THREE THEOREMS ON INTERCHANGING LIMITS WITH INTEGRALS IN CALCULUS.

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ABSTRACT. The conditions for when $\lim_{n\to\infty}\int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n$ is central to real analysis. We review three conditions for when a limit may be interchanged with an integral, characterized by a common theme of *uniformity*: **(1)** a uniformly monotone sequence of functions, **(2)** a uniformly converging sequence of functions, and, the weakest condition, **(3)** a uniformly bounded, i.e. dominated, sequence of functions. The purpose of this paper is to retrace these established theorems, and pay attention to the role *uniformity* plays. We follow A. J. Luxembourg's proof of **(3)** Arzelà's Dominated Convergence Theorem and use his method to prove the nontrivial direction of **(2)** Monotone Convergence Theorem for Riemann Integrals.

1. Introduction

Ultimately we prove three important results in real analysis for when the limit can be exchanged with an integral: **(1)** Monotone Convergence Theorem, **(2)** Uniform Convergence Theorem, and **(3)** Dominating Convergence Theorem. Even though **(3)** is the broadest result, others are involved in its proof. Thus the organization of the paper is based on the order in which the theorems are proven.

In the proceeding two sections we start by defining uniform properties and prove the easy direction of **(1)** and the popular result **(2)** found in most calculus textbooks. In the fourth section we follow Luxembourg's proof of **(3)**, and in the fifth section we adapt his ideas to prove the tough direction of **(2)**. Finally in the conclusion we naively address abstracting uniformity and wonder what deeper connection to logic these theorems have. Examples are interspersed in all sections, clarifying the necessity of assumptions in theorems.

2. Uniform Properties of Sequences of Functions.

We investigate the bounded closed interval $[a,b]$ for reals $a<b$. Our paper is based on Darboux integrals defined as upper and lower rectangular estimates over partitions of $[a,b]$ by taking suprema and infima. By convention $\mathcal{B}[a,b]$, $\mathcal{R}[a,b]$, and $\mathcal{C}[a,b]$ represent, respectively, the family of bounded, Riemann integrable, and continuous real-valued functions on $[a,b]$.

Definition 2.1 (Uniformly Monotone)**.** A sequence of functions $f_n:[a,b]\to\mathbb{R}$ is said to be *uniformly monotone* if and only if either there exists an $N\in\mathbb{N}$ such that $f_n(x)\le f_{n+1}(x)$ for every $x\in[a,b]$ and $n\ge N$, or there exists an $N\in\mathbb{N}$ such that $f_n(x)\ge f_{n+1}(x)$ for every $x\in[a,b]$ and $n\ge N$.

Remark 2.2. A sequence that is uniformly monotone is said to be either *uniformly increasing* or *uniformly decreasing* depending on which criterion is satsified.

Definition 2.3 (Uniformly Convergent)**.** A sequence of functions $f_n : [a, b] \to \mathbb{R}$ is said to be *uniformly convergent* if and only if there exists some $f : [a, b] \to \mathbb{R}$ so that $f_n \to f$ pointwise, and for every $\varepsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for every $x \in [a, b]$ and $n \geq N$.

Remark 2.4. As convergence is unique, when f_n is uniformly convergent there exists a unique f for which $f_n \to f$ pointwise, and we call the convergence *uniform*.

Definition 2.5 (Uniformly Bounded)**.** A sequence of functions $f_n : [a, b] \to \mathbb{R}$ is said to be *uniformly bounded* if and only if there exists an $N \in \mathbb{N}$ and $M > 0$ such that $|f_n(x)| \leq M$ for every $x \in [a, b]$ and $n \geq N$.

Definition 2.6 (Dominated)**.** A sequence of functions $f_n : [a, b] \to \mathbb{R}$ is said to be *dominated* if and only if there exists some $M > 0$ such that every $|f_n(x)| < M$.

Example 2.7. None of these definitions completely coincide. To each uniform property, we give a sequence of functions on $[0, 1]$ satisfying only that uniform property.

