Collections and the Classical Theory of Types

A Formal Pure Type Theory
in the Language of Objects.

Written By

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To Data

The precedence used here, in order of extent, and closed under indexing, is:

1. $\forall,\exists,\exists!$ QUANTIFIER
2. $\vdash$ 2-BINARY
3. $\Leftrightarrow$ 2-BINARY
4. $\Rightarrow$ 2-BINARY
5. $\vee$ 2-BINARY
6. $\wedge$ 2-BINARY
7. $\neg$ 2-UNARY
8. $=,\subset,:$ 2-BINARY
9. $\rightarrow$ 2-BINARY
10. $\cup,\cap,\setminus,\dot{-},\times,$**PAR,ORP** 2-BINARY, $\bigcup,\mathscr{P},\bigcap,\boldsymbol{\pi},$**SUC** 2-UNARY
11. **IND,REL,WFR,TRAN,ORD** 2-UNARY
12. $\boldsymbol{\psi},\boldsymbol{\phi},\boldsymbol{\theta},$**FUN,REC** 2-NULLARY

where it is understood that the rest are 1-NULLARY.

To keep to the economy of presentation, we introduce $\exists!$ as part of the object calculus. We introduce obvious inference rules $\exists!$EXIST, $\exists!$UNIQUE, and $\exists!$INTROS. Hereafter, we apply the restoration of parentheses to all strings seeking admissions as an object.

The prosentential semantics of the language is a function $\mathscr{R}$ (from the object language to the natural language) defined recursively as follows:

1. for every first order variable $\boldsymbol{x}$, second order variable $\boldsymbol{\psi}$, objects $\boldsymbol{x_1},\boldsymbol{x_2}$:
$\mathscr{R}[\forall \boldsymbol{x}\boldsymbol{x_1}]$ = " *for all (objects)* $\boldsymbol{x}$, $\mathscr{R}[\boldsymbol{x_1}]$ "
$\mathscr{R}[\exists \boldsymbol{x}\boldsymbol{x_1}]$ = " *there exists (object)* $\boldsymbol{x}$, $\mathscr{R}[\boldsymbol{x_1}]$ "
$\mathscr{R}[\exists! \boldsymbol{x}\boldsymbol{x_1}]$ = " *there exists a unique (object)* $\boldsymbol{x}$, $\mathscr{R}[\boldsymbol{x_1}]$ "
$\mathscr{R}\left[\forall \boldsymbol{\psi}\boldsymbol{x_1}\right]$ = " *for all (formulas)* $\boldsymbol{\psi}$, $\mathscr{R}[\boldsymbol{x_1}]$ "
$\mathscr{R}\left[\exists \boldsymbol{\psi}\boldsymbol{x_1}\right]$ = " *there exists (formula)* $\boldsymbol{\psi}$, $\mathscr{R}[\boldsymbol{x_1}]$ "
$\mathscr{R}[\boldsymbol{x_1}\vdash\boldsymbol{x_2}]$ = " *if* $\mathscr{R}[\boldsymbol{x_1}]$ *then* $\mathscr{R}[\boldsymbol{x_2}]$ "
$\mathscr{R}[\boldsymbol{x_1}=\boldsymbol{x_2}]$ = " $\mathscr{R}[\boldsymbol{x_1}]$ *equals* $\mathscr{R}[\boldsymbol{x_2}]$ "
2. for every second order variable $\boldsymbol{\psi}$ free from the signature, and for every number of objects $\boldsymbol{x_1},\boldsymbol{x_2},...\boldsymbol{x_n}$:
$\mathscr{R}\left[\boldsymbol{\psi}\boldsymbol{x_1}\boldsymbol{x_2}...\boldsymbol{x_n}\right]$ = " $\boldsymbol{\psi}$ *of* $\mathscr{R}[\boldsymbol{x_1}]$, $\mathscr{R}[\boldsymbol{x_2}]$, ... $\mathscr{R}[\boldsymbol{x_n}]$ "
$\mathscr{R}\left[\boldsymbol{x_1}\boldsymbol{\psi}\boldsymbol{x_2}...\boldsymbol{x_n}\right]$ = " $\boldsymbol{\psi}$ *of* $\mathscr{R}[\boldsymbol{x_1}]$, $\mathscr{R}[\boldsymbol{x_2}]$, ... $\mathscr{R}[\boldsymbol{x_n}]$ " ... etc.
$\mathscr{R}\left[\boldsymbol{x_1}\boldsymbol{x_2}...\boldsymbol{\psi}\boldsymbol{x_n}\right]$ = " $\boldsymbol{\psi}$ *of* $\mathscr{R}[\boldsymbol{x_1}]$, $\mathscr{R}[\boldsymbol{x_2}]$, ... $\mathscr{R}[\boldsymbol{x_n}]$ "
3. for every first order variable $\boldsymbol{x}$ free from the signature:
$\mathscr{R}[\boldsymbol{x}]$ = " $\boldsymbol{x}$ "
4. for every objects $\boldsymbol{p},\boldsymbol{q}$:
$\mathscr{R}[\neg\boldsymbol{p}]$ = " *neg.* $\mathscr{R}[\boldsymbol{p}]$ "
$\mathscr{R}[\boldsymbol{p}\Rightarrow\boldsymbol{q}]$ = " $\mathscr{R}[\boldsymbol{p}]$ *implies* $\mathscr{R}[\boldsymbol{q}]$ "
$\mathscr{R}[\boldsymbol{p}\wedge\boldsymbol{q}]$ = " $\mathscr{R}[\boldsymbol{p}]$ *and* $\mathscr{R}[\boldsymbol{q}]$ "
$\mathscr{R}[\boldsymbol{p}\vee\boldsymbol{q}]$ = " $\mathscr{R}[\boldsymbol{p}]$ *or* $\mathscr{R}[\boldsymbol{q}]$ "
$\mathscr{R}[\boldsymbol{p}\Leftrightarrow\boldsymbol{q}]$ = " $\mathscr{R}[\boldsymbol{p}]$ *iff* $\mathscr{R}[\boldsymbol{q}]$ "

5. for every objects $\boldsymbol{x},\boldsymbol{y}$:
$\mathscr{R}[\mathscr{H}]$ = " *(the collection of) (hereditary, well founded) (pure) set(s)* "
$\mathscr{R}[\boldsymbol{x}\mathbf{:}\boldsymbol{y}]$ = " $\mathscr{R}[\boldsymbol{x}]$ *is (a type) (an element) of* $\mathscr{R}[\boldsymbol{y}]$ "
$\mathscr{R}[\boldsymbol{x}\rightarrow\boldsymbol{y}]$ = " *(the collection of) function(s) from* $\mathscr{R}[\boldsymbol{x}]$ *to* $\mathscr{R}[\boldsymbol{y}]$ "
6. for every objects $\boldsymbol{x},\boldsymbol{y}$:
$\mathscr{R}[\boldsymbol{x}\cup\boldsymbol{y}]$ = " $\mathscr{R}[\boldsymbol{x}]$ *union* $\mathscr{R}[\boldsymbol{y}]$ "
$\mathscr{R}[\boldsymbol{x}\cap\boldsymbol{y}]$ = " $\mathscr{R}[\boldsymbol{x}]$ *intersect* $\mathscr{R}[\boldsymbol{y}]$ "
$\mathscr{R}[\boldsymbol{x}\setminus\boldsymbol{y}]$ = " $\mathscr{R}[\boldsymbol{x}]$ *set minus* $\mathscr{R}[\boldsymbol{y}]$ "
$\mathscr{R}[\boldsymbol{x}\dot{-}\boldsymbol{y}]$ = " $\mathscr{R}[\boldsymbol{x}]$ *symmetric difference* $\mathscr{R}[\boldsymbol{y}]$ "
$\mathscr{R}[\boldsymbol{x}\times\boldsymbol{y}]$ = " $\mathscr{R}[\boldsymbol{x}]$ *cartesian product* $\mathscr{R}[\boldsymbol{y}]$ "
$\mathscr{R}[\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}]$ = " $\mathscr{R}[\boldsymbol{x}]$ *pair* $\mathscr{R}[\boldsymbol{y}]$ "
$\mathscr{R}[\boldsymbol{x}\,\mathbf{ORP}\,\boldsymbol{y}]$ = " *the (ordered) Pair of* $\mathscr{R}[\boldsymbol{x}]$ *with* $\mathscr{R}[\boldsymbol{y}]$ "
$\mathscr{R}[\bigcup\boldsymbol{x}]$ = " *the (upper) Union of* $\mathscr{R}[\boldsymbol{x}]$ "
$\mathscr{R}[\mathscr{P}\boldsymbol{x}]$ = " *the power set of* $\mathscr{R}[\boldsymbol{x}]$ "
$\mathscr{R}[\bigcap\boldsymbol{x}]$ = " *the (upper) Intersection of* $\mathscr{R}[\boldsymbol{x}]$ "
$\mathscr{R}[\boldsymbol{\pi}_{\mathbf{1}}\boldsymbol{x}]$ = " *the first projection of* $\mathscr{R}[\boldsymbol{x}]$ "
$\mathscr{R}[\boldsymbol{\pi}_{\mathbf{2}}\boldsymbol{x}]$ = " *the second projection of* $\mathscr{R}[\boldsymbol{x}]$ "
$\mathscr{R}[\mathbf{SUC}\,\boldsymbol{x}]$ = " *the successor of* $\mathscr{R}[\boldsymbol{x}]$ "
7. for every objects $\boldsymbol{x},\boldsymbol{y}$:
$\mathscr{R}[\boldsymbol{x}\subset\boldsymbol{y}]$ = " $\mathscr{R}[\boldsymbol{x}]$ *is a subtype of* $\mathscr{R}[\boldsymbol{y}]$ "
$\mathscr{R}[\mathbf{IND}\,\boldsymbol{x}]$ = " $\mathscr{R}[\boldsymbol{x}]$ *is inductive* "
$\mathscr{R}[\mathbf{REL}\,\boldsymbol{x}]$ = " $\mathscr{R}[\boldsymbol{x}]$ *is a relation* "
$\mathscr{R}[\mathbf{WFR}\,\boldsymbol{x}]$ = " $\mathscr{R}[\boldsymbol{x}]$ *is well-formed relation* "
$\mathscr{R}[\mathbf{TRAN}\,\boldsymbol{x}]$ = " $\mathscr{R}[\boldsymbol{x}]$ *is transitive* "
$\mathscr{R}[\mathbf{ORD}\,\boldsymbol{x}]$ = " $\mathscr{R}[\boldsymbol{x}]$ *is an ordinal* "
8. nullary constants:
$\mathscr{R}[\mathbf{0}]$ = " *(the) ordinal zero (ie. empty set)* "
$\mathscr{R}[\mathbf{1}]$ = " *(the) ordinal one* "
$\mathscr{R}[\mathbf{2}]$ = " *(the) ordinal two* "
$\mathscr{R}[\boldsymbol{\omega}]$ = " *(the) (set of) counting ordinal(s)* "

Items **1.** through **3.** correspond to the object calculus; **4.** corresponds to intuionistic logic; **5.** to our set theory; and **6.** through **8.** to our (mostly natural) deductive extension.

LAWS OF INTUITIONISTIC LOGIC

Logical connectives do not themselves constitute the indivisible elements of nature. They are composed, in manner of speaking, of particular arrangements of fire and water; of the explosive characteristic of universal comprehension $\forall$, and the ebb and flow of entailment $\vdash$.

While this theory introduces the second-order binary symbol of *implication* $\Rightarrow$, it is merely of historical interest; very rarely do we use it. For implication is entailment's impostor. Let it be known who is the master and who is the fool.

$\forall p(\neg p)=(p\vdash\forall xx).$	CANONICAL DEFINITION *of Negation*	**1**
¶	DEM	
$\Box(p\vdash\forall xx)=(p\vdash\forall xx).$	EQ	**1.1**
$\Box\forall p(p\vdash\forall xx)=(p\vdash\forall xx).$	[.1] $\forall_1$GEN	**1.2**
$\Box\exists\psi\forall p\psi p=(p\vdash\forall xx).$	[.2] $\exists_2$GEN	**1.3**
$\Box\forall p(\neg p)=(p\vdash\forall xx).$	[.3] ${}_2$NAME	**1.4**
$\blacksquare.$	QED	
$\forall p\forall q(p\Rightarrow q)=(p\vdash q).$	CANONICAL DEFINITION *of Implication*	**2**
¶	DEM	
$\Box(p\vdash q)=(p\vdash q).$	EQ	**2.1**
$\Box\forall q(p\vdash q)=(p\vdash q).$	[.1] $\forall_1$GEN	**2.2**
$\Box\forall p\forall q(p\vdash q)=(p\vdash q).$	[.2] $\forall_1$GEN	**2.3**
$\Box\exists\psi\forall p\forall qp\psi q=(p\vdash q).$	[.3] $\exists_2$GEN	**2.4**
$\Box\forall p\forall q(p\Rightarrow q)=(p\vdash q).$	[.4] ${}_2$NAME	**2.5**
$\blacksquare.$	QED	
$\forall p\forall q(p\wedge q)=\forall x(p\vdash q\vdash x)\vdash x.$	CANONICAL DEFINITION *of Conjunction*	**3**
¶	DEM	
$\Box(\forall x(p\vdash q\vdash x)\vdash x)=\forall x(p\vdash q\vdash x)\vdash x.$	EQ	**3.1**
$\Box\forall q(\forall x(p\vdash q\vdash x)\vdash x)=\forall x(p\vdash q\vdash x)\vdash x.$	[.1] $\forall_1$GEN	**3.2**
$\Box\forall p\forall q(\forall x(p\vdash q\vdash x)\vdash x)=\forall x(p\vdash q\vdash x)\vdash x.$	[.2] $\forall_1$GEN	**3.3**
$\Box\exists\psi\forall p\forall qp\psi q=\forall x(p\vdash q\vdash x)\vdash x.$	[.3] $\exists_2$GEN	**3.4**
$\Box\forall p\forall q(p\wedge q)=\forall x(p\vdash q\vdash x)\vdash x.$	[.4] ${}_2$NAME	**3.5**
$\blacksquare.$	QED	
$\forall p\forall q(p\vee q)=\forall x(p\vdash x)\vdash(q\vdash x)\vdash x.$	CANONICAL DEFINITION *of Disjunction*	**4**
¶	DEM	
$\Box(\forall x(p\vdash x)\vdash(q\vdash x)\vdash x)=\forall x(p\vdash x)\vdash(q\vdash x)\vdash x.$	EQ	**4.1**
$\Box\forall q(\forall x(p\vdash x)\vdash(q\vdash x)\vdash x)=\forall x(p\vdash x)\vdash(q\vdash x)\vdash x.$	[.1] $\forall_1$GEN	**4.2**
$\Box\forall p\forall q(\forall x(p\vdash x)\vdash(q\vdash x)\vdash x)=\forall x(p\vdash x)\vdash(q\vdash x)\vdash x.$	[.2] $\forall_1$GEN	**4.3**
$\Box\exists\psi\forall p\forall qp\psi q=\forall x(p\vdash x)\vdash(q\vdash x)\vdash x.$	[.3] $\exists_2$GEN	**4.4**
$\Box\forall p\forall q(p\vee q)=\forall x(p\vdash x)\vdash(q\vdash x)\vdash x.$	[.4] ${}_2$NAME	**4.5**
$\blacksquare.$	QED	
$\forall p\forall q(p\Leftrightarrow q)=\forall x((p\vdash q)\vdash(q\vdash p)\vdash x)\vdash x.$	CANONICAL DEFINITION *of Logical Equivalence*	**5**
¶	DEM	
$\Box(\forall x((p\vdash q)\vdash(q\vdash p)\vdash x)\vdash x)=\forall x((p\vdash q)\vdash(q\vdash p)\vdash x)\vdash x.$	EQ	**5.1**
$\Box\forall q(\forall x((p\vdash q)\vdash(q\vdash p)\vdash x)\vdash x)=\forall x((p\vdash q)\vdash(q\vdash p)\vdash x)\vdash x.$	[.1] $\forall_1$GEN	**5.2**
$\Box\forall p\forall q(\forall x((p\vdash q)\vdash(q\vdash p)\vdash x)\vdash x)=\forall x((p\vdash q)\vdash(q\vdash p)\vdash x)\vdash x.$	[.2] $\forall_1$GEN	**5.3**
$\Box\exists\psi\forall p\forall qp\psi q=\forall x((p\vdash q)\vdash(q\vdash p)\vdash x)\vdash x.$	[.3] $\exists_2$GEN	**5.4**
$\Box\forall p\forall q(p\Leftrightarrow q)=\forall x((p\vdash q)\vdash(q\vdash p)\vdash x)\vdash x.$	[.4] ${}_2$NAME	**5.5**
$\blacksquare.$	QED	

Laws of Classical Type Theory

The goals of classical set theory are summarized as follows:

(1) the theory is *consistent*, ie. it is unable to prove $\forall \boldsymbol{x} \boldsymbol{x}$;

(2) the theory is *classical*, ie. it is able to prove $\forall \boldsymbol{p} \boldsymbol{p} \vee \neg \boldsymbol{p}$; and

(3) the sets are *pure*, ie. they are *hereditary* and *well-founded*–they contain only sets which themselves contain only sets, etc, and they arise (hopefully) only from axiomatic construction.

Classical type theory generalizes the notion of sets to the notion of collections, or types. Our theory has two additional goals:

(4) there is a collection of all sets $\mathscr{H}$; and

(5) there is a *function type* between any two collections, ie. whenever we may exhibit for every element of one collection $\boldsymbol{a}$, a unique element of another $\boldsymbol{b}$, we may infer the existence of a corresponding *function* $\boldsymbol{f}$ of the *function type* $\boldsymbol{a} \to \boldsymbol{b}$. We refer to this principle as ABC, the axiom of *bounded creation*.

The function so realized is an object. Thus we may define elementary set operations as symbols, within the framework of TOB, and we may naturally adopt their notation without losing syntactic specificity.

As a starting point, we keep our type theory modest. We seek to maximize what we can say of sets but minimize what we may say of types. In this way, we lay the groundwork for set-theoretic thought and leave room for further type-theoretic expansion.

Demonstrations of the Self Evident

The rest of this book is devoted to writing formal proofs. When signature preserving they are theorems, when signature extending, definitions. We attempt to write, with little deviation, a natural extension of the theory; ie. one in which defined objects arise *naturally*– they witness existentially unique statements.

We arrange these proofs tabularly with pointers to assist computer verification. The chief advantage of a computer assisted proof software is the cataloging and utilizing of tactics, ie. the meta-mathematical practice of incorporating algorithms to reduce several lines of a formal theory to a single line in an enriched higher language. In the absence of software, we rely only on three families of rule tactics (two specified now). For each pair of numbers n,m, these tactics, defined as products of elementary rules, are:

$$_n\mathrm{MINT}_m = \underbrace{\forall_1\mathrm{INST}\cdot\forall_1\mathrm{INST}\cdot\ldots\cdot\forall_1\mathrm{INST}}_{m\ times}\cdot\underbrace{\mathrm{MP}\cdot\mathrm{MP}\cdot\ldots\cdot\mathrm{MP}}_{n\ times}$$

$$\mathrm{UGEN}_m = \underbrace{\forall_1\mathrm{GEN}\cdot\forall_1\mathrm{GEN}\cdot\ldots\cdot\forall_1\mathrm{GEN}}_{m\ times}$$

Finally, our proofs are organized into sections of broad and developing material, followed by a circumlocution of exercises.

§1.1 *Logic: First-order*

21 $\forall \boldsymbol{p}\boldsymbol{p} \vdash \neg \boldsymbol{p} \vdash \forall \boldsymbol{x}\boldsymbol{x}$. .... THEOREM *of* ¬ *Elimination 1*

¶ .... DEM

21.1 $\square \boldsymbol{p} \vdash \neg \boldsymbol{p} \vdash \forall \boldsymbol{x}\boldsymbol{x}$. .... LEM

□¶ .... DEM

21.2 $\square\square \neg \boldsymbol{p} \vdash \forall \boldsymbol{x}\boldsymbol{x}$. .... LEM

□□¶ .... DEM

21.3 $\square\square\square \boldsymbol{p}$. .... **[.1]** ASM

21.4 $\square\square\square \neg \boldsymbol{p}$. .... **[.2]** ASM

21.5 $\square\square\square \forall \boldsymbol{p}(\neg \boldsymbol{p}) = (\boldsymbol{p} \vdash \forall \boldsymbol{x}\boldsymbol{x})$. .... AXM

21.6 $\square\square\square (\neg \boldsymbol{p}) = (\boldsymbol{p} \vdash \forall \boldsymbol{x}\boldsymbol{x})$. .... **[.5]** $\forall_1$INST

21.7 $\square\square\square \boldsymbol{p} \vdash \forall \boldsymbol{x}\boldsymbol{x}$. .... **[.6] [.4]** SUB

21.8 $\square\square\square \forall \boldsymbol{x}\boldsymbol{x}$. .... **[.7] [.3]** MP

□□■. .... QED

□■. .... QED

21.9 $\square \forall \boldsymbol{p}\boldsymbol{p} \vdash \neg \boldsymbol{p} \vdash \forall \boldsymbol{x}\boldsymbol{x}$. .... **[.1]** $\forall_1$GEN

■. .... QED

Uses Axioms: **1**

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22 $\forall \boldsymbol{p} \forall \boldsymbol{x}\boldsymbol{p} \vdash \neg \boldsymbol{p} \vdash \boldsymbol{x}$. .... THEOREM *of* ¬ *Elimination 2*

¶ … DEM

$\Box p \vdash \neg p \vdash x.$ … LEM **22.1**

$\Box$¶ … DEM

$\Box\Box \neg p \vdash x.$ … LEM **22.2**

$\Box\Box$¶ … DEM

$\Box\Box\Box p.$ … [.1] ASM **22.3**

$\Box\Box\Box \neg p.$ … [.2] ASM **22.4**

$\Box\Box\Box \forall p(\neg p)=(p \vdash \forall x x).$ … AXM **22.5**

$\Box\Box\Box (\neg p)=(p \vdash \forall x x).$ … [.5] $\forall_1$INST **22.6**

$\Box\Box\Box p \vdash \forall x x.$ … [.6] [.4] SUB **22.7**

$\Box\Box\Box \forall x x.$ … [.7] [.3] MP **22.8**

$\Box\Box\Box x.$ … [.8] $\forall_1$INST **22.9**

$\Box\Box\blacksquare.$ … QED

$\Box\blacksquare.$ … QED

$\Box \forall p \forall x p \vdash \neg p \vdash x.$ … [.1] UGEN_2 **22.10**

$\blacksquare.$ … QED

Uses Axioms: **1**

$\forall p(p \vdash \forall x x) \vdash \neg p.$ … THEOREM *of* $\neg$ *Introduction* **23**

¶ … DEM

$\Box (p \vdash \forall x x) \vdash \neg p.$ … LEM **23.1**

$\Box$¶ … DEM

$\Box\Box p \vdash \forall x x.$ … [.1] ASM **23.2**

$\Box\Box \forall p(\neg p)=(p \vdash \forall x x).$ … AXM **23.3**

$\Box\Box (\neg p)=(p \vdash \forall x x).$ … [.3] $\forall_1$INST **23.4**

$\Box\Box \neg p.$ … [.4] [.2] SUB **23.5**

$\Box\blacksquare.$ … QED

$\Box \forall p(p \vdash \forall x x) \vdash \neg p.$ … [.1] $\forall_1$GEN **23.6**

$\blacksquare.$ … QED

Uses Axioms: **1**

$\forall p \forall q\, p \vdash p \Rightarrow q \vdash q.$ … THEOREM *of* $\Rightarrow$ *Elimination* **24**

¶ … DEM

$\Box p \vdash p \Rightarrow q \vdash q.$ … LEM **24.1**

$\Box$¶ … DEM

$\Box\Box p \Rightarrow q \vdash q.$ … LEM **24.2**

$\Box\Box$¶ … DEM

$\Box\Box\Box p.$ … [.1] ASM **24.3**

$\Box\Box\Box p \Rightarrow q.$ … [.2] ASM **24.4**

$\Box\Box\Box \forall p \forall q(p \Rightarrow q)=(p \vdash q).$ … AXM **24.5**

$\Box\Box\Box (p \Rightarrow q)=(p \vdash q).$ … [.5] ${}_0\mathrm{MINT}_2$ **24.6**

$\Box\Box\Box p \vdash q.$ … [.6] [.4] SUB **24.7**

$\Box\Box\Box q.$ … [.7] [.3] MP **24.8**

$\square\square\blacksquare$. . . . . QED

$\square\blacksquare$. . . . . QED

24.9 $\square\forall\boldsymbol{p}\forall\boldsymbol{q}\boldsymbol{p}\vdash\boldsymbol{p}\Rightarrow\boldsymbol{q}\vdash\boldsymbol{q}$. . . . . [.1] UGEN$_2$

$\blacksquare$. . . . . QED

Uses Axioms: **2**

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25 $\forall\boldsymbol{p}\forall\boldsymbol{q}(\boldsymbol{p}\vdash\boldsymbol{q})\vdash\boldsymbol{p}\Rightarrow\boldsymbol{q}$. . . . . THEOREM *of* $\Rightarrow$ *Introduction*

¶ . . . . . DEM

25.1 $\square(\boldsymbol{p}\vdash\boldsymbol{q})\vdash\boldsymbol{p}\Rightarrow\boldsymbol{q}$. . . . . LEM

