FORMAL SEMANTICS OF CARNAP

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Abstract. The purpose of this paper is to formalize a model theory for Quantificational Logic as used in Carnap. Many pedagogical approaches of model theory feign ignorance of the ambient meta-logic the theory takes place in. In contrast, we hope to describe the model theory firmly within context of classical set theory (Zermelo-Fraenkel set theory with the axiom of choice, abbreviated ZFC). First we encode a *"toy"* version of Quantificational Logic used by Carnap within ZFC. Then we define *models* and *satisfaction* (i.e. truth with respect to models) and prove various meta-logical results: notably we show every model satisfies either a sentence or its negation. Finally, using these formal definitions we work out several examples to problems in introductory first-order logic courses and explain alternative notations (perhaps more familiar) for writing the same proofs.

§1 Introduction

Most mathematical textbooks dispense with philosophy and presume a platonic attitude towards the objects of their imagination. We find no faults with such an approach; it is easier to think of lines and points than it is to work axiomatically in *Euclid's Elements*. However in a formalists' perspective, meaning is simply a tool to find proof. A student who unknowingly aligns herself formalist, may be uneasy when first learning model theory. It is usually taught with a sense of absolutism, not relativism, and the perspective that models are really a relationship between formal systems, is usually blurred. We hope to address these concerns.

We describe a model theory of Carnap for a formalist. In general, the model theory we study does not intend to describe a process of assigning truth values by finitary methods, *"in real life"*. But it does describe a process of assigning truth values within some fixed, in this case nonconstructive, meta theory. The properties of truth are formal proofs carried out in this meta theory.

We express three notions of a *model theory* for a formalist: **(1)** a metalogical study of properties of a formal system, i.e. convincing (finitistic) arguments usually to justify the non-provability of an expression; **(2)** a metalogical method for transporting expressions and proofs in one formal system to another, i.e. a functor between syntactic categories whose objects are expressions and morphisms proofs; and **(3)** a particular case of **(2)** by encoding a formal system within ZFC and by defining a *satisfaction relation* $\models$ which assigns truth values to expressions in the *"toy"* . We follow approach **(3)**.

In section §2 we encode Quantificational Logic, as used by Carnap, in ZFC. In §3 we recursively define the satisfaction relation $\models$ which assigns truth values by generalizing the truth-table approach for propositional logic. In §4 we solve familiar problems, usually presented in introductory first-order logic courses, with the formalized toolkit.

§2 Encoding the Syntax of Quantificational Logic

Within ZFC, we encode the syntax of Quantificational Logic as used by Carnap. We assume the reader is familiar with the usual practices of set theory. When we say "for all", "for every", or "there exists", we bind a meta variable. When we say "fix", "let", or "there is some", we introduce a meta variable. The set of positive natural numbers is denoted by $\mathbb{N}^+$ and natural numbers by $\mathbb{N}$.

Definition 2.1 (Variables)**.** There is an infinite set of *variables* $\mathbb{V}$ and an injection $x : \mathbb{N} \to \mathbb{V}$. We abbreviate evaluations of x by subscripts, e.g. $x_0 := x(0)$.

Definition 2.2 (Constants)**.** There is an infinite set of *constants* $\mathbb{C}$, and an injection $c : \mathbb{N} \to \mathbb{C}$. We abbreviate evaluations of c by subscripts, e.g. $c_0 := c(0)$.

Definition 2.3 (Predicates)**.** For each $n \in \mathbb{N}^+$ there is an infinite set of *n-ary predicates* $\mathbb{P}_n$ and an injection $P^n : \mathbb{N} \to \mathbb{P}_n$. We abbreviate evaluations of P by subscripts, e.g. $P_0^1 := P^1(0)$.

Definition 2.4 (Logical)**.** There is a set $\mathbb{L}$ of ten distinguished *logical symbols*:

$$\mathbb{L} := \{\ulcorner(\urcorner, \ulcorner)\urcorner, \ulcorner\forall\urcorner, \ulcorner\exists\urcorner, \ulcorner\to\urcorner, \ulcorner\wedge\urcorner, \ulcorner\vee\urcorner, \ulcorner\leftrightarrow\urcorner, \ulcorner\neg\urcorner, \ulcorner=\urcorner\}.$$

We impose the restriction that $\mathbb{V}$, $\mathbb{C}$, $\mathbb{L}$, and all $\mathbb{P}_n$, are mutually disjoint. Indeed we can find such sets — in fact, of any infinite cardinality — but these details are unimportant.

We notate finite sequences in Quine quotes. Here are some examples:

(1) $\ulcorner abc \urcorner$ is the sequence whose first term is a, second term b, and third and last term c.
(2) $\ulcorner(\forall x_0 P_0^1 x_0)\urcorner$ is the sequence whose first term is $\ulcorner(\urcorner$, second term $\ulcorner\forall\urcorner$, third term x_0, fourth term P_0^1, fifth term x_0, and sixth and last term $\ulcorner)\urcorner$.
(3) $\ulcorner(\urcorner$ is the sequence whose first and only term is $\ulcorner(\urcorner$.
(4) Observe we omit accents of logical symbols within Quine quotes.
(5) Observe $\ulcorner()\urcorner$ is not the same as $\ulcorner c_3)\urcorner$; for the disjointness of $\mathbb{C}$ and $\mathbb{L}$ implies the two sequences disagree on the first term.

Definition 2.5 (Well-formed Formulas)**.** The set of *well-formed formulas* (or *wffs* or *formulas*) $\mathbb{WF}$ is defined inductively as follows:

(i) For each $n \in \mathbb{N}^+$, predicate $P \in \mathbb{P}_n$, and $t_1, \dots t_n \in \mathbb{V} \cup \mathbb{C}$, the sequence $\ulcorner Pt_1 \dots t_n \urcorner$ is an element of $\mathbb{WF}$.
(ii) For each $t_1, t_2 \in \mathbb{V} \cup \mathbb{C}$, the sequence $\ulcorner(t_1 = t_2)\urcorner$ is an element of $\mathbb{WF}$.
(iii) For each $\psi \in \mathbb{WF}$ and $v \in \mathbb{V}$, the sequences $\ulcorner\forall v\psi\urcorner$ and $\ulcorner\exists v\psi\urcorner$ are elements of $\mathbb{WF}$.
(iv) For each $\psi, \phi \in \mathbb{WF}$, the sequences $\ulcorner(\psi \to \phi)\urcorner$, $\ulcorner(\psi \wedge \phi)\urcorner$, $\ulcorner(\psi \vee \phi)\urcorner$, $\ulcorner(\psi \leftrightarrow \phi)\urcorner$, and $\ulcorner\neg\psi\urcorner$ are elements of $\mathbb{WF}$.

