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THREE CLASSICAL THEOREMS ON INTERCHANGING LIMITS WITH INTEGRALS IN CALCULUS

(Communicated by Sudhir Murthy, Department of Mathematics)

Abstract. The interchange of the 'limit of an integral' with the 'integral of a limit' for sequences of functions is crucial in relevant applications, such as Fourier series for decomposing periodic functions into sinusoidal components, and Fubini's theorem for changing the order of integration of multivariable functions. This expository paper reviews three classical results in real analysis for cases where the limit of an integral of a sequence of functions equals the integral of the limiting function: **(1)** Monotone Convergence Theorem, **(2)** Uniform Convergence Theorem, and the broadest result, **(3)** Dominated Convergence Theorem. While proofs of **(2)** are typically studied in undergraduate analysis, the proofs of **(1)** and **(3)** are usually reserved for graduate-level measure theory, where they are taught in a more general context. The purpose of this paper is to summarize and adapt W. A. J. Luxembourg's undergraduate-friendly proof [7] of **(3)** Arzelà's Dominated Convergence Theorem, to demonstrate the nontrivial direction of **(1)** Monotone Convergence Theorem for Riemann Integrals. Our aim is to demystify the hidden logic involved in these well-established theorems, making them more accessible for undergraduate analysis.

keywords : Real Analysis, Riemann Integrals, Arzelà's Dominated Convergence Theorem, Monotone Convergence Theorem, Uniform Convergence Theorem

1. Introduction

The conditions for when a limit may be interchanged with an integral played a crucial role in the development of Fourier Series and Real Analysis. Consider the heat equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial^2 x^2}$ subject to the constraints $u(0,t) = u(1,t)$ for all t and $u(x,0) = f(x)$ for some "well-behaved" function f. In here, $u(x,t)$ represents the temperature of a circular rod of unit length at position $x \in [0,1]$ and time $t \geq 0$. Fourier's novel solution to this equation, which was awarded the Grand Prix of the Académie des Sciences [4], is

$$u(x,t) = \sum_{n=1}^{\infty} \left(2 \int_0^1 f(x') \sin(n\pi x')\, dx' \right) \sin(n\pi x) e^{-n^2\pi^2 t}.$$

However, as shown by Cauchy [1] and discussed by Dieudonné [3], rigorously verifying the proposed solution involves exchanging the order of limits with integrals.

As another example, consider the well-known argument justifying the series expansion of $\log(1+x)$. For a real number $|t| < 1$, the geometric series $\sum_{n=0}^{\infty}(-t)^n$ converges to $\frac{1}{1+t}$. Integrating from 0 to x, where $|x| < 1$, yields:

$$\int_0^x \frac{1}{1+t}\,dt = \int_0^x \sum_{n=0}^{\infty}(-t)^n\,dt \tag{1.1}$$

$$\implies \quad \log(1+x) = \int_0^x \lim_{k\to\infty} \sum_{n=0}^{k}(-t)^k\,dt \tag{1.2}$$

$$\implies \quad \log(1+x) = \lim_{k\to\infty} \int_0^x \sum_{n=0}^{k}(-t)^n\,dt \tag{1.3}$$

$$\implies \quad \log(1+x) = \lim_{k\to\infty} \sum_{n=0}^{k} \int_0^x (-t)^n\,dt \tag{1.4}$$

$$\implies \quad \log(1+x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}. \tag{1.5}$$

The seemingly intuitive, but unjustified, step in this derivation is the interchange between the integral and the limit in (1.3).

This paper provides an overview of three conditions for when we may interchange limits with integrals of sequences of functions: **(1)** a monotone sequence of functions, **(2)** a uniformly converging sequence of functions, and the weakest condition, **(3)** a dominated sequence of functions. First, we recall some basic definitions.

We consider the bounded closed interval $[a, b]$ for real numbers $a < b$. We use $\mathbb{N}$ to denote the set of natural numbers and $\mathbb{R}$ to denote the set of real numbers.