$\square$¶ . . . . . DEM

25.2 $\square\square\boldsymbol{p}\vdash\boldsymbol{q}$. . . . . [.1] ASM

25.3 $\square\square\forall\boldsymbol{p}\forall\boldsymbol{q}(\boldsymbol{p}\Rightarrow\boldsymbol{q})=(\boldsymbol{p}\vdash\boldsymbol{q})$. . . . . AXM

25.4 $\square\square(\boldsymbol{p}\Rightarrow\boldsymbol{q})=(\boldsymbol{p}\vdash\boldsymbol{q})$. . . . . [.3] $_0$MINT$_2$

25.5 $\square\square\boldsymbol{p}\Rightarrow\boldsymbol{q}$. . . . . [.4] [.2] SUB

$\square\blacksquare$. . . . . QED

25.6 $\square\forall\boldsymbol{p}\forall\boldsymbol{q}(\boldsymbol{p}\vdash\boldsymbol{q})\vdash\boldsymbol{p}\Rightarrow\boldsymbol{q}$. . . . . [.1] UGEN$_2$

$\blacksquare$. . . . . QED

Uses Axioms: **2**

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26 $\forall\boldsymbol{p}\forall\boldsymbol{q}\boldsymbol{p}\wedge\boldsymbol{q}\vdash\boldsymbol{p}$. . . . . THEOREM *of* $\wedge$ *Elimination 1*

¶ . . . . . DEM

26.1 $\square\boldsymbol{p}\wedge\boldsymbol{q}\vdash\boldsymbol{p}$. . . . . LEM

$\square$¶ . . . . . DEM

26.2 $\square\square\boldsymbol{p}\wedge\boldsymbol{q}$. . . . . [.1] ASM

26.3 $\square\square\forall\boldsymbol{p}\forall\boldsymbol{q}(\boldsymbol{p}\wedge\boldsymbol{q})=\forall\boldsymbol{x}(\boldsymbol{p}\vdash\boldsymbol{q}\vdash\boldsymbol{x})\vdash\boldsymbol{x}$. . . . . AXM

26.4 $\square\square(\boldsymbol{p}\wedge\boldsymbol{q})=\forall\boldsymbol{x}(\boldsymbol{p}\vdash\boldsymbol{q}\vdash\boldsymbol{x})\vdash\boldsymbol{x}$. . . . . [.3] $_0$MINT$_2$

26.5 $\square\square\forall\boldsymbol{x}(\boldsymbol{p}\vdash\boldsymbol{q}\vdash\boldsymbol{x})\vdash\boldsymbol{x}$. . . . . [.4] [.2] SUB

26.6 $\square\square(\boldsymbol{p}\vdash\boldsymbol{q}\vdash\boldsymbol{p})\vdash\boldsymbol{p}$. . . . . [.5] $\forall_1$INST

26.7 $\square\square\boldsymbol{p}\vdash\boldsymbol{q}\vdash\boldsymbol{p}$. . . . . LEM

$\square\square$¶ . . . . . DEM

26.8 $\square\square\square\boldsymbol{q}\vdash\boldsymbol{p}$. . . . . LEM

$\square\square\square$¶ . . . . . DEM

26.9 $\square\square\square\square\boldsymbol{p}$. . . . . [.7] ASM

$\square\square\square\blacksquare$. . . . . QED

$\square\square\blacksquare$. . . . . QED

26.10 $\square\square\boldsymbol{p}$. . . . . [.6] [.7] MP

$\square\blacksquare$. . . . . QED

26.11 $\square\forall\boldsymbol{p}\forall\boldsymbol{q}\boldsymbol{p}\wedge\boldsymbol{q}\vdash\boldsymbol{p}$. . . . . [.1] UGEN$_2$

$\blacksquare$. . . . . QED

Uses Axioms: **3**

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27 $\forall\boldsymbol{p}\forall\boldsymbol{q}\boldsymbol{p}\wedge\boldsymbol{q}\vdash\boldsymbol{q}$. . . . . THEOREM *of* $\wedge$ *Elimination 2*

¶ ... DEM

$\square p \wedge q \vdash q$. ... LEM **27.1**

$\square$¶ ... DEM

$\square\square p \wedge q$. ... [.1] ASM **27.2**

$\square\square \forall p \forall q (p \wedge q) = \forall x (p \vdash q \vdash x) \vdash x$. ... AXM **27.3**

$\square\square (p \wedge q) = \forall x (p \vdash q \vdash x) \vdash x$. ... [.3] $_0$MINT$_2$ **27.4**

$\square\square \forall x (p \vdash q \vdash x) \vdash x$. ... [.4] [.2] SUB **27.5**

$\square\square (p \vdash q \vdash q) \vdash q$. ... [.5] $\forall_1$INST **27.6**

$\square\square p \vdash q \vdash q$. ... LEM **27.7**

$\square\square$¶ ... DEM

$\square\square\square q \vdash q$. ... LEM **27.8**

$\square\square\square$¶ ... DEM

$\square\square\square\square q$. ... [.8] ASM **27.9**

$\square\square\square\blacksquare$. ... QED

$\square\square\blacksquare$. ... QED

$\square\square p$. ... [.6] [.7] MP **27.10**

$\square\blacksquare$. ... QED

$\square \forall p \forall q\, p \wedge q \vdash q$. ... [.1] UGEN$_2$ **27.11**

$\blacksquare$. ... QED

Uses Axioms: **3**

$\forall p \forall q\, p \vdash q \vdash p \wedge q$. ... THEOREM *of* $\wedge$ *Introduction* **28**

¶ ... DEM

$\square p \vdash q \vdash p \wedge q$. ... LEM **28.1**

$\square$¶ ... DEM

$\square\square q \vdash p \wedge q$. ... LEM **28.2**

$\square\square$¶ ... DEM

$\square\square\square p$. ... [.1] ASM **28.3**

$\square\square\square q$. ... [.2] ASM **28.4**

$\square\square\square \forall p \forall q (p \wedge q) = \forall x (p \vdash q \vdash x) \vdash x$. ... AXM **28.5**

$\square\square\square (p \wedge q) = \forall x (p \vdash q \vdash x) \vdash x$. ... [.5] $_0$MINT$_2$ **28.6**

$\square\square\square (p \vdash q \vdash x) \vdash x$. ... LEM **28.7**

$\square\square\square$¶ ... DEM

$\square\square\square\square p \vdash q \vdash x$. ... [.7] ASM **28.8**

$\square\square\square\square x$. ... [.8] [.3] [.4] $_2$MINT$_0$ **28.9**

$\square\square\square\blacksquare$. ... QED

$\square\square\square \forall x (p \vdash q \vdash x) \vdash x$. ... [.7] $\forall_1$GEN **28.10**

$\square\square\square p \wedge q$. ... [.6] [.10] SUB **28.11**

$\square\square\blacksquare$. ... QED

$\square\blacksquare$. ... QED

$\square \forall p \forall q\, p \vdash q \vdash p \wedge q$. ... [.1] UGEN$_2$ **28.12**

$\blacksquare$. ... QED

Uses Axioms: **3**

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29 $\forall p \forall q \forall r (p \vdash r) \vdash (q \vdash r) \vdash p \vee q \vdash r$. .... THEOREM *of* $\vee$ *Elimination*

¶ .... DEM

29.1 □$(p \vdash r) \vdash (q \vdash r) \vdash p \vee q \vdash r$. .... LEM

□¶ .... DEM

29.2 □□$(q \vdash r) \vdash p \vee q \vdash r$. .... LEM

□□¶ .... DEM

29.3 □□□$p \vee q \vdash r$. .... LEM

□□□¶ .... DEM

29.4 □□□□$p \vdash r$ .... [.1] ASM

29.5 □□□□$q \vdash r$ .... [.2] ASM

29.6 □□□□$p \vee q$ .... [.3] ASM

29.7 □□□□$\forall p \forall q (p \vee q) = \forall x (p \vdash x) \vdash (q \vdash x) \vdash x$. .... AXM

29.8 □□□□$(p \vee q) = \forall x (p \vdash x) \vdash (q \vdash x) \vdash x$. .... [.7] $_0$MINT$_2$

29.9 □□□□$\forall x (p \vdash x) \vdash (q \vdash x) \vdash x$. .... [.8] [.6] SUB

29.10 □□□□$(p \vdash r) \vdash (q \vdash r) \vdash r$. .... [.9] $\forall_1$INST

29.11 □□□□r. .... [.10] [.4] [.5] $_2$MINT$_0$

□□□■. .... QED

□□■. .... QED

□■. .... QED

29.12 □$\forall p \forall q \forall r (p \vdash r) \vdash (q \vdash r) \vdash p \vee q \vdash r$. .... [.1] UGEN$_3$

■. .... QED

Uses Axioms: **4**

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30 $\forall p \forall q p \vdash p \vee q$. .... THEOREM *of* $\vee$ *Introduction 1*

¶ .... DEM

30.1 □$p \vdash p \vee q$. .... LEM

□¶ .... DEM

30.2 □□p. .... [.1] ASM

30.3 □□$\forall p \forall q (p \vee q) = \forall x (p \vdash x) \vdash (q \vdash x) \vdash x$. .... AXM

30.4 □□$(p \vee q) = \forall x (p \vdash x) \vdash (q \vdash x) \vdash x$. .... [.3] $_0$MINT$_2$

30.5 □□$(p \vdash x) \vdash (q \vdash x) \vdash x$. .... LEM

□□¶ .... DEM

30.6 □□□$(q \vdash x) \vdash x$. .... LEM

□□□¶ .... DEM

30.7 □□□□$p \vdash x$. .... [.5] ASM

30.8 □□□□x. .... [.7] [.2] MP

□□□■. .... QED

□□■. .... QED

30.9 □□$\forall x (p \vdash x) \vdash (q \vdash x) \vdash x$. .... [.5] $\forall_1$GEN

30.10 □□$(p \vee q)$. .... [.4] [.9] SUB

□■. .... QED

30.11 □$\forall p \forall q p \vdash p \vee q$. .... [.1] UGEN$_2$

$\blacksquare$. .... QED

Uses Axioms: **4**

$\forall p \forall q\, q \vdash p \vee q$. .... THEOREM *of* ∨ *Introduction 2* **31**

¶ .... DEM

$\square q \vdash p \vee q$. .... LEM **31.1**

$\square$¶ .... DEM

$\square\square q$. .... [.1] ASM **31.2**

$\square\square \forall p \forall q (p \vee q) = \forall x (p \vdash x) \vdash (q \vdash x) \vdash x$. .... AXM **31.3**

$\square\square (p \vee q) = \forall x (p \vdash x) \vdash (q \vdash x) \vdash x$. .... [.3] $_0\text{MINT}_2$ **31.4**

$\square\square (p \vdash x) \vdash (q \vdash x) \vdash x$. .... LEM **31.5**

$\square\square$¶ .... DEM

$\square\square\square (q \vdash x) \vdash x$. .... LEM **31.6**

$\square\square\square$¶ .... DEM

$\square\square\square\square q \vdash x$. .... [.6] ASM **31.7**

$\square\square\square\square x$. .... [.7] [.2] MP **31.8**

$\square\square\square\blacksquare$. .... QED

$\square\square\blacksquare$. .... QED

$\square\square \forall x (p \vdash x) \vdash (q \vdash x) \vdash x$. .... [.5] $\forall_1$GEN **31.9**

$\square\square (p \vee q)$. .... [.4] [.9] SUB **31.10**

$\square\blacksquare$. .... QED

$\square \forall p \forall q\, q \vdash p \vee q$. .... [.1] UGEN_2 **31.11**

$\blacksquare$. .... QED

Uses Axioms: **4**

$\forall p \forall q\, p \Leftrightarrow q \vdash p \vdash q$. .... THEOREM *of* ⇔ *Elimination 1* **32**

¶ .... DEM

$\square p \Leftrightarrow q \vdash p \vdash q$. .... LEM **32.1**

$\square$¶ .... DEM

$\square\square p \Leftrightarrow q$ .... [.1] ASM **32.2**

$\square\square \forall p \forall q (p \Leftrightarrow q) = \forall x ((p \vdash q) \vdash (q \vdash p) \vdash x) \vdash x$. .... AXM **32.3**

$\square\square (p \Leftrightarrow q) = \forall x ((p \vdash q) \vdash (q \vdash p) \vdash x) \vdash x$. .... [.3] $_0\text{MINT}_2$ **32.4**

$\square\square \forall x ((p \vdash q) \vdash (q \vdash p) \vdash x) \vdash x$. .... [.4] [.2] SUB **32.5**

$\square\square ((p \vdash q) \vdash (q \vdash p) \vdash (p \vdash q)) \vdash p \vdash q$. .... [.5] $\forall_1$INST **32.6**

$\square\square (p \vdash q) \vdash (q \vdash p) \vdash p \vdash q$. .... LEM **32.7**

$\square\square$¶ .... DEM

$\square\square\square (q \vdash p) \vdash p \vdash q$. .... LEM **32.8**

$\square\square\square$¶ .... DEM

$\square\square\square\square p \vdash q$ .... [.7] ASM **32.9**

$\square\square\square\blacksquare$. .... QED

$\square\square\blacksquare$. .... QED

$\square\square p \vdash q$. .... [.6] [.7] MP **32.10**

$\square\blacksquare$. .... QED

32.11 $\Box\forall p\forall q\, p\Leftrightarrow q\vdash p\vdash q.$ .... [.1] UGEN$_2$

$\blacksquare$. .... QED

Uses Axioms: **5**

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33 $\forall p\forall q\, p\Leftrightarrow q\vdash q\vdash p.$ .... THEOREM *of* $\Leftrightarrow$ *Elimination 2*

¶ .... DEM

33.1 $\Box p\Leftrightarrow q\vdash q\vdash p.$ .... LEM

$\Box$¶ .... DEM

33.2 $\Box\Box p\Leftrightarrow q$ .... [.1] ASM

33.3 $\Box\Box\forall p\forall q(p\Leftrightarrow q)=\forall x((p\vdash q)\vdash(q\vdash p)\vdash x)\vdash x.$ .... AXM

33.4 $\Box\Box(p\Leftrightarrow q)=\forall x((p\vdash q)\vdash(q\vdash p)\vdash x)\vdash x.$ .... [.3] $_0$MINT$_2$

33.5 $\Box\Box\forall x((p\vdash q)\vdash(q\vdash p)\vdash x)\vdash x.$ .... [.4] [.2] SUB

33.6 $\Box\Box((p\vdash q)\vdash(q\vdash p)\vdash(q\vdash p))\vdash q\vdash p.$ .... [.5] $\forall_1$INST

33.7 $\Box\Box(p\vdash q)\vdash(q\vdash p)\vdash q\vdash p.$ .... LEM

$\Box\Box$¶ .... DEM

33.8 $\Box\Box\Box(q\vdash p)\vdash q\vdash p.$ .... LEM

$\Box\Box\Box$¶ .... DEM

33.9 $\Box\Box\Box\Box q\vdash p$ .... [.8] ASM

$\Box\Box\Box\blacksquare$. .... QED

$\Box\Box\blacksquare$. .... QED

33.10 $\Box\Box q\vdash p.$ .... [.6] [.7] MP

$\Box\blacksquare$. .... QED

33.11 $\Box\forall p\forall q\, p\Leftrightarrow q\vdash q\vdash p.$ .... [.1] UGEN$_2$

$\blacksquare$. .... QED

Uses Axioms: **5**

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34 $\forall p\forall q(p\vdash q)\vdash(q\vdash p)\vdash p\Leftrightarrow q.$ .... THEOREM *of* $\Leftrightarrow$ *Introduction*

¶ .... DEM

34.1 $\Box(p\vdash q)\vdash(q\vdash p)\vdash p\Leftrightarrow q.$ .... LEM

$\Box$¶ .... DEM

34.2 $\Box\Box(q\vdash p)\vdash p\Leftrightarrow q$ .... LEM

$\Box\Box$¶ .... DEM

34.3 $\Box\Box\Box p\vdash q.$ .... [.1] ASM

34.4 $\Box\Box\Box q\vdash p.$ .... [.2] ASM

34.5 $\Box\Box\Box\forall p\forall q(p\Leftrightarrow q)=\forall x((p\vdash q)\vdash(q\vdash p)\vdash x)\vdash x.$ .... AXM

34.6 $\Box\Box\Box(p\Leftrightarrow q)=\forall x((p\vdash q)\vdash(q\vdash p)\vdash x)\vdash x.$ .... [.5] $_0$MINT$_2$

34.7 $\Box\Box\Box((p\vdash q)\vdash(q\vdash p)\vdash x)\vdash x.$ .... LEM

$\Box\Box\Box$¶ .... DEM

34.8 $\Box\Box\Box\Box(p\vdash q)\vdash(q\vdash p)\vdash x.$ .... [.7] ASM

34.9 $\Box\Box\Box\Box x.$ .... [.8] [.3] [.4] $_2$MINT$_0$

$\Box\Box\Box\blacksquare$. .... QED

34.10 $\Box\Box\Box\forall x((p\vdash q)\vdash(q\vdash p)\vdash x)\vdash x.$ .... [.7] $\forall_1$GEN

34.11 $\Box\Box\Box(p\Leftrightarrow q).$ .... [.6] [.10] SUB

□□■. . . . . QED

□■. . . . . QED

□$\forall \boldsymbol{p} \forall \boldsymbol{q}(\boldsymbol{p} \vdash \boldsymbol{q}) \vdash (\boldsymbol{q} \vdash \boldsymbol{p}) \vdash \boldsymbol{p} \Leftrightarrow \boldsymbol{q}$. . . . . [**.1**] UGEN$_{\mathbf{2}}$ **34.12**

■. . . . . QED

Uses Axioms: **5**

§1.2 *Logic: Exercises in First-order*

35 $\forall \boldsymbol{p}\boldsymbol{p}\vdash\boldsymbol{p}$. .... THEOREM *of Self Entailment*
¶ .... DEM
35.1 □$\boldsymbol{p}\vdash\boldsymbol{p}$. .... LEM
□¶ .... DEM
35.2 □□$\boldsymbol{p}$. .... **[.1]** ASM
□■. .... QED
35.3 □$\forall\boldsymbol{p}\boldsymbol{p}\vdash\boldsymbol{p}$. .... **[.1]** $\forall_1$GEN
■. .... QED
Uses Axioms: None

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36 $\forall\boldsymbol{p}\forall\boldsymbol{q}\boldsymbol{p}\vdash\boldsymbol{q}\vdash\boldsymbol{p}$. .... THEOREM *of Weakening*
¶ .... DEM
36.1 □$\boldsymbol{p}\vdash\boldsymbol{q}\vdash\boldsymbol{p}$. .... LEM
□¶ .... DEM
36.2 □□$\boldsymbol{q}\vdash\boldsymbol{p}$. .... LEM
□□¶ .... DEM
36.3 □□□$\boldsymbol{p}$. .... **[.1]** ASM
□□■. .... QED
□■. .... QED
36.4 □$\forall\boldsymbol{p}\forall\boldsymbol{q}\boldsymbol{p}\vdash\boldsymbol{q}\vdash\boldsymbol{p}$. .... **[.1]** UGEN$_2$
■. .... QED
Uses Axioms: None

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37 $\forall\boldsymbol{p}\forall\boldsymbol{q}(\boldsymbol{p}\vdash\boldsymbol{q})\vdash\neg\boldsymbol{q}\vdash\neg\boldsymbol{p}$. .... THEOREM *of Modus Tollens*
¶ .... DEM
37.1 □$(\boldsymbol{p}\vdash\boldsymbol{q})\vdash\neg\boldsymbol{q}\vdash\neg\boldsymbol{p}$. .... LEM
□¶ .... DEM
37.2 □□$\neg\boldsymbol{q}\vdash\neg\boldsymbol{p}$. .... LEM
□□¶ .... DEM
37.3 □□□$\boldsymbol{p}\vdash\boldsymbol{q}$. .... **[.1]** ASM
37.4 □□□$\neg\boldsymbol{q}$. .... **[.2]** ASM
37.5 □□□$\boldsymbol{p}\vdash\forall\boldsymbol{x}\boldsymbol{x}$. .... LEM
□□□¶ .... DEM
37.6 □□□□$\boldsymbol{p}$. .... **[.5]** ASM
37.7 □□□□$\boldsymbol{q}$. .... **[.3]** **[.6]** MP
37.8 □□□□$\forall\boldsymbol{p}\boldsymbol{p}\vdash\neg\boldsymbol{p}\vdash\forall\boldsymbol{x}\boldsymbol{x}$. .... THM
37.9 □□□□$\forall\boldsymbol{x}\boldsymbol{x}$. .... **[.8]** **[.7]** **[.4]** $_2$MINT$_1$
□□□■. .... QED
37.10 □□□$\forall\boldsymbol{p}(\boldsymbol{p}\vdash\forall\boldsymbol{x}\boldsymbol{x})\vdash\neg\boldsymbol{p}$. .... THM

$\square\square\square\neg\boldsymbol{p}$. …… [.10] [.5] ${}_{1}$MINT$_{1}$ **37.11**

$\square\square\blacksquare$. …… QED

$\square\blacksquare$. …… QED

$\square\forall\boldsymbol{p}\forall\boldsymbol{q}(\boldsymbol{p}\vdash\boldsymbol{q})\vdash\neg\boldsymbol{q}\vdash\neg\boldsymbol{p}$. …… [.1] UGEN$_{2}$ **37.12**

$\blacksquare$. …… QED

Uses Axioms: **1**

$\forall\boldsymbol{p}\boldsymbol{p}\vdash\neg\neg\boldsymbol{p}$. …… THEOREM *of the Iterated Dual* **38**

¶ …… DEM

$\square\boldsymbol{p}\vdash\neg\neg\boldsymbol{p}$. …… LEM **38.1**

$\square$¶ …… DEM

$\square\square\neg\boldsymbol{p}\vdash\forall\boldsymbol{x}\boldsymbol{x}$. …… LEM **38.2**

$\square\square$¶ …… DEM

$\square\square\square\boldsymbol{p}$. …… [.1] ASM **38.3**

$\square\square\square\neg\boldsymbol{p}$. …… [.2] ASM **38.4**

$\square\square\square\forall\boldsymbol{p}\boldsymbol{p}\vdash\neg\boldsymbol{p}\vdash\forall\boldsymbol{x}\boldsymbol{x}$. …… THM **38.5**

$\square\square\square\forall\boldsymbol{x}\boldsymbol{x}$. …… [.5] [.3] [.4] ${}_{2}$MINT$_{1}$ **38.6**

$\square\square\blacksquare$. …… QED

$\square\square\forall\boldsymbol{p}(\boldsymbol{p}\vdash\forall\boldsymbol{x}\boldsymbol{x})\vdash\neg\boldsymbol{p}$. …… THM **38.7**

$\square\square\neg\neg\boldsymbol{p}$. …… [.7] [.2] ${}_{1}$MINT$_{1}$ **38.8**

$\square\blacksquare$. …… QED

$\square\forall\boldsymbol{p}\boldsymbol{p}\vdash\neg\neg\boldsymbol{p}$. …… [.1] $\forall_{1}$GEN **38.9**

$\blacksquare$. …… QED

Uses Axioms: **1**

$\exists\boldsymbol{x}\boldsymbol{x}$. …… THEOREM *of the Existence of the Individual* **39**

¶ …… DEM

$\square\boldsymbol{p}\vdash\boldsymbol{p}$. …… LEM **39.1**

$\square$¶ …… DEM

$\square\square\boldsymbol{p}$. …… [.1] ASM **39.2**

$\square\blacksquare$. …… QED

$\square\exists\boldsymbol{x}\boldsymbol{x}$. …… [.1] $\forall_{1}$GEN **39.3**

$\blacksquare$. …… QED

Uses Axioms: None

$\forall\boldsymbol{x}\exists\boldsymbol{y}\neg\boldsymbol{x}=\boldsymbol{y}$. …… THEOREM *of Distinguishable Individuals* **40**

¶ …… DEM

$\square\forall\boldsymbol{p}(\boldsymbol{p}\vdash\forall\boldsymbol{x}\boldsymbol{x})\vdash\neg\boldsymbol{p}$. …… THM **40.1**

$\square\boldsymbol{x}=\neg\boldsymbol{x}\vdash\forall\boldsymbol{x}\boldsymbol{x}$. …… LEM **40.2**

$\square$¶ …… DEM

$\square\square\boldsymbol{x}=\neg\boldsymbol{x}$. …… [.2] ASM **40.3**

$\square\square\boldsymbol{x}\vdash\forall\boldsymbol{x}\boldsymbol{x}$. …… LEM **40.4**

$\square\square$¶ . . . DEM
40.5 $\square\square\square \boldsymbol{x}.$ . . . **[.4]** ASM
40.6 $\square\square\square \neg \boldsymbol{x}.$ . . . **[.3] [.5]** SUB
40.7 $\square\square\square \forall \boldsymbol{p} \boldsymbol{p} \vdash \neg \boldsymbol{p} \vdash \forall \boldsymbol{x} \boldsymbol{x}.$ . . . THM
40.8 $\square\square\square \forall \boldsymbol{x} \boldsymbol{x}.$ . . . **[.7] [.5] [.6]** ${}_2$MINT$_1$
$\square\square\blacksquare.$ . . . QED
40.9 $\square\square \neg \boldsymbol{x}.$ . . . **[.1] [.4]** ${}_1$MINT$_1$
40.10 $\square\square \boldsymbol{x}.$ . . . **[.2] [.9]** SUB
40.11 $\square\square \forall \boldsymbol{x} \boldsymbol{x}.$ . . . **[.4] [.10]** MP
$\square\blacksquare.$ . . . QED
40.12 $\square \neg \boldsymbol{x} = \neg \boldsymbol{x}.$ . . . **[.1] [.2]** ${}_1$MINT$_1$
40.13 $\square \exists \boldsymbol{y} \neg \boldsymbol{x} = \boldsymbol{y}.$ . . . **[.12]** $\exists_1$GEN
40.14 $\square \forall \boldsymbol{x} \exists \boldsymbol{y} \neg \boldsymbol{x} = \boldsymbol{y}.$ . . . **[.13]** $\forall_1$GEN
$\blacksquare.$ . . . QED
Uses Axioms: **1**