Remark 2.6. Inductive definitions correspond to the *smallest* set which satisfies all listed properties. That means a sequence is wff only if it can be demonstrated to be so by finitely many applications of the above rules.

In Carnap, quantifiers $\ulcorner\forall\urcorner$, $\ulcorner\exists\urcorner$, and negation $\ulcorner\neg\urcorner$ bind tightly; this is reflected by our definition. We sometimes drop outer-most parentheses of wffs, and give benefit of doubt to the writer. For example when context suggests $\ulcorner\forall yRy \to \exists zPzz\urcorner$ is wff then it must be y and z are variables, R a unary predicate, P a binary predicate, and we must reinstate outer-most parentheses. As another example, it is impossible to rectify $\ulcorner\forall yRy \to \exists zRzz\urcorner$ as wff.

We define *free* and *bound occurrences* of variables in wffs, and *capture-free substitution*, as usual. *Sentences* then are those wffs which contain no free occurrences. We also define *syntactic provability* i.e. a relation of the form $\Gamma \vdash \psi$ for some $\Gamma \subset \mathbb{WF}$ and $\psi \in \mathbb{WF}$. While explicit definitions are important for proving metalogical results, e.g. *soundness* and *completeness*, we will skip the formalism and jump straight into models.

§3 Semantical Theory of Quantificational Logic.

Definition 3.1 (Models)**.** A *model (of quantificational logic)* I consists of:

(i) a *domain* D^I,
(ii) a *valuation map* $\mathbf{val}^I : \mathbb{C} \to D^I$, and
(iii) for each $n \in \mathbb{N}^+$ and $P \in \mathbb{P}_n$, an *n-ary extension* $\mathbf{ext}^I(P) \subset \left(D^I\right)^n$.

When the underlying model is clear from context, we omit superscripts and abbreviate the domain for just D, valuation map $\mathbf{val}$, and extension $\mathbf{ext}$.

Observe how the definition of models differs from the usual setting: in place of $\mathcal{L}$ structures, the quantificational logic of Carnap provides an inexhaustible supply of all possible names. Interpretations of the model must account for all these names. Curiously, since there are infinitely many constants, we have the following result.

Corollary. *Every model has a nonempty domain.*

Proof. Every domain of a model is inhabited, namely by the valuation of $c_0 \in \mathbb{C}$. □

To simplify notation when we write models, any unspecified extensions of predicates are automatically presumed empty, and valuations of constants defaulted to $\mathbf{val}(c_0)$. For example consider the model I with domain $D := \{1, 2\}$, valuations $\mathbf{val}(c_0) := 1$ and $\mathbf{val}(c_1) = 2$, and extensions $\mathbf{ext}(P_0^2) = \{(1, 1), (2, 2)\}$. By our convention, I interprets $\mathbf{val}(c) = 1$ and $\mathbf{ext}(P) = \emptyset$ for every other $c \in \mathbb{C}$ and $P \in \mathbb{P}_n$ not listed.

A model represents truth values of certain atomic formulae. The idea is all formulas are built from atomic ones and so truth defined for the broader formula, is defined in terms of truth for the atomic. However truth of atomics with free variables are ambiguous; they depend on the assignment of free variables.

Definition 3.2 (Assignments)**.** An *assignment (of variables)* σ with respect to a model I is a function from $\mathbb{V}$ to D.

When writing assignments we omit explicit reference to the model the assignments is with respect to as long, as it is clear from context.

Theorem 3.3. *For any model I, assignment σ, variable v, and $d \in D$, there exists a unique assignment μ such that for every $u \in \mathbb{V}$:*

$$\mu(u) := \begin{cases} d & \textit{if } u = v, \\ \sigma(u) & \textit{otherwise.} \end{cases}$$

Proof. Fix a model I, assignment σ, variable $v \in \mathbb{V}$, and $d \in D$.

Existence: It suffices to show for every $u \in \mathbb{V}$, their exists a unique $o \in D$ such that $u = v \implies o = d$ and $u \neq v \implies o = \sigma(u)$. For once shown we can realize the desired function $\mu : \mathbb{V} \to D$. To this end fix $u \in \mathbb{V}$. Using the law of excluded middle either $u = v$ or $u \neq v$. In each case there exists a unique $o \in D$ such that $u = v \implies o = d$ and $u \neq v \implies o = \sigma(u)$, and hence the result.

Uniqueness: Fix two such assignments μ_1, μ_2 and let $u \in \mathbb{V}$. By excluded middle either $u = v$ or $u \neq v$; either way $\mu_1(u) = \mu_2(u)$, and therefore $\mu_1 = \mu_2$ as $u \in \mathbb{V}$ is arbitrary.

□

Remark 3.4. Our proof is the usual argument for the well-definability of piecewise functions in ZFC. The reason we reproduce the argument is to reveal the clear use of excluded middle.

We introduce bracket arrow notation to help streamline presentation. For any model I, assignment σ, variable $v \in \mathbb{V}$, and $d \in D$, we let $\sigma[u \mapsto d]$ denote the unique assignment guaranteed by the above theorem, i.e.

$$\sigma[v \mapsto d](u) := \begin{cases} d & \textit{if } u = v, \\ \sigma(v) & \textit{otherwise.} \end{cases}$$

When we write assignments in this way we associate to the left. For example $\sigma[v_1 \mapsto d_1][v_2 \mapsto d_2]$ is understood as $\left(\sigma[v_1 \mapsto d_1]\right)[v_2 \mapsto d_2]$.

Proposition 3.5 (Factorization)**.** *For every model I, assignment σ, variable $v \in \mathbb{V}$, and $d \in D$, we have $\sigma(v) = d \iff$ for all $d' \in D$, $\sigma[v \mapsto d'][v \mapsto d] = \sigma$.*

Proof. Fix model I, assignment σ, $v \in \mathbb{V}$, and $d \in D$.

