Paradoxes in Analysis:

The Strangeness of Infinity and Wonders of the Axiom of Choice

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Banach-Tarski's Paradox

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Overview

Theorem

Let "$\forall n,\ \psi(n) \implies \psi(n+1)$" and "$\psi(0)$" be true in PA. "$\psi(100)$" is true.

Proof.

1. $\psi(0)$ (context)
2. $\psi(0) \implies \psi(1)$ (forall elim)
3. $\psi(1)$ (modus ponens)
4. $\psi(1) \implies \psi(2)$ (forall elim)
201. $\psi(100)$ (modus ponens) $\square$

- To prove "$\forall n,\ \psi(n)$" we generally need induction.
- Induction chains *infinitely many* modus ponens (imp elims).

Induction vs Choice

Another perspective of choice

Theorem

Let "$\forall n \in \mathbb{N}\ \exists x,\ \psi(n, x)$" be true in ZFC. Then "there exists a finite sequence $\{x_i\}_{i=1}^{100}$ s.t. $\psi(i, x_i)$ for all $i \leq 100$" is true.

Proof.

1. $\exists x,\ \psi(1, x)$ (forall elim)
2. $\psi(1, x_1)$ (exists elim)
3. $\exists x,\ \psi(2, x)$ (forall elim)

200. $\psi(100, x_{100})$ (exists elim)

367. $\exists \{x_i\}_{i=1}^{100}\ \forall i \leq 100,\ \psi(i, x_i)$ (easy but long) □

- To prove "there exists a sequence $\{x_n\}$ s.t. $\psi(n, x_n)$ for all n" we generally need choice.
- Choice chains *infinitely many* exists elims.

Function vs Oracle

Yet another perspective of choice

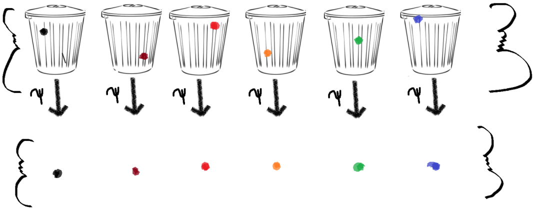
Axiom of Choice (ZFC)

Let $\mathcal{A}$ be a collection of nonempty sets. There exists a choice function $f : \mathcal{A} \to \bigcup \mathcal{A}$ s.t. $f(A) \in A$ for all A.

Axiom of Choice (2nd Order)

There exists a choice oracle ψ : Set $\to$ Set, i.e. a class function, s.t. $\psi(X) \in X$ for all nonempty sets X.

Figure: choice oracle + replacement = choice function



How is Choice Used?

Finding sets of representatives.

1. We'll define an equivalence relation $\sim$ on a set X.
2. The set of equivalence classes,
$$X/\sim := \{[x] : x \in X\},$$
partitions X.
3. We'll pick a choice function
$$f : X/\sim \to X$$
s.t. $f([x]) \in [x]$ for all $x \in X$.
4. It's image is what we care about.

Figure: Equivalence classes...literally!

Vitali's Set in $\mathbb{R}^1$

Immeasurable subset of $[0, 1]$.

Definition (Equivalence Relation on $\mathbb{R}$)

Two points are *related* iff they are rational translates.
The set of equivalence classes is denoted by $\mathbb{R}/\mathbb{Q}$.

1. There is a choice function $f : \mathbb{R}/\mathbb{Q} \to \mathbb{R}$ which picks out exactly one representative from each class.
2. A precision: we force f to pick representatives from $[0, 1]$.

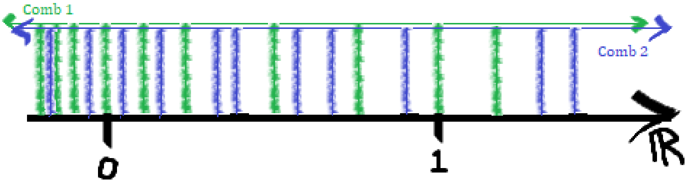
Definition (Vitali's Set)

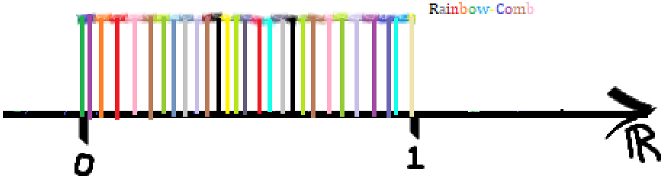
The image of f is called *Vitali's set*.

