Viewing and Transformation:

2D and 3D Geometric Transformations, 2D and 3D Viewing Transformations, Vanishing points.

[3]: Sections 2.1-2.21 (Pages 61-99),
Sections 3.1-3.17 (Pages 101-184)

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CHAPTER

TWO

TWO-DIMENSIONAL TRANSFORMATIONS

2–1 INTRODUCTION

We begin our study of the fundamentals of the mathematics underlying computer graphics by considering the representation and transformation of points and lines. Points and the lines which join them, along with an appropriate drawing algorithm, are used to represent objects or to display information graphically. The ability to transform these points and lines is basic to computer graphics. When visualizing an object, it may be desirable to scale, rotate, translate, distort or develop a perspective view of the object. All of these transformations can be accomplished using the mathematical techniques discussed in this and the next chapter.

2–2 REPRESENTATION OF POINTS

A point is represented in two dimensions by its coordinates. These two values are specified as the elements of a 1-row, 2-column matrix:

$$[x \quad y]$$

In three dimensions a 1×3 matrix

$$[x \quad y \quad z]$$

is used. Alternately, a point is represented by a 2-row, 1-column matrix

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

in two dimensions or by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

104

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in three dimensions. Row matrices like

$$[\, x \quad y \,]$$

or column matrices like

$$\begin{bmatrix} x \\ y \end{bmatrix}$$

are frequently called position vectors. In this book a row matrix formulation of the position vectors is used.

A series of points, each of which is a position vector relative to some coordinate system, is stored in a computer as a matrix or array of numbers. The position of these points is controlled by manipulating the matrix which defines the points. Lines are drawn between the points to generate lines, curves or pictures.

2–3 TRANSFORMATIONS AND MATRICES

Matrix elements can represent various quantities, such as a number store, a network or the coefficients of a set of equations. The rules of matrix algebra define allowable operations on these matrices (see Appendix B). Many physical problems lead to a matrix formulation. For models of physical systems, the problem is formulated as: given the matrices $[\,A\,]$ and $[\,B\,]$ find the solution matrix $[\,T\,]$, i.e., $[\,A\,][\,T\,] = [\,B\,]$. In this case the solution is $[\,T\,] = [\,A\,]^{-1}[\,B\,]$, where $[\,A\,]^{-1}$ is the inverse of the square matrix $[\,A\,]$ (see Ref. 2–1).

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An alternate interpretation is to treat the matrix $[\,T\,]$ as a geometric operator. Here matrix multiplication is used to perform a geometrical transformation on a set of points represented by the position vectors contained in $[\,A\,]$. The matrices $[\,A\,]$ and $[\,T\,]$ are assumed known. It is required to determine the elements of the matrix $[\,B\,]$. The interpretation of the matrix $[\,T\,]$ as a geometrical operator is the foundation of mathematical transformations useful in computer graphics.

2–4 TRANSFORMATION OF POINTS

Consider the results of the multiplication of a matrix $[\,x \quad y\,]$ containing the coordinates of a point P and a general 2×2 transformation matrix:

$$[\,X\,][\,T\,] = \underbrace{[\,x \quad y\,]}_{1\times 2}\underbrace{\overset{T}{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}}_{2\times 2} = \underbrace{[\,\underbrace{(ax + cy)}_{x^*} \quad \underbrace{(bx + dy)}_{y^*}\,]}_{1\times 2} = [\,x^* \quad y^*\,] \qquad (2-1)$$

This mathematical notation means that the initial coordinates x and y are transformed to x^* and y^*, where $x^* = (ax+cy)$ and $y^* = (bx+dy)$.† We are interested

†See Appendix B for the details of matrix multiplication.

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in the implications of considering x^* and y^* as the transformed coordinates of the point P. We begin by investigating several special cases.

Consider the case where $a = d = 1$ and $c = b = 0$. The transformation matrix $[\,T\,]$ then reduces to the identity matrix. Thus,

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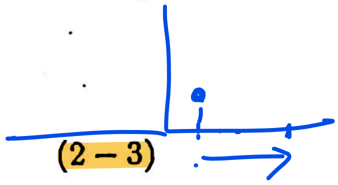
$$[\,X\,][\,T\,] = [\,x \quad y\,]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [\,x \quad y\,] = [\,x^* \quad y^*\,] \qquad (2-2)$$

and no change in the coordinates of the point P occurs. Since in matrix algebra multiplying by the identity matrix is equivalent to multiplying by 1 in ordinary algebra, this result is expected.

Next consider $d = 1,\ b = c = 0$, i.e.,

at y

$$[\,X\,][\,T\,] = [\,x \quad y\,]\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix} = [\,ax \quad y\,] = [\,x^* \quad y^*\,] \qquad (2-3)$$



which, since $x^* = ax$, produces a scale change in the x component of the position vector. The effect of this transformation is shown in Fig. 2–1a. Now consider $b = c = 0$, i.e.,

$$[\,X\,][\,T\,] = [\,x \quad y\,]\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} = [\,ax \quad dy\,] = [\,x^* \quad y^*\,] \qquad (2-4)$$

This yields a scaling of both the x and y coordinates of the original position vector P, as shown in Fig. 2–1b. If $a \neq d$, then the scalings are not equal. If $a = d > 1$, then a pure enlargement or scaling of the coordinates of P occurs. If $0 < a = d < 1$, then a compression of the coordinates of P occurs.

If a and/or d are negative, reflections through an axis or plane occur. To see this, consider $b = c = 0,\ d = 1$ and $a = -1$. Then

$$[\,X\,][\,T\,] = [\,x \quad y\,]\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = [\,-x \quad y\,] = [\,x^* \quad y^*\,] \qquad (2-5)$$

and a reflection through the y-axis results, as shown in Fig. 2–1c. If $b = c = 0,\ a = 1$, and $d = -1$, then a reflection through the x-axis occurs. If $b = c = 0,\ a = d < 0$, then a reflection through the origin occurs. This is shown in Fig. 2–1d, with $a = -1,\ d = -1$. Note that both reflection and scaling of the coordinates involve only the diagonal terms of the transformation matrix.

Now consider the effects of the off-diagonal terms. First consider $a = d = 1$ and $c = 0$. Thus,

$$[\,X\,][\,T\,] = [\,x \quad y\,]\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = [\,x \quad (bx + y)\,] = [\,x^* \quad y^*\,] \qquad (2-6)$$

Note that the x coordinate of the point P is unchanged, while y^* depends linearly on the original coordinates. This effect is called shear, as shown in Fig. 2–1e.

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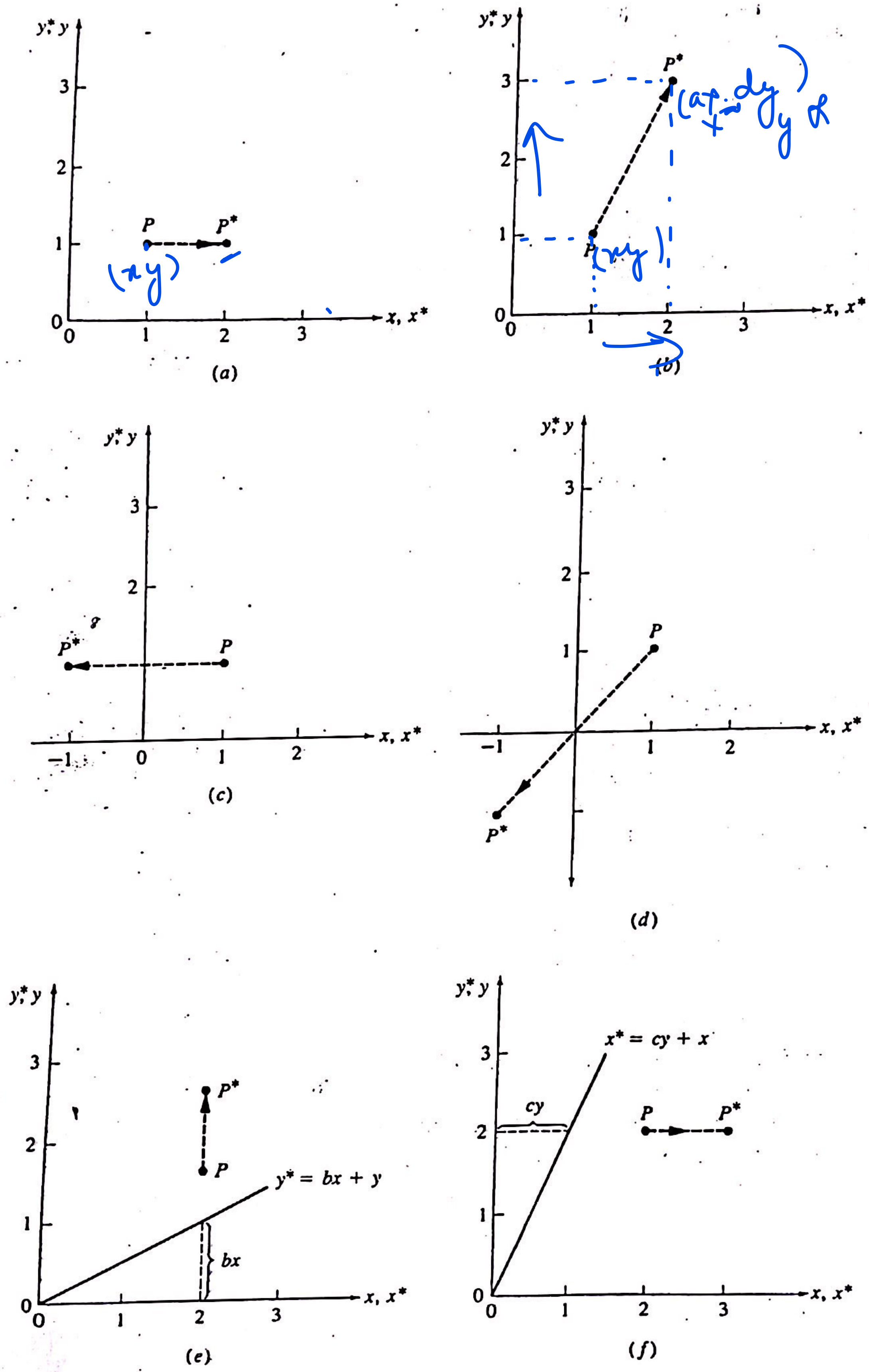

Figure 2–1 Transformation of points.

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Similarly, when $a = d = 1$, $b = 0$, the transformation produces shear proportional to the y coordinate, as shown in Fig. 2–1f. Thus, we see that the off-diagonal terms produce a shearing effect on the coordinates of the position vector for P.

Before completing our discussion of the transformation of points, consider the effect of the general 2 × 2 transformation given by Eq. (2–1) when applied to the origin, i.e.,

$$[\,x \quad y\,]\begin{bmatrix} a & b \\ c & d \end{bmatrix} = [\,(ax + cy) \quad (bx + dy)\,] = [\,x^* \quad y^*\,]$$

or for the origin,

$$[\,0 \quad 0\,]\begin{bmatrix} a & b \\ c & d \end{bmatrix} = [\,0 \quad 0\,] = [\,x^* \quad y^*\,]$$

Here we see that the origin is invariant under a general 2×2 transformation. This is a limitation which will be overcome by the use of homogeneous coordinates.

2–5 TRANSFORMATION OF STRAIGHT LINES

A straight line can be defined by two position vectors which specify the coordinates of its end points. The position and orientation of the line joining these two points can be changed by operating on these two position vectors. The actual operation of drawing a line between two points depends on the display device used. Here, we consider only the mathematical operations on the position vectors of the end points.

A straight line between two points A and B in a two-dimensional plane is drawn in Fig. 2–2. The position vectors of points A and B are $[\,A\,] = [\,0 \quad 1\,]$ and $[\,B\,] = [\,2 \quad 3\,]$, respectively. Now consider the transformation matrix

$$[\,T\,] = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} \tag{2–7}$$

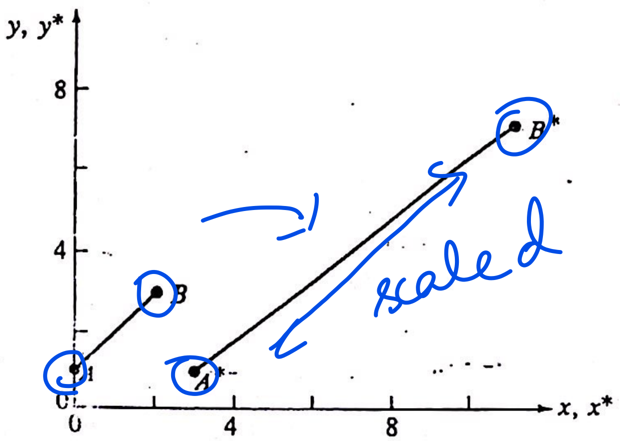

Figure 2–2 Transformation of straight lines.

