$.

**Sol.** Put $x = e^z$ so that $z = \log x$.

Let $\dfrac{d}{dz} \equiv D$, the given equation becomes

$$[D(D-1) - 3D + 1]y = z\,\frac{\sin z + 1}{e^z}$$

or

$$(D^2 - 4D + 1)y = e^{-z}\,.\,z\,(\sin z + 1)$$

Its A.E. is $\quad D^2 - 4D + 1 = 0 \quad$ whence $D = 2 \pm \sqrt{3}$

$$\therefore \qquad \text{C.F.} = c_1e^{(2+\sqrt{3})z} + c_2e^{(2-\sqrt{3})z} = e^{2z}\,(c_1e^{\sqrt{3}z} + c_2e^{-\sqrt{3}z})$$

$$\text{P.I.} = \frac{1}{D^2-4D+1}\,e^{-z}\,.\,z\,(\sin z+1)$$

$$= e^{-z}\,\frac{1}{(D-1)^2 - 4(D-1)+1}\,z\,(\sin z+1)$$

$$= e^{-z}\left[\frac{1}{D^2-6D+6}\,z\sin z + \frac{1}{D^2-6D+6}\,z\right] \qquad \text{...(1)}$$

Now $\dfrac{1}{D^2-6D+6}\,z\sin z = \text{I.P. of } \dfrac{1}{D^2-6D+6}\,ze^{iz}$

$$= \text{I.P. of } e^{iz}\,\frac{1}{(D+i)^2 - 6(D+i)+6}\,z$$

$$= \text{I.P. of } e^{iz}\,\frac{1}{D^2 + (2i-6)D + (5-6i)}\,z$$

$$= \text{I.P. of } e^{iz}\,\frac{1}{(5-6i) + (2i-6)D + D^2}\,z$$

$$= \text{I.P. of } e^{iz} \frac{1}{(5-6i)\left(1+\frac{2i-6}{5-6i}\text{D}+\frac{1}{5-6i}\text{D}^2\right)} z$$

$$= \text{I.P. of } \frac{e^{iz}}{5-6i}\left[1+\left(\frac{2i-6}{5-6i}\text{D}+\frac{1}{5-6i}\text{D}^2\right)\right]^{-1} z$$

$$= \text{I.P. of } \frac{e^{iz}}{5-6i}\left(1-\frac{2i-6}{5-6i}\text{D}+\ldots\right) z = \text{I.P. of } \frac{e^{iz}}{5-6i}\left(z-\frac{2i-6}{5-6i}\right)$$

$$= \text{I.P. of } \frac{(5+6i)\,e^{iz}}{61}\left[z-\frac{(2i-6)(5+6i)}{61}\right]$$

$$= \text{I.P. of } \frac{(5+6i)\,e^{iz}}{61}\left[z+\frac{42+26i}{61}\right]$$

$$= \text{I.P. of } \frac{e^{iz}}{61}\left[(5+6i)\,z+\frac{(5+6i)(42+26i)}{61}\right]$$

$$= \text{I.P. of } \frac{e^{iz}}{61}\left[(5+6i)\,z+\frac{54+382i}{61}\right]$$

$$= \text{I.P. of } \frac{\cos z+i\sin z}{61}\left[(5+6i)\,z+\frac{54+382i}{61}\right]$$

$$= \frac{z}{61}(6\cos z+5\sin z)+\frac{1}{3721}(382\cos z+54\sin z)$$

$$= \frac{z}{61}(6\cos z+5\sin z)+\frac{2}{3721}(191\cos z+27\sin z)$$

and $$\frac{1}{\text{D}^2-6\text{D}+6}z = \frac{1}{6-6\text{D}+\text{D}^2}z = \frac{1}{6\left(1-\text{D}+\frac{\text{D}^2}{6}\right)}z$$

$$= \frac{1}{6}\left[1-\left(\text{D}-\frac{\text{D}^2}{6}\right)\right]^{-1} z = \frac{1}{6}(1+\text{D}-\ldots)\,z = \frac{1}{6}(z+1)$$

$\therefore$ From (1), P.I. $= e^{-z}\left[\frac{z}{61}(6\cos z+5\sin z)+\frac{2}{3721}(191\cos z+27\sin z)+\frac{1}{6}(z+1)\right]$

Hence complete solution is

$$y = e^{2z}\left(c_1e^{\sqrt{3}z}+c_2e^{-\sqrt{3}z}\right)$$
$$+\,e^{-z}\left[\frac{z}{61}(6\cos z+5\sin z)+\frac{2}{3721}(191\cos z+27\sin z)+\frac{1}{6}(z+1)\right]$$

or $$y = x^2\left(c_1x^{\sqrt{3}}+c_2x^{-\sqrt{3}}\right)+\frac{1}{x}\left[\frac{\log x}{61}\{6\cos(\log x)+5\sin(\log x)\}\right.$$
$$\left.+\frac{2}{3721}\{191\cos(\log x)+27\sin(\log x)\}+\frac{1}{6}(\log x+1)\right].$$

# 6

# Trajectories

**Definitions**

**Trajectory.** A curve which cuts every member of a given family of curves according to a given law is called a trajectory of the family.

**Orthogonal Trajectory.** A curve which cuts every member of a given family of curves at right angles is called an orthogonal trajectory of the family.

**Oblique Trajectory.** A curve which cuts every member of a given family of curves at a constant angle $\alpha$ ($\neq 90°$) is called an oblique trajectory (or $\alpha$-trajectory) of the family.

**I. Rule to find the equation of $\alpha$-trajectories of a family of cartesian curves.**

Let the equation of the given family of curves be $f(x, y, c) = 0$ ...(*i*)

**Step (1)** Differentiate (*i*) and eliminate the arbitrary constant $c$ between (*i*) and the resulting equation.

That gives the differential equation of the family (*i*)

Let it be $$F\left(x, y, \frac{dy}{dx}\right) = 0 \qquad ...(ii)$$

**Step (2)** Replace $\frac{dy}{dx}$ by $\dfrac{\frac{dy}{dx} + \tan\alpha}{1 - \frac{dy}{dx} . \tan\alpha}$.

The differential equation of the $\alpha$-trajectory is

$$F\left(x, y, \frac{\frac{dy}{dx} + \tan\alpha}{1 - \frac{dy}{dx} . \tan\alpha}\right) = 0 \qquad ...(iii)$$

**Step (3)** Integrate (*iii*) to get the equation of the required trajectory.

**II. Rule to find the equation of orthogonal trajectories of a family of cartesian curves.**

Let the equation of the given family of curves be $f(x, y, c) = 0$ ...(*i*)

**Step (1)** Differentiate (*i*) and eliminate the arbitrary constant $c$ between (*i*) and the resulting equation.

That gives the differential equation of the family (*i*)

Let it be $F\left(x, y, \frac{dy}{dx}\right) = 0$ ...(*ii*)

**Step (2)** Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$.

The differential equation of the orthogonal trajectory is

$$F\left(x, y, -\frac{dx}{dy}\right) = 0 \qquad ...(iii)$$

**Step (3)** Integrate (*iii*) to get the equation of the required orthogonal trajectory.

**III. Rule to find the equation of orthogonal trajectories of a family of polar curves.**

Let the equation of the given family of polar curves be $f(r, \theta, c) = 0$ ...(*i*)

**Step (1)** Differentiate (*i*) and eliminate the arbitrary constant between (*i*) and the resulting equation.

That gives the differential equation of the family (*i*)

Let it be
$$F\left(r, \theta, \frac{dr}{d\theta}\right) = 0 \qquad ...(ii)$$

**Step (2)** Replace $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$.

The differential equation of the orthogonal trajectory is

$$F\left(r, \theta, -r^2 \frac{d\theta}{dr}\right) = 0 \qquad ...(iii)$$

**Step (3)** Integrate (*iii*) to get the equation of the required orthogonal trajectory.

**Definition. Self-orthogonal family of curves**

If each member of a given family of curves intersects all other members orthogonally, then the given family of curves is said to be self orthogonal.

In a self-orthogonal family of curves, the differential equation of the family of curves is identical with the differential equation of its orthogonal trajectories.

**Example 1.** *Find the orthogonal trajectories of the series of hyperbolas $xy = k^2$.*

**Sol.** The equation of the series of hyperbolas is $xy = k^2$ ...(*i*)

1. Differentiating (*i*) $y + x\frac{dy}{dx} = 0$ ...(*ii*)

[The arbitrary constant $k$ is eliminated]

2. Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, the differential equation of orthogonal trajectories is

$$y - x\frac{dx}{dy} = 0 \quad \text{or} \quad xdx - ydy = 0 \qquad ...(iii)$$

3. Integrating (*iii*) $\frac{x^2}{2} - \frac{y^2}{2} = c_1$

or
$$x^2 - y^2 = 2c_1 \qquad \text{or} \qquad x^2 - y^2 = c$$

which is the required equation.

**Example 2.** *Find the orthogonal trajectories of the series of parabolas whose equation is $y^2 = 4ax$.*

**Sol.** The equation of the series of parabolas is $y^2 = 4ax$ ...(i)

1. Differentiating (i) $2y \dfrac{dy}{dx} = 4a$ or $y \dfrac{dy}{dx} = 2a$ ...(ii)

Eliminating $a$ between (i) and (ii), we get

$$y^2 = 2y \frac{dy}{dx} . x \quad \text{or} \quad y = 2x \frac{dy}{dx}$$

2. Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$, the differential equation of orthogonal trajectories is

$$y = -2x \frac{dx}{dy} \quad \Rightarrow \quad ydy = -2xdx \qquad ...(iii)$$

3. Integrating (iii) $\dfrac{y^2}{2} = -x^2 + c_1$

or $$2x^2 + y^2 = 2c_1 \quad \text{or} \quad 2x^2 + y^2 = c$$

which is the required equation.

**Example 3.** *Find the orthogonal trajectories of astroids $x^{2/3} + y^{2/3} = a^{2/3}$.*

**Sol.** The equation of the series of hypo-cycloids is $x^{2/3} + y^{2/3} = a^{2/3}$ ...(i)

1. Differentiating (i)

$$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} . \frac{dy}{dx} = 0 \quad \text{or} \quad x^{-1/3} + y^{-1/3} \frac{dy}{dx} = 0 \qquad ...(ii)$$

which being independent of the parameter '$a$' is the differential equation of the family (i).

2. Replacing $\dfrac{dy}{dx}$ by $-\dfrac{dx}{dy}$, differential equation of orthogonal trajectories is

$$x^{-1/3} + y^{-1/3} \left(-\frac{dx}{dy}\right) = 0 \quad \text{or} \quad x^{-1/3} = y^{-1/3} . \frac{dx}{dy}$$

or $$x^{1/3} dx = y^{1/3} dy \qquad ...(iii)$$

3. Integrating (iii) $\dfrac{3}{4} x^{4/3} = \dfrac{3}{4} y^{4/3} + c_1$ or $x^{4/3} - y^{4/3} = \dfrac{4}{3} c_1 = c$

which is the required equation.

**Example 4.** *Find the orthogonal trajectories of $\dfrac{x^2}{a^2} + \dfrac{y^2}{a^2 + \lambda} = 1$, where $\lambda$ is arbitrary.*

**Sol.** The equation of the family of curves is

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 + \lambda} = 1 \qquad ...(i)$$

1. Differentiating (i)

$$\frac{2x}{a^2} + \frac{2y}{a^2 + \lambda} \frac{dy}{dx} = 0 \quad \text{or} \quad \frac{xy}{a^2} + \frac{y^2}{a^2 + \lambda} \frac{dy}{dx} = 0$$

or $$\frac{y^2}{a^2 + \lambda} = -\frac{xy}{a^2 p}, \qquad \text{where } p = \frac{dy}{dx}$$

$\therefore$ From (*i*), $$\frac{x^2}{a^2} - \frac{xy}{a^2 p} = 1 \quad [\lambda \text{ is eliminated}] \qquad ...(ii)$$

2. Replacing $p\left(=\frac{dy}{dx}\right)$ by $-\frac{1}{p}\left(=-\frac{dx}{dy}\right)$ the differential equation of orthogonal trajectories is

$$\frac{x^2}{a^2} + \frac{xy}{a^2} p = 1 \qquad \text{or} \quad p = \frac{a^2 - x^2}{xy}$$

or $$\frac{dy}{dx} = \frac{a^2 - x^2}{xy} \quad \text{or} \quad ydy = \left(\frac{a^2}{x} - x\right) dx \qquad ...(iii)$$

3. Integrating (*iii*) $$\frac{y^2}{2} = a^2 \log x - \frac{x^2}{2} + c_1$$

or $$x^2 + y^2 = 2a^2 \log x + 2c_1$$

or $$x^2 + y^2 = 2a^2 \log x + c$$

which is the required equation.

**Example 5.** *Find the orthogonal trajectories of a family of coaxial circles $x^2 + y^2 + 2gx + c = 0$, where g is a parameter and c constant.*

(Lucknow, 1998 ; Meerut, 1998 ; Osmania, 1997)

**Sol.** The equation of the family of coaxial circle is

$$x^2 + y^2 + 2gx + c = 0 \qquad ...(i)$$

1. Differentiating (*i*)

$$2x + 2y\frac{dy}{dx} + 2g = 0 \quad \Rightarrow \quad g = -\left(x + y\frac{dy}{dx}\right)$$

$\therefore$ From (*i*), $x^2 + y^2 - 2\left(x + y\frac{dy}{dx}\right)x + c = 0$

or $$y^2 - x^2 - 2xy\frac{dy}{dx} + c = 0 \qquad ...(ii)$$

which being free from '*g*' is the differential equation of the family (*i*).

2. Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$, the differential equation of orthogonal trajectories is

$$y^2 - x^2 + 2xy\frac{dx}{dy} + c = 0 \quad \text{or} \quad 2xy\frac{dx}{dy} - x^2 = -c - y^2$$

or $$\frac{dx}{dy} - \frac{1}{2y}\cdot x = -\frac{y^2 + c}{2xy} \qquad \text{[Bernoulli's Form]}$$

or $$2x\frac{dx}{dy} - \frac{1}{y}\cdot x^2 = -\frac{y^2 + c}{y} \qquad ...(iii)$$

3. Put $x^2 = z$ so that $$2x\frac{dx}{dy} = \frac{dz}{dy}$$

$\therefore$ From (*iii*), $\dfrac{dz}{dy} - \dfrac{1}{y} z = -\dfrac{y^2 + c}{y}$ which is linear in $z$.

$$\text{I.F.} = e^{\int -\frac{1}{y} dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1} = \frac{1}{y}$$

$\therefore$ The solution is $z \cdot \dfrac{1}{y} = \displaystyle\int -\dfrac{y^2 + c}{y} \cdot \dfrac{1}{y}\, dy - f$ ($f$ being the constant of integration)

$$= -\int (1 + cy^{-2})\, dy - f = -y - c \cdot \frac{y^{-1}}{-1} - f$$

or $$x^2 = -y^2 + c - fy \quad \text{or} \quad x^2 + y^2 + fy - c = 0$$

which is the required equation.

**Example 6.** *Find the orthogonal trajectories of the family of semi-cubical parabolas $ay^2 = x^3$.*

**Sol.** The equation of the family of semi-cubical parabolas is $ay^2 = x^3$ ...(*i*)

1. Differentiating (*i*) $$2ay \frac{dy}{dx} = 3x^2$$

$\Rightarrow$ $$a = \frac{3x^2}{2yp}, \quad \text{where} \quad p = \frac{dy}{dx}$$

$\therefore$ From (*i*), $$\frac{3x^2 y}{2p} = x^3 \quad \text{or} \quad 3y = 2xp \qquad \text{...(ii)}$$

2. Replacing $p$ by $-\dfrac{1}{p}$, the differential equation of orthogonal trajectories is

$$3y = -\frac{2x}{p} \quad \text{or} \quad 3yp + 2x = 0$$

or $$3y \frac{dy}{dx} + 2x = 0 \quad \text{or} \quad 3ydy + 2xdx = 0 \qquad \text{...(iii)}$$

3. Integrating (*iii*) $\dfrac{3y^2}{2} + x^2 = c_1$ or $2x^2 + 3y^2 + 2c_1 = c$

which is the required equation.

**Example 7.** *Prove that the system of confocal and co-axial parabolas $y^2 = 4a(x + a)$ is self-orthogonal.* (Kerala, 2001)

**Sol.** The equation of the system of confocal and co-axial parabolas is

$$y^2 = 4a(x + a) \qquad \text{...(i)}$$

1. Differentiating (*i*)

$$2y \frac{dy}{dx} = 4a \quad \Rightarrow \quad a = \frac{1}{2} y \frac{dy}{dx}$$

$\therefore$ From (*i*), $$y^2 = 2y \frac{dy}{dx}\left(x + \frac{1}{2} y \frac{dy}{dx}\right) \quad \text{or} \quad y^2 = 2xy \frac{dy}{dx} + y^2 \left(\frac{dy}{dx}\right)^2$$

**Example 9.** *Find the equation of a set of curves each member of which cuts every member of the family $xy = c^2$ at the constant angle $\frac{\pi}{4}$.*

**Sol.** The equation of the family of curves is $xy = c^2$ ...(*i*)

(1) Differentiating (*i*), $y + x\frac{dy}{dx} = 0$ or $x + yp = 0$, ...(*ii*)

where $p = \frac{dy}{dx}$ is the differential equation of the family (*i*).

(2) Replacing $p$ by

$$\frac{\frac{dy}{dx} + \tan\frac{\pi}{4}}{1 - \frac{dy}{dx} \cdot \tan\frac{\pi}{4}} = \frac{p+1}{1-p}$$

the differential equation of $\frac{\pi}{4}$ trajectories is

$$y + x \cdot \frac{p+1}{1-p} = 0 \quad \text{or} \quad y(1-p) + x(p+1) = 0$$

or $$(y - x)p = y + x \quad \text{or} \quad \frac{dy}{dx} = \frac{y+x}{y-x} \qquad \text{...(iii)}$$

which is homogeneous.

(3) Put $y = vx$ so that $\frac{dy}{dx} = v + x\frac{dv}{dx}$

(*iii*) becomes $$v + x\frac{dv}{dx} = \frac{vx + x}{vx - x} = \frac{v+1}{v-1}$$

or $$x\frac{dv}{dx} = \frac{v+1}{v-1} - v = \frac{1 + 2v - v^2}{v-1}$$

or $$\frac{v-1}{1+2v-v^2}\, dv = \frac{dx}{x} \quad \text{or} \quad \frac{2-2v}{1+2v-v^2}\, dv = -\frac{2}{x}\, dx$$

Integrating $\log(1 + 2v - v^2) = -2\log x + \log c$

or $\log(1 + 2v - v^2) + 2\log x = \log c$

or $$\log\left(1 + \frac{2y}{x} - \frac{y^2}{x^2}\right)x^2 = \log c \quad \text{or} \quad x^2 + 2xy - y^2 = c$$

which is the required equation.

**Example 10.** *Determine the 45° trajectories of the family of concentric circles $x^2 + y^2 = c^2$.*

**Sol.** The equation of the family of concentric circles is $x^2 + y^2 = c^2$ ...(*i*)

(1) Differentiating (*i*), $2x + 2y\frac{dy}{dx} = 0$ or $x + yp = 0$, where $p = \frac{dy}{dx}$ ...(*ii*)

is the differential equation of the family (*i*).

1. Taking logarithms,

$$n \log r + \log \sin n\theta = n \log a$$

Differentiating w.r.t. $\theta$,

$$\frac{n}{r} \cdot \frac{dr}{d\theta} = \frac{n \cos n\theta}{\sin n\theta} = 0 \quad \text{or} \quad \frac{dr}{d\theta} = -r \cot n\theta \qquad ...(ii)$$

which is the differential equation of the family $(i)$.

2. Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$, the differential equation of orthogonal trajectories is

$$-r^2 \frac{d\theta}{dr} = -r \cot n\theta \quad \text{or} \quad \frac{dr}{r} = \tan n\theta \, d\theta \qquad ...(iii)$$

3. Integrating $(iii)$

$$\log r = \frac{1}{n} \log \sec n\theta + \log c \quad \text{or} \quad n \log r = \log \sec n\theta + n \log c$$

or $$\log r^n = \log c^n \sec n\theta \quad \text{or} \quad \log r^n = c^n \sec n\theta \quad \text{or} \quad r^n \cos n\theta = c^n$$

which is the required equation.

**Example 15.** *Determine the orthogonal trajectories of the system of curves $r^n \cos n\theta = a^n$.*

**Sol.** Please try yourself. [**Ans.** $r^n \sin n\theta = c^n$]

**Example 16.** *Determine the orthogonal trajectories of the system of curves $r^n = a^n \cos n\theta$ and hence find the orthogonal trajectories of the series of lemniscates $r^2 = a^2 \cos 2\theta$.*

**Sol.** The equation of the system of curves is

$$r^n = a^n \cos n\theta \qquad ...(i)$$

1. Taking logarithms

$$n \log r = n \log a + \log \cos n\theta$$

Differentiating w.r.t. $\theta$,

$$\frac{n}{r} \cdot \frac{dr}{d\theta} = \frac{-n \sin n\theta}{\cos n\theta} \quad \text{or} \quad \frac{dr}{d\theta} = -r \tan n\theta \qquad ...(ii)$$

which is the differential equation of the family $(ii)$.

2. Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$, the differential equation of the orthogonal trajectories is

$$-r^2 \frac{d\theta}{dr} = -r \tan n\theta \quad \text{or} \quad \frac{dr}{r} = \cot n\theta \, d\theta \qquad ...(iii)$$

3. Integrating $(iii)$

$$\log r = \frac{\log \sin n\theta}{n} + \log c \quad \text{or} \quad n \log r = \log \sin n\theta + n \log c$$

$$\log r^n = \log c^n \sin n\theta \quad \text{or} \quad r^n = c^n \sin n\theta$$

which is the required solution.

For the second part, putting $n = 2$ the equation of the orthogonal trajectories of the family of curves

$$r^2 = a^2 \cos 2\theta \text{ is } r^2 = c^2 \sin 2\theta.$$

**Example 17.** *Find the orthogonal trajectories of $r^n = a^n \sin n\theta$.*

**Sol.** Please try yourself. [**Ans.** $r^n = c^n \cos n\theta$]

2. Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$, the differential equation of orthogonal trajectories is

$$-r^2 \frac{d\theta}{dr} = 5 \cos 5\theta \quad \text{or} \quad -\frac{5dr}{r^2} = \sec 5\theta \, d\theta \qquad ...(iii)$$

3. Integrating (*iii*), $\frac{5}{r} = \frac{1}{5} \log (\sec 5\theta + \tan 5\theta) + c_1$ or $\log (\sec 5\theta + \tan 5\theta) = \frac{25}{r} - 5c_1$

or $$\sec 5\theta + \tan 5\theta = e^{\frac{25}{r} - 5c_1} = e^{-5c_1} . e^{\frac{25}{r}} \quad \text{or} \quad \sec 5\theta + \tan 5\theta = ce^{\frac{25}{r}}$$

[Replacing $e^{-5c_1}$ by $c$]

which is the required equation.

**Example 20.** *Find the equation of the system of orthogonal trajectories of a series of confocal and coaxial parabolas*

$$r = \frac{2a}{1 + \cos\theta}.$$

**Sol.** The equation of the series of confocal and coaxial parabolas is

$$r = \frac{2a}{1 + \cos\theta} \quad \text{or} \quad r(1 + \cos\theta) = 2a \qquad ...(i)$$

1. Differentiating (*i*) w.r.t. $\theta$,

$$\frac{dr}{d\theta}(1 + \cos\theta) - r \sin\theta = 0$$

or $$\frac{dr}{d\theta} = \frac{r \sin\theta}{1 + \cos\theta} = \frac{2r \sin\frac{\theta}{2} \cos\frac{\theta}{2}}{2 \cos^2\frac{\theta}{2}} \quad \text{or} \quad \frac{dr}{d\theta} = r \tan\frac{\theta}{2} \qquad ...(ii)$$

which is the D.E. of the family (*i*).

2. Replacing $\frac{dr}{d\theta}$ by $-r^2 \frac{d\theta}{dr}$, the D.E. of orthogonal trajectories is

$$-r^2 \frac{d\theta}{dr} = r \tan\frac{\theta}{2} \quad \text{or} \quad \frac{dr}{r} = -\cot\frac{\theta}{2} \, d\theta \qquad ...(iii)$$

3. Integrating (*iii*), $\log r = -\frac{\log \sin\frac{\theta}{2}}{\frac{1}{2}} + \log c$

or $$\log r = -2 \log \sin\frac{\theta}{2} + \log c = \log \frac{c}{\sin^2\frac{\theta}{2}}$$

or $$r = \frac{c}{\sin^2\frac{\theta}{2}} \quad \text{or} \quad r = \frac{2c}{1 - \cos\theta}$$

which is the required equation.

# 7

# Equations of the First Order but not of the First Degree

**Definition.** A differential equation of the first order but not of the first degree is an equation of the form

$$P_0 p^n + P_1 p^{n-1} + P_2 p^{n-2} + \ldots\ldots + P_{n-1} p + P_n = 0$$

where $p = \dfrac{dy}{dx}$ and $P_0, P_1, P_2, \ldots\ldots, P_n$ are functions of $x$ and $y$ ($n \geq 2, n \in N$).

**Note. It is a convention to denote $\dfrac{dy}{dx}$ by p in equations of this type.**

**Types of Equations.** Differential equations of the first order but not of the first degree can be solved by one (or more) of the four methods given below. In each of the four methods, the problem is reduced to that of solving one or more equations of the first order and first degree.

**Type I. Equations solvable for p.**

Suppose the equation of the $n$th degree in $p$ is $(x, y, p) = 0$ ...(*i*)

Since it is solvable for $p$, it can be put in the form

$$[p - F_1(x, y)][p - F_2(x, y)] \ldots [p - F_n(x, y)] = 0$$

Equations each factor of the above form to zero, we get $n$ equations of the first order and first degree, *viz*.

$$p = F_1(x, y) \ ; p = F_2(x, y) \ ; \ldots\ldots \ ; p = F_n(x, y)$$

Let the solutions of these $n$ component equations be respectively

$$f_1(x, y, c_1) = 0, f_2(x, y, c_2) = 0, \ldots\ldots, f_n(x, y, c_n) = 0.$$

**Since the given equation is of the first order, its general solution cannot have more than one arbitrary constant.**

$\therefore$ Taking $c_1 = c_2 = \ldots.. = c_n = C$ (*say*)

The general solution of (*i*) is $f_1(x, y, c), f_2(x, y, c) \ldots\ldots f_n(x, y, c) = 0$.

**Note. It is to be remembered that the same arbitrary constant "C" is to be taken in the solution of each component equation of (i), since it is of the first order.**

**Example 1.** *Solve the following equations :*

(*i*) $p^2 + p = 6$ (*ii*) $p^2 - x^5 = 0$ (*iii*) $p^3 = ax^4$ *or* $\left(\dfrac{dy}{dx}\right)^3 = ax^4$.

**Sol.** (*i*) The given equation is $p^2 + p - 6 = 0$ or $(p + 3)(p - 2) = 0$

$\therefore$ either $p + 3 = 0$ | or $p - 2 = 0$

$\Rightarrow \dfrac{dy}{dx} + 3 = 0$ | $\Rightarrow \dfrac{dy}{dx} - 2 = 0$

$\Rightarrow dy + 3dx = 0$ | $\Rightarrow dy - 2dx = 0$

Integrating $y + 3x = c$ | Integrating $y - 2x = c$

$\therefore$ The complete solution of the given equation is $(y + 3x - c)(y - 2x - c) = 0$.

(*ii*) The given equation is $p^2 = x^5 \Rightarrow p = \pm x^{5/2}$

either $p = x^{5/2}$ | or $p = -x^{5/2}$

$\Rightarrow \dfrac{dy}{dx} = x^{5/2}$ | $\Rightarrow \dfrac{dy}{dx} = -x^{5/2}$

$\Rightarrow dy = x^{5/2}\, dx$ | $\Rightarrow dy = -x^{5/2}\, dx$

Integrating $y = \dfrac{2}{7} x^{7/2} + c$ | Integrating $y = -\dfrac{2}{7} x^{7/2} + c$

$\Rightarrow 7(y - c) - 2x^{7/2} = 0$ | $\Rightarrow 7(y - c) + 2x^{7/2} = 0$

$\therefore$ The complete solution of the given equation is

$$[7(y - c) - 2x^{7/2}]\,[7(y - c) + 2x^{7/2}] = 0$$

or
$$49(y - c)^2 - 4x^7 = 0 \quad \text{or} \quad 49(y - c)^2 = 4x^7.$$

(*iii*) The given equation is $p^3 = ax^4$ or $p = a^{1/3} x^{4/3}$

$$\Rightarrow \frac{dy}{dx} = a^{1/3} x^{4/3} \Rightarrow dy = a^{1/3} x^{4/3}\, dx$$

Integrating
$$y = a^{1/3} \cdot \frac{x^{7/3}}{\frac{7}{3}} + c$$

or
$$y - c = \frac{3}{7} a^{1/3} x^{7/3} \quad \text{or} \quad 7(y - c) = 3a^{1/3} x^{7/3}$$

Cubing, $343(y - c)^3 = 27\, ax^7$.

**Example 2.** *Solve the following :*

(*i*) $p^2 + p(x + y) + xy = 0$ (*ii*) $p^2 + 2px - 3x^2 = 0$

(*iii*) $x^2 \left(\dfrac{dy}{dx}\right)^2 + xy \dfrac{dy}{dx} - 6y^2 = 0$ (*iv*) $x^2 \left(\dfrac{dy}{dx}\right)^2 + 3xy \dfrac{dy}{dx} + 2y^2 = 0$

(*v*) $(p - xy)(p - x^2)(p - y^2) = 0$ (*vi*) $y^2 + xyp - x^2p^2 = 0$.

**Sol.** (*i*) The given equation is $p^2 + p(x + y) + xy = 0$ or $(p + x)(p + y) = 0$

$\therefore$ either $p + x = 0$ | or $p + y = 0$

$\Rightarrow \dfrac{dy}{dx} + x = 0$ | $\Rightarrow \dfrac{dy}{dx} + y = 0$

$\Rightarrow dy + xdx = 0$ | $\Rightarrow \dfrac{dy}{y} + dx = 0$

Integrating $y + \dfrac{x^2}{2} = c_1$ | Integrating $\log y + x = c$

$\Rightarrow 2y + x^2 = c$

$\therefore$ The complete solution of the given equation is

$$(2y + x^2 - c)(\log y + x - c) = 0.$$

(*ii*) The given equation is $p^2 + 2px - 3x^2 = 0$ or $(p + 3x)(p - x) = 0$

The component equations are

$$p + 3x = 0 \quad \text{and} \quad p - x = 0$$

$$\Rightarrow \quad \frac{dy}{dx} + 3x = 0 \quad \text{and} \quad \frac{dy}{dx} - x = 0$$

$$\Rightarrow \quad dy + 3x\,dx = 0 \quad \text{and} \quad dy - xdx = 0$$

Their solutions are $y + \dfrac{3x^2}{2} = c_1$ and $y - \dfrac{x^2}{2} = c_2$

$$\Rightarrow \quad 2y + 3x^2 = c \quad \text{and} \quad 2y - x^2 = c$$

$\therefore$ The complete solution of the given equation is $(2y + 3x^2 - c)(2y - x^2 - c) = 0.$

(*iii*) The given equation is $x^2p^2 + xy\,p - 6y^2 = 0$ $\left[\text{writing } p \text{ for } \dfrac{dy}{dx}\right]$

or $$(xp + 3y)(xp - 2y) = 0$$

The component equations are $x\,\dfrac{dy}{dx} + 3y = 0$ and $x\,\dfrac{dy}{dx} - 2y = 0$

$$\Rightarrow \quad \frac{dy}{y} + 3\,\frac{dx}{x} = 0 \quad \text{and} \quad \frac{dy}{y} - 2\,\frac{dx}{x} = 0$$

Integrating, we get $\log\, y + 3\log x = \log c$ and $\log y - 2\log x = \log c$

$$\Rightarrow \quad \log yx^3 = \log c \quad \text{and} \quad \log\frac{y}{x^2} = \log c$$

$$\Rightarrow \quad yx^3 = c \quad \text{and} \quad \frac{y}{x^2} = c$$

$\therefore$ The complete solution of the given equation is

$$(yx^3 - c)\left(\frac{y}{x^2} - c\right) = 0.$$

(*iv*) Please try yourself. **[Ans.** $(xy - c)(yx^2 - c) = 0$**]**

(*v*) The component equations are

$$\frac{dy}{dx} - xy = 0,\ \frac{dy}{dx} - x^2 = 0,\ \frac{dy}{dx} - y^2 = 0$$

$$\Rightarrow \quad \frac{dy}{y} - xdx = 0,\ dy - x^2dx = 0,\ \frac{dy}{y^2} - dx = 0$$

Integrating, we get

$$\log y - \frac{x^2}{2} = c_1,\ y - \frac{x^3}{3} = c_2,\ -\frac{1}{y} - x = c_3$$

$$\Rightarrow \quad 2\log y - x^2 = c,\ 3y - x^3 = c,\ \frac{1}{y} + x = c$$

$\therefore$ The complete solution of the given equation is

$$(2 \log y - x^2 - c)(3y - x^3 - c)\left(\frac{1}{y} + x - c\right) = 0.$$

(*vi*) The given equation is $y^2 + xyp - x^2p^2 = 0$ or $x^2p^2 - xyp - y^2 = 0$

Solving for $p$, we have

$$p = \frac{xy \pm \sqrt{x^2y^2 + 4x^2y^2}}{2x^2} = \frac{(1 \pm \sqrt{5})\, xy}{2x^2}$$

or $$\frac{dy}{dx} = \frac{(1 \pm \sqrt{5})\, y}{2x} \quad \text{or} \quad 2\,\frac{dy}{y} = (1 \pm \sqrt{5})\frac{dx}{x}$$

Integrating $\quad 2 \log y = (1 \pm \sqrt{5}) \log x + \log c$

or $$\log y^2 = \log (c x^{1 \pm \sqrt{5}}) \quad \text{or} \quad y^2 = c x^{1 \pm \sqrt{5}}$$

Hence the required solution is $(y^2 - c x^{1+\sqrt{5}})(y^2 - c x^{1-\sqrt{5}}) = 0$.

**Example 3.** *Solve the following :*

(*i*) $xp^2 + (y - x)p - y = 0$ (*ii*) $yp^2 + (x - y)p - x = 0$ (Delhi, 1998, 2000)

(*iii*) $yp^2 - (1 + xy)p + x = 0$ (*iv*) $p^2 - 2p \cosh x + 1 = 0$

(*v*) $p^2 + 2py \cot x = y^2$. (Delhi, 1996)

**Sol.** (*i*) The given equation is $xp^2 + (y - x)p - y = 0$ or $(p - 1)(xp + y) = 0$

The component equations are

$$p - 1 = 0 \quad \text{and} \quad xp + y = 0$$

$$\Rightarrow \quad \frac{dy}{dx} - 1 = 0 \quad \text{and} \quad x\frac{dy}{dx} + y = 0$$

$$\Rightarrow \quad dy - dx = 0 \quad \text{and} \quad \frac{dy}{y} + \frac{dx}{x} = 0$$

Integrating, we get

$$y - x = c \quad \text{and} \quad \log y + \log x = \log c$$

$$\Rightarrow \quad y - x = c \quad \text{and} \quad xy = c$$

$\therefore$ The complete solution of the given equation is $(y - x - c)(xy - c) = 0$.

(*ii*) Please try yourself. [**Ans.** $(y - x - c)(x^2 + y^2 - c) = 0$]

(*iii*) The given equation is $yp^2 - (1 + xy)p + x = 0$

or $$yp^2 - p - xy\,p + x = 0 \quad \text{or} \quad p(yp - 1) - x(yp - 1) = 0$$

or $$(p - x)(yp - 1) = 0$$

The component equations are

$$p - x = 0 \quad \text{and} \quad yp - 1 = 0$$

$$\Rightarrow \quad \frac{dy}{dx} - x = 0 \quad \text{and} \quad y\frac{dy}{dx} - 1 = 0$$

$$\Rightarrow \quad dy - xdx = 0 \quad \text{and} \quad ydy - dx = 0$$

**Sol.** (*i*) The given equation is $x^2p^2 - 2xyp + 2y^2 - x^2 = 0$

$$\Rightarrow \qquad p = \frac{2xy \pm \sqrt{4x^2y^2 - 4x^2(2y^2 - x^2)}}{2x^2} = \frac{2xy + 2x\sqrt{x^2 - y^2}}{2x^2}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{y + \sqrt{x^2 - y^2}}{x} \qquad \text{...}(i)$$

which is homogeneous in $x$ and $y$

Put $y = vx$ so that $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\therefore \quad \text{From (1),} \quad v + x\frac{dv}{dx} = \frac{vx \pm \sqrt{x^2 - v^2x^2}}{x} = v \pm \sqrt{1 - v^2}$$

$$\Rightarrow \qquad x\frac{dv}{dx} = \pm\sqrt{1 - v^2} \quad \Rightarrow \quad \frac{dv}{\sqrt{1 - v^2}} = \pm\frac{dx}{x}$$

Integrating, we get $\sin^{-1} v = \log x + \log c$ or $\sin^{-1} v = -\log x - \log c$

$\Rightarrow$ $\sin^{-1}\dfrac{y}{x} = \pm \log cx$ which form the required solution.

(*ii*) The given equation is $xyp^2 + p(3x^2 - 2y^2) - 6xy = 0$

$$\Rightarrow \qquad p = \frac{-(3x^2 - 2y^2) \pm \sqrt{(3x^2 - 2y^2)^2 + 24x^2y^2}}{2xy}$$

$$= \frac{-3x^2 + 2y^2 \pm (3x^2 + 2y^2)}{2xy} = \frac{4y^2}{2xy} \quad \text{or} \quad \frac{-6x^2}{2xy}$$

$$\Rightarrow \qquad \frac{dy}{dx} = \frac{2y}{x} \quad \text{or} \quad -\frac{3x}{y}$$

The component equations are

$$\frac{dy}{dx} = \frac{2y}{x} \qquad \text{and} \qquad \frac{dy}{dx} = -\frac{3x}{y}$$

$$\Rightarrow \qquad \frac{dy}{y} = \frac{2dx}{x} \qquad \text{and} \qquad ydy = -3xdx$$

Their solutions are

$$\log y = 2\log x + \log c \quad \text{and} \quad \frac{y^2}{2} = -\frac{3x^2}{2} + c_1$$

$$\Rightarrow \qquad \log y - \log x^2 = \log c \qquad \text{and} \quad y^2 + 3x^2 = 2c_1$$

$$\Rightarrow \qquad \frac{y}{x^2} = c \qquad \text{and} \quad y^2 + 3x^2 = c$$

$\therefore$ The complete solution of the given equation is $(y - cx^2)(y^2 + 3x^2 - c) = 0$.

(*iii*) The given equation is

$$xyp^2 - x^2p - y^2p + xy = 0$$

$$\Rightarrow \qquad xp(yp - x) - y(yp - x) = 0 \quad \Rightarrow \quad (yp - x)(xp - y) = 0$$

Proceed further yourself. **[Ans.** $(y^2 - x^2 - c)(y - cx) = 0$]

(*iv*) The given equation is

$$p(p^2 + 3xp - y^3p - 3xy^3) = 0$$

$$\Rightarrow \quad p[p(p + 3x) - y^3(p + 3x)] = 0$$

$$\Rightarrow \quad p(p + 3x)(p - y^3) = 0$$

The component equations are

$$\frac{dy}{dx} = 0, \quad \frac{dy}{dx} + 3x = 0, \quad \frac{dy}{dx} - y^3 = 0$$

$$\Rightarrow \quad dy = 0, \quad dy + 3xdx = 0, \quad \frac{dy}{y^3} - dx = 0$$

Their solutions are $\quad y = c, \quad y + \dfrac{3x^2}{2} = c_1, \quad -\dfrac{1}{2y^2} - x = c_2$

$$\Rightarrow \quad y = c, \quad 2y + 3x^2 = c, \quad \frac{1}{y^2} + 2x = c$$

$\therefore$ The complete solution of the given equation is

$$(y - c)(2y + 3x^2 - c)\left(\frac{1}{y^2} + 2x - c\right) = 0.$$

**Example 5.** *Solve the following :*

(*i*) $xyp^2 + (x^2 + xy + y^2)p + x^2 + xy = 0$ (*ii*) $4y^2p^2 + 2pxy(3x + 1) + 3x^3 = 0$

(*iii*) $xy^2(p^2 + 2) = 2py^3 + x^3$ (*iv*) $\left(1 - y^2 + \dfrac{y^4}{x^2}\right)p^2 - 2\dfrac{y}{x}p + \dfrac{y^2}{x^2} = 0.$

**Sol.** (*i*) The given equation can be written as

$$xyp^2 + x^2p + xyp + x^2 + y^2p + xy = 0$$

$$\Rightarrow \quad xp(yp + x) + x(yp + x) + y(yp + x) = 0$$

$$\Rightarrow \quad (yp + x)(xp + x + y) = 0$$

The component equations are

$$y\frac{dy}{dx} + x = 0 \qquad ...(1)$$

and

$$x\frac{dy}{dx} + x + y = 0 \qquad ...(2)$$

From (1), $\quad ydy + xdx = 0$

Integrating $\quad \dfrac{y^2}{2} + \dfrac{x^2}{2} = c_1 \quad \Rightarrow \quad y^2 + x^2 = c$

From (2), $(x + y)dx + xdy = 0$

which is exact $\quad \left[\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = 1\right]$

$\therefore$ Its solution is $\quad \displaystyle\int_{y\text{-constant}} (x + y)\, dx = c_1$

or $$\frac{x^2}{2} + yx = c_1 \quad \Rightarrow \quad x^2 + 2xy = c$$

$\therefore$ The complete solution of the given equation is $(x^2 + y^2 - c)(x^2 + 2xy - c) = 0$.

(*ii*) The given equation can be written as

$$4y^2p^2 + 6x^2yp + 2pxy + 3x^3 = 0$$

$$\Rightarrow \quad 2yp(2yp + 3x^2) + x(2yp + 3x^2) = 0$$

$$\Rightarrow \quad (2yp + 3x^2)(2yp + x) = 0$$

Proceed further yourself. **[Ans.** $(y^2 + x^3 - c)(2y^2 + x^3 - c) = 0$]

(*iii*) The given equation can be written as

$$xy^2p^2 - x^3 + 2xy^2 - 2py^3 = 0$$

$$\Rightarrow \quad x(y^2p^2 - x^2) - 2y^2(yp - x) = 0$$

$$\Rightarrow \quad (yp - x)[x(yp + x) - 2y^2] = 0$$

$$\Rightarrow \quad (yp - x)\,(xyp + x^2 - 2y^2) = 0$$

The component equations are $$y\,\frac{dy}{dx} - x = 0 \qquad ...(1)$$

and $$xy\,\frac{dy}{dx} + x^2 - 2y^2 = 0 \qquad ...(2)$$

From (1), $$ydy - xdx = 0$$

Integrating $$\frac{y^2}{2} - \frac{x^2}{2} = c_1 \quad \Rightarrow \quad y^2 - x^2 = c$$

From (2), $$\frac{dy}{dx} = \frac{2y^2 - x^2}{xy}$$

Put $y = vx$ so that $\dfrac{dy}{dx} = v + x\,\dfrac{dv}{dx}$

$$v + x\,\frac{dv}{dx} = \frac{2v^2x^2 - x^2}{x\,.\,vx} = \frac{2v^2 - 1}{v}$$

$$\Rightarrow \quad x\,\frac{dy}{dx} = \frac{2v^2 - 1}{v} - v = \frac{v^2 - 1}{v} \quad \Rightarrow \quad \frac{2v}{v^2 - 1}\,dv = \frac{2dx}{x}$$

Integrating $\log(v^2 - 1) = 2\log x + \log c$

or $$\log\left(\frac{y^2}{x^2} - 1\right) = \log x^2 + \log c$$

or $$\frac{y^2 - x^2}{x^2} = cx^2 \quad \Rightarrow \quad y^2 - x^2 = cx^4$$

$\therefore$ The complete solution of the given equation is

$$(y^2 - x^2 - c)(y^2 - x^2 - cx^4) = 0.$$

(*iv*) The given equation is

$$p^2 - p^2y^2 + \frac{y^4}{x^2}\,p^2 - 2\,\frac{y}{x}\,p + \frac{y^2}{x^2} = 0$$

or $$\left(p^2 - 2\frac{y}{x}p + \frac{y^2}{x^2}\right) - p^2y^2\left(1 - \frac{y^2}{x^2}\right) = 0$$

or $$\left(p - \frac{y}{x}\right)^2 - p^2y^2\left(1 - \frac{y^2}{x^2}\right) = 0$$

or $$(px - y)^2 - p^2y^2(x^2 - y^2) = 0$$

or $$(px - y + py\sqrt{x^2 + y^2})\,(px - y - py\,\sqrt{x^2 - y^2}\,) = 0$$

or $$[p(x + y\sqrt{x^2 - y^2}\,) - y][p(x - y\,\sqrt{x^2 - y^2}\,) - y] = 0$$

The component equations are

$$\left(x \pm y\,\sqrt{x^2 - y^2}\right)\frac{dy}{dx} = y \quad \text{or} \quad \frac{dx}{dy} = \frac{x \pm y\,\sqrt{x^2 - y^2}}{y}$$

Put $x = vy$ so that $\dfrac{dx}{dy} = v + y\,\dfrac{dv}{dy}$

$$\therefore \qquad v + y\,\frac{dv}{dy} = \frac{vy \pm y\,\sqrt{v^2y^2 - y^2}}{y} = v \pm y\,\sqrt{v^2 - 1}$$

$$\Rightarrow \qquad y\,\frac{dv}{dy} = \pm\, y\,\sqrt{v^2 - 1} \quad \Rightarrow \quad \frac{dv}{dy} = \pm\,\sqrt{v^2 - 1}$$

$$\Rightarrow \qquad \frac{dv}{\sqrt{v^2 - 1}} = \pm\, dy$$

Integrating $\cosh^{-1} v = \pm\, y + c$ or $\log\,(v + \sqrt{v^2 - 1}\,) = \pm\, y + c$

or $$\log\left(\frac{x}{y} + \sqrt{\frac{x^2}{y^2} - 1}\right) = \pm\, y + c \quad \text{or} \quad \log\frac{x + \sqrt{x^2 - y^2}}{y} = \pm\, y + c$$

$\therefore$ The complete solution of the given equation is

$$\left(\log\frac{x + \sqrt{x^2 - y^2}}{y} - y - c\right)\left(\log\frac{x + \sqrt{x^2 - y^2}}{y} + y - c\right) = 0.$$

**Example 6.** *Solve the following :*

*(i)* $(px + x + y)(p + x + y)(p + 2x) = 0$ *(ii)* $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$

*(iii)* $p^3 - p(x^2 + xy + y^2) + xy\,(x + y) = 0.$

**Sol.** (*i*) The component equations are

$$x\,\frac{dy}{dx} + x + y = 0 \qquad \text{...(1)}$$

$$\frac{dy}{dx} + x + y = 0 \qquad \text{...(2)}$$

and $$\frac{dy}{dx} + 2x = 0 \qquad \text{...(3)}$$

From (1), $(x + y)dx + xdy = 0$ which is exact $\left[\because \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} = 1\right]$

$\therefore$ Its solution is $\displaystyle\int_{y\text{-constant}} (x + y)dx = c_1$

or $$\frac{x^2}{2} + yx = c_1 \Rightarrow x^2 + 2xy = c$$

From (2), $\dfrac{dy}{dx} + y = -x$ which is linear in $y$.

$$\text{I.F.} = e^{\int 1dx} = e^x$$

$\therefore$ Its solution is $\displaystyle y.\, e^x = \int -xe^x\, dx + c = -(x-1)e^x + c$

or $$y = -x + 1 + ce^{-x}$$

From (3), $dy + 2xdx = 0$

Integrating $y + x^2 = c$

$\therefore$ The complete solution of the given equation is

$$(x^2 + 2xy - c)(y + x - 1 - ce^{-x})(y + x^2 - c) = 0.$$

(*ii*) The given equation can be written as

$$p(p^2 + 2xp - y^2p - 2xy^2) = 0$$

or $$p[p(p + 2x) - y^2(p + 2x)] = 0 \quad \text{or} \quad p(p + 2x)(p - y^2) = 0$$

Proceed further yourself. $\left[\textbf{Ans. } (y-c)(y+x^2-c)\left(\dfrac{1}{y}+x+c\right)=0\right]$

(*iii*) The given equations can be written as

$$p^3 - px^2 - pxy + x^2y - py^2 + xy^2 = 0$$

or $$p(p^2 - x^2) - xy(p - x) - y^2(p - x) = 0$$

or $$(p - x)[p(p + x) - xy - y^2] = 0$$

or $$(p - x)(p^2 - y^2 + px - xy) = 0$$

or $$(p - x)[(p + y)(p - y) + x(p - y)] = 0$$

or $$(p - x)(p - y)(p + x + y) = 0$$

Proceed further yourself. [**Ans.** $(2y - x^2 - c)(y - ce^x)(y + x - 1 - ce^{-x}) = 0$]

**Type II. Equations solvable for y.**

If the equation $f(x, y, p) = 0$ is solvable for $y$, we can express $y$ explicitly in terms of $x$ and $p$. Thus, an equation solvable for $y$ can be put as $y = \phi(x, p)$ ...(*i*)

Differentiating (*i*) w.r.t. $x$, we get

$$\frac{dy}{dx} = p = F\left(x, p, \frac{dp}{dx}\right) \qquad ...(ii)$$

which is a differential equation involving two variables $x$ and $p$. Let its solution be

$$\psi(x, p, c) = 0 \qquad ...(iii)$$

Eliminating $p$ between (*i*) and (*iii*), we get the required solution of (*i*).