⇒: Assume $\sigma(v) = d$ and fix $d' \in D$. We claim $\sigma[v \mapsto d'][v \mapsto d] = \sigma$. To show this let $u \in \mathbb{V}$. By the law of excluded middle, either $\sigma[v \mapsto d'][v \mapsto d](u) = \sigma(u) = d$ when $u = v$ or $\sigma[v \mapsto d'][v \mapsto d](u) = \sigma[v \mapsto d'](u) = \sigma(u)$ otherwise. Either way, $\sigma[v \mapsto d'][v \mapsto d](u) = \sigma(u)$ and, hence, the claim.

⇐: Assume for all $d' \in D$, $\sigma[v \mapsto d'][v \mapsto d] = \sigma$. In particular $\sigma[v \mapsto d][v \mapsto d] = \sigma$, and thus $\sigma(v) = \sigma[v \mapsto d][v \mapsto d](v) = d$.

□

Theorem 3.6. *For every model I, assignments σ and μ, variables v_1 and v_2, and $d_1, d_2 \in D$, we have:*

(1) $v_1 \neq v_2 \implies \sigma[v_1 \mapsto d_1][v_2 \mapsto d_2] = \sigma[v_2 \mapsto d_2][v_1 \mapsto d_1]$,
(2) $\sigma[v_1 \mapsto d_1][v_1 \mapsto d_2] = \sigma[v_1 \mapsto d_2]$,
(3) $\sigma = \sigma[v_1 \mapsto \sigma(v_1)]$, *and*
(4) $\sigma[v_1 \mapsto d_1](v_1) = \mu[v_1 \mapsto d_1](v_1)$.

Proof. It is easy to show that the function on the left agrees, on every input, with the function on the right. ☐

These theorems, and bracket arrow notation itself, are important for seamless manipulation of semantic arguments. We use the notation to define *satisfaction.*

Definition 3.7 (Satisfaction)**.** *Satisfaction* $\models$ is a relation defined recursively as follows:

(i) For any model I, assignment σ, arity $n \in \mathbb{N}^+$, predicate $P \in \mathbb{P}_n$, and $t_1, \ldots t_n \in \mathbb{V} \cup \mathbb{C}$ we have:
$$I, \sigma \models \ulcorner Pt_1 \ldots t_n \urcorner \iff (\widehat{t_1}, \ldots, \widehat{t_n}) \in \mathbf{ext}(P),$$
where each
$$\widehat{t_i} = \begin{cases} \sigma(t_i) & \text{if } t_i \in \mathbb{V}, \\ \mathbf{val}(t_i) & \text{otherwise, when } t_i \in \mathbb{C}. \end{cases}$$
This is well-defined since $\mathbb{V}$ and $\mathbb{C}$ are disjoint.

(ii) For any model I, assignment σ, and $t_1, t_2 \in \mathbb{V} \cup \mathbb{C}$ we have:
$$I, \sigma \models \ulcorner (t_1 = t_2) \urcorner \iff \widehat{t_1} = \widehat{t_2}$$
where each
$$\widehat{t_i} = \begin{cases} \sigma(t_i) & \text{if } t_i \in \mathbb{V}, \\ \mathbf{val}(t_i) & \text{otherwise, when } t_i \in \mathbb{C}. \end{cases}$$
This is well-defined for the same reason above. When $\widehat{t_1} = \widehat{t_2}$ we sometimes say $(\widehat{t_1}, \widehat{t_2})$ *belongs to the extension of identity.*

(iii) For any model I, assignment σ, variable $v \in \mathbb{V}$, and $\psi \in \mathbb{WF}$ we have:
$$\begin{aligned} I, \sigma \models \ulcorner \forall v \psi \urcorner &\iff \text{for all } d \in D \text{ we have } I, \sigma[v \mapsto d] \models \ulcorner \psi \urcorner, \text{ and} \\ I, \sigma \models \ulcorner \exists v \psi \urcorner &\iff \text{there exists } d \in D \text{ such that } I, \sigma[v \mapsto d] \models \ulcorner \psi \urcorner. \end{aligned}$$

(iv) For any model I, assignment σ, and $\psi, \phi \in \mathbb{WF}$ we have:
$$\begin{aligned} I, \sigma \models \ulcorner (\psi \to \phi) \urcorner &\iff I, \sigma \models \ulcorner \psi \urcorner \text{ implies } I, \sigma \models \ulcorner \phi \urcorner, \\ I, \sigma \models \ulcorner (\psi \wedge \phi) \urcorner &\iff I, \sigma \models \ulcorner \psi \urcorner \text{ and } I, \sigma \models \ulcorner \phi \urcorner, \\ I, \sigma \models \ulcorner (\psi \vee \phi) \urcorner &\iff I, \sigma \models \ulcorner \psi \urcorner \text{ or } I, \sigma \models \ulcorner \phi \urcorner, \\ I, \sigma \models \ulcorner (\psi \leftrightarrow \phi) \urcorner &\iff I, \sigma \models \ulcorner \psi \urcorner \text{ iff } I, \sigma \models \ulcorner \phi \urcorner \\ I, \sigma \models \ulcorner \neg \psi \urcorner &\iff I, \sigma \not\models \ulcorner \psi \urcorner. \end{aligned}$$

Remark 3.8. Note that $I, \sigma \not\models \psi$ communicates $\neg (I, \sigma \models \psi)$. This negation is a part of the ZFC meta language, not the Carnap object language. It is important to keep the two languages separate: semantical arguments about the object language are simply proofs in the meta language. Notice our notation with Quine quotes not only keeps our presentation precise, but also prevents unscrupulous intermingling of object and meta language — prevents, as Professor Nelson calls it, *"bastard children"*.

Remark 3.9. We could co-recursively define $\models$ and $\not\models$ as two interlinked relations. However it is simpler to define $\models$ as one whole. Either way $\not\models$ is the complement of $\models$ (either as a theorem or as definition). We could also package assignments within models themselves, where models would be equipped with an additional set thought of as the set of all possible assignments. That way models have all the instructions they need to assign truth values to formulas. This, however, leads to messier proofs.

Since the metalogical implication is classical, the model semantics inherits the same behavior.