106

which we recall from our previous discussion produces a shearing effect. Transforming the position vectors for A and B using $[\,T\,]$ produces new transformed position vectors A^* and B^* given by

$$[\,A\,][\,T\,] = [\,0 \quad 1\,]\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = [\,3 \quad 1\,] = [\,A^*\,] \tag{2-8}$$

and

$$[\,B\,][\,T\,] = [\,2 \quad 3\,]\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = [\,11 \quad 7\,] = [\,B^*\,] \tag{2-9}$$

Thus, the resulting coordinates for A^* are $x^* = 3$ and $y^* = 1$. Similarly, B^* is a new point with coordinates $x^* = 11$ and $y^* = 7$. More compactly the line AB may be represented by the 2×2 matrix

$$[\,L\,] = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}$$

Matrix multiplication by $[\,T\,]$ then yields

$$[\,L\,][\,T\,] = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 11 & 7 \end{bmatrix} = [\,L^*\,] \tag{2-10}$$

where the components of $[\,L^*\,]$ represent the transformed position vectors $[\,A^*\,]$ and $[\,B^*\,]$. The transformation of A to A^* and B to B^* is shown in Fig. 2–2. The initial axes are x, y and the transformed axes are x^*, y^*. Figure 2–2 shows that the shearing transformation $[\,T\,]$ increased the length of the line and changed its orientation.

2–6 MIDPOINT TRANSFORMATION

Figure 2–2 shows that the 2×2 transformation matrix (see Eq. 2–7) transforms the straight line $y = x + 1$, between points A and B, into another straight line $y = (3/4)x - 5/4$, between A^* and B^*. In fact a 2×2 matrix transforms any straight line into a second straight line. Points on the second line have a one-to-one correspondence with points on the first line. We have already shown this to be true for the end points of the line. To further confirm this we consider the transformation of the midpoint of the straight line between A and B. Letting

$$[\,A\,] = [\,x_1 \quad y_1\,] \qquad [\,B\,] = [\,x_2 \quad y_2\,] \qquad \text{and} \qquad [\,T\,] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

and transforming both end points simultaneously yields

$$\begin{bmatrix} A \\ B \end{bmatrix}[\,T\,] = \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} ax_1 + cy_1 & bx_1 + dy_1 \\ ax_2 + cy_2 & bx_2 + dy_2 \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \end{bmatrix} \tag{2-11}$$

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Hence, the end points of the transformed line A^*B^* are

$$[\,A^*\,] = [\,ax_1 + cy_1 \quad bx_1 + dy_1\,] = [\,x_1^* \quad y_1^*\,]$$

$$[\,B^*\,] = [\,ax_2 + cy_2 \quad bx_2 + dy_2\,] = [\,x_2^* \quad y_2^*\,] \qquad (2-12)$$

The midpoint of the transformed line A^*B^* calculated from the transformed end points is

$$[\,x_m^* \quad y_m^*\,] = \left[\frac{x_1^* + x_2^*}{2} \quad \frac{y_1^* + y_2^*}{2}\right]$$

$$= \left[\frac{(ax_1 + cy_1) + (ax_2 + cy_2)}{2} \quad \frac{(bx_1 + dy_1) + (bx_2 + dy_2)}{2}\right]$$

$$= \left[a\frac{(x_1 + x_2)}{2} + c\frac{(y_1 + y_2)}{2} \quad b\frac{(x_1 + x_2)}{2} + d\frac{(y_1 + y_2)}{2}\right]$$

$$(2-13)$$

Returning to the original line AB the midpoint is

$$[\,x_m \quad y_m\,] = \left[\frac{x_1 + x_2}{2} \quad \frac{y_1 + y_2}{2}\right] \qquad (2-14)$$

Using $[\,T\,]$ the transformation of the midpoint of AB is

$$[\,x_m \quad y_m\,][\,T\,] = \left[\frac{x_1 + x_2}{2} \quad \frac{y_1 + y_2}{2}\right]\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \left[a\frac{(x_1 + x_2)}{2} + c\frac{(y_1 + y_2)}{2} \quad b\frac{(x_1 + x_2)}{2} + d\frac{(y_1 + y_2)}{2}\right]$$

$$(2-15)$$

Comparing Eqs. (2–13) and (2–15) shows that they are identical. Consequently, the midpoint of the line AB transforms into the midpoint of the line A^*B^*. This process can be applied recursively to segments of the divided line. Thus, a one-to-one correspondence between points on the line AB and A^*B^* is assured.

Example 2–1 Midpoint of a Line

Consider the line AB shown in Fig. 2–2. The position vectors of the end points are

$$[\,A\,] = [\,0 \quad 1\,] \qquad [\,B\,] = [\,2 \quad 3\,]$$

The transformation

$$[\,T\,] = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}$$

yields the position vectors of the end points of the transformed line A^*B^* as

$$\begin{bmatrix} A \\ B \end{bmatrix}[\,T\,] = \begin{bmatrix} 0 & 1 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 11 & 7 \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \end{bmatrix}$$

107

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The midpoint of A^*B^* is

$$[\,x_m^* \quad y_m^*\,] = \left[\frac{3+11}{2} \quad \frac{1+7}{2}\right] = [\,7 \quad 4\,]$$

The midpoint of the original untransformed line AB is

$$[\,x_m \quad y_m\,] = \left[\frac{0+2}{2} \quad \frac{1+3}{2}\right] = [\,1 \quad 2\,]$$

Transforming this midpoint yields

$$[\,x_m \quad y_m\,][\,T\,] = [\,1 \quad 2\,]\begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix} = [\,7 \quad 4\,] = [\,x_m^* \quad y_m^*\,]$$

which is the same as our previous result.

For computer graphics applications these results show that any straight line can be transformed into any other straight line in any position by simply transforming its end points and redrawing the line between the end points.

2–7 TRANSFORMATION OF PARALLEL LINES

When a 2×2 matrix is used to transform a pair of parallel lines, the result is a second pair of parallel lines. To see this, consider a line between $[\,A\,] = [\,x_1 \quad y_1\,]$ and $[\,B\,] = [\,x_2 \quad y_2\,]$ and a line parallel to AB between E and F. To show that these lines and any transformation of them are parallel, examine the slopes of AB, EF, A^*B^* and E^*F^*. Since they are parallel, the slope of both AB and EF is

$$m = \frac{y_2 - y_1}{x_2 - x_1} \tag{2-16}$$

Transforming the end points of AB using a general 2×2 transformation yields the end points of A^*B^*:

$$\begin{aligned} \begin{bmatrix} A \\ B \end{bmatrix}[\,T\,] &= \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \begin{bmatrix} ax_1 + cy_1 & bx_1 + dy_1 \\ ax_2 + cy_2 & bx_2 + dy_2 \end{bmatrix} \\ &= \begin{bmatrix} x_1^* & y_1^* \\ x_2^* & y_2^* \end{bmatrix} = \begin{bmatrix} A^* \\ B^* \end{bmatrix} \end{aligned} \tag{2-17}$$

Using the transformed end points, the slope of A^*B^* is then

$$m^* = \frac{(bx_2 + dy_2) - (bx_1 + dy_1)}{(ax_2 + cy_2) - (ax_1 + cy_1)} = \frac{b(x_2 - x_1) + d(y_2 - y_1)}{a(x_2 - x_1) + c(y_2 - y_1)}$$

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or

$$m^* = \frac{b + d\dfrac{(y_2 - y_1)}{(x_2 - x_1)}}{a + c\dfrac{(y_2 - y_1)}{(x_2 - x_1)}} = \frac{b + dm}{a + cm} \qquad (2-18)$$

Since the slope m^* is independent of x_1, x_2, y_1 and y_2, and since m, a, b, c and d are the same for EF and AB, it follows that m^* is the same for both E^*F^* and A^*B^*. Thus, parallel lines remain parallel after transformation. This means that parallelograms transform into other parallelograms when operated on by a general 2×2 transformation matrix. These simple results begin to show the power of using matrix multiplication to produce graphical effects.

2–8 TRANSFORMATION OF INTERSECTING LINES

When a general 2×2 matrix is used to transform a pair of intersecting straight lines, the result is also a pair of intersecting straight lines. To see this consider a pair of lines, e.g., the dashed lines in Fig. 2–3, represented by

$$y = m_1 x + b_1$$

$$y = m_2 x + b_2$$

Reformulating these equations in matrix notation yields

$$[\,x \quad y\,]\begin{bmatrix} -m_1 & -m_2 \\ 1 & 1 \end{bmatrix} = [\,b_1 \quad b_2\,]$$

or

$$[\,X\,][\,M\,] = [\,B\,] \qquad (2-19)$$

If a solution to this pair of equations exists, then the lines intersect. If not, then they are parallel. A solution can be obtained by matrix inversion. Specifically,

$$[\,X_i\,] = [\,x_i \quad y_i\,] = [\,B\,][\,M\,]^{-1} \qquad (2-20)$$

The inverse of $[\,M\,]$ is

$$[\,M\,]^{-1} = \begin{bmatrix} \dfrac{1}{m_2 - m_1} & \dfrac{m_2}{m_2 - m_1} \\ \dfrac{-1}{m_2 - m_1} & \dfrac{-m_1}{m_2 - m_1} \end{bmatrix} \qquad (2-21)$$

since $[\,M\,][\,M\,]^{-1} = [\,I\,]$, the identity matrix. Hence, the intersection of the two lines is

$$[\,X_i\,] = [\,x_i \quad y_i\,] = [\,b_1 \quad b_2\,]\begin{bmatrix} \dfrac{1}{m_2 - m_1} & \dfrac{m_2}{m_2 - m_1} \\ \dfrac{-1}{m_2 - m_1} & \dfrac{-m_1}{m_2 - m_1} \end{bmatrix}$$

$$[\,X_i\,] = [\,x_i \quad y_i\,] = \left[\,\frac{b_1 - b_2}{m_2 - m_1} \quad \frac{b_1 m_2 - b_2 m_1}{m_2 - m_1}\,\right] \qquad (2-22)$$

108

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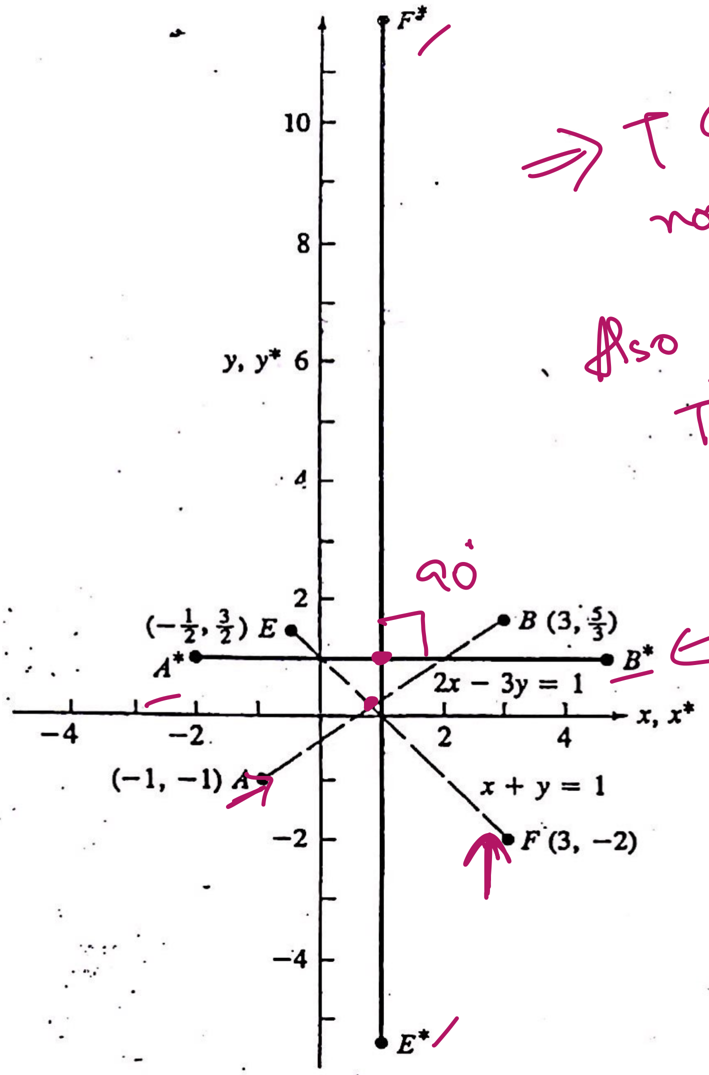

⇒ T converts a pair of intersecting non-⊥ lines into ⊥ lines

Also T^{-1} converts a pair of intersecting ⊥ lines into non-⊥ lines

Figure 2–3 Transformation of intersecting lines.