If $p$ cannot be easily eliminated, we solve the equations $(i)$ and $(iii)$ for $x$ and $y$ in terms of $p$. Then the parametric equations

$\left.\begin{array}{l} x = f_1(p, c) \\ y = f_2(p, c) \end{array}\right]$ $p$ being the parameter together constitute the solution of $(i)$.

**Note 1. If equation (ii) viz. $p - F\left(x, p, \frac{dp}{dx}\right) = 0$ can be factorised and put in the form $F_1(x, p) . F_2\left(x, p, \frac{dp}{dx}\right) = 0$ cancel the factor $F_1(x, p)$ which does not involve $\frac{dp}{dx}$ and solve the remaining equation $F_2\left(x, p, \frac{dp}{dx}\right) = 0$.**

**Let its solution be $\psi_1(x, p, c) = 0$** ...$(iv)$

**Eliminating p between (i) and (iv), we get the required solution of (i).**

**Note 2. If instead of cancelling the factor $F_1(x, p)$, we eliminate p between (i) and $F_1(x, p) = 0$, we get an equation involving no constant. This is called the singular solution of (i) and we shall discuss it in Chapter 6.**

**Special Case I of Type II.** **[Clairaut's Equation]**

**Definition.** A differential equation of the form $y = px + f(p)$ is known as Clairaut's Equation.

**To prove that the solution of the equation**

**$y = px + f(p)$ is $y = cx + f(c)$.** [*Obtained by changing p to c*]

The given equation is $y = px + f(p)$ ...$(i)$

It is solvable for $y$.

Differentiating $(i)$ w.r.t. $x$, we get

$$\frac{dy}{dx} = p + x\frac{dp}{dx} + f'(p) . \frac{dp}{dx}$$

$$\Rightarrow \quad p = p + x\frac{dp}{dx} f'(p). \frac{dp}{dx} \quad \Rightarrow \quad x\frac{dp}{dx} + f'(p). \frac{dp}{dx} = 0$$

Factorising $\frac{dp}{dx}[x + f'(p)] = 0$

Cancelling the factor $x + f'(p)$ which does not involve $\frac{dp}{dx}$, we have $\frac{dp}{dx} = 0$.

Integrating, $p = c$ ...$(ii)$

Eliminating $p$ between $(i)$ and $(ii)$, the required solution of $(i)$ is $y = cx + f(c)$.

**Special Case II of Type II** **[Lagrange's Equation]**

**To solve the equation $y = x\phi(p) + f(p)$**

The given equation is $y = x\phi(p) + f(p)$ ...$(i)$

It is solvable for $y$.

Differentiating $(i)$ w.r.t. $x$, we get

$$\frac{dy}{dx} = \phi(p) + x\phi'(p) . \frac{dp}{dx} + f'(p) . \frac{dp}{dx}$$

$$\Rightarrow \qquad p - \phi(p) = [x\phi'(p) + f'(p)] \frac{dp}{dx}$$

$$\Rightarrow \qquad [p - \phi(p)] \frac{dx}{dp} - x\phi'(p) = f'(p)$$

which is a linear equation in $x$ and hence solvable for $x$ in terms of $p$. Let the solution be

$$x = \psi(p, c) \qquad \text{...(ii)}$$

With this value of $x$, from (*i*) $y = \psi(p, c)\, \phi(p) + f(p)$ ...(*iii*)

Equations (*ii*) and (*iii*) together form the solution of (*i*) in parametric form.

**Example 1.** *Solve the following :*

(*i*) $y = px + p - p^2$ (*ii*) $y = px + a \tan^{-1} p$

(*iii*) $y = x \dfrac{dy}{dx} + \left(\dfrac{dy}{dx}\right)^2$ (*iv*) $y = px + \sqrt{a^2p^2 + b^2}$

(*v*) $y = px + e^p$ (*vi*) $px - y + p^3 = \dfrac{m^3}{p^3}$

(*vii*) $y = p(x - b) + \dfrac{a}{p}$ .

**Sol.** (*i*) The given equation is $y = px + (p - p^2)$

It is of *Clairaut's form viz.* $y = px + f(p)$.

$\therefore$ Its solution (on changing $p$ to $c$) is $y = cx + c - c^2$.

(*ii*) Please try yourself. [**Ans.** $y = cx + a \tan^{-1} c$]

(*iii*) Please try yourself. [The given equation is $y = px + p^2$] [**Ans.** $y = cx + c^2$]

(*iv*) Please try yourself. [**Ans.** $y = cx + \sqrt{a^2c^2 + b^2}$ ]

(*v*) Please try yourself. [**Ans.** $y = cx + e^c$]

(*vi*) The given equation is

$$px - y + p^3 = \frac{m^3}{p^3} \quad \text{or} \quad y = px + p^3 - \frac{m^3}{p^3}$$

It is of Clairaut's form *viz.* $y = px + f(p)$

$\therefore$ Its solution (on changing $p$ to $c$) is

$$y = cx + c^3 - \frac{m^3}{c^3} .$$

(*vii*) The given equation $y = px - pb + \dfrac{a}{p}$ is of Clairaut's form. Its solution is $y = cx - cb + \dfrac{a}{e}$ .

**Example 2.** *Solve the following :*

(*i*) $(y - px)(p - 1) = p$ (*ii*) $\sin px \cos y = \cos px \sin y + p$

(*iii*) $p = \tan (px - y)$ (*iv*) $(y - px)^2 = 1 + p^2$

(*v*) $xp^2 - yp + a = 0$ (*vi*) $p = \log (px - y)$. (Lucknow, 1998)

**Sol.** (*i*) The equation is

$$y - px = \frac{p}{p - 1} \quad \text{or} \quad y = px + \frac{p}{p - 1}$$

It is of Clairaut's form, hence its solution is

$$y = cx + \frac{c}{c-1}.$$

(*ii*) The given equation is $\sin px \cos y - \cos px \sin y = p$

or $\sin (px - y) = p$ or $px - y = \sin^{-1} p$ or $y = px - \sin^{-1} p$

It is of Clairaut's form and hence its solution is $y = cx - \sin^{-1} c$.

(*iii*) Please try yourself. [**Ans.** $y = cx - \tan^{-1} c$]

(*iv*) The given equation is

$$(y - px)^2 = 1 + p^2$$

or $$y - px = \pm \sqrt{1+p^2} \quad \text{or} \quad y = px \pm \sqrt{1+p^2}$$

Both the component equations are of Clairaut's form.

$\therefore$ The solution is $y = cx \pm \sqrt{1+c^2}$ or $y - cx = \pm \sqrt{1+c^2}$ or $(y - cx)^2 = 1 + c^2$.

(*v*) The given equation is

$$yp = xp^2 + a \quad \text{or} \quad y = px + \frac{a}{p}$$

It is of Clairaut's form and hence its solution is $y = cx + \dfrac{a}{c}$.

(*vi*) The given equation is $p = \log (px - y)$

or $$px - y = e^p \quad \text{or} \quad y = px - e^p$$

It is of Clairaut's form and hence its solution is $y = cx - e^c$.

**Example 3.** *Solve the following :*

(*i*) $\left(\dfrac{dy}{dx}\right)^2 (x^2 - a^2) - 2\left(\dfrac{dy}{dx}\right) xy + y^2 - b^2 = 0$

(*ii*) $y^2 + x^2\left(\dfrac{dy}{dx}\right)^2 - 2xy\dfrac{dy}{dx} = 4\left(\dfrac{dx}{dy}\right)^2$

(*iii*) $(x - a)p^2 + (x - y)p - y = 0$

(*iv*) $p^2x(x - 2) + p(2y - 2xy - x + 2) + y^2 + y = 0.$

**Sol.** (*i*) The given equation is $p^2(x^2 - a^2) - 2pxy + y^2 - b^2 = 0$

or $$y^2 - 2pxy + p^2x^2 = a^2p^2 + b^2$$

or $$(y - px)^2 = a^2p^2 + b^2$$

or $$y - px = \pm \sqrt{a^2p^2 + b^2}$$

or $$y = px \pm \sqrt{a^2p^2 + b^2}$$

Both the component equations are of Clairaut's form.

$\therefore$ The solution is $y = cx \pm \sqrt{a^2c^2 + b^2}$ or $(y - cx)^2 = a^2c^2 + b^2$.

(*ii*) The given equation is

$$y^2 + x^2p^2 - 2xyp = \frac{4}{p^2} \quad \text{or} \quad (y - px)^2 = \frac{4}{p^2}$$

Proceed further yourself. $\left[\textbf{Ans.}\ (y - cx)^2 = \dfrac{4}{c^2}\right]$

(*iii*) The given equation is $(x - a)p^2 + px = (1 + p)y$

or $$(1 + p)y = px(p + 1) - ap^2 \quad \text{or} \quad y = px - \frac{ap^2}{p+1}$$

which is of Clairaut's form and hence its solution is $y = cx - \dfrac{ac^2}{c+1}$.

(*iv*) The given equation is

$$p^2x^2 - 2p^2x + 2py - 2pxy - px + 2p + y^2 + y = 0$$

or $$(y^2 - 2pxy + p^2x^2) + 2p(y - px) + (y - px) + 2p = 0$$

or $$(y - px)^2 + (2p + 1)(y - px) + 2p = 0$$

or $$(y - px + 2p)(y - px + 1) = 0$$

Both the component equations are of Clairaut's form and hence the solution is

$$(y - cx + 2c)(y - cx + 1) = 0.$$

**Example 4.** *Solve the following :*

(*i*) $y = 2px - p^2$ (*ii*) $y + px = x^4 p^2$ (Delhi, 1998)

(*iii*) $y = 2px + p^4x^2$ (Delhi, 1998) (*iv*) $y = 3x + \log p$

(*v*) $y = 2px + f(xp^2)$ (*vi*) $y = 3xp + 4p^3$.

**Sol.** (*i*) The given equation is $y = 2px - p^2$ ...(*i*)

[Lagrange's Form]

Differentiating w.r.t. $x$, we have

$$p = 2p + 2x \frac{dp}{dx} - 2p \frac{dp}{dx}$$

$$\Rightarrow \quad (2x - 2p) \frac{dp}{dx} + p = 0 \quad \Rightarrow \quad p \frac{dx}{dp} + 2x - 2p = 0$$

$$\Rightarrow \quad \frac{dx}{dp} + \frac{2}{p} x = 2 \qquad \text{...(ii)}$$

which is linear in $x$

$$\text{I.F.} = e^{\int \frac{2}{p} dp} = e^{2 \log p} = e^{\log p^2} = p^2$$

$\therefore$ Solution of (*ii*) is $xp^2 = \int 2p^2 dp + c$

or $$xp^2 = \frac{2}{3} p^3 + c \quad \text{or} \quad x = \frac{2}{3} p + cp^{-2}$$

Putting this value of $x$ in (*i*)

$$y = 2p \left(\frac{2}{3} p + cp^{-2}\right) - p^2 = \frac{4}{3} p^2 + 2cp^{-1} - p^2$$

or $$y = \frac{1}{3} p^2 + 2cp^{-1}$$

Hence the solution of (*i*) in parametric form is

$$\left.\begin{aligned} x &= \frac{2}{3}p + cp^{-2} \\ y &= \frac{1}{3}p^2 + 2cp^{-1} \end{aligned}\right]$$

(*ii*) The given equation is $y + px = x^4p^2$ ...(*i*)

Differentiating w.r.t. $x$,

$$p + p + x\frac{dp}{dx} = 4x^3p^2 + 2x^4p\frac{dp}{dx}$$

$$\Rightarrow \quad \left(2p + x\frac{dp}{dx}\right) - 2px^3\left(2p + x\frac{dp}{dx}\right) = 0$$

$$\Rightarrow \quad (1 - 2px^3)\left(2p + x\frac{dp}{dx}\right) = 0$$

Rejecting the factor $(1 - 2px^3)$ which does not involve $\frac{dp}{dx}$, we get

$$2p + x\frac{dp}{dx} = 0 \quad \text{or} \quad \frac{dp}{p} + 2\frac{dx}{x} = 0$$

Integrating $\log p + 2\log x = \log c$ or $\log px^2 = \log c$ or $px^2 = c$

or $$p = \frac{c}{x^2} \qquad ...(ii)$$

Eliminating $p$ between (*i*) and (*ii*), we get

$$y = -\left(\frac{c}{x^2}\right).x + x^4\left(\frac{c^2}{x^4}\right) \quad \text{or} \quad y = -\frac{c}{x} + c^2$$

which is the required solution of (*i*).

(*iii*) The given equation is $y = 2px + p^4x^2$ ...(*i*)

Differentiating w.r.t. $x$

$$p = 2p + 2x\frac{dp}{dx} + 4p^3x^2\frac{dp}{dx} + 2p^4x$$

$$\Rightarrow \quad p + 2p^4x + 2x\frac{dp}{dx} + 4p^3x^2\frac{dp}{dx} = 0$$

$$\Rightarrow \quad p(1 + 2p^3x) + 2x\frac{dp}{dx}(1 + 2p^3x) = 0$$

$$\Rightarrow \quad (1 + 2p^3x)\left(p + 2x\frac{dp}{dx}\right) = 0$$

Rejecting the factor $(1 + 2p^3x)$ which does not involve $\frac{dp}{dx}$, we get

$$p + 2x\frac{dp}{dx} = 0 \quad \text{or} \quad \frac{2dp}{p} + \frac{dx}{x} = 0$$

Integrating $2 \log p + \log x = \log c$

or $$\log p^2x = \log c \quad \text{or} \quad p^2x = c \quad \text{or} \quad p^2 = \frac{c}{x} \qquad ...(ii)$$

The required solution of $(i)$ is obtained by eliminating $p$ between $(i)$ and $(ii)$

From $(i)$ $\quad y - p^4x^2 = 2px$

Squaring $\quad (y - p^4x^2)^2 = 4p^2x^2 \quad \text{or} \quad (y - c^2)^2 = 4 \cdot \frac{c}{x} x^2 \quad \text{or} \quad (y - c^2)^2 = 4cx$

which is the required solution of $(i)$.

**Note.** From $(ii)$ $\quad x = \frac{c}{p^2}$

Putting this value of $x$ in $(i)$

$$y = 2p \cdot \frac{c}{p^2} + p^4 \cdot \frac{c^2}{p^4} \quad \text{or} \quad y = \frac{2c}{p} + c^2$$

$\therefore$ The solution of $(i)$ in parametric form is

$$x = cp^{-2},\ y = 2cp^{-1} + c^2.$$

$(iv)$ The given equation is

$$y = 3x + \log p \qquad ...(i)$$

Differentiating w.r.t. $x$, we have

$$p = 3 + \frac{1}{p} \cdot \frac{dp}{dx} \quad \text{or} \quad p(p-3) = \frac{dp}{dx}$$

or $$dx = \frac{1}{p(p-3)} dp = \frac{1}{3}\left[\frac{1}{p-3} - \frac{1}{p}\right] dp$$

Integrating $$x = \frac{1}{3} [\log (p-3) - \log p] + \log c_1$$

or $$3x = \log \frac{p-3}{p} + 3 \log c_1 \quad \text{or} \quad 3x = \log \frac{p-3}{p} + \log {c_1}^2$$

or $$3x = \log \frac{{c_1}^3 (p-3)}{p} \quad \Rightarrow \quad \frac{{c_1}^3 (p-3)}{p} = e^{3x}$$

or $$\frac{p-3}{p} = ce^{2x} \quad \text{or} \quad 1 - \frac{3}{p} = ce^{3x}$$

or $$1 - ce^{3x} = \frac{3}{p} \quad \therefore \quad p = \frac{3}{1 - ce^{3x}}$$

Putting this value of $p$ in $(i)$, the required solution is

$$y = 3x + \log \frac{3}{1 - ce^{3x}}.$$

$(v)$ The given equation is $y = 2px + f(xp^2) \qquad ...(i)$

Differentiating $(i)$ w.r.t. $x$, we get

$$p = 2p + 2x \frac{dp}{dx} + f'(xp^2)\left[p^2 + 2xp \frac{dp}{dx}\right]$$

$$\Rightarrow \quad \left(p + 2x\frac{dp}{dx}\right) + pf'(xp^2)\left[p + 2x\frac{dp}{dx}\right] = 0$$

$$\Rightarrow \quad [1 + pf'(xp^2)]\left[p + 2x\frac{dp}{dx}\right] = 0$$

Rejecting the factor $[1 + pf'(xp^2)]$ which does not involve $\frac{dp}{dx}$, we get

$$p + 2x\frac{dp}{dx} = 0 \quad \text{or} \quad \frac{2dp}{p} + \frac{dx}{x} = 0$$

Integrating $2 \log p + \log x = \log c$

or $$\log p^2 x = \log c \quad \text{or} \quad p^2 x = c$$

$$\therefore \quad p = \sqrt{\frac{c}{x}}$$

Putting this value of $p$ in ($i$), the required solution is

$$y = 2x\sqrt{\frac{c}{x}} + f\left(x \cdot \frac{c}{x}\right) \quad \text{or} \quad y = 2\sqrt{cx} + f(c).$$

($vi$) The given equation is $y = 3xp + 4p^3$ ...(1)

Differentiating w.r.t. $x$, we have

$$p = 3p + 3x\frac{dp}{dx} + 12p^2\frac{dp}{dx}$$

$$\Rightarrow \quad (3x + 12p^2)\frac{dp}{dx} + 2p = 0 \quad \Rightarrow \quad 2p\frac{dx}{dp} + 3x + 12p^2 = 0$$

$$\Rightarrow \quad \frac{dx}{dp} + \frac{3}{2p}x = -6p \qquad ...(2)$$

which is linear in $x$.

$$\text{I.F.} = e^{\int \frac{3}{2p}dp} = e^{\frac{3}{2}\log p} = e^{\log p^{3/2}} = p^{3/2}$$

$\therefore$ Solution of (2) is

$$x \cdot p^{3/2} = \int -6p \cdot p^{3/2}\,dp + c \quad \text{or} \quad xp^{3/2} = -6\int p^{5/2}\,dp + c$$

or $$xp^{3/2} = -6 \cdot \frac{p^{7/2}}{7/2} + c \quad \text{or} \quad x = -\frac{12}{7}p^2 + cp^{-3/2}$$

Putting this value of $x$ in (1),

$$y = 3p\left(-\frac{12}{7}p^2 + cp^{-3/2}\right) + 4p^3 = \left(-\frac{36}{7} + 4\right)p^3 + 3cp^{-1/2}$$

or $$y = -\frac{8}{7}p^3 + \frac{3c}{\sqrt{p}}$$

Hence the solution of (1) in parametric form is

$$\left.\begin{aligned} x &= -\frac{12}{7}p^2 + cp^{-3/2} \\ y &= -\frac{8}{7}p^3 + \frac{3c}{\sqrt{p}} \end{aligned}\right]$$

**Example 1.** *Solve the following :*

(i) $y = 3px + 6p^2y^2$ (ii) $y = 2px + y^2p^3$

(iii) $yp^2 - 2xp + y = 0$ (iv) $y = p^2y + 2px.$

**Sol.** (i) The given equation is

$$y = 3px + 6p^2y^2 \quad \text{...(i)}$$

Solving for $x$, $$x = \frac{y}{3p} - 2py^2$$

Differentiating w.r.t. $y$,

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{3p} - \frac{y}{3p^2} \cdot \frac{dp}{dy} - 4py - 2y^2 \cdot \frac{dp}{dy}$$

$$\Rightarrow \quad 3p = p - y\frac{dp}{dy} - 12p^3y - 6p^2y^2\frac{dp}{dy}$$

$$\Rightarrow \quad 2p(1 + 6p^2y) + y\frac{dp}{dy}(1 + 6p^2y) = 0$$

$$\Rightarrow \quad (1 + 6p^2y)\left(2p + y\frac{dp}{dy}\right) = 0$$

Neglecting the first factor which does not involve $\frac{dp}{dy}$, we have

$$2p + y\frac{dp}{dy} = 0 \quad \Rightarrow \quad \frac{dp}{p} + 2\frac{dy}{y} = 0$$

Integrating $\log p + 2\log y = \log c \Rightarrow \log py^2 = \log c \Rightarrow py^2 = c$ ...(ii)

Eliminating $p$ between (i) and (ii)

$$y = 3x \cdot \frac{c}{y^2} + 6y^2 \cdot \frac{c^2}{y^4}$$

or $$y = \frac{3cx}{y^2} + \frac{6c^2}{y^2} \quad \text{or} \quad y^3 = 3cx + 6c^2$$

which is the required solution.

(ii) The given equation is

$$y = 2px + y^2p^3 \quad \text{...(i)}$$

Solving for $x$, $$x = \frac{y}{2p} - \frac{1}{2}y^2p^2$$

Differentiating w.r.t. $y$,

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2} \cdot \frac{dp}{dy} - yp^2 - y^2p \cdot \frac{dp}{dy}$$

or $$2p = p - y\frac{dp}{dy} - 2yp^3 - 2y^2p^2\frac{dp}{dy}$$

Integrating $\log p = \log y + \log c = \log cy$

$\therefore \quad p = cy \qquad \ldots(ii)$

Eliminating $p$ between $(i)$ and $(ii)$

$$y^2 \log y = xy \cdot cy + c^2y^2$$

or $\log y = cx + c^2$ which is the required solution.

$(ii)$ The given equation is $p^3 - 4xyp + 8y^2 = 0 \qquad \ldots(i)$

Solving for $x$, $\quad x = \dfrac{2y}{p} + \dfrac{p^2}{4y}$

Differentiating w.r.t. $y$,

$$\frac{dx}{dy} = \frac{1}{p} = \frac{2}{p} - \frac{2y}{p^2} \cdot \frac{dp}{dy} - \frac{p^2}{4y^2} + \frac{p}{2y} \cdot \frac{dp}{dy}$$

$$\Rightarrow \quad p = 2p - 2y \cdot \frac{dp}{dy} - \frac{p^4}{4y^2} + \frac{p^3}{2y} \cdot \frac{dp}{dy}$$

$$\Rightarrow \quad p\left(1 - \frac{p^3}{4y^2}\right) - 2y\frac{dp}{dy}\left(1 - \frac{p^3}{4y^2}\right) = 0$$

$$\Rightarrow \quad \left(1 - \frac{p^3}{4y^2}\right)\left(p - 2y \cdot \frac{dp}{dy}\right) = 0$$

Neglecting the first factor $\left(1 - \dfrac{p^3}{4y^2}\right)$ which does not involve $\dfrac{dp}{dy}$, we get

$$p - 2y \cdot \frac{dp}{dy} = 0 \qquad \Rightarrow \quad \frac{2dp}{p} - \frac{dy}{y} = 0$$

Integrating $\quad 2\log p - \log y = \log c \quad \Rightarrow \quad \log \dfrac{p^2}{y} = \log c \quad \Rightarrow \quad \dfrac{p^2}{y} = c$

$\Rightarrow \quad p^2 = cy \qquad \ldots(ii)$

Now we have to eliminate $p$ between $(i)$ and $(ii)$.

From $(i)$, $\quad p(p^2 - 4xy) = -8y^2$

Squaring $\quad p^2(p^2 - 4xy)^2 = 64y^4$ or $cy(cy - 4xy)^2 = 64y^4 \qquad$ [Using $(ii)$]

or $c(c - 4x)^2 = 64y$ which is the required solution.

$(iii)$ The given equation is $\quad x = y + a \log p \qquad \ldots(i)$

Differentiating w.r.t. $y$, $\quad \dfrac{dx}{dy} = \dfrac{1}{p} = 1 + \dfrac{a}{p} \cdot \dfrac{dp}{dy}$

or $\quad 1 = p + a \cdot \dfrac{dp}{dy}$ or $(p - 1) + a \cdot \dfrac{dp}{dy} = 0$

or $\quad dy + a \cdot \dfrac{dp}{p - 1} = 0$

Integrating $\quad y + a \log(p - 1) = c \quad \Rightarrow \quad y = c - a \log(p - 1) \qquad \ldots(ii)$

Putting this value of $y$ in ($i$), $x = c - a \log (p - 1) + a \log p$

or $$x = c + a \log \frac{p}{p-1} \qquad ...(iii)$$

Equations ($ii$) and ($iii$) together form the solution of ($i$) in parametric form.

($iv$) The given equation is $\quad x = y + p^2 \qquad ...(i)$

Differentiating w.r.t. $y$,

$$\frac{dx}{dy} = \frac{1}{p} = 1 + 2p \cdot \frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1}{p} - 1 = 2p \cdot \frac{dp}{dy} \quad \Rightarrow \quad \frac{1-p}{p} = 2p \frac{dp}{dy}$$

$$\Rightarrow \quad dy = \frac{2p^2}{1-p} dp = -2\left(p + 1 - \frac{1}{1-p}\right) dp$$

Integrating $$y = -2\left(\frac{p^2}{2} + p + \log(1-p)\right) + c$$

$$\Rightarrow \quad y = -p^2 - 2p - 2 \log (1 - p) + c \qquad ...(ii)$$

Putting this value of $y$ in ($i$)

$$x = -2p - 2 \log (1 - p) + c \qquad ...(iii)$$

Equations ($ii$) and ($iii$) together form the solution of ($i$) in parametric form.

**Example 3.** *Solve the following :*

($i$) $p^2 - 2xp + 1 = 0$ $\qquad$ ($ii$) $p = \tan\left(x - \frac{p}{1+p^2}\right)$

($iii$) $ayp^2 + (2x - b)p - y = 0$ $\qquad$ ($iv$) $x + \frac{p}{\sqrt{1+p^2}} = a.$

**Sol.** ($i$) The given equation is $p^2 - 2xp + 1 = 0$

Solving for $x$, $$x = \frac{p}{2} + \frac{1}{2p} \qquad ...(i)$$

Differentiating w.r.t. $y$,

$$\frac{dx}{dy} = \frac{1}{p} = \frac{1}{2}\frac{dp}{dy} - \frac{1}{2p^2} \cdot \frac{dp}{dy}$$

$$\Rightarrow \quad 2p = (p^2 - 1)\frac{dp}{dy} \quad \Rightarrow \quad dy = \frac{p^2-1}{2p} dp = \left(\frac{p}{2} - \frac{1}{2p}\right) dp$$

Integrating $$y = \frac{p^2}{4} - \frac{1}{2} \log p + c \qquad ...(ii)$$

Equations ($i$) and ($ii$) together form the solution of ($i$) in parametric form.

($ii$) The given equation is $p = \tan\left(x - \frac{p}{1-p^2}\right)$

or $$\tan^{-1} p = x - \frac{p}{1+p^2} \quad \text{or} \quad x = \frac{1}{1+p^2} + \tan^{-1} p \qquad ...(i)$$

Differentiating w.r.t. $y$,

$$\frac{dx}{dy}=\frac{1}{p}=\frac{(1+p^2).1-p.2p}{(1+p^2)^2}\cdot\frac{dp}{dy}+\frac{1}{1+p^2}\cdot\frac{dp}{dy}$$

$\Rightarrow$
$$\frac{1}{p}=\frac{1-p^2+1+p^2}{(1+p^2)^2}\cdot\frac{dp}{dy}\quad\Rightarrow\quad dy=\frac{2p}{(1+p^2)^2}\,dp$$

Integrating
$$y=\int(1+p^2)^{-2}.\,2pdp+c=\frac{(1+p^2)^{-1}}{-1}+c$$

or
$$y=-\frac{1}{1+p^2}+c \qquad ...(ii)$$

Equations ($i$) and ($ii$) together form the solution of ($i$) in parametric form.

($iii$) The given equation is $ayp^2+(2x-b)p-y=0$ ...($i$)

or
$$(2x-b)p=y(1-ap^2)\quad\text{or}\quad 2x-b=\frac{y}{p}-ayp$$

or
$$x=\frac{y}{2p}-\frac{1}{2}apy+\frac{b}{2}$$

Differentiating w.r.t. $y$,

$$\frac{dx}{dy}=\frac{1}{p}=\frac{1}{2p}-\frac{y}{2p^2}\cdot\frac{dp}{dy}-\frac{1}{2}ap-\frac{1}{2}ay\frac{dp}{dy}$$

or
$$2p=p-y\frac{dp}{dy}-ap^3-ayp^2\frac{dp}{dy}$$

or
$$p(1+ap^2)+y(1+ap^2)\frac{dp}{dy}=0$$

or
$$(1+ap^2)\left(p+y\frac{dp}{dy}\right)=0$$

Neglecting the factor $(1+ap^2)$ which does not involve $\frac{dp}{dy}$, we get

$$p+y\frac{dp}{dy}=0\quad\text{or}\quad\frac{dp}{p}+\frac{dy}{y}=0$$

Integrating $\log p+\log y=\log c$

$\Rightarrow$
$$\log p=\log\frac{c}{y}\quad\Rightarrow\quad p=\frac{c}{y} \qquad ...(ii)$$

Eliminating $p$ between ($i$) and ($ii$), we get

$$ay.\frac{c^2}{y^2}+(2x-b).\frac{c}{y}-y=0$$

or
$$ac^2+(2x-b)c-y^2=0$$

which is required solution.

(*iv*) The given equation is $\quad x = a - \dfrac{p}{\sqrt{1+p^2}} \quad$ ...(*i*)

Differentiating w.r.t. $y$,

$$\frac{dx}{dy} = \frac{1}{p} = -\frac{\sqrt{1+p^2}\,.\,1 - p\,\frac{1}{2}(1+p^2)^{-1/2}\,.\,2p}{1+p^2}\cdot\frac{dp}{dy}$$

$$\Rightarrow \qquad \frac{1}{p} = -\frac{\sqrt{1+p^2} - \dfrac{p^2}{\sqrt{1+p^2}}}{1+p^2}\cdot\frac{dp}{dy}$$

$$\Rightarrow \qquad \frac{1}{p} = -\frac{1}{(1+p^2)^{3/2}}\cdot\frac{dp}{dy}$$

$$\Rightarrow \qquad dy = -\frac{1}{2}\,.\,(1+p^2)^{-3/2}.\,2p\,dp$$

Integrating $\qquad y = -\dfrac{1}{2}\cdot\dfrac{(1+p^2)^{-1/2}}{-\frac{1}{2}} + c \quad$ or $\quad y = \dfrac{1}{\sqrt{1+p^2}} + c \quad$ ...(*ii*)

From (*i*) $\qquad x - a = -\dfrac{p}{\sqrt{1+p^2}}$

From (*ii*) $\qquad y - c = \dfrac{1}{\sqrt{1+p^2}}$

Squaring and adding $(x-a)^2 + (y-c)^2 = 1$ which is the required solution.

**Type IV. Equations Reducible to Clairaut's form by transformation.**

**Example 1.** *Solve the following :*

(*i*) $x^2(y - px) = yp^2$ (Allahabad, 1996)

(*ii*) $(px - y)(py + x) = a^2p$ (Delhi, 1996 ; Meerut, 1997)

(*iii*) $(px - y)(x - yp) = 2p$ $\qquad$ (*iv*) $y\left(\dfrac{dy}{dx}\right)^2 + x^2\,\dfrac{dy}{dx} - x^2y = 0.$

**Sol.** (*i*) The given equation is $x^2(y - px) = yp^2$ ...(*i*)

Put $\qquad x^2 = X \quad$ and $\quad y^2 = Y$

so that $\qquad 2x\,dx = dX \quad$ and $\quad 2y\,dy = dY$

$$\Rightarrow \qquad \frac{y}{x}\cdot\frac{dy}{dx} = \frac{dY}{dX} \quad \text{or} \quad \frac{y}{x}\,p = \frac{dY}{dX} = P\ (say) \quad \text{or} \quad p = \frac{x}{y}\,P$$

$$\therefore \quad (i) \text{ becomes } x^2\left(y - \frac{x^2}{y}P\right) = y\,.\,\frac{x^2}{y^2}\,P^2 \quad \Rightarrow \quad (y^2 - x^2P) = P^2$$

$$\Rightarrow \qquad Y = PX + P^2$$

which is of Clairaut's form.

$\therefore$ The solution is $\qquad Y = cX + c^2 \quad$ or $\quad y^2 = cx^2 + c^2.$

(*ii*) The given equation is $(px - y)(py + x) = a^2p$

With the substitution of part (*i*), it becomes

$$\left(\frac{x^2}{y}P - y\right)(xP + x) = a^2 . \frac{x}{y} P$$

$$\Rightarrow \quad (x^2P - y^2) . x(P + 1) = a^2xP \quad \Rightarrow \quad (XP - Y)(P + 1) = a^2P$$

$$\Rightarrow \quad XP - Y = \frac{a^2P}{P+1} \quad \Rightarrow \quad Y = PX - \frac{a^2P}{P+1}$$

which is of Clairaut's form.

$\therefore$ The solution is $Y = cX - \dfrac{a^2c}{c+1}$ or $y^2 = cx^2 - \dfrac{ca^2}{c+1}$.

(*iii*) Please try yourself by putting $x^2 = X$ and $y^2 = Y$. $\left[\textbf{Ans.}\ \ y^2 = cx^2 - \dfrac{2c}{1-c}\right]$

(*iv*) Please try yourself. [It is the same as part (*i*)]

[**Ans.** $y^2 = cx^2 + c^2$]

**Example 2.** *Solve the following :*

(*i*) $e^{3x}(p - 1) + p^3e^{2y} = 0$ (*ii*) $e^{4x}(p - 1) + e^{2y} . p^2 = 0$.

**Sol.** (*i*) Put $e^x = X$ and $e^y = Y$

so that $e^xdx = dX$ and $e^ydy = dY$

$$\Rightarrow \quad \frac{e^y}{e^x} . \frac{dy}{dx} = \frac{dY}{dX} \quad \text{or} \quad \frac{Y}{X}p = P\ (say) \quad \text{or} \quad p = \frac{X}{Y}P$$

$\therefore$ The given equation becomes

$$X^3\left(\frac{X}{Y}P - 1\right) + \frac{X^3}{Y^3}P^3 . Y^2 = 0$$

or $$XP - Y + P^3 = 0 \quad \text{or} \quad Y = PX + P^3$$

which is of Clairaut's form.

$\therefore$ The solution is $Y = cX + c^3$ or $e^y = ce^x + c^3$.

(*ii*) Put $e^{2x} = X$ and $e^{2y} = Y$

so that $2e^{2x}\,dx = dX$ and $2e^{2y}\,dy = dY$

$$\Rightarrow \quad \frac{e^{2y}}{e^{2x}} . \frac{dp}{dx} = \frac{dY}{dX} \quad \text{or} \quad \frac{Y}{X}p = P\ (say) \quad \text{or} \quad p = \frac{X}{Y}P$$

$\therefore$ The given equation becomes

$$X^2\left(\frac{X}{Y}P - 1\right) + Y . \frac{X^2}{Y^2}P^2 = 0$$

or $$XP - Y + P^2 = 0 \quad \text{or} \quad Y = PX + P^2$$

which is of Clairaut's form.

$\therefore$ The solution is $Y = cX + c^2$ or $e^{2y} = ce^{2x} + c^2$.

**Example 3.** *Solve the following :*

(*i*) $y^2(y - xp) = x^4p^2$ (*ii*) $(y + xp)^2 = x^2p$.

# 8

# Linear Equations of Second Order

The general form of linear equation of second order may be written as

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$$

where P, Q and R are the functions of $x$ only. There is no general method of solving this type of equations, but we will consider some particular cases in which the integral can be found.

**Complete solution in terms of known integral.** If an integral included in the complementary function of a linear equation of second order be known then the complete solution can be found. Let $y = u$ be an integral in the complementary function of the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ...(1)$$

Then put, $\quad y = uv \quad$ so that $\qquad \dfrac{dy}{dx} = u_1 v + u v_1$

and $\qquad \dfrac{d^2y}{dx^2} = u_2 v + 2u_1 v_1 + u v_2$

Putting in (1), we get

$$(u_2 v + 2u_1 v_1 + u v_2) + P(u_1 v + u v_1) + Quv = R$$

or $$uv_2 + (2u_1 + Pu)v_1 + (u_2 + Pu_1 + Qu)v = R$$

Since $y = u$ is a solution of

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \qquad \therefore \qquad u_2 + Pu_1 + Qu = 0$$

So, we have

$$uv_2 + (2u_1 + Pu)v_1 = R \qquad \text{or} \qquad v_2 + \left(\frac{2}{u}u_1 + P\right)v_1 = \frac{R}{u}$$

Putting $v_1 = P$, so that $\qquad v_2 = \dfrac{dp}{dx}$ , we get

$$\frac{dp}{dx} + \left(\frac{2}{u}.u_1 + P\right)p = \frac{R}{u} \qquad ...(2)$$

which is a linear equation in $p$.

$$\therefore \qquad \text{I.F.} = e^{\int\left(\frac{2}{u}.u_1 + P\right)dx} = e^{\int \frac{2}{u}du + \int P dx} = e^{2\log u + \int P\,dx} = u^2\, e^{\int P\,dx}$$

$\therefore$ We have $\quad p.u^2 e^{\int P\,dx} = \int \left(\frac{R}{u}.u^2\, e^{\int P\,dx}\right) dx + c_1$

$\therefore$ $$p = u^{-2}\, e^{-\int P\,dx} \int \left(Rue^{\int P\,dx}\right) dx + c_1 u^{-2}\, e^{-\int P\,dx}$$

or $$v_1 = \frac{dv}{dx} = u^{-2}\, e^{-\int P\,dx} \int \left(Rue^{\int P\,dx}\right) dx + c_1 u^{-2}\, e^{-\int P\,dx}$$

Integrating again, we have

$$v = \int \left(u^{-2}\, e^{-\int P\,dx} . \int Ru\, e^{\int P\,dx}\, dx\right) dx + c_1 \int \left(u^{-2}\, e^{-\int P\,dx}\right) dx + c_2$$

$\therefore$ The complete solution of equation (1) is

$$y = uv = u \int \left(u^{-2}\, e^{-\int P\,dx} . \int Ru\, e^{\int P\,dx}\, dx\right) dx + c_1 u \int \left(u^{-2}\, e^{-\int P\,dx}\right) dx + c_2 u$$

The above solution contains only two arbitrary constants.

**To find a particular integral of**

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0 \qquad ...(1)$$

**(I) $y = e^{mx}$ is a solution.**

If $\quad y = e^{mx}$

Then $\quad \frac{dy}{dx} = me^{mx} \quad$ and $\quad \frac{d^2y}{dx^2} = m^2e^{mx}$

$\therefore$ If $y = e^{mx}$ is a solution of (1), then $(m^2 + Pm + Q)e^{mx} = 0$ or $\mathbf{m^2 + Pm + Q = 0.}$

**Deduction.** (*i*) $y = e^x$ is a solution of (1), if $\mathbf{1 + P + Q = 0.}$

(*ii*) $y = e^{-x}$ is the solution of (1), if $1 - P + Q = 0$

(*iii*) $y = e^{ax}$ is the solution of (1), if $a^2 + Pa + Q = 0$ or $1 + \frac{P}{a} + \frac{Q}{a^2} = 0.$

**(II) $y = x^m$ is a solution.**

If $\quad y = x^m$

Then $\quad \frac{dy}{dx} = mx^{m-1} \quad$ and $\quad \frac{d^2y}{dx^2} = m(m-1)x^{m-2}$

$\therefore$ If $y = x^m$ is a solution of (1), then $m(m-1)x^{m-2} + Pmx^{m-1} + Qx^m = 0$

or $$m(m-1) + Pmx + Qx^2 = 0.$$

**Deduction.** (*i*) $y = x$ is the solution of (1), if $P + Qx = 0$.

(*ii*) $y = x^2$ is the solution of (1), if $2 + 2Px + Qx^2 = 0$.

**Note. One integral belonging to the complementary function can be found by inspection. For this following rules are observed :**

(*i*) **$y = x$ is a part of C.R., if $P + Qx = 0$**

(*ii*) **$y = e^x$ is a part of C.F., if $1 + P + Q = 0$** **(i.e., sum of the co-efficients are zero)**

(*iii*) **$y = e^{-x}$ is a part of C.F., if $1 - P + Q = 0$**

(*iv*) **$y = e^{ax}$ is a part of C.F., if $1 + \frac{P}{a} + \frac{Q}{a^2} = 0$**

(*v*) **$y = x^2$ is part of C.F., if $2 + 2Px + Qx^2 = 0.$**

**Example 1.** *Solve :* $x^2 \dfrac{d^2y}{dx^2} - 2x(1+x)\dfrac{dy}{dx} + 2(1+x)y = x^3.$ **(Meerut, 1998)**

**Sol.** The given equation can be written as

$$\frac{d^2y}{dx^2} - 2\left(\frac{1}{x}+1\right)\frac{dy}{dx} + 2\left(\frac{1}{x^2}+\frac{1}{x}\right)y = x$$

where $$P + Qx = -2\left(\frac{1}{x}+1\right) + 2x\left(\frac{1}{x^2}+\frac{1}{x}\right) = 0$$

$\therefore$ $y = x$ is a part of C.F.

Putting $y = vx$ so that

$$\frac{dy}{dx} = \frac{dv}{dx}x + v$$

and $$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2} + 2\frac{dv}{dx}\text{, we get}$$

$$\frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1 \quad \text{or} \quad \frac{dp}{dx} - 2p = 1$$

where $p = \dfrac{dv}{dx}$

which is a linear equation

$$\text{I.F.} = e^{-2\int dx} = e^{-2x}$$

$\therefore$ $$pe^{-2x} = \int 1 \,.\, e^{-2x}\,dx + c_1 = -\frac{1}{2}e^{-2x} + c_1$$

$\therefore$ $$p = \frac{dv}{dx} = -\frac{1}{2} + c_1e^{-2x}$$

Integrating, we get $v = -\dfrac{1}{2}x + \dfrac{c_1}{2}e^{2x} + c_2$

$\therefore$ The complete solution is

$$y = vx = -\frac{1}{2}x^2 + \frac{c_1}{2}xe^{2x} + c_2x.$$

**Example 2.** *Solve :* $x^2\dfrac{d^2y}{dx^2} - (x^2+2x)\dfrac{dy}{dx} + (x+2)y = x^3e^x.$ **(Delhi, 1997)**

**Sol.** The given equation can be written as

$$\frac{d^2y}{dx^2} - \left(1+\frac{2}{x}\right)\frac{dy}{dx} + \left(\frac{1}{x}+\frac{2}{x^2}\right)y = xe^x$$

Here $P = -\left(1+\dfrac{2}{x}\right)$, $Q = \dfrac{1}{x}+\dfrac{2}{x^2}$ and $R = x\,e^x$

Since $P + Qx = 0$

$\therefore$ $y = x$ is a part of the C.F.

Putting $y = vx$, so that

$$\frac{dy}{dx} = \frac{dv}{dx} \cdot x + v \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}\, x + 2\,\frac{dv}{dx}$$

We get $\dfrac{d^2v}{dx^2} - \dfrac{dv}{dx} = e^x$ or $\dfrac{dp}{dx} - p = e^x$, where $p = \dfrac{dv}{dx}$

which is a linear equation,

$$\text{I.F.} = e^{-\int dx} = e^{-x}$$

$$\therefore \qquad pe^{-x} = \int e^{-x} \cdot e^x\, dx + c_1 = x + c_1$$

$$\therefore \qquad p = \frac{dv}{dx} = xe^x + c_1 e^x$$

Integrating, we get $v = xe^x - e^x + c_1 e^x + c_2$

$\therefore$ The complete solution is

$$y = vx = x^2e^x - xe^x + c_1 x\, e^x + c_2 x.$$

**Example 3.** *Solve :* $\sin^2 x \cdot \dfrac{d^2y}{dx^2} = 2y$ *given* $y = \cot x$ *is a solution.* (Meerut, 1997)

**Sol.** Putting $y = v \cot x$, so that

$$\frac{dy}{dx} = \frac{dv}{dx} \cot x - v \operatorname{cosec}^2 x$$

and
$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2} \cot x - 2 \operatorname{cosec}^2 x\, \frac{dv}{dx} + 2v \operatorname{cosec}^2 x \cot x$$

in the given equation, we get

$$\cot x \sin^2 x\, \frac{d^2v}{dx^2} - 2\, \frac{dv}{dx} = 0$$

or
$$\frac{d^2y}{dx^2} - \frac{2}{\sin x \cos x}\, \frac{dv}{dx} = 0$$

or
$$\frac{dp}{dx} = \frac{2}{\sin x \cos x}\, p, \text{ where } p = \frac{dv}{dx}$$

or
$$\frac{dp}{p} = \frac{2}{\sin x \cos x}\, dx = \frac{2 \sec^2 x}{\tan x}\, dx$$

Integrating, we get

$$\log p = 2 \log \tan x + \log c \qquad \therefore \qquad p = c_1 \tan^2 x$$

or
$$\frac{dv}{dx} = c_1 \tan^2 x = c_1(\sec^2 x - 1)$$

Integrating, $v = c_1 (\tan x - x) + c_2$

$\therefore$ The complete solution is

$$y = v \cot x = c_1(1 - x \cot x) + c_2 \cot x.$$

**Example 4.** *Solve :* $x\, \dfrac{dy}{dx} - y = (x - 1) \left( \dfrac{d^2y}{dx^2} - x + 1 \right)$.

**Sol.** The given equation may be written as

$$\frac{d^2y}{dx^2} - \frac{x}{x-1}\frac{dy}{dx} + \frac{y}{x-1} = x - 1$$

Here $P + Qx = 0$

$\therefore$ $y = x$ is a part of C.F.

$\therefore$ Putting $y = vx$, so that $\dfrac{dy}{dx} = \dfrac{dv}{dx}x + v$ and $\dfrac{d^2y}{dx^2} = \dfrac{d^2v}{dx^2}x + 2\dfrac{dv}{dx}$

We have, $$\frac{d^2v}{dx^2} + \left(-\frac{x}{x-1} + \frac{2}{x}\right)\frac{dv}{dx} = \frac{x-1}{x}$$

or $$\frac{dp}{dx} + \left(-\frac{x}{x-1} + \frac{2}{x}\right)p = \frac{x-1}{x}, \quad \text{where } p = \frac{dv}{dx}$$

which is a linear equation.

$$\text{I.F.} = e^{-\int \frac{x}{x-1}dx + \int \frac{2}{x}dx} = e^{-\int\left(1+\frac{1}{x-1}\right)dx + \int \frac{2}{x}dx}$$

$$= e^{-x - \log(x-1) + 2\log x} = \frac{x^2}{x-1}e^{x}$$

$$\therefore \quad p\,\frac{x^2e^{-x}}{x-1} = \int \frac{x-1}{x}\cdot\frac{x^2}{x-1}e^{-x}\,dx + c_1 = \int xe^{-x}\,dx + c_1 = -xe^{-x} - e^{-x} + c_1$$

$$\therefore \quad p = \frac{dv}{dx} = -\frac{x-1}{x} - \frac{(x-1)}{x^2} + \frac{c_1(x-1)e^x}{x^2} = -1 + \frac{1}{x^2} + c_1\left(\frac{1}{x} - \frac{1}{x^2}\right)e^x$$

Integrating, $v = -x - \dfrac{1}{x} + c_1\dfrac{1}{x}e^x + c_2$

$\therefore$ The complete solution is

$$y = vx = -x^2 - 1 + c_1e^x + c_2x = c_1e^x + c_2x - (1 + x^2).$$

**Example 5.** *Solve :*

$$(x\sin x + \cos x)\frac{d^2y}{dx^2} - x\cos x\frac{dy}{dx} + y\cos x = \sin x\,(x\sin x + \cos x)^2.$$

**Sol.** The given equation may be written as

$$\frac{d^2y}{dx^2} - \frac{x\cos x}{x\sin x + \cos x}\cdot\frac{dy}{dx} + \frac{\cos x}{x\sin x + \cos x}\,y = \sin x(x\sin x + \cos x)$$

Here $P + Qx = 0$ $\quad\therefore\ y = x$ is a part of C.F.

