Review of Convergence Tests For Series.

On p-Series and Raabe's Test.

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Abstract

The comparison test serves as basis for other tests — e.g. the ratio test produces a comparison with a geometric series and Raabe's Test with a p-series. In this paper, we trace the lineage of Raabe's test and find it originates out of the success (and shortcomings) of the Ratio Test.

The purpose of this paper is primarily for the author to gain understanding by carefully reviewing and explicating, in their own words, elegant proofs of these tests. We review Yang Hansheng's alternative proof of the p-series, which separates the p-series into sums of even and odd terms, and Po-Lam Yung's proof of Raabe's Test which rewrites a series $\sum a_k$ by $\sum((k+1)-k)a_k$, and applies Abel's Lemma.

1 Ratio Test to Raabe's Test

First, we begin with the fundamental tool which allows us to determine convergence by comparing series.

Theorem 1.1 (Comparison Test for Series)**.** *Let a_n and b_n be sequences of nonnegative terms for which every $a_n \le b_n$. Then*

(1) *if Σa_n diverges then Σb_n diverges, and*

(2) *if Σb_n converges then Σa_n converges and satisfies*

$$\sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n.$$

Proof. Both items follow from the statement

$$\sum_{j=1}^{n} a_j \le \sum_{j=0}^{n} b_j \text{ for every } n \in \mathbb{N},$$

which is immediate by induction. In **(1)** if Σa_n diverges and Σb_n converges, then we are be able to bound Σa_n; monotone convergence implies the contradiction Σa_n converges. Similarly in **(2)** if Σb_n converges to b, then we are able to bound Σa_n; thus Σa_n converges to some a. Now should $b < a$, we may choose $N \in \mathbb{N}$ large which satisfies simultaneously both:

$$\begin{aligned} \left| \left(\sum_{j=1}^{N} a_j \right) - a \right| &< \frac{a-b}{2} \\ \left| \left(\sum_{j=1}^{N} b_j \right) - b \right| &< \frac{a-b}{2}. \end{aligned}$$

This supplies the contradiction $\sum_{j=1}^{N} b_j < \frac{a+b}{2} < \sum_{j=1}^{N} a_j$. ■

We shall use comparison test together with a particular series, like the geometric or p-series. In here we go over a clever approach for a p-series, by Yang Hansheng, which separates the sum into even and odd terms to estimate a bound.

Theorem 1.2 (p-Series)**.** *The series $\Sigma \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \leq 1$.*

Proof. The case when $p \leq 0$ is immediate — for $1/n^p$ fails to converge to 0. So that leaves us with two cases: $0 < p \leq 1$ and $p > 1$. Let's abbreviate the partial sums by $s_n = \sum_{k=1}^{n} \frac{1}{k^p}$. Observe since $p > 0$,

$$\begin{aligned} s_n &\leq s_{2n} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \frac{1}{(2n)^p} \\ &= 1 + \frac{1}{2^p}\left(1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots \frac{1}{n^p}\right) + \left(\frac{1}{3^p} + \frac{1}{5^p} + \cdots \frac{1}{(2n-1)^p}\right) \\ &\leq 1 + \frac{s_n}{2^p} + \left(\frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{6^p} + \cdots \frac{1}{(2n)^p}\right) \\ &\leq 1 + \frac{2s_n}{2^p}, \end{aligned}$$

and therefore $s_n \leq 1 + \frac{2s_n}{2^p}$. When $p > 0$ we can factor to obtain a bound $s_n \leq \frac{1}{\left(1 - \frac{2}{2^p}\right)}$, and thus s_n converges. When $0 < p \leq 1$ if s_n converges then we can divide to obtain the contradiction $1 \leq 1 + \frac{2}{2^p}$, and thus s_n diverges. ■