If these two lines are now transformed using a general 2×2 transformation matrix given by

$$[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then they have the form

$$y^* = m_1^* x^* + b_1^*$$
$$y^* = m_2^* x^* + b_2^*$$

It is relatively easy to show that

$$m_i^* = \frac{b + dm_i}{a + cm_i} \qquad (2-23)$$

and

$$b_i^* = b_i(d - cm_i^*) = b_i \frac{ad - bc}{a + cm_i} \qquad i = 1, 2 \qquad (2-24)$$

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The intersection of the transformed lines is obtained in the same manner as that for the untransformed lines. Thus,

$$[X_i^*] = [x_i^* \quad y_i^*]$$
$$= \begin{bmatrix} \dfrac{b_1^* - b_2^*}{m_2^* - m_1^*} & \dfrac{b_1^* m_2^* - b_2^* m_1^*}{m_2^* - m_1^*} \end{bmatrix}$$

Rewriting the components of the intersection point using Eqs. (2–23) and (2–24) yields

$$[X_i^*] = [x_i^* \quad y_i^*]$$
$$= \begin{bmatrix} \dfrac{a(b_1 - b_2) + c(b_1 m_2 - b_2 m_1)}{m_2 - m_1} & \dfrac{b(b_1 - b_2) + d(b_1 m_2 - b_2 m_1)}{m_2 - m_1} \end{bmatrix} \tag{2-25}$$

Returning now to the untransformed intersection point $[x_i \quad y_i]$ and applying the same general 2×2 transformation we have

$$[x_i^* \quad y_i^*] = [x_i \quad y_i][T]$$
$$= \begin{bmatrix} \dfrac{b_1 - b_2}{m_2 - m_1} & \dfrac{b_1 m_2 - b_2 m_1}{m_2 - m_1} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
$$= \begin{bmatrix} \dfrac{a(b_1 - b_2) + c(b_1 m_2 - b_2 m_1)}{m_2 - m_1} & \dfrac{b(b_1 - b_2) + d(b_1 m_2 - b_2 m_1)}{m_2 - m_1} \end{bmatrix} \tag{2-26}$$

Comparing Eqs. (2–25) and (2–26) shows that they are identical. Consequently, the intersection point transforms into the intersection point.

Example 2–2 Intersecting Lines

Consider the two dashed lines AB and EF shown in Fig. 2–3 with end points

$$[A] = [-1 \quad -1] \qquad [B] = [3 \quad 5/3]$$

and

$$[E] = [-1/2 \quad 3/2] \quad [F] = [3 \quad -2]$$

The equation of the line AB is $-(2/3)x + y = -(1/3)$ and of the line EF, $x + y = 1$. In matrix notation the pair of lines is represented by

$$[x \quad y]\begin{bmatrix} -2/3 & 1 \\ 1 & 1 \end{bmatrix} = [-1/3 \quad 1]$$

Using matrix inversion (see Eq. 2–21) the intersection of these lines is

$$[x_i \quad y_i] = [-1/3 \quad 1]\begin{bmatrix} -3/5 & -3/5 \\ 3/5 & 2/5 \end{bmatrix}$$
$$= [4/5 \quad 1/5]$$

109

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Now consider the transformation of these lines using

$$[T] = \begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix}$$

The resulting lines are shown as A^*B^* and E^*F^* in Fig. 2–3. In matrix form the equations of the transformed lines are

$$[x^* \quad y^*]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [1 \quad 1]$$

with intersection point at $[x_i^* \quad y_i^*] = [1 \quad 1]$.

Transforming the intersection point of the untransformed lines yields

$$[x_i^* \quad y_i^*] = [x_i \quad y_i][T]$$

$$= [4/5 \quad 1/5]\begin{bmatrix} 1 & 2 \\ 1 & -3 \end{bmatrix} = [1 \quad 1]$$

which is identical to the intersection point of the transformed lines.

Examination of Fig. 2–3 and Ex. 2–2 shows that the original pair of untransformed dashed lines AB and EF are *not* perpendicular. However, the transformed solid lines A^*B^* and E^*F^* *are* perpendicular. Thus, the transformation $[T]$ changed a pair of intersecting nonperpendicular lines into a pair of intersecting perpendicular lines. By implication, $[T]^{-1}$, the inverse of the transformation, changes a pair of intersecting perpendicular lines into a pair of intersecting nonperpendicular lines. This effect can have disastrous geometrical consequences. It is thus of considerable interest to determine under what conditions perpendicular lines transform into perpendicular lines. We will return to this question in Sec. 2–14 when a little more background has been presented.

Additional examination of Fig. 2–3 and Ex. 2–2 shows that the transformation $[T]$ involved a rotation, a reflection and a scaling. Let's consider each of these effects individually.

2–9 ROTATION

Consider the plane triangle ABC shown in Fig. 2–4. The triangle ABC is rotated through 90° about the origin in a counterclockwise sense by the transformation

$$[T] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

If we use a 3 × 2 matrix containing the x and y coordinates of the triangle's vertices, then

$$\begin{bmatrix} 3 & -1 \\ 4 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ -1 & 4 \\ -1 & 2 \end{bmatrix}$$

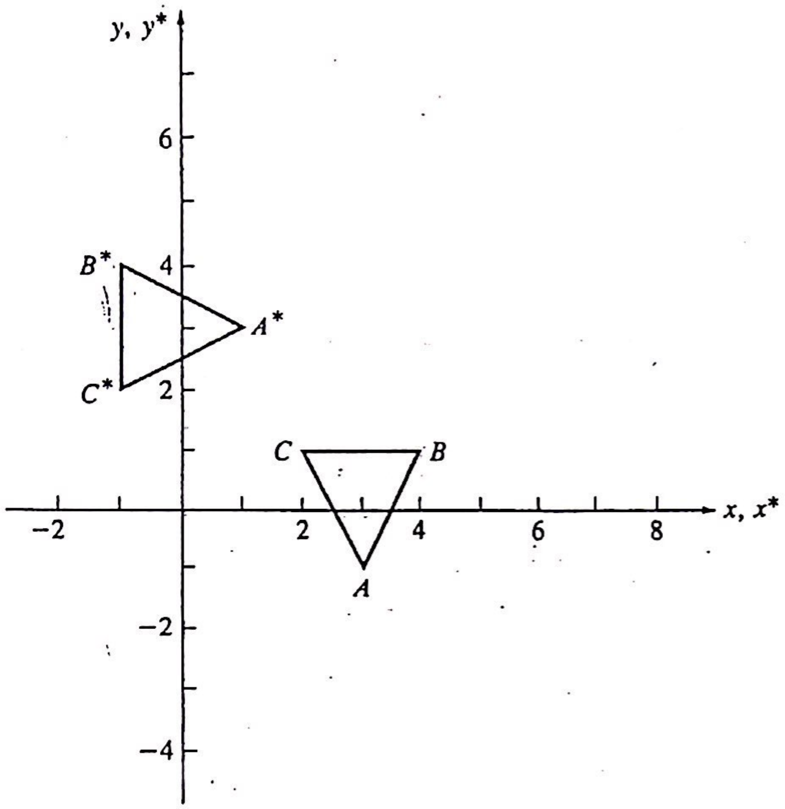

Figure 2–4 Rotation.

which produces the triangle $A^*B^*C^*$. A 180° rotation about the origin is obtained by using the transformation

$$[T]=\begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

and a 270° rotation about the origin by using

$$[T]=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

Of course, the identity matrix

$$[T]=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

corresponds to a rotation about the origin of either 0° or 360°. Note that neither scaling nor reflection has occurred in these examples.

These example transformations produce specific rotations about the origin: 0°, 90°, 180°, 270°. What about rotation about the origin by an arbitrary angle θ? To obtain this result consider the position vector from the origin to the point P shown in Fig. 2–5. The length of the vector is r at an angle ϕ to the x-axis. The position vector P is rotated about the origin by the angle θ to P^*.

110

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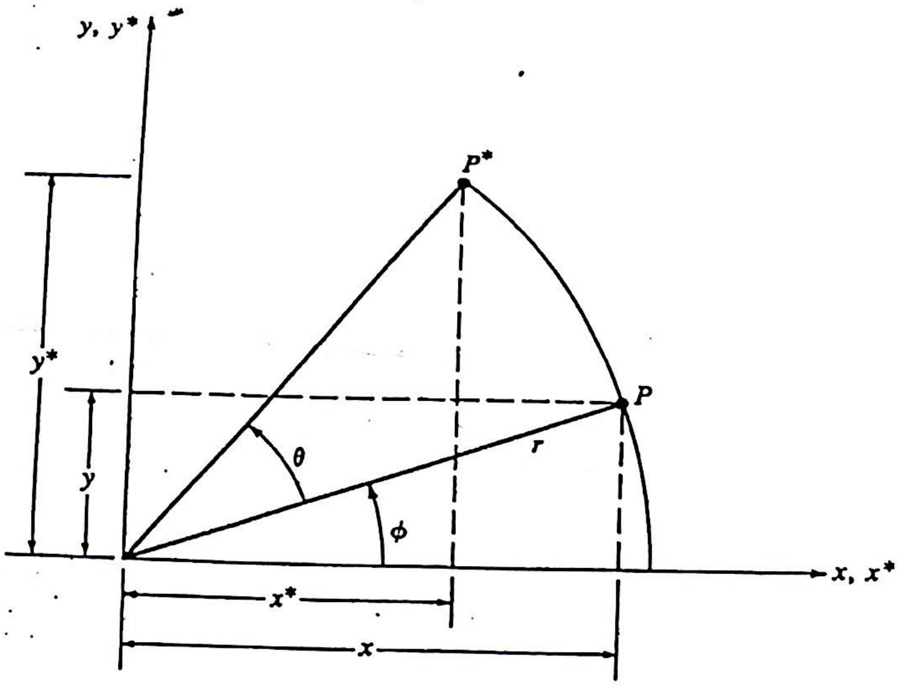

Figure 2–5 Rotation of a position vector.

Writing the position vectors for P and P^* we have

$$P = [\,x \quad y\,] = [\,r\cos\phi \quad r\sin\phi\,]$$

and

$$P^* = [\,x^* \quad y^*\,] = [\,r\cos(\phi+\theta) \quad r\sin(\phi+\theta)\,]$$

Using the sum of the angles formulas† allows writing P^* as

$$P^* = [\,x^* \quad y^*\,] = [\,r(\cos\phi\cos\theta - \sin\phi\sin\theta) \quad r(\cos\phi\sin\theta + \sin\phi\cos\theta)\,]$$

Using the definitions of x and y allows rewriting P^* as

$$P^* = [\,x^* \quad y^*\,] = [\,x\cos\theta - y\sin\theta \quad x\sin\theta + y\cos\theta\,]$$

Thus, the transformed point has components

$$x^* = x\cos\theta - y\sin\theta \qquad (2-27a)$$

$$y^* = x\sin\theta + y\cos\theta \qquad (2-27b)$$

In matrix form

$$[\,X^*\,] = [\,X\,][\,T\,] = [\,x^* \quad y^*\,]$$

$$= [\,x \quad y\,]\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad (2-28)$$

† $\cos(\phi \pm \theta) = \cos\phi\cos\theta \mp \sin\phi\sin\theta$
$\sin(\phi \pm \theta) = \cos\phi\sin\theta \pm \sin\phi\cos\theta$

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Thus, the transformation for a general rotation about the origin by an arbitrary angle θ is

$$[T] = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \qquad (2-29)$$

Rotations are positive counterclockwise about the origin, as shown in Fig. 2–5.

Evaluation of the determinant of the general rotation matrix yields

$$\det[T] = \cos^2\theta + \sin^2\theta = 1 \qquad (2-30)$$

In general, transformations with a determinant identically equal to +1 yield pure rotations.

Suppose now that we wish to rotate the point P^* back to P, i.e., perform the inverse transformation. The required rotation angle is obviously $-\theta$. From Eq. (2–29) the required transformation matrix is

$$[T]^{-1} = \begin{bmatrix} \cos(-\theta) & \sin(-\theta) \\ -\sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \qquad (2-31)$$

since $\cos(-\theta) = \cos\theta$ and $\sin(-\theta) = -\sin\theta$. $[T]^{-1}$ is a formal way of writing 'the inverse of' $[T]$. We can show that $[T]^{-1}$ is the inverse of $[T]$ by recalling that the product of a matrix and its inverse yields the identity matrix. Here,

$$[T][T]^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -\cos\theta\sin\theta + \cos\theta\sin\theta \\ -\cos\theta\sin\theta + \cos\theta\sin\theta & \cos^2\theta + \sin^2\theta \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [I]$$

where $[I]$ is the identity matrix.

Examining Eqs. (2–29) and (2–31) reveals another interesting and useful result. Recall that the transpose of a matrix is obtained by interchanging its rows and columns. Forming the transpose of $[T]$, i.e., $[T]^T$, and comparing it with $[T]^{-1}$, shows that

$$[T]^T = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = [T]^{-1} \qquad (2-32)$$

The inverse of the general rotation matrix $[T]$ is its transpose. Since formally determining the inverse of a matrix is more computationally expensive than determining its transpose, Eq. (2–32) is an important and useful result. In general, the inverse of any pure rotation matrix, i.e., one with a determinant identically equal to +1, is its transpose.†

†Such matrices are said to be orthogonal.