$\therefore$ Putting $y = vx$ the equation reduces to

$$\frac{d^2v}{dx^2} + \left(\frac{2}{x} - \frac{x\cos x}{x\sin x + \cos x}\right)\frac{dv}{dx} = \frac{\sin x\,(x\sin x + \cos x)}{x}$$

or $$\frac{dp}{dx} + \left(\frac{2}{x} - \frac{x\cos x}{x\sin x + \cos x}\right)p = \frac{\sin x}{x}(x\sin x + \cos x)$$

which is a linear equation.

$$\therefore \qquad \text{I.F.} = e^{\int\left(\frac{2}{x} - \frac{x\cos x}{x\sin x + \cos x}\right)dx} = e^{2\log x - \log(x\sin x + \cos x)} = \frac{x^2}{(x\sin x + \cos x)}$$

$$\therefore \qquad p.\frac{x^2}{(x\sin x + \cos x)} = \int x\sin x\, dx + c_1 = -x\cos x + \sin x + c_1$$

$$\therefore \qquad p = \frac{dv}{dx} = \frac{1}{x^2}(-x\cos x + \sin x)(x\sin x + \cos x) + \frac{c_1}{x^2}(x\sin x + \cos x)$$

$$\frac{dy}{dx} = -\sin x\cos x - \frac{1}{x}\cos 2x + \frac{1}{x^2}\sin x\cos x + c_1\left(\frac{1}{x}\sin x + \frac{1}{x^2}\cos x\right)$$

Integrating,

$$v = \frac{1}{2}\cos^2 x - \int \frac{1}{x}\cos 2x\, dx + \int \frac{1}{2x^2}\sin 2x\, dx + c_1\int\left(\frac{1}{x}\sin x + \frac{1}{x^2}\cos x\right)dx$$

$$= \frac{1}{2}\cos^2 x - \frac{1}{2x}\sin 2x - \frac{c_1}{x}\cos x + c_2$$

$\therefore$ The complete solution is

$$y = vx = \frac{x}{2}\cos^2 x - \frac{1}{2}\sin 2x - c_1\cos x + c_2 x.$$

**Example 6.** *Solve :* $\quad (1-x^2)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = x(1-x^2)^{3/2}.$ (Rohilkhand, 1997)

**Sol.** The given equation can be written as

$$\frac{d^2y}{dx^2} + \frac{x}{1-x^2}\frac{dv}{dx} - \frac{1}{1-x^2}y = x(1-x^2)^{1/2}$$

Here P + Qx = 0. $\therefore$ $y = x$ is a part of C.F.

Putting $y = vx$, the equation reduces to

$$\frac{d^2v}{dx^2} + \left(\frac{x}{1-x^2} + \frac{2}{x}\right)\frac{dv}{dx} = \frac{x(1-x^2)^{1/2}}{x}$$

or
$$\frac{dp}{dx} + \left(\frac{x}{1-x^2} + \frac{2}{x}\right)p = \sqrt{1-x^2}\text{, where } p = \frac{dv}{dx}$$

which is a linear equation.

$$\text{I.F.} = e^{\int\left(\frac{x}{1-x^2} + \frac{2}{x}\right)dx} = e^{-\frac{1}{2}\log(1-x^2) + 2\log x} = \frac{x^2}{\sqrt{1-x^2}}$$

$$\therefore \qquad p.\frac{x}{\sqrt{1-x^2}} = \int x^2\, dx + c_1 = \frac{x^3}{3} + c_1$$

$$\therefore \qquad p = \frac{dv}{dx} = \frac{1}{3}x\sqrt{1-x^2} + \frac{c_1}{x^2}\sqrt{1-x^2}$$

$$= \frac{1}{3}x\sqrt{1-x^2} + c_1(1-x^2)^{1/2}.\frac{1}{x^2}$$

Integrating, $$v = -\frac{1}{9}(1-x^2)^{3/2} + c_1(1-x^2)^{1/2}\left(-\frac{1}{x}\right) - c_1 \int \frac{dx}{\sqrt{1-x^2}} + c_2$$

$$= -\frac{1}{9}(1-x^2)^{3/2} - \frac{c_1}{x}(1-x^2)^{1/2} - c_1 \sin^{-1} x + c_2$$

$\therefore$ The complete solution is

$$y = vx = -\frac{1}{9}x(1-x^2)^{1/2} - c_1\{x \sin^{-1} x + \sqrt{1-x^2}\} + c_2 x.$$

**Example 7.** *Solve* $\frac{d^2y}{dx^2} - \cot x \frac{dy}{dx} - (1-\cot x)\, y = e^x \sin x.$ (Meerut, 1996)

**Sol.** From the above equation, we have $P + Q + 1 = 0$

$\therefore$ $y = e^x$ is a part of C.F.

$\therefore$ Putting $y = ve^x$ so that $\frac{dy}{dx} = \frac{dv}{dx} e^x + v.\, e^x$

and $$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2} e^x + 2\frac{dv}{dx} e^x + ve^x$$

We have, $$\frac{d^2v}{dx^2} + (2 - \cot x)\frac{dv}{dx} = \sin x$$

or $$\frac{dp}{dx} + (2-\cot x)\, p = \sin x, \text{ where } p = \frac{dv}{dx}$$

which is linear equation.

$$\text{I.F.} = e^{\int (2-\cot x)\, dx} = e^{2x - \log \sin x} = \frac{e^{2x}}{\sin x}$$

$$\therefore \quad p\frac{e^{2x}}{\sin x} = \int \frac{e^{2x}}{\sin x} \sin x\, dx + c_1 = \frac{1}{2} e^{2x} + c_1$$

$$\therefore \quad p\frac{dv}{dx} = \frac{1}{2}\sin x + c_1 e^{-2x} \sin x$$

Integrating, $v = -\frac{1}{2}\cos x + \frac{c_1}{5} e^{-2x}(-2\sin x - \cos x) + c_2$

$\therefore$ The complete solution is

$$y = ve^x = -\frac{1}{2} e^x \cos x - \frac{c_1}{5} e^{-x}(2\sin x + \cos x) + c_2 e^x.$$

**Example 8.** *Solve :* $(x+2)\frac{d^2y}{dx^2} - (2x+5)\frac{dy}{dx} + 2y = (x+1)\, e^x.$ (Vikram, 1998)

**Sol.** The above given equation may be written as

$$\frac{d^2y}{dx^2} - \frac{2x+5}{x+2}\frac{dy}{dx} + \frac{2}{x+2} y = \frac{x+1}{x+2} e^x$$

Here $$\frac{P}{2} + \frac{Q}{2^2} + 1 = 0$$

Integrating,

$$\log p = -4 \log x + \log (x^2 \cos x - 2x \sin x) + \log c_1$$

$$\therefore \qquad p = \frac{dv}{dx} = \frac{c_1 x (x \cos x - 2 \sin x)}{x^4} = \frac{c_1}{x^2} \cos x - \frac{2c_1}{x^3} \sin x$$

Integrating, $\quad v = \dfrac{c_1}{x^2} \sin x + c_2$

$\therefore$ Complete solution is

$$y = vx^2 = c_1 \sin x + c_2 x^2.$$

**Example 10.** *Solve* $\dfrac{d^2y}{dx^2} + \left(1 + \dfrac{2}{x} \cot x - \dfrac{2}{x^2}\right) y$

$= x \cos x$ *given that* $\dfrac{\sin x}{x}$ *is an integral included in C.F.*

**Sol.** Putting $\quad y = v \dfrac{\sin x}{x}$ so that

$$\frac{dy}{dx} = \frac{dv}{dx} \frac{\sin x}{x} + v \left(\frac{x \cos x - \sin x}{x^2}\right)$$

$$= \frac{dv}{dx} \frac{\sin x}{x} + v \left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right)$$

and

$$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2} \frac{\sin x}{x} + 2 \frac{dv}{dx} \left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right) + v \left(-\frac{\sin x}{x} - \frac{2 \cos x}{x^2} + \frac{2 \sin x}{x^3}\right)$$

In the given equation, we have

$$\frac{d^2v}{dx^2} \frac{\sin x}{x} + 2 \left(\frac{\cos x}{x} - \frac{\sin x}{x^2}\right) \frac{dv}{dx} + \left(-\frac{\sin x}{x} - \frac{2 \cos x}{x^2} + \frac{2 \sin x}{x^3}\right) v + \left(1 + \frac{2 \cot x}{x} - \frac{2}{x^2}\right) v \frac{\sin x}{x} = x \cos x$$

or

$$\frac{d^2v}{dx^2} + 2 \left(\cot x - \frac{1}{x}\right) \frac{dv}{dx} = x^2 \cot x$$

or

$$\frac{dp}{dx} + 2 \left(\cot x - \frac{1}{x}\right) p = x^2 \cot x, \quad \text{where } p = \frac{dv}{dx}$$

which is a linear equation

$$\text{I.F.} = e^{\int 2(\cot x - 1/x)\, dx} = e^{2 (\log \sin x - \log x)} = \frac{\sin^2 x}{x^2}$$

$$\therefore \qquad p. \frac{\sin^2 x}{x^2} = \int x^2 \cot x . \frac{\sin^2 x}{x^2}\, dx + c_1$$

$$= \frac{1}{2} \int \sin 2x\, dx + c_1 = -\frac{1}{4} \cos 2x + c_1$$

$\therefore \quad p = \dfrac{dv}{dx} = xe^x + e^x + kx,$

Integrating, $\quad v = xe^x + \dfrac{k}{2}x^2 + c_2 \quad$ or $\quad v = xe^x + c_1x^2 + c_2$

$\therefore$ The complete solution is

$$y = ve^x = xe^{2x} + c_1x^2e^x + c_2e^x.$$

## EXAMPLES FOR PRACTICE

*Solve :*

1. $x\dfrac{d^2y}{dx^2} - (2x-1)\dfrac{dy}{dx} + (x-1)y = 0.$

2. $x\dfrac{d^2y}{dx^2} - 2(x+1)\dfrac{dy}{dx} + (x+2)y = (x-2)e^x.$

3. $x\dfrac{d^2y}{dx^2} - 2(x+1)\dfrac{dy}{dx} + (x+2)y = (x-2)e^{2x}.$

4. $\dfrac{d^2y}{dx^2} - (1+x)\dfrac{dy}{dx} + xy = x.$

5. $(x+1)\dfrac{d^2y}{dx^2} - 2(x+3)\dfrac{dy}{dx} + (x+5) = e^x.$

### Answers

1. $y = e^x(c_1 \log x + c_2)$

2. $y = -\frac{1}{2}x^2e^x + xe^x + \frac{1}{3}c_1x^3e^x + c_2e^x$

3. $y = \frac{1}{3}c_1x^3e^x + c_2e^x + e^{2x}$

4. $y = -1 + c_1e^x \int e^{-x+\frac{1}{2}x\,dx + c_2e^x}$

5. $y = \frac{1}{5}c_1e^x(x+1)^5 - \frac{1}{4}xe^x + c_2e^x.$

**Removal of the First Derivative**

If the part of the complementary function is not obvious by inspection, it is sometimes useful to reduce the given equation into the form in which the term containing the first derivative is absent. For this we will change the **dependent variable** in the equation.

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} = Qy + R \qquad \text{...(1)}$$

By putting $y = uv$, where $u$ is some function of $x$, so that

$$\frac{dy}{dx} = u\frac{dv}{dx} + \frac{du}{dx}v$$

and

$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2} + 2\frac{du}{dx}\cdot\frac{dv}{dx} + \frac{d^2u}{dx^2}v$$

$\therefore$ Equation (1) reduces to

$$u\frac{d^2v}{dx^2} + \left(Pu + 2\frac{du}{dx}\right)\frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right)v = R$$

or

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} + \left(\frac{1}{u}\frac{d^2u}{dx^2} + \frac{P}{u}\frac{du}{dx} + Q\right)v = \frac{R}{u} \qquad \text{...(2)}$$

**Example 3.** *Solve* $\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = \sec xe^x$.

**Sol.** Here $P = -2\tan x$, $Q = 5$ and $R = \sec xe^x$

Putting $y = uv$ is the given equation, the equation reduces to

$$\frac{d^2v}{dx^2} + Xv = Y, \text{ where } u = e^{-\frac{1}{2}\int P dx}$$

$$= e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$$

$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$$

$$= 5 + \frac{1}{2}\, 2\sec^2 x - \frac{1}{4} \cdot 4\tan^2 x = 6$$

$$Y = Re^{\frac{1}{2}\int P dx} = e^x$$

$\therefore$ The reduced equation is $\frac{d^2v}{dx^2} + 6v = e^x$

where C.F. $= c_1 \cos\sqrt{6}x + c_2 \sin\sqrt{6}x$

and $$\text{P.I.} = \frac{1}{D^2+6}e^x = \frac{e^x}{7}$$

$\therefore$ $$v = c_1\cos\sqrt{6}x + c_2\sin\sqrt{6}x + \frac{e^x}{7}$$

$\therefore$ The solution of the given equation is

$$y = uv = \sec x\left(c_1\cos\sqrt{6}x + c_2\sin\sqrt{6}x + \frac{1}{7}e^x\right).$$

**Example 4.** *Solve* $\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 3)y = e^{x^2}$.

**Sol.** Here $P = -4x$, $Q = 4x^2 - 3$, $R = e^{x^2}$

Putting $y = uv$, the normal form is,

$$\frac{d^2v}{dx^2} + Xv = Y, \text{ where } u = e^{-\frac{1}{2}\int P dx} = e^{x^2}$$

$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P_2 = 4x^2 - 3 - \frac{1}{2}(-4) - \frac{1}{4}(16x^2) = -1$$

and $$Y = Re^{-\frac{1}{2}\int P dx} = 1$$

$\therefore$ The normal form is $\frac{d^2v}{dx^2} - v = 1$

where $$\text{C.F.} = c_1e^x + c_2e^{-x}$$

and $$\text{P.I.} = \frac{1}{D^2-1}\cdot 1 = -(1-D^2)^{-1}\cdot 1 = -1$$

$\therefore$ $$v = c_1e^x + c_2e^{-x} - 1$$

$\therefore$ The solution of the given equation is

$$y = uv = e^{x^2}\ (c_1 e^x + c_2 e^{-x} - 1).$$

**Example 5.** *Solve* $\dfrac{d^2y}{dx^2} - 4x\,\dfrac{dy}{dx} + (4x^2 - 1)\,y = -3e^{x^2}\ \sin 2x.$ (I.A.S., 2000)

**Sol.** Here $P = -4x$, $Q = 4x^2 - 1$ and $R = -3e^{x^2} \sin 2x$

Putting $y = uv$, the equation reduces to $\dfrac{d^2v}{dx^2} + Xv = Y$

where
$$u = e^{-\frac{1}{2}\int P dx} = e^{x^2}$$
$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = 4x^2 - 1 - \frac{1}{2}(-4) - \frac{1}{4}16x^2 = 1$$
$$Y = Re^{\frac{1}{2}\int P dx} = -3 \sin 2x.$$

$\therefore$ The reduced equation is $\dfrac{d^2v}{dx^2} + v = -3 \sin 2x$

whose C.F. $= c_1 \cos x + c_2 \sin x$

and
$$\text{P.I.} = \frac{1}{D^2 + 1}(-3 \sin 2x) = \frac{-3}{-2^2 + 1} \sin 2x = \sin 2x$$

$\therefore$
$$v = c_1 \cos x + c_2 \sin x + \sin 2x$$

$\therefore$ The solution of the given equation is

$$y = uv = e^{x^2}\ (c_1 \cos x + c_2 \sin x + \sin 2x).$$

**Example 6.** *Solve* $\dfrac{d^2y}{dx^2} - \dfrac{2}{x}\dfrac{dy}{dx} + \left(1 + \dfrac{2}{x^2}\right)y = xe^x.$ (Bangalore, 1997)

**Sol.** Here $P = -\dfrac{2}{x}$, $Q = 1 + \dfrac{2}{x^2}$ and $R = xe^x$

Putting $y = uv$, the normal form is $\dfrac{d^2v}{dx^2} + Xv = Y$

where $u = e^{-\frac{1}{2}\int P dx} = e^{\int 1/x\, dx} = e^{\log x} = x$

$$X = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = 1 \text{ and } Y = Re^{\frac{1}{2}\int P dx}$$
$$= xe^x\, e^{-\int dx/x} = xe^x\, e^{-\log x} = e^x$$

$\therefore$ The normal form of the given equation is

$$\frac{d^2v}{dx^2} + v = e^x \text{ whose C.F.} = c_1 \cos x + c_2 \sin x$$

and
$$\text{P.I.} = \frac{1}{D^2 + 1} e^x = \frac{e^x}{2}$$

$\therefore$
$$v = c_1 \cos x + c_2 \sin x + \tfrac{1}{2} e^x$$

$\therefore$ The solution of the given equation is

$$y = uv = x(c_1 \cos x + c_2 \sin x + \tfrac{1}{2} e^x).$$

3. $\dfrac{d^2y}{dx^2} - 2\tan x\,\dfrac{dy}{dx} + 5y = 0$ *(D.U., 1993)*

4. $\dfrac{d^2y}{dx^2} + \dfrac{2}{x}\,\dfrac{dy}{dx} + n^2y = 0$ *(Delhi, 2000)*

5. $\dfrac{d^2y}{dx^2} - 2bx\,\dfrac{dy}{dx} + b^2x^2y = 0.$

## Answers

1. $y = xc_1 \cos(ax + c_2)$
2. $y = (c_1x^4 + c_2x^{-3})/(x-2)$
3. $y = \sec x(c_1 \cos\sqrt{6}x + c_2 \sin\sqrt{6}x)$
4. $y = \dfrac{1}{x}\,c_1 \cos(nx + c_2)$
5. $y = c_1 e^{\frac{1}{2}bx^2} \cos(\sqrt{6}x + c_2)$.

**Transformation of the equation by changing the independent variable**

Sometimes the equation is transformed to an integrable form by **changing the independent variable.**

Let the equation be

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad ...(1)$$

Let the independent variable be changed from $x$ to $z$, where $z$ is a function of $x$.

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\cdot\frac{dz}{dx}\right)$$

$$= \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dx}\cdot\frac{d^2z}{dx^2}$$

Substituting in equation (1), we have

$$\left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} + \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\frac{dy}{dz} + Qy = R$$

or

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1 \qquad ...(2)$$

where $$P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2},\quad Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2} \quad \text{and} \quad R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2}$$

$P_1, Q_1, R_1$ are functions of $x$ but may be expressed as functions of $z$ by the given relation between $z$ and $x$.

We choose $z$ to make the co-efficient of $\dfrac{dy}{dx}$ zero, *i.e.*, $P_1 = 0$

*i.e.*, $$\frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0 \qquad \text{or} \qquad \frac{\dfrac{d^2z}{dx^2}}{\dfrac{dz}{dx}} = -P$$

Integrating, $\log \frac{dz}{dx} = -\int Pdx$ or $\frac{dz}{dx} = e^{-\int P\,dx}$

Then the equation (2) is reduced to

$$\frac{d^2y}{dz^2} + Q_1 y = R_1$$

which can be solved easily provided $Q_1$ comes out to be a constant or a constant multiplied by $\frac{1}{z^2}$.

Again, if we choose $z$ such that

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} = a^2 \text{ (constant)}$$

*i.e.,* $a^2\left(\frac{dz}{dx}\right)^2 = Q$ or $a\,\frac{dz}{dx} = \sqrt{Q}$

$\therefore$ $az = \int \sqrt{Q}\,dx$

Then equation (2) is reduced to

$$\frac{d^2y}{dz^2} + P_1 \frac{dx}{dz} + a^2y = R_1$$

which can be solved easily provided $P_1$ comes out to be a constant.

**Note. It is advised to remember the equation (2) and the values of $P_1$, $Q_1$ and $R_1$.**

**Example 1.** *Solve :* $\frac{d^2y}{dx^2} + \frac{2}{x}\frac{dy}{dx} + \frac{a^2}{x^4}\,y = 0.$

**Sol.** Choose $z$ such that

$$\left(\frac{dz}{dx}\right)^2 = Q = \frac{a^2}{x^4}$$

$$\therefore \qquad \frac{dz}{dx} = \pm\frac{a}{x^2}, \; z = \pm\frac{a}{x}$$

Changing the independent variable from $x$ to $z$ by the relation $z = \frac{a}{x}$, we get

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

where $$P = \frac{\frac{d^2z}{dx^2} + P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} = \frac{\frac{2a}{x^3} + \frac{2}{x}\left(-\frac{a}{x^2}\right)}{\left(-\frac{a}{x^2}\right)^2} = 0$$

$$Q_1 = \frac{Q}{\left(\frac{dz}{dx}\right)^2} \qquad \text{and} \qquad R_1 = \frac{R}{\left(\frac{dz}{dx}\right)^2} = 0$$

$\therefore$ The transformed equation is $\dfrac{d^2y}{dz^2} + y = 0$

$$\therefore \qquad y = c_1 \cos z + c_2 \sin z = c_1 \cos \frac{a}{x} + c_2 \sin \frac{a}{x}.$$

**Example 2.** *Solve* : $x \dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} + 4x^3y = x^5.$ (Bihar, 1997 ; Nagpur, 1996)

**Sol.** The given equation can be wri'ten as

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} + 4x^2y = x^4$$

Choosing $z$, such that $\left(\dfrac{dz}{dx}\right)^2 = Q = 4x^2$ or $\dfrac{dz}{dx} = 2x \quad \therefore \quad z = x^2$

Now changing the independent variable from $x$ to $z$ by the relation $z = x^2$, we get

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

where
$$P_1 = \frac{\dfrac{d^2y}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = \frac{2 + \left(-\dfrac{1}{x}\right)2x}{(2x)^2} = 0$$

$$Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2} = 1 \quad \text{and} \quad R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2} = \frac{x^4}{(2x)^2} = \frac{x^2}{4} = \frac{1}{4}z$$

The given equation is transformed to

$$\frac{d^2y}{dz^2} + y = \frac{1}{4}z$$

whose $\quad$ C.F. $= c_1 \cos z + c_2 \sin z$

and $\quad$ P.I. $= \dfrac{1}{4} \cdot \dfrac{1}{D^2+1} z = \dfrac{1}{4}(1 + D^2)^{-1} z = \frac{1}{4}(1 - D^2 + D^4 \ldots..) z = \frac{1}{4} z$

$$\therefore \qquad y = c_1 \cos z + c_2 \sin z + \tfrac{1}{4} z$$

or
$$y = c_1 \cos x^2 + c_2 \sin x^2 + \tfrac{1}{4} x^2.$$

**Example 3.** *Solve* : $\dfrac{d^2y}{dx^2} + \cot x \dfrac{dy}{dx} + 4y \operatorname{cosec}^2 x = 0.$

**Sol.** Choosing $z$, such that

$$\left(\frac{dz}{dx}\right)^2 = 4 \operatorname{cosec}^2 x, \text{ so that}$$

$$\frac{dz}{dx} = 2 \operatorname{cosec} x \text{ or } z = 2 \log \tan \frac{x}{2}.$$

Now changing the independent variable from $x$ to $z$ by the relation

$z = 2 \log \tan \dfrac{x}{2}$, we get

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

where $$P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = -\frac{2 \operatorname{cosec} x \cot x + 2 \cot x \operatorname{cosec} x}{(2 \operatorname{cosec} x)^2} = 0$$

$$Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2} = 1 \quad \text{and} \quad R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2} = 0$$

$\therefore$ The given equation is transformed to

$$\frac{d^2y}{dz^2} + y = 0$$

$\therefore$ $$y = c_1 \cos z + c_2 \sin z \quad \text{or} \quad y = k_1 \cos (z + k_2)$$

or $$y = k_1 \cos \left(2 \log \tan \frac{1}{2} x + k_2\right).$$

**Example 4.** *Solve :* $(1 + x^2)^2 \dfrac{d^2y}{dx^2} + 2x(1 + x^2) \dfrac{dy}{dx} + 4y = 0.$

**Sol.** The given equation can be written as

$$\frac{d^2y}{dx^2} + \frac{2x}{1+x^2} \frac{dy}{dx} + \frac{4}{(1+x)^2} y = 0$$

Choosing $z$, such that

$$\left(\frac{dz}{dx}\right)^2 = Q = \frac{4}{1+x^2} \quad \therefore \quad \frac{dz}{dx} = \frac{x}{1+x^2}$$

or $$z = 2 \tan^{-1} x$$

Changing the independent variable from $x$ to $z$ by the relation $z = 2 \tan^{-1} x$, we get

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

where $$P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = \frac{\dfrac{-4x}{(1-x^2)^2} + \dfrac{2x}{1+x^2} \cdot \dfrac{2}{1+x^2}}{\dfrac{4}{(1+x^2)^2}} = 0$$

$$Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2} = 1 \quad \text{and} \quad R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2} = 0$$

$\therefore$ The transformed equation is $\dfrac{d^2y}{dx^2} + y = 0$

$\therefore \qquad y = c_1 \cos z + c_2 \sin z$

or $\qquad y = c_1 \cos (2 \tan^{-1} x) + c_2 \sin (2 \tan^{-1} x)$

$$= c_1 \cos \left( \tan^{-1} \frac{2x}{1-x^2} \right) + c_2 \sin \left( \tan^{-1} \frac{2x}{1-x^2} \right)$$

$$= c_1 \frac{1-x^2}{1+x^2} + c_2 \frac{2x}{1+x^2}$$

or $\qquad y(1 + x^2) = c_1 (1 - x^2) + 2c_2 x.$

**Example 5.** *Solve* $x^6 \dfrac{d^2y}{dx^2} + 3x^5 \dfrac{dy}{dx} + a^2y = \dfrac{1}{x^2}$.

**Sol.** The given equation can be written as

$$\frac{d^2y}{dx^2} + \frac{3}{x} \frac{dy}{dx} + \frac{a^2}{x^6} y = \frac{1}{x^8}$$

Choosing $z$, such that $\qquad \left(\dfrac{dz}{dx}\right)^2 = Q = \dfrac{a^2}{x^6}$

$$\therefore \qquad \frac{dz}{dx} = \frac{a}{x^3} \qquad \text{or} \qquad z = -\frac{a}{2x^2}.$$

Changing the independent variable from $x$ to $z$ by the relation $z = -\dfrac{a}{2x^2}$, we get

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dz} + Q_1 y = R_1$$

where
$$P_1 = \frac{\dfrac{d^2z}{dx^2} + P \dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = 0, \quad Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2} = 1$$

and
$$R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2} = \frac{1}{a^2x^2} = -\frac{2z}{a^3}$$

$\therefore$ The transformed equation is

$$\frac{d^2y}{dz^2} + y = -\frac{2z}{a^3} \text{ whose C.F.} = c_1 \cos z + c_2 \sin z$$

$$= c_1 \cos \left(-\frac{a}{2x^2}\right) + c_2 \sin \left(-\frac{a}{2x^2}\right)$$

$$= c_1 \cos \frac{a}{2x^2} + c_2 \sin \frac{a}{2x^2}$$

and
$$\text{P.I.} = \frac{1}{D^2 + 1}\left(-\frac{2z}{a^3}\right) = -\frac{2}{a^3} (1 + D^2)^{-1} z$$

$$= -\frac{2}{a^3}\,(1 - D^2 + D^4 \,....)\, z = -\frac{2z}{a^3} = \frac{1}{a^2x^2}$$

$$\therefore \qquad y = c_1 \cos\frac{a}{2x^2} + c_2 \sin\frac{a}{2x^2} + \frac{1}{a^2x^2}\,.$$

**Example 6.** *Solve :* $\dfrac{d^2y}{dx^2}\cos x + \dfrac{dy}{dx}\sin x - 2y\cos^3 x = 2\cos^5 x.$ (Meerut, 1997)

**Sol.** The given equation can be written as

$$\frac{d^2y}{dx^2} + \tan x\,\frac{dy}{dx} - (2\cos^2 x)\,y = 2\cos^5 x$$

Choosing $z$ such that $\left(\dfrac{dz}{dx}\right)^2 = \cos^2 x$

$$\therefore \qquad \frac{dz}{dx} = \cos x \qquad \text{or} \qquad z = \sin x$$

Changing the independent variable from $x$ to $z$ by the relation $z = \sin x$, we have

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1$$

where $$P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = \frac{-\sin x + \tan x\cos x}{\cos^2 x} = 0, \qquad Q_1 = \frac{1}{\left(\dfrac{dz}{dx}\right)^2} = -2$$

and $$R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2} = 2\cos^2 x = 2(1 - z^2)$$

$\therefore$ The transformed equation is

$$\frac{d^2y}{dz^2} - 2y = 2(1 - z^2)$$

whose $$\text{C.F.} = c_1 e^{\sqrt{2}z} + c_2\, e^{-\sqrt{2}z}$$

and $$\text{P.I.} = \frac{1}{D^2 - 2}\,.\,2(1 - z^2) = -\left(1 - \frac{D^2}{2}\right)^{-1}(1 - z^2)$$

$$= -\left(1 + \frac{D^2}{2} + \frac{D^4}{4}\,.....\right)(1 - z^2)$$

$$= -(1 - z^2) + \tfrac{1}{2}\,(+2) = z^2$$

$$\therefore \qquad y = c_1 e^{\sqrt{2}z} + c_2\, e^{-\sqrt{2}z} + z^2$$

Required solution is $\quad = c_1 e^{\sqrt{2}\sin x} + c_2\, e^{-\sqrt{2}\sin x} + \sin^2 x.$

**Example 7.** *Solve* : $\dfrac{d^2y}{dx^2} + \left(1 - \dfrac{1}{x}\right)\dfrac{dy}{dx} + 4x^2e^{-2x}y = 4(x^2 + x^3)\, e^{-3x}$.

**Sol.** Choosing $z$, such that

$$\left(\frac{dz}{dx}\right)^2 = 4x^2 e^{-2x} \qquad \therefore \qquad \frac{dz}{dx} = 2xe^{-x} \qquad \text{or} \qquad z = -2(x+1)\, e^{-x}.$$

Changing the independent variable from $x$ to $z$ by the relation,

$x = -2(x+1)\, e^{-x}$ , we have

$$\frac{d^2y}{dz^2} + P_1 \frac{dy}{dx} + Q_1y = R_1$$

where
$$P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{\left(\dfrac{dz}{dx}\right)^2} = \frac{2(1-x)\, e^{-x} + \left(1 - \dfrac{1}{x}\right)2xe^{-x}}{4x^2\, e^{-2x}}$$

$$Q_1 = \frac{Q}{\left(\dfrac{dz}{dx}\right)^2} = 1 \qquad \text{and} \qquad R_1 = \frac{R}{\left(\dfrac{dz}{dx}\right)^2} = (1+x)\, e^{-x} = -\tfrac{1}{2} z$$

$\therefore$ Transformed equation is

$$\frac{d^2y}{dz^2} + y = -\frac{1}{2} z$$

whose $\quad$ C.F. $= c_1 \cos z + c_2 \sin z$

and
$$\text{P.I.} = \frac{1}{D^2+1}\left(-\frac{1}{2}z\right) = -\frac{1}{2}(1 + D^2)^{-1} z = -\tfrac{1}{2}(1 - D^2 + D^4 - \ldots\ldots) z = -\tfrac{1}{2} z$$

$$\therefore \qquad y = c_1 \cos z + c_2 \sin z - \tfrac{1}{2} z$$

$$y = c_1 \cos\{2(x+1)e^{-x}\} - c_2 \sin\{2(x+1)\, e^{-x}\} + (x+1)\, e^{-x}.$$

**Example 8.** *Solve* : $x\dfrac{d^2y}{dx^2} - \dfrac{dy}{dx} - 4x^3y = 8x^3 \sin x^2$.

**Sol.** The given equation can be written as

$$\frac{d^2y}{dx^2} - \frac{1}{x}\frac{dy}{dx} - 4x^2y = 8x^2 \sin x^2$$

Choosing $z$, such that

$$\left(\frac{dz}{dx}\right)^2 = 4x^2 \qquad \text{or} \qquad \frac{dz}{dx} = 2x \qquad \therefore \qquad z = x^2$$

Changing the independent variable from $x$ to $z$ by the relation $z = x^2$, we get

$$\frac{d^2y}{dx^2} + P_1 \frac{dy}{dx} + Q_1y = R_1$$

$$\therefore \qquad y = A\phi(x) + B\psi(x)$$

Satisfies the equation

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$$

$$\therefore \qquad [A\phi''(x) + B\psi''(x)] + P[A\phi'(x) + B\psi'(x) + Q[A\phi(x) + B\psi(x)] = 0$$

or
$$A[\phi''(x) + P\phi'(x) + Q\phi(x)] + B[\psi''(x) + P\psi'(x) + Q\psi(x)] = 0$$

$$\therefore \qquad \phi''(x) + P\phi'(x) + Q\phi(x) = 0 \qquad ...(2)$$

and
$$\psi''(x) + P\psi'(x) + Q\psi(x) = 0 \qquad ...(3)$$

Now let us assume that

$$y = A\phi(x) + B\psi(x) \qquad ...(4)$$

So, the complete primitive of (1) where $A$ and $B$ are functions of $x$, so chosen that (1) will be satisfied.

$$\therefore \qquad \frac{dy}{dx} = A\phi'(x) + B\psi'(x) + \frac{dA}{dx}\phi(x) + \frac{dB}{dx}\psi(x)$$

Let $A$ and $B$ satsify the equation,

$$\phi(x)\frac{dA}{dx} + \psi(x)\frac{dB}{dx} = 0 \qquad ...(5)$$

$$\therefore \qquad \frac{dy}{dx} = A\phi'(x) + B\psi'(x)$$

and
$$\frac{d^2y}{dx^2} = A\phi''(x) + B\psi''(x) + \frac{dA}{dx}\phi'(x) + \frac{dB}{dx}\psi'(x).$$

Substituting in (1), we have

$$\left[A\phi''(x) + B\psi''(x) + \frac{dA}{dx}\phi'(x) + \frac{dB}{dx}\psi'(x)\right] + P[A\phi'(x) + B\psi'(x)] + Q[A\phi(x) + B\psi(x)] = R$$

or
$$A[\phi''(x) + P\phi'(x) + Q\phi(x)] + B[\psi''(x) + P\psi'(x) + Q\psi(x)] + \phi'(x)\frac{dA}{dx} + \psi'(x)\frac{dB}{dx} = R$$

Since the co-efficient of $A$ and $B$ are zero by (2) and (3), we have

$$\phi'(x)\frac{dA}{dx} + \psi'(x)\frac{dB}{dx} = R \qquad ...(6)$$

From (5) and (6), we have

$$\frac{dA}{dx}[\phi(x)\psi'(x) - \phi'(x)\psi(x)] = -R\psi(x)$$

$$\therefore \qquad \frac{dA}{dx} = \frac{R\psi(x)}{\phi'(x)\psi(x) - \phi(x)\psi'(x)}$$

Integrating, $\quad A\int\left[\frac{R\psi(x)}{\phi'(x)\psi(x) - \phi(x)\psi'(x)}\right]dx + c_1$

Similarly B can be determined from (5) and (6).

Substituting these values of A and B in (4), we get complete primitive of (1).

**Note 1. As the solution is obtained by varying the arbitrary constants of the complementary function, the method is known as variation of parameters.**

**2. Method of variation of parameters is to be used if instructed to do so.**

**Sol.** The given equation can be written as

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(x+1)}{x^2}y = x \qquad ...(1)$$

Here $P = \frac{-2(x+1)}{x^2}$, $Q = \frac{2(x+1)}{x^2}$ or $P + Qx = 0$

$\therefore$ $y = x$ is a part of C.F.

Now to find the complementary function of the given equation, *i.e.*, the solution of the equation

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(x+1)}{x^2}y = 0 \qquad ...(2)$$

Put $y = vx$, so that

$$\frac{dy}{dx} = \frac{dv}{dx}x + v$$

and $$\frac{d^2y}{dx^2} = \frac{d^2v}{dx^2}x + 2\frac{dv}{dx}.$$

Substituting in (2), we have

$$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - \frac{2(1+x)}{x}.\left(x\frac{dv}{dx} + v\right) + \frac{2(x+1)}{x^2}.vx = 0$$

or $$\frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 0,$$

A.E. $m^2 - 2m = 0$

$\therefore$ $m = 0, 2$

$\therefore$ $v = c_1e^{0x} + c_2e^{2x} = c_1 + c_2e^{2x}$

$\therefore$ Solution of (2) is $y = vx = c_1x + c_2xe^{2x}$

Now let $y = Ax + Bxe^{2x}$ ...(3)

be the complete primitive of the given equation (1), where $A$ and $B$ are functions of $x$.

$$\frac{dy}{dx} = A + B(e^{2x} + 2xe^{2x}) + x\frac{dA}{dx} + xe^{2x}\frac{dB}{dx}$$

Now choosing A and B, such that

$$x\frac{dA}{dx} + xe^{2x}\frac{dB}{dx} = 0$$

We have $$\frac{dy}{dx} = A + B(1+2x)e^{2x}$$

$\therefore$ $$\frac{d^2y}{dx^2} = \frac{dA}{dx} + e^{2x}\frac{dB}{dx}(1+2x) + 2Be^{2x} + 2B(1+2x)e^{2x}.$$

Substituting in equation (1), we have

$$\frac{dA}{dx} + e^{2x}\frac{dB}{dx}(1+2x) + 2Be^{2x} + 2B(1+2x)e^{2x}$$
$$- \frac{2(1+x)}{x}[A + B(1+2x)e^{2x}] + \frac{2(x+1)}{x^2}[Ax + Bxe^{2x}] = x.$$

Solving (2) and (3), we have

$$\frac{dA}{dx} = -1 \qquad \therefore \qquad A = -x + c_1$$

and
$$-\frac{dB}{dx} = \cot x \qquad \therefore \qquad B = \log \sin x + c_2$$

Substituting in (1), the complete solution is

$$y = c_1 \cos x + c_2 \sin x - x \cos x + \sin x . \log \sin x.$$

**Example 5.** *Solve by the method of variation of parameters*

$$\frac{d^2y}{dx^2} + 4y = 4 \tan 2x. \qquad \text{(Delhi, 1997)}$$

**Sol.** Here the C.F. is $y = c_1 \cos 2x + c_2 \sin 2x$

Let
$$y = A \cos 2x + B \sin 2x \qquad ...(1)$$

be the complete primitive of the given equation, where $A$ and $B$ are functions of $x$.

$$\therefore \qquad \frac{dy}{dx} = -2A \sin 2x + 2B \cos 2x + \frac{dA}{dx} \cos 2x + \frac{dB}{dx} \sin 2x.$$

Now choosing A and B, such that

$$\cos 2x . \frac{dA}{dx} + \sin 2x . \frac{dB}{dx} = 0 \qquad ...(2)$$

$$\therefore \qquad \frac{dy}{dx} = -2A \sin 2x + 2B \cos 2x$$

and
$$\frac{d^2y}{dx^2} = -2 \frac{dA}{dx} \sin 2x + 2 \frac{dB}{dx} \cos 2x - 4A \cos 2x - 4B \sin 2x$$

Substituting in the given equation, we have

$$-2 \sin 2x \frac{dA}{dx} + 2 \cos 2x \frac{dB}{dx} = 4 \tan 2x$$

$$-\sin 2x \frac{dA}{dx} + \cos 2x . \frac{dB}{dx} = 2 \tan 2x \qquad ...(3)$$

Solving (2) and (3), we have

$$\frac{dA}{dx} = -2 \tan 2x . \sin 2x$$

$$= -2 \frac{\sin^2 2x}{\cos 2x} = -2 \frac{(1 - \cos^2 2x)}{\cos 2x} = -2 \sec 2x + 2 \cos 2x$$

$$\therefore \qquad A = \log (\sec 2x + \tan 2x) + \sin 2x + c_1$$

and
$$\frac{dB}{dx} = 2 \sin 2x \qquad \therefore \qquad B = -\cos 2x + c_2$$

Substituting the values of A and B in (1), the required solution

$$y = c_1 \cos 2x + c_2 \sin 2x - [\log (\sec 2x + \tan 2x)] . \cos 2x.$$

**Example 6.** *Apply the method of variation of parameters to solve the equation*

$$(1 - x) \frac{d^2y}{dx^2} + x \frac{dy}{dx} - y = (1 - x)^2.$$

**Sol.** The equations can be written as

$$\frac{d^2y}{dx^2} + \frac{x}{1-x}\frac{dy}{dx} - \frac{1}{1-x}y = (1-x) \qquad ...(1)$$

Here $\quad P + Qx = 0$

$\therefore \quad y = x$ is a part of C.F.

Now to find the C.F. of (1), *i.e.*, the solution of

$$\frac{d^2y}{dx^2} + \frac{x}{1-x}\frac{dy}{dx} - \frac{1}{1-x}y = 0 \qquad ...(2)$$

We put $y = vx$, then equation (2) reduces to

$$\frac{d^2v}{dx^2} + \left(\frac{x}{1-x} + \frac{2}{x}\right)\frac{dv}{dx} = 0$$

or $$\frac{dp}{dx} + \left(\frac{x}{1-x} + \frac{2}{x}\right)P = 0, \quad \text{where } P = \frac{dv}{dx}$$

or $$\frac{dp}{dx} + \left(-1 - \frac{x}{x-1} + \frac{2}{x}\right)p = 0$$

or $$\frac{dp}{p} = \left(1 + \frac{1}{x-1} - \frac{2}{x}\right)dx$$

Integrating, $\quad \log p = x + \log(x-1) - 2\log x + \log c_1$

or $$P = \frac{dv}{dx} = \frac{c_1(x-1)e^x}{x^2}$$

or $$\frac{dv}{dx} = c_1\left(\frac{1}{x} - \frac{1}{x^2}\right)e^x \quad \text{or} \quad dv = c_1\left(\frac{e^x}{x} - \frac{e^x}{x^2}\right)dx$$

Integrating, $$v = c_1\int\frac{e^x}{x}dx - c_1\int\frac{e^x}{x^2}dx + c_2 = \frac{c_1}{x}e^x + c_2$$

$\therefore$ C.F. of (1), *i.e.*, solution of (2) is

$$y = vx = c_1e^x + c_2x$$

Now let $\quad y = Ae^x + Bx \qquad ...(3)$

be the complete solution of (1), where A and B are functions of $x$.

$$\therefore \quad \frac{dy}{dx} = Ae^x + B + e^x.\frac{dA}{dx} + x\frac{dB}{dx}$$

Choosing A and B, such that

$$e^x.\frac{dA}{dx} + x.\frac{dB}{dx} = 0 \qquad ...(4)$$

We have $\quad \dfrac{dy}{dx} = Ae^x + B \quad$ and $\quad \dfrac{d^2y}{dx^2} = e^x\dfrac{dA}{dx} + \dfrac{dB}{dx} + e^x.A$

Substituting in equation (1), we have

$$e^x\frac{dA}{dx} + \frac{dB}{dx} = 1 - x \qquad ...(5)$$

and $$e^{-x}\frac{dB}{dx} = -\frac{1}{1+e^x} \quad \therefore \quad \frac{dB}{dx} = -\frac{e^x}{1+e^x}$$

$$B = -\int \frac{e^x\,dx}{1+e^x} = -\log(1+e^x) + c_1$$

Substituting the values of A and B in (1), the complete solution is

$$y = c_1 e^x + c_2 e^{-x} + e^x \log\frac{1+e^x}{e^x} - 1 - e^x \log(1+e^x).$$

**Example 8.** *Solve by the method of variation of parameters :*

(*i*) $\dfrac{d^2y}{dx^2} + 4y = 4\sec^2 2x$ (*ii*) $\dfrac{d^2y}{dx^2} + y = x\sin x$

(*iii*) $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = e^{2x}\sec^2 x$ (*iv*) $y'' - 2y' + y = e^x \log x.$

**Sol.** (*i*) Given equation in symbolic form is $(D^2+4)y = 4\sec^2 2x$

Its A.E. is $D^2 + 4 = 0$ whence $D = \pm 2i$

$\therefore$ C.F. is $y = c_1\cos 2x + c_2 \sin 2x$

Let $y = A\cos 2x + B\sin 2x$ ...(1)

be the complete solution of the given equation, where A and B are functions of $x$.

$$\therefore \quad \frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x + \frac{dA}{dx}.\cos 2x + \frac{dB}{dx}\sin 2x$$

Choose A and B such that $\cos 2x\dfrac{dA}{dx} + \sin 2x\dfrac{dB}{dx} = 0$ ...(2)

$$\therefore \quad \frac{dy}{dx} = -2A\sin 2x + 2B\cos 2x$$

and $$\frac{d^2y}{dx^2} = -4A\cos 2x - 4B\sin 2x - 2\sin 2x\frac{dA}{dx} + 2\cos 2x\frac{dB}{dx}$$

Substituting the values of $y$ and $\dfrac{d^2y}{dx^2}$ in the given equation, we have

$$-2\sin 2x\frac{dA}{dx} + 2\cos 2x\frac{dB}{dx} = 4\sec^2 2x$$

or $$-\sin 2x\frac{dA}{dx} + \cos 2x\frac{dB}{dx} = 2\sec^2 2x \qquad ...(3)$$

Now we solve (2) and (3) for $\dfrac{dA}{dx}$ and $\dfrac{dB}{dx}$

Multiplying (2) by $\cos 2x$, (3) by $\sin 2x$ and subtracting

$$\frac{dA}{dx} = -2\sec^2 2x\sin 2x = -2\sec 2x\tan 2x$$

Integrating, $A = -\sec 2x + c_1$

Multiplying (2) by $\sin 2x$, (3) by $\cos 2x$ and adding

$$\frac{dB}{dx} = 2\sec 2x$$

$$= \frac{1}{2}\left[x\left(-\frac{\cos 2x}{2}\right) - \int 1.\left(-\frac{\cos 2x}{2}\right)dx\right] + c_2$$

$$= -\frac{x\cos 2x}{4} + \frac{\sin 2x}{8} + c_2$$

Substituting the values of A and B in (1), the complete solution is

$$y = \left(-\frac{x^2}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8} + c_1\right)\cos x + \left(-\frac{x\cos 2x}{4} + \frac{\sin 2x}{8} + c_2\right)\sin x$$

or $$y = c_1\cos x + c_2\sin x - \frac{x^2}{4}\cos x + \frac{x}{4}(\sin 2x\cos x - \cos 2x\sin x)$$

$$+ \frac{1}{8}(\cos 2x\cos x + \sin 2x\sin x)$$

or $$y = c_1\cos x + c_2\sin x - \frac{x^2}{4}\cos x + \frac{x}{4}\sin(2x - x) + \frac{1}{8}\cos(2x - x)$$

or $$y = c_1\cos x + c_2\sin x - \frac{x^2}{4}\cos x + \frac{x}{4}\sin x + \frac{1}{8}\cos x$$

or $$y = \left(c_1 + \frac{1}{8}\right)\cos x + c_2\sin x - \frac{x^2}{4}\cos x + \frac{x}{4}\sin x$$

or $$y = C_1\cos x + C_2\sin x - \frac{x^2}{4}\cos x + \frac{x}{4}\sin x \quad \text{where} \quad C_1 = c_1 + \frac{1}{8},\ C_2 = c_2$$

(*iii*) Given equation in symbolic form is

$$(D^2 - 4D + 4)y = e^{2x}\sec^2 x$$

Its A.E. is $D^2 - 4D + 4 = 0$ or $(D-2)^2 = 0$ $\Rightarrow$ $D = 2, 2$

$\therefore$ C.F. is $y = (c_1 + c_2x)e^{2x}$ or $y = c_1e^{2x} + c_2xe^{2x}$

Let $$y = Ae^{2x} + Bxe^{2x} \quad \text{...(1)}$$

be the complete solution of the given equation, where A and B are functions of $x$.

$$\therefore \quad \frac{dy}{dx} = 2Ae^{2x} + B(e^{2x} + 2xe^{2x}) + \frac{dA}{dx}\,.\,e^{2x} + \frac{dB}{dx}\,.\,x\,e^{2x}$$

Choose A and B such that $e^{2x}\,.\,\dfrac{dA}{dx} + xe^{2x}\,.\,\dfrac{dB}{dx} = 0$

or $$\frac{dA}{dx} + x\frac{dB}{dx} = 0 \quad \text{...(2)}$$

$$\therefore \quad \frac{dy}{dx} = 2Ae^{2x} + B(e^{2x} + 2xe^{2x})$$

and $$\frac{d^2y}{dx^2} = 4Ae^{2x} + B(2e^{2x} + 2e^{2x} + 4xe^{2x}) + 2e^{2x}\,.\,\frac{dA}{dx} + (e^{2x} + 2xe^{2x})\frac{dB}{dx}$$

Substituting the values of $y$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in the given equation, we have

$$2e^{2x}\,.\,\frac{dA}{dx} + (e^{2x} + 2xe^{2x})\frac{dB}{dx} = e^{2x}\sec^2 x$$

or $$2\frac{dA}{dx} + (1+2x)\frac{dB}{dx} = \sec^2 x \quad ...(3)$$

Now we solve (2) and (3) for $\frac{dA}{dx}$ and $\frac{dB}{dx}$

Multiplying (2) by $(1 + 2x)$, (3) by $x$ and subtracting

$$\frac{dA}{dx} = -x\sec^2 x$$

Integrating $A = -\int x\sec^2 x\, dx + c_1$

$$= -\left[x\tan x - \int \tan x\, dx\right] + c_1 = -x\tan x + \log\sec x + c_1$$

Multiplying (2) by 2 and subtracting (3) from it

$$-\frac{dB}{dx} = -\sec^2 x$$

Integrating $B = \tan x + c_2$

Substituting the values of A and B in (1), the complete solution is

$$y = (-x\tan x + \log\sec x + c_1)e^{2x} + (\tan x + c_2)\,.\,xe^{2x}$$

or $$y = (c_1 + c_2 x)e^{2x} + e^{2x}\log\sec x$$

(*iv*) Given equation in symbolic form is $(D^2 - 2D + 1)y = e^x \log x$

Its A.E. is $D^2 - 2D + 1 = 0$ or $(D-1)^2 = 0$ $\Rightarrow$ $D = 1, 1$

$\therefore$ C.F. is $y = (c_1 + c_2 x)e^x$ or $y = c_1 e^x + c_2 xe^x$

Let $$y = Ae^x + Bxe^x \quad ...(1)$$

be the complete solution of the given equation, where A and B are function of $x$.

$$\therefore \quad \frac{dy}{dx} = Ae^x + B(e^x + xe^x) + \frac{dA}{dx}\,.\,e^x + \frac{dB}{dx}\,.\,xe^x$$

Choose A and B such that $e^x\,.\,\frac{dA}{dx} + xe^x\,.\,\frac{dB}{dx} = 0$

or $$\frac{dA}{dx} + x\frac{dB}{dx} = 0 \quad ...(2)$$

$$\therefore \quad \frac{dy}{dx} = Ae^x + B(e^x + xe^x)$$

and $$\frac{d^2y}{dx^2} = Ae^x + B(e^x + e^x + xe^x) + e^x\,.\,\frac{dA}{dx} + (e^x + xe^x)\frac{dB}{dx}$$

Substituting the values of $y$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we have

$$e^x\frac{dA}{dx} + (e^x + xe^x)\frac{dB}{dx} = e^x \log x$$

or $$\frac{dA}{dx} + (1+x)\frac{dB}{dx} = \log x \quad ...(3)$$

Multiplying (2) by $(1 + x)$, (3) by $x$ and subtracting

$$\frac{dA}{dx} = -x\log x$$

Integrating $\quad A = -\int (\log x) \,.\, x \, dx + c_1$

$$= -\left[(\log x)\frac{x^2}{2} - \int \frac{1}{x} \cdot \frac{x^2}{2} \, dx\right] + c_1 = -\frac{x^2}{2}\log x + \frac{x^2}{4} + c_1$$

Subtracting (2) from (3), $\quad \dfrac{dB}{dx} = \log x$

Integrating $\quad B = \int \log x \,.\, 1 \, dx + c_2$

$$= (\log x)x - \int \frac{1}{x} \,.\, x \, dx + c_2 = x \log x - x + c_2$$

Substituting the values of A and B in (1), the complete solution is

$$y = \left(-\frac{x^2}{2}\log x + \frac{x^2}{4} + c_1\right) e^x + (x \log x - x + c_2)\, x\, e^x$$

or $$y = (c_1 + c_2 x)\, e^x + \left(-\frac{x^2}{2}\log x + \frac{x^2}{4} + x^2 \log x - x^2\right) e^x$$

or $$y = (c_1 + c_2\, x)e^x + \frac{1}{4}\, x^2\, e^x\, (2 \log x - 3).$$

**Example 9.** *Solve by the method of variation of parameters :*

(*i*) $y'' - 2y' + 2y = e^x \tan x$ $\qquad$ (*ii*) $\dfrac{d^2y}{dx^2} - 6\dfrac{dy}{dx} + 9y = \dfrac{e^{3x}}{x^2}$

(*iii*) $\dfrac{d^2y}{dx^2} - y = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$.