Theorem 1.3 (Ratio Test)**.** *Let a_n be a sequence of positive terms in which*

$$\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = L.$$

(1) *If $L < 1$ then Σa_n converges.*

(2) *If $L > 1$ then Σa_n diverges.*

(3) *If $L = 1$ then the test is inconclusive.*

Proof. **(1)** Suppose $L < 1$, and observe the positivity of terms forces L nonnegative. Put $r := \frac{1+L}{2}$, so that $0 \le r < 1$ and $r - L$ is positive. In particular there is a natural N such that

$$\frac{a_{n+1}}{a_n} < r \text{ for every } n \ge N.$$

The idea is, for every $n \ge 1$,

$$a_{N+n} = a_N \left(\frac{a_{N+1}}{a_N}\right)\left(\frac{a_{N+2}}{a_{N+1}}\right)\cdots\left(\frac{a_{N+n}}{a_{N+(n-1)}}\right),$$

as all intermediate terms cancel, and therefore $a_{N+n} \le a_N r^n$. We have successfully compared the long-term behavior of the series to a geometric one. Since of course $\sum a_N r^n$ converges, it follows $\sum_{n=N}^{\infty} a_n$ — and thus $\sum_{n=1}^{\infty} a_n$ — converges.

(2) When $L > 1$ similarly we may put $r := \frac{1+L}{2}$ so that $r > 1$ and $L - r$ is positive. Again there is a natural N such that $r < \frac{a_{n+1}}{a_n}$ for every $n \ge N$. The same idea in **(1)** flips the comparison to $a_N r^n \le a_{N+n}$, for every $n \in \mathbb{N}$. Since $\Sigma a_N r^n$ diverges, it follows Σa_n diverges.

(3) The p-series $\Sigma \frac{1}{n^p}$ produces converging and diverging examples in which, independent of p, the ratio of terms $\left(\frac{n}{n+1}\right)^p$ converges to 1. Another example is the alternating series $\Sigma \frac{(-1)^n}{n}$ and $\Sigma 1$. One benefit of this example is that we exchange our dependance on p-series in favor of the alternating test. ■

Remark. Often item **(3)** in this proof, when $L = 1$, is overlooked. Imagine two competing Ratio Tests, one with **(3)** and the other without. The first communicates more information than the second; for in absence of **(3)**, a student first learning calculus may think it possible $L = 1$ guarantees divergence — but perhaps its proof so elusive that the question remains open. **(3)** ensures neither divergence nor convergence can be concluded when $L = 1$, and in this sense *"completes"* the ratio test.

In case the Ratio Test is inconclusive for a series Σa_n, obviously the sequence $1-\frac{a_{n+1}}{a_n}$ converges to 0. A family of convergence tests arise by studying how quickly $1-a_n$ converges to zero. In particular the Raabe's Test emerges when $1-\frac{a_{n+1}}{a_n}$ falls off as r/n.

Theorem 1.4 (Raabe's Test). *Let a_n be a sequence of positive terms in which*

$$\lim_{n\to\infty} n\left(1-\frac{a_{n+1}}{a_n}\right)=L.$$

(1) *If $L>1$ then Σa_n converges.*

(2) *If $L<1$ then Σa_n diverges.*

(3) *If $L=1$ then the test is inconclusive.*

Proof. **(1)** Suppose $L>1$ and put $r:=\frac{1+L}{2}$, as in the Ratio Test, so that $r>1$ and $L-r$ is positive. In particular there is an $N\in\mathbb{N}$ large such that

$$r<n\left(1-\frac{a_{n+1}}{a_n}\right),\ \text{for every } n\geq N,$$

which implies

$$\frac{a_{n+1}}{a_n}<\left(1-\frac{r}{n}\right),\ \text{for every } n\geq N.$$

Following Po-Lam Yung's proof, for all $n>N$,

$$\begin{aligned}
\sum_{k=N}^{n} a_k &= \sum_{k=N}^{n} a_k\,((k+1)-k)\\
&= a_n(n+1)-a_N N+\sum_{k=N}^{n-1}(k+1)(a_{k+1}-a_k)\\
&\geq a_n(n+1)-a_N N+\sum_{k=N}^{n-1}\frac{r(k+1)}{k}a_k
\end{aligned}$$

where the second line follows from Abel's Lemma. Hence for every $n>N$,

$$\sum_{k=N}^{n}(r-1)a_k\leq\sum_{k=N}^{n}\left(r\left(\frac{k+1}{k}\right)-1\right)a_k\leq a_N N-a_n(n+1)\leq a_N N.$$

Looking at the head and tail (since $r-1$ is positive) we can bound the eventual partial sums, $\sum_{k=N}^{n}a_k\leq a_N N/(r-1)$ for each $n>N$, and therefore Σa_k converges.