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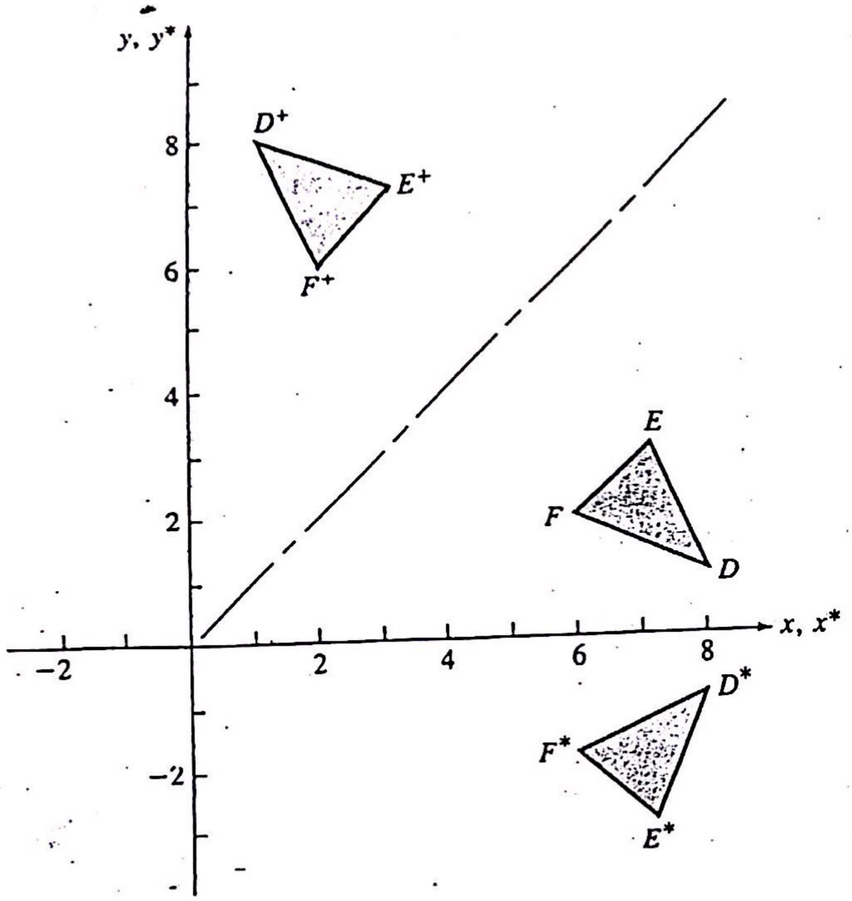

Figure 2–6 Reflection.

2–10 REFLECTION

Whereas a pure two-dimensional rotation in the xy plane occurs entirely in the two-dimensional plane about an axis normal to the xy plane, a reflection is a 180° rotation out into three space and back into two space about an axis in the xy plane. Two reflections of the triangle DEF are shown in Fig. 2–6. A reflection about $y = 0$, the x-axis, is obtained by using

$$[T] = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \tag{2-33}$$

In this case the new vertices $D^*E^*F^*$ for the triangle are given by

$$\begin{bmatrix} 8 & 1 \\ 7 & 3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 8 & -1 \\ 7 & -3 \\ 6 & -2 \end{bmatrix}$$

Similarly reflection about $x = 0$, the y-axis, is given by

$$[T] = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \tag{2-34}$$

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A reflection about the line $y = x$ occurs for

$$[T] = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{2-35}$$

The transformed, new vertices $D^+E^+F^+$ are given by

$$\begin{bmatrix} 8 & 1 \\ 7 & 3 \\ 6 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 8 \\ 3 & 7 \\ 2 & 6 \end{bmatrix}$$

Similarly, a reflection about the line $y = -x$ is given by

$$[T] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \tag{2-36}$$

Each of these reflection matrices has a determinant that is identically -1. In general, if the determinant of a transformation matrix is identically -1, then the transformation produces a pure reflection.

If two pure reflection transformations about lines passing through the origin are applied successively, the result is a pure rotation about the origin. To see this, consider the following example.

Example 2–3 Reflection and Rotation

Consider the triangle ABC shown in Fig. 2–7, first reflected about the x axis (see Eq. 2–33) and then about the line $y = -x$ (see Eq. 2–36). Specifically, the result of the reflection about the x-axis is

$$[X^*] = [X][T_1] = \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 5 & -2 \\ 4 & -3 \end{bmatrix}$$

Reflecting the triangle $A^*B^*C^*$ about the line $y = -x$ yields

$$[X^+] = [X^*][T_2] = \begin{bmatrix} 4 & -1 \\ 5 & -2 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 2 & -5 \\ 3 & -4 \end{bmatrix}$$

Rotation about the origin by an angle $\theta = 270°$ (see Eq. 2–29) yields the identical result, i.e.,

$$[X^+] = [X][T_3] = \begin{bmatrix} 4 & 1 \\ 5 & 2 \\ 4 & 3 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -4 \\ 2 & -5 \\ 3 & -4 \end{bmatrix}$$

112

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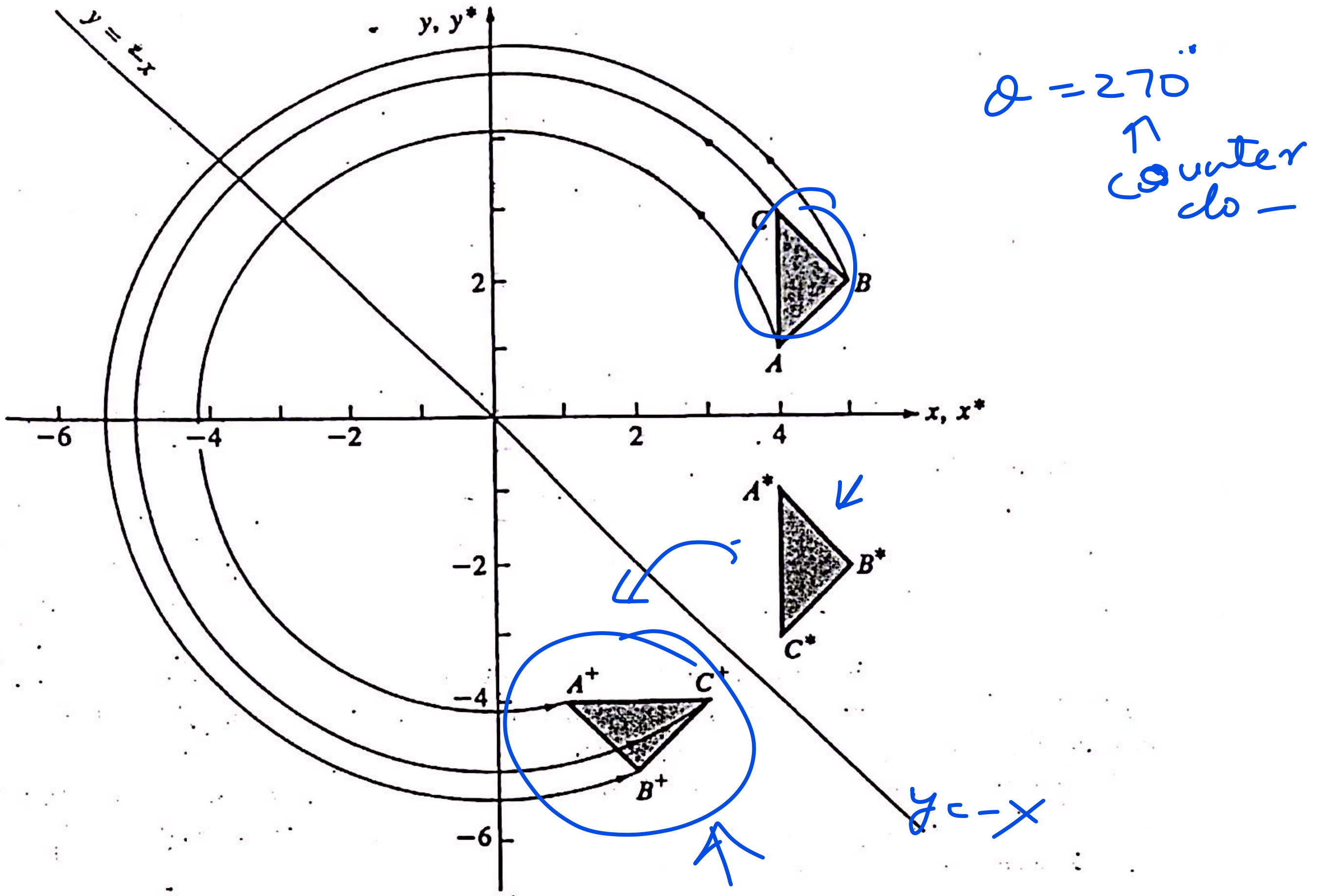

Figure 2–7 Combined reflections yield rotations.

Note that the reflection matrices given above in Eqs. (2–33) and (2–36) are orthogonal; i.e., the transpose is also the inverse. For example,

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^T = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}^{-1}$$

2–11 SCALING

Recalling our discussion of the transformation of points, we see that scaling is controlled by the magnitude of the two terms on the primary diagonal of the matrix. If the matrix

$$[T] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

is used as an operator on the vertices of a triangle, a '2-times' enlargement, or uniform scaling, occurs about the origin. If the magnitudes are unequal, a distortion occurs. These effects are shown in Fig. 2–8. Triangle ABC is transformed by

$$[T] = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

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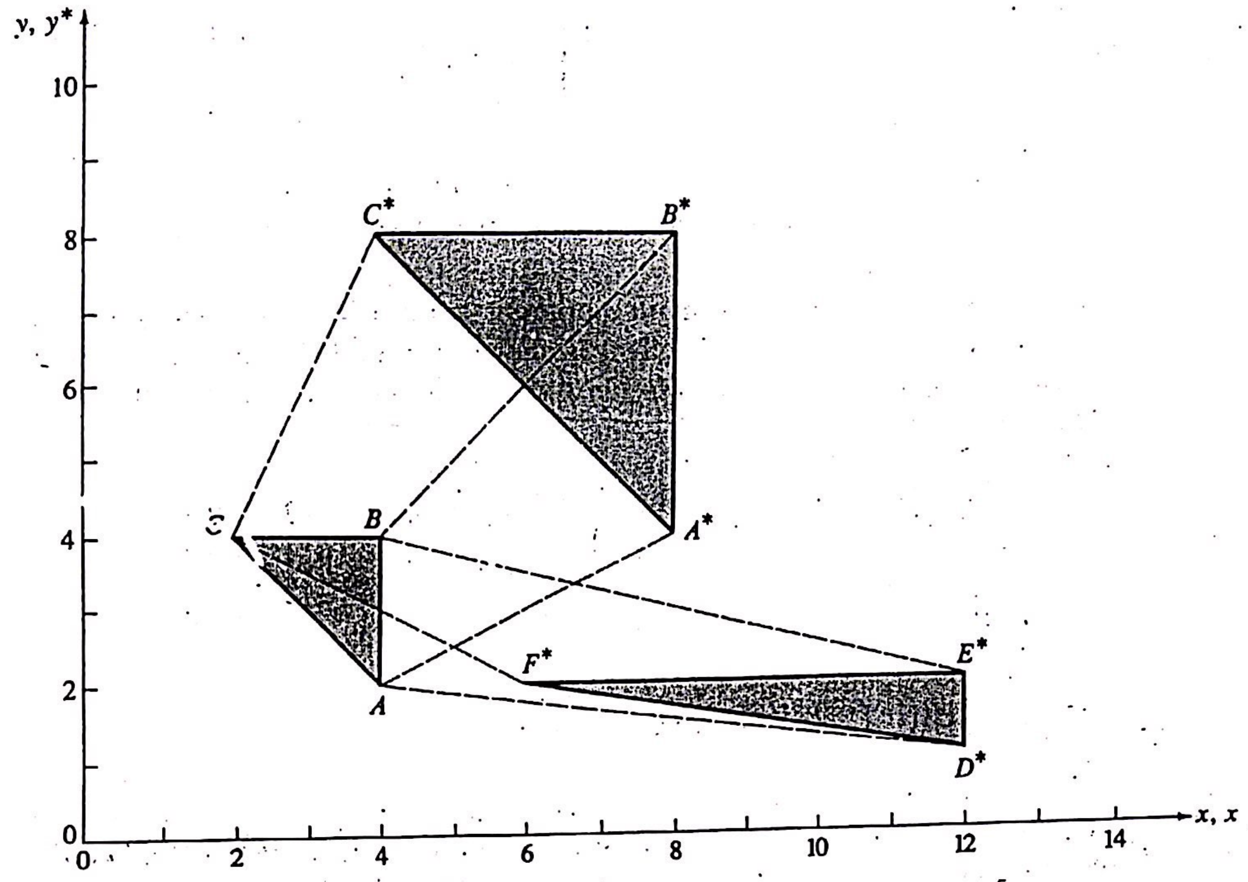

Figure 2–8 Uniform and nonuniform scaling or distortion.

to yield $A^*B^*C^*$ where a uniform scaling occurs. Transforming triangle ABC by

$$\begin{bmatrix} 1/2 & 0 \\ 0 & 3 \end{bmatrix}$$

to $D^*E^*F^*$ shows distortion due to the nonuniform scale factors.

In general, if

$$[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2-37}$$

with $a = d$, $b = c = 0$, a uniform scaling occurs; and if $a \neq d$, $b = c = 0$, a nonuniform scaling occurs. For a uniform scaling, if $a = d > 1$, a uniform expansion occurs; i.e., the figure gets larger. If $a = d < 1$, then a uniform compression occurs; i.e., the figure gets smaller. Nonuniform expansions and compressions occur, depending on whether a and d are individually > 1 or < 1.

Figure 2–8 also reveals what at first glance is an apparent translation of the transformed triangles. This apparent translation is easily understood if we recall that the *position vectors*, not the *points*, are scaled with respect to the origin. To see this more clearly examine the transformation of ABC to $D^*E^*F^*$ more closely. Specifically,

113

$$[X^*] = [X][T] = \begin{bmatrix} 4 & 2 \\ 4 & 4 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 6 \\ 2 & 12 \\ 1 & 12 \end{bmatrix}$$

Note that each of the x components of the *position vectors* of DEF is increased by a scale factor of 3 and the y components of the *position vectors* by a scale factor of 2.