**Sol.** (*i*) Given equation in symbolic form is $(D^2 - 2D + 2)y = e^x \tan x$

Its A.E. is $\quad D^2 - 2D + 2 = 0 \quad$ whence $\quad D = 1 \pm i$

$\therefore$ C.F. is $\quad y = e^x (c_1 \cos x + c_2 \sin x)$

or $\quad y = c_1 e^x \cos x + c_2 e^x \sin x$

Let $\quad y = Ae^x \cos x + Be^x \sin x \quad$ ...(1)

be the complete solution of the given equation, where A and B are functions of $x$.

$$\therefore \quad \frac{dy}{dx} = A(e^x \cos x - e^x \sin x) + B(e^x \sin x + e^x \cos x) + \frac{dA}{dx} \,.\, e^x \cos x + \frac{dB}{dx} \,.\, e^x \sin x$$

Choose A and B such that $\quad e^x \cos x \dfrac{dA}{dx} + e^x \sin x \dfrac{dB}{dx} = 0$

or $$\cos x \frac{dA}{dx} + \sin x \frac{dB}{dx} = 0 \quad \text{...(2)}$$

$$\therefore \quad \frac{dy}{dx} = A(e^x \cos x - e^x \sin x) + B(e^x \sin x + e^x \cos x)$$

$$= Ae^x (\cos x - \sin x) + Be^x (\sin x + \cos x)$$

and $$\frac{d^2y}{dx^2} = A\,[e^x(\cos x - \sin x) + e^x(-\sin x - \cos x)]$$
$$+ B\,[e^x(\sin x + \cos x) + e^x(\cos x - \sin x)]$$
$$+ \frac{dA}{dx}\,.\,e^x(\cos x - \sin x) + \frac{dB}{dx}\,e^x(\sin x + \cos x)$$

Substituting the values of $y$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation, we have

$$\frac{dA}{dx}\,e^x(\cos x - \sin x) + \frac{dB}{dx}\,e^x(\sin x + \cos x) = e^x \tan x$$

or $$(\cos x - \sin x)\frac{dA}{dx} + (\sin x + \cos x)\frac{dB}{dx} = \tan x \qquad ...(3)$$

Multiplying (2) by $(\sin x + \cos x)$, (3) by $\sin x$ and subtracting

$$\frac{dA}{dx} = -\sin x \tan x$$

or $$\frac{dA}{dx} = -\frac{\sin^2 x}{\cos x} = -\frac{1-\cos^2 x}{\cos x} = -\sec x + \cos x$$

Integrating $\quad A = -\log(\sec x + \tan x) + \sin x + c_1$

Multiplying (2) by $(\cos x - \sin x)$, (3) by $\cos x$ and subtracting

$$-\frac{dB}{dx} = -\sin x \qquad \text{or} \qquad \frac{dB}{dx} = \sin x$$

Integrating $\quad B = -\cos x + c_2$

Substituting the values of A and B in (1), the complete solution is

$$y = [-\log(\sec x + \tan x) + \sin x + c_1]\,e^x \cos x + (-\cos x + c_2)\,e^x \sin x$$

or $$y = e^x(c_1 \cos x + c_2 \sin x) + e^x[-\cos \log(\sec x + \tan x) + \sin x \cos x - \cos x \sin x]$$

or $$y = e^x(c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)$$

(*ii*) Given equation in symbolic form is $(D^2 - 6D + 9)y = \frac{e^{3x}}{x^2}$

Its A.E. is $\quad D^2 - 6D + 9 = 0 \quad$ or $\quad (D-3)^2 = 0 \quad \Rightarrow \quad D = 3, 3$

$\therefore \quad$ C.F. is $y = (c_1 + c_2 x)\,e^{3x} \quad$ or $\quad y = c_1 e^{3x} + c_2 x\, e^{3x}$

Let $$y = A\,e^{3x} + Bx\,e^{3x} \qquad ...(1)$$

be the complete solution of the given equation, where A and B are functions of $x$.

$$\therefore \quad \frac{dy}{dx} = A\,.\,3e^{3x} + B(e^{3x} + x\,.\,3e^{3x}) + \frac{dA}{dx}\,e^{3x} + \frac{dB}{dx}\,.\,x\,e^{3x}$$

Choose A and B such that $\quad e^{3x}\frac{dA}{dx} + x\,e^{3x}\frac{dB}{dx} = 0$

or $$\frac{dA}{dx} + x\frac{dB}{dx} = 0 \qquad ...(2)$$

$\therefore \qquad \dfrac{dy}{dx} = 3Ae^{3x} + B\,(1 + 3x)e^{3x}$

and $\qquad \dfrac{d^2y}{dx^2} = 9Ae^{3x} + B[3e^{3x} + (1 + 3x) \,.\, 3e^{3x})] + 3e^{3x}\dfrac{dA}{dx} + (1 + 3x)\,e^{3x}\dfrac{dB}{dx}$

Substituting the values of $y$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in the given equation, we have

$$3e^{3x}\frac{dA}{dx} + (1 + 3x)\,e^{3x}\frac{dB}{dx} = \frac{e^{3x}}{x^2}$$

or $$3\frac{dA}{dx} + (1 + 3x)\frac{dB}{dx} = \frac{1}{x^2} \qquad \text{...(3)}$$

Multiplying (2) by $(1 + 3x)$, (3) by $x$ and subtracting

$$\frac{dA}{dx} = -\frac{1}{x}$$

Integrating, $\qquad A = -\log x + c_1$

Multiplying (2) by 3 and subtracting (3)

$$-\frac{dB}{dx} = -\frac{1}{x^2}$$

Integrating, $\qquad B = -\dfrac{1}{x} + c_2$

Substituting the values of A and B in (1), the complete solution is

$$y = (-\log x + c_1)\,e^{3x} + \left(-\frac{1}{x} + c_2\right)x\,e^{3x}$$

or $\qquad y = (c_1 + c_2x)\,e^{3x} - e^{3x}\log x - e^{3x}$

or $\qquad y = (c_1 - 1 + c_2x)\,e^{3x} - e^{3x}\log x$

or $\qquad y = (C_1 + c_2x - \log x)\,e^{3x} \qquad$ where $\quad C_1 = c_1 - 1$.

(*iii*) Given equation in symbolic form is $(D^2 - 1)y = e^{-x}\sin(e^{-x}) + \cos(e^{-x})$

Its A.E. is $\qquad D^2 - 1 = 0 \quad$ whence $\quad D = \pm 1$

$\therefore$ C.F. is $\qquad y = c_1\,e^{x} + c_2e^{-x}$

Let $\qquad y = Ae^{x} + Be^{-x} \qquad$ ...(1)

be the complete solution of the given equation, where A and B are functions of $x$.

$\therefore \qquad \dfrac{dy}{dx} = Ae^{x} - Be^{-x} + e^{x}\dfrac{dA}{dx} + e^{-x}\dfrac{dB}{dx}$

Choose A and B such that

$$e^{x}\frac{dA}{dx} + e^{-x}\frac{dB}{dx} = 0 \qquad \text{...(2)}$$

$\therefore \qquad \dfrac{dy}{dx} = Ae^{x} - Be^{-x}$

and $\qquad \dfrac{d^2y}{dx^2} = Ae^{x} + Be^{-x} + e^{x}\dfrac{dA}{dx} - e^{-x}\dfrac{dB}{dx}$

Substituting the values of $y$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the given equation,

we have $$e^x \frac{dA}{dx} - e^{-x} \frac{dB}{dx} = e^{-x} \sin(e^{-x}) + \cos(e^{-x}) \qquad ...(3)$$

Adding (2) and (3), $$2e^x \frac{dA}{dx} = e^{-x} \sin(e^{-x}) + \cos(e^{-x})$$

or $$\frac{dA}{dx} = \frac{1}{2} e^{-x} [e^{-x} \sin(e^{-x}) + \cos(e^{-x})]$$

Integrating $$A = \frac{1}{2} \int e^{-x} [e^{-x} \sin(e^{-x}) + \cos(e^{-x})]\, dx + c_1$$

$$= -\frac{1}{2} \int (t \sin t + \cos t)\, dt + c_1, \text{ where } t = e^{-x}$$

$$= -\frac{1}{2} \left[\int t \sin t\, dt + \int \cos t\, dt\right] + c_1$$

$$= -\frac{1}{2}\left[t(-\cos t) - \int 1(-\cos t)\, dt + \sin t\right] + c_1$$

$$= -\frac{1}{2}(-t \cos t + 2 \sin t) + c_1 = \frac{1}{2} e^{-x} \cos(e^{-x}) - \sin(e^{-x}) + c_1$$

Subtracting (3) from (2), $$2e^{-x} \frac{dB}{dx} = -e^{-x} \sin(e^{-x}) - \cos(e^{-x})$$

or $$\frac{dB}{dx} = \frac{1}{2} e^x [-e^{-x} \sin(e^{-x}) - \cos(e^{-x})] = -\frac{1}{2} [\sin(e^{-x}) + e^x \cos(e^{-x})]$$

Integrating $$B = -\frac{1}{2} \left[\int \cos(e^{-x}) . e^x\, dx + \int \sin(e^{-x})\, dx\right] + c_2$$

$$= -\frac{1}{2} \left[\cos(e^{-x}) . e^x - \int -\sin(e^{-x}) . e^{-x} + c_2 e^x\, dx + \int \sin(e^{-x})\, dx\right] + c_2$$

$$= -\frac{1}{2} \left[e^x \cos(e^{-x}) - \int \sin(e^{-x})\, dx + \int \sin(e^{-x})\, dx\right] + c_2$$

$$= -\frac{1}{2} e^x \cos(e^{-x}) + c_2$$

Substituting the values of A and B in (1), the complete solution is

$$y = \left[\frac{1}{2} e^{-x} \cos(e^{-x}) - \sin(e^{-x}) + c_1\right] e^x + \left[-\frac{1}{2} e^x \cos(e^{-x}) + c_2\right] e^{-x}$$

or $$y = c_1 e^x + c_2 e^{-x} + \frac{1}{2} \cos(e^{-x}) - e^x \sin(e^{-x}) - \frac{1}{2} \cos(e^{-x})$$

or $$y = c_1 e^x + c_2 e^{-x} - e^x \sin(e^{-x}).$$

# 9

# Simultaneous Differential Equations

## Introduction

In this chapter, we shall discuss differential equations in which there is one independent variable and two or more than two dependent variables. To solve such equations completely, we shall require as many simultaneous equations as are the number of dependent variables.

## Simultaneous Linear Differential Equations with Constant Co-efficients

There are two methods for the solution of simultaneous linear differential equation with constant co-efficients. Let $x$ and $y$ be the dependent variables and $t$ be the independent variable. Thus, in such equations there occur differential co-efficients w.r.t. '$t$'.

## First Method : Symbolic Method (Use of operator D)

Let $$f_1(D)x + f_2(D)y = T_1 \qquad ...(1)$$

and $$\phi_1(D)x + \phi_2(D)y = T_2 \qquad ...(2)$$

where D denotes the operator $\dfrac{d}{dt}$, $T_1$, $T_2$ and functions of the independent variable $t$ and $f_1(D)$, $f_2(D)$, $\phi_1(D)$, $\phi_2(D)$ are all rational integral functions of D with constant co-efficients.

Now operating on both sides of (1) by $\phi_2(D)$ and on both the sides of (2) by $f_2(D)$ and subtracting, we have

$$[f_1(D)\,\phi_1(D) - \phi_1(D)f_2(D)]\,x = \phi_2(D)T_1 - f_2(D)T_2 \qquad ...(3)$$

which is a linear equation in $x$ and $t$ and can be solved to give the value of $x$.

Substituting this value of $x$ in either (1) or (2), we get value of $y$.

**Note 1. We can also eliminate x to get a linear equation in y and t which can be solved for y and x can be obtained from (1) or (2) after putting the value of y.**

**Note 2. Since $f_2(D)$ and $\phi_2(D)$ are functions with constant co-efficients, therefore,**

$$f_2(D)\phi_2(D) = \phi_2(D)f_2(D).$$

## Second Method : Method of Differentiation

Sometimes, $x$ or $y$ can be conveniently eliminated if we differentiate (1) or (2). From the resulting equations after eliminating one dependent variable ($x$ or $y$), we can solve for the second variable and then the value of the remaining variable can be found.

**Example 1.** *Solve the simultaneous equations*

$$\frac{d^2x}{dt^2} - 3x - 4y = 0\ ;\ \frac{d^2y}{dt^2} + x + y = 0. \qquad \text{(Meerut, 1998)}$$

**Sol.** Writing D for $\frac{d}{dt}$ , the equations are

$$(D^2 - 3)\,x - 4y = 0 \qquad ...(1)$$

or
$$x + (D^2 + 1)y = 0 \qquad ...(2)$$

Eliminating $y$, we have $\quad [(D^2 + 1)(D^2 - 3) + 4]x = 0$

or $\quad (D^4 - 2D^2 + 1)x = 0 \quad$ or $\quad (D^2 - 1)^2\, x = 0$

$$\therefore \qquad x = (c_1 + c_2t)\, e^{-t} + (c_3 + c_4t)e^t \qquad ...(3)$$

Now
$$\frac{dx}{dt} = -(c_1 + c_2t)e^{-t} + c_2e^{-t} + (c_3 + c_4t) + c_3e^t$$

$$\frac{d^2x}{dt^2} = (c_1 + c_2t)e^{-t} - 2c_2e^{-t} + (c_3 + c_4t)e^t + 2c_4e^t$$

Substituting in (1), we have

$$4y = D^2x - 3x = (c_1 + c_2t)e^{-t} - 2c_2e^{-t} + (c_3 + c_4t)e^t + 2c_4e^t - 3(c_1 + c_2t)e^{-t} - 3(c_3 - c_4t)e^t$$

$$= -(2c_1 + 2c_2 + 2c_2t)e^{-t} + (-2c_3 + 2c_4 - 2c_4t)e^t$$

$$\therefore \qquad y = -\frac{1}{2}(c_1 + c_2 + c_2t)e^{-t} + \frac{1}{2}(c_4 - c_3 - c_4t)e^t$$

and
$$y = (c_1 + c_2t)e^{-t} + (c_3 + c_4t)e^t.$$

**Example 2.** *Solve* $\frac{dx}{dt} - 7x + y = 0\,;\ \frac{dy}{dt} - 2x - 5y = 0.$

**Sol.** Writing D for $\frac{d}{dt}$, the equations are

$$(D - 7)x + y = 0 \qquad ...(1)$$

and
$$-2x + (D - 5)y = 0 \qquad ...(2)$$

Eliminating $y$, we get $[(D - 5)(D - 7) + 2]x = 0$

or
$$(D^2 - 12D + 37)x = 0$$

A.E. is $\quad m^2 - 12m + 37 = 0$

$$m = \frac{12 \pm \sqrt{144 - 148}}{2} = 6 \pm i$$

$$\therefore \qquad x = e^{6t}(c_1 \cos t + c_2 \sin t)$$

$$\therefore \qquad \frac{dx}{dt} = 6e^{6t}(c_1 \cos t + c_2 \sin t) + e^{6t}(-c_1 \sin t + c_2 \cos t)$$

Putting in (1), we get $\; y = 7x - Dx$

$$= 7e^{6t}(c_1 \cos t + c_2 \sin t) - 6e^{6t}(c_1 \cos t + c_2 \sin t) - e^{6t}(-c_1 \sin t + c_2 \cos t)$$

$$= e^{6t}[(c_1 - c_2) \cos t + (c_2 + c_1) \sin t]$$

Hence the solution is, $\; x = e^{6t}(c_1 \cos t + c_2 \sin t)$

$$y = e^{6t}[(c_1 - c_2) \cos t + (c_1 + c_2) \sin t].$$

**Example 3.** *Solve* $\frac{dx}{dt} + 2\frac{dy}{dt} - 2x + 2y = 3e^t$

$$3\frac{dx}{dt} + \frac{dy}{dt} + 2x + y = 4e^{2t}.$$

**Sol.** Writing D for $\frac{d}{dt}$, the equations are

$$(D-2)x + 2(D+1)y = 3e^t \qquad ...(1)$$

and

$$(3D+2)x + (D+1)y = 4e^{2t} \qquad ...(2)$$

Eliminating $y$, we get

$$[2(3D+2)-(D-2)]x = 8e^{2t} - 3e^t$$

or

$$(5D+6)x = 8e^{2t} - 3e^t$$

or

$$\frac{dx}{dt} + \frac{6}{5}x = \frac{8}{5}e^{2t} - \frac{3}{5}e^t$$

which is linear differential equation,

$$\text{I.F.} = e^{\int \frac{6}{5} dt} + e^{6t/5}$$

$$\therefore \quad x.e^{6t/5} = \int e^{6t/5}\left(\frac{8}{5}e^{2t} - \frac{3}{5}e^t\right)dt + c_1 = \frac{8}{5}\int e^{\frac{16}{5}t}\,dt - \frac{3}{5}\int e^{\frac{11}{5}t}\,dt + c_1$$

$$= \frac{1}{2}e^{\frac{16}{5}t} - \frac{3}{11}e^{\frac{11}{5}t} + c_1$$

$$\therefore \quad x = \frac{1}{2}e^{2t} - \frac{3}{11}e^t + c_1e^{-6t/5}$$

Now $\quad \frac{dx}{dt} = e^{2t} - \frac{3}{11}e^t - \frac{6}{5}c_1e^{-6t/5}$

Substituting in (1), we get

$$2Dy + 2y + e^{2t} - \frac{3}{11}e^t - \frac{6}{5}c_1e^{-6t/5} - e^{2t} + \frac{6}{11}e^t - 2c_1e^{-6t/5} = 3e^t$$

or

$$\frac{2dy}{dt} + 2y = \frac{30}{11}e^t + \frac{16}{5}c_1e^{-6t/5}$$

or

$$\frac{dy}{dt} + y = \frac{15}{11}e^t + \frac{8}{5}c_1\,e^{-6t/5}$$

which is a linear differential equation.

$$\text{I.F.} = e^{\int dt} = e^t$$

$$\therefore \quad y.e^t = \frac{15}{11}\int e^{2t}\,dt + \frac{8}{5}c_1\int e^{-t/5}\,dt + c_2$$

Hence the solution is $\quad x = \frac{1}{2}e^{2t} - \frac{3}{11}e^t + c_1\,e^{-6t/5}$

$$y = \frac{15}{22}\,e^t - 8\,c_1\,e^{-6t/5} + c_2\,e^{-t}.$$

**Example 4.** *Solve* $\frac{dx}{dt} + \frac{dy}{dt} - 2y = 2\cos t - 7\sin t$

$$\frac{dx}{dt} - \frac{dy}{dt} + 2x = 4\cos t - 3\sin t.$$

(Kanpur, 1998)

**Sol.** Writing D for $\dfrac{d}{dt}$ the equations are

$$Dx + (D-2)\,y = 2\cos t - 7\sin t$$
$$(D+2)x - Dy = 4\cos t - 4\sin t$$

Eliminating $y$, we get

$$[D + (D-2)(D+2)]x = D(2\cos t - 7\sin t) + (D-2)(4\cos t - 3\sin t)$$

or $$(D^2 - 2)x = -9\cos t$$

$\therefore$ $$\text{C.F.} = c_1\, e^{\sqrt{2}t} + c_2\, e^{-\sqrt{2}t}$$

and $$\text{P.I.} = -9.\,\frac{1}{D^2-2}\cos t = \frac{-9\cos t}{-1^2-2} = 3\cos t$$

$\therefore$ $$x = c_1\, e^{\sqrt{2}t} + c_2\, e^{-\sqrt{2}t} + 3\cos t$$

$$\frac{dx}{dt} = 2c_1\, e^{\sqrt{2}t} - c_2\sqrt{2}\, e^{-\sqrt{2}t} - 3\sin t$$

Adding (1) and (3), we get

$$2\,Dx + 2x - 2y = 6\cos t - 10\sin t$$

$$y = \frac{dx}{dt} + x - 3\cos t + 5\sin t$$
$$= \sqrt{2}\, c_1 e^{\sqrt{2}t} - c_2\sqrt{2}\, e^{-\sqrt{2}t} - 3\sin t + c_1\, e^{\sqrt{2}t} + c_2\, e^{-\sqrt{2}t} + 3\cos t - 3\cos t + 5\sin t$$
$$= (\sqrt{2}+1)\, c_1\, e^{\sqrt{2}t} + (1-\sqrt{2})\, c_2\, e^{-\sqrt{2}t} + 2\sin t$$

Hence the solution is $x = c_1\, e^{\sqrt{2}t} + c_2\, e^{-\sqrt{2}t} + 3\cos t$

$$y = (\sqrt{2}+1)\, c_1 e^{\sqrt{2}t} + (1-\sqrt{2})\, c_2\, e^{-\sqrt{2}t} + 2\sin t.$$

**Example 5.** *Solve* $\dfrac{dx}{dt} = ax + by$ ; $\dfrac{dy}{dt} = a'x + b'y.$

**Sol.** Writing D for $\dfrac{d}{dt}$, the equations are

$$(D-a)\,x - by = 0 \quad \text{...(1)}$$

and $$-a'x + (D-b')y = 0 \quad \text{...(2)}$$

Eliminating $y$, we have $[(D-a)(D-b') - a'b]x = 0$

or $$[D^2 - (a+b)D + (ab' - a'b)]x = 0$$

$\therefore$ A.E. is $$m^2 - (a+b')m + (ab' - a'b) = 0$$

$$m = \frac{(a+b') \pm \sqrt{(a-b')^2 - 4(ab' - a'b)}}{2}$$
$$= \frac{(a+b') \pm \sqrt{[(a-b')^2 + 4a'b]}}{2} = m_1, m_2$$

where $$\left.\begin{aligned} m_1 &= \frac{(a+b') + \sqrt{[(a-b')^2 + 4a'b]}}{2} \\ m_2 &= \frac{(a+b') - \sqrt{[(a-b')^2 + 4a'b]}}{2} \end{aligned}\right\} \quad \text{...(3)}$$

and

$\therefore \qquad x = c_1 e^{m_1 t} + c_2 e^{m_2 t}$

$$\frac{dx}{dt} = c_1 m_1 e^{m_1 t} + c_1 m_2 e^{m_2 t}$$

$\therefore$ From (1), $\qquad y = \frac{1}{b}\left[\frac{dx}{dt} - ax\right] = \frac{1}{b}[(m_1 - a) c_1 e^{m_1 t} + (m_2 - a) c_2 e^{m_2 t}]$

Hence the solution is $x = c_1 e^{m_1 t} + c_2 e^{m_2 t}$

$$y = \frac{1}{b}[(m_1 - a) c_1 e^{m_1 t} + (m_2 - a) c_2 e^{m_2 t}]$$

where $m_1$ and $m_2$ are given as above.

**Example 6.** *Solve* $\frac{dx}{dt} + \omega y = 0$

$$\frac{dy}{dt} - \omega x = 0.$$

**Sol. First method**

Writing D for $\frac{d}{dt}$, the equations are

$$Dx + \omega y = 0 \qquad ...(1)$$

$$-\omega x + Dy = 0 \qquad ...(2)$$

Eliminating $y$, we have $\qquad (D^2 + \omega^2)x = 0$

$\therefore \qquad x = -A \cos \omega t + B \sin \omega t,$

so that $\qquad \frac{dx}{dt} = -A\omega \sin \omega t + B\omega \cos \omega t$

$\therefore$ From (1), $\qquad y = -\frac{1}{\omega}\frac{dx}{dt}$

$\therefore \qquad y = A \sin \omega t - B \cos \omega t$

and $\qquad x = A \cos \omega t + B \sin \omega t.$

**Second method**

Differentiating (1) w.r.t. '$t$', we have $D^2x + \omega Dy = 0$

$\therefore \qquad D^2x + \omega(\omega x) = 0$ with the help of (2) $\quad$ or $\quad (D^2 + \omega^2)x = 0$

Now proceed as in first method.

**Example 7.** *Solve* $\frac{dx}{dt} + 4x + 3y = t$

$$\frac{dy}{dt} + 2x + 5y = e^t. \qquad \text{(Delhi, 1999)}$$

**Sol.** Writing D for $\frac{d}{dt}$, the equations are

$$(D + 4)x + 3y = t \qquad ...(1)$$

and $\qquad 2x + (D + 5)y = e^t \qquad ...(2)$

Eliminating $y$, we have

$$[(D + 4)(D + 5) - 6]x = (D + 5)t - 3e^t$$

or $\qquad (D^2 + 9D + 14)x = 1 + 5t - 3e^t$

Putting the value of $t\dfrac{dy}{dt}$ from (2) and (3), we have

$$t^2\frac{d^2x}{dt^2}+3t\frac{dx}{dt}+2x+10y-2t^2=t$$

Now putting the value of $y$ from (1),

$$t^2\frac{d^2x}{dt^2}+3t\frac{dx}{dt}+2x+5\left(t\frac{dx}{dt}+2x-t\right)-2t^2=t$$

or
$$t\frac{d^2x}{dt^2}+8t\frac{dx}{dt}+12x=2t^2+6t \qquad ...(4)$$

which is a homogeneous linear equation.

Putting $t=e^z$ so that $\dfrac{dx}{dt}=\dfrac{dx}{dz}\cdot\dfrac{dz}{dt}=\dfrac{dx}{dz}\cdot\dfrac{1}{t}$

or
$$t\frac{dx}{dt}=\frac{dx}{dz} \quad \text{or} \quad t\frac{d}{dt}=\frac{d}{dz}$$

Denoting the operator $\dfrac{d}{dz}$ by D

$$t\frac{d}{dt}\left(t\frac{dx}{dt}\right)=t^2\frac{d^2x}{dt^2}+t\frac{dx}{dt}$$

$$\therefore \qquad t^2\frac{d^2x}{dt^2}=\left(t\frac{d}{dt}-1\right)t\frac{dx}{dt}=(\mathrm{D}-1)\mathrm{D}x$$

$\therefore$ Equation (4) gives $[(\mathrm{D}-1)\mathrm{D}+8\mathrm{D}+12]x=2e^{2z}+6e^z$

or
$$(\mathrm{D}^2+7\mathrm{D}+12)x=3e^{2z}+6e^z$$

A.E. is $m^2+7m+12=0$

$$\therefore \qquad m=-3,-4$$

$$\text{C.F.}=c_1e^{-3z}+c_2e^{-4z}$$

$$\text{P.I.}=\frac{2}{\mathrm{D}^2+7\mathrm{D}+12}e^{2z}+\frac{6}{\mathrm{D}^2+7\mathrm{D}+12}e^z$$

$$=\frac{2}{30}e^{2z}+\frac{6}{20}e^z=\frac{1}{15}e^z+\frac{3}{10}e^z$$

$$\therefore \qquad x=c_1e^{-3z}+c_2e^{-4z}+\frac{1}{15}e^{2z}+\frac{3}{10}e^z$$

or
$$x=\frac{c_1}{t^3}+\frac{c_2}{t^4}+\frac{t^2}{15}+\frac{3}{10}t$$

so that
$$\frac{dx}{dt}=-\frac{3c_1}{t^4}-\frac{4c^2}{t^5}+\frac{2}{15}t+\frac{3}{10}$$

Putting in (1), we get

$$2y=-\frac{3c_1}{t^3}-\frac{4c_2}{t^4}+\frac{2}{15}t^2+\frac{3}{10}t+\frac{2c_1}{t^3}+\frac{2c^2}{t^4}+\frac{2t^2}{15}+\frac{6}{10}t-t$$

$\therefore \qquad y = (3c_2 - c_1 - c_2 t)\, e^t - \frac{c_3}{6} e^{-3/2t} - \frac{1}{3}$

and $\qquad x = (c_1 + c_2 t)e^t + c_3 e^{-3/2t} - t.$

**Example 10.** *Solve :* $\frac{d^2x}{dt^2} + 4x + y = te^{3t}$, $\frac{d^2x}{dt^2} + y - 2x = \cos^2 t$.

**Sol.** Writing D for $\frac{d}{dt}$ the equations are

$$(D^2 + 4)x + y = te^{3t} \qquad ...(1)$$

and
$$-2x + (D^2 + 1)y = \cos^2 t$$

Eliminating $y$, we get $\qquad [(D^2 + 1)(D^2 + 4) + 2]x = (D^2 + 1)te^{3t} - \cos^2 t$

or
$$(D^4 + 5D^2 + 6)x = 10te^{3t} - \cos^2 t + 6e^{3t}$$

A.E. is $\qquad m^4 + 5m^2 + 6 = 0 \quad$ or $\quad (m^2 + 3)(m^2 + 2) = 0$

$\therefore \qquad m = \pm \sqrt{3}\, i, \pm \sqrt{2}\, i$

$$\text{C.F.} = (c_1 \cos \sqrt{3}\, t + c_2 \sin \sqrt{3}\, t) + (c_3 \cos \sqrt{2}\, t + c_4 \sin \sqrt{2}\, t)$$

$$\text{P.I.} = \frac{10}{D^4 + 5D^2 + 6} te^{3t} + \frac{6}{D^4 + 5D^2 + 6} e^{3t} - \frac{1}{D^4 + 5D^2 + 6} \cos^2 t$$

$$= 10e^{3t} \frac{1}{(D+3)^4 + 5(D+3)^2 + 6} t + \frac{6e^{3t}}{3^4 + 5.3^2 + 6}$$
$$- \frac{1}{D^4 + 5D^2 + 6} \frac{1}{2} (1 + \cos 2t)$$

$$= 10e^{3t} \frac{1}{132 + 138D + 59D^2 \,......} t + \frac{1}{22} e^{3t}$$
$$- \frac{1}{6 + 5D^2 + D^4} . \frac{1}{2} - \frac{1}{D^4 + 5D^2 + 6} \times \frac{1}{2} \cos 2t$$

$$= 10e^{3t} \frac{1}{132} \left(1 + \frac{23}{22} D + \frac{59}{132} D^2 \,......\right)^{-1} t$$
$$+ \frac{e^{3t}}{22} - \frac{1}{6}\left(1 + \frac{5D^2}{6} + \frac{D^4}{6}\right)^{-1} . \frac{1}{2} - \frac{\frac{1}{2}\cos 2t}{(-2^2)^2 - 5(-2^2) + 6}$$

$$= \frac{5e^{3t}}{66}\left(t - \frac{23}{22}\right) + \frac{e^{3t}}{22} - \frac{1}{6} . \frac{1}{2} - \frac{\cos 2t}{4}$$

$$= \frac{5}{66} te^{3t} + \frac{49}{1452} e^{3t} - \frac{1}{12} - \frac{1}{4} \cos 2t$$

$$x = (c_1 \cos \sqrt{3}\, t + c_2 \sin \sqrt{3}\, t) + (c_3 \cos \sqrt{2} t + c_4 \sin \sqrt{2} t) + \frac{5}{66} te^{3t}$$
$$- \frac{49}{1452} e^{3t} - \frac{1}{12} - \frac{1}{4} \cos 2t \qquad ...(3)$$

or $$x^2 \frac{d^2y}{dx^2} = \left(x\frac{d}{dx} - 1\right) x\frac{dy}{dx} = (D-1)\,Dy$$

$\therefore$ From (3), we get $\{(D-1)D + D - 1\}\,y = 0$

or $$(D^2-1)\,y = 0$$

$\therefore$ $$y = c_1e^t + c_2e^{-t} \quad \text{or} \quad y = c_1x + c_2x^{-1} \qquad ...(4)$$

so that $$\frac{dy}{dx} = c_1 - \frac{c_2}{x^2}$$

$\therefore$ From (1), we have $z = -x\,\dfrac{dy}{dx}$

or $$z = -c_1x + c_2x^{-1} \qquad ...(5)$$

The required solution is given by (4) and (5).

**Example 15.** *Solve* $\dfrac{d^2x}{dt^2} + 4x + 5y = t^2$

$$\frac{d^2y}{dt^2} + 5x + 4y = t + 1.$$

**Sol.** Writing D for $\dfrac{d}{dt}$, the given equations become

$$(D^2+4)x + 5y = t^2 \qquad ...(1)$$

and $$5x + (D^2+4)y = t + 1 \qquad ...(2)$$

To eliminate $y$, operating on both sides of (1) by $(D^2+4)$ and on both sides of (2) by 5 and subtracting,

we get $$[(D^2+4)^2 - 25]x = (D^2+4)t^2 - 5(t+1)$$

or $$(D^4 + 8D^2 - 9)x = 2 + 4t^2 - 5t - 5$$

or $$(D^4 + 8D^2 - 9)x = 4t^2 - 5t - 3$$

Its A.E. is $D^4 + 8D^2 - 9 = 0$

or $$(D^2+9)(D^2-1) = 0 \quad \therefore \quad D = \pm 1, \pm 3i$$

$$\text{C.F.} = c_1e^t + c_2e^{-t} + c_3\cos 3t + c_4 \sin 3t$$

$$\text{P.I.} = \frac{1}{D^4+8D^2-9}(4t^2-5t-3) = \frac{1}{-9\left(1-\frac{8D^2}{9}-\frac{D^4}{9}\right)}(4t^2-5t-3)$$

$$= -\frac{1}{9}\left[1-\left(\frac{8D^2}{9}+\frac{D^4}{9}\right)\right]^{-1}(4t^2-5t-3)$$

$$= -\frac{1}{9}\left[1+\left(\frac{8D^2}{9}+\frac{D^4}{9}\right)+\ldots\right](4t^2-5t-3)$$

$$= -\frac{1}{9}\left[4t^2-5t-3+\frac{8}{9}(8)\right] = -\frac{1}{9}\left(4t^2-5t+\frac{37}{9}\right)$$

$\therefore \qquad x = c_1e^t + c_2e^{-t} + c_3 \cos 3t + c_4 \sin 3t - \frac{4}{9}t^2 + \frac{5}{9}t - \frac{37}{81}$

Now $\qquad \frac{dx}{dt} = c_1e^t - c_2e^{-t} - 3c_3 \sin 3t + 3c_4 \cos 3t - \frac{8}{9}t + \frac{5}{9}$

$$\frac{d^2x}{dt^2} = c_1e^t + c_2e^{-t} - 9c_3 \cos 3t - 9c_4 \sin 3t - \frac{8}{9}$$

Substituting the values of $x$ and $\frac{d^2x}{dt^2}$ in (1), we have

$$\begin{aligned} 5y &= t^2 - 4x - \frac{d^2x}{dt^2} \\ &= t^2 - 4c_1e^t - 4c_2e^{-t} - 4c_3 \cos 3t - 4c_4 \sin 3t \\ &\quad + \frac{16}{9}t^2 - \frac{20}{9}t + \frac{148}{81} - c_1e^t - c_2e^{-t} + 9c_3 \cos 3t + 9c_4 \sin 3t + \frac{8}{9} \end{aligned}$$

$\therefore \qquad y = \frac{1}{5}\left[-5c_1e^t - 5c_2e^{-t} + 5c_3 \cos 3t + 5c_4 \sin 3t + \frac{25}{9}t^2 - \frac{20}{9}t + \frac{220}{81}\right]$

Hence $\qquad x = c_1e^t + c_2e^{-t} + c_3 \cos 3t + c_4 \sin 3t - \frac{1}{9}\left(4t^2 - 5t + \frac{37}{9}\right)$

$$y = -c_1e^t - c_2e^{-t} + c_3 \cos 3t + c_4 \sin 3t + \frac{1}{9}\left(5t^2 - 4t + \frac{44}{9}\right).$$

**Example 16.** *Solve the simultaneous equations :*

$$t\frac{dx}{dt} + y = 0,\ t\frac{dy}{dt} + x = 0 \text{ given } x(1) = 1,\ y(-1) = 0.$$

**Sol.** The given equations are $\quad t\frac{dx}{dt} + y = 0 \qquad$ ...(1)

$$t\frac{dy}{dt} + x = 0 \qquad \text{...(2)}$$

Differentiating (1) w.r.t. $t$, we have

$$t\frac{d^2x}{dt^2} + \frac{dx}{dt} + \frac{dy}{dt} = 0$$

Multiplying throughout by $t$

$$t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} + t\frac{dy}{dt} = 0$$

or $\qquad t^2\frac{d^2x}{dt^2} + t\frac{dx}{dt} - x = 0 \qquad$ [Using (2)] $\qquad$ ...(3)

which is Cauchy's homogeneous linear equation.

Putting $t = e^u$ *i.e.*, $u = \log t$, so that $t\frac{d}{dt} = \frac{d}{du} = \text{D}$, equation (3) becomes

$$[\text{D}(\text{D} - 1) + \text{D} - 1]x = 0 \quad \text{or} \quad (\text{D}^2 - 1)x = 0$$

Its A.E. is $D^2 - 1 = 0$ whence $D = \pm 1$

$$\therefore \qquad x = c_1 e^{u} + c_2 e^{-u} = c_1 t + \frac{c_2}{t} \qquad ...(4)$$

From (1), $$y = -t\frac{dx}{dt} = -t\left(c_1 - \frac{c_2}{t^2}\right) = -c_1 t + \frac{c_2}{t} \qquad ...(5)$$

Since $x(1) = 1$, $\therefore$ from (4), we have $1 = c_1 + c_2$

Also $y(-1) = 0$, $\therefore$ from (5), we have $0 = c_1 - c_2$

Solving $$c_1 = c_2 = \frac{1}{2}$$

Hence $$x = \frac{1}{2}\left(t + \frac{1}{t}\right), y = \frac{1}{2}\left(-t + \frac{1}{t}\right).$$

**Example 17.** *Solve the following simultaneous equations :*

$$\frac{dx}{dt} = 2y, \frac{dy}{dt} = 2z, \frac{dz}{dt} = 2x.$$

**Sol.** The given equations are

$$\frac{dx}{dt} = 2y \quad ...(1) \qquad \frac{dy}{dt} = 2z \quad ...(2) \qquad \frac{dz}{dt} = 2x \quad ...(3)$$

Differentiating (1) w.r.t. $t$, $$\frac{d^2x}{dt^2} = 2\frac{dy}{dt} = 2(2z) \qquad \text{[Using (2)]}$$

Differentiating again w.r.t. $t$, $$\frac{d^3x}{dt^3} = 4\frac{dz}{dt} = 4(2x)$$

or $$(D^3 - 8)x = 0 \quad \text{where } D = \frac{d}{dt}$$

Its A.E. is $D^3 - 8 = 0$ or $(D-2)(D^2 + 2D + 4) = 0$

whence $D = 2, \dfrac{-2 \pm 2i\sqrt{3}}{2}$ or $D = 2, -1 \pm i\sqrt{3}$

$$\therefore \quad x = c_1 e^{2t} + c_2 e^{-t}\cos(\sqrt{3}t - c_3)$$

From (1), $$y = \frac{1}{2}\frac{dx}{dt}$$

$$= \frac{1}{2}\left[2c_1 e^{2t} - c_2 e^{-t}\cos(\sqrt{3}t - c_3) - c_2\sqrt{3}e^{-t}\sin(\sqrt{3}t - c_3)\right]$$

$$= c_1 e^{2t} + c_2 e^{-t}\left[\cos\frac{2\pi}{3}\cos(\sqrt{3}t - c_3) - \sin\frac{2\pi}{3}\sin(\sqrt{3}t - c_3)\right]$$

$$= c_1 e^{2t} + c_2 e^{-t}\cos\left(\sqrt{3}t - c_3 + \frac{2\pi}{3}\right)$$

From (2), $$z = \frac{1}{2}\frac{dy}{dt}$$

The above equations can be written in the form,

$$P_1 \frac{dx}{dz} + Q_1 \frac{dy}{dz} + R_1 = 0$$

$$P_2 \frac{dx}{dz} + Q_2 \frac{dy}{dz} + R_2 = 0$$

Solving these by the method of cross-multiplication, we get

$$\frac{dx/dz}{Q_1R_2 - Q_2R_1} = \frac{dy/dz}{R_1P_2 - R_2P_1} = \frac{1}{P_1Q_2 - P_2Q_1}$$

or
$$\frac{dx}{Q_1R_2 - Q_2R_1} = \frac{dy}{R_1P_2 - R_2P_1} = \frac{dz}{P_1Q_2 - P_2Q_1}$$

which is of the form
$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad ...(3)$$

where P, Q, R are functions of $x, y, z$.

Thus simultaneous equation (1) and (2) can always be put in the form (3).

**Solution of Simultaneous Equations of the Form**

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \quad ...(1)$$

The following methods can be used for the solution of simultaneous equations of the form (1).

**First method.** First take any two members of the equation (1).

$\frac{dx}{P} = \frac{dz}{R}$ (say). Integrating, we obtain an equation.

Again, take other two members of equation (1),

$\frac{dy}{Q} = \frac{dz}{R}$ (say).

Integrating this we obtain another equation.

These two equations so obtained form the complete solution.

**Note. One solution so obtained can be used to simplify the other differential equations in the integrable form.**

**Second method.** We may be able to find $l$, $m$, $n$ and L, M, N such that one of the equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} = \frac{l dx + m dy + n dz}{lP + mQ + nR}$$

$$= \frac{L dx + M dy + N dz}{LP + MQ + NR}$$

can be easily integrated.

If $l$, $m$, $n$ are such that $lP + mQ + nR = 0$ then, we get $l dx + m dy + n dz = 0$ which give another equation on integration.

These two equations so obtained form the complete solution.

**Example 1.** *Solve :* $\dfrac{dx}{x^2}=\dfrac{dy}{y^2}=\dfrac{dz}{nxy}$.

**Sol.** From $\dfrac{dx}{x^2}=\dfrac{dy}{y^2}$

Integrating, $\dfrac{1}{x}=\dfrac{1}{y}+c_1$

$\therefore$ $$y-x=c_1xy \quad \text{...(1)}$$

Again, we have $$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{nxy}=\frac{\dfrac{dx}{x}-\dfrac{1}{y}dy+\dfrac{c_1}{n}dz}{x-y+c_1xy}$$

$$=\frac{\dfrac{dx}{x}-\dfrac{1}{y}dy+\dfrac{c_1}{n}dz}{0} \quad \text{[From (1)]}$$

$\therefore$ $\dfrac{1}{x}dx-\dfrac{1}{y}dy+\dfrac{c_1}{n}dz=0$

Integrating, $\log x-\log y+\dfrac{c_1}{n}z=c_1$

or $$\frac{c_1}{n}z=\log\frac{y}{x}+c \quad \text{or} \quad z=\frac{n}{c_1}\log\frac{y}{x}+c_2$$

or $$z=\frac{nxy}{y-x}\log\frac{y}{x}+c_2 \quad \text{...(2)}$$

(1) and (2) form the solution of the given equation.

**Example 2.** *Solve :* $\dfrac{dx}{xy}=\dfrac{dy}{y^2}=\dfrac{dz}{xyz-2x^2}$.

**Sol.** Taking the first two members, we get

$$\frac{dx}{xy}=\frac{dy}{y^2} \quad \text{or} \quad \frac{dx}{x}=\frac{dy}{y}$$

Integrating, $\log x=\log y+\log c_1$

$\therefore$ $$x=c_1y \quad \text{...(1)}$$

Again, taking the last two members, we get

$$\frac{dy}{y^2}=\frac{dz}{xyz-2x^2} \quad \text{or} \quad \frac{dy}{y^2}=\frac{dy}{zc_1y^2-2c_2y^2} \quad [\because \text{ of (1)}]$$

or $$dy=\frac{dz}{zc_1-2{c_1}^2} \quad \text{or} \quad c_1dy=\frac{dz}{z-2c_1}$$

Integrating, $c_1y=\log(z-2c_1)+c_2$

or $$x=\log\left(z-\frac{2x}{y}\right)+c_2 \quad [\because c_1y=x \text{ from (1)}]$$

or $$x=\log(xy-2)-\log y+c_2 \quad \text{...(2)}$$

Hence (1) and (2) form the complete solution of the equation.

