STRONG RECURSION THEOREM AND THE AXIOM OF STRONG DEPENDENT CHOICE

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Abstract. The Recursion Theorem is instrumental in foundations; it secures the process of generating a sequence whose $n+1$th term depends only on index n and the nth term. The Strong Recursion Theorem extends this principle and allows reference to all previous terms — not just the immediate predecessor. The purpose of this paper is to use dependent products to reformulate the Strong Recursion Theorem in modern language, to prove it in set-theoretic foundations, and, using the generalization as a guideline, to reformulate the Axiom of Strong Dependent Choice. We apply these principles to solve two simple questions in undergraduate mathematics: (1) we show every infinite group contains infinitely many subgroups with Strong Recursion, and (2) we prove for every real $\varepsilon > 0$ and sequentially compact metric space, there exists a finite ε-net using Strong Dependent Choice.

Keywords: Logic, Set Theory, Dependent Product, Strong Induction, Strong Recursion, Strong Dependent Choice, Modern Algebra, Topology

§1 Introduction

When we usually apply set theory, we are rarely concerned with their foundations from axioms. From a pedagogical point of view the axiomatic method is unwieldy; we'd much rather think of sets platonically than from axiomatic construction, operating under the trust our intuition is captured by the formalism. As a result, details are often swept under the rug. In this paper we bridge that gap to show how those subconscious activities arise when forming sequences. We reformulate Strong Recursion Theorem and Axiom of Dependent Choice in set-theoretic foundations, and see how they are used in two popular problems in undergraduate mathematics. We work in ZF (Zermelo-Frankel set theory without the Axiom of Choice) described by Kunen[1].

Our first challenge is stating the Strong Recursion Theorem in a natural way. We recall the original Recursion Theorem (Halmos, pp. 48-49)[2].

Theorem 1.1 (Recursion Theorem)**.** *To any set X, element $a \in X$, and function $F : \mathbb{N} \times X \to X$, there exists a sequence $f : \mathbb{N} \to X$ such that:*

(i) $f(0) = a$*, and*
(ii) $f(n+1) = F(n, f(n))$ *for each* $n \in \mathbb{N}$*.*

F serves as the explicit relationship between the successive term and its immediate predecessor. When curried, F is of type $\mathbb{N} \to (X \to X)$; this gives a clue for the generalization to Strong Recursion Theorem. We would like to consider a function F which assigns any $k \in \mathbb{N}$ and a $k+1$-tuple of elements in X, to an element of X. Observe such an F is dependently typed, namely $\Pi_{k\in\mathbb{N}}\left(X^{k+1} \to X\right)$.

We clarify on notation above. In set-theoretic foundations, $f : A \to B$ is an abbreviation for $f \in B^A$ where B^A is the set of all functions from A to B when regarded as a subset of $A \times B$. The evaluation $f(a)$ abbreviates the unique b such that $(a, b) \in f$. The dependent product $f : \Pi_{a \in A} B(a)$ abbreviates $f \in \left(\bigcup_{a \in A} B(a)\right)^A$ where each $f(a)$ is an element of $B(a)$. It represents dependently typed functions whose output type depends on the input parameter.

Using the dependent product, in §2 we state and prove the Strong Recursion Theorem and in §3 we express the Axiom of Strong Dependent Choice. We apply these results in §4 to solve an algebra and topology problem. Finally in §5 we explore how Strong Recursion and Strong Dependent Choice arise in type-theoretic foundations.

§2 Strong Recursion

Proving the original Recursion Theorem is done in one of two ways: *"collapsing down"* by taking the intersection of all sets which contain the desired, recursive sequence; or *"building up"* by taking unions of all finite segments of the intended recursive sequence[2]. We present both methods starting by *"collapsing down"*.

Theorem 2.1 (Strong Recursion Theorem)**.** *Let X be a set, $a \in X$, and $F : \Pi_{k \in \mathbb{N}} \left(X^{k+1} \to X\right)$. Then there exists a sequence $f : \mathbb{N} \to X$ such that:*

(i) $f(0) = a$, and

(ii) $f(n+1) = F(n)\overrightarrow{f}(n)$ for each $n \in \mathbb{N}$,

where each $\overrightarrow{f}(n)$ is defined as the $n+1$-tuple $(f(0), f(1), \dots, f(n))$.