To obtain a pure scaling without apparent translation, the centroid of the figure must be at the origin. This effect is shown in Fig. 2–9, where the triangle ABC with the centroid coordinates (1/3 the base and 1/3 the height) at the origin is scaled by a factor of 2. Specifically,

$$[X^*] = [X][T] = \begin{bmatrix} -1 & -1 \\ 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 4 & -2 \\ -2 & 4 \end{bmatrix}$$

2–12 COMBINED TRANSFORMATIONS

The power of the matrix methods described in the previous sections is clear. By performing matrix operations on the position vectors which define the vertices, the shape and position of the surface can be controlled. However, a desired orientation may require more than one transformation. Since matrix multiplication is noncommutative, the order of application of the transformations is important.

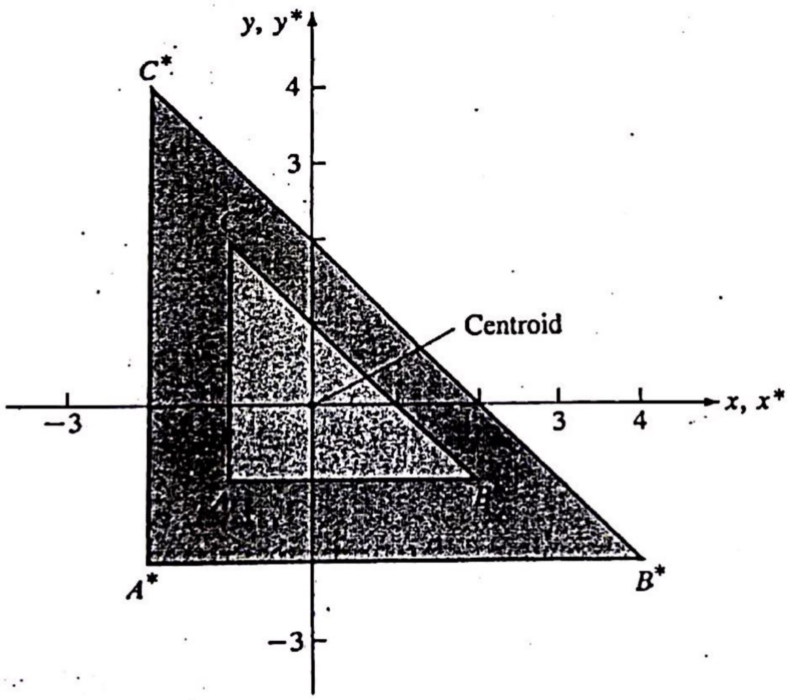

Figure 2–9 Uniform scaling without apparent translation.

In order to illustrate the effect of noncommutative matrix multiplication, consider the operations of rotation and reflection on the position vector $[x \quad y]$. If a 90° rotation, $[T_1]$, is followed by reflection through the line $y = -x$, $[T_2]$, these two consecutive transformations give

$$[X'] = [X][T_1] = [x \quad y]\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = [-y \quad x]$$

and then

$$[X^*] = [X'][T_2] = [-y \quad x]\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = [-x \quad y]$$

On the other hand, if reflection is followed by rotation, the results given by

$$[X'] = [X][T_2] = [x \quad y]\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = [-y \quad -x]$$

and

$$[X^*] = [X'][T_1] = [-y \quad -x]\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = [x \quad -y]$$

are obtained. The results are different, confirming that the order of application of matrix transformations is important.

Another important point is illustrated by the above results and by the example given below. Above, the individual transformation matrices were successively applied to the successively obtained position vectors, e.g.,

$$[x \quad y][T_1] \rightarrow [x' \quad y']$$

and

$$[x' \quad y'][T_2] \rightarrow [x^* \quad y^*]$$

In the example below the individual transformations are first combined or concatenated and then the *concatenated* transformation is applied to the original position vector, e.g., $[T_1][T_2] \rightarrow [T_3]$ and $[x \quad y][T_3] \rightarrow [x^* \quad y^*]$.

Example 2–4 Combined Two-Dimensional Transformations

Consider the triangle ABC shown in Fig. 2–10. The two transformations are a +90° rotation about the origin:

$$[T_1] = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

and a reflection through the line $y = -x$

$$[T_2] = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

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114

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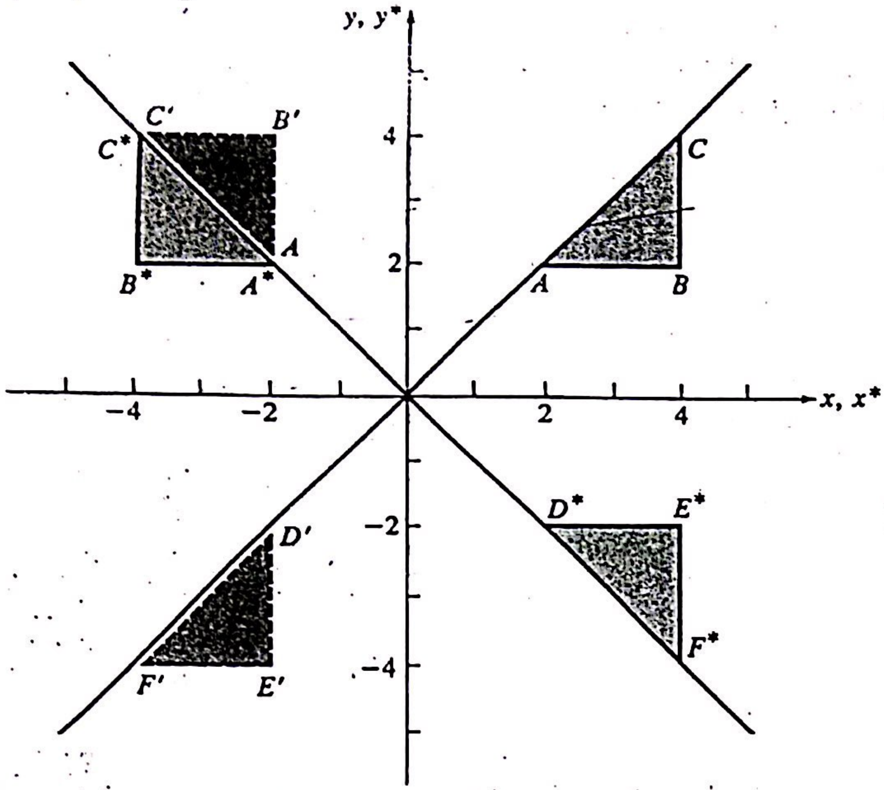

Figure 2–10 Combined two-dimensional transformations.

The effect of the combined transformation $[T_3] = [T_1][T_2]$ on the triangle ABC is

$$[X^*] = [X][T_1][T_2] = [X][T_3]$$

or

$$\begin{bmatrix} 2 & 2 \\ 4 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -4 & 2 \\ -4 & 4 \end{bmatrix}$$

The final result is shown as $A^*B^*C^*$ and the intermediate result as $A'B'C'$ in Fig. 2–10.

Reversing the order of application of the transformations yields

$$[X^*] = [X][T_2][T_1] = [X][T_4]$$

or

$$\begin{bmatrix} 2 & 2 \\ 4 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 2 \\ 4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & -2 \\ 4 & -4 \end{bmatrix}$$

The final result is shown as $D^*E^*F^*$ and the intermediate result as $D'E'F'$ in Fig. 2–10.

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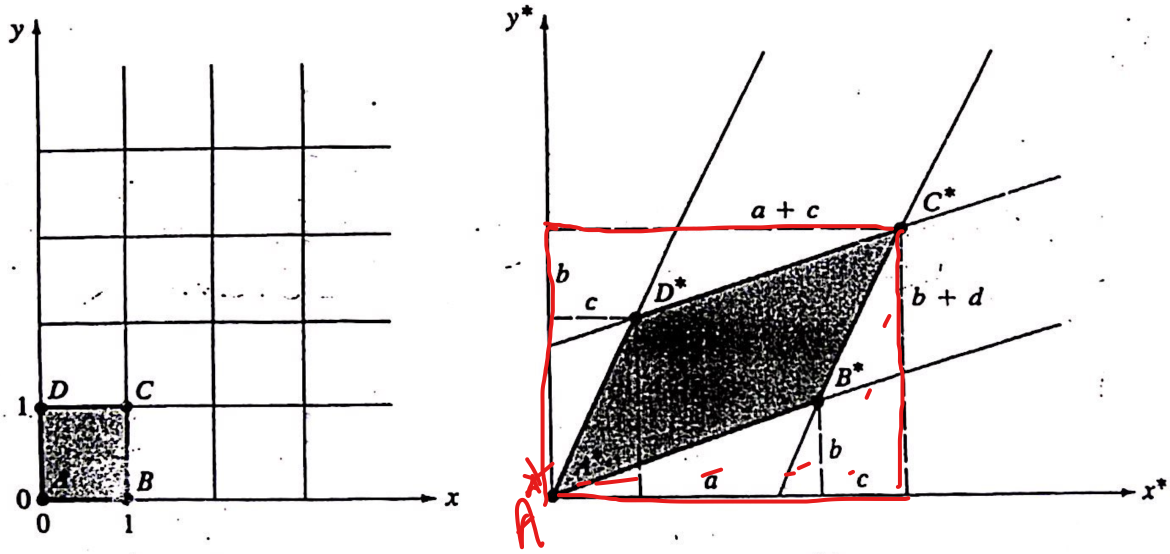
The results are different, again confirming that the order of application of the transformations is important. Note also that $\det[T_3] = -1$ and $\det[T_4] = -1$, indicating that both results can be obtained by a single reflection. $A^*B^*C^*$ can be obtained from ABC by reflection through the y-axis (see $[T_3]$ and Eq. 2–34). $D^*E^*F^*$ can be obtained from ABC by reflection through the x-axis (see $[T_4]$ and Eq. 2–33).

2–13 TRANSFORMATION OF THE UNIT SQUARE

So far we have concentrated on the behavior of points and lines to determine the effect of simple matrix transformations. However, the matrix is correctly considered to operate on *every* point in the plane. As has been shown, the only point that remains invariant under a 2×2 matrix transformation is the origin. All other points within the plane are transformed. This transformation may be interpreted as a stretching of the original plane and coordinate system into a new shape. More formally, we say that the transformation causes a mapping from one coordinate space into a second.

Consider a square-grid network consisting of unit squares in the xy plane as shown in Fig. 2–11. The four position vectors of a unit square with one corner at the origin of the coordinate system are

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{array}{l} \text{origin of the coordinates} - A \\ \text{unit point on the } x\text{-axis} - B \\ \text{outer corner} - C \\ \text{unit point on the } y\text{-axis} - D \end{array}$$



(a) (b)

Figure 2–11 General transformation of unit square. (a) Before transformation; (b) after transformation.

$[X][T] = [X^*]$ 115

$A(X^*) = A(X) * \det[T]$

This unit square is shown in Fig. 2–11a. Application of a general 2 × 2 matrix transformation to the unit square yields

$$\begin{array}{c} A \\ B \\ C \\ D \end{array}\begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ a & b \\ a+c & b+d \\ c & d \end{bmatrix}\begin{array}{c} A^* \\ B^* \\ C^* \\ D^* \end{array} \tag{2-38}$$

The results of this transformation are shown in Fig. 2–11b. First notice from Eq. (2–38) that the origin is not affected by the transformation, i.e., $[A] = [A^*] = [0 \quad 0]$. Further, notice that the coordinates of B^* are equal to the first row in the general transformation matrix, and the coordinates of D^* are equal to the second row in the general transformation matrix. Thus, once the coordinates of B^* and D^* (the transformed unit vectors $[1 \quad 0]$ and $[0 \quad 1]$, respectively) are known, the general transformation matrix is determined. Since the sides of the unit square are originally parallel, and since we have previously shown that parallel lines transform into parallel lines, the transformed figure is a parallelogram.

The effect of the terms a, b, c and d in the 2×2 matrix can be identified separately. The terms b and c cause a shearing (see Sec. 2–4) of the initial square in the y and x directions, respectively, as can be seen in Fig. 2–11b. The terms a and d act as scale factors, as noted earlier. Thus, the general 2×2 matrix produces a combination of shearing and scaling.

It is also possible to easily determine the area of $A^*B^*C^*D^*$, the parallelogram shown in Fig. 2–11b. The area within the parallelogram can be calculated as follows:

$$A_p = (a+c)(b+d) - \frac{1}{2}(ab) - \frac{1}{2}(cd) - \frac{c}{2}(b+b+d) - \frac{b}{2}(c+a+c)$$

which yields

$$A_p = ad - bc = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{2-39}$$

It can be shown that the area of any parallelogram A_p, formed by transforming a square, is a function of the transformation matrix determinant and is related to the area of the initial square A_s by the simple relationship

$$A_p = A_s(ad - bc) = A_s \det[T] \tag{2-40}$$

In fact, since the area of a general figure is the sum of unit squares, the area of any transformed figure A_t is related to the area of the initial figure A_i by

$$A_t = A_i(ad - bc) \tag{2-41}$$

This is a useful technique for determining the areas of arbitrary shapes.

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Example 2–5 Area Scaling

The triangle ABC with position vectors $[1 \quad 0]$, $[0 \quad 1]$ and $[-1 \quad 0]$, is transformed by

$$[T] = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$$

to create a second triangle $A^*B^*C^*$ as shown in Fig. 2–12.