**Example 3.** *Solve :* $\dfrac{adx}{(b-c)\,yz} = \dfrac{bdy}{(c-a)\,zx} = \dfrac{cdz}{(a-b)\,xy}$.

**Sol.** Choosing $x, y, z$ as multipliers, each function

$$= \frac{axdx + bydy + czdz}{0}$$

$\therefore$ $$axdx + bydy + czdz = 0$$

Integrating, $$ax^2 + by^2 + cz^2 = c_1 \qquad ...(1)$$

Again, choosing $ax, by, cz$ as multipliers, each fraction

$$= \frac{a^2xdx + b^2ydy + c^2zdz}{0}$$

$\therefore$ $$a^2xdx + b^2ydy + c^2zdz = 0$$

Integrating, $$a^2x^2 + b^2y^2 + c^2z^2 = c_2 \qquad ...(2)$$

(1) and (2) form the complete solution, or the general solution is

$$\phi\,(ax^2 + by^2 + cz^2,\, a^2x^2 + b^2y^2 + c^2z^2) = 0.$$

**Example 4.** *Solve :* $\dfrac{dx}{x^2 - yz} = \dfrac{dy}{y^2 - zx} = \dfrac{dz}{z^2 - xy}$. (Delhi, 1998)

**Sol.** From the above given equations, we get

$$\frac{dx - dy}{(x-z)(x+y+z)} = \frac{dy - dz}{(y-z)(x+y+z)}$$

or $$\frac{dx - dy}{x - y} = \frac{dy - dz}{y - z}$$

Integrating, $$\log(x - y) = \log(y - z) + \log a$$

or $$\frac{x-y}{y-z} = a \qquad ...(1)$$

Using $x, y, z$ as multipliers, each given fraction

$$= \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{xdx + ydy + zdz}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)} \qquad ...(2)$$

Also, each given fraction

$$= \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx} \qquad ...(3)$$

From (2) and (3) $$\frac{xdx + ydy + zdz}{x+y+z} = dx + dy + dz$$

or $$xdx + ydy + zdz = (x+y+z)\,d(x+y+z)$$

Integrating, $$\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{(x+y+z)^2}{2} + c$$

or $$x^2 + y^2 + z^2 = (x+y+z)^2 + 2c$$

or $$2xy + 2yz + 2zx + 2c = 0$$

or $$xy + yz + zx = b \quad \text{where} \quad b = -c \qquad ...(4)$$

**Sol.** Taking the first two members, we get

$$dx + dy = 0$$

$$\therefore \quad x + y = c_1 \qquad \text{...(1)}$$

Again, taking the first and the last members, we get

$$\frac{dx}{z} = \frac{dz}{z^2 + (x+y)^2}$$

or $$2dx = \frac{2zdz}{z^2 + c_1^2} \qquad \text{[From (1)]}$$

or $$2x + c_2 = \log (z^2 + c_1^2)$$

or $$\log \{z^2 + (x+y)^2\} - 2x = c_2 \qquad \text{...(2)}$$

(1) and (2) give the required solution.

**Example 9.** *Solve :* $\dfrac{dx}{y+z} = \dfrac{dy}{z+x} = \dfrac{dz}{x+y}$. (Delhi, 1999)

**Sol.** From the above given equation, we get

$$\frac{dx - dy}{y - x} = \frac{dy - dz}{z - y} = \frac{dx + dy + dz}{2(x+y+z)} \qquad \text{...(1)}$$

From the first two members, we get

$$\log (y - x) = \log (z - y) + \log c_1$$

$$\therefore \quad \frac{y-x}{z-y} = c_1$$

Again, from the first and the last members, we get

$$-\log (x - y) = \frac{1}{2} \log (x + y + z) - \log c_2$$

or $$(x + y)^2(x + y + z) = c_2 \qquad \text{...(2)}$$

(1) and (2) together give the complete solution.

**Example 10.** *Solve :* $\dfrac{dx}{1} = \dfrac{dy}{-2} = \dfrac{dz}{3x^2 \sin (y+x)}$.

**Sol.** From the first two members, we have

$$2dx + dy = 0$$

$$\therefore \quad 2x + y = c_1. \qquad \text{...(1)}$$

Again, from the first and the last members, we get $\quad dx = \dfrac{dz}{3x^2 \sin (y + 2x)}$

or $$3x^2 \sin c_1 \, dx = dz \qquad [\because \quad y + 2x = c_1 \text{ from (1)}]$$

Integrating, $c_2 + x^3 \sin c_1 = z$

or $$z - x^3 \sin (y + 2x) = c_2 \qquad \text{...(2)}$$

(1) and (2) together give the complete solution.

**Example 11.** *Solve :* $\dfrac{dx}{y^2 + z^2 - x^2} = \dfrac{dy}{-2xy} = \dfrac{dz}{-2xz}$. (Delhi, 2000)

**Sol.** From the last two members, we get

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\therefore \quad y = c_1 z \qquad \text{...(1}$$

Now using $x, y, z$ as multipliers, we get

Each fraction $= \dfrac{dz}{-2xz} = \dfrac{xdx + ydy + zdz}{-x(x^2+y^2+z^2)}$

or $$\frac{dz}{z} = \frac{2xdx + 2ydy + 2zdz}{x^2+y^2+z^2}$$

Integrating, $\log z + \log c_2 = \log (x^2 + y^2 + z^2)$

or $$x^2 + y^2 + z^2 = c_2 z \qquad ...(2)$$

(1) and (2) give the complete solution.

**Example 12.** *Solve :* $\dfrac{dx}{xz(z^2+xy)} = \dfrac{dy}{-yz(z^2+xy)} = \dfrac{dz}{x^4}$.

**Sol.** From the first two members, we get $\dfrac{dx}{x} + \dfrac{dy}{y} = 0$

Integrating, $\log x + \log y = \log c_1$ or $xy = c_1$ ...(1)

Again, taking the first and the last members, we get

$$\frac{dx}{xz(z^2+xy)} = \frac{dz}{x^4}$$

or $$x^3 dx = z\,(z^2 + c_1)\,dz \qquad [\because \; xy = c_1 \text{ from (1)}]$$

Integrating, $x^4 + c_2 = (z^2 + c_1)^2$

$$(z^2 + xy)^2 - x^4 = c_2 \qquad ...(2)$$

(1) and (2) give the required solution.

**Example 13.** *Solve :* $\dfrac{dx}{z(x+y)} = \dfrac{dx}{z(x-y)} = \dfrac{dz}{x^2+y^2}$. **(Meerut, 1998)**

**Sol.** Using $x, -y, -z$ as multipliers, we get

$$\text{Each fraction} = \frac{xdx - ydy - zdz}{xz(x+y) - yz(x-y) - z(x^2+y^2)} = \frac{xdx - ydy - zdz}{0}$$

$\therefore \quad xdx - ydy - zdz = 0$

Integrating, $x^2 - y^2 - z^2 = c_1$ ...(1)

Similarly, using $y, x, -z$ as multipliers, we get

$$\text{Each fraction} = \frac{ydx + xdy - zdz}{yz(x+y) + xz(x-y) - z(x^2+y^2)} = \frac{ydx + xdy - zdz}{0} \quad \therefore \; ydx + xdy - zdz = 0$$

Integrating, $2xy - x^2 = c_2$ ...(2)

(1) and (2) give the required solution.

**Example 14.** *Solve :* $\dfrac{dx}{x^2+y^2+yz} = \dfrac{dy}{x^2+y^2-xz} = \dfrac{dz}{z(x+y)}$ .

**Sol.** From the given equations, we get

$$\frac{dx - dy}{z(x+y)} = \frac{dz}{z(x+y)}$$

or $$dx - dy = dz$$

From (1) and (2), the general solution is

$$\phi\,(x^2 + y^2 + z^2, xyz) = 0.$$

**Example 17.** *Solve* : $\dfrac{dx}{y^3x - 2x^4} = \dfrac{dy}{2y^4 - x^3y} = \dfrac{dz}{9z(x^3 - y^3)}$.

**Sol.** The given equations can be written as

$$\frac{dx}{x(y^3 - 2x^3)} = \frac{dy}{y(2y^3 - x^3)} = \frac{dz}{9z(x^3 - y^3)}$$

Using $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{3z}$ as multipliers, each fraction

$$= \frac{\dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{3z}}{(y^3 - 2x^3) + (2y^3 - x^3) + 3(x^3 - y^3)} = \frac{\dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{3z}}{0}$$

$$\therefore \qquad \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{3z} = 0$$

Integrating, $\log x + \log y + \dfrac{1}{3}\log z = \log a$

or $$xyz^{1/3} = a \qquad \text{...(1)}$$

Now, from the first two fractions, we have

$$(2y^4 - x^3y)dx = (xy^3 - 2x^4)\,dy$$

Dividing by $x^3y^3$, $\left(\dfrac{2y}{x^3} - \dfrac{1}{y^2}\right) dx = \left(\dfrac{1}{x^2} - \dfrac{2x}{y^3}\right) dy$

or $$\left(\frac{1}{x^2}\,dy - \frac{2y}{x^3}\,dx\right) + \left(\frac{1}{y^2}\,dx - \frac{2x}{y^3}\,dy\right) = 0$$

or $$\frac{x^2dy - 2xydx}{x^4} + \frac{y^2dx - 2xydy}{y^4} = 0$$

or $$d\left(\frac{y}{x^2}\right) + d\left(\frac{x}{y^2}\right) = 0$$

Integrating, $\dfrac{y}{x^2} + \dfrac{x}{y^2} = b$ ...(2)

From (1) and (2), the general solution is

$$\phi\left(xyz^{1/3}, \frac{x}{y^2} + \frac{y}{x^2}\right) = 0.$$

**Example 18.** *Solve* : $\dfrac{dx}{x(y^2 + z)} = \dfrac{dy}{-y(x^2 + z)} = \dfrac{dz}{z(x^2 - y^2)}$

**Sol.** Using $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ as multipliers, each fraction

$$= \frac{\dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{z}}{y^2 + z - (x^2 + z) + x^2 - y^2} = \frac{\dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{z}}{0}$$

$$\therefore \qquad \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating, $\qquad \dfrac{l^2x^2}{2} + \dfrac{m^2y^2}{2} + \dfrac{n^2z^2}{2} = c$

or $\qquad l^2x^2 + m^2y^2 + n^2z^2 = b \quad \text{where} \quad b = 2c \qquad \ldots(2)$

(1) and (2) together give the general solution.

**Example 21.** *Solve* : $\dfrac{dx}{y - zx} = \dfrac{dy}{x + yz} = \dfrac{dz}{x^2 + y^2}$. (Delhi, 1998)

**Sol.** Using $x, -y, z$ as multipliers, each fraction

$$= \frac{xdy - ydy + zdz}{x(y - zx) - y(x + yz) + z(x^2 + y^2)} = \frac{xdx - ydy + zdz}{0}$$

$\therefore \qquad xdx - ydy + zdz = 0$

Integrating, $\qquad \dfrac{x^2}{2} - \dfrac{y^2}{2} + \dfrac{z^2}{2} = c$

or $\qquad x^2 - y^2 + z^2 = a \text{ where } \quad a = 2c \qquad \ldots(1)$

Using $y, x, -1$ as multipliers, each fraction

$$= \frac{ydx + xdy - dz}{y(y - zx) + x(x + yz) - (x^2 + y^2)} = \frac{ydx + xdy - dz}{0}$$

$\therefore \qquad ydx + xdy - dz = 0 \quad \text{or} \quad d(xy) - dz = 0$

Integrating, $\qquad xy - z = b \qquad \ldots(2)$

(1) and (2) together give the general solution.

**Example 22.** *Solve* : $\dfrac{dx}{y^2 + z^2} = \dfrac{dy}{-xy} = \dfrac{dz}{-xz}$.

**Sol.** From the last two fractions, we have $\dfrac{dy}{y} = \dfrac{dz}{z}$

Integrating, $\qquad \log y - \log z = \log a \quad \text{or} \quad \dfrac{y}{z} = a \qquad \ldots(1)$

Using $x, y, z$ as multipliers, each fraction

$$= \frac{xdx + ydy + dz}{x(y^2 + z^2) - xy^2 - xz^2} = \frac{xdx + ydy + zdz}{0}$$

$\therefore \qquad xdx + ydy + zdz = 0$

Integrating, $\qquad \dfrac{x^2}{2} + \dfrac{y^2}{2} + \dfrac{z^2}{2} = c$

or $\qquad x^2 + y^2 + z^2 = b \quad \text{where} \quad b = 2c \qquad \ldots(2)$

From (1) and (2), the general solution is

$$\phi\left(\frac{y}{z}, x^2 + y^2 + z^2\right) = 0.$$

**Example 23.** *Solve* : $\dfrac{dx}{y^2(x - y)} = \dfrac{dy}{-x^2(x - y)} = \dfrac{dz}{z(x^2 + y^2)}$

**Sol.** From the first two fractions, we have

$$x^2dx + y^2dy = 0$$

Integrating $\qquad \dfrac{x^3}{3} + \dfrac{y^3}{3} = c$

$$x^3 + y^3 = a \text{ where } a = 3c \qquad ...(1)$$

Also $$\frac{dx - dy}{y^2(x-y) + x^2(x-y)} = \frac{dz}{z(x^2+y^2)}$$

or $$\frac{dx - dy}{(x-y)(x^2+y^2)} = \frac{dz}{z(x^2+y^2)}$$

or $$\frac{dx - dy}{x-y} = \frac{dz}{z}$$

Integrating, $\log(x-y) - \log z = \log b$

or $$\frac{x-y}{z} = b \qquad ...(2)$$

From (1) and (2), the general solution is

$$\phi\left(x^3 + y^3, \frac{x-y}{z}\right) = 0.$$

**Example 24.** *Solve :* $\dfrac{dx}{x^2(y-z)} = \dfrac{dy}{y^2(z-x)} = \dfrac{dz}{z(x^2-y^2)}$.

**Sol.** Using $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$ as multipliers, each fraction

$$= \frac{\dfrac{dx}{x} + \dfrac{dy}{y} + \dfrac{dz}{z}}{0}$$

$\therefore$ $$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0$$

Integrating, $\log x + \log y + \log z = \log a$

or $$xyz = a \qquad ...(1)$$

Using $\dfrac{1}{x^2}, \dfrac{1}{y^2}, \dfrac{1}{z^2}$ as multipliers, each fraction

$$= \frac{\dfrac{dx}{x^2} + \dfrac{dy}{y^2} + \dfrac{dz}{z^2}}{0}$$

$\therefore$ $$\frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0$$

Integrating, $-\dfrac{1}{x} - \dfrac{1}{y} - \dfrac{1}{z} = c$

or $$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = b \text{ where } b = -c \qquad ...(2)$$

From (1) and (2), the general solution is

$$\phi\left(xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) = 0.$$

**Example 25.** *Solve :* $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}$.

**Sol.** From the last two fractions, we have $\frac{dy}{y} = \frac{dz}{z}$

Integrating, $\log y - \log z = \log a$

or $$\frac{y}{z} = a \qquad ...(1)$$

Using $x, y, z$ as multipliers, each fraction

$$= \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)}$$

$$\therefore \quad \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)} = \frac{dz}{2xz} \quad \text{or} \quad \frac{2xdx + 2ydy + 2zdz}{x^2 + y^2 + z^2} = \frac{dz}{z}$$

or $$\frac{d(x^2 + y^2 + z^2)}{x^2 + y^2 + z^2} = \frac{dz}{z}$$

Integrating, $\log (x^2 + y^2 + z^2) = \log z + \log b$.

or $$\frac{x^2 + y^2 + z^2}{z} = b \qquad ...(2)$$

From (1) and (2), the general solutions is

$$\phi\left(\frac{y}{z}, \frac{x^2 + y^2 + z^2}{z}\right) = 0.$$

**Example 26.** *Solve :* $\frac{dx}{\cos (x + y)} = \frac{dy}{\sin (x + y)} = \frac{dz}{z}$.

**Sol.** Using 1, 1, 0 as multipliers, each fraction

$$= \frac{dx + dy}{\cos (x + y) + \sin (x + y)} \qquad ...(1)$$

Using 1, – 1, 0 as multipliers, each fraction

$$= \frac{dx - dy}{\cos (x + y) - \sin (x + y)} \qquad ...(2)$$

Combining (1) with the third given fraction, we have

$$\frac{dz}{z} = \frac{dx + dy}{\cos (x + y) + \sin (x + y)}$$

or $$\frac{dz}{z} = \frac{dt}{\cos t + \sin t} \quad \text{where } t = x + y$$

$$= \frac{dt}{\sqrt{2}\left(\frac{1}{\sqrt{2}} \cos t + \frac{1}{\sqrt{2}} \sin t\right)} = \frac{dt}{\sqrt{2}\left(\sin \frac{\pi}{4} \cos t + \cos \frac{\pi}{4} \sin t\right)}$$

$$= \frac{dt}{\sqrt{2} \sin\left(t + \frac{\pi}{4}\right)}$$

or $$\sqrt{2}\, \frac{dz}{z} = \text{cosec}\left(t + \frac{\pi}{4}\right) dt$$

Integrating, $\sqrt{2}\log z = \log\tan\left(\frac{t}{2}+\frac{\pi}{8}\right)+\log a$

or $$z^{\sqrt{2}} = a\tan\left(\frac{t}{2}+\frac{\pi}{8}\right)$$

or $$z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right) = a \qquad ...(3)$$

From (1) and (2), $\dfrac{dx-dy}{\cos(x+y)-\sin(x+y)} = \dfrac{dx+dy}{\cos(x+y)+\sin(x+y)}$

or $$dx-dy = \frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}(dx+dy)$$

or $$dx-dy = \frac{\cos t-\sin t}{\cos t+\sin t}\,dt \quad \text{where } t = x+y$$

Integrating, $x-y = \log(\cos t+\sin t)+\log c$

or $$x-y = \log c(\cos t+\sin t)$$

or $$e^{x-y} = c(\cos t+\sin t)$$

or $$e^{y-x}(\cos t+\sin t) = \frac{1}{c}$$

or $$e^{y-x}[\cos(x+y)+\sin(x+y)] = b \text{ where } b = \frac{1}{c} \qquad ...(4)$$

(3) and (4) together give the general solution.

## EXAMPLES FOR PRACTICE

*Solve :*

1. $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{z}$

2. $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{z}$

3. $\dfrac{dx}{x} = \dfrac{dy}{z} = -\dfrac{dz}{y}$

4. $\dfrac{xdx}{y^2z} = \dfrac{dy}{xz} = \dfrac{dz}{y^2}$

5. $\dfrac{dx}{y^2} = \dfrac{dy}{x^2} = \dfrac{dz}{x^2y^2z^2}$ *(Delhi, 2000)*

6. $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$.

### Answers

1. $\dfrac{x}{a} = \dfrac{y}{b} = z$

2. $x^2-y^2 = c_1 y + x = c_2 z$

3. $y^2+z^2 = c_1, \log x = \tan^{-1}\dfrac{y}{z}$

4. $x^2-y^2 = c_1, x^3-y^3 = c_2$

5. $x^3-y^3 = c_1, x^3+\dfrac{3}{z} = c_2$

6. $x+y+z = c_1, xyz = c_2$.

# 10

# Series Solution of Differential Equations

## 1. Introduction

We have already discussed the solution of linear differential equations with constant co-efficients. The solution involves elementary functions such as polynomials, rational functions, trigonometric functions, logarithmic functions, exponential functions, hyperbolic functions etc. However, linear differential equations with variable co-efficients, which arise from physical problems, do not always admit solutions which are expressible in terms of elementary functions. Such equations can be solved by numerical methods, but in many cases it is easier to find a solution in the form of an infinite convergent series. The series solution of certain differential equations give rise to special functions such as Bessel's function, Legendre's function etc. which have many applications.

## 2. Power Series

An infinite series of the form

$$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1(x - x_0) + a_2 (x - x_0)^2 + \ldots\ldots$$

is called a power series in $(x - x_0)$.

Here the co-efficients $a_0, a_1, a_2, \ldots\ldots$ are constants and $x$ is a variable. The fixed number $x_0$ is called the centre of the power series.

In particular, if $x_0 = 0$, then the power series in $x$ is

$$\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \ldots\ldots.$$

For example, $1 + (x-2) + \dfrac{(x-2)^2}{2!} + \dfrac{(x-2)^3}{3!} + \ldots\ldots$ is a power series in $(x - 2)$. The centre of this power series is 2.

## 3. Convergence of Power Series

Let $$\sum_{m=0}^{\infty} a_m (x - x_0)^m = a_0 + a_1 (x - x_0) + a_2(x - x_0)^2 + \ldots\ldots \qquad \ldots(1)$$

be a power series with centre $x_0$.

Let $$S_n(x) = \sum_{m=0}^{n} a_m (x - x_0)^m$$
$$= a_0 + a_1 (x - x_0) + a_2 (x - x_0)^2 + \ldots\ldots + a_n (x - x_0)^n$$

The power series (1) is said to be **convergent at x = c** if $\lim_{n \to \infty} S_n(c)$ where $S_n(c) = a_0 + a_1 (c - x_0) + a_2(c - x_0)^2 + \ldots\ldots + a_n (c - x_0)^n$, exists finitely. The finite value of the limit is called the sum of the power series $\sum_{m=0}^{\infty} a_m (c - x_0)^m$.

Clearly, the power series (1) is always convergent at $x = x_0$ because in this case
$$S_n (x_0) = a_0 + 0 + 0 + \ldots.. + 0 = a_0$$
and $$\lim_{n \to \infty} S_n(x_0) = \lim_{n \to \infty} a_0 = a_0 \text{ which is finite.}$$

The set of all points (*i.e.*, values of $x$) for which (1) is convergent is called the **interval of convergence.** If the interval of convergence is finite then it is of the form $| x - x_0 | < R$, *i.e.*, $x_0 - R < x < x_0 + R$ or $(x_0 - R, x_0 + R)$. The constant R is called the **radius of convergence.**

Clearly, if $x_0 = 0$, then the interval of convergence of the power series $\sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \ldots\ldots$ is $(- R, R)$.

The radius of convergence can be determined by the formula
$$\frac{1}{R} = \lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right| \quad \text{or} \quad \frac{1}{R} = \lim_{m \to \infty} \sqrt[m]{| a_m |}$$

If $R = \infty$, then the interval of convergence is $(- \infty, \infty)$ and the power series converges for all $x$.

If $R = 0$, then the power series converges only at $x = x_0$.

**4. Working Rule for finding Radius of Convergence and Interval of Convergence**

Let the given power series be $\sum_{m=k}^{\infty} a_m (x - x_0)^m$ where $x_0$ may or may not be zero and $k$ is a non-negative integer.

(*a*) Find $\left| \frac{a_{m+1}}{a_m} \right|$

(*b*) Let $\lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right| = l$

(*i*) If $l = 0$, then $R = \infty$ and the interval of convergence is $(- \infty, \infty)$.

(*ii*) If $l = \infty$, then $R = 0$ and the power series converges only at $x = x_0$.

(*iii*) If $l$ is non-zero and finite, then $R = \frac{1}{l}$ and the interval of convergence is $(x_0 - R,\ x_0 + R)$.

**Remark.** If $a_m$ involves $m$ in the index, then use

$$\frac{1}{R} = \lim_{m \to \infty} \sqrt[m]{|a_m|}.$$

**Note.** Consider $\sum_{m=0}^{\infty} m(m+3)^2 x^{m+2}$

Suppose we want to express it in terms of $x^m$.
Put $m + 2 = k$ so that $m = k - 2$.
When $m = 0$, $k = 2$. As $m \to \infty$, $k \to \infty$

$$\therefore \quad \sum_{m=0}^{\infty} m(m+3)^2 x^{m+2} = \sum_{k=2}^{\infty} (k-2)(k+1)^2 x^k, \text{ Replacing } k \text{ by } m$$

$$= \sum_{k=2}^{\infty} (m-2)(m+1)^2 x^m.$$

**Example 1.** *Find the radius of convergence of the following power series :*

(*i*) $\sum_{m=0}^{\infty} (m+1)!\, x^m$ (*ii*) $\sum_{m=0}^{\infty} \frac{x^m}{(m+2)!}$

(*iii*) $\sum_{m=0}^{\infty} \frac{x^m}{5^m}$ (*iv*) $\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} x^m$.

**Sol.** (*i*) Comparing $\sum_{m=0}^{\infty} (m+1)!\, x^n$ with $\sum_{m=0}^{\infty} a_m x^m$, we get

$$a_m = (m+1)! \quad \therefore \quad a_{m+1} = (m+2)!$$

and $$\left| \frac{a_{m+1}}{a_m} \right| = \left| \frac{(m+2)!}{(m+1)!} \right| = m + 2$$

$$\therefore \quad \frac{1}{R} = \lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right| = \lim_{m \to \infty} (m+2) = \infty$$

$\Rightarrow$ Radius of convergence R = 0.

(*ii*) Comparing $\sum_{m=0}^{\infty} \frac{x^m}{(m+2)!}$ with $\sum_{m=0}^{\infty} a_m x^m$, we get

$$a_m = \frac{1}{(m+2)!} \quad \therefore \quad a_{m+1} = \frac{1}{(m+3)!}$$

and $$\left| \frac{a_{m+1}}{a_m} \right| = \left| \frac{(m+2)!}{(m+3)!} \right| = \frac{1}{m+3}$$

$$\therefore \quad \frac{1}{R} = \lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right| = \lim_{m \to \infty} \frac{1}{m+3} = 0$$

$\Rightarrow$ Radius of convergence R = $\infty$.

(*iii*) Comparing $\sum_{m=0}^{\infty} \frac{x^m}{5^m}$ with $\sum_{m=0}^{\infty} a_m x^m$, we get

$$a_m = \frac{1}{5^m} \quad \therefore \quad a_{m+1} = \frac{1}{5^{m+1}}$$

and
$$\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{5^m}{5^{m+1}}\right| = \frac{1}{5}$$

$\therefore$
$$\frac{1}{R} = \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\frac{1}{5} = \frac{1}{5}$$

$\Rightarrow$ Radius of convergence R = 5.

(*iv*) Comparing $\sum_{m=0}^{\infty} \frac{(-1)^m}{8^m} x^m$ with $\sum_{m=0}^{\infty} a_m x^m$, we get

$$a_m = \frac{(-1)^m}{8^m} \quad \therefore \quad a_{m+1} = \frac{(-1)^{m+1}}{8^{m+1}}$$

and
$$\left|\frac{a_{m+1}}{a_m}\right| = \left|\frac{-8^m}{8^{m+1}}\right| = \frac{1}{8}$$

$\therefore$
$$\frac{1}{R} = \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\frac{1}{8} = \frac{1}{8}$$

$\Rightarrow$ Radius of convergence R = 8.

**Example 2.** *Find the radius of convergence of the following power series :*

(*i*) $\sum_{m=0}^{\infty} (m+1)^2 x^m$ (*ii*) $\sum_{m=0}^{\infty} \frac{x^m}{m!}$

(*iii*) $\sum_{m=1}^{\infty} (2m-1)!\, x^m$ (*iv*) $\sum_{m=0}^{\infty} \frac{(3m)!}{(m!)^3} x^m$

**Sol.** (*i*) Comparing $\sum_{m=0}^{\infty} (m+1)^2 x^m$ with $\sum_{m=0}^{\infty} a_m x^m$, we get

$$a_m = (m+1)^2 \quad \therefore \quad a_{m+1} = (m+2)^2$$

and
$$\left|\frac{a_{m+1}}{a_m}\right| = \frac{(m+2)^2}{(m+1)^2} = \left(\frac{1+\frac{2}{m}}{1+\frac{1}{m}}\right)^2$$

$\therefore$
$$\frac{1}{R} = \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty}\left(\frac{1+\frac{2}{m}}{1+\frac{1}{m}}\right)^2 = \left(\frac{1+0}{1+0}\right)^2 = 1$$

$\Rightarrow$ Radius of convergence R = 1.

(*ii*) Please try yourself. **[Ans.** $R = \infty$]

(*iii*) Comparing $\sum_{m=1}^{\infty} (2m-1)!\, x^m$ with $\sum_{m=1}^{\infty} a_m x^m$, we get

$$a_m = (2m-1)! \quad \therefore \quad a_{m+1} = (2m+1)!$$

and
$$\left|\frac{a_{m+1}}{a_m}\right| = \frac{(2m+1)!}{(2m-1)!} = (2m+1)(2m)$$

$$\therefore \quad \frac{1}{R} = \lim_{m \to \infty} \left|\frac{a_{m+1}}{a_m}\right| = \lim_{m \to \infty} (2m+1)(2m) = \infty$$

$\Rightarrow$ Radius of convergence $R = 0$

$\Rightarrow$ The power series converges only at its centre $x = 0$.

(*iv*) Comparing $\sum_{m=0}^{\infty} \frac{(3m)!}{(m!)^3} x^m$ with $\sum_{m=0}^{\infty} a_m x^m$, we get

$$a_m = \frac{(3m)!}{(m!)^3} \quad \therefore \quad a_{m+1} = \frac{(3m+3)!}{((m+1)!)^3}$$

and
$$\left|\frac{a_{m+1}}{a_m}\right| = \frac{(3m+3)!}{((m+1)!)^3} \times \frac{(m!)^3}{(3m)!}$$

$$= \frac{(3m+3)(3m+2)(3m+1)}{(m+1)^3} = \frac{3(3m+2)(3m+1)}{(m+1)^2}$$

$$= \frac{3\left(3+\frac{2}{m}\right)\left(3+\frac{1}{m}\right)}{\left(1+\frac{1}{m}\right)^2}$$

$$\therefore \quad \frac{1}{R} = \lim_{m \to \infty} \left|\frac{a_{m+1}}{a_m}\right| = \lim_{m \to \infty} \frac{3\left(3+\frac{2}{m}\right)\left(3+\frac{1}{m}\right)}{\left(1+\frac{1}{m}\right)^2} = \frac{3(3)(3)}{(1)^2} = 27$$

$\Rightarrow$ Radius of convergence $R = \frac{1}{27}$.

**Example 3.** *Find the radius of convergence of the following power series :*

(*i*) $\sum_{m=0}^{\infty} (m+2)^m x^m$ (*ii*) $\sum_{m=2}^{\infty} m^m \cdot x^m$

(*iii*) $\sum_{m=0}^{\infty} (m+2)(m+3)x^{m+1}$.

**Sol.** (*i*) Comparing $\sum_{m=0}^{\infty} (m+2)^m x^m$ with $\sum_{m=0}^{\infty} a_m x^m$, we get

$$a_m = (m+2)^m \qquad \text{(involves } m \text{ in the index)}$$

$\therefore \quad \sqrt[m]{|a_m|} = [(m+2)^m]^{1/m} = m+2$

$$\frac{1}{R} = \lim_{m\to\infty} \sqrt[m]{|a_m|} = \lim_{m\to\infty} (m+2) = \infty$$

$\Rightarrow$ Radius of convergence R = 0.

(*ii*) Please try yourself by comparing with $\sum_{m=2}^{\infty} a_m x^m$. **[Ans. R = 0.]**

(*iii*) The given power series $\sum_{m=0}^{\infty} (m+2)(m+3)\, x^{m+1}$ is in terms of $x^{m+1}$.

We express it in terms of $x^m$.

Putting $m+1 = k$ so that $m = k-1$

When $m = 0$, $k = 1$. As $m \to \infty$, $k \to \infty$

$$\therefore \quad \sum_{m=0}^{\infty} (m+2)(m+3)\, x^{m+1} = \sum_{k=1}^{\infty} (k+1)(k+2)\, x^k$$

$$= \sum_{m=1}^{\infty} (m+1)(m+2) x^m \qquad \text{[Replacing } k \text{ by } m]$$

Comparing $\sum_{m=1}^{\infty} (m+1)(m+2) x^m$ with $\sum_{m=1}^{\infty} a_m x^m$, we get

$$a_m = (m+1)(m+2) \quad \therefore \quad a_{m+1} = (m+2)(m+3)$$

and
$$\left|\frac{a_{m+1}}{a_m}\right| = \frac{m+3}{m+1} = \frac{1+\frac{3}{m}}{1+\frac{1}{m}}$$

$$\therefore \quad \frac{1}{R} = \lim_{m\to\infty}\left|\frac{a_{m+1}}{a_m}\right| = \lim_{m\to\infty} \frac{1+\frac{3}{m}}{1+\frac{1}{m}} = \frac{1+0}{1+0} = 1$$

$\Rightarrow$ Radius of convergence R = 1.

**Example 4.** *Find the radius of convergence of the following power series :*

(*i*) $\sum_{m=0}^{\infty} \frac{x^{2m}}{3^m}$ (*ii*) $\sum_{m=0}^{\infty} (-1)^m x^{2m}$

(*iii*) $\sum_{m=0}^{\infty} m^m \,.\, x^{2m}$ (*iv*) $\sum_{m=0}^{\infty} \frac{(-1)^m}{k^m} x^{2m}$.

**Sol.** (*i*) The given power series is

$$\sum_{m=0}^{\infty} \frac{x^{2m}}{3^m} = \sum_{m=0}^{\infty} \frac{(x^2)^m}{3^m} = \sum_{m=0}^{\infty} \frac{y^m}{3^m}, \text{ where } y = x^2$$

Comparing it with $\sum_{m=0}^{\infty} a_m y^m$, we get

$$a_m = \frac{1}{3^m} \quad \therefore \quad a_{m+1} = \frac{1}{3^{m+1}}$$

and
$$\left| \frac{a_{m+1}}{a_m} \right| = \frac{3^m}{3^{m+1}} = \frac{1}{3}$$

$\therefore$
$$\frac{1}{R} = \lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right| = \lim_{m \to \infty} \frac{1}{3} = \frac{1}{3}$$

$\Rightarrow$ Radius of convergence for the power series $\sum_{m=0}^{\infty} \frac{y^m}{3^m}$ is 3

$\Rightarrow$ The power series $\sum_{m=0}^{\infty} \frac{y^m}{3^m}$ converges for $|y| < 3$

*i.e.,*
$$|x^2| < 3 \quad \text{or} \quad |x|^2 < 3 \quad \text{or} \quad |x| < \sqrt{3}$$

$\Rightarrow$ Radius of convergence for the given series is $\sqrt{3}$.

**Remark.** Here $a_m = \frac{1}{3^m}$ involves $m$ in the index. Therefore,

we can also use
$$\frac{1}{R} = \lim_{m \to \infty} \sqrt[m]{|a_m|}.$$

(*ii*) The given power series is

$$\sum_{m=0}^{\infty} (-1)^m x^{2m} = \sum_{m=0}^{\infty} (-1)^m (x^2)^m = \sum_{m=0}^{\infty} (-1)^m y^m, \quad \text{where } y = x^2.$$

Comparing it with $\sum_{m=0}^{\infty} a_m y^m$, we get

$$a_m = (-1)^m \quad \therefore \quad a_{m+1} = (-1)^{m+1}$$

and
$$\left| \frac{a_{m+1}}{a_m} \right| = \left| \frac{(-1)^{m+1}}{(-1)^m} \right| = |(-1)| = 1$$

$\therefore$
$$\frac{1}{R} = \lim_{m \to \infty} \left| \frac{a_{m+1}}{a_m} \right| = \lim_{m \to \infty} 1 = 1$$

$\Rightarrow$ Radius of convergence for the power series $\sum_{m=0}^{\infty} (-1)^m y^m$ is 1

$\Rightarrow$ The power series $\sum_{m=0}^{\infty} (-1)^m y^m$ converges for $|y| < 1$

*i.e.,*
$$|x^2| < 1 \quad \text{or} \quad |x|^2 < 1 \quad \text{or} \quad |x| < 1$$

$\Rightarrow$ Radius of convergence for the given series is 1.

In other words, $f(x)$ is analytic at $x = x_0$ if it possesses derivatives of all orders in some nbd of $x = x_0$.

All polynomial functions, $e^x$, $\sin x$, $\cos x$, $\sinh x$ and $\cosh x$ are analytic everywhere (*i.e.*, for all values of $x$). A rational function (*i.e.*, quotient of two polynomials) is analytic everywhere except at points at which the denominator is zero.

For example, the rational function $\dfrac{x+1}{x^2-3x+2}$ is analytic everywhere except at points where the denominator $x^2 - 3x + 2 = 0$ *i.e.*, $(x-1)(x-2) = 0 \Rightarrow x = 1$ or 2.

**6. Ordinary and Singular Points of Differential Equations**

Consider the differential equation

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \qquad ...(1)$$

of second order with variable co-efficients $P(x)$ and $Q(x)$.

A point $x_0$ is called an **ordinary point** of the differential equation (1) if both $P(x)$ and $Q(x)$ are analytic at $x_0$. Thus $x_0$ is an ordinary point of (1) if both $P(x)$ and $Q(x)$ possess derivatives of all orders in some nbd of $x_0$.

If the point $x_0$ is not an ordinary point of the differential co-efficient (1), then it is called a **singular point** of the differential equation (1). Thus at a singular point either $P(x)$ or $Q(x)$ or both are not analytic.

Singular points are of two types :

(*i*) Regular Singular point (*ii*) Irregular Singular point

If $x_0$ is a singular point of differential equation (1) and the functions $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are **both** analytic at $x_0$ [*i.e.*, both $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ possess derivatives of all orders in a nbd of $x_0$] then $x_0$ is called a **regular singular point** of the differential equation (1).

A singular point which is not regular is called an **irregular singular point.**

**Example 1.** *Show that $x = 0$ is an ordinary point of each of the following differential equations :*

(*i*) $(x^2+1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - xy = 0$ (*ii*) $\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + 2y = 0.$

**Sol.** (*i*) Dividing the given D.E. by $(x^2 + 1)$ to make the co-efficient of $\dfrac{d^2y}{dx^2}$ unity, we have

$$\frac{d^2y}{dx^2} + \frac{x}{x^2+1}\frac{dy}{dx} - \frac{x}{x^2+1}y = 0$$

Comparing it with $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} + Q(x)y = 0$, we have

$$P(x) = \frac{x}{x^2+1} \quad \text{and} \quad Q(x) = -\frac{x}{x^2+1}$$

Both P($x$) and Q($x$) possess derivatives of all orders in a nbd of $x = 0$.

$\Rightarrow$ Both P($x$) and Q($x$) are analytic at $x = 0$

$\therefore$ $x = 0$ is an ordinary point of the D.E.

(*ii*) Please try yourself.

**Example 2.** *Show that $x = 0$ is an ordinary point of*

$$(x^2 - 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$

*but $x = 1$ is a regular singular point.*

**Sol.** Dividing the given D.E. by $(x^2 - 1)$ to make the co-efficient of $\frac{d^2y}{dx^2}$ unity, we have

$$\frac{d^2y}{dx^2} + \frac{x}{x^2-1}\frac{dy}{dx} - \frac{1}{x^2-1}y = 0$$

Comparing it with $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$, we have

$$P(x) = \frac{x}{x^2-1} \quad \text{and} \quad Q(x) = \frac{-1}{x^2-1}$$

Both P($x$) and Q($x$) are analytic at $x = 0$

$\therefore$ $x = 0$ is an ordinary point of the D.E.

Since both P($x$) and Q($x$) are not defined at $x = 1$, so they are not analytic at $x = 1$. Thus $x = 1$ is a singular point.

Now $\quad (x-1)\,P(x) = (x-1) \times \frac{x}{x^2-1} = \frac{x}{x+1}$

and $$(x-1)^2\,Q(x) = (x-1)^2 \times \frac{-1}{x^2-1} = -\frac{x-1}{x+1}$$

Clearly both $(x - 1)$ P($x$) and $(x - 1)^2$ Q($x$) are analytic at $x = 1$.

$\therefore$ $x = 1$ is a regular singular point of the D.E.

**Example 3.** *Determine whether $x = 0$ is an ordinary point or a regular singular point of the differential equation*

$$2x^2\frac{d^2y}{dx^2} + 7x(x+1)\frac{dy}{dx} - 3y = 0. \qquad \text{(Delhi, 2000)}$$

**Sol.** Dividing the given D.E. by $2x^2$ to make the co-efficient of $\frac{d^2y}{dx^2}$ unity, we have

$$\frac{d^2y}{dx^2} + \frac{7(x+1)}{2x}\frac{dy}{dx} - \frac{3}{2x^2}y = 0$$

Comparing it with $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$, we have

$$P(x) = \frac{7(x+1)}{2x} \quad \text{and} \quad Q(x) = -\frac{3}{2x^2}$$

Since both $P(x)$ and $Q(x)$ are not defined at $x = 0$, so they are not analytic at $x = 0$. Thus $x = 0$ is not an ordinary point of the D.E.

$\Rightarrow$ $x = 0$ is a singular point.

Now $$(x - 0)\,P(x) = x \cdot \frac{7(x+1)}{2x} = \frac{7}{2}(x+1)$$

and $$(x - 0)^2\,Q(x) = x^2 \cdot \left(-\frac{3}{2x^2}\right) = -\frac{3}{2}$$

Clearly both $(x - 0)\,P(x)$ and $(x - 0)^2\,Q(x)$ are analytic at $x = 0$

$\therefore$ $x = 0$ is a regular singular point.

**Example 4.** *Verify that origin is a regular singular point of the equation*

$$2x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x+1)\,y = 0.$$

**Sol.** Please try yourself.

**Example 5.** *Show that $x = 0$ is a regular singular point of the equation*

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + \left(x^2 - \frac{1}{4}\right)y = 0.$$

**Sol.** Please try yourself.

**Example 6.** *Determine whether $x = 0$ is an ordinary point or a regular singular point for the differential equation*

$$2x^2 \frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x-5)y = 0.$$

**Sol.** Please try yourself. [**Ans.** Regular singular point.]

**Example 7.** *Show that $x = 0$ and $x = -1$ are singular points of*

$$x^2(x+1)^2 \frac{d^2y}{dx^2} + (x^2 - 1)\frac{dy}{dx} + 2y = 0$$

*where the first is irregular and the other is regular.*

**Sol.** Dividing the given D.E. by $x^2(x+1)^2$ to make the co-efficient of $\frac{d^2y}{dx^2}$ unity, we have

$$\frac{d^2y}{dx^2} + \frac{x-1}{x^2(x+1)}\frac{dy}{dx} + \frac{2}{x^2(x+1)^2}\,y = 0$$

Comparing it with $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$, we have

$$P(x) = \frac{x-1}{x^2(x+1)} \quad \text{and} \quad Q(x) = \frac{2}{x^2(x+1)^2}$$

Both $P(x)$ and $Q(x)$ are not defined at $x = 0$ and $x = -1$

$\therefore$ Both $P(x)$ and $Q(x)$ are not analytic at $x = 0$ and $x = -1$

$\Rightarrow$ $x = 0$ and $x = -1$ are singular points.

Now $$(x - 0)\,P(x) = \frac{x-1}{x(x+1)} \quad \text{and} \quad (x-0)^2\,Q(x) = \frac{2}{(x+1)^2}$$

which shows $(x - 0)\,P(x)$ is not analytic at $x = 0$

$\therefore$ $x = 0$ is an irregular singular point.

Also $\quad (x+1)\,P(x) = \dfrac{x-1}{x^2}$ and $(x+1)^2\,Q(x) = \dfrac{2}{x^2}$

which shows both $(x+1)\,P(x)$ and $(x+1)^2\,Q(x)$ are analytic at $x = -1$

$\therefore\ x = -1$ is a regular singular point.

**Example 8.** *Show that $x = 0$ is a regular singular point and $x = 1$ is an irregular singular point of the differential equation*

$$x(x-1)^3 \frac{d^2y}{dx^2} + 2(x-1)^2 \frac{dy}{dx} + 3y = 0.$$

**Sol.** Dividing the given D.E. by $x(x-1)^3$ to make the co-efficient of $\dfrac{d^2y}{dx^2}$ unity, we have

$$\frac{d^2y}{dx^2} + \frac{2}{x(x-1)}\frac{dy}{dx} + \frac{3}{x(x-1)^3}\,y = 0$$

Comparing it with $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} + Q(x)y = 0$, we have

$$P(x) = \frac{2}{x(x-1)} \quad \text{and} \quad Q(x) = \frac{3}{x(x-1)^3}$$

Both $P(x)$ and $Q(x)$ are not defined at $x = 0$ and $x = 1$

$\therefore$ Both $P(x)$ and $Q(x)$ are not analytic at $x = 0$ and $x = 1$

$\Rightarrow x = 0$ and $x = -1$ are singular points.

Now $\quad (x-0)\,P(x) = \dfrac{2}{x-1}$ and $(x-0)^2\,Q(x) = \dfrac{3x}{(x-1)^3}$

which shows both $(x-0)\,P(x)$ and $(x-0)^2\,Q(x)$ are analytic at $x = 0$

$\therefore\ x = 0$ is a regular singular point.

Also $\quad (x-1)\,P(x) = \dfrac{2}{x}$ and $(x-1)^2\,Q(x) = \dfrac{3}{x(x-1)}$

which shows $(x-1)^2\,Q(x)$ is not analytic at $x = 1$

$\therefore\ x = 1$ is an irregular singular point.

**Example 9.** *Show that $x = 0$ is a regular singular point of the differential equation $xy'' + y \sin x = 0$.*

**Sol.** Please try yourself.

**Example 10.** *Find the ordinary points, regular singular points and irregular singular points of the following differential equations :*

(*i*) $\dfrac{d^2y}{dx^2} + (x+2)\dfrac{dy}{dx} + (x^3 - 5x + 3)y = 0$ (*ii*) $x\dfrac{d^2y}{dx^2} + 4x^2\dfrac{dy}{dx} - 3xy = 0$

(*iii*) $(x-1)\dfrac{d^2y}{dx^2} + (2x-3)\dfrac{dy}{dx} + 5xy = 0$ (*iv*) $\dfrac{d^2y}{dx^2} + \dfrac{1}{x-2}\dfrac{dy}{dx} + 4y = 0$

(*v*) $(x+1)\dfrac{d^2y}{dx^2} + 3\dfrac{dy}{dx} + 7xy = 0.$

**Sol.** (*i*) The given D.E. is

$$\frac{d^2y}{dx^2} + (x+2)\frac{dy}{dx} + (x^3 - 5x + 3)y = 0$$

Comparing it with $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$, we have

$$P(x) = x + 2 \quad \text{and} \quad Q(x) = x^3 - 5x + 3$$

Both $P(x)$ and $Q(x)$ are polynomial functions and therefore, analytic everywhere.

$\therefore$ **Every point is an ordinary point** of this D.E. As such, it has no singular point, regular or irregular.

(*ii*) Divide the given D.E. by $x$ and proceed further yourself.

**Ans.** Every point is an ordinary point.

(*iii*) Dividing the given D.E. by $(x - 1)$

$$\frac{d^2y}{dx^2} + \frac{2x-3}{x-1}\frac{dy}{dx} + \frac{5x}{x-1}y = 0$$

Comparing it with $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$, we have

$$P(x) = \frac{2x-3}{x-1} \quad \text{and} \quad Q(x) = \frac{5x}{x-1}$$

Both $P(x)$ and $Q(x)$ are rational functions, analytic everywhere except at $x = 1$.

$\Rightarrow$ **Every point except $x = 1$ is an ordinary point** and $x = 1$ is a singular point.

Now $\quad (x - 1)P(x) = 2x - 3 \quad \text{and} \quad (x - 1)^2 Q(x) = 5x(x - 1)$

which shows that both $(x - 1)P(x)$ and $(x - 1)^2 Q(x)$ are polynomial functions, analytic everywhere.

$\therefore$ **$x = 1$ is a regular singular point of the D.E.** There is no irregular singular point.

(*iv*) Please try yourself. **Ans.** Every point except $x = 2$ is an ordinary point, $x = 2$ is a regular singular point, there is no irregular singular point.

(*v*) Please try yourself. **Ans.** Every point except $x = -1$ is an ordinary point, $x = -1$ is a regular singular point, there is no irregular singular point.

**Example 11.** *Find the ordinary points, regular singular points and irregular singular points of the following differential equations :*

(*i*) $x^3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 6(x + 1)y = 0$ (*ii*) $(x - 1)^4\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$

(*iii*) $(x - 1)(x + 2)\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + xy = 0$ (*iv*) $2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x + 1)y = 0.$

**Sol.** (*i*) [**Hint.** Divide by $x^3$.]

Here $\quad P(x) = \frac{4}{x^3} \quad \text{and} \quad Q(x) = \frac{6(x+1)}{x^3}$

Both are analytic everywhere except at $x = 0$.