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41 $\forall \boldsymbol{x} \boldsymbol{x} = \boldsymbol{x}.$ . . . THEOREM *of Reflexivity, Equality Identity 1*
¶ . . . DEM
41.1 $\square \boldsymbol{x} = \boldsymbol{x}.$ . . . EQ
41.2 $\square \forall \boldsymbol{x} \boldsymbol{x} = \boldsymbol{x}.$ . . . **[.1]** $\forall_1$GEN
$\blacksquare.$ . . . QED
Uses Axioms: None

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42 $\forall \boldsymbol{x} \forall \boldsymbol{y} \boldsymbol{x} = \boldsymbol{y} \vdash \boldsymbol{y} = \boldsymbol{x}.$ . . . THEOREM *of Symmetry, Equality Identity 2*
¶ . . . DEM
42.1 $\square \boldsymbol{x} = \boldsymbol{y} \vdash \boldsymbol{y} = \boldsymbol{x}.$ . . . LEM
$\square$¶ . . . DEM
42.2 $\square\square \boldsymbol{x} = \boldsymbol{y}.$ . . . **[.1]** ASM
42.3 $\square\square \boldsymbol{y} = \boldsymbol{x}.$ . . . **[.2] [.2]** SUB
$\square\blacksquare.$ . . . QED
42.4 $\square \forall \boldsymbol{x} \forall \boldsymbol{y} \boldsymbol{x} = \boldsymbol{y} \vdash \boldsymbol{y} = \boldsymbol{x}.$ . . . **[.1]** UGEN$_2$
$\blacksquare.$ . . . QED
Uses Axioms: None

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43 $\forall \boldsymbol{x} \forall \boldsymbol{y} \forall \boldsymbol{z} \boldsymbol{x} = \boldsymbol{y} \vdash \boldsymbol{y} = \boldsymbol{z} \vdash \boldsymbol{x} = \boldsymbol{z}.$ . . . THEOREM *of Transitivity, Equality Identity 3*
¶ . . . DEM
43.1 $\square \boldsymbol{x} = \boldsymbol{y} \vdash \boldsymbol{y} = \boldsymbol{z} \vdash \boldsymbol{x} = \boldsymbol{z}.$ . . . LEM
$\square$¶ . . . DEM
43.2 $\square\square \boldsymbol{y} = \boldsymbol{z} \vdash \boldsymbol{x} = \boldsymbol{z}.$ . . . LEM
$\square\square$¶ . . . DEM
43.3 $\square\square\square \boldsymbol{x} = \boldsymbol{y}.$ . . . **[.1]** ASM
43.4 $\square\square\square \boldsymbol{y} = \boldsymbol{z}.$ . . . **[.2]** ASM

$\Box\Box\Box x=z$. .... [.3] [.4] SUB **43.5**
$\Box\Box\blacksquare$. .... QED
$\Box\blacksquare$. .... QED
$\Box\forall x\forall y\forall z x=y\vdash y=z\vdash x=z$. .... [.1] UGEN$_3$ **43.6**
$\blacksquare$. .... QED
Uses Axioms: None

$\forall p\neg\neg(p\vee\neg p)$. .... THEOREM *of the Translated Law of Excluded Middle* **44**
¶ .... DEM
$\Box\forall p p\vdash\neg p\vdash\forall x x$. .... THM **44.1**
$\Box\forall p(p\vdash\forall x x)\vdash\neg p$. .... THM **44.2**
$\Box\forall p\forall q p\vdash p\vee q$. .... THM **44.3**
$\Box\forall p\forall q q\vdash p\vee q$. .... THM **44.4**
$\Box\neg(p\vee\neg p)\vdash\forall x x$. .... LEM **44.5**
$\Box$¶ .... DEM
$\Box\Box\neg(p\vee\neg p)$. .... [.1] ASM **44.6**
$\Box\Box p\vdash\forall x x$. .... LEM **44.7**
$\Box\Box$¶ .... DEM
$\Box\Box\Box p$. .... [.3] ASM **44.8**
$\Box\Box\Box p\vee\neg p$. .... [.3] [.8] $_1$MINT$_2$ **44.9**
$\Box\Box\Box\forall x x$. .... [.1] [.9] [.6] $_2$MINT$_1$ **44.10**
$\Box\Box\blacksquare$. .... QED
$\Box\Box\neg p$. .... [.2] [.7] $_1$MINT$_1$ **44.11**
$\Box\Box p\vee\neg p$. .... [.4] [.11] $_1$MINT$_2$ **44.12**
$\Box\Box\forall x x$. .... [.1] [.12] [.6] $_2$MINT$_1$ **44.13**
$\Box\blacksquare$. .... QED
$\Box\neg\neg(p\vee\neg p)$. .... [.2] [.5] $_1$MINT$_1$ **44.14**
$\Box\forall p\neg\neg(p\vee\neg p)$. .... [.5] $\forall_1$GEN **44.15**
$\blacksquare$. .... QED
Uses Axioms: **1, 4**

If q follows from some instance of LEM, then $\neg\neg q$. PROOF: Suppose $p\vee\neg p\vdash q$. Applying *modus tollens* twice gives: $\neg\neg(p\vee\neg p)\vdash\neg\neg q$. $\neg\neg(p\vee\neg p)$; hence $\neg\neg q$. QED.
This theorem establishes that truths from classical prop. logic are injectable to intuitionstic prop. logic.

$\forall p\forall q(p\vee\neg p\vdash q)\vdash\neg\neg q$. .... THEOREM *of Glivenko's Propositional Translation 1* **45**
¶ .... DEM
$\Box(p_1\vee\neg p_1\vdash q_1)\vdash\neg\neg q_1$. .... LEM **45.1**
$\Box$¶ .... DEM
$\Box\Box p_1\vee\neg p_1\vdash q_1$. .... [.1] ASM **45.2**
$\Box\Box\forall p\forall q(p\vdash q)\vdash\neg q\vdash\neg p$. .... THM **45.3**
$\Box\Box\neg q_1\vdash\neg(p_1\vee\neg p_1)$. .... [.3] [.2] $_1$MINT$_2$ **45.4**

45.5 $\Box\Box\forall \boldsymbol{p}\neg\neg(\boldsymbol{p}\vee\neg\boldsymbol{p})$. .... THM
45.6 $\Box\Box\neg\neg(\boldsymbol{p}_1\vee\neg\boldsymbol{p}_1)$. .... [.5] $\forall_1$INST
45.7 $\Box\Box\neg\neg\boldsymbol{q}_1$. .... [.3] [.4] [.6] $_2$MINT$_2$
$\Box\blacksquare$. .... QED
45.8 $\Box\forall \boldsymbol{p}\forall \boldsymbol{q}(\boldsymbol{p}\vee\neg\boldsymbol{p}\vdash \boldsymbol{q})\vdash\neg\neg\boldsymbol{q}$. .... [.1] UGEN$_2$
$\blacksquare$. .... QED
Uses Axioms: **1, 4**

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46 $\forall \boldsymbol{p}\forall \boldsymbol{q}\,\boldsymbol{p}\wedge(\boldsymbol{p}\vdash \boldsymbol{q})\vdash \boldsymbol{q}$. .... THEOREM *of Conjunctive Entailment*
¶ .... DEM
46.1 $\Box \boldsymbol{p}\wedge(\boldsymbol{p}\vdash \boldsymbol{q})\vdash \boldsymbol{q}$. .... LEM
$\Box$¶ .... DEM
46.2 $\Box\Box \boldsymbol{p}\wedge(\boldsymbol{p}\vdash \boldsymbol{q})$. .... [.1] ASM
46.3 $\Box\Box\forall \boldsymbol{p}\forall \boldsymbol{q}\,\boldsymbol{p}\wedge \boldsymbol{q}\vdash \boldsymbol{p}$. .... THM
46.4 $\Box\Box \boldsymbol{p}$. .... [.3] [.2] $_1$MINT$_2$
46.5 $\Box\Box\forall \boldsymbol{p}\forall \boldsymbol{q}\,\boldsymbol{p}\wedge \boldsymbol{q}\vdash \boldsymbol{q}$. .... THM
46.6 $\Box\Box \boldsymbol{p}\vdash \boldsymbol{q}$. .... [.5] [.2] $_1$MINT$_2$
46.7 $\Box\Box \boldsymbol{q}$. .... [.6] [.4] MP
$\Box\blacksquare$. .... QED
46.8 $\Box\forall \boldsymbol{p}\forall \boldsymbol{q}\,\boldsymbol{p}\wedge(\boldsymbol{p}\vdash \boldsymbol{q})\vdash \boldsymbol{q}$. .... [.1] UGEN$_2$
$\blacksquare$. .... QED
Uses Axioms: **3**

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47 $\forall \boldsymbol{p}\forall \boldsymbol{q}\forall \boldsymbol{r}\,\boldsymbol{p}\wedge \boldsymbol{q}\vdash \boldsymbol{r}\vdash \boldsymbol{p}\wedge \boldsymbol{r}$. .... THEOREM *of Conjunctive Disassociation*
¶ .... DEM
47.1 $\Box \boldsymbol{p}\wedge \boldsymbol{q}\vdash \boldsymbol{r}\vdash \boldsymbol{p}\wedge \boldsymbol{r}$. .... LEM
$\Box$¶ .... DEM
47.2 $\Box\Box \boldsymbol{p}\wedge \boldsymbol{q}$. .... [.1] ASM
47.3 $\Box\Box \boldsymbol{r}\vdash \boldsymbol{p}\wedge \boldsymbol{r}$. .... LEM
$\Box\Box$¶ .... DEM
47.4 $\Box\Box\Box \boldsymbol{r}$. .... [.3] ASM
47.5 $\Box\Box\Box\forall \boldsymbol{p}\forall \boldsymbol{q}\,\boldsymbol{p}\wedge \boldsymbol{q}\vdash \boldsymbol{p}$. .... THM
47.6 $\Box\Box\Box \boldsymbol{p}$. .... [.5] [.2] $_1$MINT$_2$
47.7 $\Box\Box\Box\forall \boldsymbol{p}\forall \boldsymbol{q}\,\boldsymbol{p}\vdash \boldsymbol{q}\vdash \boldsymbol{p}\wedge \boldsymbol{q}$. .... THM
47.8 $\Box\Box\Box \boldsymbol{p}\wedge \boldsymbol{r}$. .... [.7] [.6] [.4] $_2$MINT$_2$
$\Box\Box\blacksquare$. .... QED
$\Box\blacksquare$. .... QED
47.9 $\Box\forall \boldsymbol{p}\forall \boldsymbol{q}\forall \boldsymbol{r}\,\boldsymbol{p}\wedge \boldsymbol{q}\vdash \boldsymbol{r}\vdash \boldsymbol{p}\wedge \boldsymbol{r}$. .... [.1] UGEN$_3$
$\blacksquare$. .... QED
Uses Axioms: **3**

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48 $\forall \boldsymbol{p}\forall \boldsymbol{q}\forall \boldsymbol{r}(\boldsymbol{p}\vdash \boldsymbol{q}\vdash \boldsymbol{r})\Leftrightarrow(\boldsymbol{p}\wedge \boldsymbol{q}\vdash \boldsymbol{r})$. .... THEOREM *of Conjunctive Weakening*

¶ . . . . . . DEM

$\square\forall p\forall q\, p\wedge q\vdash p$. . . . . . . THM **48.1**

$\square\forall p\forall q\, p\wedge q\vdash q$. . . . . . . THM **48.2**

$\square\forall p\forall q\, p\vdash q\vdash p\wedge q$. . . . . . . THM **48.3**

$\square\forall p\forall q\forall r(p\vdash q)\vdash(q\vdash p)\vdash p\Leftrightarrow q$. . . . . . . THM **48.4**

$\square(p_1\vdash q_1\vdash r_1)\vdash p_1\wedge q_1\vdash r_1$. . . . . . . LEM **48.5**

$\square$¶ . . . . . . DEM

$\square\square p_1\vdash q_1\vdash r_1$. . . . . . . **[.5]** ASM **48.6**

$\square\square p_1\wedge q_1\vdash r_1$. . . . . . . LEM **48.7**

$\square\square$¶ . . . . . . DEM

$\square\square\square p_1\wedge q_1$. . . . . . . **[.7]** ASM **48.8**

$\square\square\square p_1$. . . . . . . **[.1] [.8]** $_1$MINT$_2$ **48.9**

$\square\square\square q_1$. . . . . . . **[.2] [.8]** $_1$MINT$_2$ **48.10**

$\square\square\square r_1$. . . . . . . **[.6] [.9] [.10]** $_2$MINT$_0$ **48.11**

$\square\square\blacksquare$. . . . . . . QED

$\square\blacksquare$. . . . . . . QED

$\square(p_1\wedge q_1\vdash r_1)\vdash p_1\vdash q_1\vdash r_1$. . . . . . . LEM **48.12**

$\square$¶ . . . . . . DEM

$\square\square p_1\wedge q_1\vdash r_1$. . . . . . . **[.12]** ASM **48.13**

$\square\square p_1\vdash q_1\vdash r_1$. . . . . . . LEM **48.14**

$\square\square$¶ . . . . . . DEM

$\square\square\square p_1$. . . . . . . **[.14]** ASM **48.15**

$\square\square\square q_1\vdash r_1$. . . . . . . LEM **48.16**

$\square\square\square$¶ . . . . . . DEM

$\square\square\square\square q_1$. . . . . . . **[.16]** ASM **48.17**

$\square\square\square\square r_1$. . . . . . . **[.3] [.15] [.17] [.13]** $_3$MINT$_2$ **48.18**

$\square\square\square\blacksquare$. . . . . . . QED

$\square\square\blacksquare$. . . . . . . QED

$\square\blacksquare$. . . . . . . QED

$\square(p_1\vdash q_1\vdash r_1)\Leftrightarrow(p_1\wedge q_1\vdash r_1)$. . . . . . . **[.4] [.5] [.12]** $_2$MINT$_3$ **48.19**

$\square\forall p\forall q\forall r(p\vdash q\vdash r)\Leftrightarrow(p\wedge q\vdash r)$. . . . . . . **[.19]** UGEN$_3$ **48.20**

$\blacksquare$. . . . . . . QED

Uses Axioms: **3, 5**

$\forall p\forall q\forall r\, p\vdash q\vdash p\wedge(q\vee r)$. . . . . . . THEOREM *of Conjunctive Disjunctive Weakening* **49**

¶ . . . . . . DEM

$\square p_1\vdash q_1\vdash p_1\wedge(q_1\vee r_1)$. . . . . . . LEM **49.1**

$\square$¶ . . . . . . DEM

$\square\square p_1$. . . . . . . **[.1]** ASM **49.2**

$\square\square q_1\vdash p_1\wedge(q_1\vee r_1)$. . . . . . . LEM **49.3**

$\square\square$¶ . . . . . . DEM

$\square\square\square q_1$. . . . . . . **[.3]** ASM **49.4**

$\square\square\square\forall p\forall q\, p\vdash p\vee q$. . . . . . . THM **49.5**

$\square\square\square q_1\vee r_1$. . . . . . . **[.5] [.4]** $_1$MINT$_2$ **49.6**

49.7 $\Box\Box\Box\forall p\forall q\, p \vdash q \vdash p \wedge q$. .... THM
49.8 $\Box\Box\Box p_1 \wedge (q_1 \vee r_1)$. .... [.7] [.2] [.6] $_2$MINT$_2$
$\Box\Box\blacksquare$. .... QED
$\Box\blacksquare$. .... QED
49.9 $\Box\forall p\forall q\forall r\, p \vdash q \vdash p \wedge (q \vee r)$. .... [.1] UGEN$_3$
$\blacksquare$. .... QED
Uses Axioms: **3, 4**

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50 $\forall p\, p \Leftrightarrow p$. .... THEOREM *of* $\Leftrightarrow$ *Reflexivity*
¶ .... DEM
50.1 $\Box\forall p\, p \vdash p$. .... THM
50.2 $\Box p \vdash p$. .... [.1] $\forall_1$INST
50.3 $\Box\forall p\forall q (p \vdash q) \vdash (q \vdash p) \vdash p \Leftrightarrow q$. .... THM
50.4 $\Box p \Leftrightarrow p$. .... [.3] [.2] [.2] $_2$MINT$_3$
50.5 $\Box\forall p\, p \Leftrightarrow p$. .... [.4] $\forall_1$GEN
$\blacksquare$. .... QED
Uses Axioms: **5**

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51 $\forall p\forall q\, p \Leftrightarrow q \vdash q \Leftrightarrow p$. .... THEOREM *of* $\Leftrightarrow$ *Symmetry*
¶ .... DEM
51.1 $\Box\forall p\forall q\, p \Leftrightarrow q \vdash p \vdash q$. .... THM
51.2 $\Box\forall p\forall q\, p \Leftrightarrow q \vdash q \vdash p$. .... THM
51.3 $\Box\forall p\forall q (p \vdash q) \vdash (q \vdash p) \vdash p \Leftrightarrow q$. .... THM
51.4 $\Box p_1 \Leftrightarrow q_1 \vdash q_1 \Leftrightarrow p_1$. .... LEM
$\Box$¶ .... DEM
51.5 $\Box\Box p_1 \Leftrightarrow q_1$. .... [.4] ASM
51.6 $\Box\Box q_1 \vdash p_1$. .... LEM
$\Box\Box$¶ .... DEM
51.7 $\Box\Box\Box q_1$. .... [.6] ASM
51.8 $\Box\Box\Box p_1$. .... [.2] [.5] [.7] $_2$MINT$_2$
$\Box\Box\blacksquare$. .... QED
51.9 $\Box\Box p_1 \vdash q_1$. .... LEM
$\Box\Box$¶ .... DEM
51.10 $\Box\Box\Box p_1$. .... [.9] ASM
51.11 $\Box\Box\Box q_1$. .... [.1] [.5] [.10] $_2$MINT$_2$
$\Box\Box\blacksquare$. .... QED
51.12 $\Box\Box q_1 \Leftrightarrow p_1$. .... [.3] [.6] [.9] $_2$MINT$_2$
$\Box\blacksquare$. .... QED
51.13 $\Box\forall p\forall q\, p \Leftrightarrow q \vdash p \vdash q$. .... [.4] UGEN$_2$
$\blacksquare$. .... QED
Uses Axioms: **5**

$\forall p \forall q \forall r\, p \Leftrightarrow q \vdash q \Leftrightarrow r \vdash p \Leftrightarrow r$. …… THEOREM *of* ⇔ *Transitivity 1* **52**
¶ …… DEM
□$\forall p \forall q\, p \Leftrightarrow q \vdash p \vdash q$. …… THM **52.1**
□$\forall p \forall q\, p \Leftrightarrow q \vdash q \vdash p$. …… THM **52.2**
□$\forall p \forall q (p \vdash q) \vdash (q \vdash p) \vdash p \Leftrightarrow q$. …… THM **52.3**
□$p_1 \Leftrightarrow q_1 \vdash q_1 \Leftrightarrow r_1 \vdash p_1 \Leftrightarrow r_1$. …… LEM **52.4**
□¶ …… DEM
□□$p_1 \Leftrightarrow q_1$. …… [.4] ASM **52.5**
□□$q_1 \Leftrightarrow r_1 \vdash p_1 \Leftrightarrow r_1$. …… LEM **52.6**
□□¶ …… DEM
□□□$q_1 \Leftrightarrow r_1$. …… [.6] ASM **52.7**
□□□$p_1 \vdash r_1$. …… LEM **52.8**
□□□¶ …… DEM
□□□□p_1. …… [.8] ASM **52.9**
□□□□q_1. …… [.1] [.5] [.9] $_2$MINT$_2$ **52.10**
□□□□r_1. …… [.1] [.7] [.10] $_2$MINT$_2$ **52.11**
□□□■. …… QED
□□□$r_1 \vdash p_1$. …… LEM **52.12**
□□□¶ …… DEM
□□□□r_1. …… [.12] ASM **52.13**
□□□□q_1. …… [.2] [.7] [.13] $_2$MINT$_2$ **52.14**
□□□□p_1. …… [.2] [.5] [.14] $_2$MINT$_2$ **52.15**
□□□■. …… QED
□□□$p_1 \Leftrightarrow r_1$. …… [.3] [.8] [.12] $_2$MINT$_2$ **52.16**
□□■. …… QED
□■. …… QED
□$\forall p \forall q \forall r\, p \Leftrightarrow q \vdash q \Leftrightarrow r \vdash p \Leftrightarrow r$. …… [.4] UGEN$_3$ **52.17**
■. …… QED
Uses Axioms: **5**

$\forall p \forall q \forall r\, p \Leftrightarrow q \vdash r \Leftrightarrow q \vdash p \Leftrightarrow r$. …… THEOREM *of* ⇔ *Transitivity 2* **53**
¶ …… DEM
□$p \Leftrightarrow q \vdash r \Leftrightarrow q \vdash p \Leftrightarrow r$. …… LEM **53.1**
□¶ …… DEM
□□$p \Leftrightarrow q$. …… [.1] ASM **53.2**
□□$r \Leftrightarrow q \vdash p \Leftrightarrow r$. …… LEM **53.3**
□□¶ …… DEM
□□□$r \Leftrightarrow q$. …… [.3] ASM **53.4**
□□□$\forall p \forall q\, p \Leftrightarrow q \vdash q \Leftrightarrow p$. …… THM **53.5**
□□□$q \Leftrightarrow r$. …… [.5] [.4] $_1$MINT$_2$ **53.6**
□□□$\forall p \forall q \forall r\, p \Leftrightarrow q \vdash q \Leftrightarrow r \vdash p \Leftrightarrow r$. …… THM **53.7**
□□□$p \Leftrightarrow r$. …… [.7] [.2] [.6] $_2$MINT$_3$ **53.8**

$\square\square\blacksquare$. . . . . . QED
$\square\blacksquare$. . . . . . QED
53.9 $\square\forall p \forall q \forall r p \Leftrightarrow q \vdash r \Leftrightarrow q \vdash p \Leftrightarrow r$. . . . . . **[.1]** UGEN$_3$
$\blacksquare$. . . . . . QED
Uses Axioms: **5**

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54 $\forall p \forall q p \Leftrightarrow q \vdash \neg p \vdash \neg q$. . . . . . THEOREM *of* $\Leftrightarrow$ *Negativity 1*
¶. . . . . . DEM
54.1 $\square p \Leftrightarrow q \vdash \neg p \vdash \neg q$. . . . . . LEM
$\square$¶. . . . . . DEM
54.2 $\square\square p \Leftrightarrow q$. . . . . . **[.1]** ASM
54.3 $\square\square\neg p \vdash \neg q$. . . . . . LEM
$\square\square$¶. . . . . . DEM
54.4 $\square\square\square\neg p$. . . . . . **[.3]** ASM
54.5 $\square\square\square q \vdash \forall x x$. . . . . . LEM
$\square\square\square$¶. . . . . . DEM
54.6 $\square\square\square\square q$. . . . . . **[.5]** ASM
54.7 $\square\square\square\square\forall p \forall q p \Leftrightarrow q \vdash q \vdash p$. . . . . . THM
54.8 $\square\square\square\square p$. . . . . . **[.7] [.2] [.6]** $_2$MINT$_2$
54.9 $\square\square\square\square\forall p p \vdash \neg p \vdash \forall x x$. . . . . . THM
54.10 $\square\square\square\square\forall x x$. . . . . . **[.9] [.8] [.4]** $_2$MINT$_1$
$\square\square\square\blacksquare$. . . . . . QED
54.11 $\square\square\square\forall p (p \vdash \forall x x) \vdash \neg p$. . . . . . THM
54.12 $\square\square\square\neg q$. . . . . . **[.11] [.5]** $_1$MINT$_1$
$\square\square\blacksquare$. . . . . . QED
$\square\blacksquare$. . . . . . QED
54.13 $\square\forall p \forall q p \Leftrightarrow q \vdash \neg p \vdash \neg q$. . . . . . **[.1]** UGEN$_2$
$\blacksquare$. . . . . . QED
Uses Axioms: **1, 5**

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55 $\forall p \forall q p \Leftrightarrow q \vdash \neg q \vdash \neg p$. . . . . . THEOREM *of* $\Leftrightarrow$ *Negativity 2*
¶. . . . . . DEM
55.1 $\square p \Leftrightarrow q \vdash \neg q \vdash \neg p$. . . . . . LEM
$\square$¶. . . . . . DEM
55.2 $\square\square p \Leftrightarrow q$. . . . . . **[.1]** ASM
55.3 $\square\square\neg q \vdash \neg p$. . . . . . LEM
$\square\square$¶. . . . . . DEM
55.4 $\square\square\square\neg q$. . . . . . **[.3]** ASM
55.5 $\square\square\square p \vdash \forall x x$. . . . . . LEM
$\square\square\square$¶. . . . . . DEM
55.6 $\square\square\square\square p$. . . . . . **[.5]** ASM
55.7 $\square\square\square\square\forall p \forall q p \Leftrightarrow q \vdash p \vdash q$. . . . . . THM
55.8 $\square\square\square\square q$. . . . . . **[.7] [.2] [.6]** $_2$MINT$_2$