(1) $f_n(x) := n$ is uniformly monotone but neither uniformly convergent nor uniformly bounded.
(2) $f_n(x) := {}^{(-1)^n}/_n$ when $x = 0$ and $f_n(x) := {}^1/_x$ otherwise, uniformly converges to $f(x) := 0$ when $x = 0$ and $f(x) := {}^1/_x$ otherwise. But f_n is neither uniformly monotone due to its alternating behavior at 0, nor uniformly bounded.
(3) Define $f_1(x) := 0$ when $x = 0$ and $f_1(x) := {}^1/_x$ otherwise, and $f_n(x) := \sin(nx)$ for $n > 1$. Then f_n is uniformly bounded but neither uniformly monotone nor uniformly convergent.

Domination implies uniformly bounded but the converse is not true, e.g. see (3) in the above example. Although this example demonstrates that the uniformity conditions are separate, there is a sense in which domination is the weakest. We make this precise.

Theorem 2.8. *Let $f_n \in \mathcal{B}[a, b]$ be so that $f_n \to f$ pointwise.*

(i) f_n is uniformly monotone and $f \in \mathcal{B}[a, b] \implies f_n$ is uniformly bounded.
(ii) f_n is uniformly convergent $\implies f_n$ is uniformly bounded.
(iii) f_n is uniformly bounded $\iff f_n$ is dominated.

Proof. (i) Without loss of generality, suppose f_n is uniformly increasing and pick N so that $f_n(x) \leq f_{n+1}(x)$ for all x and $n \geq N$. Since f and f_N are bounded, there is an $M > 0$ such that every $|f(x)| < M$ and $|f_N(x)| < M$. Therefore, by induction

$$-M < f_N(x) \leq f_n(x) \leq f(x) < M$$

for all x and $n \geq N$.

(ii) Suppose f_n is uniformly convergent. Then $f_n \to f$ uniformly and so in particular, there is an $N \in \mathbb{N}$ for which $|f_n(x) - f(x)| < 1$ for all $x \in [a, b]$ and $n \geq N$. By picking a bound $M > 0$ of f_N, we have

$$|f_n(x)| < 1 + |f(x)| < 2 + |f_N(x)| < 2 + M$$

for all x and $n \geq N$.

(iii) ($\Longleftarrow$) Straightforward. *($\Longrightarrow$)* Suppose f_n is uniformly bounded and pick $N \in \mathbb{N}$ and $M > 0$ so that $|f_n(x)| < M$ for all x and $n \geq N$. As $f_1, \ldots, f_N$ are bounded, we can pick bounds $M_1, \ldots, M_N > 0$ for each of those functions. That way

$$|f_n(x)| \leq \max\{M, M_1, \ldots, M_N\}$$

for all x and n. □

Remark 2.9. *(iii)* despite appearances does not use the axiom of choice. Even though $N \in \mathbb{N}$ is arbitrary, we can still make finitely many picks of M_i's by inducting over $\mathbb{N}$. To improve the cost of the proof, we can avoid induction by explicitly defining each M_i as the infimum of the set of all bounds of f_i.

Example 2.10. This theorem cannot be strengthened. Example 2.7 shows the converse of (ii) and (iii) are false, so we provide counter examples for (i). Consider $f_n(x) := (-x)^n$ on [0,1] so that f_n converges pointwise to a bounded function. f_n is uniformly bounded but f_n is not uniformly monotone; thus the $\Longleftarrow$ direction of (i) is false. Moreover, the extra condition on (i) that f be bounded, cannot be weakened. For consider $f_n(x) := 0$ when $x = 0$, and $f_n(x) := {}^1\!/\!{}_x$ otherwise. Each f_n is uniformly monotone since at fixed x, the sequence $f_n(x)$ is stationary; however f_n is not uniformly bounded.