Theorem 3.10. *For every model I, assignment σ, and $\psi, \phi \in \mathbb{WF}$,*

$$I, \sigma \models \ulcorner(\psi \to \phi)\urcorner \iff I, \sigma \not\models \ulcorner\psi\urcorner \text{ or } I, \sigma \models \ulcorner\phi\urcorner.$$

Proof. Fix model I, assignment σ, and $\psi, \phi \in \mathbb{WF}$.

⇒: Assume $I, \sigma \models \ulcorner(\psi \to \phi)\urcorner$. Then $I, \sigma \models \psi$ implies $I, \sigma \models \phi$. By the law of excluded middle either $I, \sigma \models \psi$ or $I, \sigma \not\models \psi$. In either case $I, \sigma \not\models \psi$ or $I, \sigma \models \phi$.

⇐: Assume $I, \sigma \not\models \psi$ or $I, \sigma \models \phi$. In either case it is clear $I, \sigma \models \psi$ implies $I, \sigma \models \phi$; for supposing $I, \sigma \models \psi$ we can contradict $I, \sigma \not\models \psi$, or just end with $I, \sigma \models \phi$.

□

We present powerful, fundamental theorems about satisfaction and independence of assignments for free variables.

Theorem 3.11. *For every model I, variable $v \in \mathbb{V}$, and $d \in D$ we have for all $\psi \in \mathbb{WF}$ that if v does not occur free in ψ then for all assignments σ:*

$$I, \sigma \models \psi \iff I, \sigma[v \mapsto d] \models \psi.$$

Proof. Fix model I, variable $v \in \mathbb{V}$, and $d \in D$. Since wffs are inductively defined, they come equipped with their own natural induction rules. We proceed by induction over formula construction.

<u>Base Cases</u>

1. *Extension:* Take any n-ary predicate P with $t_1, \ldots, t_n \in \mathbb{V} \cup \mathbb{C}$ such that v is not among t_i, and fix assignment σ.

⇒: Suppose $I, \sigma \models \ulcorner Pt_1 \ldots t_n\urcorner$. So $(\widehat{t_1}, \ldots, \widehat{t_n}) \in \mathbf{ext}(P)$ where each

$$\widehat{t_i} = \begin{cases} \sigma(t_i) & \text{if } t_i \in \mathbb{V}, \\ \mathbf{val}(t_i) & \text{otherwise, when } t_i \in \mathbb{C}. \end{cases}$$

Since v is not among t_i, it follows each

$$\widehat{t_i} = \begin{cases} \sigma[v \mapsto d](t_i) & \text{if } t_i \in \mathbb{V}, \\ \mathbf{val}(t_i) & \text{otherwise, when } t_i \in \mathbb{C}. \end{cases}$$

Thus $I, \sigma[v \mapsto d] \models \psi$, as desired.

⇐: Symmetric.

2. *Equality:* Similar to **1.**

<u>Construction Cases</u>

Assume formulas ψ and ϕ have the property that if they have no free occurrences of v, then $I, \sigma \models \psi \iff I, \sigma[v \to d] \models \psi$ and $I, \sigma \models \phi \iff I, \sigma[v \to d] \models \phi$ for all assignments σ.

3. *Universal:* Take any $u \in \mathbb{V}$ and assume the formula $\ulcorner \forall u \psi \urcorner$ has no free occurrences of v. Now fix assignment σ. By excluded middle either $u = v$ or $u \neq v$.

Case 1: Assume $u = v$ so that ψ may contain free occurrences of v.

$\Rightarrow$**:** Suppose $I, \sigma \models \ulcorner \forall v \psi \urcorner$. To any $d_1 \in D$, we have $I, \sigma[v \mapsto d_1] \models \psi$. The idea is we can factorize so that $I, \sigma[v \mapsto d][v \mapsto d_1] \models \psi$. Since $d_1 \in D$ is arbitrary, it follows $I, \sigma[v \mapsto d] \models \ulcorner \forall v \psi \urcorner$.

$\Leftarrow$**:** Suppose $I, \sigma[v \mapsto d] \models \ulcorner \forall v \psi \urcorner$. To any $d_1 \in D$, we have $I, \sigma[v \mapsto d][v \mapsto d_1] \models \psi$. We can absorb the factor so that $I, \sigma[v \mapsto d_1] \models \psi$. Since $d_1 \in D$ is arbitrary, it follows $I, \sigma \models \ulcorner \forall v \psi \urcorner$.

Case 2: Assume $u \neq v$; ψ has no free occurrences of v. By hypothesis, $I, \sigma \models \psi \iff I, \sigma[v \mapsto d] \models \psi$ for all assignments σ. Now fix assignment σ.

$\Rightarrow$**:** Suppose $I, \sigma \models \ulcorner \forall u \psi \urcorner$ and fix $d_1 \in D$. In particular $I, \sigma[u \mapsto d_1] \models \psi$. A specific instance of the hypothesis guarantees $I, \sigma[u \mapsto d_1][v \mapsto d] \models \psi$. The idea is the order can be interchanged so that $I, \sigma[v \mapsto d][u \mapsto d_1] \models \psi$. As $d_1 \in D$ is arbitrary, we have $I, \sigma[v \mapsto d] \models \ulcorner \forall u \psi \urcorner$.

$\Leftarrow$**:** Suppose $I, \sigma[v \mapsto d] \models \ulcorner \forall u \psi \urcorner$ and fix $d_1 \in D$. In particular $I, \sigma[v \mapsto d][u \mapsto d_1] \models \psi$. We can interchange order to get $I, \sigma[u \mapsto d_1][v \mapsto d] \models \psi$, and use a specific instance of the hypotheses to obtain $I, \sigma[u \mapsto d_1] \models \psi$. As $d_1 \in D$ is arbitrary, we have $I, \sigma \models \ulcorner \forall u \psi \urcorner$ as desired.

4. *Implication:* Assume the formula $\ulcorner (\psi \to \phi) \urcorner$ has no free occurrences of v. Then ψ and ϕ have no free occurrences. By hypothesis, $I, \sigma \models \psi \iff I, \sigma[v \mapsto d] \models \psi$ and $I, \sigma \models \phi \iff I, \sigma[v \mapsto d] \models \phi$ for all assignments σ. Now fix assignment σ.