$\square\square\square\square\forall \boldsymbol{p}\boldsymbol{p}\vdash\neg\boldsymbol{p}\vdash\forall \boldsymbol{x}\boldsymbol{x}$. .... THM **55.9**

$\square\square\square\square\forall \boldsymbol{x}\boldsymbol{x}$. .... [.9] [.8] [.4] $_2$MINT$_1$ **55.10**

$\square\square\square\blacksquare$. .... QED

$\square\square\square\forall \boldsymbol{p}(\boldsymbol{p}\vdash\forall \boldsymbol{x}\boldsymbol{x})\vdash\neg\boldsymbol{p}$. .... THM **55.11**

$\square\square\square\neg\boldsymbol{p}$. .... [.11] [.5] $_1$MINT$_1$ **55.12**

$\square\square\blacksquare$. .... QED

$\square\blacksquare$. .... QED

$\square\forall \boldsymbol{p}\forall \boldsymbol{q}\boldsymbol{p}\Leftrightarrow\boldsymbol{q}\vdash\neg\boldsymbol{q}\vdash\neg\boldsymbol{p}$. .... [.1] UGEN$_2$ **55.13**

$\blacksquare$. .... QED

Uses Axioms: **1, 5**

$\forall \boldsymbol{p}\boldsymbol{p}\vee\boldsymbol{p}\Leftrightarrow\boldsymbol{p}$. .... THEOREM *of* $\Leftrightarrow$ *Disjunctive Symmetry* **56**

¶ .... DEM

$\square\boldsymbol{p}\vee\boldsymbol{p}\vdash\boldsymbol{p}$. .... LEM **56.1**

$\square$¶ .... DEM

$\square\square\boldsymbol{p}\vee\boldsymbol{p}$. .... [.1] ASM **56.2**

$\square\square\forall \boldsymbol{p}\boldsymbol{p}\vdash\boldsymbol{p}$. .... THM **56.3**

$\square\square\boldsymbol{p}\vdash\boldsymbol{p}$. .... [.3] $\forall_1$INST **56.4**

$\square\square\forall \boldsymbol{p}\forall \boldsymbol{q}\forall \boldsymbol{r}(\boldsymbol{p}\vdash\boldsymbol{r})\vdash(\boldsymbol{q}\vdash\boldsymbol{r})\vdash\boldsymbol{p}\vee\boldsymbol{q}\vdash\boldsymbol{r}$. .... THM **56.5**

$\square\square\boldsymbol{p}$. .... [.5] [.4] [.4] [.2] $_3$MINT$_3$ **56.6**

$\square\blacksquare$. .... QED

$\square\boldsymbol{p}\vdash\boldsymbol{p}\vee\boldsymbol{p}$. .... LEM **56.7**

$\square$¶ .... DEM

$\square\square\boldsymbol{p}$. .... [.7] ASM **56.8**

$\square\square\forall \boldsymbol{p}\forall \boldsymbol{q}\boldsymbol{p}\vdash\boldsymbol{p}\vee\boldsymbol{q}$. .... THM **56.9**

$\square\square\boldsymbol{p}\vee\boldsymbol{p}$. .... [.9] [.8] $_1$MINT$_2$ **56.10**

$\square\blacksquare$. .... QED

$\square\forall \boldsymbol{p}\forall \boldsymbol{q}(\boldsymbol{p}\vdash\boldsymbol{q})\vdash(\boldsymbol{q}\vdash\boldsymbol{p})\vdash\boldsymbol{p}\Leftrightarrow\boldsymbol{q}$. .... THM **56.11**

$\square\boldsymbol{p}\vee\boldsymbol{p}\Leftrightarrow\boldsymbol{p}$. .... [.11] [.1] [.7] $_2$MINT$_2$ **56.12**

$\square\forall \boldsymbol{p}\boldsymbol{p}\vee\boldsymbol{p}\Leftrightarrow\boldsymbol{p}$. .... [.12] $\forall_1$GEN **56.13**

$\blacksquare$. .... QED

Uses Axioms: **4, 5**

$\forall \boldsymbol{p}\forall \boldsymbol{q}\forall \boldsymbol{r}\boldsymbol{p}\Leftrightarrow\boldsymbol{q}\vee\boldsymbol{r}\vdash\boldsymbol{q}\vdash\boldsymbol{p}$. .... THEOREM *of* $\Leftrightarrow$ *Disjunctive Elimination 1* **57**

¶ .... DEM

$\square\boldsymbol{p}_1\Leftrightarrow\boldsymbol{q}_1\vee\boldsymbol{r}_1\vdash\boldsymbol{q}_1\vdash\boldsymbol{p}_1$. .... LEM **57.1**

$\square$¶ .... DEM

$\square\square\boldsymbol{p}_1\Leftrightarrow\boldsymbol{q}_1\vee\boldsymbol{r}_1$. .... [.1] ASM **57.2**

$\square\square\boldsymbol{q}_1\vdash\boldsymbol{p}_1$. .... LEM **57.3**

$\square\square$¶ .... DEM

$\square\square\square\boldsymbol{q}_1$. .... [.3] ASM **57.4**

$\square\square\square\forall \boldsymbol{p}\forall \boldsymbol{q}\boldsymbol{p}\vdash\boldsymbol{p}\vee\boldsymbol{q}$. .... THM **57.5**

$\square\square\square\boldsymbol{q}_1\vee\boldsymbol{r}_1$. .... [.5] [.4] $_1$MINT$_2$ **57.6**

57.7 $\Box\Box\Box\forall \boldsymbol{p}\forall \boldsymbol{q}\boldsymbol{p}\Leftrightarrow \boldsymbol{q}\vdash \boldsymbol{q}\vdash \boldsymbol{p}$. .... THM
57.8 $\Box\Box\Box \boldsymbol{p}_1$. .... **[.7] [.2] [.6]** $_2$MINT$_2$
$\Box\Box\blacksquare$. .... QED
$\Box\blacksquare$. .... QED
57.9 $\Box\forall \boldsymbol{p}\forall \boldsymbol{q}\forall \boldsymbol{r}\boldsymbol{p}\Leftrightarrow \boldsymbol{q}\vee \boldsymbol{r}\vdash \boldsymbol{q}\vdash \boldsymbol{p}$. .... **[.1]** UGEN$_3$
$\blacksquare$. .... QED
Uses Axioms: **4, 5**

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58 $\forall \boldsymbol{p}\forall \boldsymbol{q}\forall \boldsymbol{r}\boldsymbol{p}\Leftrightarrow \boldsymbol{q}\vee \boldsymbol{r}\vdash \boldsymbol{r}\vdash \boldsymbol{p}$. .... THEOREM *of* $\Leftrightarrow$ *Disjunctive Elimination 2*
¶ .... DEM
58.1 $\Box \boldsymbol{p}_1\Leftrightarrow \boldsymbol{q}_1\vee \boldsymbol{r}_1\vdash \boldsymbol{r}_1\vdash \boldsymbol{p}_1$. .... LEM
$\Box$¶ .... DEM
58.2 $\Box\Box \boldsymbol{p}_1\Leftrightarrow \boldsymbol{q}_1\vee \boldsymbol{r}_1$. .... **[.1]** ASM
58.3 $\Box\Box \boldsymbol{r}_1\vdash \boldsymbol{p}_1$. .... LEM
$\Box\Box$¶ .... DEM
58.4 $\Box\Box\Box \boldsymbol{r}_1$. .... **[.3]** ASM
58.5 $\Box\Box\Box\forall \boldsymbol{p}\forall \boldsymbol{q}\boldsymbol{q}\vdash \boldsymbol{p}\vee \boldsymbol{q}$. .... THM
58.6 $\Box\Box\Box \boldsymbol{q}_1\vee \boldsymbol{r}_1$. .... **[.5] [.4]** $_1$MINT$_2$
58.7 $\Box\Box\Box\forall \boldsymbol{p}\forall \boldsymbol{q}\boldsymbol{p}\Leftrightarrow \boldsymbol{q}\vdash \boldsymbol{q}\vdash \boldsymbol{p}$. .... THM
58.8 $\Box\Box\Box \boldsymbol{p}_1$. .... **[.7] [.2] [.6]** $_2$MINT$_2$
$\Box\Box\blacksquare$. .... QED
$\Box\blacksquare$. .... QED
58.9 $\Box\forall \boldsymbol{p}\forall \boldsymbol{q}\forall \boldsymbol{r}\boldsymbol{p}\Leftrightarrow \boldsymbol{q}\vee \boldsymbol{r}\vdash \boldsymbol{r}\vdash \boldsymbol{p}$. .... **[.1]** UGEN$_3$
$\blacksquare$. .... QED
Uses Axioms: **4, 5**

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59 $\forall \boldsymbol{p}\forall \boldsymbol{q}\boldsymbol{p}=\boldsymbol{q}\vdash \boldsymbol{p}\Leftrightarrow \boldsymbol{q}$. .... THEOREM *of, “To Be Fulfilled Is To Be Satisfied.”*
¶ .... DEM
59.1 $\Box \boldsymbol{p}=\boldsymbol{q}\vdash \boldsymbol{p}\Leftrightarrow \boldsymbol{q}$. .... LEM
$\Box$¶ .... DEM
59.2 $\Box\Box \boldsymbol{p}=\boldsymbol{q}$. .... **[.1]** ASM
59.3 $\Box\Box\forall \boldsymbol{p}\boldsymbol{p}\Leftrightarrow \boldsymbol{p}$. .... THM
59.4 $\Box\Box \boldsymbol{p}\Leftrightarrow \boldsymbol{p}$. .... **[.3]** $\forall_1$INST
59.5 $\Box\Box \boldsymbol{p}\Leftrightarrow \boldsymbol{q}$. .... **[.2] [.4]** SUB
$\Box\blacksquare$. .... QED
59.6 $\Box\forall \boldsymbol{p}\forall \boldsymbol{q}\boldsymbol{p}=\boldsymbol{q}\vdash \boldsymbol{p}\Leftrightarrow \boldsymbol{q}$. .... **[.1]** UGEN$_2$
$\blacksquare$. .... QED
Uses Axioms: **5**

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60 $\neg\exists \boldsymbol{x}\boldsymbol{x}=\neg \boldsymbol{x}$. .... THEOREM *of Unfulfillability, a Russell’s Paradox 1*
¶ .... DEM
60.1 $\Box(\exists \boldsymbol{x}\boldsymbol{x}=\neg \boldsymbol{x})\vdash\forall \boldsymbol{x}\boldsymbol{x}$. .... LEM

$\Box\P$ . . . . . DEM

$\Box\Box\exists x x{=}\neg x$. . . . . [.1] ASM **60.2**

$\Box\Box x{=}\neg x$. . . . . [.2] $\exists_1$INST **60.3**

$\Box\Box x\vdash\forall x x$. . . . . LEM **60.4**

$\Box\Box\P$ . . . . . DEM

$\Box\Box\Box x$. . . . . [.4] ASM **60.5**

$\Box\Box\Box\neg x$. . . . . [.3] [.5] SUB **60.6**

$\Box\Box\Box\forall p p\vdash\neg p\vdash\forall x x$. . . . . THM **60.7**

$\Box\Box\Box\forall x x$. . . . . [.7] [.5] [.6] ${}_2$MINT$_1$ **60.8**

$\Box\Box\blacksquare$. . . . . QED

$\Box\Box\forall p(p\vdash\forall x x)\vdash\neg p$. . . . . THM **60.9**

$\Box\Box\neg x$. . . . . [.9] [.4] ${}_1$MINT$_1$ **60.10**

$\Box\Box x$. . . . . [.3] [.10] SUB **60.11**

$\Box\Box\forall p p\vdash\neg p\vdash\forall x x$. . . . . THM **60.12**

$\Box\Box\forall x x$. . . . . [.12] [.11] [.10] ${}_2$MINT$_1$ **60.13**

$\Box\blacksquare$. . . . . QED

$\Box\forall p(p\vdash\forall x x)\vdash\neg p$. . . . . THM **60.14**

$\Box\neg\exists x x{=}\neg x$. . . . . [.14] [.1] ${}_1$MINT$_1$ **60.15**

$\blacksquare$. . . . . QED

Uses Axioms: **1**

$\neg\exists p\neg p\Leftrightarrow p$. . . . . THEOREM *of Unsatisfiability, a Russell's Paradox 2* **61**

$\P$ . . . . . DEM

$\Box(\exists p\neg p\Leftrightarrow p)\vdash\forall x x$. . . . . LEM **61.1**

$\Box\P$ . . . . . DEM

$\Box\Box\exists p\neg p\Leftrightarrow p$. . . . . [.1] ASM **61.2**

$\Box\Box\neg p\Leftrightarrow p$. . . . . [.2] $\exists_1$INST **61.3**

$\Box\Box p\vdash\forall x x$. . . . . LEM **61.4**

$\Box\Box\P$ . . . . . DEM

$\Box\Box\Box p$. . . . . [.4] ASM **61.5**

$\Box\Box\Box\forall p\forall q p\Leftrightarrow q\vdash q\vdash p$. . . . . THM **61.6**

$\Box\Box\Box\neg p$. . . . . [.6] [.3] [.5] ${}_2$MINT$_2$ **61.7**

$\Box\Box\Box\forall p p\vdash\neg p\vdash\forall x x$. . . . . THM **61.8**

$\Box\Box\Box\forall x x$. . . . . [.8] [.5] [.7] ${}_2$MINT$_1$ **61.9**

$\Box\Box\blacksquare$. . . . . QED

$\Box\Box\forall p(p\vdash\forall x x)\vdash\neg p$. . . . . THM **61.10**

$\Box\Box\neg p$. . . . . [.10] [.4] ${}_1$MINT$_1$ **61.11**

$\Box\Box\forall p\forall q p\Leftrightarrow q\vdash p\vdash q$. . . . . THM **61.12**

$\Box\Box p$. . . . . [.12] [.3] [.11] ${}_2$MINT$_2$ **61.13**

$\Box\Box\forall p p\vdash\neg p\vdash\forall x x$. . . . . THM **61.14**

$\Box\Box\forall x x$. . . . . [.14] [.13] [.11] ${}_2$MINT$_1$ **61.15**

$\Box\blacksquare$. . . . . QED

$\Box\forall p(p\vdash\forall x x)\vdash\neg p$. . . . . THM **61.16**

$\Box\neg\exists p\neg p\Leftrightarrow p$. . . . . [.16] [.1] ${}_1$MINT$_1$ **61.17**

■. . . . . QED

Uses Axioms: **1, 5**

Iff there exists an $\boldsymbol{x}$ st. $\boldsymbol{x}$ and $\boldsymbol{p}\wedge\boldsymbol{x}\vdash\boldsymbol{q}$, does $\boldsymbol{p}\vdash\boldsymbol{q}$. PROOF: Suppose $\exists\boldsymbol{x}\boldsymbol{x}\wedge(\boldsymbol{p}\wedge\boldsymbol{x}\vdash\boldsymbol{q})$ and assume $\boldsymbol{p}$. We may existentially witness object $\boldsymbol{x}$ so that both $\boldsymbol{x}$ and $\boldsymbol{p}\wedge\boldsymbol{x}\vdash\boldsymbol{q}$ are true. Conversely suppose $\boldsymbol{p}\vdash\boldsymbol{q}$. Then there does exist an object $\boldsymbol{x}$ st. $\boldsymbol{x}$ and $\boldsymbol{p}\wedge\boldsymbol{x}\vdash\boldsymbol{q}$– namely $\boldsymbol{p}\vdash\boldsymbol{q}$. QED.

This is might not have been the original intent of Goldblatt and Kane, on their *enthymematic conditional*, but in our quantification theory (of objects), it is a fact.

62 $\forall\boldsymbol{p}\forall\boldsymbol{q}(\exists\boldsymbol{x}\boldsymbol{x}\wedge(\boldsymbol{p}\wedge\boldsymbol{x}\vdash\boldsymbol{q}))\Leftrightarrow\boldsymbol{p}\vdash\boldsymbol{q}$. . THEOREM *of Goldblatt's and Kane's Enthymematic Conditional*

¶. . . . . DEM

62.1 $\Box(\exists\boldsymbol{x}\boldsymbol{x}\wedge(\boldsymbol{p}\wedge\boldsymbol{x}\vdash\boldsymbol{q}))\vdash\boldsymbol{p}\vdash\boldsymbol{q}$. . . . . LEM

□¶. . . . . DEM

62.2 $\Box\Box\exists\boldsymbol{x}\boldsymbol{x}\wedge(\boldsymbol{p}\wedge\boldsymbol{x}\vdash\boldsymbol{q})$. . . . . [.1] ASM

62.3 $\Box\Box\boldsymbol{x}\wedge(\boldsymbol{p}\wedge\boldsymbol{x}\vdash\boldsymbol{q})$. . . . . [.2] $\exists_1$INST

62.4 $\Box\Box\boldsymbol{p}\vdash\boldsymbol{q}$. . . . . LEM

□□¶. . . . . DEM

62.5 $\Box\Box\Box\boldsymbol{p}$. . . . . [.4] ASM

62.6 $\Box\Box\Box\forall\boldsymbol{p}\forall\boldsymbol{q}\boldsymbol{p}\wedge\boldsymbol{q}\vdash\boldsymbol{p}$. . . . . THM

62.7 $\Box\Box\Box\boldsymbol{x}$. . . . . [.6] [.3] $_1$MINT$_2$

62.8 $\Box\Box\Box\forall\boldsymbol{p}\forall\boldsymbol{q}\boldsymbol{p}\wedge\boldsymbol{q}\vdash\boldsymbol{q}$. . . . . THM

62.9 $\Box\Box\Box\boldsymbol{p}\wedge\boldsymbol{x}\vdash\boldsymbol{q}$. . . . . [.8] [.3] $_1$MINT$_2$

62.10 $\Box\Box\Box\forall\boldsymbol{p}\forall\boldsymbol{q}\boldsymbol{p}\vdash\boldsymbol{q}\vdash\boldsymbol{p}\wedge\boldsymbol{q}$. . . . . THM

62.11 $\Box\Box\Box\boldsymbol{p}\wedge\boldsymbol{x}$. . . . . [.10] [.5] [.7] $_2$MINT$_2$

62.12 $\Box\Box\Box\boldsymbol{q}$. . . . . [.9] [.11] MP

□□■. . . . . QED

□■. . . . . QED

62.13 $\Box(\boldsymbol{p}\vdash\boldsymbol{q})\vdash\exists\boldsymbol{x}\boldsymbol{x}\wedge(\boldsymbol{p}\wedge\boldsymbol{x}\vdash\boldsymbol{q})$. . . . . LEM

□¶. . . . . DEM

62.14 $\Box\Box\boldsymbol{p}\vdash\boldsymbol{q}$. . . . . [.13] ASM

62.15 $\Box\Box\forall\boldsymbol{p}\forall\boldsymbol{q}\boldsymbol{p}\wedge(\boldsymbol{p}\vdash\boldsymbol{q})\vdash\boldsymbol{q}$. . . . . THM

62.16 $\Box\Box\boldsymbol{p}\wedge(\boldsymbol{p}\vdash\boldsymbol{q})\vdash\boldsymbol{q}$. . . . . [.15] $_0$MINT$_2$

62.17 $\Box\Box\forall\boldsymbol{p}\forall\boldsymbol{q}\boldsymbol{p}\vdash\boldsymbol{q}\vdash\boldsymbol{p}\wedge\boldsymbol{q}$. . . . . THM

62.18 $\Box\Box\forall\boldsymbol{p}_1\forall\boldsymbol{q}_1\boldsymbol{p}_1\vdash\boldsymbol{q}_1\vdash\boldsymbol{p}_1\wedge\boldsymbol{q}_1$. . . . . [.17] QNT

62.19 $\Box\Box\boldsymbol{p}\vdash\boldsymbol{q}\wedge(\boldsymbol{p}\wedge(\boldsymbol{p}\vdash\boldsymbol{q})\vdash\boldsymbol{q})$. . . . . [.18] [.14] [.16] $_2$MINT$_2$

62.20 $\Box\Box\exists\boldsymbol{x}\boldsymbol{x}\wedge(\boldsymbol{p}\wedge\boldsymbol{x}\vdash\boldsymbol{q})$. . . . . [.19] $\exists_1$GEN

□■. . . . . QED

62.21 $\Box\forall\boldsymbol{p}\forall\boldsymbol{q}(\boldsymbol{p}\vdash\boldsymbol{q})\vdash(\boldsymbol{q}\vdash\boldsymbol{p})\vdash\boldsymbol{p}\Leftrightarrow\boldsymbol{q}$. . . . . THM

62.22 $\Box\forall\boldsymbol{p}_1\forall\boldsymbol{q}_1(\boldsymbol{p}_1\vdash\boldsymbol{q}_1)\vdash(\boldsymbol{q}_1\vdash\boldsymbol{p}_1)\vdash\boldsymbol{p}_1\Leftrightarrow\boldsymbol{q}_1$. . . . . [.21] QNT

62.23 $\Box(\exists\boldsymbol{x}\boldsymbol{x}\wedge(\boldsymbol{p}\wedge\boldsymbol{x}\vdash\boldsymbol{q}))\Leftrightarrow\boldsymbol{p}\vdash\boldsymbol{q}$. . . . . [.22] [.1] [.13] $_2$MINT$_2$

62.24 $\Box\forall\boldsymbol{p}\forall\boldsymbol{q}(\exists\boldsymbol{x}\boldsymbol{x}\wedge(\boldsymbol{p}\wedge\boldsymbol{x}\vdash\boldsymbol{q}))\Leftrightarrow\boldsymbol{p}\vdash\boldsymbol{q}$. . . . . [.23] UGEN$_3$

■. . . . . QED

Uses Axioms: **3, 5**

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§1.3 *Logic: Exercises in Second-order*

If $\forall \boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x} \vdash \exists \boldsymbol{y}\boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}$ then $\forall \boldsymbol{x}\exists \boldsymbol{y}\boldsymbol{\psi}\boldsymbol{x} \vdash \boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}$. PROOF: Suppose $\forall \boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x} \vdash \exists \boldsymbol{y}\boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}$. Take any $\boldsymbol{x}$. We claim $\boldsymbol{\psi}\boldsymbol{x} \vdash \boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}$. For from $\boldsymbol{\psi}\boldsymbol{x}$, we know their exists some $\boldsymbol{y}$ st. $\boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}$. By virtue of defining, it may be so named. Importantly the claim is true for *some* $\boldsymbol{y}$– not *any*– and so we can not generalize universally by $\boldsymbol{x}$ before we generalize existentially by $\boldsymbol{y}$. QED.
This theorem was unprovable in older systems of TOB. It gave rise the notion of *protected variables* and the rule of inference of *naming*. For this reason we present it first.