The area of the triangle ABC is

$$A_i = \frac{1}{2}(base)(height) = \frac{1}{2}(2)(1) = 1$$

Using Eq. (2–41) the area of the transformed triangle $A^*B^*C^*$ is

$$A_t = A_i(ad - bc) = 1(6 + 2) = 8$$

Now the vertices of the transformed triangle $A^*B^*C^*$ are

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 2 \\ -3 & -2 \end{bmatrix}$$

Calculating the area from the transformed vertices yields

$$A_t = \frac{1}{2}(base)(height) = \frac{1}{2}(4)(4) = 8$$

which confirms the previous result.

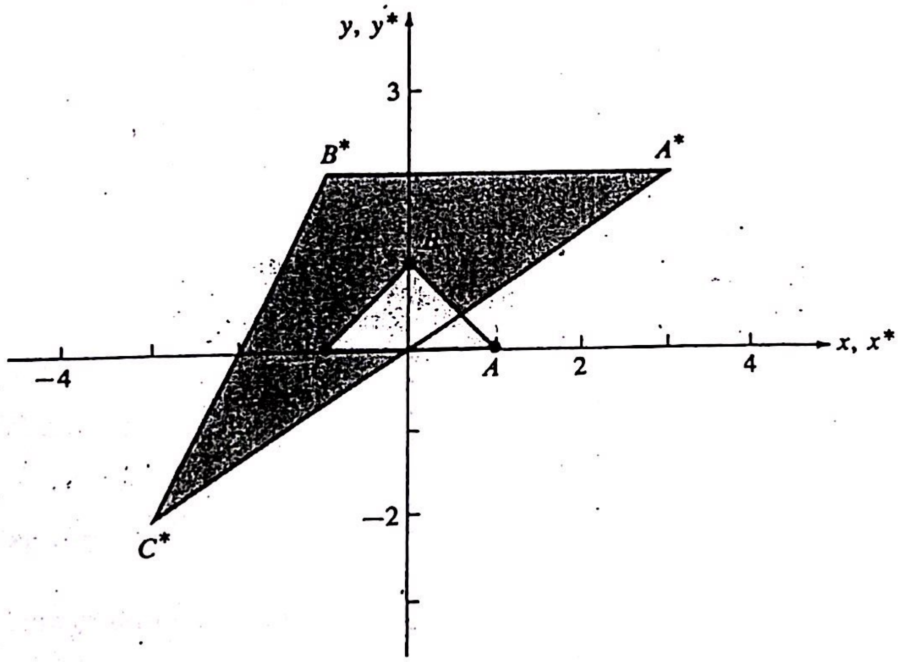

Figure 2–12 Area scaling.

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2–14 SOLID BODY TRANSFORMATIONS

We now return to the question posed in Sec. 2–8, i.e., when do perpendicular lines transform as perpendicular lines? First consider the somewhat more general question of when is the angle between intersecting lines preserved?

Recall that the dot or scalar product of two vectors is

$$\bar{V}_1 \cdot \bar{V}_2 = V_{1x}V_{2x} + V_{1y}V_{2y} = |\bar{V}_1|\,|\bar{V}_2|\cos\theta \tag{2-42}$$

and the cross product of two vectors confined to the two-dimensional xy plane is

$$\bar{V}_1 \times \bar{V}_2 = (V_{1x}V_{2y} - V_{2x}V_{1y})\bar{k} = |\bar{V}_1||\bar{V}_2|\bar{k}\sin\theta \tag{2-43}$$

where the subscripts x, y refer to the x and y components of the vector, θ is the acute angle between the vectors and $\bar{k}$ is the unit vector perpendicular to the xy plane.

Transforming $\bar{V}_1$ and $\bar{V}_2$ using a general 2×2 transformation yields

$$\begin{bmatrix} \bar{V}_1 \\ \bar{V}_2 \end{bmatrix} [T] = \begin{bmatrix} V_{1x} & V_{1y} \\ V_{2x} & V_{2y} \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} aV_{1x} + cV_{1y} & bV_{1x} + dV_{1y} \\ aV_{2x} + cV_{2y} & bV_{2x} + dV_{2y} \end{bmatrix} = \begin{bmatrix} \bar{V}_1^* \\ \bar{V}_2^* \end{bmatrix} \tag{2-44}$$

The cross product of $\bar{V}_1^*$ and $\bar{V}_2^*$ is

$$\bar{V}_1^* \times \bar{V}_2^* = (ad - cb)(V_{1x}V_{2y} - V_{2x}V_{1y})\bar{k} = |\bar{V}_1^*||\bar{V}_2^*|\bar{k}\sin\theta \tag{2-45}$$

Similarly the scalar product is

$$\begin{aligned} \bar{V}_1^* \cdot \bar{V}_2^* &= (a^2 + b^2)V_{1x}V_{2x} + (c^2 + d^2)V_{1y}V_{2y} + (ac + bd)(V_{1x}V_{2y} + V_{1y}V_{2x}) \\ &= |\bar{V}_1^*||\bar{V}_2^*|\cos\theta \end{aligned} \tag{2-46}$$

Requiring that the magnitude of the vectors, as well as the angle between them, remains unchanged, comparing Eqs. (2–42) and (2–46) and Eqs. (2–43) and (2–45) and equating coefficients of like terms yields

$$a^2 + b^2 = 1 \tag{2-47a}$$

$$c^2 + d^2 = 1 \tag{2-47b}$$

$$ac + bd = 0 \tag{2-47c}$$

$$ad - bc = +1 \tag{2-48}$$

Equations (2–47a, b, c) correspond to the conditions that a matrix be orthogonal, i.e.,

$$[T][T]^{-1} = [T][T]^T = [I]$$

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or

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Equation (2–48) requires that the determinant of the transformation matrix be +1.

Thus, the angles between intersecting lines are preserved by pure rotation. Since reflective transformations are also orthogonal with a determinant of -1, these results are easily extended. In this case the magnitude of the vectors is preserved, but the angle between the transformed vectors is technically $2\pi - \theta$. Hence, the angle is technically not preserved. Still, perpendicular lines transform as perpendicular lines. Since $\sin(2\pi - \theta) = -\sin\theta$, $ad - bc = -1$. Pure rotations and reflections are called rigid body transformations. In addition, a few minutes' thought or experimentation reveals that uniform scalings also preserve the angle between intersecting lines but not the magnitudes of the transformed vectors.†

2–15 TRANSLATIONS AND HOMOGENEOUS COORDINATES

A number of transformations governed by the general 2×2 transformation matrix, e.g., rotation, reflection, scaling, shearing etc., were discussed in the previous sections. As noted previously, the origin of the coordinate system is invariant with respect to all of these transformations. However, it is necessary to be able to modify the position of the origin, i.e., to transform every point in the two-dimensional plane. This can be accomplished by translating the origin or any other point in the two-dimensional plane, i.e.,

$$x^* = ax + cy + m$$
$$y^* = bx + dy + n$$

Unfortunately, it is not possible to introduce the constants of translation m, n into the general 2×2 transformation matrix; there is no room!

This difficulty can be overcome by introducing homogeneous coordinates. The homogeneous coordinates of a nonhomogeneous position vector $[\,x \quad y\,]$ are $[\,x' \quad y' \quad h\,]$ where $x = x'/h$ and $y = y'/h$ and h is any real number. Note that $h = 0$ has special meaning. One set of homogeneous coordinates is always of the form $[\,x \quad y \quad 1\,]$. We choose this form to represent the position vector $[\,x \quad y\,]$ in the physical xy plane. All other homogeneous coordinates are of the form $[\,hx \quad hy \quad h\,]$. There is no unique homogeneous coordinate representation, e.g., $[\,6 \quad 4 \quad 2\,]$, $[\,12 \quad 8 \quad 4\,]$, $[\,3 \quad 2 \quad 1\,]$ all represent the physical point $(3, 2)$.

The general transformation matrix is now 3×3. Specifically,

$$[T] = \begin{bmatrix} a & b & 0 \\ c & d & 0 \\ m & n & 1 \end{bmatrix} \qquad (2-49)$$

†Since an orthogonal matrix preserves *both* the angle between the vectors and their magnitudes, the uniform scaling transformation matrix is *not* orthogonal.

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where the elements a, b, c, d of the upper left 2×2 submatrix have exactly the same effects revealed by our previous discussions. m, n are the translation factors in the x and y directions, respectively. The pure two-dimensional translation matrix is

$$[x^* \quad y^* \quad 1] = [x \quad y \quad 1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & n & 1 \end{bmatrix} = [x+m \quad y+n \quad 1] \qquad (2-50)$$

Notice that now every point in the two-dimensional plane, even the origin ($x = y = 0$), can be transformed.

2–16 ROTATION ABOUT AN ARBITRARY POINT

Previously we have considered rotations as occurring about the origin. Homogeneous coordinates provide a mechanism for accomplishing rotations about points other than the origin. In general, a rotation about an arbitrary point can be accomplished by first translating the point to the origin, performing the required rotation, and then translating the result back to the original center of rotation. Thus, rotation of the position vector $[x \quad y \quad 1]$ about the point m, n through an arbitrary angle can be accomplished by

$$[x^* \quad y^* \quad 1] = [x \quad y \quad 1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -m & -n & 1 \end{bmatrix} \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ m & n & 1 \end{bmatrix} \qquad (2-51)$$

By carrying out the two interior matrix products we can write

$$[x^* \quad y^* \quad 1] = [x \quad y \quad 1] \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ \left\{ \begin{matrix} -m(\cos\theta - 1) \\ +n\sin\theta \end{matrix} \right\} & \left\{ \begin{matrix} -n(\cos\theta - 1) \\ -m\sin\theta \end{matrix} \right\} & 1 \end{bmatrix} \qquad (2-52)$$

An example illustrates this result.

Example 2–6 Rotation About an Arbitrary Point.

Suppose the center of an object is at $[4 \quad 3]$ and it is desired to rotate the object 90° counterclockwise about its center. Using the matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

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causes a rotation about the origin, not the object center. The necessary procedure is to first translate the object so that the desired center of rotation is at the origin by using the translation matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix}$$

Next apply the rotation matrix, and finally translate the results of the rotation back to the original center by means of the inverse translation matrix. The entire operation

$$[x^* \quad y^* \quad 1] = [x \quad y \quad 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & -3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 4 & 3 & 1 \end{bmatrix}$$

can be combined into one matrix operation by concatenating the transformation matrices, i.e.,

$$[x^* \quad y^* \quad 1] = [x \quad y \quad 1]\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 7 & -1 & 1 \end{bmatrix}$$

2–17 REFLECTION THROUGH AN ARBITRARY LINE

Previously (see Sec. 2–10) reflection through lines that passed through the origin was discussed. Occasionally reflection of an object through a line that does not pass through the origin is required. This can be accomplished using a procedure similar to that for rotation about an arbitrary point. Specifically,

Translate the line and the object so that the line passes through the origin.

Rotate the line and the object about the origin until the line is coincident with one of the coordinate axes.

Reflect through the coordinate axis.

Apply the inverse rotation about the origin.

Translate back to the original location.

In matrix notation the resulting concatenated matrix is

$$[T] = [T'][R][R'][R]^{-1}[T']^{-1} \qquad (2-53)$$

where

T' is the translation matrix
R is the rotation matrix about the origin
R' is the reflection matrix

The translations, rotations and reflections are also applied to the figure to be transformed. An example is given below.

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Example 2–7 Reflection Through an Arbitrary Line

Consider the line L and the triangle ABC shown in Fig. 2–13a. The equation of the line L is

$$y = \frac{1}{2}(x+4)$$

The position vectors $[2 \quad 4 \quad 1]$, $[4 \quad 6 \quad 1]$ and $[2 \quad 6 \quad 1]$ describe the vertices of the triangle ABC.

The line L will pass through the origin by translating it -2 units in the y direction. The resulting line can be made coincident with the x-axis by rotating it by $-\tan^{-1}(\frac{1}{2}) = -26.57°$ about the origin. Equation (2–33) is then used to reflect the triangle through the x-axis. The transformed position vectors of the triangle are then rotated and translated back to the original orientation. The combined transformation is

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \times$$

$$\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

$$[T] = \begin{bmatrix} 3/5 & 4/5 & 0 \\ 4/5 & -3/5 & 0 \\ -8/5 & 16/5 & 1 \end{bmatrix}$$

and the transformed position vectors for the triangle $A^*B^*C^*$ are

$$\begin{bmatrix} 2 & 4 & 1 \\ 4 & 6 & 1 \\ 2 & 6 & 1 \end{bmatrix} \begin{bmatrix} 3/5 & 4/5 & 0 \\ 4/5 & -3/5 & 0 \\ -8/5 & 16/5 & 1 \end{bmatrix} = \begin{bmatrix} 14/5 & 12/5 & 1 \\ 28/5 & 14/5 & 1 \\ 22/5 & 6/5 & 1 \end{bmatrix}$$

as shown in Fig. 2–13a. Figures 2–13b through 2–13e show the various steps in the transformation.