Definition 1.1. A sequence of functions $f_n : [a, b] \to \mathbb{R}$ is said to be:

(i) *Monotone* if and only if either $f_{n+1}(x) \geq f_n(x)$ for every $x \in [a, b]$ and $n \in \mathbb{N}$ (*monotone increasing*), or $f_n(x) \geq f_{n+1}(x)$ for every $x \in [a, b]$ and $n \in \mathbb{N}$ (*monotone decreasing*).

(ii) *Uniformly convergent* if and only if there exists some $f : [a, b] \to \mathbb{R}$ such that $f_n \to f$ pointwise and for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \varepsilon$ for all $x \in [a, b]$ and $n \geq N$.

(iii) *Dominated* if and only if there exists a positive, real M such that $|f_n(x)| \leq M$ for every $x \in [a, b]$.

One could weaken conditions **(1)** and **(3)** to "uniformly monotone" and "uniformly bounded", conveying the notions of sequences of functions that eventually behave in a monotone or bounded manner, that is, for all sufficiently large indices. The advantage of such an approach is the insight that conditions for when a limit may be interchanged with an integral require some form of "uniformity". The cost, however, is lengthier proofs. For example, proofs involving the limiting behavior of uniformly monotone sequences are mostly the same for monotone sequences, but with the added step of taking a sufficiently large m-tail subsequence to ensure monotonicity. By acknowledging uniformity, here we hope to provide the reader with this insight while maintaining simplicity of the original conditions.

In section two, we review the Uniform Convergence Theorem, usually taught as an exercise in undergraduate analysis, and the easy direction of Monotone Convergence Theorem. In section three, we follow Luxembourg's proof [7] of Dominated Convergence Theorem intended for undergraduate-level analysis. In the fourth section, we adapt his ideas to present a proof of the challenging direction of Monotone Convergence Theorem. Finally, in the conclusion, we highlight the importance of understanding the logic underlying these proofs and their utility for computer verified mathematics. Although the digital formalization of real analysis has largely been achieved, it seems to lack undergraduate-friendly proofs of the Dominated Convergence Theorem for Riemann Integrals. This paper aims to take the first steps towards bridging this gap, by elucidating the logic underpinning these paper-pen proofs.

2. Integrals of Monotone and Uniformly Convergent Sequences

We partially prove the result that limits can be interchanged with integrals for monotone sequences. The full result borrows the set-up in Luxembourg's paper. We prove the easy direction for now and save the other as Theorem 4.1.

Our paper is based on Darboux integrals defined as upper and lower rectangular estimates over partitions of $[a, b]$ by taking suprema and infima. By convention $\mathcal{B}[a, b]$, $\mathbb{R}[a, b]$, and $\mathcal{C}[a, b]$ represent, respectively, the family of bounded, Riemann integrable, and continuous real-valued functions on $[a, b]$.

Theorem 2.1 (Monotone Convergence Theorem Part 1)**.** *Let $f, f_n \in \mathcal{R}[a, b]$ with $f_n \to f$ pointwise and suppose f_n is monotone increasing. Then the sequence $\int_a^b f_n$ converges and, moreover,*

$$\lim_{n\to\infty} \int_a^b f_n \leq \int_a^b f.$$

Proof. Since $f_n(x) \leq f_{n+1}(x)$ for all $x \in [a, b]$ and $n \in \mathbb{N}$, by monotony of integration,

$$\forall n \in \mathbb{N} \quad \int_a^b f_n \leq \int_a^b f_{n+1}.$$

It follows from induction

$$\forall n \in \mathbb{N} \quad \int_a^b f_n \leq \int_a^b f,$$

and consequently the result follows. □

Example 2.1, by Bradley [5], provides a remarkable application of this theorem to produce a family of non-trivial results.