63 $\forall \boldsymbol{\psi}\forall \boldsymbol{\phi}(\forall \boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x} \vdash \exists \boldsymbol{y}\boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}) \vdash \forall \boldsymbol{x}\exists \boldsymbol{y}\boldsymbol{\psi}\boldsymbol{x} \vdash \boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}$. .... THEOREM *of Unprotected Quantification*
¶ .... DEM
63.1 $\Box(\forall \boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x} \vdash \exists \boldsymbol{y}\boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}) \vdash \forall \boldsymbol{x}\exists \boldsymbol{y}\boldsymbol{\psi}\boldsymbol{x} \vdash \boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}$. .... LEM
$\Box$¶ .... DEM
63.2 $\Box\Box\forall \boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x} \vdash \exists \boldsymbol{y}\boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}$. .... [.1] ASM
63.3 $\Box\Box\boldsymbol{\psi}\boldsymbol{x} \vdash \boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}$. .... LEM
$\Box\Box$¶ .... DEM
63.4 $\Box\Box\Box\boldsymbol{\psi}\boldsymbol{x}$. .... [.3] ASM
63.5 $\Box\Box\Box\exists \boldsymbol{y}\boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}$. .... [.2] [.4] $_1$MINT$_1$
63.6 $\Box\Box\Box\boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}$. .... [.5] $_1$NAME
$\Box\Box\blacksquare$. .... QED
63.7 $\Box\Box\exists \boldsymbol{y}\boldsymbol{\psi}\boldsymbol{x} \vdash \boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}$. .... [.3] $\exists_1$GEN
63.8 $\Box\Box\forall \boldsymbol{x}\exists \boldsymbol{y}\boldsymbol{\psi}\boldsymbol{x} \vdash \boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}$. .... [.7] $\forall_1$GEN
$\Box\blacksquare$. .... QED
63.9 $\Box\forall \boldsymbol{\phi}(\forall \boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x} \vdash \exists \boldsymbol{y}\boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}) \vdash \forall \boldsymbol{x}\exists \boldsymbol{y}\boldsymbol{\psi}\boldsymbol{x} \vdash \boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}$. .... [.1] $\forall_2$GEN
63.10 $\Box\forall \boldsymbol{\psi}\forall \boldsymbol{\phi}(\forall \boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x} \vdash \exists \boldsymbol{y}\boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}) \vdash \forall \boldsymbol{x}\exists \boldsymbol{y}\boldsymbol{\psi}\boldsymbol{x} \vdash \boldsymbol{\phi}\boldsymbol{x}\boldsymbol{y}$. .... [.9] $\forall_2$GEN
$\blacksquare$. .... QED
Uses Axioms: None

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64 $\forall \boldsymbol{\psi}(\forall \boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x}) \vdash \forall \boldsymbol{y}\boldsymbol{\psi}\boldsymbol{y}$. .... THEOREM *of Symmetry 1*
¶ .... DEM
64.1 $\Box(\forall \boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x}) \vdash \forall \boldsymbol{y}\boldsymbol{\psi}\boldsymbol{y}$. .... LEM
$\Box$¶ .... DEM
64.2 $\Box\Box\forall \boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x}$. .... [.1] ASM
64.3 $\Box\Box\boldsymbol{\psi}\boldsymbol{x}$. .... [.2] $\forall_1$INST
64.4 $\Box\Box\forall \boldsymbol{y}\boldsymbol{\psi}\boldsymbol{y}$. .... [.3] $\forall_1$GEN
$\Box\blacksquare$. .... QED
64.5 $\Box\forall \boldsymbol{\psi}(\forall \boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x}) \vdash \forall \boldsymbol{y}\boldsymbol{\psi}\boldsymbol{y}$. .... [1.] $\forall_2$GEN
$\blacksquare$. .... QED
Uses Axioms: None

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65 $\forall \boldsymbol{\psi}(\exists \boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x}) \vdash \exists \boldsymbol{y}\boldsymbol{\psi}\boldsymbol{y}$. .... THEOREM *of Symmetry 2*

¶ . . . . . . DEM

$\square(\exists x\psi x)\vdash\exists y\psi y$. . . . . . . LEM **65.1**

$\square$¶ . . . . . . DEM

$\square\square\exists x\psi x$. . . . . . . **[.1]** ASM **65.2**

$\square\square\psi x$. . . . . . . **[.2]** $\exists_1$INST **65.3**

$\square\square\exists y\psi y$. . . . . . . **[.3]** $\exists_1$GEN **65.4**

$\square\blacksquare$. . . . . . . QED

$\square\forall\psi(\exists x\psi x)\vdash\exists y\psi y$. . . . . . . **[1.]** $\forall_2$GEN **65.5**

$\blacksquare$. . . . . . . QED

Uses Axioms: None

$\forall\psi(\forall x\forall y\psi xy)\vdash\forall y\forall x\psi xy$. . . . . . . THEOREM *of Quantifier Exchange 1* **66**

¶ . . . . . . DEM

$\square(\forall x\forall y\psi xy)\vdash\forall y\forall x\psi xy$. . . . . . . LEM **66.1**

$\square$¶ . . . . . . DEM

$\square\square\forall x\forall y\psi xy$. . . . . . . **[.1]** ASM **66.2**

$\square\square\psi xy$. . . . . . . **[.2]** $_0$MINT$_2$ **66.3**

$\square\square\forall y\forall x\psi xy$. . . . . . . **[.3]** UGEN$_2$ **66.4**

$\square\blacksquare$. . . . . . . QED

$\square\forall\psi(\forall x\forall y\psi xy)\vdash\forall y\forall x\psi xy$. . . . . . . **[1.]** $\forall_2$GEN **66.5**

$\blacksquare$. . . . . . . QED

Uses Axioms: None

$\forall\psi(\exists x\exists y\psi xy)\vdash\exists y\exists x\psi xy$. . . . . . . THEOREM *of Quantifier Exchange 2* **67**

¶ . . . . . . DEM

$\square(\exists x\exists y\psi xy)\vdash\exists y\exists x\psi xy$. . . . . . . LEM **67.1**

$\square$¶ . . . . . . DEM

$\square\square\exists x\exists y\psi xy$. . . . . . . **[.1]** ASM **67.2**

$\square\square\exists y\psi xy$. . . . . . . **[.2]** $\exists_1$INST **67.3**

$\square\square\psi xy$. . . . . . . **[.3]** $\exists_1$INST **67.4**

$\square\square\exists y\psi xy$. . . . . . . **[.4]** $\exists_1$GEN **67.5**

$\square\square\exists x\exists y\psi xy$. . . . . . . **[.5]** $\exists_1$GEN **67.6**

$\square\blacksquare$. . . . . . . QED

$\square\forall\psi(\exists x\exists y\psi xy)\vdash\exists y\exists x\psi xy$. . . . . . . **[1.]** $\forall_2$GEN **67.7**

$\blacksquare$. . . . . . . QED

Uses Axioms: None

$\forall\psi(\exists x\forall y\psi xy)\vdash\forall y\exists x\psi xy$. . . . . . . THEOREM *of Quantifier Exchange 3* **68**

¶ . . . . . . DEM

$\square(\exists x\forall y\psi xy)\vdash\forall y\exists x\psi xy$. . . . . . . LEM **68.1**

$\square$¶ . . . . . . DEM

$\square\square\exists x\forall y\psi xy$. . . . . . . **[.1]** ASM **68.2**

68.3 $\Box\Box\forall y\psi xy$. .... [.2] $\exists_1$INST

68.4 $\Box\Box\psi xy$. .... [.3] $\forall_1$INST

68.5 $\Box\Box\exists x\psi xy$. .... [.4] $\exists_1$GEN

68.6 $\Box\Box\forall y\exists y\psi xy$. .... [.5] $\forall_1$GEN

$\Box\blacksquare$. .... QED

68.7 $\Box\forall\psi(\exists x\forall y\psi xy)\vdash\forall y\exists x\psi xy$. .... [1.] $\forall_2$GEN

$\blacksquare$. .... QED

Uses Axioms: None

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69 $\forall\psi(\neg\exists x\psi x)\vdash\forall x\neg\psi x$. .... THEOREM *of the Negative Contour 1*

¶ .... DEM

69.1 $\Box(\neg\exists x\psi x)\vdash\forall x\neg\psi x$. .... LEM

$\Box$¶ .... DEM

69.2 $\Box\Box\psi x\vdash\forall xx$. .... LEM

$\Box\Box$¶ .... DEM

69.3 $\Box\Box\Box\neg\exists x\psi x$. .... [.1] ASM

69.4 $\Box\Box\Box\psi x$. .... [.2] ASM

69.5 $\Box\Box\Box\exists x\psi x$. .... [.4] $\exists_1$GEN

69.6 $\Box\Box\Box\forall pp\vdash\neg p\vdash\forall xx$. .... THM

69.7 $\Box\Box\Box\forall xx$. .... [.6] [.5] [.3] ${}_2$MINT$_1$

$\Box\Box\blacksquare$. .... QED

69.8 $\Box\Box\forall p(p\vdash\forall xx)\vdash\neg p$. .... THM

69.9 $\Box\Box\neg\psi x$. .... [.8] [.2] ${}_1$MINT$_1$

69.10 $\Box\Box\forall x\neg\psi x$. .... [.9] $\forall_1$GEN

$\Box\blacksquare$. .... QED

69.11 $\Box\forall\psi(\neg\exists x\psi x)\vdash\forall x\neg\psi x$. .... [1.] $\forall_2$GEN

$\blacksquare$. .... QED

Uses Axioms: **1**

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70 $\forall\psi(\forall x\neg\psi x)\vdash\neg\exists x\psi x$. .... THEOREM *of the Negative Contour 2*

¶ .... DEM

70.1 $\Box(\forall x\neg\psi x)\vdash\neg\exists x\psi x$. .... LEM

$\Box$¶ .... DEM

70.2 $\Box\Box\forall x\neg\psi x$. .... [.1] ASM

70.3 $\Box\Box\exists x\psi x\vdash\forall xx$. .... LEM

$\Box\Box$¶ .... DEM

70.4 $\Box\Box\Box\exists x\psi x$. .... [.3] ASM

70.5 $\Box\Box\Box\psi x$. .... [.4] $\exists_1$INST

70.6 $\Box\Box\Box\neg\psi x$. .... [.2] $\forall_1$INST

70.7 $\Box\Box\Box\forall pp\vdash\neg p\vdash\forall xx$. .... THM

70.8 $\Box\Box\Box\forall xx$. .... [.7] [.5] [.6] ${}_2$MINT$_1$

$\Box\Box\blacksquare$. .... QED

70.9 $\Box\Box\forall p(p\vdash\forall xx)\vdash\neg p$. .... THM

□□¬∃$\boldsymbol{x\psi x}$. . . . . [.9] [.3] $_1$MINT$_1$ **70.10**
□■. . . . . QED
□∀$\boldsymbol{\psi}$(∀$\boldsymbol{x}$¬$\boldsymbol{\psi x}$)⊢¬∃$\boldsymbol{x\psi x}$. . . . . [.1] ∀$_2$GEN **70.11**
■. . . . . QED
Uses Axioms: **1**

∀$\boldsymbol{\psi}$(∃$\boldsymbol{x\psi x}$)⊢¬∀$\boldsymbol{x}$¬$\boldsymbol{\psi x}$. . . . . THEOREM *of the Negative Contour 3* **71**
¶. . . . . DEM
□(∃$\boldsymbol{x\psi x}$)⊢¬∀$\boldsymbol{x}$¬$\boldsymbol{\psi x}$. . . . . LEM **71.1**
□¶. . . . . DEM
□□(∀$\boldsymbol{x}$¬$\boldsymbol{\psi x}$)⊢∀$\boldsymbol{xx}$. . . . . LEM **71.2**
□□¶. . . . . DEM
□□□∃$\boldsymbol{x\psi x}$. . . . . [.1] ASM **71.3**
□□□∀$\boldsymbol{x}$¬$\boldsymbol{\psi x}$. . . . . [.2] ASM **71.4**
□□□$\boldsymbol{\psi x}$. . . . . [.3] ∃$_1$INST **71.5**
□□□¬$\boldsymbol{\psi x}$. . . . . [.4] ∀$_1$INST **71.6**
□□□∀$\boldsymbol{pp}$⊢¬$\boldsymbol{p}$⊢∀$\boldsymbol{xx}$. . . . . THM **71.7**
□□□∀$\boldsymbol{xx}$. . . . . [.7] [.5] [.6] $_2$MINT$_1$ **71.8**
□□■. . . . . QED
□□∀$\boldsymbol{p}$($\boldsymbol{p}$⊢∀$\boldsymbol{xx}$)⊢¬$\boldsymbol{p}$. . . . . THM **71.9**
□□¬∀$\boldsymbol{x}$¬$\boldsymbol{\psi x}$. . . . . [.9] [.2] $_1$MINT$_1$ **71.10**
□■. . . . . QED
□∀$\boldsymbol{\psi}$(∃$\boldsymbol{x\psi x}$)⊢¬∀$\boldsymbol{x}$¬$\boldsymbol{\psi x}$. . . . . [1.] ∀$_2$GEN **71.11**
■. . . . . QED
Uses Axioms: **1**

∀$\boldsymbol{\psi}$∀$\boldsymbol{x}$∀$\boldsymbol{yx=y}$⊢$\boldsymbol{\psi x=\psi y}$. . . . . THEOREM *of Application, Equality Identity 4* **72**
¶. . . . . DEM
□$\boldsymbol{x=y}$⊢$\boldsymbol{\psi x=\psi y}$. . . . . LEM **72.1**
□¶. . . . . DEM
□□$\boldsymbol{x=y}$. . . . . [.1] ASM **72.2**
□□$\boldsymbol{\psi x=\psi x}$. . . . . EQ **72.3**
□□$\boldsymbol{\psi x=\psi y}$. . . . . [.2] [.3] SUB **72.4**
□■. . . . . QED
□∀$\boldsymbol{x}$∀$\boldsymbol{yx=y}$⊢$\boldsymbol{\psi x=\psi y}$. . . . . [.1] UGEN$_2$ **72.5**
□∀$\boldsymbol{\psi}$∀$\boldsymbol{x}$∀$\boldsymbol{yx=y}$⊢$\boldsymbol{\psi x=\psi y}$. . . . . [.5] ∀$_2$GEN **72.6**
■. . . . . QED
Uses Axioms: None

∀$\boldsymbol{\psi}$(∀$\boldsymbol{x}$∀$\boldsymbol{y\psi xy}$)⊢∀$\boldsymbol{\phi}$∀$\boldsymbol{x\psi x}$($\boldsymbol{\phi x}$). . . . . THEOREM *of Second Order Quantifier Replacement 1* **73**
¶. . . . . DEM
□(∀$\boldsymbol{x}$∀$\boldsymbol{y\psi xy}$)⊢∀$\boldsymbol{\phi}$∀$\boldsymbol{x\psi x}$($\boldsymbol{\phi x}$). . . . . LEM **73.1**

$\Box\P$ . . . DEM

73.2 $\Box\Box\forall x\forall y\psi xy$. . . . [.1] ASM

73.3 $\Box\Box\forall y\psi xy$. . . . [.2] $\forall_1$INST

73.4 $\Box\Box\psi x(\phi x)$. . . . [.3] $\forall_1$INST

73.5 $\Box\Box\forall x\psi x(\phi x)$. . . . [.4] $\forall_1$GEN

73.6 $\Box\Box\forall\phi\forall x\psi x(\phi x)$. . . . [.5] $\forall_2$GEN

$\Box\blacksquare$. . . . QED

73.7 $\Box\forall\psi(\forall x\forall y\psi xy)\vdash\forall\phi\forall x\psi x(\phi x)$. . . . [.1] $\forall_2$GEN

$\blacksquare$. . . . QED

Uses Axioms: None

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74 $\forall\psi(\forall\phi\forall x\psi x(\phi x))\vdash\forall x\forall y\psi xy$. . . . THEOREM *of Second Order Quantifier Replacement 2*

$\P$ . . . DEM

74.1 $\Box(\forall\phi\forall x\psi x(\phi x))\vdash\forall x\forall y\psi xy$. . . . LEM

$\Box\P$ . . . DEM

74.2 $\Box\Box\forall\phi\forall x\psi x(\phi x)$. . . . [.1] ASM

74.3 $\Box\Box\forall x\psi xy$. . . . [.2] $\forall_2$INST

74.4 $\Box\Box\psi xy$. . . . [.3] $\forall_1$INST

74.5 $\Box\Box\forall y\psi xy$. . . . [.4] $\forall_1$GEN

74.6 $\Box\Box\forall x\forall y\psi xy$. . . . [.5] $\forall_1$GEN

$\Box\blacksquare$. . . . QED

74.7 $\Box\forall\psi(\forall\phi\forall x\psi x(\phi x))\vdash\forall x\forall y\psi xy$. . . . [.1] $\forall_2$GEN

$\blacksquare$. . . . QED

Uses Axioms: None

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75 $\forall\psi\forall p(p\vee\neg p\vdash\psi p)\vdash\neg\neg\psi p$. . . . THEOREM *of Glivenko's Propositional Translation 2*

$\P$ . . . DEM

75.1 $\Box\forall\psi(\forall x\forall y\psi xy)\vdash\forall\phi\forall x\psi x(\phi x)$. . . . THM

75.2 $\Box(\forall x\forall y(x\vee\neg x\vdash y)\vdash\neg\neg y)\vdash\forall\phi\forall x(x\vee\neg x\vdash\phi x)\vdash\neg\neg\phi x$. . . . [.1] $\forall_2$INST

75.3 $\Box\forall p\forall q(p\vee\neg p\vdash q)\vdash\neg\neg q$. . . . THM

75.4 $\Box(p\vee\neg p\vdash q)\vdash\neg\neg q$. . . . [.3] $_0$MINT$_2$

75.5 $\Box\forall x\forall y(x\vee\neg x\vdash y)\vdash\neg\neg y$. . . . [.4] UGEN$_2$

75.6 $\Box\forall\phi\forall x(x\vee\neg x\vdash\phi x)\vdash\neg\neg\phi x$. . . . [.2] [.5] MP

75.7 $\Box\forall x(x\vee\neg x\vdash\phi x)\vdash\neg\neg\phi x$. . . . [.6] $\forall_2$INST

75.8 $\Box(x\vee\neg x\vdash\phi x)\vdash\neg\neg\phi x$. . . . [.7] $\forall_1$INST

75.9 $\Box\forall p(p\vee\neg p\vdash\phi p)\vdash\neg\neg\phi p$. . . . [.8] $\forall_1$GEN

75.10 $\Box\forall\psi\forall p(p\vee\neg p\vdash\psi p)\vdash\neg\neg\psi p$. . . . [.9] $\forall_2$GEN

$\blacksquare$. . . . QED

Uses Axioms: **1, 4**

If p is provable in classical logic, then $(\forall\psi(\neg\forall x\psi x)\vdash\exists x\neg\psi x)\vdash\neg\neg p$ is provable in intuitionistic logic. PROOF: We can show $\neg\neg(\forall p\vee\neg p)$. For $\forall p\neg\neg(p\vee\neg p)$ entails $\neg\exists p\neg(p\vee\neg p)$; yet from

$\forall\psi(\neg\forall x\psi x)\vdash\exists x\neg\psi x$ and $\neg(\forall p\vee\neg p)$, it follows $\exists p\neg(p\vee\neg p)$, thus giving a contradiction. Applying *Modus Tollens* twice on $(\forall p\vee\neg p)\vdash p$, gives $\neg\neg p$. QED.

$\forall p((\forall pp\vee\neg p)\vdash p)\vdash(\forall\psi(\neg\forall x\psi x)\vdash\exists x\neg\psi x)\vdash\neg\neg p.$ .... THEOREM *of Predicate Translation* **76**

¶ .... DEM

$\square\forall pp\vdash\neg p\vdash\forall xx.$ .... THM **76.1**

$\square\forall p(p\vdash\forall xx)\vdash\neg p.$ .... THM **76.2**

$\square((\forall pp\vee\neg p)\vdash p)\vdash(\forall\psi(\neg\forall x\psi x)\vdash\exists x\neg\psi x)\vdash\neg\neg p$ .... LEM **76.3**

$\square$¶ .... DEM

$\square\square(\forall pp\vee\neg p)\vdash p.$ .... **[.3]** ASM **76.4**

$\square\square(\forall\psi(\neg\forall x\psi x)\vdash\exists x\neg\psi x)\vdash\neg\neg p$ .... LEM **76.5**

$\square\square$¶ .... DEM

$\square\square\square\forall\psi(\neg\forall x\psi x)\vdash\exists x\neg\psi x.$ .... **[.5]** ASM **76.6**

$\square\square\square(\neg\forall pp\vee\neg p)\vdash\forall xx$ .... LEM **76.7**

$\square\square\square$¶ .... DEM

$\square\square\square\square\neg\forall pp\vee\neg p.$ .... **[.7]** ASM **76.8**

$\square\square\square\square(\neg\forall xx\vee\neg x)\vdash\exists x\neg(x\vee\neg x).$ .... **[.6]** $\forall_2$INST **76.9**

$\square\square\square\square(\forall xx\vee\neg x)\vdash\forall xx$ .... LEM **76.10**

$\square\square\square\square$¶ .... DEM

$\square\square\square\square\square\forall xx\vee\neg x.$ .... **[.10]** ASM **76.11**

$\square\square\square\square\square x\vee\neg x.$ .... **[.11]** $\forall_1$INST **76.12**

$\square\square\square\square\square\forall pp\vee\neg p.$ .... **[.12]** $\forall_1$GEN **76.13**

$\square\square\square\square\square\forall xx.$ .... **[.1] [.13] [.8]** $_2$MINT$_1$ **76.14**

$\square\square\square\square\blacksquare.$ .... QED

$\square\square\square\square\exists x\neg(x\vee\neg x).$ .... **[.2] [.10] [.9]** $_2$MINT$_1$ **76.15**

$\square\square\square\square\forall p\neg\neg(p\vee\neg p).$ .... THM **76.16**

$\square\square\square\square\neg\neg(x\vee\neg x).$ .... **[.16]** $\forall_1$INST **76.17**

$\square\square\square\square\forall x\neg\neg(x\vee\neg x).$ .... **[.17]** $\forall_1$GEN **76.18**

$\square\square\square\square\forall\psi(\forall x\neg\psi x)\vdash\neg\exists x\psi x.$ .... THM **76.19**

$\square\square\square\square(\forall x\neg\neg(x\vee\neg x))\vdash\neg\exists x\neg(x\vee\neg x).$ .... **[.19]** $\forall_2$INST **76.20**

$\square\square\square\square\neg\exists x\neg(x\vee\neg x).$ .... **[.20] [.18]** MP **76.21**

$\square\square\square\square\forall xx.$ .... **[.1] [.15] [.21]** $_2$MINT$_1$ **76.22**

$\square\square\square\blacksquare.$ .... QED

$\square\square\square\neg\neg\forall pp\vee\neg p.$ .... **[.2] [.7]** $_1$MINT$_1$ **76.23**

$\square\square\square\forall p\forall q(p\vdash q)\vdash\neg q\vdash\neg p.$ .... THM **76.24**

$\square\square\square\neg p\vdash\neg(\forall pp\vee\neg p).$ .... **[.24] [.4]** $_1$MINT$_2$ **76.25**

$\square\square\square\neg\neg(\forall pp\vee\neg p)\vdash\neg\neg p.$ .... **[.24] [.25]** $_1$MINT$_2$ **76.26**

$\square\square\square\neg\neg p.$ .... **[.26] [.23]** MP **76.27**

$\square\square\blacksquare.$ .... QED

$\square\blacksquare.$ .... QED

$\square\forall p((\forall pp\vee\neg p)\vdash p)\vdash(\forall\psi(\neg\forall x\psi x)\vdash\exists x\neg\psi x)\vdash\neg\neg p.$ .... **[.3]** $\forall_1$GEN **76.28**

$\blacksquare.$ .... QED

Uses Axioms: **1, 4**

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77 $\forall\psi(\exists!x\psi x)\Leftrightarrow(\exists x\psi x)\wedge\forall x_1\forall x_2\psi x_1\vdash\psi x_2\vdash x_1=x_2$. . . . . THEOREM *of* ∃! *Equivalence 1*
¶ . . . . . DEM
77.1 $\Box(\exists!x\psi x)\vdash(\exists x\psi x)\wedge\forall x_1\forall x_2\psi x_1\vdash\psi x_2\vdash x_1=x_2$. . . . . LEM
□¶ . . . . . DEM
77.2 $\Box\Box\exists!x\psi x$. . . . . [.1] ASM
77.3 $\Box\Box\exists x\psi x$. . . . . [.2] ∃!EXISTS
77.4 $\Box\Box\forall x_1\forall x_2\psi x_1\vdash\psi x_2\vdash x_1=x_2$. . . . . [.2] ∃!UNIQUE
77.5 $\Box\Box\forall p\forall q p\vdash q\vdash p\wedge q$. . . . . THM
77.6 $\Box\Box(\exists x\psi x)\wedge\forall x_1\forall x_2\psi x_1\vdash\psi x_2\vdash x_1=x_2$. . . . . [.5] [.3] [.4] $_2$MINT$_2$
□■. . . . . QED
77.7 $\Box(\exists x\psi x)\wedge(\forall x_1\forall x_2\psi x_1\vdash\psi x_2\vdash x_1=x_2)\vdash\exists!x\psi x$. . . . . LEM
□¶ . . . . . DEM
77.8 $\Box\Box(\exists x\psi x)\wedge\forall x_1\forall x_2\psi x_1\vdash\psi x_2\vdash x_1=x_2$. . . . . [.7] ASM
77.9 $\Box\Box\forall p\forall q p\wedge q\vdash p$. . . . . THM
77.10 $\Box\Box\exists x\psi x$. . . . . [.9] [.8] $_1$MINT$_2$
77.11 $\Box\Box\forall p\forall q p\wedge q\vdash q$. . . . . THM
77.12 $\Box\Box\forall x_1\forall x_2\psi x_1\vdash\psi x_2\vdash x_1=x_2$. . . . . [.11] [.8] $_1$MINT$_2$
77.13 $\Box\Box\exists!x\psi x$. . . . . [.10] [.12] ∃!INTROS
□■. . . . . QED
77.14 $\Box\forall p\forall q(p\vdash q)\vdash(q\vdash p)\vdash p\Leftrightarrow q$. . . . . THM
77.15 $\Box(\exists!x\psi x)\Leftrightarrow(\exists x\psi x)\wedge\forall x_1\forall x_2\psi x_1\vdash\psi x_2\vdash x_1=x_2$. . . . . [.14] [.1] [.7] $_2$MINT$_2$
77.16 $\Box\forall\psi(\exists!x\psi x)\Leftrightarrow(\exists x\psi x)\wedge\forall x_1\forall x_2\psi x_1\vdash\psi x_2\vdash x_1=x_2$. . . . . [.1] $\forall_2$GEN
■. . . . . QED
Uses Axioms: **3, 5**

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78 $\forall\psi(\exists!x\psi x)\Leftrightarrow\exists x\forall x_1\psi x_1\Leftrightarrow x_1=x$. . . . . THEOREM *of* ∃! *Equivalence 2*
¶ . . . . . DEM
78.1 $\Box(\exists!x\psi x)\vdash\exists x\forall x_1\psi x_1\Leftrightarrow x_1=x$. . . . . LEM
□¶ . . . . . DEM
78.2 $\Box\Box\exists!x\psi x$. . . . . [.1] ASM
78.3 $\Box\Box\exists x\psi x$. . . . . [.2] ∃!EXISTS
78.4 $\Box\Box\forall x_1\forall x_2\psi x_1\vdash\psi x_2\vdash x_1=x_2$. . . . . [.2] ∃!UNIQUE
78.5 $\Box\Box\psi x$. . . . . [.3] $\exists_1$INST
78.6 $\Box\Box\psi x_1\vdash x_1=x$. . . . . LEM
□□¶ . . . . . DEM
78.7 $\Box\Box\Box\psi x_1$. . . . . [.6] ASM
78.8 $\Box\Box\Box x_1=x$. . . . . [.4] [.7] [.5] $_2$MINT$_2$
□□■. . . . . QED
78.9 $\Box\Box x_1=x\vdash\psi x_1$. . . . . LEM
□□¶ . . . . . DEM
78.10 $\Box\Box\Box x_1=x$. . . . . [.9] ASM
78.11 $\Box\Box\Box\psi x_1$. . . . . [.10] [.7] SUB