$(x - 0)P(x)$ and $(x - 0)^2 Q(x)$ are not analytic at $x = 0$.]

**Ans.** Every point except $x = 0$ is an ordinary point. $x = 0$ is an irregular singular point, there is no regular singular point.

(*ii*) Please try yourself. **Ans.** Every point except $x = 1$ is an ordinary point, $x = 1$ is an irregular singular point, there is no regular singular point.

(*iii*) Please try yourself. **Ans.** Every point except $x = 1$ and $x = -2$ is an ordinary point, both $x = 1$ and $x = -2$ are regular singular points, there is no irregular singular point.

**Sol.** The given D.E. is $$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + x^2y = 0 \quad ...(1)$$

Comparing it with $$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$$

we have $$P(x) = x \quad \text{and} \quad Q(x) = x^2$$

Both $P(x)$ and $Q(x)$ are analytic at $x = 0$

$\Rightarrow$ 0 is an ordinary point of (1).

Let $$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1x + a_2x^2 + a_3x^3 + \ldots.. \quad ...(2)$$

be the power series solution of (1) so that

$$\frac{dy}{dx} = \sum_{m=1}^{\infty} m\,a_m x^{m-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}$$

Putting the values of $y$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{m=1}^{\infty} m\,a_m x^{m-1} + x^2\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \quad \sum_{m=2}^{\infty} m\,(m-1)\,a_m x^{m-2} + \sum_{m=1}^{\infty} m\,a_m x^m + \sum_{m=0}^{\infty} a_m x^{m+2} = 0 \quad ...(3)$$

Let $k = m - 2$ so that $m = k + 2$

$$\therefore \quad \sum_{m=2}^{\infty} m\,(m-1)\,a_m x^{m-2} = \sum_{k=0}^{\infty} (k+2)\,(k+1)\,a_{k+2}\,x^k$$

$$= \sum_{m=0}^{\infty} (m+1)\,(m+2)\,a_{m+2}\,x^m \qquad \text{(by replacing } k \text{ by } m)$$

Let $k = m + 2$ so that $m = k - 2$

$$\therefore \quad \sum_{m=0}^{\infty} a_m x^{m+2} = \sum_{k=2}^{\infty} a_{k-2}x^k = \sum_{m=2}^{\infty} a_{m-2}\,x^m \qquad \text{(by replacing } k \text{ by } m)$$

$\therefore$ From (3), we have

$$\sum_{m=0}^{\infty} (m+1)\,(m+2)\,a_{m+2}\,x^m + \sum_{m=1}^{\infty} m\,a_m x^m + \sum_{m=2}^{\infty} a_{m-2}\,x^m = 0$$

$$\Rightarrow \quad \left[1.2\,a_2 + 2.3a_3\,x + \sum_{m=2}^{\infty}(m+1)(m+2)\,a_{m+2}\,x^m\right] + \left[a_1x + \sum_{m=2}^{\infty} ma_m\,x^m\right] + \sum_{m=2}^{\infty} a_{m-2}\,x^m = 0$$

$$\Rightarrow \quad 2a_2 + (a_1 + 6a_3)x + \sum_{m=2}^{\infty} [(m+1)\,(m+2)\,a_{m+2} + ma_m + a_{m-2}]\,x^m = 0$$

Putting the values of $y$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$\sum_{m=2}^{\infty} m(m-1)\, a_m\, x^{m-2} + x \sum_{m=1}^{\infty} m\, a_m\, x^{m-1} + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \qquad \sum_{m=2}^{\infty} m(m-1)\, a_m\, x^{m-2} + \sum_{m=1}^{\infty} m\, a_m\, x^m + \sum_{m=0}^{\infty} a_m x^m = 0 \qquad \text{...(3)}$$

Let $k = m - 2$ so that $m = k + 2$

$$\therefore \quad \sum_{m=2}^{\infty} m(m-1)\, a_m\, x^{m-2} = \sum_{k=0}^{\infty} (k+2)(k+1)\, a_{k+2}\, x^k$$

$$= \sum_{m=0}^{\infty} (m+1)(m+2)\, a_{m+2}\, x^m \qquad \text{(by replacing } k \text{ by } m)$$

$\therefore$ From (3), we have

$$\sum_{m=0}^{\infty} (m+1)(m+2)\, a_{m+2}\, x^m + \sum_{m=1}^{\infty} m\, a_m\, x^m + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \quad \left[1.2\, a_2 + \sum_{m=1}^{\infty} (m+1)(m+2)\, a_{m+2}\, x^m\right] + \sum_{m=1}^{\infty} m a_m\, x^m + \left[a_0 + \sum_{m=1}^{\infty} a_m\, x^m\right] = 0$$

$$\Rightarrow \quad (a_0 + 2a_2) + \sum_{m=1}^{\infty} [(m+1)(m+2)\, a_{m+2} + m a_m + a_m]\, x^m = 0$$

$$\Rightarrow \quad (a_0 + 2a_2) + \sum_{m=1}^{\infty} [(m+1)(m+2)\, a_{m+2} + (m+1)\, a_m]\, x^m = 0$$

Equating to zero the co-efficients of various powers of $x$, we have

$$a_0 + 2a_2 = 0 \quad \Rightarrow \quad a_2 = -\frac{a_0}{2}$$

$(m+1)(m+2)a_{m+2} + (m+1)a_m = 0$ **for all m ≥ 1** $\qquad (\because \quad m + 1 \neq 0)$

$$\Rightarrow \quad (m+2)a_{m+2} + a_m = 0$$

$$\Rightarrow \quad a_{m+2} = -\frac{a_m}{m+2} \qquad \text{...(4)}$$

Putting $m = 1$ in (4) $\quad a_3 = -\frac{a_1}{3}$

Putting $m = 2$ in (4) $\quad a_4 = -\frac{a_2}{4} = -\frac{1}{4}\left(-\frac{a_0}{2}\right) = \frac{a_0}{2.4}$

Putting $m = 3$ in (4) $\quad a_5 = -\frac{a_3}{5} = -\frac{1}{5}\left(-\frac{a_1}{3}\right) = \frac{a_1}{3.5}$

Putting $m = 4$ in (4) $\quad a_6 = -\frac{a_4}{6} = -\frac{1}{6}\left(\frac{a_0}{2.4}\right) = -\frac{a_0}{2.4.6}$

$$\therefore \quad \sum_{m=0}^{\infty} a_m x^{m+1} = \sum_{k=1}^{\infty} a_{k-1} x^k = \sum_{m=1}^{\infty} a_{m-1} x^m \qquad \text{(by replacing } k \text{ by } m\text{)}$$

$\therefore$ From (3), we have

$$\sum_{m=0}^{\infty} (m+1)(m+2) a_{m+2} x^m + \sum_{m=1}^{\infty} a_{m-1} x^m = 0$$

$$\Rightarrow \quad 1.2\, a_2 + \sum_{m=1}^{\infty} (m+1)(m+2) a_{m+2} x^m + \sum_{m=1}^{\infty} a_{m-1} x^m = 0$$

$$\Rightarrow \quad 2\, a_2 + \sum_{m=1}^{\infty} [(m+1)(m+2) a_{m+2} + a_{m-1}] x^m = 0$$

Equating to zero the co-efficients of various powers of $x$, we have

$$2a_2 = 0 \quad \Rightarrow \quad a_2 = 0$$

$$(m+1)(m+2) a_{m+2} + a_{m-1} = 0 \quad \text{for all } m \geq 1$$

$$\Rightarrow \quad a_{m+2} = -\frac{a_{m-1}}{(m+1)(m+2)} \qquad \text{...(4)}$$

Putting $m = 1$ in (4) $\quad a_3 = -\dfrac{a_0}{2.3} = -\dfrac{a_0}{3!}$

Putting $m = 2$ in (4) $\quad a_4 = -\dfrac{a_1}{3.4} = -\dfrac{2a_1}{4!}$

Putting $m = 3$ in (4) $\quad a_5 = -\dfrac{a_2}{4.5} = 0 \qquad (\because \ a_2 = 0)$

Putting $m = 4$ in (4) $\quad a_6 = -\dfrac{a_3}{5.6} = -\dfrac{1}{5.6}\left(-\dfrac{a_0}{3!}\right) = \dfrac{4}{6!} a_0$

Putting $m = 5$ in (4) $\quad a_7 = -\dfrac{a_4}{6.7} = -\dfrac{1}{6.7}\left(-\dfrac{2a_1}{4!}\right) = \dfrac{2.5}{7!} a_1$

Putting $m = 6$ in (4) $\quad a_8 = -\dfrac{a_5}{7.8} = 0 \qquad (\because \ a_5 = 0)$

Putting $m = 7$ in (4) $\quad a_9 = -\dfrac{a_6}{8.9} = -\dfrac{1}{8.9} \cdot \dfrac{4}{6!} a_0 = -\dfrac{1.4.7}{9!} a_0$

Putting $m = 8$ in (4) $\quad a_{10} = -\dfrac{a_7}{9.10} = -\dfrac{1}{9.10} \cdot \left(\dfrac{2.5}{7!} a_1\right) = -\dfrac{2.5.8}{10!} a_1$

..........................................................................................................

..........................................................................................................

Putting these values in (2), the required solution is

$$y = a_0 + a_1 x + 0 - \frac{a_0}{3!} x^3 - \frac{2a_1}{4!} x^4 + 0 + \frac{4a_0}{6!} x^6 + \frac{2.5\, a_1}{7!} x^7 + 0$$
$$- \frac{1.4.7 a_0}{9!} x^9 - \frac{2.5.8 a_1}{10!} x^{10} + \ldots..$$

Comparing it with $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} + Q(x)\,y = 0$

we have $P(x) = x$ and $Q(x) = 2x^2 + 1$

Both $P(x)$ and $Q(x)$ are analytic at $x = 0$

$\Rightarrow$ 0 is an ordinary point of (1).

Let $$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots \qquad \ldots(2)$$

be the power series solution of (1) so that

$$\frac{dy}{dx} = \sum_{m=1}^{\infty} m a_m x^{m-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

Putting the values of $y$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we get

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{m=1}^{\infty} m a_m x^{m-1} + (2x^2+1)\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \quad \sum_{m=2}^{\infty} m(m-1)\, a_m x^{m-2} + \sum_{m=1}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} 2a_m x^{m+2} + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} (m+2)(m+1)\, a_{m+2} x^m + \sum_{m=1}^{\infty} m a_m x^m + \sum_{m=2}^{\infty} 2a_{m-2} x^m + \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \quad \left[2.1a_2 + 3.2a_3 x + \sum_{m=2}^{\infty}(m+2)(m+1)a_{m+2}x^m\right] + \left[a_1 x + \sum_{m=2}^{\infty} m a_m x^m\right] + \sum_{m=2}^{\infty} 2a_{m-2}x^m$$
$$+ \left[a_0 + a_1 x + \sum_{m=2}^{\infty} a_m x^m\right] = 0$$

$$\Rightarrow \quad (a_0 + 2a_2) + (2a_1 + 6a_3)x + \sum_{m=2}^{\infty}\left[(m+2)(m+1)a_{m+2} + (m+1)a_m + 2a_{m-2}\right]x^m = 0$$

Equating to zero the co-efficients of various powers of $x$, we have

$$a_0 + 2a_2 = 0 \qquad \Rightarrow \quad a_2 = -\frac{a_0}{2}$$

$$2a_1 + 6a_3 = 0 \qquad \Rightarrow \quad a_3 = -\frac{a_1}{3}$$

$$(m+2)(m+1)a_{m+2} + (m+1)a_m + 2a_{m-2} = 0 \quad \text{for all } m \ge 2 \qquad \ldots(3)$$

Putting $m = 2$ in (3) $\quad 4.3\, a_4 + 3a_2 + 2a_0 = 0$

$$\Rightarrow \quad 12a_4 + 3\left(-\frac{a_0}{2}\right) + 2a_0 = 0$$

$$\Rightarrow \quad 12a_4 = -\frac{a_0}{2} \qquad \Rightarrow \quad a_4 = -\frac{a_0}{24}$$

Putting $m = 3$ in (3) $\quad 5.4a_5 + 4a_3 + 2a_1 = 0$

$$\Rightarrow \quad 20a_5 + 4\left(-\frac{a_1}{3}\right) + 2a_1 = 0$$

$$\frac{dy}{dx} = \sum_{m=1}^{\infty} ma_m x^{m-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}$$

Putting the values of $y$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - x\sum_{m=1}^{\infty} ma_m x^{m-1} - p\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=1}^{\infty} ma_m x^m - \sum_{m=0}^{\infty} pa_m x^m = 0$$

$$\Rightarrow \quad \left[2.1a_2 + \sum_{m=1}^{\infty}(m+2)(m+1)a_{m+2}x^m\right] - \sum_{m=1}^{\infty} ma_m x^m - \left[pa_0 + \sum_{m=1}^{\infty} pa_m x^m\right] = 0$$

$$\Rightarrow \quad (2a_2 - pa_0) + \sum_{m=1}^{\infty} [(m+2)(m+1)a_{m+2} - (m+p)a_m]x^m = 0$$

Equating to zero the co-efficients of various powers of $x$, we have

$$2a_2 - pa_0 = 0 \quad \Rightarrow \quad a_2 = \frac{pa_0}{2} = \frac{pa_0}{2!}$$

$$(m+2)(m+1)a_{m+2} - (m+p)a_m = 0 \quad \Rightarrow \quad a_{m+2} = \frac{m+p}{(m+2)(m+1)}a_m \text{ for all } m \geq 1. \quad ...(3)$$

Putting $m = 1$ in (3) $\qquad a_3 = \frac{1+p}{3.2}a_1 = \frac{p+1}{3!}a_1$

Putting $m = 2$ in (3) $\qquad a_4 = \frac{2+p}{4.3}a_2 = \frac{p+2}{4.3}\left(\frac{pa_0}{2!}\right) = \frac{p(p+2)}{4!}a_0$

Putting $m = 3$ in (3) $\qquad a_5 = \frac{3+p}{5.4}a_3 = \frac{p+3}{5.4}\left(\frac{p+1}{3!}a_1\right) = \frac{(p+1)(p+3)}{5!}a_1$

..........................................................................................................

..........................................................................................................

Putting these values in (2), the required solution is

$$y = a_0 + a_1 x + \frac{pa_0}{2!}x^2 + \frac{p+1}{3!}a_1 x^3 + \frac{p(p+2)}{4!}a_0 x^4 + \frac{(p+1)(p+3)}{5!}a_1 x^5 + \ldots\ldots$$

or $$y = a_0\left[1 + \frac{p}{2!}x^2 + \frac{p(p+2)}{4!}x^4 + \ldots\ldots\right] + a_1\left[x + \frac{p+1}{3!}x^3 + \frac{(p+1)(p+3)}{5!}x^5 + \ldots\ldots\right].$$

**Example 8.** (*a*) *Find the power series solution of the equation*

$$(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - xy = 0$$

*in powers of x.* **(D.U., 2000)**

(*b*) *Find the power series solution of the following differential equations in powers of x :*

(*i*) $(x^2+1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + xy = 0$ (*ii*) $(1+x^2)y'' + xy' - y = 0.$ **(Delhi, 1999)**

Putting the values of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$(1-x^2)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + 2x\sum_{m=1}^{\infty} m a_m x^{m-1} - \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \quad \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1)a_m x^m + \sum_{m=1}^{\infty} 2m a_m x^m - \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} (m+2)(m+1)\, a_{m+2} x^m - \sum_{m=2}^{\infty} m(m-1)\, a_m x^m + \sum_{m=1}^{\infty} 2m a_m x^m - \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \quad \left[2.1a_2 + 3.2a_3 x + \sum_{m=2}^{\infty} (m+2)(m+1)a_{m+2}x^m\right] - \sum_{m=2}^{\infty} m(m-1)a_m x^m + \left[2a_1 x + \sum_{m=2}^{\infty} 2m a_m x^m\right] - \left[a_0 + a_1 x + \sum_{m=2}^{\infty} a_m x^m\right] = 0$$

$$\Rightarrow \quad (2a_2 - a_0) + (6a_3 + a_1)x + \sum_{m=2}^{\infty} \left[(m+2)(m+1)a_{m+2} - \{m(m-1) - 2m + 1\}a_m\right]x^m = 0$$

$$\Rightarrow \quad (2a_2 - a_0) + (6a_3 + a_1)x + \sum_{m=2}^{\infty} \left[(m+2)(m+1)a_{m+2} - \left(m^2 - 3m + 1\right)a_m\right]x^m = 0$$

Equating to zero the co-efficients of various powers of $x$, we have

$$2a_2 - a_0 = 0 \qquad \Rightarrow \quad a_2 = \frac{a_0}{2}$$

$$6a_3 + a_1 = 0 \qquad \Rightarrow \quad a_3 = -\frac{a_1}{6}$$

$$(m+2)(m+1)a_{m+2} - (m^2 - 3m + 1)a_m = 0 \quad \text{for all } m \geq 2$$

$$\Rightarrow \quad a_{m+2} = \frac{m^2 - 3m + 1}{(m+2)(m+1)}a_m \qquad \text{...(3)}$$

Putting $m = 2$ in (3) $\quad a_4 = \frac{4-6+1}{4.3}a_2 = \frac{-1}{4.3}\left(\frac{a_0}{2}\right) = -\frac{a_0}{4.3.2}$

Putting $m = 3$ in (3) $\quad a_5 = \frac{9-9+1}{5.4}a_3 = \frac{1}{5.4}\left(-\frac{a_1}{6}\right) = -\frac{a_1}{5.4.3.2}$

..........................................................................................

..........................................................................................

Putting these values in (2), the required solution is

$$y = a_0 + a_1 x + \frac{a_0}{2}x^2 - \frac{a_1}{6}x^3 - \frac{a_0}{4.3.2}x^4 - \frac{a_1}{5.4.3.2}x^5 + \ldots\ldots$$

or

$$y = a_0\left(1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \ldots\ldots\right) + a_1\left(x - \frac{x^3}{3!} - \frac{x^5}{5!}\ldots\ldots\right).$$

$$\Rightarrow \qquad a_{m+2} = \frac{m^2 - 2m + 2}{(m+2)(m+1)} a_{m-1} \qquad \ldots(3)$$

Putting $m = 3$ in (3) $\quad a_5 = \frac{9-6+2}{5.4} a_2 = 0 \qquad (\because\ a_2 = 0)$

Putting $m = 4$ in (3) $\quad a_6 = \frac{16-8+2}{6.5} a_3 = \frac{1}{3}\left(\frac{a_0}{6}\right) = \frac{a_0}{18}$

Putting $m = 5$ in (3) $\quad a_7 = \frac{25-10+2}{7.6} a_4 = \frac{17}{42}\left(\frac{a_1}{6}\right) = \frac{17}{252} a_1$ and so on.

Putting these values in (2), the required solution is

$$y = a_0 + a_1 x + 0 + \frac{a_0}{6} x^3 + \frac{a_1}{6} x^4 + 0 + \frac{a_0}{18} x^6 + \frac{17}{252} a_1 x^7 + \ldots$$

or
$$y = a_0 \left(1 + \frac{x^3}{6} + \frac{x^6}{18} + \ldots\right) + a_1 \left(x + \frac{x^4}{6} + \frac{17}{252} x^7 + \ldots\right).$$

**Example 11.** *Find the power series solution of the following initial value problems :*

(i) $\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0,\ y(0) = 1,\ y'(0) = 0$

(ii) $(x^2 + 1)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 2xy = 0,\ y(0) = 2,\ y'(0) = 3.$

**Sol.** (i) The given D.E. is $\quad \frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0 \qquad \ldots(1)$

Comparing it with $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$

we have $\qquad P(x) = -x \quad$ and $\quad Q(x) = -1$

Since the initial values of $y$ and $y'$ are given at $x = 0$, we must have power series solution of (1) in powers of $x - 0$ *i.e.*, $x$.

Both P($x$) and Q($x$) are analytic at $x = 0$

$\Rightarrow$ 0 is an ordinary point of (1).

Let
$$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \ldots\ldots \qquad \ldots(2)$$

be the power series solution of (1) so that

$$\frac{dy}{dx} = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots\ldots \qquad \ldots(3)$$

and
$$\frac{d^2y}{d^2x} = \sum_{m=2}^{\infty} m(m-1)\, a_m\, x^{m-2}$$

When $\quad x = 0, \quad y = 1 \qquad$ (given)

$\therefore$ From (2), $\mathbf{a_0 = 1}$

When $x = 0$, $\frac{dy}{dx} = 0$ (given)

$\therefore$ From (3), $\mathbf{a_1 = 0}$

Putting the values of $y$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - x\sum_{m=1}^{\infty} m a_m x^{m-1} - \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m - \sum_{m=1}^{\infty} m a_m x^m - \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \quad \left[2.1a_2 + 3.2a_3x + \sum_{m=2}^{\infty}(m+2)(m+1)a_{m+2}x^m\right] - \left[a_1x + \sum_{m=2}^{\infty} m a_m x^m\right] - \left[a_0 + a_1x + \sum_{m=2}^{\infty} a_m x^m\right] = 0$$

$$\Rightarrow \quad (2a_2 - a_0) + (6a_3 - 2a_1)x + \sum_{m=2}^{\infty}\left[(m+2)(m+1)a_{m+2} - (m+1)a_m\right]x^m = 0$$

$$\Rightarrow \quad (2a_2 - 1) + 6a_3x + \sum_{m=2}^{\infty}\left[(m+2)(m+1)a_{m+2} - (m+1)a_m\right]x^m = 0$$

($\because$ $a_0 = 1$ and $a_1 = 0$)

Equating to zero the co-efficients of various powers of $x$, we get

$$2a_2 - 1 = 0 \quad \Rightarrow \quad a_2 = \frac{1}{2}$$

$$6a_3 = 0 \quad \Rightarrow \quad a_3 = 0$$

$$(m+2)(m+1)a_{m+2} - (m+1)\,a_m = 0 \quad \text{for all } m \geq 2$$

$$\Rightarrow \quad (m+1)\left[(m+2)a_{m+2} - a_m\right] = 0$$

$$\Rightarrow \quad (m+2)a_{m+2} - a_m = 0 \qquad (\because \; m+1 \neq 0 \text{ since } m \geq 2)$$

$$\Rightarrow \quad a_{m+2} = \frac{a_m}{m+2} \qquad \text{...(4)}$$

Putting $m = 2$ in (4), $a_4 = \frac{a_2}{4} = \frac{1}{8}$

Putting $m = 3$ in (4), $a_5 = \frac{a_3}{5} = 0$

Putting $m = 4$ in (4), $a_6 = \frac{a_4}{6} = \frac{1}{48}$ and so on.

Putting these values in (2), the required solution is

$$y = 1 + 0 + \frac{1}{2}x^2 + 0 + \frac{1}{8}x^4 + 0 + \frac{1}{48}x^6 + \ldots\ldots$$

or $$y = 1 + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^6}{48} + \ldots\ldots$$

(*ii*) The given D.E. is $(x^2+1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + 2xy = 0$ ...(1)

or $$\frac{d^2y}{dx^2} + \frac{x}{x^2+1}\frac{dy}{dx} + \frac{2x}{x^2+1}y = 0$$

Comparing with $$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

we have $$P(x) = \frac{x}{x^2+1} \quad \text{and} \quad Q(x) = \frac{2x}{x^2+1}$$

Since the initial values of $y$ and $y'$ are given at $x = 0$, we must have power series solution of (1) in powers of $x - 0$ *i.e.*, $x$.

Both P($x$) and Q($x$) are analytic at $x = 0$

$\Rightarrow$ 0 is an ordinary point of (1).

Let $$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \ldots\ldots \qquad \text{...(2)}$$

be the power series solution of (1) so that

$$\frac{dy}{dx} = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots\ldots \qquad \text{...(3)}$$

and $$\frac{d^2y}{dx^2} = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}$$

When $x = 0, y = 2$ (given)

$\therefore$ From (2) $\mathbf{a_0 = 2}$

When $x = 0, \dfrac{dy}{dx} = 3$ (given)

$\therefore$ From (3) $\mathbf{a_1 = 3}$

Putting the values of $y$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we get

$$(x^2+1)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{m=1}^{\infty} m a_m x^{m-1} + 2x\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1)a_m x^m + \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=1}^{\infty} m a_m x^m + \sum_{m=0}^{\infty} 2a_m x^{m+1} = 0$$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1)a_m x^m + \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m + \sum_{m=1}^{\infty} m a_m x^m + \sum_{m=1}^{\infty} 2a_{m-1} x^m = 0$$

$$\Rightarrow \quad \sum_{m=2}^{\infty} m(m-1)a_m x^m + \left[2.1a_2 + 3.2a_3 x + \sum_{m=2}^{\infty}(m+2)(m+1)a_{m+2}x^m\right]$$

$$+\left[a_1 x + \sum_{m=2}^{\infty} m a_m x^m\right] + \left[2a_0 x + \sum_{m=2}^{\infty} 2a_{m-1}x^m\right] = 0$$

$$\Rightarrow \quad 2a_2 + (6a_3 + a_1 + 2a_0)x + \sum_{m=2}^{\infty}\left[(m+2)(m+1)a_{m+2} + \{m(m-1)+m\}a_m + 2a_{m-1}\right]x^m = 0$$

$$\Rightarrow \quad 2a_2 + (6a_3 + 7)x + \sum_{m=2}^{\infty}\left[(m+2)(m+1)\,a_{m+2} + m^2 a_m + 2a_{m-1}\right]x^m = 0$$

$$(\because \quad a_0 = 2 \quad \text{and} \quad a_1 = 3)$$

Equating to zero the co-efficients of various powers of $x$, we have

$$2a_2 = 0 \qquad \Rightarrow \quad a_2 = 0$$

$$6a_3 + 7 = 0 \qquad \Rightarrow \quad a_3 = -\frac{7}{6}$$

$$(m+2)(m+1)a_{m+2} + m^2 a_m + 2a_{m-1} = 0 \quad \text{for all } m \geq 2 \qquad \text{...(4)}$$

Putting $m = 2$ in (4) $\quad 4.3a_4 + 4a_2 + 2a_1 = 0$

$$\Rightarrow \quad 12a_4 + 0 + 6 = 0 \qquad (\because \quad a_2 = 0\,,\, a_1 = 3)$$

$$\Rightarrow \quad a_4 = -\frac{1}{2}$$

Putting $m = 3$ in (4) $\quad 5.4a_5 + 9a_3 + 2a_2 = 0$

$$\Rightarrow \quad 20a_5 + 9\left(-\frac{7}{6}\right) + 0 = 0$$

$$a_5 = \frac{21}{40} \quad \text{and so on.}$$

Putting these values in (2), the required solution is

$$y = 2 + 3x - \frac{7x^3}{6} - \frac{x^4}{2} + \frac{21x^5}{40} + \ldots\ldots$$

**Example 12.** *Find the power series solution of the following initial value problems :*

(*i*) $(x^2 - 1)\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + xy = 0, \quad y(0) = 4,\; y'(0) = 6$

(*ii*) $(1 - x^2)\dfrac{d^2y}{dx^2} + 2y = 0, \quad y(0) = 4,\; y'(0) = 5.$

**Sol.** (*i*) The given D.E is $\quad (x^2 - 1)\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + xy = 0 \qquad$ ...(1)

or

$$\frac{d^2y}{dx^2} + \frac{3x}{x^2-1}\frac{dy}{dx} + \frac{x}{x^2-1}y = 0$$

Comparing it with $\quad \dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} + Q(x)y = 0$

we have $$P(x) = \frac{3x}{x^2-1} \quad \text{and} \quad Q(x) = \frac{x}{x^2-1}$$

Since the initial values of $y$ and $y'$ are given at $x = 0$, we must have power series solution of (1) in powers of $x - 0$ *i.e.*, $x$.

Both $P(x)$ and $Q(x)$ are analytic at $x = 0$

$\Rightarrow$ 0 is an ordinary point of (1).

Let $$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1 x + a_2 x^2 + \ldots\ldots \quad \ldots(2)$$

be the power series solution of (1) so that

$$\frac{dy}{dx} = \sum_{m=1}^{\infty} m a_m x^{m-1} = a_1 + 2a_2 x + 3a_3 x^2 + \ldots\ldots \quad \ldots(3)$$

and $$\frac{d^2y}{dx^2} = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$$

When $x = 0, y = 4$ (given)

$\therefore$ From (2) $\mathbf{a_0 = 4}$

When $x = 0$, $\frac{dy}{dx} = 6$ (given)

$\therefore$ From (3) $\mathbf{a_1 = 6}$

Putting the values of $y$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$(x^2-1)\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + 3x \sum_{m=1}^{\infty} m a_m x^{m-1} + x\sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=1}^{\infty} 3m a_m x^m + \sum_{m=0}^{\infty} a_m x^{m+1} = 0$$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1)a_m x^m - \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m + \sum_{m=1}^{\infty} 3m a_m x^m + \sum_{m=1}^{\infty} a_{m-1} x^m = 0$$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1)a_m x^m - \left[2.1a_2 + 3.2a_3 x + \sum_{m=2}^{\infty}(m+2)(m+1)a_{m+2}x^m\right]$$
$$+ \left[3a_1 x + \sum_{m=2}^{\infty} 3m a_m x^m\right] + \left[a_0 x + \sum_{m=2}^{\infty} a_{m-1}x^m\right] = 0$$

$$\Rightarrow -2a_2 + (a_0 + 3a_1 - 6a_3)x$$
$$+ \sum_{m=2}^{\infty}\left[\{m(m-1) + 3m\}a_m + a_{m-1} - (m+2)(m+1)a_{m+2}\right]x^m = 0$$

$$\Rightarrow -2a_2 + (22 - 6a_3)x + \sum_{m=2}^{\infty}\left[m(m+2)a_m + a_{m-1} - (m+2)(m+1)a_{m+2}\right]x^m = 0$$

$(\because \ a_0 = 4, a_1 = 6)$

**Example 13.** *Apply power series method to solve the following equations in powers of x :*

(i) $y'' - y = x$ (ii) $y'' - 2x^2y' + 4xy = x^2 + 2x + 4$

(iii) $y'' + x^2y = 2 + x + x^2$ (iv) $y'' - xy' = e^{-x}$, $y(0) = 2$, $y'(0) = -3$.

**Sol.** (i) The given D.E. is $y'' - y = x$ ...(1)

Clearly $x = 0$ is an ordinary point of (1).

Let $$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1x + a_2x^2 + \ldots\ldots \quad ...(2)$$

be the power series solution of (1) so that

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \quad \text{and} \quad y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}$$

Putting the values of $y$ and $y''$ in (1), we have

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=0}^{\infty} a_m x^m = x$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m - \sum_{m=0}^{\infty} a_m x^m - x = 0$$

$$\Rightarrow \quad \left[2.1a_2 + 3.2a_3x + \sum_{m=2}^{\infty} (m+2)(m+1)a_{m+2}x^m\right] - \left[a_0 + a_1x + \sum_{m=2}^{\infty} a_m x^m\right] - x = 0$$

$$\Rightarrow \quad (2a_2 - a_0) + (6a_3 - a_1 - 1)x + \sum_{m=2}^{\infty} \left[(m+2)(m+1)a_{m+2} - a_m\right]x^m = 0$$

Equating to zero the co-efficients of various powers of $x$, we get

$$2a_2 - a_0 = 0 \quad \Rightarrow \quad a_2 = \frac{a_0}{2} = \frac{a_0}{2!}$$

$$6a_3 - a_1 - 1 = 0 \quad \Rightarrow \quad a_3 = \frac{1+a_1}{6} = \frac{1+a_1}{3!}$$

$$(m+2)(m+1)a_{m+2} - a_m = 0 \quad \text{for all } m \geq 2$$

$$\Rightarrow \quad a_{m+2} = \frac{a_m}{(m+2)(m+1)} \quad ...(3)$$

Putting $m = 2$ in(3) $\quad a_4 = \dfrac{a_2}{4.3} = \dfrac{a_0}{4.3.2} = \dfrac{a_0}{4!}$

Putting $m = 3$ in (3) $\quad a_5 = \dfrac{a_3}{5.4} = \dfrac{1+a_1}{5.4.3!} = \dfrac{1+a_1}{5!}$ and so on.

Putting these values in (2), the required solution is

$$y = a_0 + a_1x + \frac{a_0}{2!}x^2 + \frac{1+a_1}{3!}x^3 + \frac{a_0}{4!}x^4 + \frac{1+a_1}{5!}x^5 + \ldots\ldots$$

or $$y = a_0\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \ldots\ldots\right) + a_1\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\right) + \left(\frac{x^3}{3!} + \frac{x^5}{5!} + \ldots\ldots\right).$$

(*ii*) The given D.E. is $y'' - 2x^2y' + 4xy = x^2 + 2x + 4$ ...(1)

Clearly $x = 0$ is an ordinary point of (1)

Let $$y = \sum_{m=0}^{\infty} a_m x^m = a_0 + a_1x + a_2x^2 + \ldots\ldots \qquad \ldots(2)$$

be the power series solution of (1) so that

$$y' = \sum_{m=1}^{\infty} m a_m x^{m-1} \quad \text{and} \quad y'' = \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2}$$

Putting the values of $y$, $y'$ and $y''$ in (1), we have

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - 2x^2 \sum_{m=1}^{\infty} m a_m x^{m-1} + 4x \sum_{m=0}^{\infty} a_m x^m = x^2 + 2x + 4$$

$$\Rightarrow \quad \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} - \sum_{m=1}^{\infty} 2m\, a_m x^{m+1} + \sum_{m=0}^{\infty} 4a_m x^{m+1} - x^2 - 2x - 4 = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} x^m - \sum_{m=2}^{\infty} 2(m-1)a_{m-1} x^m + \sum_{m=1}^{\infty} 4a_{m-1} x^m - x^2 - 2x - 4 = 0$$

$$\Rightarrow \quad \left[2.1a_2 + 3.2a_3x + 4.3a_4x^2 + \sum_{m=3}^{\infty} (m+2)(m+1)a_{m+2}x^m\right]$$

$$- \left[2.1a_1x^2 + \sum_{m=3}^{\infty} 2(m-1)a_{m-1}x^m\right] + \left[4a_0x + 4a_1x^2 + \sum_{m=3}^{\infty} 4a_{m-1}x^m\right] - x^2 - 2x - 4 = 0$$

$$\Rightarrow \quad (2a_2 - 4) + (6a_3 + 4a_0 - 2)x + (12a_4 - 2a_1 + 4a_1 - 1)x^2$$

$$+ \sum_{m=3}^{\infty} \left[(m+2)(m+1)a_{m+2} - 2(m-1)a_{m-1} + 4a_{m-1}\right]x^m = 0$$

Equating to zero the co-efficients of various powers of $x$, we get

$$2a_2 - 4 = 0 \quad \Rightarrow \quad a_2 = 2$$

$$6a_3 + 4a_0 - 2 = 0 \quad \Rightarrow \quad a_3 = \frac{1 - 2a_0}{3}$$

$$12a_4 + 2a_1 - 1 = 0 \quad \Rightarrow \quad a_4 = \frac{1 - 2a_1}{12}$$

$$(m+2)(m+1)a_{m+2} - \left[2(m-1) - 4\right]a_{m-1} = 0 \quad \text{for all } m \geq 3$$

$$\Rightarrow \quad (m+2)(m+1)a_{m+2} - 2(m-3)a_{m-1} = 0$$

$$\Rightarrow \quad a_{m+2} = \frac{2(m-3)}{(m+2)(m+1)} a_{m-1} \qquad \ldots(3)$$

Putting $m = 3$ in (3) $\quad a_5 = 0$

Puttting $m = 4$ in (3) $\quad a_6 = \dfrac{2.1}{6.5} a_3 = \dfrac{1}{15}\left(\dfrac{1 - 2a_0}{3}\right) = \dfrac{1 - 2a_0}{45}$ and so on.

$$12a_4 - 2a_2 - \frac{1}{2} = 0 \Rightarrow 12a_4 - 1 - \frac{1}{2} = 0 \qquad \left(\because \ a_2 = \frac{1}{2}\right)$$

$$\Rightarrow \qquad a_4 = \frac{1}{8}$$

$$20a_5 - 3a_3 + \frac{1}{6} = 0 \Rightarrow 20a_5 + 2 + \frac{1}{6} = 0 \qquad \left(\because \ a_3 = -\frac{2}{3}\right)$$

$$\Rightarrow \qquad a_5 = -\frac{13}{120} \quad \text{and so on.}$$

Putting these values in (2), the required solution is

$$y = 2 - 3x + \frac{1}{2}x^2 - \frac{2}{3}x^3 + \frac{1}{8}x^4 - \frac{13}{120}x^5 \ ......$$

**Example 14.** *Find the power series solution of the follwing differential equations :*

*(i)* $y'' - xy' + 2y = 0$ *about* $x = 1$

*(ii)* $\dfrac{d^2y}{dx^2} + (x-1)\dfrac{dy}{dx} + y = 0$ *about* $x = 1$

*(iii)* $x\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + 2y = 0$ *in powers of* $(x-1)$.

**Sol.** (*i*) The given D.E. is $\quad y'' - xy' + 2y = 0 \qquad$ ...(1)

Comparing it with $\quad y'' + P(x)y' + Q(x)y = 0$

we have $\quad P(x) = -x \quad$ and $\quad Q(x) = 2$

But $P(x)$ and $Q(x)$ are analytic at $x = 1$

$\Rightarrow$ 1 is an ordinary point of (1).

Let $\qquad y = \sum_{m=0}^{\infty} a_m (x-1)^m = a_0 + a_1(x-1) + a_2(x-1)^2 + \ ......\qquad$ ...(2)

be the power series solution of (1) about $x = 1$ so that

$$y' = \sum_{m=1}^{\infty} m a_m (x-1)^{m-1} \quad \text{and} \quad y'' = \sum_{m=2}^{\infty} m(m-1)a_m (x-1)^{m-2}$$

Putting the values of $y$, $y'$ and $y''$ in (1), we get

$$\sum_{m=2}^{\infty} m(m-1)a_m(x-1)^{m-2} - x\sum_{m=1}^{\infty} m a_m (x-1)^{m-1} + 2\sum_{m=0}^{\infty} a_m (x-1)^m = 0$$

$$\Rightarrow \quad \sum_{m=2}^{\infty} m(m-1)a_m(x-1)^{m-2} - [(x-1)+1]\sum_{m=1}^{\infty} m a_m (x-1)^{m-1} + 2\sum_{m=0}^{\infty} a_m (x-1)^m = 0$$

$$\Rightarrow \quad \sum_{m=2}^{\infty} m(m-1)\, a_m(x-1)^{m-2} - \sum_{m=1}^{\infty} m\, a_m (x-1)^{m} - \sum_{m=1}^{\infty} m\, a_m (x-1)^{m-1}$$

$$+ \sum_{m=0}^{\infty} 2a_m (x-1)^m = 0$$

Let $$y = \sum_{m=0}^{\infty} a_m (x-1)^m = a_0 + a_1(x-1) + a_2(x-1)^2 + \ldots \quad \ldots(2)$$

be the power series solution of (1) about $x = 1$ so that

$$\frac{dy}{dx} = \sum_{m=1}^{\infty} m\, a_m (x-1)^{m-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{m=2}^{\infty} m(m-1) a_m (x-1)^{m-2}$$

[**Note.** Upto this, the solution is same by both methods.]

Now we shift the origin to $x = 1$ by writing $t = x - 1$ so that $x = t + 1$

Also $$\frac{dy}{dx} = \frac{dy}{dt} \cdot \frac{dt}{dx} = \frac{dy}{dt} \cdot 1 = \frac{dy}{dt} \quad \Rightarrow \quad \frac{d}{dx} \equiv \frac{d}{dt}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

$\therefore$ (1) reduces to $$\frac{d^2y}{dt^2} - (t+1)\frac{dy}{dt} + 2y = 0 \quad \ldots(3)$$

and (2) reduces to $$y = \sum_{m=0}^{\infty} a_m t^m = a_0 + a_1 t + a_2 t^2 + \ldots \quad \ldots(4)$$

so that $$\frac{dy}{dt} = \sum_{m=1}^{\infty} m\, a_m t^{m-1} \quad \text{and} \quad \frac{d^2y}{dt^2} = \sum_{m=2}^{\infty} m(m-1)\, a_m t^{m-2}$$

Putting the values of $y$, $\frac{dy}{dt}$ and $\frac{d^2y}{dt^2}$ in (3), we have

$$\sum_{m=2}^{\infty} m(m-1)\, a_m t^{m-2} - (t+1)\sum_{m=1}^{\infty} m\, a_m t^{m-1} + 2\sum_{m=0}^{\infty} a_m t^m = 0$$

$$\Rightarrow \quad \sum_{m=2}^{\infty} m(m-1)\, a_m t^{m-2} - \sum_{m=1}^{\infty} m\, a_m t^m - \sum_{m=1}^{\infty} m\, a_m t^{m-1} + \sum_{m=0}^{\infty} 2a_m t^m = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} (m+2)(m+1)\, a_{m+2} t^m - \sum_{m=1}^{\infty} m\, a_m t^m - \sum_{m=0}^{\infty} (m+1)\, a_{m+1} t^m + \sum_{m=0}^{\infty} 2a_m t^m = 0$$

$$\Rightarrow \quad \left[2.1\, a_2 + 3.2\, a_3 t + \sum_{m=2}^{\infty} (m+2)(m+1) a_{m+2}\, t^m\right] - \left[a_1 t + \sum_{m=2}^{\infty} m a_m t^m\right]$$

$$- \left[a_1 + 2\, a_2 t + \sum_{m=2}^{\infty} (m+1) a_{m+1}\, t^m\right] + \left[2a_0 + 2a_1 t + \sum_{m=2}^{\infty} 2a_m t^m\right] = 0$$

$$\Rightarrow \quad (2a_2 - a_1 + 2a_0) + (6a_3 - a_1 - 2a_2 + 2a_1)t$$

$$+ \sum_{m=2}^{\infty} [(m+2)(m+1)\, a_{m+2} - m a_m - (m+1)\, a_{m+1} + 2a_m]\, t^m = 0$$

Equating to zero the co-efficients of various powers of $t$, we have

$$2a_2 - a_1 + 2a_0 = 0 \quad \Rightarrow \quad a_2 = \frac{a_1 - 2a_0}{2}$$

$$6a_3 - 2a_2 + a_1 = 0 \quad \Rightarrow \quad a_3 = \frac{2a_2 - a_1}{6} = \frac{a_1 - 2a_0 - a_1}{6} = -\frac{a_0}{3}$$

$$(m+2)(m+1)a_{m+2} - (m+1)a_{m+1} - (m-2)a_m = 0 \quad \text{for all } m \geq 2 \qquad \text{...(5)}$$

Putting $m = 2$ in (5) $\quad 4.3a_4 - 3a_3 = 0$

$$\Rightarrow \qquad a_4 = \frac{a_3}{4} = -\frac{a_0}{12} \qquad \text{and so on.}$$

Putting these values in (4), the required solution is

$$y = a_0 + a_1 t + \frac{a_1 - 2a_0}{2} t^2 - \frac{a_0}{3} t^3 - \frac{a_0}{12} t^4 + \text{.......}$$

or $$y = a_0 \left(1 - t^2 - \frac{1}{3} t^3 - \frac{1}{12} t^4 + \text{....}\right) + a_1 \left(t + \frac{1}{2} t^2 + \text{...}\right)$$

or $$y = a_0 \left[1 - (x-1)^2 - \frac{1}{3}(x-1)^3 - \frac{1}{12}(x-1)^4 + \text{...}\right] + a_1 \left[(x-1) + \frac{1}{2}(x-1)^2 + \text{...}\right].$$

$(\because \quad t = x - 1)$

(*ii*) Please try yourself.

**Ans.** $y = a_0 \left[1 - \frac{1}{2}(x-1)^2 + \frac{1}{8}(x-1)^4 + \text{...}\right] + a_1 \left[(x-1) - \frac{1}{6}(x-1)^3 - \frac{1}{12}(x-1)^4 + \text{...}\right].$

(*iii*) The given D.E. is $\quad x \dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} + 2y = 0 \qquad$ ...(1)

or $$\frac{d^2y}{dx^2} + \frac{1}{x} \frac{dy}{dx} + \frac{2}{x} y = 0$$

Comparing it with $\quad \dfrac{d^2y}{dx^2} + P(x) \dfrac{dy}{dx} + Q(x)y = 0$

we have $\quad P(x) = \dfrac{1}{x} \quad$ and $\quad Q(x) = \dfrac{2}{x}$

Both $P(x)$ and $Q(x)$ are analytic at $x = 1$

$\Rightarrow$ 1 is an ordinary point of (1).

Let $$y = \sum_{m=0}^{n} a_m (x-1)^m = a_0 + a_1(x-1) + a_2(x-1)^2 + \text{.....} \qquad \text{...(2)}$$

be the power series solution of (1) about $x = 1$ so that

$$\frac{dy}{dx} = \sum_{m=1}^{\infty} m a_m (x-1)^{m-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{m=2}^{\infty} m(m-1) a_m (x-1)^{m-2}$$

Putting the values of $y$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we have

Putting these values in (2), the required solution is

$$y = a_0 + a_1 (x-1) - \frac{2a_0 + a_1}{2} (x-1)^2 + \frac{2}{3} a_0 (x-1)^3 + \frac{a_1 - 4a_0}{12} (x-1)^4 + \ldots$$

or
$$y = a_0 \left[1 - (x-1)^2 + \frac{2}{3}(x-1)^3 - \frac{1}{3}(x-1)^4 + \ldots\right]$$
$$+ a_0 \left[(x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{12}(x-1)^4 + \ldots\right].$$

**Example 15.** *Find the power series solution of the following differential equations :*

(i) $\dfrac{d^2y}{dx^2} + (x-3)\dfrac{dy}{dx} + y = 0$ *about* $x = 2$

(ii) $y'' + (x-1)y' + y = 0$ *in powers of* $(x-2)$

(iii) $(x^2 - 4x + 5)\dfrac{d^2y}{dx^2} + (x-2)\dfrac{dy}{dx} - (x-2)y = 0$ *in powers of* $(x-2)$

(iv) $(x^2 + 2x)\dfrac{d^2y}{dx^2} + (x+1)\dfrac{dy}{dx} - y = 0$ *about* $x = -1$.

**Sol.** (i) The given D.E. is $\dfrac{d^2y}{dx^2} + (x-3)\dfrac{dy}{dx} + y = 0$ ...(1)

Comparing it with $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} + Q(x)y = 0$

we have $P(x) = x - 3$ and $Q(x) = 1$

Both P(x) and Q(x) are analytic at $x = 2$

$\Rightarrow$ 2 is an ordinary point of (1).

Let $y = \sum_{m=0}^{\infty} a_m (x-2)^m = a_0 + a_1(x-2) + a_2(x-2)^2 + \ldots$ ...(2)

be the power series solution of (1) about $(x-2)$ so that

$$\frac{dy}{dx} = \sum_{m=1}^{\infty} m a_m (x-2)^{m-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{m=2}^{\infty} m(m-1) a_m (x-2)^{m-2}$$

Putting the values of $y$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we get

$$\sum_{m=2}^{\infty} m(m-1) a_m (x-2)^{m-2} + [(x-2) - 1] \sum_{m=1}^{\infty} m a_m (x-2)^{m-1} + \sum_{m=0}^{\infty} a_m (x-2)^m = 0$$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1) a_m (x-2)^{m-2} + \sum_{m=1}^{\infty} m a_m (x-2)^m - \sum_{m=1}^{\infty} m a_m (x-2)^{m-1} + \sum_{m=0}^{\infty} a_m (x-2)^m = 0$$

(*iii*) The given D.E. is $(x^2 - 4x + 5)\dfrac{d^2y}{dx^2} + (x-2)\dfrac{dy}{dx} - (x-2)y = 0$ ...(1)

or
$$\frac{d^2y}{dx^2} + \frac{x-2}{x^2-4x+5}\frac{dy}{dx} - \frac{x-2}{x^2-4x+5}\,y = 0$$

Comparing it with
$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)\,y = 0$$

we have
$$P(x) = \frac{x-2}{x^2-4x+5} \quad \text{and} \quad Q(x) = -\frac{x-2}{x^2-4x+5}$$

Both P($x$) and Q($x$) are analytic at $x = 2$

$\Rightarrow$ 2 is an ordinary point of (1).