Combs!

Visualization of Vitali's set

1. Imagine $\mathbb{R}/\mathbb{Q}$ as a set of 2-d *combs* distinguished by color.
2. Elements of a comb are its teeth which bite the line.
3. Vitali's set is a rainbow comb, all its teeth have different colors.





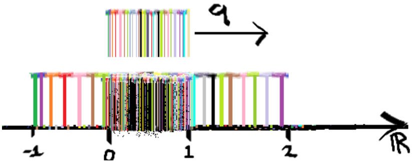
Theorem (Vitali)

There is no set function $\mu : \mathcal{P}(\mathbb{R}) \to [0, +\infty]$ which is translation invariant, assigns intervals their usual lengths, and is σ-additive.

Sketch of Proof:

$$[0,1] \subset \bigcup_{q \in \mathbb{Q} \cap [-1,1]} (f(\mathbb{R}/\mathbb{Q}) + q) \subset [-1,2]$$

Figure: Rational translates of rainbow combs



Immeasurable set in R^d

Visualized as Brushes, or higher dimensional combs

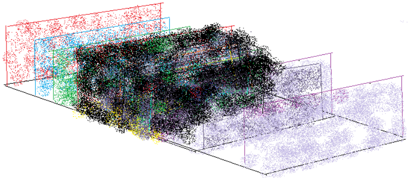
Theorem (Extended Vitali)

There is no set function $\mu : \mathbb{R}^d \to [0, \infty]$ which is translation invariant, assigns rectangles their usual volume, and is σ-additive. (all d $\in \mathbb{N}$)

Sketch of Proof: For $d \geq 2$,

$$[0,1]^d \subset \bigcup_{q \in \mathbb{Q} \cap [-1,1]} (f(\mathbb{R}/\mathbb{Q}) + q) \times [0,1]^{d-1} \subset [-1,2] \times [0,1]^{d-1}.$$

Figure: Rational translates of rainbow brushes



Theorem (Extended Vitali)

There is no set function $\mu : \mathcal{P}(\mathbb{R}^d) \to [0, \infty]$ which is translation invariant, assigns rectangles their usual volume, and is σ-additive.

Theorem (Corollary of Paradox)

There is no set function $\mu : \mathcal{P}(\mathbb{R}^3) \to [0, \infty]$ which is **rotation** *invariant, assigns rectangles their usual volume, and is* **finitely** *additive.*

Theorem (Banach-Tarski's Paradox)

D^3 is equidecomposable with two identical copies of D^3.

Definitions and Set-up

Preparation

Definition (Equidecomposable)

Two subsets of $\mathbb{R}^3$ are *equidecomposable* if we can partition one into finitely many pieces and then reassemble them, by rigid transformations, into the other. No overlaps or unused pieces.

Definition (Elementary Rotations)

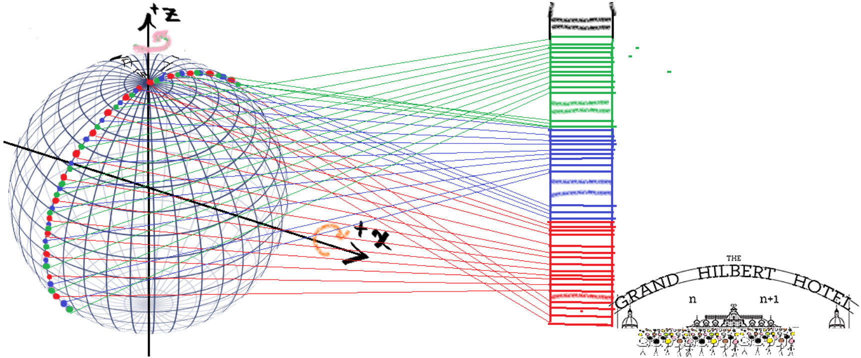
Fix θ as an irrational multiple of π. Define $x, z : \mathbb{R}^3 \to \mathbb{R}^3$ as clockwise rotations of the $+x$ and $+z$ axes, resp., by θ.