$\square\square\blacksquare$. . . . . . QED

$\square\square\forall p\forall q(p\vdash q)\vdash(q\vdash p)\vdash p\Leftrightarrow q$. . . . . . THM **78.12**

$\square\square\psi x_1\Leftrightarrow x_1=x$. . . . . . **[.12] [.6] [.9]** $_2$MINT$_2$ **78.13**

$\square\square\forall x_1\psi x_1\Leftrightarrow x_1=x$. . . . . . **[.13]** $\forall_1$GEN **78.14**

$\square\square\exists x\forall x_1\psi x_1\Leftrightarrow x_1=x$. . . . . . **[.14]** $\exists_1$GEN **78.15**

$\square\blacksquare$. . . . . . QED

$\square(\exists x\forall x_1\psi x_1\Leftrightarrow x_1=x)\vdash\exists!x\psi x$. . . . . . LEM **78.16**

$\square$¶ . . . . . . DEM

$\square\square\exists x\forall x_1\psi x_1\Leftrightarrow x_1=x$. . . . . . **[.16]** ASM **78.17**

$\square\square\forall x_1\psi x_1\Leftrightarrow x_1=x$. . . . . . **[.17]** $\exists_1$INST **78.18**

$\square\square\psi x\Leftrightarrow x=x$. . . . . . **[.18]** $\forall_1$INST **78.19**

$\square\square x=x$. . . . . . EQ **78.20**

$\square\square\forall p\forall q p\Leftrightarrow q\vdash q\vdash p$. . . . . . THM **78.21**

$\square\square\psi x$. . . . . . **[.21] [.19] [.20]** $_2$MINT$_2$ **78.22**

$\square\square\exists x\psi x$. . . . . . **[.22]** $\exists_1$GEN **78.23**

$\square\square\psi x_1\vdash\psi x_2\vdash x_1=x_2$. . . . . . LEM **78.24**

$\square\square$¶ . . . . . . DEM

$\square\square\square\psi x_1$. . . . . . **[.24]** ASM **78.25**

$\square\square\square\psi x_2\vdash x_1=x_2$. . . . . . LEM **78.26**

$\square\square\square$¶ . . . . . . DEM

$\square\square\square\square\psi x_2$. . . . . . **[.26]** ASM **78.27**

$\square\square\square\square\psi x_1\Leftrightarrow x_1=x$. . . . . . **[.18]** $\forall_1$INST **78.28**

$\square\square\square\square\psi x_2\Leftrightarrow x_2=x$. . . . . . **[.18]** $\forall_1$INST **78.29**

$\square\square\square\square\forall p\forall q p\Leftrightarrow q\vdash p\vdash q$. . . . . . THM **78.30**

$\square\square\square\square x_1=x$. . . . . . **[.30] [.28] [.25]** $_2$MINT$_2$ **78.31**

$\square\square\square\square x_2=x$. . . . . . **[.30] [.29] [.27]** $_2$MINT$_2$ **78.32**

$\square\square\square\square x_1=x_2$. . . . . . **[.32] [.31]** SUB **78.33**

$\square\square\square\blacksquare$. . . . . . QED

$\square\square\blacksquare$. . . . . . QED

$\square\square\forall x_1\forall x_2\psi x_1\vdash\psi x_2\vdash x_1=x_2$. . . . . . **[.24]** UGEN$_2$ **78.34**

$\square\square\exists!x\psi x$. . . . . . **[.23] [.34]** $\exists!$INTROS **78.35**

$\square\blacksquare$. . . . . . QED

$\square\forall p\forall q(p\vdash q)\vdash(q\vdash p)\vdash p\Leftrightarrow q$. . . . . . THM **78.36**

$\square(\exists!x\psi x)\Leftrightarrow\exists x\forall x_1\psi x_1\Leftrightarrow x_1=x$. . . . . . **[.36] [.1] [.16]** $_2$MINT$_2$ **78.37**

$\square\forall\psi(\exists!x\psi x)\Leftrightarrow\exists x\forall x_1\psi x_1\Leftrightarrow x_1=x$. . . . . . **[.37]** $\forall_2$GEN **78.38**

$\blacksquare$. . . . . . QED

Uses Axioms: **3, 5**

$\forall x\forall y x=y\vdash\forall\psi\psi x\Leftrightarrow\psi y$. . . . . . THEOREM *of the Indiscernibility of Identicals* **79**

¶ . . . . . . DEM

$\square x=y\vdash\forall\psi\psi x\Leftrightarrow\psi y$. . . . . . LEM **79.1**

$\square$¶ . . . . . . DEM

$\square\square x=y$. . . . . . **[.1]** ASM **79.2**

$\square\square\psi x=\psi x$. . . . . . EQ **79.3**

79.4 $\Box\Box \psi x = \psi y$. ... [.2] [.3] SUB

79.5 $\Box\Box \forall p p \Leftrightarrow p$. ... THM

79.6 $\Box\Box \psi x \Leftrightarrow \psi x$. ... [.5] $\forall_1$INST

79.7 $\Box\Box \psi x \Leftrightarrow \psi y$. ... [.4] [.6] SUB

79.8 $\Box\Box \forall \psi \psi x \Leftrightarrow \psi y$. ... [.7] $\forall_2$GEN

$\Box\blacksquare$. ... QED

79.9 $\Box \forall x \forall y x = y \vdash \forall \psi \psi x \Leftrightarrow \psi y$. ... [.1] $_2$UGEN

$\blacksquare$. ... QED

Uses Axioms: **5**

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80 $\forall x \forall y (\forall \psi \psi x \Leftrightarrow \psi y) \vdash x = y$. ... THEOREM *of the Identity of Indiscernibles*

¶ ... DEM

80.1 $\Box(\forall \psi \psi x \Leftrightarrow \psi y) \vdash x = y$. ... LEM

$\Box$¶ ... DEM

80.2 $\Box\Box \forall \psi \psi x \Leftrightarrow \psi y$. ... [.1] ASM

80.3 $\Box\Box x = x \Leftrightarrow x = y$. ... [.2] $\forall_2$INST

80.4 $\Box\Box x = x$. ... EQ

80.5 $\Box\Box \forall p \forall q p \Leftrightarrow q \vdash p \vdash q$. ... THM

80.6 $\Box\Box x = y$. ... [.5] [.3] [.4] $_2$MINT$_2$

$\Box\blacksquare$. ... QED

80.7 $\Box \forall x \forall y (\forall \psi \psi x \Leftrightarrow \psi y) \vdash x = y$. ... [.1] $_2$UGEN

$\blacksquare$. ... QED

Uses Axioms: **5**

Not all formulas ψ are *realizable* as objects. PROOF: Suppose all formulas were; $\forall \psi \exists f \forall x \psi x = f x$. Particularly we may choose the formula $\neg x x$. Then $\forall x \neg x x = f x$, for some object f. Instantiating f gives a contradiction. QED.

81 $\neg \forall \psi \exists f \forall x \psi x = f x$. ... THEOREM *of Unrealizability 1, a Russell's Paradox 3*

¶ ... DEM

81.1 $\Box(\forall \psi \exists f \forall x \psi x = f x) \vdash \forall x x$. ... LEM

$\Box$¶ ... DEM

81.2 $\Box\Box \forall \psi \exists f \forall x \psi x = f x$. ... [.1] ASM

81.3 $\Box\Box \exists f \forall x \neg x x = f x$. ... [.2] $\forall_2$INST

81.4 $\Box\Box \forall x \neg x x = f x$. ... [.3] $\exists_1$INST

81.5 $\Box\Box \neg f f = f f$. ... [.4] $\forall_1$INST

81.6 $\Box\Box \exists p \neg p = p$. ... [.5] $\exists_1$GEN

81.7 $\Box\Box \neg \exists p \neg p = p$. ... THM

81.8 $\Box\Box \forall p p \vdash \neg p \vdash \forall x x$. ... THM

81.9 $\Box\Box \forall x x$. ... [.8] [.6] [.7] $_2$MINT$_1$

$\Box\blacksquare$. ... QED

81.10 $\Box \forall p (p \vdash \forall x x) \vdash \neg p$. ... THM

81.11 $\Box \neg \forall \psi \exists f \forall x \psi x = f x$. ... [.10] [.1] $_1$MINT$_1$

$\blacksquare$. . . . . QED

Uses Axioms: **1**

$\neg\forall\boldsymbol{\psi}\exists\boldsymbol{f}\forall\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x}\Leftrightarrow\boldsymbol{f}\boldsymbol{x}$. . . . . THEOREM *of Unrealizability 2, a Russell's Paradox 4* **82**

¶ . . . . DEM

$\Box(\forall\boldsymbol{\psi}\exists\boldsymbol{f}\forall\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x}\Leftrightarrow\boldsymbol{f}\boldsymbol{x})\vdash\forall\boldsymbol{x}\boldsymbol{x}$. . . . . LEM **82.1**

$\Box$¶ . . . . DEM

$\Box\Box\forall\boldsymbol{\psi}\exists\boldsymbol{f}\forall\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x}\Leftrightarrow\boldsymbol{f}\boldsymbol{x}$. . . . . [.1] ASM **82.2**

$\Box\Box\exists\boldsymbol{f}\forall\boldsymbol{x}\neg\boldsymbol{x}\boldsymbol{x}\Leftrightarrow\boldsymbol{f}\boldsymbol{x}$. . . . . [.2] $\forall_2$INST **82.3**

$\Box\Box\forall\boldsymbol{x}\neg\boldsymbol{x}\boldsymbol{x}\Leftrightarrow\boldsymbol{f}\boldsymbol{x}$. . . . . [.3] $\exists_1$INST **82.4**

$\Box\Box\neg\boldsymbol{f}\boldsymbol{f}\Leftrightarrow\boldsymbol{f}\boldsymbol{f}$. . . . . [.4] $\forall_1$INST **82.5**

$\Box\Box\exists\boldsymbol{p}\neg\boldsymbol{p}\Leftrightarrow\boldsymbol{p}$. . . . . [.5] $\exists_1$GEN **82.6**

$\Box\Box\neg\exists\boldsymbol{p}\neg\boldsymbol{p}\Leftrightarrow\boldsymbol{p}$. . . . . THM **82.7**

$\Box\Box\forall\boldsymbol{p}\boldsymbol{p}\vdash\neg\boldsymbol{p}\vdash\forall\boldsymbol{x}\boldsymbol{x}$. . . . . THM **82.8**

$\Box\Box\forall\boldsymbol{x}\boldsymbol{x}$. . . . . [.8] [.6] [.7] ${}_2$MINT$_1$ **82.9**

$\Box\blacksquare$. . . . . QED

$\Box\forall\boldsymbol{p}(\boldsymbol{p}\vdash\forall\boldsymbol{x}\boldsymbol{x})\vdash\neg\boldsymbol{p}$. . . . . THM **82.10**

$\Box\neg\forall\boldsymbol{\psi}\exists\boldsymbol{f}\forall\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x}\Leftrightarrow\boldsymbol{f}\boldsymbol{x}$. . . . . [.10] [.1] ${}_1$MINT$_1$ **82.11**

$\blacksquare$. . . . . QED

Uses Axioms: **1, 5**

$\forall\boldsymbol{\psi}(\forall\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x})=\forall\boldsymbol{y}\boldsymbol{\psi}\boldsymbol{y}$. . . . . THEOREM *of Quantifier Equality 1* **83**

¶ . . . . DEM

$\Box(\forall\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x})=\forall\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x}$. . . . . EQ **83.1**

$\Box(\forall\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x})=\forall\boldsymbol{y}\boldsymbol{\psi}\boldsymbol{y}$. . . . . [.1] QNT **83.2**

$\Box\forall\boldsymbol{\psi}(\forall\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x})=\forall\boldsymbol{y}\boldsymbol{\psi}\boldsymbol{y}$. . . . . [.2] $\forall_2$GEN **83.3**

$\blacksquare$. . . . . QED

Uses Axioms: None

$\forall\boldsymbol{\psi}(\exists\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x})=\exists\boldsymbol{y}\boldsymbol{\psi}\boldsymbol{y}$. . . . . THEOREM *of Quantifier Equality 2* **84**

¶ . . . . DEM

$\Box(\exists\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x})=\exists\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x}$. . . . . EQ **84.1**

$\Box(\exists\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x})=\exists\boldsymbol{y}\boldsymbol{\psi}\boldsymbol{y}$. . . . . [.1] QNT **84.2**

$\Box\forall\boldsymbol{\psi}(\exists\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x})=\exists\boldsymbol{y}\boldsymbol{\psi}\boldsymbol{y}$. . . . . [.2] $\forall_2$GEN **84.3**

$\blacksquare$. . . . . QED

Uses Axioms: None

$\forall\boldsymbol{\psi}(\exists!\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x})=\exists!\boldsymbol{y}\boldsymbol{\psi}\boldsymbol{y}$. . . . . THEOREM *of Quantifier Equality 3* **85**

¶ . . . . DEM

$\Box(\exists!\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x})=\exists!\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x}$. . . . . EQ **85.1**

$\Box(\exists!\boldsymbol{x}\boldsymbol{\psi}\boldsymbol{x})=\exists!\boldsymbol{y}\boldsymbol{\psi}\boldsymbol{y}$. . . . . [.1] QNT **85.2**

85.3 $\Box\forall\boldsymbol{\psi}(\exists!\boldsymbol{x\psi x})=\exists!\boldsymbol{y\psi y}$. .... [.2] $\forall_2$GEN
$\blacksquare$. .... QED
Uses Axioms: None

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86 $\forall\boldsymbol{\psi}(\forall\boldsymbol{x}\exists\boldsymbol{y\psi xy})=\forall\boldsymbol{x}\exists\boldsymbol{yx\psi y}$. .... THEOREM *of Second Order Variation 1*
¶ .... DEM
86.1 $\Box(\forall\boldsymbol{x}\exists\boldsymbol{y\psi xy})=\forall\boldsymbol{x}\exists\boldsymbol{y\psi xy}$. .... EQ
86.2 $\Box(\forall\boldsymbol{x}\exists\boldsymbol{y\psi xy})=\forall\boldsymbol{x}\exists\boldsymbol{yx\psi y}$. .... [.1] VAR
86.3 $\Box\forall\boldsymbol{\psi}(\forall\boldsymbol{x}\exists\boldsymbol{y\psi xy})=\forall\boldsymbol{x}\exists\boldsymbol{yx\psi y}$. .... [.2] $\forall_2$GEN
$\blacksquare$. .... QED
Uses Axioms: None

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87 $\forall\boldsymbol{\psi}(\forall\boldsymbol{x}\exists\boldsymbol{y\psi xy})=\forall\boldsymbol{x}\exists\boldsymbol{y\psi xy}$. .... THEOREM *of Second Order Variation 2*
¶ .... DEM
87.1 $\Box(\forall\boldsymbol{x}\exists\boldsymbol{y\psi xy})=\forall\boldsymbol{x}\exists\boldsymbol{y\psi xy}$. .... EQ
87.2 $\Box(\forall\boldsymbol{x}\exists\boldsymbol{y\psi xy})=\forall\boldsymbol{x}\exists\boldsymbol{y\psi xy}$. .... [.1] VAR
87.3 $\Box\forall\boldsymbol{\psi}(\forall\boldsymbol{x}\exists\boldsymbol{y\psi xy})=\forall\boldsymbol{x}\exists\boldsymbol{y\psi xy}$. .... [.2] $\forall_2$GEN
$\blacksquare$. .... QED
Uses Axioms: None

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88 $\forall\boldsymbol{\psi}(\forall\boldsymbol{\phi}\forall\boldsymbol{x\psi x}\vdash\exists\boldsymbol{y\phi y})\vdash\forall\boldsymbol{\theta}\forall\boldsymbol{x\psi x}\vdash\exists\boldsymbol{y\theta xy}$. .... THEOREM *of Second Order Extension*
¶ .... DEM
88.1 $\Box(\forall\boldsymbol{\phi}\forall\boldsymbol{x\psi x}\vdash\exists\boldsymbol{y\phi y})\vdash\forall\boldsymbol{\theta}\forall\boldsymbol{x\psi x}\vdash\exists\boldsymbol{y\theta xy}$. .... LEM
$\Box$¶ .... DEM
88.2 $\Box\Box\forall\boldsymbol{\phi}\forall\boldsymbol{x\psi x}\vdash\exists\boldsymbol{y\phi y}$. .... [.1] ASM
88.3 $\Box\Box\forall\boldsymbol{x\psi x}\vdash\exists\boldsymbol{y\theta ty}$. .... [.2] $\forall_2$INST
88.4 $\Box\Box\boldsymbol{\psi t}\vdash\exists\boldsymbol{y\theta ty}$. .... [.3] $\forall_1$INST
88.5 $\Box\Box\forall\boldsymbol{x\psi x}\vdash\exists\boldsymbol{y\theta xy}$. .... [.4] $\forall_1$GEN
88.6 $\Box\Box\forall\boldsymbol{\theta}\forall\boldsymbol{x\psi x}\vdash\exists\boldsymbol{y\theta xy}$. .... [.5] $\forall_2$GEN
$\Box\blacksquare$. .... QED
88.7 $\Box\forall\boldsymbol{\psi}(\forall\boldsymbol{\phi}\forall\boldsymbol{x\psi x}\vdash\exists\boldsymbol{y\phi y})\vdash\forall\boldsymbol{\theta}\forall\boldsymbol{x\psi x}\vdash\exists\boldsymbol{y\theta xy}$. .... [.1] $\forall_2$GEN
$\blacksquare$. .... QED
Uses Axioms: None

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89 $\neg\forall\boldsymbol{\psi}(\forall\boldsymbol{x}\exists!\boldsymbol{y\psi xy})\vdash\exists\boldsymbol{f}\forall\boldsymbol{x}\forall\boldsymbol{y\psi xy}\Leftrightarrow\boldsymbol{fx}=\boldsymbol{y}$. ....
.... THEOREM *on the Failure of Unbound Creation, a Russell's Paradox 5*
¶ .... DEM
89.1 $\Box(\forall\boldsymbol{\psi}(\forall\boldsymbol{x}\exists!\boldsymbol{y\psi xy})\vdash\exists\boldsymbol{f}\forall\boldsymbol{x}\forall\boldsymbol{y\psi xy}\Leftrightarrow\boldsymbol{fx}=\boldsymbol{y})\vdash\forall\boldsymbol{xx}$. .... LEM
$\Box$¶ .... DEM
89.2 $\Box\Box\forall\boldsymbol{\psi}(\forall\boldsymbol{x}\exists!\boldsymbol{y\psi xy})\vdash\exists\boldsymbol{f}\forall\boldsymbol{x}\forall\boldsymbol{y\psi xy}\Leftrightarrow\boldsymbol{fx}=\boldsymbol{y}$. .... [.1] ASM
89.3 $\Box\Box(\forall\boldsymbol{x}\exists!\boldsymbol{yy}=\neg\boldsymbol{xx})\vdash\exists\boldsymbol{f}\forall\boldsymbol{x}\forall\boldsymbol{yy}=\neg\boldsymbol{xx}\Leftrightarrow\boldsymbol{fx}=\boldsymbol{y}$. .... [.2] $\forall_2$INST

$\square\square(\neg xx)=\neg xx.$ .... EQ **89.4**

$\square\square\exists yy=\neg xx.$ .... [.4] $\exists_1$GEN **89.5**

$\square\square y_1=\neg xx\vdash y_2=\neg xx\vdash y_1=y_2.$ .... LEM **89.6**

$\square\square$¶ .... DEM

$\square\square\square y_2=\neg xx\vdash y_1=y_2.$ .... LEM **89.7**

$\square\square\square$¶ .... DEM

$\square\square\square\square y_1=\neg xx.$ .... [.6] ASM **89.8**

$\square\square\square\square y_2=\neg xx.$ .... [.7] ASM **89.9**

$\square\square\square\square y_1=y_2.$ .... [.8] [.9] ASM **89.10**

$\square\square\square\blacksquare.$ .... QED **89.11**

$\square\square\blacksquare.$ .... QED

$\square\square\forall y_1\forall y_2 y_1=\neg xx\vdash y_2=\neg xx\vdash y_1=y_2.$ .... [.6] UGEN$_2$ **89.12**

$\square\square\exists!yy=\neg xx.$ .... [.5] [.12] $\exists!$INTROS **89.13**

$\square\square\forall x\exists!yy=\neg xx.$ .... [.13] $\forall_1$GEN **89.14**

$\square\square\exists f\forall x\forall yy=\neg xx\Leftrightarrow fx=y.$ .... [.3] [.14] MP **89.15**

$\square\square\forall x\forall yy=\neg xx\Leftrightarrow fx=y.$ .... [.15] $\exists_1$INST **89.16**

$\square\square(\neg ff)=\neg ff\Leftrightarrow ff=\neg ff.$ .... [.16] ${}_0$MINT$_2$ **89.17**

$\square\square(\neg ff)=\neg ff.$ .... EQ **89.18**

$\square\square\forall p\forall qp\Leftrightarrow q\vdash p\vdash q.$ .... THM **89.19**

$\square\square ff=\neg ff.$ .... [.19] [.17] [.18] ${}_2$MINT$_2$ **89.20**

$\square\square\exists xx=\neg x.$ .... [.20] $\exists_1$GEN **89.21**

$\square\square\neg\exists xx=\neg x.$ .... THM **89.22**

$\square\square\forall pp\vdash\neg p\vdash\forall xx.$ .... THM **89.23**

$\square\square\forall xx.$ .... [.23] [.21] [.22] ${}_2$MINT$_1$ **89.24**

$\square\blacksquare.$ .... QED

$\square\forall p(p\vdash\forall xx)\vdash\neg p.$ .... THM **89.25**

$\square\neg\forall\psi(\forall x\exists!y\psi xy)\vdash\exists f\forall x\forall y\psi xy\Leftrightarrow fx=y.$ .... [.25] [.1] ${}_1$MINT$_1$ **89.26**

$\blacksquare.$ .... QED

Uses Axioms: **1, 5**

Already it is cumbersome to explicitly reference instances of logical theorems. After all, our work is type / set theoertic; we should have greater freedom to manipulate logic. So we introduce the third and final *rule tactic*: TAUT. For each pair of numbers n,m, this tactic is defined as:

$$\mathrm{TAUT}{=}\mathrm{THM}\cdot_n\mathrm{MINT}_m$$

Due to unending labor, we do not pass explicit pointer reference to prior theorems. We simply assume, when using TAUT, the correct references are passed, ie. the correct theorems are used. Many times, the intended theorem is not proven. We do this for brevity– to keep to the positive practice of applying one TAUT as opposed to applying multiple successive TAUTs. If needed, we ask the readers to fill in these blanks.

We pledge to only apply TAUT for logical theorems, ie. those which keep to the signature of *Intuitionstic Logic*.

§1.4 *Logic: Laws of Spontaneous Choice and the Excluded Middle*

Their exists a unique set which contains no elements. PROOF: *Existence.* Axiomatically, their exists such a set. *Uniqueness.* Take any two empty sets $\boldsymbol{x_1},\boldsymbol{x_2}$. For any $\boldsymbol{t}{:}\boldsymbol{x_1}$, it is vacuously true $\boldsymbol{t}{:}\boldsymbol{x_2}$. Likewise for any $\boldsymbol{t}{:}\boldsymbol{x_2}$, $\boldsymbol{t}{:}\boldsymbol{x_1}$. Thus $\forall \boldsymbol{t}\,\boldsymbol{t}{:}\boldsymbol{x_1}\Leftrightarrow \boldsymbol{t}{:}\boldsymbol{x_2}$. By *extensionality*, $\boldsymbol{x_1}=\boldsymbol{x_2}$. QED.