$\Rightarrow$**:** Assume $I, \sigma \models \ulcorner (\psi \to \phi) \urcorner$. We claim $I, \sigma[v \mapsto d] \models \ulcorner (\psi \to \phi) \urcorner$. To show this suppose $I, \sigma[v \mapsto d] \models \psi$. One instance of the hypotheses gives $I, \sigma \models \psi$, which implies $I, \sigma \models \phi$. Another instance of the hypotheses gives $I, \sigma[v \mapsto d] \models \phi$ and hence the claim.

$\Leftarrow$**:** Symmetric.

5. The rest of the construction cases (existential, conjunction, disjunction, equivalence, and negation) follow a similar approach to the above two. $\square$

Remark 3.12. This theorem allows us to delete individual assignments for variables which do not occur free.

Theorem 3.13. *Let I be a model and $v \in \mathbb{V}$. For every $\psi \in \mathbb{WF}$, if ψ is free in at most $v_1, \ldots, v_n \in \mathbb{V}$ for any $n \in \mathbb{N}^+$, then*

$$I, \sigma[v_1 \mapsto d_1] \ldots [v_n \mapsto d_n] \models \psi \implies I, \mu[v_1 \mapsto d_1] \ldots [v_n \mapsto d_n] \models \psi$$

for all $d_1, \ldots, d_n \in D$ and assignments σ, μ.

Proof. We induct over formula construction.

<u>Base Cases</u>

1. *(Extension)* Take any m-ary predicate P and $t_1, \ldots, t_m \in \mathbb{V} \cup \mathbb{C}$ so that $\ulcorner P t_1 \ldots t_m \urcorner$ has at most free occurrences in $v_1, \ldots, v_n \in \mathbb{V}$. Fix $d_1, \ldots, d_n \in D$, assignments σ and μ, and assume $I, \sigma[v_1 \mapsto d_1]\dots[v_n \mapsto d_n] \models \ulcorner Pt_1 \dots t_m \urcorner$. By definition $(\widehat{t_1}, \dots, \widehat{t_n}) \in \mathbf{ext}(P)$ where each

$$\widehat{t_i} = \begin{cases} \sigma[v_1 \mapsto d_1]\dots[v_n \mapsto d_n](t_i) & \text{if } t_i \in \mathbb{V}, \\ \mathbf{val}(t_i) & \text{otherwise, when } t_i \in \mathbb{C}. \end{cases}$$

Since every t_i that is a variable is among v_j, it follows

$$\widehat{t_i} = \begin{cases} \mu[v_1 \mapsto d_1]\dots[v_n \mapsto d_n](t_i) & \text{if } t_i \in \mathbb{V}, \\ \mathbf{val}(t_i) & \text{otherwise, when } t_i \in \mathbb{C}. \end{cases}$$

Thus $I, \mu[v_1 \mapsto d_1]\dots[v_n \mapsto d_n] \models \ulcorner Pt_1 \dots t_m \urcorner$.

2. *(Equality)* Similar to **1.**

<u>Construction Cases</u>

Assume formulas ψ and ϕ have the property if they are free in at most $v_1, \dots, v_n \in \mathbb{V}$ for any $n \in \mathbb{N}^+$, then

$$I, \sigma[v_1 \mapsto d_1]\dots[v_n \mapsto d_n] \models \psi \implies I, \mu[v \mapsto d]\dots[v_n \mapsto d_n] \models \psi, \text{ and}$$

$$I, \sigma[v_1 \mapsto d_1]\dots[v_n \mapsto d_n] \models \phi \implies I, \mu[v_1 \mapsto d_1]\dots[v_n \mapsto d_n] \models \phi$$

for all $d_1, \dots, d_n \in D$ and assignments σ, μ.

3. *(Universal)* Take any $u \in \mathbb{V}$ and assume $\ulcorner \forall u \psi \urcorner$ is free in at most $v_1, \dots, v_n \in \mathbb{V}$; then ψ is free in at most $u, v_1, \dots, v_n$. Fix $d, d_1, \dots, d_n \in D$, assignments σ and μ, and assume $I, \sigma[v_1 \mapsto d_1]\dots[v_n \mapsto d_n] \models \ulcorner \forall u \psi \urcorner$. In particular $I, \sigma[v_1 \mapsto d_1]\dots[v_n \mapsto d_n][u \mapsto d] \models \psi$. By hypothesis we may conclude $I, \mu[v_1 \mapsto d_1]\dots[v_n \mapsto d_n][u \mapsto d] \models \psi$. As $d \in D$ is arbitrary, $I, \mu[v_1 \mapsto d_1]\dots[v_n \mapsto d_n] \models \ulcorner \forall u \psi \urcorner$.

4. *(Implication)* Assume the formula $\ulcorner (\psi \to \phi) \urcorner$ is free in at most $v_1, \dots, v_n \in \mathbb{V}$; then ψ and ϕ are free in at most $u, v_1, \dots, v_n \in$. Fix $d_1, \dots, d_n \in D$, assignments σ and μ, and assume $I, \sigma[v_1 \mapsto d_1]\dots[v_n \mapsto d_n] \models \ulcorner (\psi \to \phi) \urcorner$. We claim $I, \mu[v_1 \mapsto d_1]\dots[v_n \mapsto d_n] \models \ulcorner (\psi \to \phi) \urcorner$. To show this suppose $I, \mu[v_1 \mapsto d_1]\dots[v_n \mapsto d_n] \models \psi$. Using the hypothesis we have $I, \sigma[v_1 \mapsto d_1]\dots[v_n \mapsto d_n] \models \psi$, which implies $I, \sigma[v_1 \mapsto d_1]\dots[v_n \mapsto d_n] \models \phi$. Using another instance of the hypothesis gives us $I, \mu[v_1 \mapsto d_1]\dots[v_n \mapsto d_n] \models \phi$ as needed.

5. *(Existential, Conjunction, Disjunction, Equivalence, and Negation)* The rest of the construction cases follow a similar approach to the above two. □

Remark 3.14. This theorem justifies ignorance for assignments outside the domain of variables which appear free in the wff; other assignations do not matter.