Proof. For brevity let $\psi(n, S)$ abbreviate *"for all $g : \mathbb{N} \to X$, if $(k, g(k)) \in S$ for every $k \le n$, then $(n+1, F(n)\overrightarrow{g}(n)) \in S$"* where n intends to be a natural and S a subset of $N \times X$. Consider the family:

$$\mathcal{F} := \left\{ S \subset N \times X \;\middle|\; (0, a) \in S \wedge \forall n \in \mathbb{N} \;\; \psi(n, S) \right\}.$$

It is non-empty, e.g. inhabited by $N \times X$, and the intersection $f := \bigcap \mathcal{F}$ we claim is the desired function.

First, we show by strong induction it is a function from N to X.

(Base Case) We show there is a unique y s.t. $(0, y) \in f$. It's clear $(0, a) \in f$ so let $(0, y) \in f$. In fact, the set formed by starting with $N \times X$ and removing all pairs of the form $(0, x)$ except $(0, a)$, is indeed an element of $\mathcal{F}$. Thus $y = a$.

(Successor Case) Suppose there exists a unique y s.t. $(k, y) \in f$ for every $k \le n$. We must show there exists a unique y s.t. $(n+1, y) \in f$. First define the finite sequence $h(k)$ which picks out the unique y with $(k, y) \in f$ for each $k \le n$. It's easy to show $(n+1, F(n)\overrightarrow{h}(n)) \in f$. In fact the set formed by starting with $N \times X$ and removing all pairs of the form (k, x) for $k \le n$ except $(k, h(k))$, and pairs $(n+1, x)$ except $(n+1, F(n)\overrightarrow{h}(n))$, is indeed an element of $\mathcal{F}$. Thus the only $(n+1, y) \in f$ is when $y = F(n)\overrightarrow{h}(n)$.

Now we show by strong induction the function f has the desired properties. *(Base Case)* Obviously $(0, a) \in f$ and $(1, F(0)(a)) \in f$. *(Successor Case)* Suppose

$(k+1, F(k)\overrightarrow{f}(k)) \in f$ for every $k \leq n$. Then since every element of $\mathcal{F}$ contains all such $(k+1, F(k)\overrightarrow{f}(k))$ for $k \leq n$, it contains $(n+2, F(n+1)\overrightarrow{f}(n+1))$. $\square$

Remark 2.2. The sequence generated by strong recursion is unique (just another application of strong induction). We note the similar role strong induction plays in Strong Recursion Theorem, to the role induction plays in Recursion Theorem.

Strong Recursion Theorem can also be seen as a special case of regular Recursion Theorem. For we can apply Recursion Theorem on finite segments to obtain a family of finite segments which approximate the intended sequence. The union of this family creates the desired function. This pathway represents our other approach to the proof by *"building up"*.

Theorem 2.3. *Strong Recursion Theorem is an instance of Recursion Theorem.*

Proof. Let X be a set, $a \in X$, and $F : \Pi_{k\in\mathbb{N}}(X^{k+1} \to X)$. Define F' : $(\mathbb{N} \times \Pi_{k\in\mathbb{N}}(\{0, \dots, k\} \to X)) \to \Pi_{k\in\mathbb{N}}(\{0, \dots, k\} \to X)$ by

$$F'(n, f)(k)(m) := \begin{cases} F(n)\overrightarrow{f}(n) & \text{if } m = n+1 \\ f(m) & \text{otherwise} \end{cases}$$

where each $\overrightarrow{f}(n)$ is defined as the $n+1$-tuple $(f(0), f(1), \dots, f(n))$. By applying Recursion Theorem on F' over the initial $A : \Pi_{k\in\mathbb{N}}(\{0, \dots, k\} \to X)$ which assigns each $A(k)(m) := a$, we obtain a sequence of families of finite sequences $f' : \mathbb{N} \to \Pi_{k\in\mathbb{N}}(\{0, \dots, k\} \to X)$. The union of diagonals $\bigcup_{k\in\mathbb{N}} f'(k)(k)$ gives the desired function which would have been generated by Strong Recursion. $\square$

Remark 2.4. For an easier proof we can similarly define $F' : \left(\mathbb{N} \times X^{\mathbb{N}}\right) \to X^{\mathbb{N}}$ preferring sequences instead of finite sequences. We do this in Theorem 3.3.