90 $\exists!\boldsymbol{x}\,\boldsymbol{x}{:}\mathscr{H}\wedge\forall\boldsymbol{y}\neg\boldsymbol{y}{:}\boldsymbol{x}$. .... THEOREM *of the Unique Empty Set*
¶ .... DEM
90.1 $\Box\exists\boldsymbol{x}\,\boldsymbol{x}{:}\mathscr{H}\wedge\forall\boldsymbol{y}\neg\boldsymbol{y}{:}\boldsymbol{x}$. .... AXM
90.2 $\Box(\boldsymbol{x_1}{:}\mathscr{H}\wedge\forall\boldsymbol{y}\neg\boldsymbol{y}{:}\boldsymbol{x_1})\vdash(\boldsymbol{x_2}{:}\mathscr{H}\wedge\forall\boldsymbol{y}\neg\boldsymbol{y}{:}\boldsymbol{x_2})\vdash\boldsymbol{x_1}=\boldsymbol{x_2}$. .... LEM
$\Box$¶ .... DEM
90.3 $\Box\Box\boldsymbol{x_1}{:}\mathscr{H}\wedge\forall\boldsymbol{y}\neg\boldsymbol{y}{:}\boldsymbol{x_1}$. .... **[.2]** ASM
90.4 $\Box\Box(\boldsymbol{x_2}{:}\mathscr{H}\wedge\forall\boldsymbol{y}\neg\boldsymbol{y}{:}\boldsymbol{x_2})\vdash\boldsymbol{x_1}=\boldsymbol{x_2}$. .... LEM
$\Box\Box$¶ .... DEM
90.5 $\Box\Box\Box\boldsymbol{x_2}{:}\mathscr{H}\wedge\forall\boldsymbol{y}\neg\boldsymbol{y}{:}\boldsymbol{x_2}$. .... **[.4]** ASM
90.6 $\Box\Box\Box\boldsymbol{x_1}{:}\mathscr{H}$. .... **[.3]** TAUT
90.7 $\Box\Box\Box\boldsymbol{x_2}{:}\mathscr{H}$. .... **[.5]** TAUT
90.8 $\Box\Box\Box\forall\boldsymbol{y}\neg\boldsymbol{y}{:}\boldsymbol{x_1}$. .... **[.3]** TAUT
90.9 $\Box\Box\Box\forall\boldsymbol{y}\neg\boldsymbol{y}{:}\boldsymbol{x_2}$. .... **[.5]** TAUT
90.10 $\Box\Box\Box\neg\boldsymbol{t}{:}\boldsymbol{x_1}$. .... **[.8]** $\forall_1$INST
90.11 $\Box\Box\Box\neg\boldsymbol{t}{:}\boldsymbol{x_2}$. .... **[.9]** $\forall_1$INST
90.12 $\Box\Box\Box\boldsymbol{t}{:}\boldsymbol{x_1}\vdash\boldsymbol{t}{:}\boldsymbol{x_2}$. .... LEM
$\Box\Box\Box$¶ .... DEM
90.13 $\Box\Box\Box\Box\boldsymbol{t}{:}\boldsymbol{x_1}$. .... **[.12]** ASM
90.14 $\Box\Box\Box\Box\boldsymbol{t}{:}\boldsymbol{x_2}$. .... **[.10]** **[.13]** TAUT
$\Box\Box\Box\blacksquare$. .... QED
90.15 $\Box\Box\Box\boldsymbol{t}{:}\boldsymbol{x_2}\vdash\boldsymbol{t}{:}\boldsymbol{t_1}$. .... LEM
$\Box\Box\Box$¶ .... DEM

$\square\square\square\square t{:}x_2$. .... [.15] ASM **90.16**

$\square\square\square\square t{:}x_1$. .... [.11] [.16] TAUT **90.17**

$\square\square\square\blacksquare$. .... QED

$\square\square\square t{:}x_1 \Leftrightarrow t{:}x_2$. .... [.12] [.15] TAUT **90.18**

$\square\square\square\forall t t{:}x_1 \Leftrightarrow t{:}x_2$. .... [.18] $\forall_1$GEN **90.19**

$\square\square\square\forall x\forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash (\forall t t{:}x \Leftrightarrow t{:}y) \vdash x = y$. .... AXM **90.20**

$\square\square\square x_1 = x_2$. .... [.20] [.6] [.7] [.19] ${}_3$MINT$_2$ **90.21**

$\square\square\blacksquare$. .... QED

$\square\blacksquare$. .... QED

$\square\forall x_1\forall x_2(x_1{:}\mathscr{H} \wedge \forall y\neg y{:}x_1) \vdash (x_2{:}\mathscr{H} \wedge \forall y\neg y{:}x_2) \vdash x_1 = x_2$. .... [.2] UGEN$_2$ **90.22**

$\square\exists! x x{:}\mathscr{H} \wedge \forall y\neg y{:}x$. .... [.1] [.22] $\exists!$INTROS **90.23**

$\blacksquare$. .... QED

Uses Axioms: **1, 3, 5, 9, 11**

$0{:}\mathscr{H} \wedge \forall x\neg x{:}0$. .... NATURAL DEFINITION *of the Empty Set* **91**

¶ .... DEM

$\square\exists x x{:}\mathscr{H} \wedge \forall y\neg y{:}x$. .... AXM **91.1**

$\square 0{:}\mathscr{H} \wedge \forall y\neg y{:}0$. .... [.1] ${}_1$NAME **91.2**

$\square\forall y\neg y{:}0$. .... [.2] TAUT **91.3**

$\square\neg y{:}0$. .... [.3] $\forall_1$INST **91.4**

$\square\forall x\neg x{:}0$. .... [.4] $\forall_1$GEN **91.5**

$\square 0{:}\mathscr{H} \wedge \forall x\neg x{:}0$. .... [.2] [.5] TAUT **91.6**

$\blacksquare$. .... QED

Uses Axioms: **1, 3, 11**

$\exists! x x{:}\mathscr{H} \wedge \forall y y{:}x \Leftrightarrow y = 0$. .... THEOREM *of the Unique Ordinal* **1** **92**

¶ .... DEM

$\square 0{:}\mathscr{H} \wedge \forall x\neg x{:}0$. .... DEF **92.1**

$\square 0{:}\mathscr{H}$. .... [.1] TAUT **92.2**

$\square\forall x\forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \exists z z{:}\mathscr{H} \wedge \forall z_1 z_1{:}z \Leftrightarrow z_1 = x \vee z_1 = y$. .... AXM **92.3**

$\square\exists z z{:}\mathscr{H} \wedge \forall z_1 z_1{:}z \Leftrightarrow z_1 = 0 \vee z_1 = 0$. .... [.3] [.2] [.2] ${}_2$MINT$_2$ **92.4**

$\square x{:}\mathscr{H} \wedge \forall z_1 z_1{:}x \Leftrightarrow z_1 = 0 \vee z_1 = 0$. .... [.4] $\exists_1$INST **92.5**

$\square\forall z_1 z_1{:}x \Leftrightarrow z_1 = 0 \vee z_1 = 0$. .... [.5] TAUT **92.6**

$\square y{:}x \Leftrightarrow y = 0$. .... LEM **92.7**

$\square$¶ .... DEM

$\square\square y{:}x \Leftrightarrow y = 0 \vee y = 0$. .... [.6] $\forall_1$INST **92.8**

$\square\square y = 0 \vee y = 0 \Leftrightarrow y = 0$. .... TAUT **92.9**

$\square\square y{:}x \Leftrightarrow y = 0$. .... [.8] [.9] TAUT **92.10**

$\square\blacksquare$. .... QED

$\square\forall y y{:}x \Leftrightarrow y = 0$. .... [.7] $\forall_1$GEN **92.11**

$\square x{:}\mathscr{H} \wedge \forall y y{:}x \Leftrightarrow y = 0$. .... [.5] [.11] TAUT **92.12**

$\square\exists x x{:}\mathscr{H} \wedge \forall y y{:}x \Leftrightarrow y = 0$. .... [.12] $\exists_1$GEN **92.13**

$\square(x_1{:}\mathscr{H} \wedge \forall y y{:}x_1 \Leftrightarrow y = 0) \vdash (x_2{:}\mathscr{H} \wedge \forall y y{:}x_2 \Leftrightarrow y = 0) \vdash x_1 = x_2$. .... LEM **92.14**

$\Box\P$ .... DEM

92.15 $\Box\Box x_1{:}\mathscr{H}\wedge\forall yy{:}x_1\Leftrightarrow y=0.$ .... [.14] ASM

92.16 $\Box\Box(x_2{:}\mathscr{H}\wedge\forall yy{:}x_2\Leftrightarrow y=0)\vdash x_1=x_2.$ .... LEM

$\Box\Box\P$ .... DEM

92.17 $\Box\Box\Box x_2{:}\mathscr{H}\wedge\forall yy{:}x_2\Leftrightarrow y=0.$ .... [.16] ASM

92.18 $\Box\Box\Box x_1{:}\mathscr{H}.$ .... [.15] TAUT

92.19 $\Box\Box\Box x_2{:}\mathscr{H}.$ .... [.17] TAUT

92.20 $\Box\Box\Box\forall yy{:}x_1\Leftrightarrow y=0.$ .... [.15] TAUT

92.21 $\Box\Box\Box\forall yy{:}x_2\Leftrightarrow y=0.$ .... [.17] TAUT

92.22 $\Box\Box\Box t{:}x_1\Leftrightarrow t=0.$ .... [.20] $\forall_1$INST

92.23 $\Box\Box\Box t{:}x_2\Leftrightarrow t=0.$ .... [.21] $\forall_1$INST

92.24 $\Box\Box\Box t{:}x_1\Leftrightarrow t{:}x_2.$ .... [.22] [.23] TAUT

92.25 $\Box\Box\Box\forall tt{:}x_1\Leftrightarrow t{:}x_2.$ .... [.24] $\forall_1$GEN

92.26 $\Box\Box\Box\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash(\forall tt{:}x\Leftrightarrow t{:}y)\vdash x=y.$ .... AXM

92.27 $\Box\Box\Box x_1=x_2.$ .... [.26] [.18] [.19] [.25] $_3$MINT$_2$

$\Box\Box\blacksquare.$ .... QED

$\Box\blacksquare.$ .... QED

92.28 $\Box\forall x_1\forall x_2(x_1{:}\mathscr{H}\wedge\forall yy{:}x_1\Leftrightarrow y=0)\vdash(x_2{:}\mathscr{H}\wedge\forall yy{:}x_2\Leftrightarrow y=0)\vdash x_1=x_2.$ .... [.14] UGEN$_2$

92.29 $\Box\exists!xx{:}\mathscr{H}\wedge\forall yy{:}x\Leftrightarrow y=0.$ .... [.13] [.28] $\exists!$INTROS

$\blacksquare.$ .... QED

Uses Axioms: **1, 3, 4, 5, 9, 11, 12**

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93 $1{:}\mathscr{H}\wedge\forall xx{:}1\Leftrightarrow x=0.$ .... NATURAL DEFINITION *of Ordinal* **1**

$\P$ .... DEM

93.1 $\Box\exists!xx{:}\mathscr{H}\wedge\forall yy{:}x\Leftrightarrow y=0.$ .... THM

93.2 $\Box\exists xx{:}\mathscr{H}\wedge\forall yy{:}x\Leftrightarrow y=0.$ .... [.1] $\exists!$EXISTS

93.3 $\Box 1{:}\mathscr{H}\wedge\forall yy{:}1\Leftrightarrow y=0.$ .... [.2] $_1$NAME

93.4 $\Box\forall yy{:}1\Leftrightarrow y=0.$ .... [.3] TAUT

93.5 $\Box y{:}1\Leftrightarrow y=0.$ .... [.4] $\forall_1$INST

93.6 $\Box\forall xx{:}1\Leftrightarrow x=0.$ .... [.5] $\forall_1$GEN

93.7 $\Box 1{:}\mathscr{H}\wedge\forall xx{:}1\Leftrightarrow x=0.$ .... [.3] [.6] TAUT

$\blacksquare.$ .... QED

Uses Axioms: **1, 3, 4, 5, 9, 11, 12**

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94 $\exists!xx{:}\mathscr{H}\wedge\forall yy{:}x\Leftrightarrow y=0\vee y=1.$ .... THEOREM *of the Unique Ordinal* **2**

$\P$ .... DEM

94.1 $\Box 0{:}\mathscr{H}\wedge\forall x\neg x{:}0.$ .... DEF

94.2 $\Box 1{:}\mathscr{H}\wedge\forall xx{:}1\Leftrightarrow x=0.$ .... DEF

94.3 $\Box 0{:}\mathscr{H}.$ .... [.1] TAUT

94.4 $\Box 1{:}\mathscr{H}.$ .... [.2] TAUT

94.5 $\Box\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists zz{:}\mathscr{H}\wedge\forall z_1z_1{:}z\Leftrightarrow z_1=x\vee z_1=y.$ .... AXM

94.6 $\Box\exists zz{:}\mathscr{H}\wedge\forall z_1z_1{:}z\Leftrightarrow z_1=0\vee z_1=1.$ .... [.5] [.3] [.4] $_2$MINT$_2$

94.7 $\Box x{:}\mathscr{H}\wedge\forall z_1z_1{:}x\Leftrightarrow z_1=0\vee z_1=1.$ .... [.6] $\exists_1$INST

$\Box\forall z_1 z_1{:}x \Leftrightarrow z_1=0 \vee z_1=1.$ .... [.7] TAUT **94.8**
$\Box y{:}x \Leftrightarrow y=0 \vee y=1.$ .... [.9] $\forall_1$INST **94.9**
$\Box\forall yy{:}x \Leftrightarrow y=0 \vee y=1.$ .... [.10] $\forall_1$GEN **94.10**
$\Box x{:}\mathscr{H} \wedge \forall yy{:}x \Leftrightarrow y=0 \vee y=1.$ .... [.7] [.10] TAUT **94.11**
$\Box\exists xx{:}\mathscr{H} \wedge \forall yy{:}x \Leftrightarrow y=0 \vee y=1.$ .... [.11] $\exists_1$GEN **94.12**
$\Box(x_1{:}\mathscr{H} \wedge \forall yy{:}x_1 \Leftrightarrow y=0 \vee y=1) \vdash (x_2{:}\mathscr{H} \wedge \forall yy{:}x_2 \Leftrightarrow y=0 \vee y=1) \vdash x_1=x_2.$ .... LEM **94.13**
$\Box$¶ .... DEM
$\Box\Box x_1{:}\mathscr{H} \wedge \forall yy{:}x_1 \Leftrightarrow y=0 \vee y=1.$ .... [.13] ASM **94.14**
$\Box\Box(x_2{:}\mathscr{H} \wedge \forall yy{:}x_2 \Leftrightarrow y=0 \vee y=1) \vdash x_1=x_2.$ .... LEM **94.15**
$\Box\Box$¶ .... DEM
$\Box\Box\Box x_2{:}\mathscr{H} \wedge \forall yy{:}x_2 \Leftrightarrow y=0 \vee y=1.$ .... [.15] ASM **94.16**
$\Box\Box\Box x_1{:}\mathscr{H}.$ .... [.14] TAUT **94.17**
$\Box\Box\Box x_2{:}\mathscr{H}.$ .... [.16] TAUT **94.18**
$\Box\Box\Box\forall yy{:}x_1 \Leftrightarrow y=0 \vee y=1.$ .... [.14] TAUT **94.19**
$\Box\Box\Box\forall yy{:}x_2 \Leftrightarrow y=0 \vee y=1.$ .... [.16] TAUT **94.20**
$\Box\Box\Box t{:}x_1 \Leftrightarrow t=0 \vee t=1.$ .... [.19] $\forall_1$INST **94.21**
$\Box\Box\Box t{:}x_2 \Leftrightarrow t=0 \vee t=1.$ .... [.20] $\forall_1$INST **94.22**
$\Box\Box\Box t{:}x_1 \Leftrightarrow t{:}x_2.$ .... [.21] [.22] TAUT **94.23**
$\Box\Box\Box\forall tt{:}x_1 \Leftrightarrow t{:}x_2.$ .... [.23] $\forall_1$GEN **94.24**
$\Box\Box\Box\forall x\forall yx{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash (\forall tt{:}x \Leftrightarrow t{:}y) \vdash x=y.$ .... AXM **94.25**
$\Box\Box\Box x_1=x_2.$ .... [.25] [.17] [.18] [.24] $_3$MINT$_2$ **94.26**
$\Box\Box\blacksquare.$ .... QED
$\Box\blacksquare.$ .... QED
$\Box\forall x_1\forall x_2(x_1{:}\mathscr{H} \wedge \forall yy{:}x_1 \Leftrightarrow y=0) \vdash (x_2{:}\mathscr{H} \wedge \forall yy{:}x_2 \Leftrightarrow y=0) \vdash x_1=x_2.$ .... [.13] UGEN$_2$ **94.27**
$\Box\exists!xx{:}\mathscr{H} \wedge \forall yy{:}x \Leftrightarrow y=0 \vee y=1.$ .... [.12] [.27] $\exists!$INTROS **94.28**
$\blacksquare.$ .... QED
Uses Axioms: **1, 3, 4, 5, 9, 11, 12**

$2{:}\mathscr{H} \wedge \forall xx{:}2 \Leftrightarrow x=0 \vee x=1.$ .... NATURAL DEFINITION *of Ordinal* **2** **95**
¶ .... DEM
$\Box\exists!xx{:}\mathscr{H} \wedge \forall yy{:}x \Leftrightarrow y=0 \vee y=1.$ .... THM **95.1**
$\Box\exists xx{:}\mathscr{H} \wedge \forall yy{:}x \Leftrightarrow y=0 \vee y=1.$ .... [.1] $\exists!$EXISTS **95.2**
$\Box 2{:}\mathscr{H} \wedge \forall yy{:}2 \Leftrightarrow y=0 \vee y=1.$ .... [.2] $_1$NAME **95.3**
$\Box\forall yy{:}2 \Leftrightarrow y=0 \vee y=1.$ .... [.3] TAUT **95.4**
$\Box y{:}2 \Leftrightarrow y=0 \vee y=1.$ .... [.4] $\forall_1$INST **95.5**
$\Box\forall xx{:}2 \Leftrightarrow x=0 \vee x=1.$ .... [.5] $\forall_1$GEN **95.6**
$\Box 2{:}\mathscr{H} \wedge \forall xx{:}2 \Leftrightarrow x=0 \vee x=1.$ .... [.3] [.6] TAUT **95.7**
$\blacksquare.$ .... QED
Uses Axioms: **1, 3, 4, 5, 9, 11, 12**

$0{:}1.$ .... THEOREM *of Basic Ordinal Truths 1* **96**
¶ .... DEM
$\Box 1{:}\mathscr{H} \wedge \forall xx{:}1 \Leftrightarrow x=0.$ .... DEF **96.1**

96.2 $\Box\forall x\,x{:}1\Leftrightarrow x=0.$ .... [.1] TAUT
96.3 $\Box 0{:}1\Leftrightarrow 0=0.$ .... [.2] $\forall_1$INST
96.4 $\Box 0=0.$ .... EQ
96.5 $\Box 0{:}1.$ .... [.3] [.4] TAUT
$\blacksquare.$ .... QED
Uses Axioms: **1, 3, 4, 5, 9, 11, 12**

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97 $0{:}2.$ .... THEOREM *of Basic Ordinal Truths 2*
¶ .... DEM
97.1 $\Box 2{:}\mathscr{H}\wedge\forall x\,x{:}2\Leftrightarrow x=0\vee x=1.$ .... DEF
97.2 $\Box\forall x\,x{:}2\Leftrightarrow x=0\vee x=1.$ .... [.1] TAUT
97.3 $\Box 0{:}2\Leftrightarrow 0=0\vee 0=1.$ .... [.2] $\forall_1$INST
97.4 $\Box 0=0.$ .... EQ
97.5 $\Box 0{:}2.$ .... [.3] [.4] TAUT
$\blacksquare.$ .... QED
Uses Axioms: **1, 3, 4, 5, 9, 11, 12**

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98 $1{:}2.$ .... THEOREM *of Basic Ordinal Truths 3*
¶ .... DEM
98.1 $\Box 2{:}\mathscr{H}\wedge\forall x\,x{:}2\Leftrightarrow x=0\vee x=1.$ .... DEF
98.2 $\Box\forall x\,x{:}2\Leftrightarrow x=0\vee x=1.$ .... [.1] TAUT
98.3 $\Box 1{:}2\Leftrightarrow 1=0\vee 1=1.$ .... [.2] $\forall_1$INST
98.4 $\Box 1=1.$ .... EQ
98.5 $\Box 1{:}2.$ .... [.3] [.4] TAUT
$\blacksquare.$ .... QED
Uses Axioms: **1, 3, 4, 5, 9, 11, 12**

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99 $\neg 1=0.$ .... THEOREM *of Basic Ordinal Truths 4*
¶ .... DEM
99.1 $\Box 1=0\vdash\forall x\,x.$ .... LEM
$\Box$¶ .... DEM
99.2 $\Box\Box 1=0.$ .... [.1] ASM
99.3 $\Box\Box 0{:}1.$ .... THM
99.4 $\Box\Box 1{:}0.$ .... [.2] [.3] SUB
99.5 $\Box\Box 0{:}\mathscr{H}\wedge\forall x\neg x{:}0.$ .... DEF
99.6 $\Box\Box\forall x\neg x{:}0.$ .... [.5] TAUT
99.7 $\Box\Box\neg 1{:}0.$ .... [.6] $\forall_1$INST
99.8 $\Box\Box\forall x\,x.$ .... [.3] [.7] TAUT
$\Box\blacksquare.$ .... QED
99.9 $\Box\neg 1=0.$ .... [.1] TAUT
$\blacksquare.$ .... QED
Uses Axioms: **1, 3, 4, 5, 9, 11, 12**

$\neg\mathbf{2}=\mathbf{0}$. . . . . THEOREM *of Basic Ordinal Truths 4* **100**

¶ . . . . DEM

$\square\mathbf{2}=\mathbf{0}\vdash\forall\boldsymbol{x}\boldsymbol{x}$. . . . . LEM **100.1**

$\square$¶ . . . . DEM

$\square\square\mathbf{2}=\mathbf{0}$. . . . . **[.1]** ASM **100.2**

$\square\square\mathbf{0}:\mathbf{2}$. . . . . THM **100.3**

$\square\square\mathbf{2}:\mathbf{0}$. . . . . **[.2] [.3]** SUB **100.4**

$\square\square\mathbf{0}:\mathscr{H}\wedge\forall\boldsymbol{x}\neg\boldsymbol{x}:\mathbf{0}$. . . . . DEF **100.5**

$\square\square\forall\boldsymbol{x}\neg\boldsymbol{x}:\mathbf{0}$. . . . . **[.5]** TAUT **100.6**

$\square\square\neg\mathbf{2}:\mathbf{0}$. . . . . **[.6]** $\forall_1$INST **100.7**

$\square\square\forall\boldsymbol{x}\boldsymbol{x}$. . . . . **[.3] [.7]** TAUT **100.8**

$\square\blacksquare$. . . . . QED

$\square\neg\mathbf{2}=\mathbf{0}$. . . . . **[.1]** TAUT **100.9**

$\blacksquare$. . . . . QED

Uses Axioms: **1, 3, 4, 5, 9, 11, 12**

$\neg\mathbf{2}=\mathbf{1}$. . . . . THEOREM *of Basic Ordinal Truths 4* **101**

¶ . . . . DEM

$\square\mathbf{2}=\mathbf{1}\vdash\forall\boldsymbol{x}\boldsymbol{x}$. . . . . LEM **101.1**

$\square$¶ . . . . DEM

$\square\square\mathbf{2}=\mathbf{1}$. . . . . **[.1]** ASM **101.2**

$\square\square\mathbf{1}:\mathbf{2}$. . . . . THM **101.3**

$\square\square\mathbf{2}:\mathbf{1}$. . . . . **[.2] [.3]** SUB **101.4**

$\square\square\mathbf{1}:\mathscr{H}\wedge\forall\boldsymbol{x}\boldsymbol{x}:\mathbf{1}\Leftrightarrow\boldsymbol{x}=\mathbf{0}$. . . . . DEF **101.5**

$\square\square\forall\boldsymbol{x}\boldsymbol{x}:\mathbf{1}\Leftrightarrow\boldsymbol{x}=\mathbf{0}$. . . . . **[.5]** TAUT **101.6**

$\square\square\mathbf{2}:\mathbf{1}\Leftrightarrow\mathbf{2}=\mathbf{0}$. . . . . **[.6]** $\forall_1$INST **101.7**

$\square\square\neg\mathbf{2}=\mathbf{0}$. . . . . THM **101.8**

$\square\square\neg\mathbf{2}:\mathbf{1}$. . . . . **[.7] [.8]** TAUT **101.9**

$\square\square\forall\boldsymbol{x}\boldsymbol{x}$. . . . . **[.4] [.9]** TAUT **101.10**

$\square\blacksquare$. . . . . QED

$\square\neg\mathbf{2}=\mathbf{1}$. . . . . **[.1]** TAUT **101.11**

$\blacksquare$. . . . . QED

Uses Axioms: **1, 3, 4, 5, 9, 11, 12**

For any set of non-empty sets, $\boldsymbol{x}$, there is a *choice* function $\boldsymbol{f}:\boldsymbol{x}\to\mathscr{H}$ st. $\boldsymbol{f}\boldsymbol{x}_1:\boldsymbol{x}_1$ for each $\boldsymbol{x}_1:\boldsymbol{x}$.
PROOF: Take any set of non-empty sets $\boldsymbol{x}$ and denote the predetermined choice function as $\boldsymbol{g}$. Since $\boldsymbol{x}$ is a set, by *heredity* each of its elements $\boldsymbol{x}_1$ are also sets; by *application*, each of its evaluations $\boldsymbol{g}\boldsymbol{x}_1$ are also sets. Thus, by *creation*, we may define the function $\boldsymbol{f}:\boldsymbol{x}\to\mathscr{H}$ which pointwise evaluates by $\boldsymbol{g}$. It has precisely the desired property. QED.
In usual ZFC set theories, the *axiom of predetermined choice* is stronger than the usual forms of choice. So we prove this particular form and uphold it as law.