3. Integrals of Uniformly Monotone and Convergent Sequences.

We partially prove the result that limits can be interchanged with integrals for uniformly monotone sequences. The full result requires the set-up in Luxembourg's paper. We prove the easy direction for now and save the other as Theorem 5.1.

Theorem 3.1 (Monotone Convergence Theorem Part 1)**.** *Let $f, f_n \in \mathcal{R}[a, b]$ so that $f_n \to f$ pointwise and f_n is uniformly increasing. Then*

$$\lim_{n\to\infty} \int_a^b f_n \leq \int_a^b f.$$

Proof. Pick N so that $f_n(x) \leq f_{n+1}(x)$ for all x and $n \geq N$. By induction $\int_a^b f_n \leq \int_a^b f$ for every $n \geq N$ and consequently,

$$\lim_{n\to\infty} \int_a^b f_n \leq \int_a^b f.$$

□

A remarkable application of this theorem to produce a family of non-trivial examples is due to Tai-Danae Bradley [1].

Example 3.2. Let $f \in \mathcal{R}[0,1]$ be nonnegative. Then

$$\lim_{n\to\infty} \int_0^1 n \log\left(1 + {}^{f}\!/\!{}_{n}\right) \leq \int_0^1 f.$$

For the sequence of functions $n \log(1 + {}^{f}\!/\!{}_{n})$ converges to $\log(e^f) = f$. Since f is nonnegative, $n \log(1 + {}^{f}\!/\!{}_{n})$ is uniformly increasing and the above theorem applies. Indeed by Theorem 5.1, we can strengthen this result with equality.

Now we prove the classical result found in calculus textbooks: the limit can be interchanged with an integral under uniform convergence.

Theorem 3.3 (Uniform Convergence Theorem)**.** *Let $f_n \in \mathcal{R}[a,b]$ converge to $f : [a,b] \to \mathbb{R}$. If the convergence is uniform, then $f \in \mathcal{R}[a,b]$ and moreover,*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Proof. Fix $\varepsilon > 0$ and pick $N \in \mathbb{N}$ so that $|f_n(x) - f(x)| < \frac{\varepsilon}{(b-a)}$ for all x and $n \geq N$. In particular, $f_N(x) - \frac{\varepsilon}{(b-a)} < f(x) < f_N(x) + \frac{\varepsilon}{(b-a)}$ for all x, and as f_N is bounded it follows f is bounded. Thus f possesses lower and upper integrals, and

$$\text{(i)} \quad \overline{\int_a^b} f - \int_a^b \frac{\varepsilon}{b-a} \leq \int_a^b f_n \leq \overline{\int_a^b} f + \int_a^b \frac{\varepsilon}{b-a}, \text{ and}$$

$$\text{(ii)} \quad \underline{\int_a^b} f - \int_a^b \frac{\varepsilon}{b-a} \leq \int_a^b f_n \leq \underline{\int_a^b} f + \int_a^b \frac{\varepsilon}{b-a}$$

for every $n \geq N$. As $\varepsilon > 0$ is arbitrary, $\int_a^b f_n \to \overline{\int_a^b} f$ and $\int_a^b f_n \to \underline{\int_a^b} f$, so f is Riemann integrable on $[a,b]$ and $\int_a^b f_n$ converges to the common value $\int_a^b f$. $\square$

Pointwise convergence is not a strong enough condition to justify the interchange of limit with integral. The following example [2] makes this clear.

Example 3.4. Define the sequence of functions on $[0,1]$ by $f_1(x) = 1$ and

$$f_n(x) := \begin{cases} n^2 x & \text{if } 0 \leq x < {}^{1}\!/\!{}_{n} \\ 2n - n^2 x & \text{if } {}^{1}\!/\!{}_{n} \leq x < {}^{2}\!/\!{}_{n} \\ 0 & \text{otherwise,} \end{cases}$$

for $n > 1$. Each sequence represents an isosceles triangle with height n and base ${}^{2}\!/\!{}_{n}$, and thus has unit area. However the sequence converges pointwise to $f(x) = 0$, and thus we have a disagreement of integrals $\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$.