Let
$$y = \sum_{m=0}^{\infty} a_m (x-2)^m = a_0 + a_1(x-2) + a_2(x-2)^2 + \ldots. \quad \ldots(2)$$

be the power series solution of (1) in powers of $(x-2)$ so that
$$\frac{dy}{dx} = \sum_{m=1}^{\infty} m\,a_m (x-2)^{m-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{m=2}^{\infty} m(m-1)\,a_m (x-2)^{m-2}$$

Also $\quad x^2 - 4x + 5 = (x^2 - 4x + 4) + 1 = (x-2)^2 + 1$

Putting the values of $y$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we have
$$[(x-2)^2+1]\sum_{m=2}^{\infty} m(m-1)\,a_m (x-2)^{m-2} + (x-2)\sum_{m=1}^{\infty} m\,a_m (x-2)^{m-1}$$
$$- (x-2)\sum_{m=0}^{\infty} a_m (x-2)^m = 0$$

$$\Rightarrow \quad \sum_{m=2}^{\infty} m(m-1)\,a_m (x-2)^m + \sum_{m=2}^{\infty} m(m-1)a_m (x-2)^{m-2}$$
$$+ \sum_{m=1}^{\infty} m\,a_m (x-2)^m - \sum_{m=0}^{\infty} a_m (x-2)^{m+1} = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} m(m-1)a_m (x-2)^m + \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2} (x-2)^m$$
$$+ \sum_{m=1}^{\infty} m\,a_m(x-2)^m - \sum_{m=1}^{\infty} a_{m-1} (x-2)^m = 0$$

$$\Rightarrow \quad \sum_{m=2}^{\infty} m(m-1)a_m (x-2)^m + \left[2.1a_2 + 3.2a_3(x-2) + \sum_{m=2}^{\infty}(m+2)(m+1)a_{m+2}(x-2)^m\right]$$
$$+ \left[a_1(x-2) + \sum_{m=2}^{\infty} ma_m (x-2)^m\right] - \left[a_0(x-2) + \sum_{m=2}^{\infty} a_{m-1}(x-2)^m\right] = 0$$

$\Rightarrow \qquad 3a_3 + 2a_2 + a_1 = 0 \qquad \Rightarrow \quad 3a_3 + 2(-2) + 2 = 0$

$\Rightarrow \qquad a_3 = \dfrac{2}{3}$

Putting $m = 2$ in (4) $\quad 4.3a_4 + 9a_3 + 2a_2 = 0$

$\Rightarrow \qquad 12a_4 + 9\left(\dfrac{2}{3}\right) + 2(-2) = 0 \quad \Rightarrow \quad a_4 = -\dfrac{1}{6}$

Putting $m = 3$ in (4) $\quad 5.4a_5 + 16a_4 + 2a_3 = 0$

$\Rightarrow \qquad 10a_5 + 8\left(-\dfrac{1}{6}\right) + \left(\dfrac{2}{3}\right) = 0 \quad \Rightarrow \quad a_5 = \dfrac{1}{15}$ and so on.

Putting these values in (2), the required solution is

$$y = 1 + 2(x-1) - 2(x-1)^2 + \frac{2}{3}(x-1)^3 - \frac{1}{6}(x-1)^4 + \frac{1}{15}(x-1)^5 + \ldots..$$

(*ii*) The given D.E. is $\qquad (x^2-1)\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + xy = 0 \qquad \ldots(1)$

or
$$\frac{d^2y}{dx^2} + \frac{3x}{x^2-1}\frac{dy}{dx} + \frac{x}{x^2-1}y = 0$$

Comparing it with
$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

we have
$$P(x) = \frac{3x}{x^2-1} \quad \text{and} \quad Q(x) = \frac{x}{x^2-1}$$

Since the initial values of $y$ and $y'$ are given at $x = 2$, we must have power series solution of (1) in powers of $(x-2)$.

Both P($x$) and Q($x$) are analytic at $x = 2$

$\Rightarrow$ 2 is an ordinary point of (1).

Let
$$y = \sum_{m=0}^{\infty} a_m (x-2)^m = a_0 + a_1(x-2) + a_2(x-2)^2 + \ldots. \qquad \ldots(2)$$

be the power series solution of (1) in powers of $(x-2)$ so that

$$\frac{dy}{dx} = \sum_{m=1}^{\infty} m a_m (x-2)^{m-1} = a_1 + 2a_2(x-2) + 3a_3(x-2)^2 + \ldots \qquad \ldots(3)$$

and
$$\frac{d^2y}{dx^2} = \sum_{m=2}^{\infty} m(m-1)a_m(x-2)^{m-2}$$

Also $\qquad x^2 - 1 = (x^2 - 4x + 4) + 4x - 5 = (x-2)^2 + 4(x-2) + 3$

When $\quad x = 2, \quad y = 4$ (given)

$\therefore$ From (2) $\quad \mathbf{a_0 = 4}$

When $\quad x = 2, \quad y' = 6$ (given)

$\therefore$ From (3) $\quad \mathbf{a_1 = 6}$

Putting the values of $y$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$[(x-2)^2+4(x-2)+3]\sum_{m=2}^{\infty} m(m-1)\,a_m\,(x-2)^{m-2}+3[(x-2)+2]\sum_{m=1}^{\infty} ma_m\,(x-2)^{m-1}$$

$$+[(x-2)+2]\sum_{m=0}^{\infty} a_m\,(x-2)^m=0$$

$$\Rightarrow\left[\sum_{m=2}^{\infty} m(m-1)\,a_m(x-2)^m+\sum_{m=2}^{\infty} 4m(m-1)a_m(x-2)^{m-1}+\sum_{m=2}^{\infty} 3m(m-1)a_m(x-2)^{m-2}\right]$$

$$+\left[\sum_{m=1}^{\infty} 3ma_m(m-2)^m+\sum_{m=1}^{\infty} 6ma_m(x-2)^{m-1}\right]$$

$$+\left[\sum_{m=0}^{\infty} a_m(m-2)^{m+1}+\sum_{m=0}^{\infty} 2a_m(x-2)^m\right]=0$$

$$\Rightarrow\quad\sum_{m=2}^{\infty} m(m-1)a_m\,(x-2)^m+\sum_{m=1}^{\infty} 4(m+1)ma_{m+1}\,(x-2)^m$$

$$+\sum_{m=0}^{\infty} 3(m+2)(m+1)a_{m+2}\,(x-2)^m$$

$$+\sum_{m=1}^{\infty} 3ma_m\,(x-2)^m+\sum_{m=0}^{\infty} 6(m+1)a_{m+1}\,(x-2)^m$$

$$+\sum_{m=1}^{\infty} a_{m-1}\,(x-2)^m+\sum_{m=0}^{\infty} 2a_m\,(x-2)^m=0$$

$$\Rightarrow\quad\sum_{m=2}^{\infty} m(m-1)a_m\,(x-2)^m+\left[4.2.1a_2\,(x-2)+\sum_{m=2}^{\infty} 4(m+1)ma_{m+1}\,(x-2)^m\right]$$

$$+\left[3.2.1a_2+3.3.2a_3\,(x-2)+\sum_{m=2}^{\infty} 3(m+2)(m+1)a_{m+2}\,(x-2)^m\right]$$

$$+\left[3a_1\,(x-2)+\sum_{m=2}^{\infty} 3ma_m(x-2)^m\right]+\left[6a_1+6.2a_2(x-2)+\sum_{m=2}^{\infty} 6(m+1)a_{m+1}\,(x-2)^m\right]$$

$$+\left[a_0\,(x-2)+\sum_{m=2}^{\infty} a_{m-1}(x-2)^m\right]+\left[2a_0+2a_1(x-2)+\sum_{m=2}^{\infty} 2a_m(x-2)^m\right]=0$$

$$\Rightarrow \quad (6a_2 + 6a_1 + 2a_0) + (8a_2 + 18a_3 + 3a_1 + 12\, a_2 + a_0 + 2a_1)\,(x - 2)$$

$$+ \sum_{m=2}^{\infty} [m(m-1)a_m + 4(m+1)\, ma_{m+1} + 3(m+2)(m+1)a_{m+2}$$

$$+ 3\, ma_m + 6(m+1)a_{m+1} + a_{m-1} + 2a_m]\,(x-2)^m = 0$$

$$\Rightarrow \quad (6a_2 + 44) + (18a_3 + 20a_2 + 34)(x-2) + \sum_{m=2}^{\infty} [3(m+2)(m+1)a_{m+2}$$

$$+ (m+1)\,(4m+6)a_{m+1} + (m^2 + 2m + 2)a_m + a_{m-1}]\,(x-2)^m = 0 \quad (\because \quad a_0 = 4, a_1 = 6)$$

Equating to zero the co-efficients of various powers of $(x - 2)$, we have

$$6a_2 + 44 = 0 \quad \Rightarrow \quad a_2 = -\frac{22}{3}$$

$$18a_3 + 20a_2 + 34 = 0 \quad \Rightarrow \quad 9a_3 + 10\left(-\frac{22}{3}\right) + 17 = 0$$

$$\Rightarrow \quad 9a_3 = \frac{169}{3} \quad \Rightarrow \quad a_3 = \frac{169}{27}$$

$$3(m+2)(m+1)a_{m+2} + (m+1)(4m+6)a_{m+1} + (m^2 + 2m + 2)\, a_m + a_{m-1} = 0$$

for all $m \geq 2$

Putting $m = 2$, $\quad 3.4.3a_4 + 3.14a_3 + 10a_2 + a_1 = 0$

$$\Rightarrow \quad 36a_4 + 42\left(\frac{169}{27}\right) + 10\left(-\frac{22}{3}\right) + 6 = 0$$

$$\Rightarrow \quad 36a_4 = -\frac{1760}{9} \quad \Rightarrow \quad a_4 = -\frac{440}{81} \quad \text{and so on.}$$

Putting these values in (2), the required solution is

$$y = 4 + 6\,(x-2) - \frac{22}{3}\,(x-2)^2 + \frac{169}{27}\,(x-2)^3 - \frac{440}{81}\,(x-2)^4 + \ldots\ldots$$

**Note.** Try this question by shifting the origin to $x = 2$ by writing $t = x - 2$ so that $x = t + 2$.

$$\frac{dy}{dx} = \frac{dy}{dt}\cdot\frac{dt}{dx} = \frac{dy}{dt}\,.\,1 = \frac{dy}{dt} \quad \Rightarrow \quad \frac{d}{dx} \equiv \frac{d}{dt}$$

and

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dt}\right) = \frac{d^2y}{dt^2}$$

$$x^2 - 1 = (t+2)^2 - 1 = t^2 + 4t + 3$$

When $x = 2$, $y = 4$ $\quad \Rightarrow$ when $t = 0$, $y = 4$

When $x = 2$, $\dfrac{dy}{dx} = 6$ $\quad \Rightarrow$ when $t = 0$, $\dfrac{dy}{dt} = 6$.

## 9. Frobenius Method

**(Series solution about regular singular point x = 0)**

If $x = x_0$ is a regular singular point of the differential equation

$$P_0(x)\,\frac{d^2y}{dx^2} + P_1(x)\,\frac{dy}{dx} + P_2(x)\,y = 0 \qquad \ldots(1)$$

then **Frobenius Method** is used to find the series solution of the differential equation in powers of $(x - x_0)$.

If $x_0 \neq 0$, then we first shift the origin to the point $x = x_0$ so that '0' is a regular singular point of the transformed equation.

Therefore, it is sufficient to discuss this method for the regular singular point '0'.

(If 0 is an irregular singular point of the differential equation then the discussion of solution of the differential equation is beyond the scope of this book.)

**Working Rule**

I. Verify that $x = 0$ is a regular singular point of the given D.E. (1).

II. Let $y = x^k \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+k}$, $\quad a_0 \neq 0 \quad$ ...(2)

be the series solution of the D.E. (1) so that

$$\frac{dy}{dx} = \sum_{m=0}^{\infty} (m+k)a_m x^{m+k-1} \text{ and } \frac{d^2y}{dx^2} = \sum_{m=0}^{\infty} (m+k)(m+k-1)a_m x^{m+k-2}.$$

III. Put the values of $y$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1).

IV. Equate to zero the co-efficient of smallest power of $x$. This gives a quadratic equation in $k$ which is called the **indicial equation** of (1).

V. Solve the indicial equation. Let its roots be $k_1, k_2$.

VI. Equate to zero the co-efficients of next higher powers of $x$ and find a relation between $a_m, a_{m-2}$ or $a_m, a_{m-1}$ in terms of $k$.

This relation is called the **recurrence relation**.

Further solution depends on the nature of roots of the indicial equation. Following cases arise :

(*i*) $k_1 \neq k_2$ and $k_1 - k_2$ is not an integer.

(*ii*) $k_1 \neq k_2$ and $k_1 - k_2$ is an integer making a co-efficient of $y$ indeterminate.

(*iii*) $k_1 \neq k_2$ and $k_1 - k_2$ is an integer making a co-efficient of $y$ infinite.

(*iv*) $k_1 = k_2$

The necessary working in the four different cases together with certain modifications, wherever necessary, is discussed one by one in the following examples.

**Case I. Roots of indicial equation are unequal and do not differ by an integer.**

Let the roots of the indicial equation be $k_1$ and $k_2$ where $k_1 \neq k_2$ and $k_1 - k_2$ is not an integer.

By putting $k = k_1$, find $a_1, a_2, a_3, \ldots\ldots$ in terms of $a_0$. Put these values in (2) to get

$$y = a_0 x^{k_1} \sum_{m=0}^{\infty} \lambda_m x^m.$$

Therefore one solution of the D.E. is $u = x^{k_1} \sum_{m=0}^{\infty} \lambda_m x^m$ ...(3)

By putting $k = k_2$, find $a_1, a_2, a_3, \ldots\ldots$ in terms of $a_0$. Put these values in (2) to get

$$y = a_0 x^{k_2} \sum_{m=0}^{\infty} \mu_m x^m$$

Therefore another independent solution of the D.E. is

$$v = x^{k_2} \sum_{m=0}^{\infty} \mu_m x^m \qquad \ldots(4)$$

Thus we obtain two **independent solutions** $u$ and $v$. The general solution of the D.E. (1) is

$$y = au + bv$$

where $a$ and $b$ are arbitrary constant.

**Example 1.** *Solve the following differential equations in series :*

(i) $4x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0$ (ii) $9x(1-x)\dfrac{d^2y}{dx^2} - 12\dfrac{dy}{dx} + 4y = 0$

(iii) $2x^2\dfrac{d^2y}{dx^2} - x\dfrac{dy}{dx} + (1-x^2)y = 0.$

**Sol.** (*i*) The given D.E. is $4x\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + y = 0 \qquad \ldots(1)$

or
$$\frac{d^2y}{dx^2} + \frac{1}{2x}\frac{dy}{dx} + \frac{1}{4x}y = 0$$

Comparing it with $\dfrac{d^2y}{dx^2} + P(x)\dfrac{dy}{dx} + Q(x)y = 0$

we have
$$P(x) = \frac{1}{2x} \quad \text{and} \quad Q(x) = \frac{1}{4x}$$

Neither $P(x)$ nor $Q(x)$ is analytic at $x = 0$

$\Rightarrow$ 0 is not an ordinary point of (1).

$$xP(x) = \frac{1}{2} \quad \text{and} \quad x^2Q(x) = \frac{x}{4}$$

Both $xP(x)$ and $x^2Q(x)$ are analytic at $x = 0$

$\Rightarrow$ 0 is a regular singular point of (1).

Let
$$y = x^k \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+k}, \quad a_0 \neq 0 \qquad \ldots(2)$$

be a series solution of (1) so that

$$\frac{dy}{dx} = \sum_{m=0}^{\infty} (m+k)a_m x^{m+k-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{m=0}^{\infty} (m+k)(m+k-1)a_m x^{m+k-2}$$

Putting the values of $y$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we get

$$4x\sum_{m=0}^{\infty} (m+k)(m+k-1)a_m x^{m+k-2} + 2\sum_{m=0}^{\infty} (m+k)a_m x^{m+k-1} + \sum_{m=0}^{\infty} a_m x^{m+k} = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} 4(m+k)(m+k-1)a_m x^{m+k-1} + \sum_{m=0}^{\infty} 2(m+k)a_m x^{m+k-1} + \sum_{m=0}^{\infty} a_m x^{m+k} = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} 2(m+k)[2(m+k-1)+1]\, a_m x^{m+k-1} + \sum_{m=0}^{\infty} a_m x^{m+k} = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} (2m+2k)(2m+2k-1)a_m x^{m+k-1} + \sum_{m=0}^{\infty} a_m x^{m+k} = 0$$

$$\Rightarrow \quad \sum_{m=-1}^{\infty} [2(m+1)+2k][2(m+1)+2k-1]\, a_{m+1} x^{m+k} + \sum_{m=0}^{\infty} a_m x^{m+k} = 0$$

$$\Rightarrow \quad [2k(2k-1)\, a_0 x^{k-1} + \sum_{m=0}^{\infty} (2m+2k+2)(2m+2k+1)\, a_{m+1} x^{m+k}] + \sum_{m=0}^{\infty} a_m x^{m+k} = 0$$

$$\Rightarrow \quad 2k(2k-1)a_0 x^{k-1} + \sum_{m=0}^{\infty} \left[(2m+2k+2)(2m+2k+1)\, a_{m+1} + a_m\right] x^{m+k} = 0$$

Equating to zero the co-efficient of lowest power of $x$, *i.e.*, $x^{k-1}$, we have

$$2k(2k-1)a_0 = 0 \quad \Rightarrow \quad k(2k-1) = 0 \qquad (\because\ a_0 \neq 0)$$

This is the indicial equation of the given D.E.

Solving $k = 0, \dfrac{1}{2}$ (distinct, not differing by an integer)

Equating to zero the co-efficient of $x^{m+k}$, $m \geq 0$, we have

$$(2m+2k+2)(2m+2k+1)a_{m+1} + a_m = 0$$

$$\Rightarrow \quad a_{m+1} = \frac{-a_m}{(2m+2k+2)(2m+2k+1)} \quad \text{for all } m \geq 0 \qquad \text{...(3)}$$

**When k = 0**, from (3)

$$a_{m+1} = \frac{-a_m}{(2m+2)(2m+1)} \quad \text{for all } m \geq 0$$

Putting $m = 0, 1, 2, \ldots\ldots$

$$a_1 = \frac{-a_0}{2.1} = -\frac{a_0}{2!}$$

$$a_2 = \frac{-a_1}{4.3} = \frac{a_0}{4.3.2!} = \frac{a_0}{4!}$$

$$a_3 = \frac{-a_2}{6.5} = -\frac{a_0}{6.5.4!} = -\frac{a_0}{6!} \quad \text{and so on.}$$

Putting these values in (2), *i.e.*, in

$$y = x^k\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\right), \text{ we have}$$

$$y = x^0\left(a_0 - \frac{a_0}{2!}x + \frac{a_0}{4!}x^2 - \frac{a_0}{6!}x^3 + \ldots\ldots\right)$$

or

$$y = a_0\left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \ldots\ldots\right)$$

$\Rightarrow$ One solution of the given D.E. is

$$u = 1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \ldots\ldots \qquad \text{(taking } a_0 = 1\text{)}$$

**When k = $\frac{1}{2}$**, from (3)

$$a_{m+1} = \frac{-a_m}{(2m+3)(2m+2)} \quad \text{for all } m \geq 0$$

Putting $m = 0, 1, 2, \ldots\ldots$

$$a_1 = \frac{-a_0}{3.2} = -\frac{a_0}{3!}$$

$$a_2 = \frac{-a_1}{5.4} = \frac{a_0}{5.4.3!} = \frac{a_0}{5!}$$

$$a_3 = \frac{-a_2}{7.6} = -\frac{a_0}{7.6.5!} = -\frac{a_0}{7!} \quad \text{and so on.}$$

Putting these values in (2), *i.e.*, in

$$y = x^k\left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots\ldots\right), \text{ we have}$$

$$y = x^{1/2}\left(a_0 - \frac{a_0}{3!}x + \frac{a_0}{5!}x^2 - \frac{a_0}{7!}x^3 + \ldots\ldots\right)$$

or
$$y = a_0\sqrt{x}\left(1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \ldots..\right)$$

$\Rightarrow$ Another independent solution of the given D.E. is

$$v = \sqrt{x}\left(1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \ldots..\right) \qquad \text{(taking } a_0 = 1\text{)}$$

Hence the general solution of the D.E. is

$$y = au + bv$$

or
$$y = a\left(1 - \frac{x}{2!} + \frac{x^2}{4!} - \frac{x^3}{6!} + \ldots..\right) + b\sqrt{x}\left(1 - \frac{x}{3!} + \frac{x^2}{5!} - \frac{x^3}{7!} + \ldots..\right).$$

(*ii*) The given D.E. is $9x(1-x)\frac{d^2y}{dx^2} - 12\frac{dy}{dx} + 4y = 0$ ...(1)

or
$$\frac{d^2y}{dx^2} - \frac{4}{3x(1-x)}\frac{dy}{dx} + \frac{4}{9x(1-x)}y = 0$$

Comparing it with $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$

we have $P(x) = \frac{-4}{3x(1-x)}$ and $Q(x) = \frac{4}{9x(1-x)}$

Neither $P(x)$ nor $Q(x)$ is analytic at $x = 0$

$\Rightarrow$ 0 is not an ordinary point of (1).

$$xP(x) = \frac{-4}{3(1-x)} \quad \text{and} \quad x^2Q(x) = \frac{4x}{9(1-x)}$$

Both $x$ P($x$) and $x^2$ Q($x$) are analytic at $x = 0$

$\Rightarrow$ 0 is a regular singular point of (1).

Let $$y = x^k \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+k}, \; a_0 \neq 0 \qquad \text{...(2)}$$

be a series solution of (1) so that

$$\frac{dy}{dx} = \sum_{m=0}^{\infty} (m+k)a_m x^{m+k-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{m=0}^{\infty} (m+k)(m+k-1)a_m x^{m+k-2}$$

Putting the values of $y$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$(9x-9x^2)\sum_{m=0}^{\infty} (m+k)(m+k-1)a_m x^{m+k-2} - 12\sum_{m=0}^{\infty} (m+k)a_m x^{m+k-1} + 4\sum_{m=0}^{\infty} a_m x^{m+k} = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} 9(m+k)(m+k-1)a_m x^{m+k-1} - \sum_{m=0}^{\infty} 9(m+k)(m+k-1)a_m x^{m+k}$$

$$- \sum_{m=0}^{\infty} 12(m+k)a_m x^{m+k-1} + \sum_{m=0}^{\infty} 4a_m x^{m+k} = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} 3(m+k)\,[3(m+k-1)-4]a_m x^{m+k-1}$$

$$- \sum_{m=0}^{\infty} [9(m+k)(m+k-1)-4]a_m x^{m+k} = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} (3m+3k)(3m+3k-7)a_m x^{m+k-1} - \sum_{m=0}^{\infty} [9(m+k)(m+k-1)-4]a_m x^{m+k} = 0$$

$$\Rightarrow \quad \sum_{m=-1}^{\infty} [3(m+1)+3k][3(m+1)+3k-7]a_{m+1} x^{m+k}$$

$$- \sum_{m=0}^{\infty} [9(m+k)(m+k-1)-4]a_m x^{m+k} = 0$$

$$\Rightarrow \quad \sum_{m=1}^{\infty} (3m+3k+3)(3m+3k-4)\, a_{m+1} x^{m+k}$$

$$- \sum_{m=0}^{\infty} [9\,(m+k)\,(m+k-1)-4]\, a_m x^{m+k} = 0$$

$$\Rightarrow \quad \left[(k-1)(2k-1)a_0x^k + k(2k+1)a_1x^{k+1} + \sum_{m=2}^{\infty}(m+k-1)(2m+2k-1)a_mx^{m+k}\right] - \sum_{m=2}^{\infty} a_{m-2}x^{m+k} = 0$$

$$\Rightarrow [(k-1)(2k-1)a_0x^k + k(2k+1)a_1x^{k+1} + \sum_{m=2}^{\infty}[(m+k-1)(2m+2k-1)a_m - a_{m-2}]\,x^{m+k} = 0$$

Equating to zero the co-efficient of lowest power of $x$, *i.e.*, $x^k$, we have

$$(k-1)(2k-1)a_0 = 0 \quad \Rightarrow \quad (k-1)(2k-1) = 0 \qquad (\because\ a_0 \neq 0)$$

This is the indicial equation of the given D.E.

Solving $k = 1, \frac{1}{2}$ (distinct, not differing by an integer)

Equating to zero the co-efficient of $x^{k+1}$, we have

$$k(2k+1)a_1 = 0 \quad \Rightarrow \quad \mathbf{a_1 = 0} \qquad \left(\because k \neq 0 \text{ or } -\frac{1}{2}\right)$$

Equating to zero the co-efficient of $x^{m+k}$, $m \geq 2$, we have

$$(m+k-1)(2m+2k-1)a_m - a_{m-2} = 0$$

$$\Rightarrow \qquad a_m = \frac{a_{m-2}}{(m+k-1)(2m+2k-1)} \quad \text{for all } m \geq 2 \qquad \text{...(3)}$$

**When k = 1,** from (3) $a_m = \dfrac{a_{m-2}}{m(2m+1)}$

Putting $m = 2, 3, 4, ......$

$$a_2 = \frac{a_0}{2.5}$$

$$a_3 = \frac{a_1}{3.7} = 0 \qquad (\because\ a_1 = 0)$$

$$a_4 = \frac{a_2}{4.9} = \frac{a_0}{(2.4)(5.9)}$$

$$a_5 = \frac{a_3}{5.11} = 0 \qquad (\because\ a_3 = 0)$$

$$a_6 = \frac{a_4}{6.13} = \frac{a_0}{(2.4.6)(5.9.13)} \text{ and so on.}$$

Putting these values in (2), *i.e.*, in

$$y = x^k\left(a_0 + a_1x + a_2x^2 + ......\right), \text{ we have}$$

$$y = x\left[a_0 + \frac{a_0}{2.5}x^2 + \frac{a_0}{(2.4)(5.9)}x^4 + \frac{a_0}{(2.4.6)(5.9.13)}x^6 + ......\right]$$

or

$$y = a_0x\left[1 + \frac{x^2}{2.5} + \frac{x^4}{(2.4)(5.9)} + \frac{x^6}{(2.4.6)(5.9.13)} + ......\right]$$

**Sol.** (*i*) The given D.E. is

$$2x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} + (x-5)y = 0 \qquad ...(1)$$

or

$$\frac{d^2y}{dx^2} - \frac{1}{2x}\frac{dy}{dx} + \frac{x-5}{2x^2}y = 0$$

Comparing it with $\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$

we have

$$P(x) = -\frac{1}{2x} \quad \text{and} \quad Q(x) = \frac{x-5}{2x^2}$$

Neither $P(x)$ nor $Q(x)$ is analytic at $x = 0$

$\Rightarrow$ 0 is not an ordinary point of (1).

$$xP(x) = -\frac{1}{2} \quad \text{and} \quad x^2Q(x) = \frac{1}{2}(x-5)$$

Both $xP(x)$ and $x^2\,Q(x)$ are analytic at $x = 0$

$\Rightarrow$ 0 is a regular singular point of (1).

Let

$$y = x^k\sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+k}, \quad a_0 \neq 0 \qquad ...(2)$$

be a series solution of (1) so that

$$\frac{dy}{dx} = \sum_{m=0}^{\infty} (m+k)a_m x^{m+k-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{m=0}^{\infty} (m+k)(m+k-1)a_m x^{m+k-2}$$

Putting the values of $y$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$2x^2\sum_{m=0}^{\infty} (m+k)(m+k-1)a_m x^{m+k-2} - x\sum_{m=0}^{\infty} (m+k)\,a_m x^{m+k-1} + (x-5)\sum_{m=0}^{\infty} a_m x^{m+k} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} 2(m+k)(m+k-1)\,a_m x^{m+k} - \sum_{m=0}^{\infty} (m+k)\,a_m x^{m+k} + \sum_{m=0}^{\infty} a_m x^{m+k+1} - \sum_{m=0}^{\infty} 5a_m x^{m+k} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} [2\,(m+k)\,(m+k-1) - (m+k) - 5]a_m x^{m+k} + \sum_{m=0}^{\infty} a_{m-1}x^{m+k} = 0$$

$$\Rightarrow [2k(k-1) - k - 5]\,a_0 x^k + \sum_{m=1}^{\infty} [2(m+k)(m+k-1) - (m+k) - 5]\,a_m x^{m+k} + \sum_{m=1}^{\infty} a_{m-1}x^{m+k} = 0$$

$$\Rightarrow \quad (2k^2 - 3k - 5)\, a_0 x^k + \sum_{m=1}^{\infty} [\{2(m+k)(m+k-1) - (m+k) - 5\}a_m + a_{m-1}]x^{m+k} = 0$$

Equating to zero the co-efficient of lowest power of $x$, *i.e.*, $x^k$,

we get $\qquad (2k^2 - 3k - 5)a_0 = 0 \;\Rightarrow\; 2k^2 - 3k - 5 = 0 \qquad (\because\ a_0 \neq 0)$

This is the indicial equation of the given D.E.

Solving $\qquad k = \dfrac{3 \pm \sqrt{9+40}}{4} = \dfrac{3 \pm 7}{4} = -1, \dfrac{5}{2}$ (distinct, not differing by an integer.)

Equating to zero the co-efficient of $x^{m+k}$, $m \geq 1$, we get

$$[2(m+k)(m+k-1) - (m+k) - 5]a_m + a_{m-1} = 0$$

$$\Rightarrow \qquad a_m = \frac{-a_{m-1}}{2(m+k)(m+k-1) - (m+k) - 5} \quad \text{for all } m \geq 1. \qquad ...(3)$$

**When k = 1,** from (3) $\quad a_m = \dfrac{-a_{m-1}}{2(m-1)(m-2) - (m-1) - 5}$

$$= \frac{-a_{m-1}}{2m^2 - 7m} = \frac{-a_{m-1}}{m(2m-7)}$$

Putting $m = 1, 2, 3, \ldots\ldots$

$$a_1 = \frac{-a_0}{-5} = \frac{a_0}{5}$$

$$a_2 = \frac{-a_1}{-6} = \frac{a_0}{30}$$

$$a_3 = \frac{-a_2}{-3} = \frac{a_0}{90} \quad \text{and so on.}$$

Putting these values in (2), *i.e.*, in

$$y = x^k(a_0 + a_1 x + a_2 x^2 + \ldots\ldots), \text{ we have}$$

$$y = x^{-1}\left(a_0 + \frac{a_0}{5}x + \frac{a_0}{30}x^2 + \frac{a_0}{90}x^3 + \ldots\ldots\right)$$

or

$$y = a_0 x^{-1}\left(1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} + \ldots\ldots\right)$$

$\Rightarrow$ One solution of the given D.E. is

$$u = x^{-1}\left(1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} + \ldots\ldots\right) \qquad (\text{taking } a_0 = 1)$$

**When k = $\frac{5}{2}$**, from (3) $\quad a_m = \dfrac{-a_{m-1}}{2\left(m+\frac{5}{2}\right)\left(m+\frac{3}{2}\right) - \left(m+\frac{5}{2}\right) - 5}$

$$= \frac{-a_{m-1}}{2\left(m^2 + 4m + \frac{15}{4}\right) - \left(m + \frac{5}{2}\right) - 5}$$

$$= \frac{-a_{m-1}}{2m^2 + 7m} = \frac{-a_{m-1}}{m(2m+7)}$$

Putting $m = 1, 2, 3, \ldots\ldots\ldots$

$$a_1 = \frac{-a_0}{9}, a_2 = \frac{-a_1}{22} = \frac{a_0}{198}$$

$$a_3 = \frac{-a_2}{39} = \frac{-a_0}{7722} \quad \text{and so on.}$$

Putting these values in (2), *i.e.*, in

$$y = x^k (a_0 + a_1 x + a_2 x^2 + \ldots..), \text{ we have}$$

$$y = x^{5/2}\left(a_0 - \frac{a_0}{9}x + \frac{a_0}{198}x^2 - \frac{a_0}{7722}x^3 + \ldots\ldots.\right)$$

or $$y = a_0 x^{5/2}\left(1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \ldots..\right)$$

$\Rightarrow$ Another independent solution of the given D.E. is

$$v = x^{5/2}\left(1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \ldots..\right) \qquad \text{(taking } a_0 = 1\text{)}$$

Hence the general solution of the D.E. is

$$y = au + bv$$

or $$y = ax^{-1}\left(1 + \frac{x}{5} + \frac{x^2}{30} + \frac{x^3}{90} + \ldots..\right) + bx^{5/2}\left(1 - \frac{x}{9} + \frac{x^2}{198} - \frac{x^3}{7722} + \ldots..\right).$$

(*ii*) Please try yourself.

**Ans.** $y = ax\left(1 + \dfrac{x}{5} + \dfrac{x^2}{70} + \ldots..\right) + bx^{-1/2}\left(1 - x - \dfrac{x^2}{2} + \ldots..\right).$

(*iii*) The given D.E. is $$2x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - 1)y = 0 \qquad \ldots(1)$$

or $$\frac{d^2y}{dx^2} + \frac{1}{2x}\frac{dy}{dx} + \frac{x^2 - 1}{2x^2}y = 0$$

Comparing it with $$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

we have $$P(x) = \frac{1}{2x} \quad \text{and} \quad Q(x) = \frac{x^2 - 1}{2x^2}$$

Neither $P(x)$ nor $Q(x)$ is analytic at $x = 0$

$\Rightarrow$ 0 is not an ordinary point of (1).

$$x\,P(x) = \frac{1}{2} \quad \text{and} \quad x^2 Q(x) = \frac{1}{2}(x^2 - 1)$$

Both $x\,P(x)$ and $x^2 Q(x)$ are analytic at $x = 0$

$\Rightarrow$ 0 is a regular singular point of (1).

Let $$y = x^k \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+k}, \quad a_0 \neq 0 \qquad \ldots(2)$$

be a series solution of (1) so that

Putting $m = 2, 3, 4, \ldots\ldots$

$$a_2 = \frac{-a_0}{14}$$

$$a_3 = \frac{-a_1}{27} = 0 \qquad (\because \quad a_1 = 0)$$

$$a_4 = \frac{-a_2}{44} = \frac{a_0}{616} \qquad \text{and so on.}$$

Putting these values in (2)

$$y = x\left(a_0 - \frac{a_0}{14}x^2 + \frac{a_0}{616}x^4 - \ldots\ldots\right)$$

or
$$y = a_0 x\left(1 - \frac{x^2}{14} + \frac{x^4}{616} - \ldots\ldots\ldots\right)$$

$\Rightarrow$ One solution of the given D.E. is

$$u = x\left(1 - \frac{x^2}{14} + \frac{x^4}{616} - \ldots\ldots\ldots\right) \qquad \text{(taking } a_0 = 1\text{)}$$

**When k = $-\frac{1}{2}$,** from (3)

$$a_m = \frac{-a_{m-2}}{2\left(m - \frac{1}{2}\right)\left(m - \frac{3}{2}\right) + \left(m - \frac{1}{2}\right) - 1}$$

$$= \frac{-a_{m-2}}{2\left(m^2 - 2m + \frac{3}{4}\right) + m - \frac{3}{2}} = \frac{-a_{m-2}}{2m^2 - 3m} = \frac{-a_{m-2}}{m(2m-3)}$$

Putting $m = 2, 3, 4, \ldots\ldots\ldots$

$$a_2 = \frac{-a_0}{2}$$

$$a_3 = \frac{-a_1}{9} = 0 \qquad (\because \quad a_1 = 0)$$

$$a_4 = \frac{-a_2}{20} = \frac{a_0}{40} \qquad \text{and so on.}$$

Putting these values in (2)

$$y = x^{-1/2}\left(a_0 - \frac{a_0}{2}x^2 + \frac{a_0}{40}x^4 - \ldots\ldots\right)$$

or
$$y = a_0 x^{-1/2}\left(1 - \frac{x^2}{2} + \frac{x^4}{40} - \ldots\ldots\right)$$

$\Rightarrow$ Another independent solution of the given D.E. is

$$v = x^{-1/2}\left(1 - \frac{x^2}{2} + \frac{x^4}{40} - \ldots\ldots\right) \qquad \text{(taking } a_0 = 1\text{)}$$

Hence the general solution of the D.E. is

$$y = au + bv$$

or
$$y = ax\left(1-\frac{x^2}{14}+\frac{x^4}{616}-\ldots\ldots\right)+bx^{-1/2}\left(1-\frac{x^2}{2}+\frac{x^4}{40}-\ldots\ldots\right).$$

(*iv*) Please try yourself.

**Ans.**
$$y = ax\left(1-\frac{x^2}{10}+\frac{x^4}{360}-\ldots\ldots\right)+b\sqrt{x}\left(1-\frac{x^2}{6}+\frac{x^4}{168}-\ldots\ldots\right)$$

(*v*) The given D.E. is $2x^2\,\frac{d^2y}{dx^2}+x\,\frac{dy}{dx}+(x^2-3)y=0$ ...(1)

or
$$\frac{d^2y}{dx^2}+\frac{1}{2x}\frac{dy}{dx}+\frac{x^2-3}{2x^2}\,y=0$$

Comparing it with
$$\frac{d^2y}{dx^2}+P(x)\,\frac{dy}{dx}+Q(x)y=0$$

we have
$$P(x)=\frac{1}{2x} \quad \text{and} \quad Q(x)=\frac{x^2-3}{2x^2}$$

Neither $P(x)$ nor $Q(x)$ is analytic at $x = 0$

$\Rightarrow$ 0 is not an ordinary point of (1).

$$x\,P(x)=\frac{1}{2} \quad \text{and} \quad x^2\,Q(x)=\frac{1}{2}\,(x^2-3)$$

Both $xP(x)$ and $x^2Q(x)$ are analytic at $x = 0$

$\Rightarrow$ 0 is a regular singular point of (1).

Let
$$y = x^k\sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+k}, \quad a_0 \neq 0 \qquad \text{...(2)}$$

be a series solution of (1) so that

$$\frac{dy}{dx}=\sum_{m=0}^{\infty}\,(m+k)\,a_m\,x^{m+k-1}$$

and
$$\frac{d^2y}{dx^2}=\sum_{m=0}^{\infty}\,(m+k)(m+k-1)\,a_m\,x^{m+k-2}$$

Putting the values of $y$, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$2x^2\sum_{m=0}^{\infty}\,(m+k)(m+k-1)\,a_m\,x^{m+k-2}+x\sum_{m=0}^{\infty}\,(m+k)\,a_m\,x^{m+k-1}$$
$$+(x^2-3)\sum_{m=0}^{\infty}a_m\,x^{m+k}=0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} 2(m+k)(m+k-1)\, a_m\, x^{m+k} + \sum_{m=0}^{\infty} (m+k)\, a_m\, x^{m+k} + \sum_{m=0}^{\infty} a_m\, x^{m+k+2} - \sum_{m=0}^{\infty} 3a_m\, x^{m+k} = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} [2(m+k)(m+k-1)+(m+k)-3]\, a_m\, x^{m+k} + \sum_{m=2}^{\infty} a_{m-2}\, x^{m+k} = 0$$

$$\Rightarrow \quad [2k\,(k-1)+k-3]a_0\, x^k + [2(k+1)(k)+(k+1)-3]\, a_1\, x^{k+1} + \sum_{m=2}^{\infty} [2(m+k)(m+k-1)+(m+k)-3]\, a_m\, x^{m+k} + \sum_{m=2}^{\infty} a_{m-2}\, x^{m+k} = 0$$

$$\Rightarrow \quad (2k^2-k-3)a_0\, x^k + (2k^2+3k-2)a_1\, x^{k+1} + \sum_{m=2}^{\infty} [\{2(m+k)(m+k-1)+(m+k)-3\}\, a_m + a_{m-2}]\, x^{m+k} = 0$$

Equating to zero the co-efficient of lowest power of $x$, *i.e.*, $x^k$, we get

$$(2k^2-k-3)\, a_0 = 0 \quad \Rightarrow \quad 2k^2-k-3=0 \qquad (\because \quad a_0 \neq 0)$$

This is the indicial equation of the given D.E.

Solving $k = -1, 3/2$ (distinct, not differing by an integer)

Equating to zero the co-efficient of $x^{k+1}$

$$(2k^2+3k-2)a_1 = 0 \quad \Rightarrow \quad (k+2)(2k-1)a_1 = 0$$

$$\Rightarrow \qquad \mathbf{a_1 = 0} \qquad (\because \quad k \neq -2 \text{ or } 1/2)$$

Equating to zero the co-efficient of $x^{m+k}$, $m \geq 2$, we get

$[2(m+k)(m+k-1)+(m+k)-3]\, a_m + a_{m-2} = 0$

$$\Rightarrow \quad a_m = \frac{-a_{m-2}}{2(m+k)(m+k-1)+(m+k)-3} \quad \text{for all } m \geq 2 \qquad \text{...(3)}$$

**When k = – 1,** from (3) $a_m = \dfrac{-a_{m-2}}{2(m-1)(m-2)+(m-1)-3}$

$$= \frac{-a_{m-2}}{2m^2-5m} = \frac{-a_{m-2}}{m(2m-5)}$$

Putting $m = 2, 3, 4, \ldots\ldots$

$$a_2 = \frac{-a_0}{-2} = \frac{a_0}{2}$$

$$a_3 = \frac{-a_1}{3} = 0 \qquad (\because \quad a_1 = 0)$$

$$a_4 = \frac{-a_2}{12} = -\frac{a_0}{24} \quad \text{and so on.}$$

Putting these values in (2)

$$y = x^{-1}\left(a_0 + \frac{a_0}{2}x^2 - \frac{a_0}{24}x^4 + \ldots\ldots\right) \quad \text{or} \quad y = a_0\, x^{-1}\left(1 + \frac{x^2}{2} - \frac{x^4}{24} + \ldots\ldots\right)$$

$\Rightarrow$ One solution of the given D.E. is

$$u = x^{-1}\left(1+\frac{x^2}{2}-\frac{x^4}{24}+.....\right) \qquad \text{(taking } a_0 = 1\text{)}$$

**When k = $\frac{3}{2}$**, from (3)

$$a_m = \frac{-a_{m-2}}{2\left(m+\frac{3}{2}\right)\left(m+\frac{1}{2}\right)+\left(m+\frac{3}{2}\right)-3}$$

$$= \frac{-a_{m-2}}{2\left(m^2+2m+\frac{3}{4}\right)+m-\frac{3}{2}} = \frac{-a_{m-2}}{2m^2+5m} = \frac{-a_{m-2}}{m(2m+5)}$$

Putting $m = 2, 3, 4, .......$

$$a_2 = \frac{-a_0}{18}$$

$$a_3 = \frac{-a_1}{33} = 0 \qquad (\because \quad a_1 = 0)$$

$$a_4 = \frac{-a_2}{52} = \frac{a_0}{936} \quad \text{and so on.}$$

Putting these values in (2)

$$y = x^{3/2}\left(a_0 - \frac{a_0}{18}x^2 + \frac{a_0}{936}x^4 + .....\right)$$

or
$$y = a_0\, x^{3/2}\left(1-\frac{x^2}{18}+\frac{x^4}{936}+.....\right)$$

$\Rightarrow$ Another independent solution of the given D.E. is

$$v = x^{3/2}\left(1-\frac{x^2}{18}+\frac{x^4}{936}+.....\right). \qquad \text{(taking } a_0 = 1\text{)}$$

Hence the general solution of the D.E. is

$$y = au + bv$$

or
$$y = ax^{-1}\left(1+\frac{x^2}{2}-\frac{x^4}{24}+.....\right)+bx^{3/2}\left(1-\frac{x^2}{18}+\frac{x^4}{936}+.....\right).$$

**Example 3.** *Find the series solution of the following differential equations :*

(*i*) $3x\,\frac{d^2y}{dx^2} - (x-2)\,\frac{dy}{dx} - 2y = 0$ (*ii*) $2x\,\frac{d^2y}{dx^2} + (x+1)\,\frac{dy}{dx} + 3y = 0$

(*iii*) $3x\,\frac{d^2y}{dx^2} + (1-x)\,\frac{dy}{dx} - y = 0$ (*iv*) $4x\,\frac{d^2y}{dx^2} + 2(1-x)\,\frac{dy}{dx} - y = 0$

(*v*) $4x\,y'' + 2y' + y = 0.$

**Sol.** (*i*) The given D.E. is $3x\dfrac{d^2y}{dx^2} - (x-2)\dfrac{dy}{dx} - 2y = 0$ ...(1)

or $$\frac{d^2y}{dx^2} - \frac{x-2}{3x}\frac{dy}{dx} - \frac{2}{3x}y = 0$$

Comparing it with $$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0$$

we have $$P(x) = \frac{2-x}{3x} \quad \text{and} \quad Q(x) = -\frac{2}{3x}$$

Neither $P(x)$ nor $Q(x)$ is analytic at $x = 0$

$\Rightarrow$ 0 is not an ordinary point of (1).

$$xP(x) = \frac{1}{3}(2-x) \quad \text{and} \quad x^2Q(x) = -\frac{2x}{3}$$

Both $xP(x)$ and $x^2Q(x)$ are analytic at $x = 0$

$\Rightarrow$ 0 is a regular singular point of (1).

Let $$y = x^k \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+k}, \; a_0 \neq 0 \qquad \text{...(2)}$$

be a series solution of (1) so that

$$\frac{dy}{dx} = \sum_{m=0}^{\infty} (m+k)a_m x^{m+k-1} \quad \text{and} \quad \frac{d^2y}{dx^2} = \sum_{m=0}^{\infty} (m+k)(m+k-1)a_m x^{m+k-2}$$

Putting the values of $y$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we get

$$3x\sum_{m=0}^{\infty} (m+k)(m+k-1)a_m x^{m+k-2} - (x-2)\sum_{m=0}^{\infty} (m+k)a_m x^{m+k-1} - 2\sum_{m=0}^{\infty} a_m x^{m+k} = 0$$

$$\Rightarrow \sum_{m=0}^{\infty} 3(m+k)(m+k-1)a_m x^{m+k-1} - \sum_{m=0}^{\infty} (m+k)a_m x^{m+k} + \sum_{m=0}^{\infty} 2(m+k)a_m x^{m+k-1} - \sum_{m=0}^{\infty} 2a_m x^{m+k} = 0$$

$$\Rightarrow \sum_{m=-1}^{\infty} 3(m+1+k)(m+k)a_{m+1} x^{m+k} - \sum_{m=0}^{\infty} (m+k)a_m x^{m+k} + \sum_{m=-1}^{\infty} 2(m+1+k)a_{m+1} x^{m+k} - \sum_{m=0}^{\infty} 2a_m x^{m+k} = 0$$

$$\Rightarrow \sum_{m=-1}^{\infty} [3(m+k+1)(m+k) + 2(m+k+1)]a_{m+1} x^{m+k} - \sum_{m=0}^{\infty} (m+k+2)a_m x^{m+k} = 0$$

$$\Rightarrow \sum_{m=-1}^{\infty} (m+k+1)(3m+3k+2)a_{m+1} x^{m+k} - \sum_{m=0}^{\infty} (m+k+2)a_m x^{m+k} = 0$$

$$\Rightarrow \quad k(3k-1)a_0 x^{k-1} + \sum_{m=0}^{\infty} (m+k+1)(3m+3k+2)a_{m+1}x^{m+k} - \sum_{m=0}^{\infty} (m+k+2)a_m x^{m+k} = 0$$

$$\Rightarrow \quad k(3k-1)a_0 x^{k-1} + \sum_{m=0}^{\infty} [(m+k+1)(3m+3k+2)a_{m+1} - (m+k+2)a_m]\, x^{m+k} = 0$$

Equating to zero the co-efficient of lowest power of $x$ *i.e.*, $x^{k-1}$, we get

$$k(3k-1)a_0 = 0 \quad \Rightarrow \quad k(3k-1) = 0 \qquad (\because \quad a_0 \neq 0)$$

This is the indicial equation of the given D.E.