In this approach Strong Recursion Theorem for sequences in X, is just an application of Recursion Theorem on the *"higher space"* of finite sequences in X. This leads us to an interesting question: can we prove Strong Recursion on X as an instance of regular Recursion on X? We believe so. Perhaps we can define a sequence of naturals, with Recursion, where each $n(k)$ intends to encode Gödel values of all $x(0), x(1), \dots x(k)$. Once formed, we imagine *"unzipping"* $n(k)$'s to obtain $x(k)$'s.

§3 Strong Dependent Choice

The proof system accommodates finitely many arbitrary *"picks"*, i.e. witnesses to existential formulas. It is when we have infinitely many arbitrary picks that we require some form of Choice. Strong Dependent Choice arises when the next arbitrary pick depends on all the previous picks. The notation established in Theorems 1.1 and 2.1 allows us to make this precise. First we recall Dependent Choice.

Axiom 3.1 (Dependent Choice)**.** Let X be a set, $a \in X$, and $F : \mathbb{N} \times X \to \mathcal{P}(X) \setminus \emptyset$, where $\mathcal{P}(X)$ represents the power set of X and $\emptyset$ is the empty set. There exists a sequence $f : \mathbb{N} \to X$ such that:

(i) $f(0) = a$, and
(ii) $f(n+1) \in F(n, f(n))$ for each $n \in \mathbb{N}$.

We start by assigning $f(0) = a$, and then we assign $f(1)$ as an element of the nonempty set $F(0, f(0))$, and so on. The arbitrary selection of the $n+1$th term depends on the index n and the arbitrary selection of the nth term.

Axiom 3.2 (Strong Dependent Choice)**.** Let X be a set, $a \in X$, and $F : \Pi_{k\in\mathbb{N}}\big(X^{k+1} \to \mathcal{P}(X) \setminus \emptyset\big)$. There exists a sequence $f : \mathbb{N} \to X$ such that:

(i) $f(0) = a$, and
(ii) $f(n+1) \in F(n)\overrightarrow{f}(n)$ for each $n \in \mathbb{N}$,

where each $\overrightarrow{f}(n)$ is defined as the $n+1$-tuple $(f(0), f(1), \dots, f(n))$.

$F(n)\overrightarrow{f}(n)$ describes a family of nonempty sets for each $n \in \mathbb{N}$ and $n+1$ tuple of points in X. The arbitrary selection of the $n+1$th term depends on the index n and all arbitrary selections of the previous terms.

Similar to before, Strong Dependent Choice is a special case of Dependent Choice.

Theorem 3.3. *Dependent Choice implies Strong Dependent Choice.*

Proof. Let X be a set, $a \in X$, and $F : \Pi_{k\in\mathbb{N}}(X^{k+1} \to \mathcal{P}(X) \setminus \emptyset)$. Define $F' : \big(\mathbb{N} \times X^{\mathbb{N}}\big) \to \mathcal{P}(X^{\mathbb{N}}) \setminus \emptyset$ by

$$F'(n, f) := \{f' : \mathbb{N} \to X \mid f'(n+1) \in F(n)\overrightarrow{f}(n) \wedge \forall k \leq n\ f'(k) = f(k)\}$$

where each $\overrightarrow{f}(n)$ is defined as the $n+1$-tuple $(f(0), f(1), \dots, f(n))$. By applying Dependent Choice on F' over the initial constant sequence $A : \mathbb{N} \to X$ which assigns each $A(n) := a$, we obtain a sequence of sequences $f' : \mathbb{N} \to (\mathbb{N} \to X)$. The diagonal sequence $f : \mathbb{N} \to X$ defined by $f(n) := f'(n)(n)$ gives the desired function which would have been created from Strong Dependent Choice. $\square$

Often times while the Axiom of Choice may appear useful, it's unnecessary (e.g. Bolzano–Weierstrass and Dini's Uniform Convergence Theorem in Real Analysis). For example, if we had a property $\psi(n, x, y)$ which depends on $n \in \mathbb{N}$ and $x, y \in X$, and a proof of the proposition $\forall n \forall x \exists! y\, (\psi(n, x, y) \wedge y \in F(n, x))$, then we can define a function $F' : \mathbb{N} \times X \to X$ which realizes this property. Applying F' with Recursion Theorem allows us to build our sequence constructively. It is precisely when we are unable to find such a property, that we need Dependent Choice.