(2) When $L<1$, again put $r:=\frac{1+L}{2}$ so that $r<1$ and $r-L$ is positive. There is an $N\in\mathbb{N}$ such that

$$n\left(1-\frac{a_{n+1}}{a_n}\right)<r,\ \text{for every } n\geq N,$$

which implies

$$\left(1+\frac{1}{n}\right)\left(1-\frac{r}{n}\right) < \frac{(n+1)a_{n+1}}{na_n}, \text{ for every } n \geq N.$$

Observe the left hand side is $1+\frac{n(1-r)-r}{n^2}$, which we would like to say is greater than or equal to 1. In case r is nonpositive, this conclusion is clear. But in case r is positive, to achieve this, choose $M \in \mathbb{N}$ by the Archimedean such that $M(1-r) > 1$. In this way, if r is nonpositive then $na_n \leq (n+1)a_{n+1}$ for every $n \geq N$, and if r is positive, then $na_n \leq (n+1)a_{n+1}$ for every $n \geq \max\{N, M\}$, and therefore in either case the sequence na_n eventually increases. That is, there is some $L \in \mathbb{N}$ such that $(a_L L)^{1/n} \leq a_n$ for every $n \geq L$. Therefore Σa_n diverges by the Comparison Test against the harmonic series.

(3) The family of series $\Sigma \frac{1}{n(\log n)^p}$, for each $p \geq 1$, produces converging and diverging examples with limit $n\left(1-\frac{n(\log n)^p}{(n+1)\log(n+1)^p}\right) \to 1$. For example, by Cauchy's Condensation Test, when $p=1$ the series converges and when $p=2$ the series diverges. ■

Remark. First, while our proof does not explicitly draw a comparison with the p-series in the convergent case, such proofs are possible but they use more involved arguments which estimate logarithms. Second, Raabe's test is usually presented by considering the limit of $n\left(\frac{a_n}{a_{n+1}}-1\right)$. However, we find it more natural to consider the limit $n\left(1-\frac{a_{n+1}}{a_n}\right)$. Indeed these two limits are the same.

As mentioned, whenever the limit used in Raabe's Test exists, we must be a part of the inconclusive case of the Ratio Test. We conclude with two simple examples: one where Raabe's Test guarantees divergence, and another where it guarantees convergence.

Example 1.1. *Apply Raabe's Test to the series* $\sum \frac{4\cdot 7\cdot \ldots (3n+1)}{n!}\left(\frac{1}{3}\right)^n$.

Begin.

$$\begin{aligned}
& \lim_{n\to\infty} n\left(1-\frac{n!(4\cdot 7\cdot ...(3n+1)\cdot(3n+4))}{3(n+1)!(4\cdot 7\cdot ...(3n+1))}\right) \\
= & \lim_{n\to\infty} n\left(1-\frac{3n+4}{3n+3}\right) \\
= & \lim_{n\to\infty} \left(\frac{-n}{3n+3}\right) \\
= & -1/3.
\end{aligned}$$

Therefore the series diverges by Raabe's Test. *End.*

Example 1.2. *Apply Raabe's Test to the series* $\sum \frac{3^2\cdot 5^2\cdot ...(2n+1)^2}{6^2\cdot 8^2\cdot ...(2n+4)^2}$.

Begin.

$$\begin{aligned}
& \lim_{n\to\infty} n\left(1-\frac{3^2\cdot 5^2\cdot ...(2n+3)^2\cdot 6^2\cdot 8^2\cdot ...(2n+4)^2}{3^2\cdot 5^2\cdot ...(2n+1)^2\cdot 6^2\cdot 8^2\cdot ...(2n+6)^2}\right) \\
= & \lim_{n\to\infty} n\left(1-\frac{(2n+3)^2}{(2n+6)^2}\right) \\
= & \lim_{n\to\infty} \left(\frac{12n^2+25n}{(4n^2+24n+36}\right) \\
= & 3.
\end{aligned}$$

Therefore the series converges by Raabe's Test. *End.*

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