4. Arzelà's Dominated Convergence Theorem.

Uniform convergence is a sufficient condition for interchanging a limit with an integral — however, Dominated Convergence Theorem weakens the condition. We follow A. J. Luxembourg's proof [3] of Arzelà's Dominated Convergence Theorem. We set the stage for our main arguments by proving two useful principles.

Theorem 4.1 (Dini's Uniform Convergence Theorem)**.** *Let $g_n, g \in \mathcal{C}[a,b]$ so that $g_n \to g$ pointwise and g_n is uniformly monotone. Then the convergence is uniform.*

Proof. It is enough to show a decreasing sequence $f_n \in \mathcal{C}[a,b]$ that converges pointwise to 0, converges uniformly. For let $N \in \mathbb{N}$ witness g_n uniformly monotone; we can transform the general problem $g_n \to g$ by putting $f_n := g_{n+N} - g$ or $f_n := g - g_{n+N}$ — whichever ensures $(f_n)_{n\in\mathbb{N}}$ decreasing.

Given $\varepsilon > 0$, first define $N : [a,b] \to \mathbb{N}$ and then $\delta : [a,b] \to \mathbb{R}$ by:

$$\begin{aligned} N(x) &:= \min\{N' \in \mathbb{N} \mid \forall n \geq N'.\ |f_n(x)| < \varepsilon/2\}\ \text{, and} \\ \delta(x) &:= \sup\{\delta' \in [0, b-a] \mid \forall t \in \mathcal{B}_{\delta'}(x) \cap [a,b].\ |f_{N(x)}(t) - f_{N(x)}(x)| < \varepsilon/2\}. \end{aligned}$$

These are well-defined by the pointwise convergence of $f_n(x)$ and the continuity of $f_{N(x)}(x)$; in fact the supremum is the maximum in the definition of $\delta(x)$.

For each $x \in [a,b]$ observe as $(f_n)_{n\in\mathbb{N}}$ is decreasing and nonnegative,

$$|f_n(t)| \leq |f_{N(x)}(t)| \leq |f_{N(x)}(t) - f_{N(x)}(x)| + |f_{N(x)}(x)| < \varepsilon$$

for every $n \geq N(x)$ and $t \in \mathcal{B}_{\delta(x)}(x) \cap [a,b]$. The idea is the family $\{\mathcal{B}_{\delta(x)}(x) : x \in [a,b]\}$ is an open cover of $[a,b]$, and thus by compactness contains a finite subcover, say $\{\mathcal{B}_{\delta(x_1)}(x_1), \ldots, \mathcal{B}_{\delta(x_n)}(x_n)\}$ for some $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in [a,b]$. By taking $N := \max\{N(x_1), \ldots N(x_n)\}$, we can solve the ε, N criterion regarding the uniform convergence $f_n \to f$. $\square$

Remark 4.2. We constructively define $N(x)$ and $\delta(x)$ and therefore avoid accidental invocations of the axiom of choice. Notice the comment that the supremum is the maximum in the definition of $\delta(x)$, is not superficial. It is precisely this topological property we use in our proof.

The condition $g \in \mathcal{C}[a,b]$ in the theorem is needed. For example consider $g_n(x) := x^n$ defined on $[0,1]$. The sequence uniformly decreases to the discontinuous function $g(x) := 1$ at $x = 1$ and $g(x) := 0$ otherwise. The convergence however is not uniform, as the uniform norm $\|g_n - g\|_u = \lim_{n\to\infty} \sup_{x\in[0,1]}\{|g_n(x) - g(x)|\} = 1$.

Example 4.3. This theorem together with Theorem 3.3, provides a powerful means of computing integrals. Consider $\displaystyle\lim_{n\to\infty} \int_0^1 \sin(x^2/n)\, dx$. With regular calculus we would struggle to find a closed-form anti-derivative. But observe the sequence of functions $\sin(x^2/n)$ is continuous and converges pointwise to 0. Thus the convergence is uniform and the integral computes to 0.