§4 Applications

The uses of Strong Recursion Theorem and Strong Dependent Choice are broad and varied. While they are important principles, in many textbooks their involvement is purposefully hidden to not let logic interfere with idea. We look under-the-hood, so to speak, to see explicitly how these principles can be used to solve undergraduate mathematics problems in algebra and topology.

§4.1 Algebra

We provide an alternative proof, using Strong Recursion Theorem, that every infinite group has infinitely many subgroups.

Definition 4.1 (Strongly Injective Sequences)**.** Let G be a group. A sequence $g : \mathbb{N} \to G$ is said to be *strongly injective* if and only if it is injective and $g^{-1}(n) = g(m)$ implies $n = m$, for every $n, m \in \mathbb{N}$.

Lemma 4.2. *Let G be an infinite group. There exists a strongly injective sequence $g : \mathbb{N} \to G$.*

Proof. As G is infinite, there is an injection $g : \mathbb{N} \to G$. For brevity let $\psi(k,m)$ abbreviate *"for all $j \leq k$, both $g^{-1}(n(j))$ and $g(m)$ do not equal"* where k and m intend to be naturals. Define a strictly increasing sequence of indices $n(k)$ by:

$$n(0) = 0, \text{ and}$$
$$n(k+1) = \min\left\{m \in \mathbb{N} \;\middle|\; m > n(k) \wedge \psi(k,m)\right\} \text{ for each } k \in \mathbb{N}.$$

Using strong induction, the subsequence $g(n(k))$ is strongly injective and hence the result. Formally we apply Strong Recursion Theorem on the function $F : \Pi_{k\in\mathbb{N}} \left(\mathbb{N}^{k+1} \to \mathbb{N}\right)$ defined by

$$F(k)(x(0), \ldots, x(k)) := \min\left\{m \in \mathbb{N} \;\middle|\; m > x(k) \wedge \psi(k,m)\right\}.$$

This is possible since the sets $\left\{m \in \mathbb{N} \;\middle|\; m > x(k) \wedge \psi(k,m)\right\}$ are nonempty by using a pigeon hole argument alongside the injectivity of g. □

Remark 4.3. Intuitively, we pick $g(0)$ and then write down the inverse of $g(0)$ on a *"blacklist"*. We continue picking in order $g(1)$, $g(2)$, and so on, avoiding any names on the list, selecting the next available candidate and appending the list by its inverse. This is possible because of Strong Recursion Theorem.

Theorem 4.4. *Every infinite group has infinitely many subgroups.*

Proof. Let G be an infinite group; by Lemma 4.2 there is a strongly injective sequence $g : \mathbb{N} \to G$. For brevity let $\psi(k,m)$ abbreviate *"for all $j \leq k$, the cyclic group generated by $g(n(j))$ does not equal the cyclic group generated by $g(m)$"* where k and m intend to be naturals. Define a strictly increasing sequence of indices by:

$$n(0) = 0, \text{ and}$$
$$n(k+1) = \min\left\{m \in \mathbb{N} \;\middle|\; m > n(k) \wedge \psi(k,m)\right\} \text{ for each } k \in \mathbb{N}.$$

Using strong induction, the sequence of cyclic subgroups $\langle g_{n_k} \rangle$ is injective and hence the result.

The challenge is demonstrating the sequence $n(k)$ is well-formed. We apply Strong Recursion Theorem on the function $F : \Pi_{k\in\mathbb{N}} \left(\mathbb{N}^{k+1} \to \mathbb{N}\right)$ defined by

$$F(k)(x(0), \ldots, x(k)) := \min\left\{m \in \mathbb{N} \;\middle|\; m > x(k) \wedge \psi(k,m)\right\}.$$

This is only well-defined when sets $\left\{m \in \mathbb{N} \;\middle|\; m > x(k) \wedge \psi(k,m)\right\}$ are nonempty. The idea is we can pick $m \in \mathbb{N}$ large so that:

(1) $m > x(j)$ for all $j \leq k$, and

(2) $g(m) \notin \langle g(x_j) \rangle$ for all $j \leq k$ provided $\langle g(x_j) \rangle$ is finite.