Example 2.1. For $f \in \mathcal{R}[0, 1]$ nonnegative, we show

$$\lim_{n\to\infty} \int_0^1 n \log\left(1 + \frac{f}{n}\right) \leq \int_0^1 f.$$

Note that the sequence of functions $n \log\left(1 + {}^{f}\!/_{n}\right) = \log\left(1 + {}^{f}\!/_{n}\right)^n$ converges to $\log(e^f) = f$. Moreover, since f is nonnegative, the sequence $\left(1 + {}^{f}\!/_{n}\right)^n$ is monotone increasing. As the logarithm is an increasing function, it follows that $\log\left(1 + f/n\right)^n$ is also monotone increasing, and thus Theorem 2.1 applies. Indeed, after proving Theorem 4.1, we can strengthen this example with equality.

We now turn our attention to the classical result found in calculus textbooks: the limit can be interchanged with an integral under uniform convergence.

Theorem 2.2 (Uniform Convergence Theorem)**.** *Let $f_n \in \mathcal{R}[a,b]$ converge uniformly to $f : [a,b] \to \mathbb{R}$. Then $f \in \mathcal{R}[a,b]$ and, moreover,*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Proof. Fix $\varepsilon > 0$ and pick $N \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\varepsilon}{(b-a)}$ for all x and $n \geq N$. In particular, $f_N(x) - \frac{\varepsilon}{(b-a)} < f(x) < f_N(x) + \frac{\varepsilon}{(b-a)}$ for all x, and as f_N is bounded it follows f is bounded. Thus f possesses lower and upper integrals, and

$$\textit{(i)}\ \forall n \geq N \quad \overline{\int_a^b} f - \int_a^b \frac{\varepsilon}{b-a} \leq \int_a^b f_n \leq \overline{\int_a^b} f + \int_a^b \frac{\varepsilon}{b-a}, \text{ and}$$
$$\textit{(ii)}\ \forall n \geq N \quad \underline{\int_a^b} f - \int_a^b \frac{\varepsilon}{b-a} \leq \int_a^b f_n \leq \underline{\int_a^b} f + \int_a^b \frac{\varepsilon}{b-a}.$$

As $\varepsilon > 0$ is arbitrary, $\int_a^b f_n \to \overline{\int_a^b} f$ and $\int_a^b f_n \to \underline{\int_a^b} f$, so f is Riemann integrable on $[a,b]$ and $\int_a^b f_n$ converges to the common value $\int_a^b f$. $\square$

Pointwise convergence is not a strong enough condition to justify the interchange of limit with integral, as shown by Example 2.2.

Example 2.2. Define the sequence of functions on $[0,1]$ by $f_1(x) = 1$ and

$$\forall n \geq 2 \quad f_n(x) := \begin{cases} n^2 x & \text{if } 0 \leq x < 1/n \\ 2n - n^2 x & \text{if } 1/n \leq x < 2/n \\ 0 & \text{otherwise.} \end{cases}$$

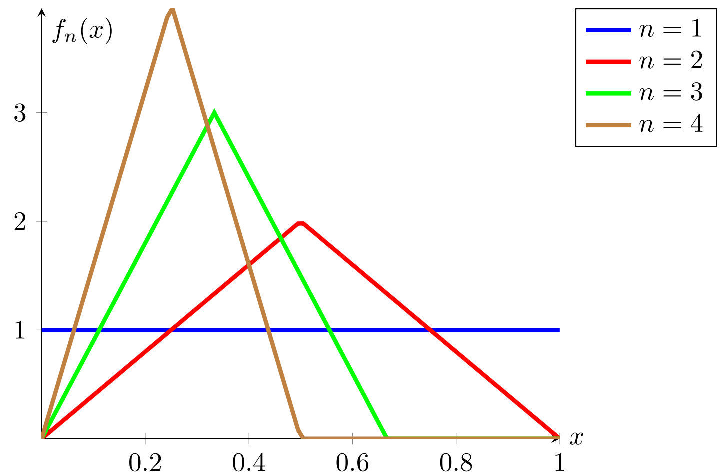

FIGURE 2.1. Graphs of the first four terms of the sequence of functions $f_n(x)$.