Lemma 4.4. *For any $0 \leq f \in \mathcal{B}[a,b]$ and $\varepsilon > 0$, there exists $g \in \mathcal{C}[a,b]$ such that every $0 \leq g(x) \leq f(x)$ and*

$$\underline{\int_a^b} f - \varepsilon \leq \int_a^b g.$$

Proof. Fix $0 \leq f \in \mathcal{B}[a,b]$ and $\varepsilon > 0$. By definition of lower integral, there must be a partition $P := \{x_0, \ldots, x_n\}$ of $[a,b]$, where $a = x_0 < \cdots < x_n = b$ and

$$\underline{\int_a^b} f - \varepsilon/2 < L(P, f).$$

Using this partition, define a step function s on $[a, b]$ by

$$s(x) := \inf_{t \in [x_{i-1}, x_i]} f(t)$$

where $[x_{i-1}, x_i]$ is the first interval from the partition which contains x. By design, $0 \le s(x) \le f(x)$ and $\int_a^b s = L(P, f)$. We can define g as a continuous extension of s by connecting lines between its steps, so that:

(1) Each $0 \le g(x) \le s(x)$ by adjusting the points for which g breaks into a line, and
(2) the magnitude of the integral underneath the lines is in total less than $\varepsilon/2$ by adjusting their widths to be as small as needed.

This is possible since the lines are bounded and has at most $n + 1$ steps. Thus $0 \le g(x) \le s(x) \le f(x)$ and

$$\underline{\int_a^b} f - \varepsilon < \int_a^b s - \varepsilon/2 < \int_a^b g.$$

□

The stage is set; we are ready for Luxembourg's main proof.

Theorem 4.5 (Dominated Convergence Theorem)**.** *Let $f_n, f \in \mathcal{R}[a, b]$ so that $f_n \to f$ pointwise and f_n is dominated. Then*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Proof. Without loss of generality, we assume $f_n \to 0$ pointwise with all $0 \le f_n(x) \le M$ for some fixed M, and try to show $\lim_{n\to\infty} \displaystyle\int_a^b f_n = 0$. For we can always take the f_n in the problem and redefine a new sequence of functions by $|f_n - f|$.

Set $p_n(x) := \sup_{k \ge n} f_k(x)$; note this is well-defined since each sequence $f_n(x)$ is bounded by M. Moreover observe p_n monotonically decreases to 0 and $0 \le f_n(x) \le p_n(x)$. Lemma 4.4 implies there is a sequence of continuous functions g_n such that:

(1) every $0 \le g_n(x) \le p_n(x)$,
(2) every $\underline{\int_a^b} p_n - \varepsilon \le \int_a^b g_n$, and
(3) g_n monotonically decreases to 0.

Note that the domination condition is used in (2): each function p_n is lower integrable because it is bounded by M. While (1) and (2) are clear from Lemma 4.4, the "magic step" is (3). We reveal how Luxembourg obtains such a function in the next section.

We can use Theorem 4.1 to conclude $\displaystyle\int_a^b g_n = 0$. Since $0 \le \displaystyle\int_a^b f_n \le \underline{\int_a^b} p_n \le \int_a^b g_n + \varepsilon$, in the limit we have $0 \le \lim_{n\to\infty} \displaystyle\int_a^b f_n \le \varepsilon$. As ε is arbitrary, $\lim_{n\to\infty} \displaystyle\int_a^b f_n = 0$ our and our proof is complete. □

Remark 4.6. We can generalize the result to uniformly bounded sequences; eventually some tail of uniformly bounded sequence is dominated. Since the distinction does not matter, we shall stick with Dominating Convergence Theorem.