Corollary. *Let I be a model I and ψ a sentence. Either $I, \sigma \models \psi$ for every assignment σ, or $I, \sigma \models \ulcorner \neg\psi \urcorner$ for every assignment σ.*

Proof. Consider the assignment defined by $\sigma(v) = \mathbf{val}(c_0)$. Either $I, \sigma \models \psi$ or $I, \sigma \models \ulcorner \neg\psi \urcorner$; either way ψ has no free occurrences and hence the result. □

We present a final list of abbreviations to help facilitate proof writing and smoothen the back-and-forth, seesaw passage from syntax to semantics.

(1) When $I, \sigma \models \psi$ we say *"ψ is true in I under σ"*.
(2) When $I, \sigma \models \psi$ for all assignments σ, we say *"$I \models \psi$"*, *"ψ is true in I"*, and *"I satisfies ψ"*.
(3) When $I, \sigma[v_1 \mapsto d_1] \dots [v_n \mapsto d_n] \models \psi$ for all assignments σ, we say *"ψ is true in I under $[v_1 \mapsto d_1] \dots [v_n \mapsto d_n]$"*.
(4) When *"$I \models \theta$"* for all $\theta \in \Gamma$ implies *"$I \models \psi$"*, we say *"$\Gamma \models \psi$"*.
(5) When there exists model I such that *"$I \models \theta$"* for every $\theta \in \Gamma$, we say *"Γ is satisfiable"* or sometimes, due to Henkin, *"Γ is consistent"*.

Remark 3.15. When Γ is finite, we may list out each of the formulas separated by commas, as opposed to a set of formulas. E.g. $\ulcorner P^2_0 c_0 c_1 \urcorner \models \ulcorner \exists x_0 P^2_0 x_0 x_0 \urcorner$ represents $\{\ulcorner P^2_0 c_0 c_1 \urcorner\} \models \ulcorner \exists x_0 P^2_0 x_0 x_0 \urcorner$ and $\models \ulcorner (x_0 = x_0) \urcorner$ represents $\{\} \models \ulcorner (x_0 = x_0) \urcorner$. Negations are defined as usual, e.g. $I \not\models \psi$ communicates $\neg (I \models \psi)$. Note $I \not\models \psi$ is not the same as $I \models \neg\psi$. For the first suggests the existence of an assignment σ so that $I, \sigma \models \neg\psi$, whereas the second says for all assignments σ, we have $I, \sigma \models \neg\psi$.

§4 Solutions to Selected Exercises.

Example 4.1. $\ulcorner \forall x_0 P^1_0 x_0 \urcorner \models \ulcorner (P^1_0 c_0 \wedge P^1_0 c_1) \urcorner$.

Proof. Here is a step-by-step breakdown of the proof.

(1) Our goal is to show for every model I and assignment σ that
$$I, \sigma \models \ulcorner \forall x_0 P^1_0 x_0 \urcorner \implies I, \sigma \models \ulcorner (P^1_0 c_0 \wedge P^1_0 c_1) \urcorner.$$
(2) To this end fix a model I and assignment σ, and assume $I, \sigma \models \ulcorner \forall x_0 P^1_0 x_0 \urcorner$.
(3) By definition of $\models$ using the truth of the $\ulcorner \forall \urcorner$ clause, $I, \sigma[x_0 \mapsto d] \models \ulcorner P^1_0 x_0 \urcorner$ for every $d \in D$.
(4) As $\mathbf{val}(c_0) \in D$, in particular $I, \sigma[x_0 \mapsto \mathbf{val}(c_0)] \models \ulcorner P^1_0 x_0 \urcorner$.
(5) By definition of $\models$ using the truth of the primitive extension clause, since $\sigma[x_0 \mapsto \mathbf{val}(c_0)](x_0) = \mathbf{val}(c_0)$ we have $\mathbf{val}(c_0) \in \mathbf{ext}(P^1_0)$.
(6) By definition of $\models$ using the truth of the primitive extension clause, since $\mathbf{val}(c_0) \in \mathbf{ext}(P^1_0)$ we have $I, \sigma \models \ulcorner P^1_0 c_0 \urcorner$.
(7) We retrace steps (3)-(6) to conclude $I, \sigma \models \ulcorner P^1_0 c_1 \urcorner$.
(8) By definition of $\models$ using the truth of the $\ulcorner \wedge \urcorner$ clause, $I, \sigma \models \ulcorner (P^1_0 c_0 \wedge P^1_0 c_1) \urcorner$
(9) Detaching the assumption $I, \sigma \models \ulcorner \forall x_0 P^1_0 x_0 \urcorner$ from (2)-(8) we deduce
$$I, \sigma \models \ulcorner \forall x_0 P^1_0 x_0 \urcorner \implies I, \sigma \models \ulcorner (P^1_0 c_0 \wedge P^1_0 c_1) \urcorner.$$
(10) As model I and assignment σ are arbitrary, taking the universal closure completes the proof.

□

Example 4.2. $\ulcorner \forall y F y \urcorner \models \ulcorner F a \wedge F b \urcorner$.

Proof. Giving benefit of the doubt to the author, we must interpret $y \in \mathbb{V}$, $F \in \mathbb{P}_1$, and $a, b \in \mathbb{V} \cup \mathbb{C}$. Regardless of the ambiguity, whether a, b are variables, constants, or some combination, the same proof from above works.

(1) Fix a model I, assignment σ, and assume $I, \sigma \models \ulcorner \forall x_0 P^1_0 x_0 \urcorner$.
(2) By definition $I, \sigma[y \mapsto d] \models \ulcorner F y \urcorner$ for every $d \in D$.
(3) Define $\widehat{a} := \sigma(a)$ if a is variable and $\widehat{a} := \mathbf{val}(a)$ otherwise if a is constant.
(4) In particular $I, \sigma[y \mapsto \widehat{a}] \models \ulcorner F y \urcorner$.

(5) By definition $\widehat{a} \in \mathbf{ext}(F)$, and thus $I, \sigma \models \ulcorner Fa \urcorner$.
(6) We retrace steps (3)-(5) to conclude $I, \sigma \models \ulcorner Fb \urcorner$.
(7) By definition $I, \sigma \models \ulcorner (Fa \wedge Fb) \urcorner$ and this completes the proof.

□

Example 4.3. Let x and y be distinct variables. Then $\ulcorner \exists y \forall x Pxy \urcorner \models \ulcorner \forall x \exists y Pxy \urcorner$.