As illustrated by Figure 2.1, each function represents an isosceles triangle with height n and base $2/n$ and thus has unit area. However, f_n converges pointwise to the zero function and, thus, we have the disagreement $\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 f$.

3. Arzelà's Dominated Convergence Theorem.

Uniform convergence is a sufficient condition for interchanging a limit with an integral — however, Dominated Convergence Theorem weakens the condition. We follow W. A. J. Luxembourg's proof [7] of Arzelà's Dominated Convergence Theorem. We set the stage for his main arguments by proving Lemmas 3.1 and 3.2.

Lemma 3.1 (Dini's Uniform Convergence Theorem)**.** *Let* $g_n, g \in \mathcal{C}[a, b]$ *where* $g_n \to g$ *pointwise and* g_n *is monotone. Then the convergence is uniform.*

Proof. It is enough to show a monotone decreasing sequence $f_n \in \mathcal{C}[a, b]$ that converges pointwise to 0, converges uniformly. For we can transform the general problem $g_n \to g$ by putting $f_n := g_n - g$ or $f_n := g - g_n$ — whichever ensures the sequence of functions f_n monotonically decreases.

Given $\varepsilon > 0$, first define $N : [a,b] \to \mathbb{N}$ and then $\delta : [a,b] \to \mathbb{R}$ by:

$$\begin{aligned} N(x) \quad &:= \quad \min\{N' \in \mathbb{N} \mid \forall n \geq N',\ |f_n(x)| < {}^{\varepsilon}\!/2\} \text{ , and} \\ \delta(x) \quad &:= \quad \sup\{\delta' \in [0, b-a] \mid \forall t \in \mathcal{B}_{\delta'}(x) \cap [a,b],\ |f_{N(x)}(t) - f_{N(x)}(x)| < {}^{\varepsilon}\!/2\}. \end{aligned}$$

Observe that $N(x)$ is well-defined since the pointwise convergence of $f_n(x)$ guarantees the corresponding set of natural numbers is nonempty. Using continuity of $f_{N(x)}$ at x guarantees $\delta(x)$ is well-defined by completeness, since the corresponding set of real numbers is bounded and nonempty. In fact this supremum is also an element of the set and, thus, is the maximum.

For each $x \in [a,b]$, since $\{f_n\}_{n \in \mathbb{N}}$ is decreasing and nonnegative,

$$|f_n(t)| \leq |f_{N(x)}(t)| \leq |f_{N(x)}(t) - f_{N(x)}(x)| + |f_{N(x)}(x)| < \varepsilon \tag{1}$$

for every $n \geq N(x)$ and $t \in \mathcal{B}_{\delta(x)}(x) \cap [a,b]$. The idea is the family of open intervals $\{\mathcal{B}_{\delta(x)}(x) : x \in [a,b]\}$ is an open cover of $[a,b]$, and thus by compactness contains a finite subcover, say $\{\mathcal{B}_{\delta(x_1)}(x_1), \ldots, \mathcal{B}_{\delta(x_n)}(x_n)\}$ for some $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in [a,b]$. By taking $N := \max\{N(x_1), \ldots N(x_n)\}$, we can solve our current ε, N criterion as shown by Figure 3.1. $\square$