2–18 PROJECTION – A GEOMETRIC INTERPRETATION OF HOMOGENEOUS COORDINATES

The general 3×3 transformation matrix for two-dimensional homogeneous coordinates can be subdivided into four parts:

$$[T] = \left[\begin{array}{cc:c} a & b & p \\ c & d & q \\ \hdashline m & n & s \end{array}\right] \qquad (2-54)$$

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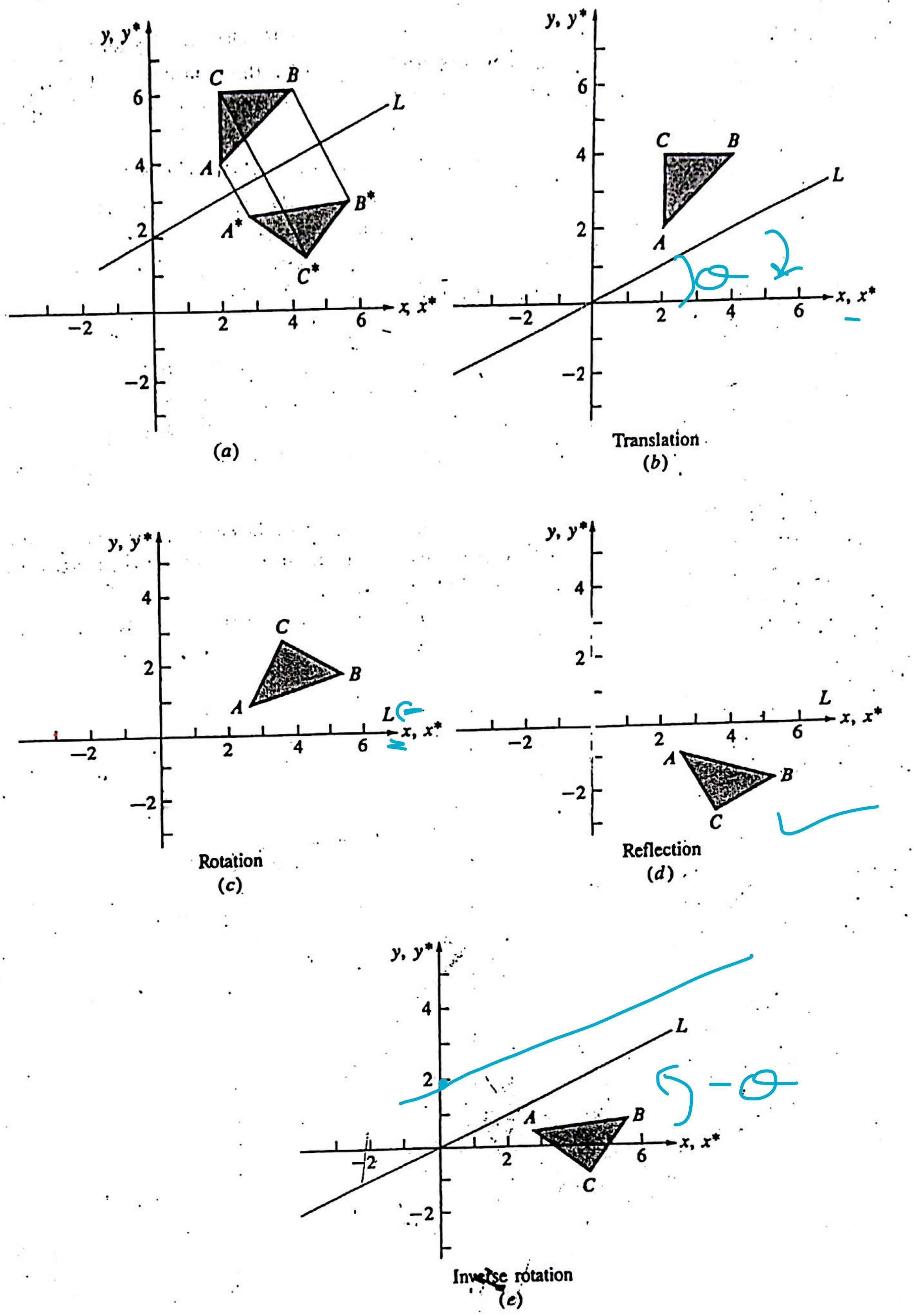

Figure 2–13 Reflection through an arbitrary line. (a) Original and final position; (b) translate line through origin; (c) rotate line to x-axis; (d) reflect about x-axis; (e) undo rotation; (a) undo translation.

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Recall that a, b, c and d produce scaling, rotation, reflection and shearing; and m and n produce translation. In the previous two sections $p = q = 0$ and $s = 1$. Suppose p and q are not zero. What are the effects? A geometric interpretation is useful.

When $p = q = 0$ and $s = 1$, the homogeneous coordinate of the transformed position vectors is always $h = 1$. Geometrically this result is interpreted as confining the transformation to the $h = 1$ physical plane.

To show the effect of $p \neq 0$, $q \neq 0$ in the third column in the general 3×3 transformation matrix, consider the following:

$$[X \quad Y \quad h] = [hx \quad hy \quad h] = [x \quad y \quad 1]\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & q \\ 0 & 0 & 1 \end{bmatrix}$$

$$= [x \quad y \quad (px + qy + 1)] \tag{2-55}$$

Here $X = hx$, $Y = hy$ and $h = px + qy + 1$. The transformed position vector expressed in homogeneous coordinates now lies in a plane in three-dimensional space defined by $h = px + qy + 1$. This transformation is shown in Fig. 2–14, where the line AB in the physical $(h = 1)$ plane is transformed to the line CD in the $h \neq 1$ plane, i.e., $pX + qY - h + 1 = 0$.

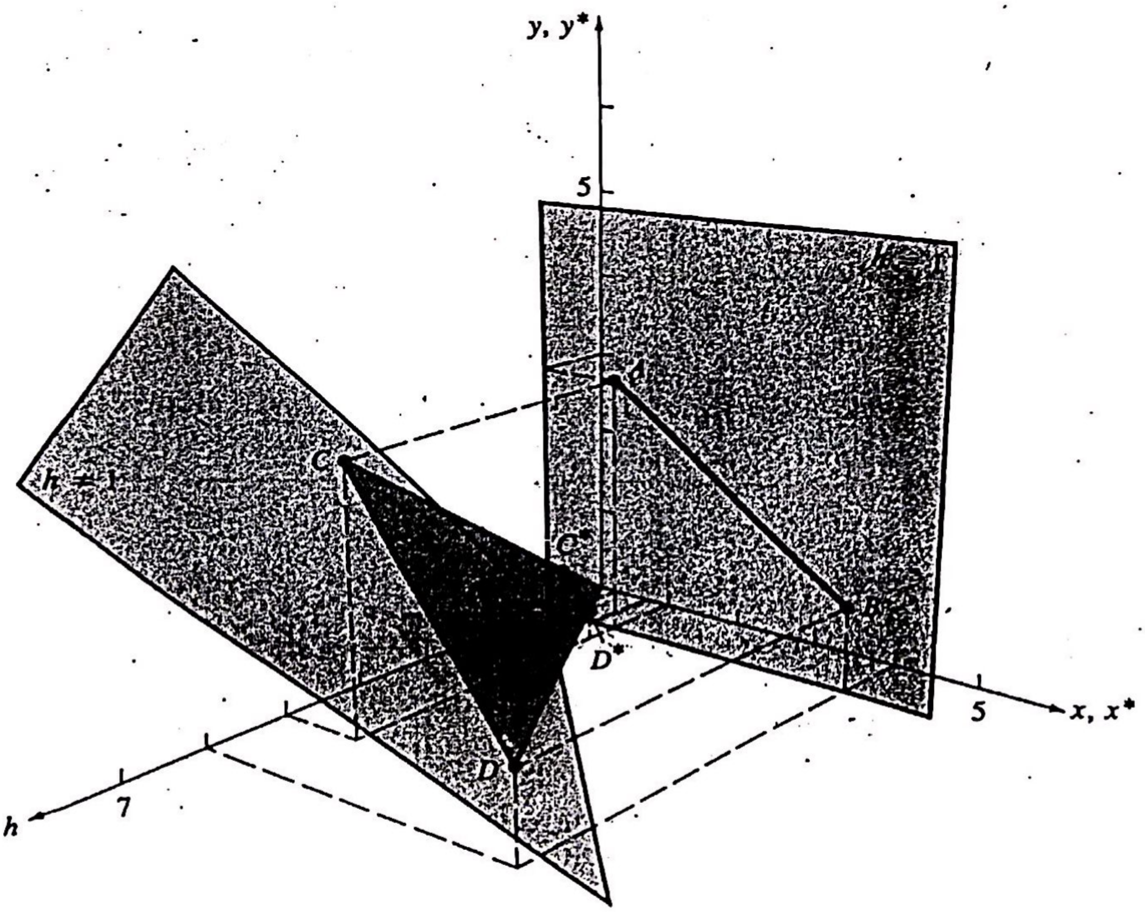

Figure 2–14 Transformation from the physical $(h = 1)$ plane into the $h \neq 1$ plane and projection from the $h \neq 1$ plane back into the physical plane.

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However, the results of interest are those in the physical plane corresponding to $h = 1$. These results can be obtained by geometrically projecting CD from the $h \neq 1$ plane back onto the $h = 1$ plane using a pencil of rays through the origin. From Fig. 2–14, using similar triangles,

$$x^* = \frac{X}{h} \qquad y^* = \frac{Y}{h}$$

or in homogeneous coordinates

$$[x^* \quad y^* \quad 1] = \left[\frac{X}{h} \quad \frac{Y}{h} \quad 1\right]$$

Now, normalizing Eq. (2–55) by dividing through by the homogeneous coordinate value h yields

$$[x^* \quad y^* \quad 1] = \left[\frac{X}{h} \quad \frac{Y}{h} \quad 1\right] = \left[\frac{x}{px+qy+1} \quad \frac{y}{px+qy+1} \quad 1\right] \qquad (2-56)$$

or

$$x^* = \frac{X}{h} = \frac{x}{px+qy+1} \qquad (2-57a)$$

$$y^* = \frac{Y}{h} = \frac{y}{px+qy+1} \qquad (2-57b)$$

The details are given in the example below.

Example 2–8 Projection in Homogeneous Coordinates

For the line AB in Fig. 2–14 we have, with $p = q = 1$, $[A] = [1 \quad 3 \quad 1]$ and $[B] = [4 \quad 1 \quad 1]$,

$$\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} A \\ B \end{bmatrix}[T] = \begin{bmatrix} 1 & 3 & 1 \\ 4 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 5 \\ 4 & 1 & 6 \end{bmatrix}$$

Thus, $[C] = [1 \quad 3 \quad 5]$ and $[D] = [4 \quad 1 \quad 6]$ in the plane $h = x + y + 1$. Projecting back onto the $h = 1$ physical plane by dividing through by the homogeneous coordinate factor yields the two-dimensional transformed points

$$[C^*] = [1 \quad 3 \quad 5] = [1/5 \quad 3/5 \quad 1]$$

$$[D^*] = [4 \quad 1 \quad 6] = [2/3 \quad 1/6 \quad 1]$$

The result is shown in Fig. 2–14.

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2–19 OVERALL SCALING

The remaining unexplained element in the general 3×3 transformation matrix (see Eq. 2–54), s, produces overall scaling; i.e., all components of the position vector are equally scaled. To show this, consider the transformation

$$[X \quad Y \quad h] = [x \quad y \quad 1] \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix} = [x \quad y \quad s] \qquad (2-58)$$

Here, $X = x$, $Y = y$ and $h = s$. After normalizing, this yields

$$X^* = \frac{x}{s} \qquad \text{and} \qquad Y^* = \frac{y}{s}$$

Thus, the transformation is $[x \quad y \quad 1][T] = [\frac{x}{s} \quad \frac{y}{s} \quad 1]$, a uniform scaling of the position vector. If $s < 1$, then an expansion occurs; and if $s > 1$, a compression occurs.

Note that this is also a transformation out of the $h = 1$ plane. Here, $h = s =$ constant. Hence, the $h \neq 1$ plane is parallel to the $h = 1$ plane. A geometric interpretation of this effect is shown in Fig. 2–15. If $s < 1$, then the $h =$ constant plane lies between the $h = 1$ and $h = 0$ planes. Consequently, when the transformed line AB is projected back onto the $h = 1$ plane to A^*B^*, it becomes larger. Similarly, if $s > 1$, then the $h =$ constant plane lies beyond the $h = 1$ plane along the h-axis. When the transformed line CD is projected back onto the $h = 1$ plane to C^*D^*, it becomes smaller.