Definition (Paths)

A *path* is a finite sequence of rotations x, x^{-1}, z, and z^{-1} without consecutive occurrences of xx^{-1}, $x^{-1}x$, zz^{-1}, or $z^{-1}z$. A path $\{p_i\}$ *connects* $a \in \mathbb{R}^3$ to $b \in \mathbb{R}^3$ only when $(p_n \circ \ldots p_2 \circ p_1)(a) = b$.

Some Examples

1. Are $(3,4,5)$ and $(1,2,6)$ connected? (Hint: x, z preserve radius!)
2. Consider points $(0,0,1)$, $x(0,0,1)$, $x^2(0,0,1)$, $\ldots$, etc. Are they all different? (Hint: θ is an irrational multiple of π)
3. Remarkably, if we compose along a nonempty path, then we'll never get the identity map. (Important!)

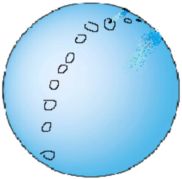


Proof.

Proof of Equidecomposision

1. We show D^3 equidecomposes to D^3 w/o the north pole $N := (0, 0, 1)$.
2. Partition into two pieces $\{D^3 - E, E\}$, where $E := \{N, xN, x^2N, \ldots\}$.

$\square$



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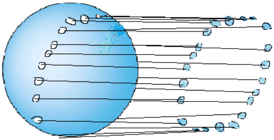

D^3 Equidecomposes to D^3 Without the Origin

Hilbert's Hotel Trick

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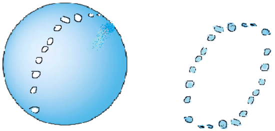

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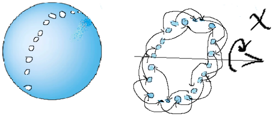
2. Partition into two pieces $\{D^3 - E, E\}$, where $E := \{N, xN, x^2N, \ldots\}$.
3. Rotate E by x.

$\square$



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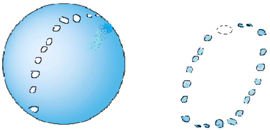
2. Partition into two pieces $\{D^3 - E, E\}$, where $E := \{N, xN, x^2N, \ldots\}$.
3. Rotate E by x. By design $xE = \{xN, x^2N, \ldots\} = E - N$.

$\square$



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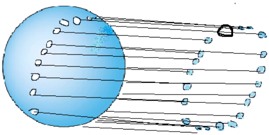
3. Rotate E by x. By design $xE = \{xN, x^2N, \dots\} = E - N$.
4. The two pieces $\{D^3 - E, xE\}$ partition $D^3 - N$. □



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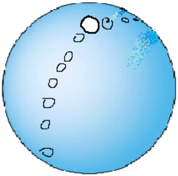


1. It's enough to equidecompose S^2 into two identical copies.
2. Imagine D^3, without the center, as a spherical cake.
3. Using a knife, cut radially along the partition of S^2.
4. Apply the same reassembly process on these volume pieces.

Initial Strategy to Double S^2

Definition (Equivalence Relation $\sim$ on S^2)

Given by $a \sim b$ iff there exists a path which connects a to b. The equivalence class at a point $a \in S^2$, is called the *orbit* of a.

Attempt breaking into 5 pieces!

1. M of representatives, one from each orbit, called *initial points*.

   All points correspond to a unique word!
   (Precision: except for so called *fixed points*.)

2. $S(x)$ of points connected by a path, from an initial point, that ends with x, called *x-endpoints*.
3. $S(x^{-1})$ of *x^{-1}-endpoints*.
4. $S(z)$ of *z-endpoints*.
5. $S(z^{-1})$ of *z^{-1}-endpoints*.