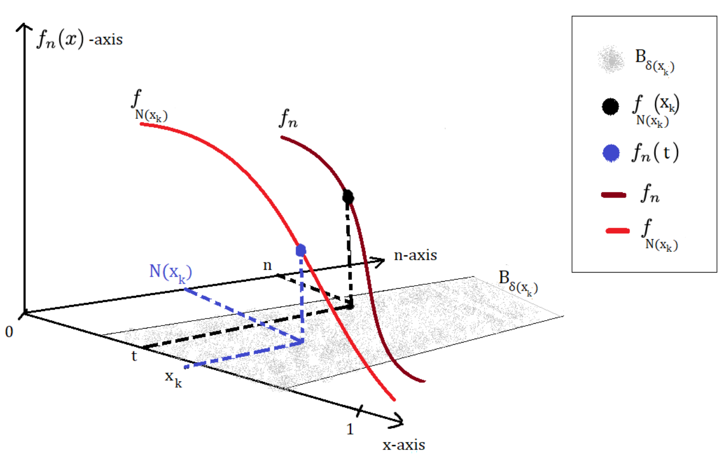

FIGURE 3.1. Any $t \in [a,b]$ belongs to an open interval $\mathcal{B}_{\delta(x_k)}(x_k)$ for some $k \in \mathbb{N}$. Since any $n \geq N$ is greater than or equal to $N(x_k)$, we can use an instance of (1) to estimate $|f_n(t)| < \varepsilon$.

Remark. We explicitly construct $N(x)$ and $\delta(x)$, and therefore avoid accidental applications of the axiom of choice. Notice our comment, that the supremum is the maximum in the definition of $\delta(x)$, is not superficial. It is precisely this topological property we use in our proof.

The condition $g \in \mathcal{C}[a,b]$ in the theorem is needed. For example consider $g_n(x) := x^n$ defined on $[0,1]$. The sequence monotonically decreases to the discontinuous function $g(x) := 1$ if $x = 1$ and $g(x) := 0$ otherwise. The convergence, however, is not uniform.

Lemma 3.2. *For all nonnegative $f \in \mathcal{B}[a,b]$ and $\varepsilon > 0$, there exists $g \in \mathcal{C}[a,b]$ such that $0 \leq g(x) \leq f(x)$ for all $x \in [a,b]$ and*

$$\underline{\int_a^b} f - \varepsilon \leq \int_a^b g.$$

Proof. Fix nonnegative $f \in \mathcal{B}[a,b]$ and $\varepsilon > 0$. By definition of lower integral, there must be a partition $P := \{x_0, \dots, x_n\}$ of $[a,b]$, where $a = x_0 < \cdots < x_n = b$ and

$$\underline{\int_a^b} f - \varepsilon/2 < L(P,f).$$

Using this partition, define a step function s on $[a,b]$ by

$$s(x) := \inf_{t \in [x_{i-1}, x_i]} f(t)$$

where $[x_{i-1}, x_i]$ is the first interval from the partition which contains x.

By design, $0 \leq s(x) \leq f(x)$ and $\int_a^b s = L(P,f)$. We can define g as a continuous extension of s by connecting lines between its steps (see Figure 3.2), to ensure:

(1) $0 \leq g(x) \leq s(x)$ for every $x \in [a,b]$, by adjusting the points for which g breaks into a line, and
(2) the magnitude of the integral underneath the lines is in total less than $\varepsilon/2$ by adjusting their widths to be as small as needed.