Proof. This example is written carelessly; not only must we interpret P as a binary predicate, but we must accept the symbol x as a variable. This conflicts with x defined as a function to label variables. We ignore the problem, pretending to change the x to x_0 or some unused indexing number.

(1) Fix a model I, and assume $\ulcorner \exists y \forall x Pxy \urcorner$ is true in I.
(2) Fix assignment σ; there is some $d \in D$ s.t. $I, \sigma[y \mapsto d] \models \ulcorner \forall x Pxy \urcorner$.
(3) Fix $d_1 \in D$; in particular $I, \sigma[y \mapsto d][x \mapsto d_1] \models \ulcorner Pxy \urcorner$.
(4) Since $x \neq y$, $\sigma[y \mapsto d][x \mapsto d_1] = \sigma[x \mapsto d_1][y \mapsto d]$. Thus the order can be interchanged, i.e. $I, \sigma[x \mapsto d_1][y \mapsto d] \models \ulcorner Pxy \urcorner$.
(5) By definition $I, \sigma[x \mapsto d_1] \models \ulcorner \exists y Pxy \urcorner$.
(6) As $d_1 \in D$ is arbitrary, $I, \sigma \models \ulcorner \forall x \exists y Pxy \urcorner$.

□

Example 4.4. $\ulcorner \exists x_0 \forall x_0 P_0^2 x_0 x_0 \urcorner \models \ulcorner \forall x_1 \exists x_1 P_0^2 x_1 x_1 \urcorner$.

Proof. We present two proofs: one which keeps track of the assignments and another which takes advantage of our conventions to hide their presence.

(1) Let $I \models \ulcorner \exists x_0 \forall x_0 P_0^2 x_0 x_0 \urcorner$.
(2) Fix an assignment σ; in particular there is some $d \in D$ s.t. $I, \sigma[x_0 \mapsto d] \models \ulcorner \forall x_0 P_0^2 x_0 x_0 \urcorner$.
(3) Fix $d_1 \in D$; in particular $I, \sigma[x_0 \mapsto d][x_0 \mapsto d_1] \models \ulcorner P_0^2 x_0 x_0 \urcorner$.
(4) Thus $(d_1, d_1) \in \mathbf{ext}(P_0^2)$ and so $I, \sigma[x_1 \mapsto d_1][x_1 \mapsto d_1] \models \ulcorner P_0^2 x_1 x_1 \urcorner$.
(5) There exists $d \in D$, namely d_1, s.t. $I, \sigma[x_1 \mapsto d_1][x_1 \mapsto d] \models \ulcorner P_0^2 x_1 x_1 \urcorner$.
(6) So by definition $I, \sigma[x_1 \mapsto d_1] \models \ulcorner \exists x_1 P_0^2 x_1 x_1 \urcorner$.
(7) Generalizing over $d_1 \in D$ and then σ gives us $I \models \ulcorner \forall x_0 \exists x_0 P_0^2 x_0 x_0 \urcorner$.

Observe the interplay between both variables. Here is the same proof in words:

(1) Let I be a model which satisfies $\ulcorner \exists x_0 \forall x_0 P_0^2 x_0 x_0 \urcorner$.
(2) There is some $d \in D$ s.t. $\ulcorner \forall x_0 P_0^2 x_0 x_0 \urcorner$ is true in I under $[x_0 \mapsto d]$.
(3) Fix $d_1 \in D$; in particular $\ulcorner P_0^2 x_0 x_0 \urcorner$ is true in I under $[x_0 \mapsto d][x_0 \mapsto d_1]$.
(4) $(d_1, d_1) \in \mathbf{ext}(P_0^2)$ and so $\ulcorner P_0^2 x_1 x_1 \urcorner$ is true in I under $[x_1 \mapsto d_1][x_1 \mapsto d_1]$.
(5) There exists $d \in D$ s.t. $\ulcorner P_0^2 x_1 x_1 \urcorner$ is true in I under $[x_1 \mapsto d_1][x_1 \mapsto d]$.
(6) Thus $\ulcorner \exists x_1 P_0^2 x_1 x_1 \urcorner$ is true in I under $[x_1 \mapsto d_1]$.
(7) As $d_1 \in D$ is arbitrary, we have $\ulcorner \forall x_1 \exists x_1 P_0^2 x_1 x_1 \urcorner$ is true in I.

□

Remark 4.5. This example represents the previous when one forces x to equal y. Interestingly, the ill practice of reusing variables when binding quantifiers is still well-behaved. Observe the x_0 in $\ulcorner \exists x_0 \ \forall x_0 \ P_0^2 x_0 x_0 \urcorner$ which occurs with $P_0^2 x_0 x_0$, is bound by the universal, the innermost quantifier — not the outermost.

Example 4.6. $\ulcorner \forall x_0 (F x_0 \to G c_0) \urcorner, \ulcorner \exists x_0 F x_0 \urcorner \models \ulcorner (\neg G c_0 \to F c_0) \urcorner$.

Proof. Again we present two proofs.

(1) Let $I \models \ulcorner \forall x_0 (F x_0 \to G c_0) \urcorner$ and $I \models \ulcorner \exists x_0 F x_0 \urcorner$, and fix assignment σ.

(2) There must be some $d \in D$ which witnesses $I, \sigma[x_0 \mapsto d] \models \ulcorner Fx_0 \urcorner$.
(3) Since $I \models \ulcorner \forall x_0(Fx_0 \to Gc_0)\urcorner$, in particular $I, \sigma[x_0 \mapsto d] \models \ulcorner(Fx_0 \to Gc_0)\urcorner$.
(4) From (2) and (3) it follows $I, \sigma[x_0 \mapsto d] \models \ulcorner Gc_0 \urcorner$.
(5) Therefore $\mathbf{val}(c_0) \in \mathbf{ext}(G)$ and so $I, \sigma \models \ulcorner Gc_0 \urcorner$.
(6) Assume $I, \sigma \models \ulcorner \neg Gc_0 \urcorner$; by definition $I, \sigma \not\models \ulcorner Gc_0 \urcorner$.
(7) Contradiction; hence we may conclude vacuously $I, \sigma \models \ulcorner Fc_0 \urcorner$.
(8) Detaching from the assumption $I, \sigma \models \ulcorner Gc_0 \urcorner$ in (6)-(8) gives
$$I, \sigma \models \ulcorner \neg Gc_0 \urcorner \implies I, \sigma \models \ulcorner Fc_0 \urcorner.$$
(9) Therefore $I, \sigma \models \ulcorner(\neg Gc_0 \to Fc_0)\urcorner$ and as σ is arbitrary, $I \models \ulcorner(\neg Gc_0 \to Fc_0)\urcorner$.