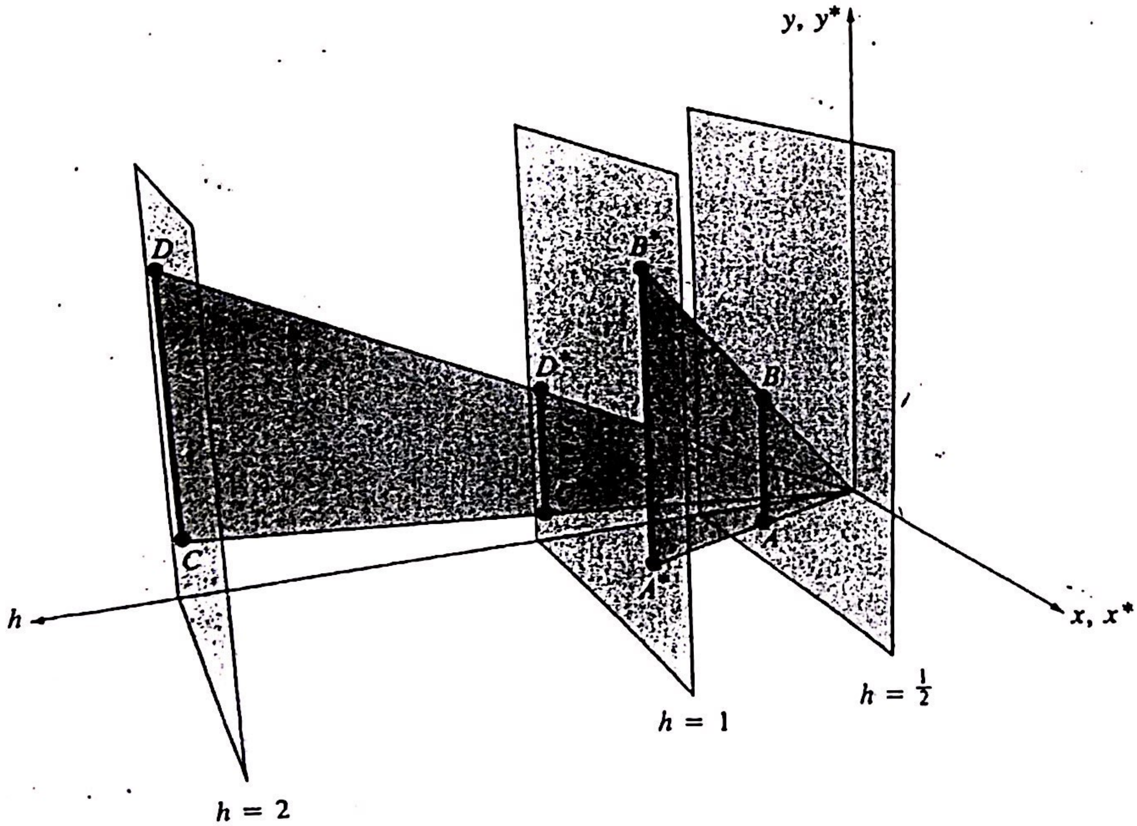

Figure 2–15 A geometric interpretation of overall scaling.

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2–20 POINTS AT INFINITY

Homogeneous coordinates provide a convenient and efficient technique for mapping a set of points from one coordinate system into a corresponding set in an alternate coordinate system. Frequently, an infinite range in one coordinate system is mapped into a finite range in an alternate coordinate system. Unless the mappings are carefully chosen, parallel lines may not map into parallel lines. However, intersection points map into intersection points. This property is used to determine the homogeneous coordinate representation of a point at infinity.

We begin by considering the pair of intersecting lines given by

$$x + y = 1$$
$$2x - 3y = 0$$

which have an intersection point at $x = 3/5$, $y = 2/5$. Writing the equations as $x + y - 1 = 0$ and $2x - 3y = 0$ and casting them in matrix form yields

$$[x \quad y \quad 1]\begin{bmatrix} 1 & 2 \\ 1 & -3 \\ -1 & 0 \end{bmatrix} = [0 \quad 0]$$

or

$$[X][M'] = [R]$$

If $[M']$ were square, the intersection could be obtained by matrix inversion. This can be accomplished by slightly rewriting the system of original equations. Specifically,

$$x + y - 1 = 0$$
$$2x - 3y = 0$$
$$1 = 1$$

In matrix form this is

$$[X][M] = [R]$$

i.e.,

$$[x \quad y \quad 1]\begin{bmatrix} 1 & 2 & 0 \\ 1 & -3 & 0 \\ -1 & 0 & 1 \end{bmatrix} = [0 \quad 0 \quad 1]$$

The inverse of this square matrix is†

$$[M]^{-1} = \begin{bmatrix} 3/5 & 2/5 & 0 \\ 1/5 & -1/5 & 0 \\ 3/5 & 2/5 & 1 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 0 \\ 3 & 2 & 5 \end{bmatrix}$$

†Matrix inversion techniques are discussed in Ref. 2-1 or any good linear algebra book.

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Multiplying both sides of the equation by $[M]^{-1}$ and noting that $[M][M]^{-1} = [I]$, the identity matrix, yields

$$[x \quad y \quad 1] = \frac{1}{5}[0 \quad 0 \quad 1]\begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 0 \\ 3 & 2 & 5 \end{bmatrix} = [3/5 \quad 2/5 \quad 1]$$

Thus, the intersection point is again $x = 3/5$ and $y = 2/5$.

Now consider two parallel lines defined by

$$x + y = 1$$
$$x + y = 0$$

By definition, in Euclidean (common) geometric space, the intersection point of this pair of parallel lines occurs at infinity. Proceeding, as above, to calculate the intersection point of these lines leads to the matrix formulation

$$[x \quad y \quad 1]\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} = [0 \quad 0 \quad 1]$$

However, even though the matrix is square it does not have an inverse, since two rows are identical. The matrix is said to be singular. Another alternate formulation is possible which does have an invertible matrix. This is obtained by rewriting the system of equations as

$$x + y - 1 = 0$$
$$x + y = 0$$
$$x = x$$

In matrix form this is

$$[x \quad y \quad 1]\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} = [0 \quad 0 \quad x]$$

Here, the matrix is not singular; the inverse exists and is

$$[M]^{-1} = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix}.$$

Multiplying both sides of the equation by the inverse yields

$$[x \quad y \quad 1] = [0 \quad 0 \quad x]\begin{bmatrix} 0 & 1 & -1 \\ 0 & 1 & 1 \\ 1 & -1 & 0 \end{bmatrix} = [x \quad -x \quad 0] = x[1 \quad -1 \quad 0]$$

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The resulting homogeneous coordinates $[1 \quad -1 \quad 0]$ represent the 'point of intersection' for the two parallel lines, i.e., a point at infinity. Specifically it represents the point at infinity in the direction $[1 \quad -1]$ in the two-dimensional plane. In general, the two-dimensional homogeneous vector $[a \quad b \quad 0]$ represents the point at infinity on the line $ay - bx = 0$. Some examples are:

$[1 \quad 0 \quad 0]$ on the positive x-axis
$[-1 \quad 0 \quad 0]$ on the negative x-axis
$[0 \quad 1 \quad 0]$ on the positive y-axis
$[0 \quad -1 \quad 0]$ on the negative y-axis
$[1 \quad 1 \quad 0]$ along the line $y = x$ in the direction $[1 \quad 1]$

The fact that a vector with the homogeneous component $h = 0$ does indeed represent a point at infinity can also be illustrated by the limiting process shown in Table 2–1. Consider the line $y^* = (3/4)x^*$ and the point $[X \quad Y \quad h] = [4 \quad 3 \quad 1]$. Recalling that a unique representation of a position vector does not exist in homogeneous coordinates, the point $[4 \quad 3 \quad 1]$ is represented in homogeneous coordinates in all the ways shown in Table 2–1. Note that in Table 2–1 as $h \to 0$, the ratio of y^*/x^* remains at 3/4, as is required by the governing equation. Further, note that successive pairs of (x^*, y^*) all of which fall on the line $y^* = (3/4)x^*$, become closer to infinity. Thus, in the limit as $h \to 0$, the point at infinity is given by $[X \quad Y \quad h] = [4 \quad 3 \quad 0]$ in homogeneous coordinates.

By recalling Fig. 2–15, a geometrical interpretation of the limiting process as $h \to 0$ is also easily illustrated. Consider a line of unit length from $x = 0$, $y = 0$ in the direction $[1 \quad 0]$, in the plane $h = s$ $(s < 1)$. As $s \to 0$ the projection of this line back onto the $h = 1$ physical plane by a pencil of rays through the origin becomes of infinite length. Consequently, the end point of the line must represent the point at infinity on the x-axis.

Table 2–1 Homogeneous Coordinates for the Point $[4 \quad 3]$

h	x^*	y^*	X	Y
1	4	3	4	3
1/2	8	6	4	3
1/3	12	9	4	3
.				
.				
.				
1/10	40	30	4	3
.				
.				
.				
1/100	400	300	4	3
.				
.				
.				

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2-21 TRANSFORMATION CONVENTIONS

Various conventions are used to represent data and to perform transformations with matrix multiplication. Extreme care is necessary in defining the problem and interpreting the results. For example, before performing a rotation the following decisions must be made:

Are the position vectors (vertices) to be rotated defined relative to a right-hand coordinate or a left-hand coordinate system?

Is the object or the coordinate system being rotated?

How are positive and negative rotations defined?

Are the position vectors stored as a row matrix or as a column matrix?

About what line, or axis, is rotation to occur?

In this text a right-hand coordinate system is used, the object is rotated in a fixed coordinate system, positive rotation is defined using the right-hand rule, i.e., clockwise about an axis as seen by an observer at the origin looking outward along the positive axis, and position vectors are represented as row matrices.

Equation (2-29) gives the transformation for positive rotation about the origin or about the z-axis. Since position vectors are represented as row matrices, the transformation matrix appears *after* the data or position vector matrix. This is a post-multiplication transformation. Using homogeneous coordinates for positive rotation by an angle θ of an object about the origin (z-axis) using a post-multiplication transformation gives

$$[X^*] = [X][R]$$

$$[x^* \quad y^* \quad 1] = [x \quad y \quad 1]\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2-59}$$

If we choose to represent the position vectors in homogeneous coordinates as a column matrix, then the same rotation is performed using

$$[X^*] = [R]^{-1}[X]$$

$$\begin{bmatrix} x^* \\ y^* \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \tag{2-60}$$

Equation (2-60) is called a premultiplication transformation because the transformation matrix appears *before* the column position vector or data matrix. Notice that the 3×3 matrix in Eq. (2-60) is also the transpose of the 3×3 matrix in Eq. (2-59). That is, the rows and columns have been interchanged.

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To rotate the coordinate system and keep the position vectors fixed, simply replace θ with $-\theta$ in Eq. (2–59). Recall that $\sin\theta = -\sin(-\theta)$ and $\cos\theta = \cos(-\theta)$. Equation (2–59) is then

$$[x^* \quad y^* \quad 1] = [x \quad y \quad 1]\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2-61}$$

Notice that the 3×3 matrix is again the inverse and also the transpose of that in Eq. (2–59).

If the coordinate system is rotated *and* a left-hand coordinate system used, then the replacement of θ with $-\theta$ is made *twice* and Eq. (2–59) is again valid, assuming a post-multiplication transformation is used on a row data matrix.

Note that, as shown in Fig. 2–16, a counterclockwise rotation of the vertices which represent an object is identical to a clockwise rotation of the coordinate axes for a fixed object. Again, no change occurs in the 3×3 transformation

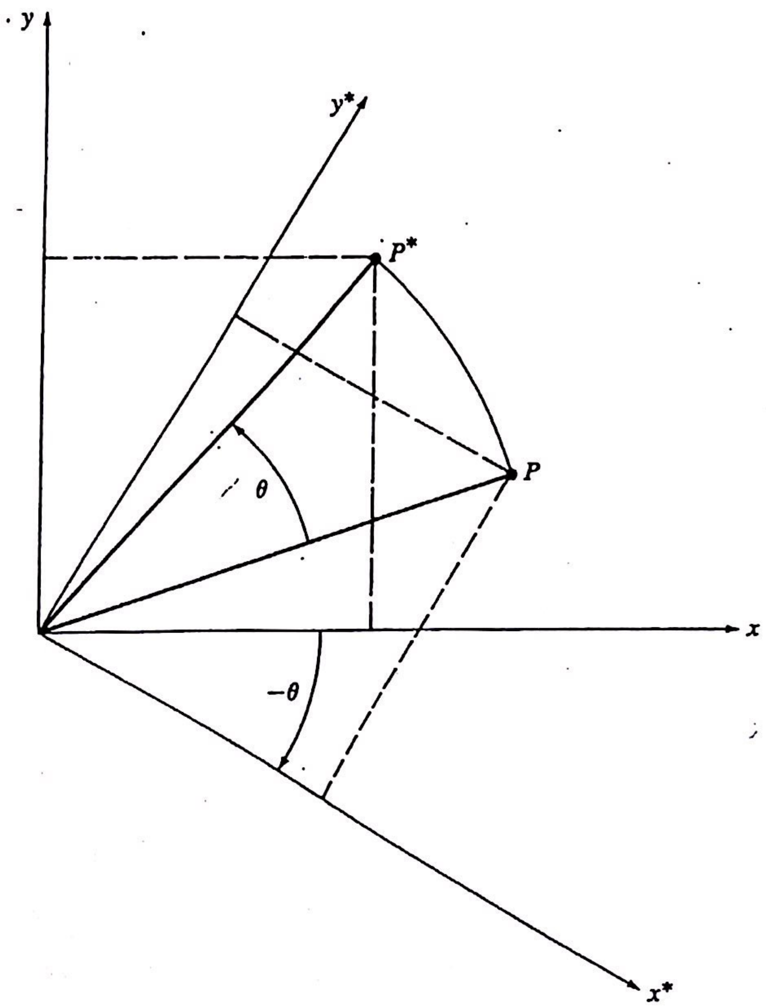

Figure 2–16 Equivalence of position vector and coordinate system rotation.

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