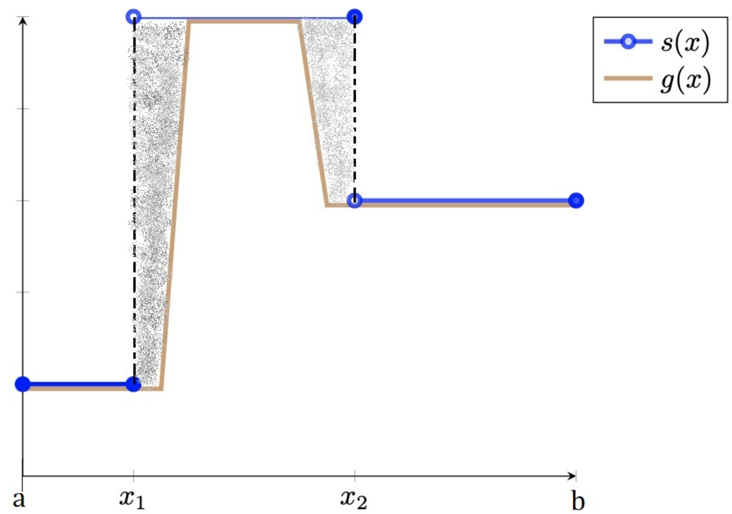

FIGURE 3.2. The blue graph $s(x)$ represents the step function and the brown graph $g(x)$ is its continuous approximation as used in the proof of Lemma 3.2. Since there are finitely many jumps, we can make the total shaded area smaller than $\varepsilon/2$ while preserving the inequality $0 \leq g(x) \leq f(x)$ for all $x \in [a, b]$.

This is possible since the lines are bounded and have at most $n+1$ steps. Thus $0 \leq g(x) \leq s(x) \leq f(x)$ and

$$\underline{\int_a^b} f - \varepsilon < \int_a^b s - \varepsilon/2 < \int_a^b g.$$

$\square$

The stage is set; we are ready for Luxembourg's main proof from [7].

Theorem 3.1 (Dominated Convergence Theorem)**.** *Let $f_n, f \in \mathcal{R}[a, b]$ where $f_n \to f$ pointwise and f_n is dominated. Then*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Proof. Without loss of generality, we may assume $f_n \to 0$ pointwise and that there is some $M \in \mathbb{R}$ such that $0 \leq f_n(x) \leq M$ for all $x \in [a, b]$. For we can always redefine a new sequence of functions $|f_n - f|$, and apply the implication

$$\lim_{n\to\infty} \int_a^b |f_n - f| = 0 \implies \lim_{n\to\infty} \int_a^b f_n = \int_a^b f$$

to recover a proof for the original problem.

Set $p_n(x) := \sup_{k\geq n} f_k(x)$ for all $x \in [a,b]$ and $n \in \mathbb{N}$; note this is well-defined since for $x \in [a,b]$ fixed, the sequence $f_n(x)$ is bounded by M. Moreover, observe that p_n monotonically decreases to 0 and $0 \leq f_n(x) \leq p_n(x)$ for all x and n.

The key technique used in this proof, which we will show in the next section, involves constructing a sequence of continuous functions h_n such that:

$$\forall x \in [a,b]\ \forall n \in \mathbb{N} \quad 0 \leq h_n(x) \leq p_n(x), \tag{1}$$

$$\forall n \in \mathbb{N} \quad \underline{\int_a^b} p_n - \varepsilon \leq \int_a^b h_n, \text{ and} \tag{2}$$

$$h_n \text{ monotonically decreases to } 0. \tag{3}$$

Note that p_n is lower integrable because it is bounded by M, so condition (2) is sensible. While (1) and (2) are clearly implied by Lemma 3.2, the "magic step" is (3). We reveal in the next section how Luxembourg constructs such a sequence.

By Lemma 3.1, we conclude that $\displaystyle\int_a^b h_n = 0$. Since

$$0 \leq \int_a^b f_n \leq \underline{\int_a^b} p_n \leq \int_a^b h_n + \varepsilon,$$

in the limit we have

$$0 \leq \lim_{n\to\infty} \int_a^b f_n \leq \varepsilon.$$

As $\varepsilon > 0$ is arbitrary, $\displaystyle\lim_{n\to\infty} \int_a^b f_n = 0$ and our proof is complete. □

This theorem provides an efficient means for computing limits of integrals.

Example 3.1. Consider $\displaystyle\lim_{n\to\infty} \int_0^1 \sin(x^2/n)\,dx$. Without the theorem, we would struggle to find a closed-form anti-derivative. But it is quick to check the sequence of functions $\sin(x^2/n)$ is dominated and converges pointwise to 0. Thus the limit computes to 0.

4. Full Monotone Convergence Theorem for Integrals

We return to the study of monotone sequences and aim to extend Theorem 2.1 to the full Monotone Convergence Theorem. Our proof adapts Luxembourg's justification of the "magic step" employed in Theorem 3.1.