Notice the contradiction we reach in (5) and (6) is with $I, \sigma \models \ulcorner Gc_0 \urcorner$ and $I, \sigma \not\models \ulcorner Gc_0 \urcorner$. We could have striped this down one more level to contradict $\mathbf{val}(c_0) \in \mathbf{ext}(G)$ with $\mathbf{val}(c_0) \notin \mathbf{ext}(G)$, but that is unnecessary; just as membership in extensions is part of our set theory meta language, so is satisfaction.

Also observe that in (5) we infer $I, \sigma \models \ulcorner Gc_0 \urcorner$ from $\mathbf{val}(c_0) \in \mathbf{ext}(G)$ applying basic definitions. But we could infer $I, \sigma \models \ulcorner Gc_0 \urcorner$ directly from (4) applying **Theorem 3.11** since x_0 does not occur free in $\ulcorner Gc_0 \urcorner$.

Here is the same proof disguised by words.

(1) Let $\ulcorner \forall x_0(Fx_0 \to Gc_0)\urcorner$ and $\ulcorner \exists x_0\ Fx_0 \urcorner$ be true in I.
(2) There must be some $d \in D$ which witnesses $\ulcorner Fx_0 \urcorner$ true in I under $[x_0 \mapsto d]$.
(3) Since $\ulcorner \forall x_0(Fx_0 \to Gc_0)\urcorner$ is true in I, in particular $\ulcorner(Fx_0 \to Gc_0)\urcorner$ is true in I under $[x_0 \mapsto d]$.
(4) From (2) and (3) it follows $\ulcorner Gc_0 \urcorner$ is true in I under $[x_0 \mapsto d]$.
(5) Therefore $\mathbf{val}(c_0) \in \mathbf{ext}(G)$ and so $\ulcorner Gc_0 \urcorner$ is true in I. (Alternatively, by **Theorem 3.11** since x_0 does not occur free in $\ulcorner Gc_0 \urcorner$, we have $\ulcorner Gc_0 \urcorner$ is true in I.)
(6) If $\ulcorner \neg Gc_0 \urcorner$ were true in I then we would would reach contradiction and vacuously conclude $\ulcorner Fc_0 \urcorner$ is true in I.
(7) Therefore $\ulcorner(\neg Gc_0 \to Fc_0)\urcorner$ is true in I.

□

Example 4.7. $\ulcorner \neg\forall x_0 \psi \urcorner \models \ulcorner \exists x_0 \neg\psi \urcorner$ for all $\psi \in \mathbb{WF}$.

Proof. Fix wff ψ.

(1) Fix a model I, assignment σ, and assume $I, \sigma \models \ulcorner \neg\forall x_0 \psi \urcorner$.
(2) By definition $I, \sigma \not\models \ulcorner \forall x_0 \psi \urcorner$, so it is false that $I, \sigma[x_0 \mapsto d] \models \ulcorner \psi \urcorner$ for every $d \in D$.
(3) Negating the universal quantifier in ZFC gives some $d_1 \in D$ such that $I, \sigma[x_0 \mapsto d_1] \not\models \ulcorner \psi \urcorner$.
(4) Thus there exists $d \in D$, namely d_1, such that $I, \sigma[x_0 \mapsto d] \models \ulcorner \neg\psi \urcorner$.
(5) $I, \sigma \models \ulcorner \exists x_0 \neg\psi \urcorner$ and hence the result.

Here is the same proof.

(1) Take any model I for which $\ulcorner \neg\forall x_0 \psi \urcorner$ is true in I.
(2) Then $\ulcorner \forall x_0 \psi \urcorner$ is false in I.
(3) There is some $d_1 \in D$ such that $\ulcorner \psi \urcorner$ is false in I under $[x_0 \mapsto d_1]$.
(4) $\ulcorner \neg\psi \urcorner$ is true in I under $[x_0 \mapsto d_1]$.
(5) Thus there exists $d \in D$, namely d_1, such that $\ulcorner \neg\psi \urcorner$ is true in I under $\sigma[x_0 \mapsto d]$.

(6) $\ulcorner \exists x_0 \neg \psi \urcorner$ is true in I, and hence the result.

$\square$

Every proof we have given can be equivalently transformed into a proof denying the existence of a counter models. For we can assume the existence of the counter model, run it through our proof, and find a contradiction. The converse is also true: every proof denying the existence of a counter model can be transformed into a direct one. The following theorem makes this clear.

Theorem 4.8. *For all $\Gamma \subset \mathbb{WF}$ and $\psi \in \mathbb{WF}$, if there does not exist a model I such that $I \models \theta$ for all $\theta \in \Gamma$ and $I \not\models \psi$, then $\Gamma \models \psi$.*

Proof. Let Γ be a set of wffs and ψ a wff. Assume there does not exist such counter models. Now fix a model I and assume all formulas in Γ are true in I. By the law of excluded middle either $I \models \psi$ or $I \not\models \psi$. It cannot be the second case because then we find a counter a model. So it must be $I \models \psi$ and hence our proof. $\square$

Remark 4.9. One might the think the law of excluded middle is required here. Even in a constructive meta theory, our proof can still be salvaged but weakened to the condition ψ is a sentence. For we covet the property $I \models \psi$ or $I \models \ulcorner \neg \psi \urcorner$. Thus we would be able to conclude $I \models \ulcorner \neg\neg \psi \urcorner$. Despite a constructive meta theory, the encoded Carnap is classical and so we would expect $I \models \psi$ to follow.

§5 Conclusion

We have carefully worked out a formal model theory for Carnap's Quantificational Logic. Our main goal was to demonstrate how the formalist may interpret a model theory: "truth" is simply "proof" in a broader meta language. A natural question to ask is how might a model theory of Carnap look like in an intuitionistic meta language. We have seen exorbitant use of the law of excluded middle; what additional assumptions are needed to relinquish their use — which constructions hold and which fail?

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