Now we claim m inhabits the set. Let $j \leq k$; either $\langle g(x_j) \rangle$ is infinite or finite. In the first case it is isomorphic to $\mathbb{Z}$ and thus can only be generated by $g(x_j)$ or its inverse. By (1) and g being strongly injective, $g(m)$ is neither $g(x_j)$ nor its inverse, and therefore $\langle g(m) \rangle \neq \langle g(x_j) \rangle$. In the second case, (2) together with $g(m) \in \langle g(m) \rangle$ implies $\langle g(m) \rangle \neq \langle g(x_j) \rangle$. In both cases m is an element of the set, as needed. □

§4.2 Topology

We dissect a popular proof for a topology problem to see the hidden use of Strong Dependent Choice.

Definition 4.5 (ε-nets)**.** Let X be a metric space. A subset $A \subset X$ is said to be an *ε-net* for $\varepsilon > 0$, if and only if $\{\mathcal{B}_\varepsilon(a) : a \in A\}$ is an open cover of X where $\mathcal{B}_\varepsilon(a)$ represents the open ball of radius ε centered at a.

Theorem 4.6. *Let X be a sequentially compact metric space. Given the Axiom of Strong Dependent Choice, for every $\varepsilon > 0$ there exists a finite ε-net.*

Proof. Fix $\varepsilon > 0$ and suppose, for sake of contradiction, there is no finite ε-net. Then there is a point $a \in X$ s.t. $\mathcal{B}_\varepsilon(a)$ does not cover the space; otherwise we'd have a finite ε-net. Thus we can pick a point outside the ball and continue this process, of picking points outside the union of balls, to form a sequence $f : \mathbb{N} \to X$ with the property $f(0) = a$ and $f(n+1) \notin \bigcup_{k \leq n} \mathcal{B}_\varepsilon(f(k))$ for all $n \in \mathbb{N}$. Since X is sequentially compact, there is a converging subsequence of f; however it's clear f has no accumulation points and thus has no converging subsequences. We have reached a contradiction as needed.

Formally we use the Axiom of Strong Dependent Choice on the function $F : \Pi_{k\in\mathbb{N}}\big(X^{k+1} \to \mathcal{P}(X) \setminus \emptyset\big)$ defined by

$$F(k)(x(0), \ldots, x(k)) := X \setminus \bigcup_{j \leq k} \mathcal{B}_\varepsilon(x(j)).$$

This is possible since the family $\{\mathcal{B}_\varepsilon(x(j))\}_{j\leq k}$ does not cover X. $\square$

§5 Conclusion

We've shown that dependent products offer a convenient way to describe the Strong Recursion Theorem and Axiom of Strong Dependent Choice. This convenience is not limited to efficient notation; dependent products are the heart of type theories. In fact, our results on Strong Recursion can be extended to type-theoretic foundations like Lean. For inductive types come equipped with an associated recursor function which captures its elimination rules. Using the built-in recursor, our proof of Theorem 2.3 shows how to implement strong recursion in Lean. In Remark 2.4 we mention there are other methods of implementing strong recursion. While the two implementations are extensionally equal, can we prove their equality without axiom of extensionality in Lean? We think the answer is no. What weaker forms of equality do we need to answer yes?

As we've seen, Strong Induction, Strong Recursion, and Strong Dependent Choice are not actually stronger than their regular counterparts — they are provable instances of one another. They are stronger *forms* only in the sense of appearances, not in the sense of content. Perhaps a more suitable adjective in place of *"strong"* is *"dependent"*, and instead of Dependent Choice and Strong Dependent Choice, we can say Recursive Choice and Dependent Recursive Choice.

Finally, we applied Strong Recursion Theorem and Strong Dependent Choice to solve problems in undergraduate mathematics. Our hope is to increase awareness of when these sequence-forming principles are invoked.

There is a valid criticism of formal mindfulness. If we are too conscious about which principles are invoked, then we may fail to see the bigger picture — we may jeopardize cohesion of the whole. At some level we go by *"feeling"*. It is our ongoing duty as a mathematician to negotiate this compromise of formality and naivety, of rigor and intuition. Perhaps truth is like polished sand; when clutched too tightly it slips, too loosely it falls. Only from the balance in our grip can we seize enlightenment.

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