Theorem 4.1 (Monotone Convergence for Riemann Integrals Part 2)**.** *Let $f, f_n \in \mathcal{R}[a,b]$ with $f_n \to f$ pointwise and suppose f_n is monotone decreasing. Then*

$$\lim_{n\to\infty} \int_a^b f_n \leq \int_a^b f.$$

Proof. Without loss of generality, we may assume f monotonically decreases to 0 and, therefore, is nonnegative. For we can redefine a new sequence of functions by $f_n - f$, which monotonically decreases to 0, and show instead $\lim\limits_{n\to\infty} \displaystyle\int_a^b f_n \leq 0$.

Fix $\varepsilon > 0$. Its is enough to construct a sequence $h_n \in \mathcal{C}[a,b]$ such that:

$$\forall x \in [a,b]\ \forall n \in \mathbb{N} \quad 0 \leq h_n(x) \leq f_n(x), \tag{1}$$

$$\forall n \in \mathbb{N} \quad \underline{\int_a^b} f_n - \varepsilon \leq \int_a^b h_n, \text{ and} \tag{2}$$

$$h_n \text{ monotonically decreases to } 0. \tag{3}$$

For once constructed, by Lemma 3.1 it follows $\lim\limits_{n\to\infty} \displaystyle\int_a^b h_n = 0$, and therefore $\lim\limits_{n\to\infty} \displaystyle\int_a^b f_n \leq \varepsilon$. As $\varepsilon > 0$ is arbitrary, the proof is finished.

By Lemma 3.2, we form a sequence $g_n \in \mathcal{C}[a,b]$ such that $0 \leq g_n \leq f_n$ and $\underline{\int_a^b} f_n - \varepsilon/2^n \leq \int_a^b g_n$ for all n. Now for each n, define $h_n := \min\{g_1, \ldots, g_n\}$. It is easy to verify that h_n is a sequence of continuous functions which satisfies properties (1) and (3); the challenge is demonstrating (2).

It is clear that $\underline{\int_a^b} f_1 - \varepsilon \leq \int_a^b h_1$, so it remains to show $\underline{\int_a^b} f_n - \varepsilon \leq \int_a^b h_n$ for $n \geq 2$. From h_n being defined as a minimum, it follows

$$0 \leq g_n \leq h_n + \sum_{k=1}^{n-1} \left(\max\{g_k, \ldots, g_n\} - g_k\right) \leq h_n + \sum_{k=1}^{n-1} \left(f_k - g_k\right).$$

Note that the final inequality comes from $g_k \leq f_k$ for all k and the sequence f_k monotonically decreasing. So by taking the lower integral, we have

$$\begin{aligned} \int_a^b g_n &\leq \int_a^b h_n + \underline{\int_a^b} \left(\sum_{k=1}^{n-1} f_k - g_k\right) \\ &\leq \int_a^b h_n + \sum_{k=1}^{n-1} \left(\underline{\int_a^b} f_k - \int_a^b g_k\right) \\ &\leq \int_a^b h_n + \sum_{k=1}^{n-1} \frac{\varepsilon}{2^k}. \end{aligned}$$

Since $\underline{\int_a^b} f_n - \varepsilon/2^n \leq \int_a^b g_n$, it follows

$$\underline{\int_a^b} f_n \leq \int_a^b g_n + \varepsilon/2^n \leq \int_a^b h_n + \sum_{k=1}^{n} \frac{\varepsilon}{2^k} \leq \int_a^b h_n + \varepsilon,$$

as desired. $\square$

Remark. While the process of generating g_n suggests the use of axiom of countable choice, we can avoid it. By reinserting the mechanism of how each g was obtained in the proof of Lemma 3.2, we can constructively generate the sequence g_n. This method of obtaining h_n, by taking the minimum of $g_1, \ldots, g_n$ is the "magic step" omitted earlier in Theorem 3.1.