Solving $\qquad k = 0, \dfrac{1}{3}$ (distinct, not differing by an integer)

Equating to zero the co-efficient of $x^{m+k}$, $m \geq 0$, we get

$$(m+k+1)(3m+3k+2)a_{m+1} - (m+k+2)a_m = 0$$

$$\Rightarrow \quad a_{m+1} = \frac{m+k+2}{(m+k+1)(3m+3k+2)}\, a_m \quad \text{for all } m \geq 0 \qquad ...(3)$$

**When k = 0**, from (3) $\quad a_{m+1} = \dfrac{m+2}{(m+1)(3m+2)}\, a_m$

Putting $m = 0, 1, 2, ......$

$$a_1 = \frac{2}{1.2}\, a_0 = a_0$$

$$a_2 = \frac{3}{2.5}\, a_1 = \frac{3}{10}\, a_0$$

$$a_3 = \frac{4}{3.8}\, a_2 = \frac{1}{6}\cdot\frac{3}{10}\, a_0 = \frac{1}{20}\, a_0 \quad \text{and so on.}$$

Putting these values in (2)

$$y = x^0 \left( a_0 + a_0 x + \frac{3}{10} a_0 x^2 + \frac{1}{20} a_0 x^3 + ...... \right)$$

or

$$y = a_0 \left( 1 + x + \frac{3x^2}{10} + \frac{x^3}{20} + ...... \right)$$

$\Rightarrow$ One solution of the given D.E. is

$$u = 1 + x + \frac{3x^2}{10} + \frac{x^3}{20} + ..... \qquad (\text{taking } a_0 = 1)$$

**When k = $\frac{1}{3}$**, from (3)

$$a_{m+1} = \frac{m + \frac{7}{3}}{\left(m + \frac{4}{3}\right)(3m+3)}\, a_m = \frac{3m+7}{3(m+1)(3m+4)}\, a_m$$

$$\Rightarrow \quad 2k(2k-1)a_0 x^{k-1} + \sum_{m=0}^{\infty} 2(m+k+1)(2m+2k+1)\, a_{m+1} x^{m+k}$$

$$- \sum_{m=0}^{\infty} (2m+2k+1)a_m x^{m+k} = 0$$

$$\Rightarrow \quad 2k(2k-1)a_0 x^{k-1} + \sum_{m=0}^{\infty} (2m+2k+1)[(2m+2k+2)\, a_{m+1} - a_m]\, x^{m+k} = 0$$

Roots of indicial equation are $0, \frac{1}{2}$.

Equating to zero the co-efficient of $x^{m+k}$, $m \geq 0$, we have

$$(2m+2k+1)\,[(2m+2k+2)a_{m+1} - a_m] = 0$$

$$\Rightarrow \quad (2m+2k+2)a_{m+1} - a_m = 0$$

(since $2m + 2k + 1 \neq 0$ for $k = 0$ or $\frac{1}{2}$ and $m \geq 0$)

$$\Rightarrow \quad a_{m+1} = \frac{a_m}{2m+2k+2} \quad \text{for all } m \geq 0$$

**Ans.** $y = a\left(1 + \frac{x}{2} + \frac{x^2}{2^2\, 2!} + \frac{x^3}{2^3\, 3!} + \ldots\ldots\right) + b\sqrt{x}\left(1 + \frac{x}{3} + \frac{x^2}{3.5} + \frac{x^3}{3.5.7} + \ldots\ldots\right)$.

($v$) Please try yourself.

**Ans.** $y = a\left(1 - \frac{x}{(2)!} + \frac{x^2}{(4)!} - \ldots\ldots\right) + b\sqrt{x}\left(1 - \frac{x}{(3)!} + \frac{x^2}{(5)!} - \ldots\ldots\right)$.

**Example 4.** *Solve the following equation in series:*

$$2x(1-x)\frac{d^2y}{dx^2} + (1-x)\frac{dy}{dx} + 3y = 0.$$

**Sol.** The given D.E. is $\quad 2x(1-x)\dfrac{d^2y}{dx^2} + (1-x)\dfrac{dy}{dx} + 3y = 0 \qquad$ ...(1)

0 is a regular singular point of (1). (Prove it)

Let $\quad y = x^k \sum_{m=0}^{\infty} a_m x^m = \sum_{m=0}^{\infty} a_m x^{m+k}, \qquad a_0 \neq 0 \qquad$ ...(2)

be a series solution of (1).

Putting the values of $y$, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we get

$$(2x - 2x^2)\sum_{m=0}^{\infty} (m+k)(m+k-1)a_m x^{m+k-2} + (1-x)\sum_{m=0}^{\infty} (m+k)\, a_m x^{m+k-1}$$

$$+ 3\sum_{m=0}^{\infty} a_m x^{m+k} = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} 2(m+k)(m+k-1)a_m x^{m+k-1} - \sum_{m=0}^{\infty} 2(m+k)(m+k-1)\, a_m x^{m+k}$$

$$+ \sum_{m=0}^{\infty} (m+k)\, a_m x^{m+k-1} - \sum_{m=0}^{\infty} (m+k)\, a_m x^{m+k} + \sum_{m=0}^{\infty} 3a_m x^{m+k} = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} [2(m+k)(m+k-1) + (m+k)]a_m x^{m+k-1}$$

$$- \sum_{m=0}^{\infty} [2(m+k)(m+k-1) + (m+k) - 3]a_m x^{m+k} = 0$$

$$\Rightarrow \quad \sum_{m=0}^{\infty} (m+k)(2m+2k-1)a_m x^{m+k-1} - \sum_{m=0}^{\infty} [(m+k)(2m+2k-1) - 3]\, a_m x^{m+k} = 0$$

$$\Rightarrow \sum_{m=-1}^{\infty} (m+1+k)[2(m+1)+2k-1]\, a_{m+1}x^{m+k} - \sum_{m=0}^{\infty} [(m+k)(2m+2k-1)-3]\, a_m x^{m+k} = 0$$

$$\Rightarrow \quad \sum_{m=-1}^{\infty} (m+k+1)(2m+2k+1)a_{m+1} x^{m+k} - \sum_{m=0}^{\infty} [(m+k)(2m+2k-1)-3]\, a_m x^{m+k} = 0$$

$$\Rightarrow \quad k(2k-1)\, a_0 x^{k-1} + \sum_{m=0}^{\infty} [(m+k+1)(2m+2k+1)\; a_{m+1}$$

$$- \{(m+k)(2m+2k-1)-3\}\, a_m]\, x^{m+k} = 0$$

Equating to zero the co-efficient of lowest power of $x$, *i.e.*, $x^{k-1}$, we get

$$k(2k-1)a_0 = 0 \quad \Rightarrow \quad k(2k-1) = 0 \qquad (\because \quad a_0 \neq 0)$$

This is the indicial equation of the given D.E.

Solving $k = 0, \dfrac{1}{2}$ (distinct, not differing by an integer)

Equating to zero the co-efficient of $x^{m+k}$, $m \geq 0$, we get

$$(m+k+1)(2m+2k+1)a_{m+1} - [(m+k)(2m+2k-1)-3]\, a_m = 0$$

$$\Rightarrow \quad a_{m+1} = \frac{(m+k)(2m+2k-1)-3}{(m+k+1)(2m+2k+1)}\, a_m \quad \text{for all } m \geq 0 \qquad ...(3)$$

**When k = 0,** from (3)

$$a_{m+1} = \frac{m(2m-1)-3}{(m+1)(2m+1)}\, a_m = \frac{(m+1)(2m-3)}{(m+1)(2m+1)}\, a_m$$

$$\Rightarrow \quad a_{m+1} = \frac{2m-3}{2m+1}\, a_m \qquad (\because \quad m \neq -1)$$

Putting $m = 0, 1, 2, 3, \ldots\ldots$

$$a_1 = -3a_0$$

$$a_2 = -\frac{1}{3}\, a_1 = a_0$$

# 11

# Legendre's Equation

## Solution of Legendre's Equation

The differential equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)\,y = 0 \qquad \text{...}(i)$$

is called Legendre's differential equation. It can be solved in series of ascending or descending powers of $x$. The solution in descending powers of $x$ is more important than the one in ascending powers.

Let us assume,

$$y = \sum_{r=0}^{\infty} a_r x^{k-r}$$

$$\frac{dy}{dx} = \sum_{r=0}^{\infty} a_r (k-r)\, x^{k-r-1}$$

and

$$\frac{d^2y}{dx^2} = \sum_{r=0}^{\infty} a_r (k-r)(k-r-1)\, x^{k-r-2}$$

Putting in $(i)$, we have

$$(1-x^2)\sum_{r=0}^{\infty} a_r (k-r)(k-r-1)\, x^{k-r-2} - 2x\sum_{r=0}^{\infty} a_r (k-r)\, x^{k-r-1} + n(n+1)\sum_{r=0}^{\infty} a_r x^{k-r} = 0$$

or

$$\sum_{r=0}^{\infty} a_r[(k-r)(k-r-1)\, x^{k-r-2} + \{n(n+1) - (k-r)(k-r-1) - 2(k-r)\}\, x^{k-r}] = 0$$

or

$$\sum_{r=0}^{\infty} a_r[(k-r)(k-r-1)\, x^{k-r-2} + \{n(n+1) - (k-r)(k-r+1)\}\, x^{k-r} = 0$$

or

$$\sum_{r=0}^{\infty} a_r[(k-r)(k-r-1)\, x^{k-r-2} + \{(n^2 - (k-r)^2 + n - (k-r)\}\, x^{k-r}] = 0$$

or

$$\sum_{r=0}^{\infty} a_r[(k-r)(k-r-1)\, x^{k-r-2} + (n-k+r)(n+k-r+1)\, x^{k-r}] = 0 \qquad \text{...}(ii)$$

Now (*ii*) being an identity, we can equate to zero the co-efficients of various powers of $x$.

$\therefore$ Equating to zero the co-efficient of highest power of $x$ *i.e.*, $x^k$, we have $a_0\,(n-k)(n+k+1) = 0$.

Now $a_0 \neq 0$ as it the co-efficient of the first term with which we start to write series

$$\left.\begin{array}{ll}\text{either} & k = n \\ \text{or} & k = -(n+1)\end{array}\right\} \qquad \text{...}(iii)$$

Equating to zero the co-efficient of the next lower power of $x$ *i.e.*, of $x^{k-1}$, we have

$$a_1(n-k+1)(n+k) = 0$$

$\therefore$ $a_1 = 0$, since $(n-k+1)$ or $(n+k)$ is not zero by virtue of (*iii*).

Again equating to zero the co-efficient of the general term *i.e.*, $x^{k-r}$, we have

$$a_{r-2}\,(k-r+2)(k-r+1) + (n-k+r)(n+k-r+1)a_r = 0$$

$$\therefore \qquad a_r = -\frac{(k-r+2)(k-r+1)}{(n-k+r)(n+k-r+1)}\,a_{r-2} \qquad \text{...}(iv)$$

Putting $r = 3$,
$$a_3 = -\frac{(k-1)(k-2)}{(n-k+3)(n+k-2)}\,a_1$$
$$= 0, \quad \text{since } a_1 = 0$$

$\therefore$ We have $\qquad a_1 = a_3 = a_5 = \ldots. = 0$ (each)

Now two cases arises :

**Case I.** When $k = n$ from (*iv*), we have

$$c_r = -\frac{(n-r+2)(n-r+1)}{r(2n-r+1)}\,a_{r-2}$$

Putting $r = 2, 4, \ldots..$

$$a_2 = -\frac{n(n-1)}{2(2n-1)}\,a_0$$

$$a_4 = -\frac{(n-2)(n-3)}{4(2n-3)}\,.\,a_2 = \frac{n(n-1)(n-2)(n-3)}{2\,.\,4\,.\,(2n-1)(2n-3)}\,a_0 \text{ etc.}$$

$$\therefore \qquad y = a_0x^n + a_2x^{n-2} + a_4x^{n-1} + \ldots\ldots.$$

$$= a_0\left[x^n - \frac{n(n-1)}{2(2n-1)}\,.\,x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\,.\,4\,.\,(2n-1)(2n-3)}\,x^{n-4}\ \ldots\ldots\right] \qquad \text{...}(v)$$

which is one solution of Legendre's equation.

**Case II.** When $\qquad k = -(n+1)$

$$a_r = \frac{(n+r-1)(n+r)}{r(2n+r+1)}\,a_{r-2}$$

Putting $r = 2, 4$ etc.

$$a_2 = \frac{(n+1)(n+2)}{2(2n+3)}\,a_0$$

$$a_4 = \frac{(n+3)(n+4)}{4(2n+5)}\,a_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{2\,.\,4\,.\,(2n+3)(2n+5)}\,a_0 \text{ etc.}$$

$$y = \sum_{r=0}^{\infty} a_r x^{-n-1+r} + a_0 x^{-n-1} + a_2 x^{-n-3} + a_4 x^{-n-5} + \ldots$$

$$= a_0 \left[ x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4.(2n+3)(2n+5)} x^{-x-5} + \ldots.. \right] \quad \ldots(iv)$$

which is other solution of Legendre's equation.

**Definitions of $P_n(x)$ and $Q_n(x)$**

The solution of Legendre's equation are called Legendre's functions.

When $n$ is a positive intger

and $$a_0 = \frac{1.3.5\ldots\ldots(2n-1)}{n!},$$

the solution of $(v)$ of the equation $(i)$ is denoted by $P_n(x)$ and is called Legendre's function of the first kind.

$$\therefore \quad P_n(x) = \frac{1.3.5\ldots\ldots(2n-1)}{n!} \left[ x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)} x^{n-4} \ldots.. \right]$$

$P_n(x)$ is a terminating series and gives what are called Legendre's polynomials for different values of $n$.

We can write

$$P_n(x) = \sum_{r=0}^{(n/2)} (-1)^r \frac{(2n-2r)!}{2^n . r!(n-2r)!(n-r)!} x^{n-2r}$$

where $$\left(\frac{n}{2}\right) = \begin{cases} \dfrac{n}{2} & \text{if } n \text{ is even} \\ \dfrac{(n-1)}{2} & \text{if } n \text{ is odd} \end{cases}$$

Again, when $n$ is a positive integer

and $$a_0 = \frac{n!}{1.3.5\ldots\ldots(2n+1)}$$

the solution $(vi)$ of equation $(i)$ is denoted by $Q_n(x)$ and is called the Legendre's function of the second kind.

$$\therefore \quad Q_n(x) = \frac{n!}{1.3\ldots\ldots(2n+1)} \left[ x^{-n-1} + \frac{(n+1)(n+2)}{2.(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)} x^{-x-5} + \ldots.. \right]$$

$Q_n(x)$ is terminating series as $n$ is positive.

The most general solution of the Legendre's equation is

$$y = AP_n(x) + BQ_n(x)$$

where A and B are arbitrary constants.

**To show that $P_n(x)$ is the co-efficient of $h^n$ in the expansion in ascending powers of h of $(1 - 2xh + h^2)^{-1/2}$.** (Meerut, 1998)

We have $(1 - 2xh + h^2)^{-1/2} = \{1 - h(2x - h)\}^{-1/2} = 1 + \frac{1}{2} h(2x - h) + \frac{1.3}{2.4}$

$$h^2(2x - h^2) + \frac{1.3 \ldots\ldots (2n-3)}{2.4 \ldots\ldots (2n-2)} h^{n-1} (2x - h)^{n-1} + \frac{1.3 \ldots\ldots (2n-1)}{2.4 \ldots\ldots (2n)} h^n (2x - h)^n + \ldots$$

$\therefore$ Co-efficient of

$$h^n = \frac{1.3 \ldots\ldots (2n-1)}{2.4 \ldots\ldots (2n)} (2n)^n - \frac{1.3 \ldots\ldots (2n-3)}{2.4 \ldots\ldots (2n-2)} {}^{n-1}C_1 (2x)^{n-2} + \frac{1.3 \ldots\ldots (2n-5)}{2.4 \ldots\ldots (2n-4)} {}^{n-2}C_2 (2x)^{n-4} + \ldots\ldots$$

$$= \frac{1.3 \ldots\ldots (2n-1)}{(n)!} \left[ x^n - \frac{2n}{2n-1} (n-1) \cdot \frac{x^{n-2}}{2^2} + \frac{2n(2n-2)}{(2n-1)(2n-3)} \cdot \frac{(n-1)(n-3)}{2!} \cdot \frac{x^{n-4}}{2^4} - \ldots\ldots \right]$$

$$= \frac{1.3 \ldots\ldots (2n-1)}{n!} \left[ x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4(2n-1)(2n-3)} x^{x-4} - \ldots\ldots \right]$$

$= P_n(x)$. Thus we can say that

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2hx + h^2)^{-1/2}.$$

**Laplace's Definite Integrals for $P_n(x)$**

(*i*) **Laplace's first integral for $P_n(x)$. When n is positive integer.**

$$\mathbf{P_n(x) = \frac{1}{\pi} \int_0^p [x \pm \sqrt{x^2 - 1} \cos \phi]^n \, dx.}$$

**Proof.** From Integral Calculus, we have

$$\int_0^\pi \frac{d\phi}{a \pm b \cos \phi} = \frac{\pi}{\sqrt{a^2 - b^2}}, \text{ where } a^2 > b^2$$

Putting $a = 1 - hx$ and $b = h\sqrt{x^2 - 1}$

so that $a^2 - b^2 = (1 - hx)^2 - h^2(x^2 - 1) = 1 - 2xh + h^2$

We have $\pi(1 - 2xh + h^2)^{-1/2} = \int_0^\pi [1 - hx \pm h \sqrt{x^2 - 1} \cos \phi]^{-1} \, d\phi$

$$= \int_0^\pi [1 - hx \pm \sqrt{x^2 - 1} \cos \phi]^{-1} \, d\phi = \int_0^\pi [1 - ht]^{-1} \, d\phi$$

where $t = \pm \sqrt{x^2 - 1} \cos \phi$

or $$\pi \sum_{n=0}^{\infty} h^n P_n(x) = \int_0^{\pi} (1 + ht + h^2t^2 + \ldots\ldots + h^n t + \ldots.)\, d\phi$$

Equating the co-efficients of $h^n$, we have

$$\pi P_n(x) = \int_0^{\pi} t^n\, d\phi = \int_0^{\pi} [x \pm \sqrt{x^2 - 1} \cos \phi]^n\, d\phi$$

$\therefore$ $$P_n(x) = \frac{1}{\pi} \int_0^{\pi} [x \pm \sqrt{x^2 - 1} \cos \phi]^n\, d\phi.$$

**Note.** Putting $x = \cos \theta$, we get

$$P_n(\cos \theta) = \frac{1}{\pi} \int_0^{\pi} (\cos \theta + i \sin \theta \cos \phi)^n\, d\phi.$$

(*ii*) **Laplace's second integral $P_n(x)$. When n is a positive integer.**

$$\mathbf{P_n(x) = \frac{1}{\pi} \int_0^{\pi} \left[\frac{d\phi}{[x \pm \sqrt{x^2 - 1} \cos \phi]}\right]^{n+1}}.$$

**Proof.** From Integral Calculus, we get

$$\int_0^{\pi} \frac{d\phi}{a \pm b \cos \phi} = \frac{\pi}{\sqrt{a^2 - b^2}}, \text{ where } a^2 > b^2$$

Putting $a = xh - 1$ and $b = h\sqrt{x^2 - 1}$

so that $a^2 - b^2 = 1 - 2xh + h^2$

We have $$\pi(1 - 2hx + h^2)^{-1/2} = \int_0^{\pi} [-1 + xh \pm h\sqrt{x^2 - 1} \cos \phi]^{-1}\, d\phi$$

or $$\frac{\pi}{h}\left(1 - 2x\frac{1}{h} + \frac{1}{h^2}\right)^{-1/2} = \int_0^{\pi} [h\{x \pm \sqrt{x^2 - 1} \cos \phi\} - 1]^{-1}\, d\phi$$

or $$\frac{\pi}{h} \sum_{n=0}^{\infty} \frac{1}{h^n} P_n(x) = \int_0^{\pi} (t - 1)^{-1}\, d\phi$$

where $$t = h\{x + \sqrt{x^2 - 1} \cos \phi\}$$

$$= \int_0^{\pi} \frac{1}{t}\left(t - \frac{1}{t}\right)^{-1} d\phi = \int_0^{\pi} \frac{1}{t}\left(1 + \frac{1}{t} + \frac{1}{t^2} + \ldots\ldots + \frac{1}{t^n}\right) d\phi$$

$$= \int_0^{\pi} \left(\frac{1}{t} + \frac{1}{t^2} + \frac{1}{t^3} + \ldots\ldots + \frac{1}{t^{n+1}} + \ldots\ldots\right) d\phi = \int_0^{\pi} \sum_{n=0}^{\infty} \frac{1}{t^{n+1}}\, d\phi$$

$$= \int_0^{\pi} \sum_{n=0}^{\infty} \frac{1}{h^{n+1}\left\{x \pm \sqrt{x^2 - 1} \cos \phi\right\}^{n+1}}\, d\phi$$

$\therefore$ Equating the coefficients of $\frac{1}{h^{n+1}}$, we get

$$\pi P_n(x) = \int_0^{\pi} \frac{d\phi}{\{x^2 \pm \sqrt{x^2+1}\cos\phi\}^{n+1}}.$$

or
$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \frac{d\phi}{\{x \pm \sqrt{x^2+1}\cos\phi\}^{n+1}}.$$

**Note.** Replacing $n$ by $-(n+1)$ in Laplace second integral, we get

$$P_{-(n+1)}(x) = \frac{1}{x}\int_0^{\pi} \{x \pm \sqrt{x^2-1}\cos\phi\}^n \, d\phi = P_n(x)$$

from Laplace's first integral.

Hence $\qquad P_{-(n+1)} = P_n.$

**Orthogonal Properties of Legendre's Polynomials**

(*i*) $\displaystyle\int_{-1}^{1} \mathbf{P_m(x)\, P_n(x)\, dx = 0 \text{ if } m \neq n}$

(*ii*) $\displaystyle\int_{-1}^{+1} [\mathbf{P_n(x)}]^2\, \mathbf{dx} = \frac{\mathbf{2}}{\mathbf{2n+1}} \text{ if } \mathbf{m = n}.$

**Proof.** (*i*) Legendre's equation can be written as

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + n(n+1)\,y = 0$$

$$\therefore \qquad \frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} + n(n+1)\,P_n = 0 \qquad ...(i)$$

and
$$\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + m(m+1)\,P_m = 0 \qquad ...(ii)$$

Multiplying (*i*) by $P_m$ and (*ii*) by $P_n$ and then subtracting, we get

$$P_m\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\} - P_n\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\} + \{n(n+1) - m(m+1)\}\,P_nP_m = 0$$

Integrating between the limits $-1$ to $+1$, we have

$$\int_{-1}^{+1} P_m\frac{d}{dx}\left\{(1-x^2)\frac{dP_n}{dx}\right\}dx + \int_{-1}^{+1} P_n\frac{d}{dx}\left\{(1-x^2)\frac{dP_m}{dx}\right\}dx$$
$$+ \{n(n+1) - m(m+1)\}\int_{-1}^{+1} P_mP_n\,dx = 0$$

Integrating by parts

$$\left[P_m\,(1-x^2)\frac{dP_n}{dx}\right]_{-1}^{+1} - \int_{-1}^{+1}\left\{\frac{d}{dx}P_m\right\}\left\{(1-x^2)\frac{d}{dx}P_n\right| dx$$
$$-\left[P_n\,(1-x^2)\frac{d}{dx}P_m\right]_{-1}^{+1} + \int_{-1}^{+1}\left\{\frac{d}{dx}P_n\right\}\left\{(1-x^2)\frac{d}{dx}P_m\right\}dx$$
$$+ \{n(n+1) - m\,(m+1)\}\int_{-1}^{+1} P_m\,P_n\,dx = 0$$

$$\therefore \quad \{n\,(n+1) - m\,(m+1)\}\int_{-1}^{+1} P_m\,P_n\,dx = 0$$

From Recurrence formulae II, we have

$$xP_n' = nP_n + P'_{n-1} \qquad ...(ii)$$

Eliminating $xP'_n$ from (*i*) and (*ii*), we have

$$(2n+1)(nP_n + P'_{n-1}) + (2n+1)\,P_n = (n+1)\,P'_{n+1} + nP'_{n-1}$$

or
$$(2n+1)(n+1)\,P_n = (n+1)\,P'_{n+1} + nP'_{n-1} - (2n+1)\,P'_{n-1}$$

or
$$(2n+1)(n+1)\,P_n = (n+1)\,P'_{n+1} - (n+1)\,P'_{n-1}$$

$$\therefore \qquad \mathbf{(2n+1)P = P'_{n+1} - P'_{n-1}}.$$

**(IV)** $\mathbf{(n+1)P_n = P'_{n+1} - xP_n'}$.

**Proof.** Writing Recurrence formulae II and III, we have

$$nP_n = xP_n' - P'_{n-1} \qquad ...(i)$$

and
$$(2n+1)P_n = P'_{n+1} - P'_{n-1} \qquad ...(ii)$$

Subtracting (*i*) from (*ii*), we have

$$\mathbf{(n+1)P_n = P'_{n+1} - xP'_n}.$$

**(V)** $\mathbf{(1-x^2)P'_n = n(P_{n-1} - xP_n)}$.

**Proof.** Replacing $n$ by $(n-1)$ in Recurrence formula IV, we have

$$nP_{n-1} = P'_n - xP'_{n-1} \qquad ...(i)$$

Writing II Recurrence formulae, we have

$$nP_n = xP'_n - xP'_{n-1} \qquad ...(ii)$$

Multiplying (*ii*) by $x$ and then subtracting from (*i*), we have

$$n(P_{n-1} - xP_n) = (1-x^2)P'_n$$

*i.e.,*
$$\mathbf{(1-x^2)P'_n = n(P_{n-1} - xP_n)}.$$

**Aliter.** From Laplace's first Integral, we have

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} \{x + \sqrt{(x^2-1)}\cos\phi\}^n\,d\phi$$

Replacing $n$ by $(n-1)$, we have

$$P_{n-1}(x) = \frac{1}{\pi}\int_0^{\pi} \{x + \sqrt{(x^2-1)}\cos\phi\}^{n-1}\,d\phi$$

$$\therefore \qquad P_{n-1} - xP_n = \frac{1}{\pi}\int_0^{\pi} [\{x + \sqrt{(x^2-1)} - 1)\cos\phi\}^{n-1} - x\{x + \sqrt{(x^2-1)}\cos\phi\}^n]\,d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} \{x + \sqrt{(x^2-1)}\cos\phi\}^{n-1}\,[1 - \{x + \sqrt{(x^2-1)}\cos\phi\}\,x]\,d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} \{x + \sqrt{(x^2-1)}\cos\phi\}^{n-1}\,[1 - x\{x + \sqrt{(x^2-1)}\cos\phi\}]\,d\phi$$

$$= \frac{1}{\pi}\int_0^{\pi} \{x + \sqrt{(x^2-1)}\cos\phi\}^{n-1}\,[(1-x^2) - x + \sqrt{(x^2-1)}\cos\phi]\,d\phi$$

$$= -\frac{(x^2-1)}{\pi}\int_0^{\pi} \{x + \sqrt{(x^2-1)}\cos\phi\}^{n-1}\left[1 + \frac{x}{\sqrt{(x^2-1)}}\cos\phi\right]d\phi$$

$$= -\frac{(x^2-1)}{\pi}\int_0^{\pi} \left[\{x + \sqrt{(x^2-1)}\cos\phi\}^{n-1} \times \frac{d}{dx}\left\{x + \sqrt{(x^2-1)}\cos\phi\right\}\right]d\phi$$

Replacing $n$ by $(n-2)$, $(n-4)$, ... in $(i)$, we have

$$P'_{n-2} = (2n-5)\,P_{n-3} + P'_{n-4} \qquad ...(ii)$$

$$P'_{n-4} = (2n-9)\,P_{n-5} + P'_{n-6} \qquad ...(iii)$$

..................................................

..................................................

$$P_2' = 3P_1 + P_0$$

*when n is even.*

Adding $(i)$, $(ii)$, $(iii)$, etc., we have

$$P_n' = (2n-1)\,P_{n-1} + (2n-5)\,P_{n-3} + (2n-9)\,P_{n-5} + \ldots + 3P_1 + P_0'$$

$$= (2n-1)\,P_{n-1} + (2n-5)\,P_{n-3} + (2n-9)P_{n-5} + \ldots + 3P_1$$

as $P_0' = 0$ [See Ex. 1]

*when n is odd*

$$P_n' = (2n-1)\,P_{n-1} + (2n-5)\,P_{n-3} + \ldots + 5P_2 + P_1'$$

$$= (2n-1)\,P_{n-1} + (2n-5)\,P_{n-3} + \ldots + P_0 \qquad \text{as } P_1' = 1 = P_0 \text{ [See Ex. 1]}$$

Hence $\quad P_n' = (2n-1)\,P_{n-1} + (2n-5)\,P_{n-2} + \ldots\ldots$

the last term of the series being $3P_1$ or $P_0$ according as $n$ is even or odd.

**Christoffel's Summation Formula. To prove that**

$$\sum_{r=1}^{n} (2r+1)\,P_r(x)\,P_r(y) = (n+1)\,\frac{P_{n+1}(x)\,P_n(y) - P_{n-1}(y)\,P_n(x)}{(x-y)}$$

**Proof.** From Recurrence Summation formula, we have

$$(2r+1)\,xP_r(x) = (r+1)\,P_{n-1}(x) + r\,P_{r-1}\,(x) \qquad ...(i)$$

and

$$(2r+1)\,yP_r(y) = (r+1)\,P_{r-1}(y) + r\,P_{r-1}\,(y) \qquad ...(ii)$$

Multiplying $(i)$ by $P_r(y)$ and $(ii)$ by $P_r(x)$ and then subtracting, we have

$$(2r+1)\,(x-y)\,P_r(x)\,P_r(y) = (r+1)\,[P_{r+1}\,(x)\,P_r(y) - P_{r+1}(y)\,P_r(x)]$$
$$- r\,[P_{r-1}\,(y).\,P_r(x) - P_{r-1}(y)\,P_r(x)]$$

Putting $r = 0, 1, 2, 3, \ldots, (n-1), n$, we have

$$(x-y)\,P_0(x)\,P_0(y) = [P_1(x)\,P_0(y) - P_1\,(y).\,P_0(x)] + 0 \qquad ...(A_0)$$

$$3.(x-y)P_1(x)P_1(y) = 2.[P_2(x)\,P_1(y) - P_2(y)\,P_1(x)]$$
$$-1\,[P_0(y)\,P_1(x) - P_0(x)\,P_1(x)] \qquad ...(A_1)$$

$$5.(x-y)\,P_2(x)\,P_2(y) = 3\,[P_3(x)\,P_2(y) - P_3(y)\,P_2(x)]$$
$$-2\,[P_1(y)\,P_2(x) - P_1(x)\,P_2(y)] \qquad ...(A_2)$$

..........................................................................................

..........................................................................................

$$(2n-1)\,(x-y)\,P_{n-1}(x)\,P_{n+1}(y) = n\,[P_n(x)\,P_{n-1}(y) - P_n(y)\,P_{n-1}(x)]$$
$$-(n-1)\,[P_{n-2}(y)\,P_{n-1}(x) - P_{n-2}(x)\,P_{n-1}(y)] \qquad ...(A_{n-1})$$

$$(2n+1)\,(x-y)\,P_{n-1}(x)\,P_{n-1}(y) = (n+1)\,[P_{n+1}(x)\,P_n(y) - P_{n+1}(y)\,P_n(x)]$$
$$-n\,[P_{n-1}(y)\,P_n(x) - P_{n-1}(y)\,P_1(y)] \qquad ...(A_n)$$

Adding $(A_0)$, $(A_1)$, $(A_2)$, ...., $(A_{n-1})$ and $(A_n)$, we have

$$(x-y)\sum_{r=0}^{n}\;(2r+1)\,P_r(x)\,P_r(y) = (n+1)\,P_{n+1}\,(x)\,P_n(y) - P_{n+1}(y)\,P_n(x)]$$

$$= (x-1)^n . n! + n \frac{n!}{1!} . (x+1) n (x-1)^{n-1} + ...$$

$$+ n . n (x+1)^{n-1} . \frac{n!}{1!} (x-1) + (x+1)^n . n!$$

Putting $x = 1$

$$\left(\frac{d^n y}{dx^n}\right)_{x=1} = (1+1)^n . n! = 2^n . n!$$

$\therefore$ From (*i*), we get $P_n(x) = \frac{1}{c}\frac{d^n y}{dx^n}$ or $P_n(x) = \frac{2n}{2^n\, n!} . \frac{d^n (x^2-1)}{dx^n}$

which is Rodrigue's Formula.

**Example 1.** *Show that*

$$P_0(x) = 1,\ P_1(x) = x,\ P_2(x) = \frac{(3x^2-1)}{2},$$

$$P_3(x) = \tfrac{1}{2}(5x^3 - 3x).$$

**Sol.** We know that

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2xh + h^2)^{-1/2} = \{1 - h(2x-h)\}^{-1/2}$$

$$= 1 + \frac{h}{2}(2x-h) + \frac{1.3}{2.4} h^2 (2x-h)^2 + \frac{1.3.5}{2.4.6} h^3 (2x-h)^3$$

or $$P_0(x) + h\, P_1(x) + h^2 P_2(x) + h^3 P_3(x) + ......$$

$$= 1 + x.\, h + \tfrac{1}{2}(3x^2 - 1) h^2 + \tfrac{1}{2}(5x^3 - 3x)h^3 + ......$$

Equating the coefficients of like power of $h$, we get

$$P_0(x) = 1,\ P_1(x) = xP_2(x) = \tfrac{1}{2}(3x^2-1),\ P_3(x) = (5x^3 - 3x) \text{ etc.}$$

**Example 2.** *Show that*

(*i*) $P_n(1) = 1$ (*ii*) $P_n(-1) = (-1)^n P_n(x)$

*Hence deduce that* $P_n(-1) = (-1)^n$.

**Sol.** (*i*) We know that

$$\sum_{n=0}^{\infty} h^n P_n(x) = (1 - 2xh + h^2)^{-1/2}.$$

Putting $x = 1$

$$\sum_{n=0}^{\infty} h^n P_n(1) = (1 - h + h^2)^{-1/2} = (1-h)^{-1}$$

$$= 1 + h + h^2 + ... + h^n + ... = \sum_{n=0}^{\infty} h^n$$

Equating the coefficient of $h^n$, we get $P_n(1) = 1$.

(*ii*) We have $(1-2xh+h^2)^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(x)$

$\therefore \quad (1+2xh+h^2)^{-1/2} = \{1-2x(-h)+(-h)^2\}^{-1/2} = \sum_{n=0}^{\infty} (-h)^n P_n(x)$

$$= \sum_{n=0}^{\infty} (-1)^n h^n P_n(x) \quad ...(1)$$

Again, $(1+2xh+h^2)^{-1/2} = \{1-2(-x)h+h^2\}^{-1/2} = \sum_{n=0}^{\infty} h^n P_n(-x) \quad ...(2)$

From (1) and (2), we get

$$\sum_{n=0}^{\infty} h^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n h^n P_n(x)$$

Equating the coefficients of $h^n$,

$$\mathbf{P_n(-x) = (-1)^n P_n(x).}$$

**Deduction.** Putting $x = 1$, we get

$$P_n(-1) = (-1)^n P_n(1) = (-1)^n$$

Since $P_n(1) = 1$ refer Ex. 2 (*i*).

**Example 3.** *Prove that*

(*i*) $P_{2m+1}(0) = 0$ *and* (*ii*) $P_{2m}(0) = (-1)^m \dfrac{2m!}{2^{2m}(m!)^2}$.

**Sol.** (*i*) We know that

$$P_n(x) = \frac{1.3.5.....(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4} .....\right]$$

$$\therefore \quad P_{2m+1}(x) = \frac{1.3.5 ......\{2(2m+1)-1\}}{(2m+1)!} \times \left[x^{2m+1} - \frac{(2m+1)(2m+1-1)}{2.\{2(2m+1)-1\}}x^{2m+1-2} + ......\right]$$

Putting $x = 0$, $P_{2m+1}(0) = 0$

(*ii*) We have $\sum_{n=0}^{\infty} h^n P_n(x) = (1-2xh+h^2)^{-1/2}$

$$= \sum_{n=0}^{\infty} h^n P_n(0) = \{1+h^2\}^{-1/2} = \{1-(-h^2)\}^{-1/2}$$

$$= 1 + \frac{1}{2}(-h^2) + \frac{1.3}{2.4}(-h^2)^2 + \frac{1.3.5}{2.4.6}(-h^2)^3 + ... + \frac{1.3.5 .......(2r-1)}{2.4......2r}(-h^2)^r + ......$$

Equating the coefficient of $h^{2m}$ on both sides, we get

$$P_{2m}(0) = \frac{1.3.5 .......(2m-1)}{2.4.6.....2m}(-1)^m = (-1)^m \frac{(2m)!}{2^m . m!}$$

$$= \frac{1}{z}(1-2xz+z^2)^{-1/2} + (1-2xz+z^2)^{-1/2} - \frac{1}{z}$$

$$= \frac{1}{z}\sum_{n=0}^{\infty} z^n P_n(x) + \sum_{n=0}^{\infty} z^n P_n(x) - \frac{1}{z}$$

$$= \frac{1}{z}\left[P_0(x) + \sum_{n=1}^{\infty} z^n P_n(x)\right] + \sum_{n=0}^{\infty} z^n P_n(x) - \frac{1}{z}$$

$$= \frac{1}{z} + \frac{1}{z}\sum_{n=1}^{\infty} z^n P_n(x) + \sum_{n=0}^{\infty} z^n P_n(x) - \frac{1}{z}$$

Since $\qquad P_0(x) = 1$

$$= \sum_{n=1}^{\infty} z^{n-1} P_n(x) + \sum_{n=0}^{\infty} z^n P_n(x) = \sum_{n=0}^{\infty} z^n P_{n+1}(x) + \sum_{n=0}^{\infty} z^n P_n(x)$$

$$= \sum_{n=0}^{\infty} [P_{n+1}(x) + P_n(x)]z^n.$$

**Example 7.** *Show that*

$$\frac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1) P_n(x) z^n.$$

**Sol.** We have $\qquad (1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x) \qquad$ ...(*i*)

Differentiating w.r.t. $z$, we get

$$(x-z)(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} nz^{n-1} P_n(x)$$

$$\therefore \qquad 2(x-z)z(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} 2nz^n P_n(x) \qquad \text{...(ii)}$$

Adding (*i*) and (*ii*), we get

$$\frac{1-2xz+z^2+2(x-z)z}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1) z^n P_n(x)$$

or

$$\frac{1-z^2}{(1-2xz+z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n+1) z^n P_n(x).$$

**Example 8.** *Prove that*

$$P'_{n+1} + P_n' = P_0 + 3P_1 + 5P_2 + \dots + (2n+1)P_n.$$

**Sol.** Writing Recurrence formula III, we get

$$(2n+1) P_n = P'_{n+1} - P_{n-1}$$

Putting $n = 1, 2, 3, \dots\dots, n$, we get

$$3P_1 = P_2' - P_0'$$
$$5P_2 = P_3' - P_1'$$

$$7P_3 = P_4' - P_2'$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$(2n-3)\,P_{n-3} = P'_{n-1} - P'_{n-3}$$
$$(2n-1)\,P_{n-1} = P_n' - P'_{n-2}$$
$$(2n+1)P_n = P'_{n+1} + P'_{n-1}\,.$$

Adding all, we get

$$3P_1 + 5P_2 + \ldots + (2n+1)\,P_n = P_n' + P'_{n+1} - P_0' - P_1' = P_n' + P'_{n+1} - 0 - P_0$$

Since $\quad P_0 = 1 \quad \text{and} \quad P = x$

$\therefore \quad P_1' = 1 = P_0$

Hence $P_0 + 3P_1 + 5P_2 + \ldots + (2n+1)\,P_n = P'_{n+1} + P_n'$.

**Example 9.** *Prove that*

$$\int_{-1}^{+1} (x^2-1)\,P_{n+1}\,P_n'\,dx = \frac{2n(n+1)}{(2n+1)(2n+3)}.$$

**Sol.** From Recurrence formula V, we get

$$(x^2-1)\,P_n' = n(xP_n - P_{n-1})$$

$$\therefore \quad \int_{-1}^{+1} (x^2-1)\,P_{n+1}\,P_n'\,dx$$

$$= \int_{-1}^{+1} n(x\,P_n - P_{n-1})\,P_{n+1}\,dx$$

$$= n\int_{-1}^{+1} x\,P_n\,P_{n+1}\,dx$$

[the other integral being zero, since $\int_{-1}^{+1} P_m\,P_n\,dx = 0$, if $m \neq n$]

$$= n\int_{-1}^{+1} \frac{(n+1)P_{n+1} + nP_{n-1}}{2n+1}\,P_{n+1}\,dx.$$

(from Recurrence formula)

$$= \frac{n(n+1)}{2n+1}\int_{-1}^{+1} P^2_{n+1}\,dx + \frac{n^2}{2n+1}\int_{-1}^{+1} P_{n-1}\,P_{n+1}\,dx$$

$$= \frac{n(n+1)}{(2n+1)}\cdot\frac{2}{2(n+1)+1} + 0 = \frac{2n(n+1)}{(2n+1)(2n+3)}.$$

**Example 10.** *Prove that*

*(i)* $\int_{-1}^{+1} P_n(x)\,dx = 0,\ n \neq 0$ *(ii)* $\int_{-1}^{+1} P_0(x)\,dx = 2.$

**Sol.** From Rodrigue's formula, we get

$$P_n(x) = \frac{1}{2^n\,n!}\frac{d^n}{dx^n}(x^2-1)^n$$

$$\therefore \quad \int_{-1}^{+1} P_n(x)\,dx = \frac{1}{2^n\,n!}\int_{-1}^{+1}\frac{d^n}{dx^n}(x^2-1)^n\,dx$$

$$= \frac{1}{2^n\,n!}\left[\frac{d^{n-1}}{dx^{n-1}}(x^2-1)^n\right]_{-1}^{+1} \qquad \ldots(1)$$

Now $\quad \dfrac{d^{n-1}}{dx^{n-1}}(x^2-1)^n = \dfrac{d^{n-1}}{dx^{n-1}}(x+1)^n(x-1)^n = (x+1)^n \dfrac{d^{n-1}}{dx^{n-1}}(x-1)^n$

$$+ (n-1)\,n(n+1)^{n-1}\frac{d^{n-2}}{dx^{n-2}}(x-1)^n + \ldots + (x-1)^n \frac{d^{n-1}}{dx^{n-1}}(x+1)^n$$

$$= (x+1)^n \frac{n!}{1!}(x-1) + (n-1)\,n(x+1)^{n-1}\frac{n!}{2!}(x-1)^2$$

$$+ \ldots\ldots + (x-1)^n\, n!\,(x+1) = 0$$

When $x = -1$ or $1$ since each term contains $(x-1)$ and $(x+1)$.

$\therefore$ From (1), $\displaystyle\int_{-1}^{+1} P_n(x)\,dx = 0.$

(*ii*) We know that $\quad P_0(x) = 1$

$$\therefore \qquad \int_{-1}^{+1} P_0(x)\,dx = \int_{-1}^{+1} dx = \Big[x\Big]_{-1}^{+1} = 2.$$

**Example 11.** *Prove that*

$$\int_{-1}^{+1} x^2 P_{n+1} P_{n-1}\,dx = \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}.$$

**Sol.** From Recurrence formula I, we get

$$(2n+1)x\,P_n = (n+1)\,P_{n+1} + n\,P_{n-1}$$

Replacing $n$ by $(n-1)$ and $(n+1)$ respectively, we get

$$(2n-1)x\,P_{n-1} = n\,P_n + (n-1)\,P_{n-2}$$

and $\qquad (2n+3)x\,P_{n+1} = (n+2)\,P_{n+2} + (n+1)\,P_n$

Multiplying, $(2n-1)(2n+3)x^2\,P_{n+1}\,P_{n-1}$

$$= n(n+1)\,P_n^{\,2} + n(n+2)\,P_n\,P_{n+1} + (n-1)(n+2)\,P_{n-2}\,P_{n+2}$$

$$+ (n-1)(n+1)\,P_{n-1}\,P_n$$

Integrating between the limits $-1$ to $+1$, we get

$$(2n-1)(2n+3)\int_{-1}^{+1} x^2\,P_{n+1}\,P_{n-1}\,dx = n(n+1)\int_{-1}^{+1} P_n^{\,2}\,dx$$

all other integrals being zero

$$= n(n+1)\frac{2}{(2n+1)}, \quad \therefore \quad \int_{-1}^{+1} x^2\,P_{n+1}\,P_{n-1}\,dx$$

$$= \frac{2n(n+1)}{(2n-1)(2n+1)(2n+3)}.$$

**Example 12.** *Prove that*

$$\int_{-2}^{+1} (1-x^2)\,P_m'\,P_n'\,dx = 0$$

*where m and n are distinct positive integer.*

**Sol.** $\displaystyle\int_{-1}^{+1} (1-x^2)\,P_m'\,P_n'\,dx = \Big[(1-x^2)\,P_m'\,P_n\,dx\Big]_{-1}^{+1} - \int_{-1}^{+1}\Big[P_n \frac{d}{dx}\{(1-x^2)\,P_m'\}\Big]dx$

[Integrating by parts taking $P_n'$ as II function

$$= -\int_{-1}^{+1}\Big[P_n \frac{d}{dx}\{(1-x^2)\}\,P_m'\Big]dx \qquad \ldots(1)$$

$\therefore \qquad m' = \dfrac{n}{2}$

Now $\qquad (P_n')^2 = (2n-1)^2 P_{n-1}^2 + (2n-5)^2 P_{n-3}^2 + \ldots\ldots$

$$- 2(2n-1)(2n-5) P_{n-1} P_{n-3} + \ldots.$$

$$\therefore \qquad \int_{-1}^{+1} (P_n')^2\,dx = (2n-1)^2 \int_{-1}^{+1} (P_{n-1})^2\,dx + (2n-5)^2 \int_{-1}^{+1} P_{n-3}^2\,dx + \ldots\ldots$$

Other integrals are zero.

$$= (2n-1)^2 \frac{2}{2(n-1)+1} + (2n-5)^2 \frac{2}{2(n-3)+1} + (2n-9)^2 \frac{2}{2(n-5)+1} + \ldots.$$

$$= 2[(2n-1) + (2n-5) + (2n-9) + \ldots] \qquad \ldots(2)$$

The last term being 1 or 2 according as $n$ is odd or even.

**Case I.** When $n$ is even no. of terms on the R.H.S. of (2) is $\dfrac{n}{2}$.

$$\therefore \qquad \int_{-1}^{+1} (P_n')^2\,dx = 2.\,\frac{1}{2}\,.\,\frac{n}{2}\left[2(2n-1) + \left(\frac{n}{2}-1\right)(-4)\right] = n(n+1).$$

**Case II.** When $n$ is odd, no. of terms on the R.H.S. of (2) is $\dfrac{n+1}{2}$.

$$\int_{-1}^{+1} (P_n')^2\,dx = 2.\,\frac{1}{2}\,.\left(\frac{n+1}{2}\right)\left[2(2n-1) + \left(\frac{n+1}{2}-1\right)(-4)\right] = n(n+1).$$

Hence $\qquad \displaystyle\int_{-1}^{+1} (P_n')^2\,dx = n\,(n+1).$

**Example 15.** *Show that all the roots of $P_n(x) = 0$ are real and lie between $-1$ and $+1$.*

**Sol.** We have $\qquad P_n(x) = \dfrac{1}{2^n\,n!}\dfrac{d^n}{dx^n}(x^2-1)^n$

Now $\qquad (x^2-1)^n = (x-1)^n\,(n+1)^n.$

Hence $(x^2-1)^n$ vanishes $n$ times at $x = 1$ and $n$ times at $x = -1$. Therefore by the theory of equations, $\dfrac{d^n}{dx^n}(x^2-1)^n$ will have $n$ roots all real and lying between $-1$ and $+1$.

Hence $\qquad P_n(x) = \dfrac{1}{2^n\,n!}\dfrac{d^n}{dx^n}(x^2-1)^n$

has $n$ roots all real and lying between $-1$ and $+1$.

**Example 16.** *Prove that*

*(i)* $P_n'(1) = \frac{1}{2}\,n(n+1)$ *and* $\qquad$ *(ii)* $P_n'(-1) = (-1)^{n-1}.\frac{1}{2}\,n(n+1).$

**About the Book**

The book has been designed for the use of degree students of various universities. It will also be found useful by the students preparing for various competitive examinations.

Many new examples have been added in the book. A new chapter on integration in the series (Frobenius Method) enhances the utility of the book.

The author has endeavoured to present the fundamental concepts in a comprehensive and lucid manner.

**About the Author**

**N.P. Bali** is a prolific author of over 100 books for degree and engineering students. He has been writing books for nearly forty years.

His books on the following topics are well-known for their easy comprehension and lucid presentation:

Algebra, Trigonometry, Differential Calculus, Integral Calculus, Real Analysis , Co-ordinate Geometry, Statics, Dynamics, etc.

**Level :** Undergraduate, Graduate

**Price : Rs. 250.00**

ISBN 81-7008-939-5

9788170089391

FIREWALL MEDIA