Why These Pieces?

Crux of the Paradox!

Lemma

Rotations of $S(x^{-1})$ by x and $S(z^{-1})$ by z, satisfy:

$$\begin{aligned} xS(x^{-1}) = \{x(a) : a \in S(x^{-1})\} &= M \cup S(x^{-1}) \cup S(z) \cup S(z^{-1}), \\ zS(z^{-1}) = \{z(a) : a \in S(z^{-1})\} &= M \cup S(x) \cup S(x^{-1}) \cup S(z^{-1}). \end{aligned}$$

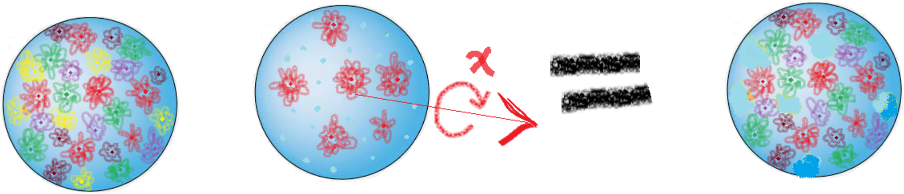
Sketch of Proof of $xS(x^{-1}) \subset M \cup S(x^{-1}) \cup S(z) \cup S(z^{-1})$:

1. Let a be an x^{-1}-endpoint.
2. Then $(x^{-1} \circ p_{n-1} \circ \cdots \circ p_1)(m) = a$ for some initial point m.
3. $(p_{n-1} \circ \cdots \circ p_1)(m) = x(a)$
4. Is $p_1, \ldots, p_{n-1}$ a bonafide path? Can $p_{n-1} = x$? What if $n = 1$?

What's so Magical?

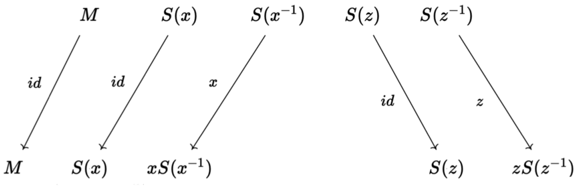
Visualized As Bouquets

1. View S^2 as a spherical bouquet of flowers.
2. There are five varieties corresponding to the decomposition.
3. Twirl the $S(x^{-1})$ variety of flowers by x.
4. Morphs into four varieties (all except $S(x)$)!



Identifying the Problem!

Figure: Here's how we'd want to decompose S^2



Problems:

1. $\{M, S(x), S(x^{-1}), S(z), S(z^{-1})\}$ isn't a partition.
2. $\{M, xS(x^{-1}), S(x)\}$ forms one sphere but isn't disjoint.
3. $\{zS(z^{-1}), S(z)\}$ forms the other, but also isn't disjoint.

The problematic set D consists of all *fixed points* in S^2 and elements of their orbits. E.g. the north pole is a fixed point of z^{-1}.

Resolution And Strategy for the Entire Paradox

We prove three equidecompositions (numbers signify *importance*):

$$
\begin{tikzcd}[column sep=large]
 & S^2 - D \arrow[r, leftrightsquigarrow, "1"] & (S^2 - D) \sqcup (S^2 - D) \arrow[dr, leftrightsquigarrow, bend left] & \\
S^2 \arrow[ur, leftrightsquigarrow, bend left, "3"] \arrow[rrr, leftrightsquigarrow, "\text{decompose sphere into two}"] \arrow[drrr, phantom, "\Downarrow \text{cake!}"] & & & S^2 \sqcup S^2 \\
D^3 - 0 \arrow[rrr, leftrightsquigarrow, "\text{decompose centerless ball into two}"'] \arrow[d, leftrightsquigarrow, "2"'] & & & (D^3 - 0) \sqcup (D^3 - 0) \arrow[d, leftrightsquigarrow] \\
D^3 \arrow[rrr, leftrightsquigarrow, "\text{decompose ball into two}"'] & & & D^3 \sqcup D^3
\end{tikzcd}
$$

References and Questions?

Thanks!

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