Example 4.7. Dominated Convergence gives us a fast way to compute integrals. Define the sequence of functions on $[0,1]$ by

$$f_n(x) := \begin{cases} 1 + \frac{\sin(nx^2)}{n\cos(n)} & \text{if } 0 \leq x < 1/2 \\ x^n - 1 + \frac{\sin(nx^2)}{n} & \text{if } 1/2 \leq x \leq 1. \end{cases}$$

While it is not a continuous sequence of functions, it is quick to see it converges almost everywhere to a step function with steps valued at ± 1. (Only $f_n(1)$ converges 0.) It seems easier to verify f_n is dominated than to check uniform convergence, and so we can quickly confirm the integral vanishes, i.e. $\lim\limits_{n\to\infty} \displaystyle\int_0^1 f_n = 0$.

5. Full Monotone Convergence Theorem for Integrals

We return to the study of uniformly monotone sequences and aim to extend Theorem 3.1 to the Monotone Convergence Theorem. Our proof adapts Luxembourg's justification of the "magic step" employed in Theorem 4.5.

Theorem 5.1 (Monotone Convergence for Riemann Integrals Part 2)**.** *Let $f, f_n \in \mathcal{R}[a,b]$ so that $f_n \to f$ pointwise and f_n is uniformly decreasing. Then*

$$\lim_{n\to\infty} \int_a^b f_n \leq \int_a^b f.$$

Proof. Without loss of generality, we may assume f monotonically decreases to 0; for we can choose a sufficiently large tail of the sequence $f_n - f$ and pass the analysis back. With these new assumptions in mind, our goal is to show $\lim\limits_{n\to\infty} \displaystyle\int_a^b f_n \leq 0$.

Fix $\varepsilon > 0$ and choose a sequence of continuous functions g_n such that:

(1) every $0 \leq g_n(x) \leq f_n(x)$,
(2) every $\int_a^b f_n - \varepsilon \leq \int_a^b g_n$, and
(3) g_n uniformly decreases to 0.

Intuitively, first we choose g_1 by Lemma 4.4 so that $0 \leq g_1(x) \leq f_1(x)$. Then we find a temporary h by Lemma 4.4, so that $0 \leq h(x) \leq f_2(x)$, and define $g_2(x) := \min\{h(x), g_1(x)\}$. Then we find a temporary h again with Lemma 4.4 so that $0 \leq h(x) \leq f_3(x)$, and define $g_3(x) := \min\{h(x), g_2(x)\}$. We continue this process to generate the sequence g_n that satisfies the listed properties.

By Theorem 4.1, it follows $\lim\limits_{n\to\infty} \displaystyle\int_a^b g_n = 0$, and therefore $\lim\limits_{n\to\infty} \displaystyle\int_a^b f_n \leq \varepsilon$. As ε is arbitrary, the result follows. □

Remark 5.2. While the process of generating g_n suggests the use of axiom of dependent choice we can avoid it. By reinserting the mechanism of how each temporary

h was obtained in the proof of Lemma 4.4, we can constructively generate the sequence by recursion. This method of obtaining g_n is the "magic" step omitted earlier.

Both parts of monotone convergence, i.e. Theorems 3.1 and 5.1, prove Monotone Convergence Theorem for integrals in general.

Corollary 5.3 (Monotone Convergence Theorem)**.** *Let $f, f_n \in \mathcal{R}[a,b]$ so that $f_n \to f$ pointwise and f_n is uniformly monotone. Then*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Proof. When f_n is uniformly increasing, then $-f_n$ is uniformly decreasing so Theorem 5.1 proves the missing inequality

$$\lim_{n\to\infty} \int_a^b f_n \geq \int_a^b f.$$

Similarly when f_n is uniformly decreasing, Theorem 3.1 proves

$$\lim_{n\to\infty} \int_a^b f_n \geq \int_a^b f.$$

$\square$

There is an interesting generalization of this result for sequences of integrable functions which need not fully converge pointwise.