Both parts of monotone convergence, i.e. Theorems 2.1 and 4.1, prove Monotone Convergence Theorem for integrals in general.

Corollary 4.1 (Monotone Convergence Theorem)**.** *Let* $f, f_n \in \mathcal{R}[a,b]$ *with* $f_n \to f$ *pointwise and* f_n *monotone. Then*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f.$$

Proof. When f_n is monotonically increasing, then $-f_n$ is monotonically decreasing so Theorem 4.1 proves the missing inequality

$$\lim_{n\to\infty}\int_a^b f_n \geq \int_a^b f.$$

Similarly when f_n is monotonically decreasing, Theorem 2.1 proves

$$\lim_{n\to\infty}\int_a^b f_n \geq \int_a^b f.$$

□

Corollary 4.2 is an interesting generalization of this result for sequences of integrable functions which need not fully converge pointwise.

Corollary 4.2 (Fatou's Lemma)**.** *Let $f_n \in \mathcal{R}[a,b]$ and suppose, for $x \in [a,b]$ fixed, the sequence $f_n(x)$ is bounded so that limit superiors exist. Given $\limsup_{n\to\infty} f_n \in \mathcal{R}[a,b]$ and $\sup_{k\geq n} f_k \in \mathcal{R}[a,b]$, then*

$$\limsup_{n\to\infty}\int_a^b f_n \leq \int_a^b \limsup_{n\to\infty} f_n.$$

Proof. Observe that $\sup_{k\geq n} f_k$ is a decreasing sequence of functions and converges pointwise to $\limsup_{n\to\infty} f_n$. Thus

$$\limsup_{n\to\infty}\int_a^b f_n \leq \lim_{n\to\infty}\int_a^b \sup_{k\geq n} f_k = \int_a^b \limsup_{n\to\infty} f_n,$$

where the equality comes from Theorem 4.1, and the inequality comes from the observation that $f_n \leq \sup_{k\geq n} f_k$ for all n. □

The inequality in Fatou's Lemma cannot be strengthened.

Example 4.1. Define a sequence of functions on $[0,1]$ by

$$f_n(x) := \begin{cases} (-1)^n & \text{if } x < 1/2 \\ (-1)^{n+1} & \text{otherwise.} \end{cases}$$

Observe that all integrals vanish, i.e. $\int_0^1 f_n = 0$ for all n. However, for fixed n, the suprema $\sup_{x\in[a,b]} f_n(x) = 1$. Thus $\lim_{n\to\infty} \sup \int_a^b f_n = 0$ and $\int_0^1 \lim_{n\to\infty} \sup f_n = 1$.

5. Conclusion

Understanding when a limit and integral commute is important for grasping the scope of various techniques in calculus. We have re-proven three classical theorems on the interplay of limits with integrals: **(1)** Monotone Convergence Theorem, **(2)** Uniform Convergence Theorem, and **(3)** Dominated Convergence Theorem. While **(2)** was a familiar exercise, we have shown how the methods used by Luxembourg's proof of **(3)** can be adapted to prove **(1)**. We have explicitly discussed the hidden logic underlying these proofs in the hopes to demystify them and make them more accessible to undergraduate students in real analysis. Our goal, albeit modest, is to help strengthen foundations and foster enthusiasm for more advanced topics in measure theory.

A benefit of demystification is its application to the formalization of mathematics through digital proof systems such as Lean [6]. Founded on Thierry Coquand's pioneering work [2] on the Calculus of Constructions, Lean is an interactive theorem prover for mathematicians to write and check proofs. Through the improvements of AI technologies and proof automation, we believe computer-verified mathematics will become more mainstream in the nearby future. By presenting proofs written closer to the axiomatic level, computer scientists, mathematicians, and logicians may find it easier to transport them into Lean. Our paper is a humble attempt in this direction.

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Mentor

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