Corollary 5.4 (Fatou's Lemma)**.** *Let $f_n \in \mathcal{R}[a,b]$ and suppose the sequence $f_n(x)$ is bounded for every $x \in [a,b]$ so that limit superiors exist. Given $\lim_{n\to\infty} \sup f_n \in \mathcal{R}[a,b]$ and $\sup_{k\geq n} f_k \in \mathcal{R}[a,b]$, then*

$$\lim_{n\to\infty} \sup \int_a^b f_n \leq \int_a^b \lim_{n\to\infty} \sup f_n.$$

Proof. Observe $\sup_{k\geq n} f_k(x)$ is a decreasing sequence of functions and converges pointwise to $\lim_{n\to\infty} \sup f_n$. Thus,

$$\lim_{n\to\infty} \sup \int_a^b f_n \leq \lim_{n\to\infty} \int_a^b \sup_{k\geq n} f_k = \int_a^b \lim_{n\to\infty} \sup f_n$$

where the equality comes from Theorem 5.3, and the inequality comes from the observation $f_n \leq \sup_{k\geq n} f_k$. $\square$

Example 5.5. The inequality in the theorem cannot be strengthened. Define a sequence of functions on $[0,1]$ by

$$f_n(x) := \begin{cases} (-1)^n & \text{if } x < 1/2 \\ (-1)^{n+1} & \text{otherwise.} \end{cases}$$

Observe all integrals vanish, i.e. $\int_0^1 f_n = 0$. However all suprema of $f_n(x)$ is 1. Thus $\lim_{n\to\infty} \sup \int_a^b f_n = 0$ and $\int_0^1 \lim_{n\to\infty} \sup f_n = 1$.

6. Conclusion

We have proven three conditions for when $\lim_{n\to\infty}\int_a^b f_n = \int_a^b \lim_{n\to\infty} f_n$: (1) when f_n is uniformly convergent, (2) when f_n is uniformly bounded, i,e. dominated, and (3) when f_n is uniformly monotone. While (1) was a familiar exercise, we have shown how the methods used by Luxembourg's proof of (2) Dominating Convergence Theorem can be extended to prove (3) Monotone Convergence Theorem.

We observe how all three conditions express some uniform property. In general the *uniform* version of a *pointwise property* represents some interchange of the universal with the existential quantifier. For example, the pointwise property expressing f_n is convergent is captured by a $\exists f \forall \varepsilon \forall x \exists N \forall n$ formula. However the uniform version of the property is captured by a $\exists f \forall \varepsilon \exists N \forall x \forall n$ formula. The fact that uniform properties imply their pointwise versions is self-evident since $\exists\forall$ statements imply $\forall\exists$ counterparts. However the other direction is non-trivial.

For example consider Theorem 3.1's conclusion:

$$\lim_{n\to\infty}\int_a^b f_n \leq \int_a^b \lim_{n\to\infty} f$$

where f_n is monotone increasing. Suppose we associate the $\leq$ symbol with implication, the $\lim\limits_{n\to\infty}$ symbol with existential quantification over a variable n, and the $\int_a^b dx$ symbol with universal quantification over x, and we associate monotone increasing sequence of functions $f_n(x)$ with properties of the form $P(n,x)$. This corresponds to the trivial statement

$$(\exists n \forall x\, P(n,x)) \implies \forall x \exists n\, P(x,n).$$

The nontrivial direction of Monotone Convergence, i.e. Theorem 5.1, tells us

$$\lim_{n\to\infty}\int_a^b f_n \leq \int_a^b \lim_{n\to\infty} f$$

when f_n is monotone decreasing. According to our association it represents,

$$\forall x \exists n\, P(x,n) \implies (\exists n \forall x\, P(n,x))\,.$$

Somehow the condition f_n is monotone decreasing corresponds to the non trivial interchange of the universal quantifier with existential. Perhaps the relationship between the interchange of quantifiers with the interchange of limit with integral, is a clue at some deeper connection between analysis and logic.

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