102 $\forall x x{:}\mathscr{H} \vdash (\forall x_1 x_1{:}x \vdash \exists x_2 x_2{:}x_1) \vdash \exists f f{:}x \to \mathscr{H} \wedge \forall x_1 x_1{:}x \vdash f x_1{:}x_1.$ .... LAW *of Spontaneous Choice for Sets* **[21]**

$\P$ .... DEM

102.1 $\square a{:}\mathscr{H} \vdash (\forall x_1 x_1{:}a \vdash \exists x_2 x_2{:}x_1) \vdash \exists f f{:}a \to \mathscr{H} \wedge \forall x_1 x_1{:}a \vdash f x_1{:}x_1.$ .... LEM

$\square\P$ .... DEM

102.2 $\square\square a{:}\mathscr{H}.$ .... **[.1]** ASM

102.3 $\square\square(\forall x_1 x_1{:}a \vdash \exists x_2 x_2{:}x_1) \vdash \exists f f{:}a \to \mathscr{H} \wedge \forall x_1 x_1{:}a \vdash f x_1{:}x_1.$ .... LEM

$\square\square\P$ .... DEM

102.4 $\square\square\square\forall x_1 x_1{:}a \vdash \exists x_2 x_2{:}x_1.$ .... **[.3]** ASM

102.5 $\square\square\square\exists f f{:}\mathscr{H} \to \mathscr{H} \wedge \forall x x{:}\mathscr{H} \vdash (\exists x_1 x_1{:}x) \vdash f x{:}x.$ .... AXM

102.6 $\square\square\square g{:}\mathscr{H} \to \mathscr{H} \wedge \forall x x{:}\mathscr{H} \vdash (\exists x_1 x_1{:}x) \vdash g x{:}x.$ .... **[.5]** $\exists_1$INST

102.7 $\square\square\square\forall a \forall b \forall \psi(\forall x x{:}a \vdash \exists! y y{:}b \wedge \psi x y) \vdash \exists f f{:}a \to b \wedge \forall x \forall y x{:}a \vdash y{:}b \vdash \psi x y \Leftrightarrow f x = y.$ .... AXM

102.8 $\square\square\square\forall \psi(\forall x x{:}a \vdash \exists! y y{:}\mathscr{H} \wedge \psi x y) \vdash \exists f f{:}a \to \mathscr{H} \wedge \forall x \forall y x{:}a \vdash y{:}\mathscr{H} \vdash \psi x y \Leftrightarrow f x = y.$ ... **[.7]** $_0$MINT$_2$

102.9 $\square\square\square(\forall x x{:}a \vdash \exists! y y{:}\mathscr{H} \wedge g x = y) \vdash \exists f f{:}a \to \mathscr{H} \wedge \forall x \forall y x{:}a \vdash y{:}\mathscr{H} \vdash g x = y \Leftrightarrow f x = y.$ .... **[.8]** $\forall_2$INST

102.10 $\square\square\square x{:}a \vdash \exists! y y{:}\mathscr{H} \wedge g x = y.$ .... LEM

$\square\square\square\P$ .... DEM

102.11 $\square\square\square\square x{:}a.$ .... **[.10]** ASM

102.12 $\square\square\square\square\forall x \forall y x{:}y \vdash y{:}\mathscr{H} \vdash x{:}\mathscr{H}.$ .... AXM

102.13 $\square\square\square\square x{:}\mathscr{H}.$ .... **[.12] [.11] [.2]** $_2$MINT$_2$

102.14 $\square\square\square\square\forall a \forall b \forall f f{:}a \to b \vdash \forall x x{:}a \vdash f x{:}b.$ .... AXM

102.15 $\square\square\square\square g{:}\mathscr{H} \to \mathscr{H}.$ .... **[.6]** TAUT

102.16 $\square\square\square\square\forall x x{:}\mathscr{H} \vdash g x{:}\mathscr{H}.$ .... **[.14] [.15]** $_1$MINT$_3$

102.17 $\square\square\square\square g x{:}\mathscr{H}.$ .... **[.16] [.13]** $_1$MINT$_1$

102.18 $\square\square\square\square g x = g x.$ .... EQ

102.19 $\square\square\square\square g x{:}\mathscr{H} \wedge g x = g x.$ .... **[.17] [.18]** TAUT

102.20 $\square\square\square\square\exists y y{:}\mathscr{H} \wedge g x = y.$ .... **[.19]** $\exists_1$GEN

102.21 $\square\square\square\square y_1{:}\mathscr{H} \wedge g x = y_1 \vdash y_2{:}\mathscr{H} \wedge g x = y_2 \vdash y_1 = y_2.$ .... LEM

$\square\square\square\square\P$ .... DEM

102.22 $\square\square\square\square\square y_1{:}\mathscr{H} \wedge g x = y_1.$ .... **[.21]** ASM

102.23 $\square\square\square\square\square y_2{:}\mathscr{H} \wedge g x = y_2 \vdash y_1 = y_2.$ .... LEM

$\square\square\square\square\square\P$ .... DEM

102.24 $\square\square\square\square\square\square y_2{:}\mathscr{H} \wedge g x = y_2.$ .... **[.23]** ASM

102.25 $\square\square\square\square\square\square g x = y_1.$ .... **[.22]** TAUT

102.26 $\square\square\square\square\square\square g x = y_2.$ .... **[.24]** TAUT

102.27 $\square\square\square\square\square\square y_1 = y_2.$ .... **[.25] [.26]** SUB

$\square\square\square\square\square\blacksquare.$ .... QED

$\square\square\square\square\blacksquare.$ .... QED

102.28 $\square\square\square\square\forall y_1 \forall y_2 y_1{:}\mathscr{H} \wedge g x = y_1 \vdash y_2{:}\mathscr{H} \wedge g x = y_2 \vdash y_1 = y_2.$ .... **[.21]** UGEN$_2$

102.29 $\square\square\square\square\exists! y y{:}\mathscr{H} \wedge g x = y.$ .... **[.20] [.28]** $\exists!$INTROS

$\square\square\square\blacksquare.$ .... QED

102.30 $\square\square\square\forall x x{:}a \vdash \exists! y y{:}\mathscr{H} \wedge g x = y.$ .... **[.10]** $\forall_1$GEN

102.31 $\square\square\square\exists f f{:}a \to \mathscr{H} \wedge \forall x \forall y x{:}a \vdash y{:}\mathscr{H} \vdash g x = y \Leftrightarrow f x = y.$ .... **[.9] [.30]** MP

102.32 $\square\square\square f{:}a \to \mathscr{H} \wedge \forall x \forall y x{:}a \vdash y{:}\mathscr{H} \vdash g x = y \Leftrightarrow f x = y.$ .... **[.31]** $\exists_1$INST

$\square\square\square x{:}a\vdash fx{:}x.$ .... LEM **102.33**

$\square\square\square\P$ .... DEM

$\square\square\square\square x{:}a.$ .... **[.10]** ASM **102.34**

$\square\square\square\square\forall x\forall yx{:}y\vdash y{:}\mathscr{H}\vdash x{:}\mathscr{H}.$ .... AXM **102.35**

$\square\square\square\square x{:}\mathscr{H}.$ .... **[.35] [.34] [.2]** $_2\mathrm{MINT}_2$ **102.36**

$\square\square\square\square\forall x\forall yx{:}a\vdash y{:}\mathscr{H}\vdash gx=y\Leftrightarrow fx=y.$ .... **[.32]** TAUT **102.37**

$\square\square\square\square\forall a\forall b\forall ff{:}a\rightarrow b\vdash\forall xx{:}a\vdash fx{:}b.$ .... AXM **102.38**

$\square\square\square\square g{:}\mathscr{H}\rightarrow\mathscr{H}.$ .... **[.6]** TAUT **102.39**

$\square\square\square\square\forall xx{:}\mathscr{H}\vdash gx{:}\mathscr{H}.$ .... **[.38] [.39]** $_1\mathrm{MINT}_3$ **102.40**

$\square\square\square\square gx{:}\mathscr{H}.$ .... **[.40] [.36]** $_1\mathrm{MINT}_1$ **102.41**

$\square\square\square\square gx=gx.$ .... EQ **102.42**

$\square\square\square\square fx=gx.$ .... **[.37] [.34] [.41] [.42]** TAUT **102.43**

$\square\square\square\square\forall xx{:}\mathscr{H}\vdash(\exists x_1x_1{:}x)\vdash gx{:}x.$ .... **[.6]** TAUT **102.44**

$\square\square\square\square\exists x_2x_2{:}x.$ .... **[.4] [.34]** $_1\mathrm{MINT}_1$ **102.45**

$\square\square\square\square x_1{:}x.$ .... **[.45]** $\exists_1$INST **102.46**

$\square\square\square\square\exists x_1x_1{:}x.$ .... **[.46]** $\exists_1$GEN **102.47**

$\square\square\square\square gx{:}x.$ .... **[.44] [.36] [.47]** $_2\mathrm{MINT}_1$ **102.48**

$\square\square\square\square fx{:}x.$ .... **[.43] [.48]** SUB **102.49**

$\square\square\square\blacksquare.$ .... QED

$\square\square\square\forall xx{:}a\vdash fx{:}x.$ .... **[.33]** $\forall_1$GEN **102.50**

$\square\square\square f{:}a\rightarrow\mathscr{H}\wedge\forall xx{:}a\vdash fx{:}x.$ .... **[.32] [.50]** TAUT **102.51**

$\square\square\square\exists ff{:}a\rightarrow\mathscr{H}\wedge\forall xx{:}a\vdash fx{:}x.$ .... **[.51]** $\exists_1$GEN **102.52**

$\square\square\blacksquare.$ .... QED

$\square\blacksquare.$ .... QED

$\square\forall xx{:}\mathscr{H}\vdash(\forall x_1x_1{:}x\vdash\exists x_2x_2{:}x_1)\vdash\exists ff{:}x\rightarrow\mathscr{H}\wedge\forall x_1x_1{:}x\vdash fx_1{:}x_1.$ .... **[.1]** $\forall_1$GEN **102.53**

$\blacksquare.$ .... QED

Uses Axioms: **3, 5, 6, 7, 10, 19**

$p\vee\neg p$ for each p. PROOF: Take any object p. We may create sets a,b by *specifying* over $\mathbf{2}$ as follows: an element $x{:}\mathbf{2}$ is in a iff. $x=\mathbf{0}\vee p$, and is in b iff. $x=\mathbf{1}\vee p$. Obviously a and b are non-empty. So consider the *paired* set c, whose elements are exactly a and b. By the *law of spontaneous choice*, their is a function $f{:}c\rightarrow\mathscr{H}$, st. $fa{:}a$ and $fb{:}b$. We work out $p\vee\neg fb=fa$ and with *extensionality*, it follows $p\vee\neg p$. QED.

The constructivist's bane is the classicist's boon. In order to distinguish uses of *choice* from uses of *lem*, we uphold this theorem as law.

$\forall pp\vee\neg p.$ .... LAW *of the Excluded Middle* **[22]** **103**

$\P$ .... DEM

$\square\forall x\forall\psi x{:}\mathscr{H}\vdash\exists yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}x\wedge\psi y_1.$ .... AXM **103.1**

$\square\forall\psi\mathbf{2}{:}\mathscr{H}\vdash\exists yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}\mathbf{2}\wedge\psi y_1.$ .... **[.1]** $\forall_1$INST **103.2**

$\square\mathbf{2}{:}\mathscr{H}\vdash\exists yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}\mathbf{2}\wedge(y_1=\mathbf{0}\vee p).$ .... **[.2]** $\forall_2$INST **103.3**

$\square\mathbf{2}{:}\mathscr{H}\vdash\exists yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}\mathbf{2}\wedge(y_1=\mathbf{1}\vee p).$ .... **[.2]** $\forall_2$INST **103.4**

$\square\mathbf{2}{:}\mathscr{H}\wedge\forall xx{:}\mathbf{2}\Leftrightarrow x=\mathbf{0}\vee x=\mathbf{1}.$ .... DEF **103.5**

$\square\mathbf{2}{:}\mathscr{H}.$ .... **[.5]** TAUT **103.6**

103.7 $\square\exists y y{:}\mathscr{H}\wedge\forall y_1 y_1{:}y\Leftrightarrow y_1{:}2\wedge(y_1=0\vee p)$. .... [.3] [.6] MP
103.8 $\square a{:}\mathscr{H}\wedge\forall y_1 y_1{:}a\Leftrightarrow y_1{:}2\wedge(y_1=0\vee p)$. .... [.7] $\exists_1$INST
103.9 $\square\exists y y{:}\mathscr{H}\wedge\forall y_1 y_1{:}y\Leftrightarrow y_1{:}2\wedge(y_1=1\vee p)$. .... [.4] [.6] MP
103.10 $\square b{:}\mathscr{H}\wedge\forall y_1 y_1{:}b\Leftrightarrow y_1{:}2\wedge(y_1=1\vee p)$. .... [.9] $\exists_1$INST
103.11 $\square\exists x_2 x_2{:}a$. .... LEM
$\square$¶ .... DEM
103.12 $\square\square\forall y_1 y_1{:}a\Leftrightarrow y_1{:}2\wedge(y_1=0\vee p)$. .... [.8] TAUT
103.13 $\square\square 0{:}a\Leftrightarrow 0{:}2\wedge(0=0\vee p)$. .... [.12] $\forall_1$INST
103.14 $\square\square 0{:}2$. .... THM
103.15 $\square\square 0=0$. .... EQ
103.16 $\square\square 0{:}2\wedge(0=0\vee p)$. .... [.14] [.15] TAUT
103.17 $\square\square 0{:}a$. .... [.13] [.16] TAUT
103.18 $\square\square\exists x_2 x_2{:}a$. .... [.17] $\exists_1$GEN
$\square\blacksquare$. .... QED
103.19 $\square\exists x_2 x_2{:}b$. .... LEM
$\square$¶ .... DEM
103.20 $\square\square\forall y_1 y_1{:}b\Leftrightarrow y_1{:}2\wedge(y_1=1\vee p)$. .... [.10] TAUT
103.21 $\square\square 1{:}b\Leftrightarrow 1{:}2\wedge(1=1\vee p)$. .... [.20] $\forall_1$INST
103.22 $\square\square 1{:}2$. .... THM
103.23 $\square\square 1=1$. .... EQ
103.24 $\square\square 1{:}2\wedge(1=1\vee p)$. .... [.22] [.23] TAUT
103.25 $\square\square 1{:}b$. .... [.21] [.24] TAUT
103.26 $\square\square\exists x_2 x_2{:}b$. .... [.25] $\exists_1$GEN
$\square\blacksquare$. .... QED
103.27 $\square\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists z z{:}\mathscr{H}\wedge\forall z_1 z_1{:}z\Leftrightarrow z_1=x\vee z_1=y$. .... AXM
103.28 $\square a{:}\mathscr{H}$. .... [.8] TAUT
103.29 $\square b{:}\mathscr{H}$. .... [.10] TAUT
103.30 $\square\exists z z{:}\mathscr{H}\wedge\forall z_1 z_1{:}z\Leftrightarrow z_1=a\vee z_1=b$. .... [.27] [.28] [.29] ${}_2$MINT$_2$
103.31 $\square c{:}\mathscr{H}\wedge\forall z_1 z_1{:}c\Leftrightarrow z_1=a\vee z_1=b$. .... [.30] $\exists_1$INST
103.32 $\square x{:}c\vdash\exists x_2 x_2{:}x$. .... LEM
$\square$¶ .... DEM
103.33 $\square\square x{:}c$. .... [.32] ASM
103.34 $\square\square\forall z_1 z_1{:}c\Leftrightarrow z_1=a\vee z_1=b$. .... [.31] TAUT
103.35 $\square\square x{:}c\Leftrightarrow x=a\vee x=b$. .... [.34] $\forall_1$INST
103.36 $\square\square x=a\vee x=b$. .... [.35] [.33] TAUT
103.37 $\square\square x=a\vdash\exists x_2 x_2{:}x$. .... LEM
$\square\square$¶ .... DEM
103.38 $\square\square\square x=a$. .... [.37] ASM
103.39 $\square\square\square\exists x_2 x_2{:}x$. .... [.38] [.11] SUB
$\square\square\blacksquare$. .... QED
103.40 $\square\square x=b\vdash\exists x_2 x_2{:}x$. .... LEM
$\square\square$¶ .... DEM
103.41 $\square\square\square x=b$. .... [.40] ASM
103.42 $\square\square\square\exists x_2 x_2{:}x$. .... [.41] [.19] SUB

$\square\square\blacksquare$. ... QED

$\square\square\exists x_2 x_2{:}x$. ... [.36] [.37] [.40] TAUT **103.43**

$\square\blacksquare$. ... QED

$\square\forall x_1 x_1{:}c \vdash \exists x_2 x_2{:}x_1$. ... [.32] $\forall_1$GEN **103.44**

$\square\forall xx{:}\mathscr{H} \vdash (\forall x_1 x_1{:}x \vdash \exists x_2 x_2{:}x_1) \vdash \exists f f{:}x \to \mathscr{H} \wedge \forall x_1 x_1{:}x \vdash f x_1{:}x_1$. ... LAW **103.45**

$\square c{:}\mathscr{H}$. ... [.31] TAUT **103.46**

$\square\exists f f{:}c \to \mathscr{H} \wedge \forall x_1 x_1{:}c \vdash f x_1{:}x_1$. ... [.45] [.46] [.44] ${}_2$MINT$_1$ **103.47**

$\square f{:}c \to \mathscr{H} \wedge \forall x_1 x_1{:}c \vdash f x_1{:}x_1$. ... [.47] $\exists_1$INST **103.48**

$\square\forall x_1 x_1{:}c \vdash f x_1{:}x_1$. ... [.48] TAUT **103.49**

$\square a{:}c \vdash f a{:}a$. ... [.49] $\forall_1$INST **103.50**

$\square b{:}c \vdash f b{:}b$. ... [.49] $\forall_1$INST **103.51**

$\square\forall z_1 z_1{:}c \Leftrightarrow z_1 = a \vee z_1 = b$. ... [.31] TAUT **103.52**

$\square a{:}c \Leftrightarrow a = a \vee a = b$. ... [.52] $\forall_1$INST **103.53**

$\square a = a$. ... EQ **103.54**

$\square a{:}c$. ... [.53] [.54] TAUT **103.55**

$\square b{:}c \Leftrightarrow b = a \vee b = b$. ... [.52] $\forall_1$INST **103.56**

$\square b = b$. ... EQ **103.57**

$\square b{:}c$. ... [.56] [.57] TAUT **103.58**

$\square f a{:}a$. ... [.50] [.55] MP **103.59**

$\square f b{:}b$. ... [.51] [.58] MP **103.60**

$\square\forall y_1 y_1{:}a \Leftrightarrow y_1{:}2 \wedge (y_1 = 0 \vee p)$. ... [.8] TAUT **103.61**

$\square f a{:}a \Leftrightarrow f a{:}2 \wedge (f a = 0 \vee p)$. ... [.61] $\forall_1$INST **103.62**

$\square f a = 0 \vee p$. ... [.62] [.59] TAUT **103.63**

$\square p \vdash p \vee \neg p$. ... LEM **103.64**

$\square$¶ ... DEM

$\square\square p$. ... [.64] ASM **103.65**

$\square\square p \vee \neg p$. ... [.65] TAUT **103.66**

$\square\blacksquare$. ... QED

$\square f a = 0 \vdash p \vee \neg p$. ... LEM **103.67**

$\square$¶ ... DEM

$\square\square f a = 0$. ... [.64] ASM **103.68**

$\square\square\forall y_1 y_1{:}b \Leftrightarrow y_1{:}2 \wedge (y_1 = 1 \vee p)$. ... [.10] TAUT **103.69**

$\square\square f b{:}b \Leftrightarrow f b{:}2 \wedge (f b = 1 \vee p)$. ... [.69] $\forall_1$INST **103.70**

$\square\square f b = 1 \vee p$. ... [.70] [.60] TAUT **103.71**

$\square\square f b = 1 \vdash p \vee \neg p$. ... LEM **103.72**

$\square\square$¶ ... DEM

$\square\square\square f b = 1$. ... [.72] ASM **103.73**

$\square\square\square p \vdash \forall xx$. ... LEM **103.74**

$\square\square\square$¶ ... DEM

$\square\square\square\square p$. ... [.74] ASM **103.75**

$\square\square\square\square x{:}a \Leftrightarrow x{:}2 \wedge (x = 0 \vee p)$. ... [.61] $\forall_1$INST **103.76**

$\square\square\square\square x{:}a \Leftrightarrow x{:}2$. ... [.76] [.75] TAUT **103.77**

$\square\square\square\square\forall tt{:}a \Leftrightarrow t{:}2$. ... [.77] $\forall_1$GEN **103.78**

$\square\square\square\square\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash (\forall tt{:}x \Leftrightarrow t{:}y) \vdash x = y$. ... AXM **103.79**

103.80 □□□□$\boldsymbol{a}=2$. .... **[.79]** **[.28]** **[.6]** **[.78]** $_3$MINT$_2$

103.81 □□□□$\boldsymbol{x}:\boldsymbol{b}\Leftrightarrow\boldsymbol{x}:2\wedge(\boldsymbol{x}=1\vee\boldsymbol{p})$. .... **[.69]** $\forall_1$INST

103.82 □□□□$\boldsymbol{x}:\boldsymbol{b}\Leftrightarrow\boldsymbol{x}:2$. .... **[.81]** **[.75]** TAUT

103.83 □□□□$\forall\boldsymbol{t}\,\boldsymbol{t}:\boldsymbol{b}\Leftrightarrow\boldsymbol{t}:2$. .... **[.82]** $\forall_1$GEN

103.84 □□□□$\boldsymbol{b}=2$. .... **[.79]** **[.29]** **[.6]** **[.83]** $_3$MINT$_2$

103.85 □□□□$\boldsymbol{b}=\boldsymbol{a}$. .... **[.80]** **[.84]** SUB

103.86 □□□□$\boldsymbol{f}\,\boldsymbol{b}=\boldsymbol{f}\,\boldsymbol{b}$. .... EQ

103.87 □□□□$\boldsymbol{f}\,\boldsymbol{b}=\boldsymbol{f}\,\boldsymbol{a}$. .... **[.85]** **[.86]** SUB

103.88 □□□□$1=0$. .... **[.73]** **[.68]** **[.87]** TAUT

103.89 □□□□$\neg 1=0$. .... THM

103.90 □□□□$\forall\boldsymbol{x}\,\boldsymbol{x}$. .... **[.89]** **[.88]** TAUT

□□□■. .... QED

103.91 □□□$\boldsymbol{p}\vee\neg\boldsymbol{p}$. .... **[.74]** TAUT

103.92 □□■. .... QED

103.93 □□$\boldsymbol{p}\vee\neg\boldsymbol{p}$. .... **[.71]** **[.72]** **[.64]** TAUT

103.94 □■. .... QED

103.95 □$\boldsymbol{p}\vee\neg\boldsymbol{p}$. .... **[.63]** **[.67]** **[.64]** TAUT

103.96 □$\forall\boldsymbol{p}\,\boldsymbol{p}\vee\neg\boldsymbol{p}$. .... **[.95]** $\forall_1$GEN

■. .... QED

Uses Axioms: **1, 3, 4, 5, 6, 9, 11, 12, 15, 21**

Laws are deductions which we regard as having the same status as axioms. In proofs, when making reference to a law, we use the assigned axiom number; eg. *excluded middle* is **21**.

In this way, we can explore alternative axiomatizations of the theory, and comfortably express the cumulative pedigree of axiomatic entailment. Importantly, we gain the insight of what is constructive and what is classical.

§1.5 *Logic: Exercises in Classical Logic*

$\forall \boldsymbol{p} \neg\neg \boldsymbol{p} \vdash \boldsymbol{p}$. .... THEOREM *of Propositional Duality* **104**

¶ .... DEM

$\Box \neg\neg \boldsymbol{p} \vdash \boldsymbol{p}$. .... LEM **104.1**

$\Box$¶ .... DEM

$\Box\Box \neg\neg \boldsymbol{p}$. .... **[.1]** ASM **104.2**

$\Box\Box \neg \boldsymbol{p} \vdash \boldsymbol{p}$. .... LEM **104.3**

$\Box\Box$¶ .... DEM

$\Box\Box\Box \neg \boldsymbol{p}$. .... **[.3]** ASM **104.4**

$\Box\Box\Box \forall \boldsymbol{x}\boldsymbol{x}$. .... **[.2] [.4]** TAUT **104.5**

$\Box\Box\Box \boldsymbol{p}$. .... **[.5]** $\forall_1$INST **104.6**

$\Box\Box$■. .... QED

$\Box\Box \forall \boldsymbol{p}\boldsymbol{p} \vee \neg \boldsymbol{p}$. .... LAW **104.7**

$\Box\Box \boldsymbol{p} \vee \neg \boldsymbol{p}$. .... **[.7]** $\forall_1$INST **104.8**

$\Box\Box \boldsymbol{p}$. .... **[.8] [.3]** TAUT **104.9**

$\Box$■. .... QED

$\Box \forall \boldsymbol{p} \neg\neg \boldsymbol{p} \vdash \boldsymbol{p}$. .... **[.1]** $\forall_1$GEN **104.10**

■. .... QED

Uses Axioms: **1, 4, 22**

$\forall \boldsymbol{p} \forall \boldsymbol{q} (\boldsymbol{p} \vdash \boldsymbol{q}) \vdash \neg \boldsymbol{p} \vee \boldsymbol{q}$. .... THEOREM *of Material Entailment 1* **105**

¶ .... DEM

$\Box (\boldsymbol{p} \vdash \boldsymbol{q}) \vdash \neg \boldsymbol{p} \vee \boldsymbol{q}$. .... LEM **105.1**

$\Box$¶ .... DEM

$\Box\Box \boldsymbol{p} \vdash \boldsymbol{q}$. .... **[.1]** ASM **105.2**

$\Box\Box \forall \boldsymbol{p}\boldsymbol{p} \vee \neg \boldsymbol{p}$. .... LAW **105.3**

$\Box\Box \boldsymbol{p} \vee \neg \boldsymbol{p}$. .... **[.3]** $\forall_1$INST **105.4**

$\Box\Box \neg \boldsymbol{p} \vee \boldsymbol{q}$. .... **[.4] [.2]** TAUT **105.5**

$\Box$■. .... QED

$\Box \forall \boldsymbol{p} \forall \boldsymbol{q} (\boldsymbol{p} \vdash \boldsymbol{q}) \vdash \neg \boldsymbol{p} \vee \boldsymbol{q}$. .... **[.1]** UGEN$_2$ **105.6**

■. .... QED

Uses Axioms: **4, 22**

$\forall \boldsymbol{p} \forall \boldsymbol{q} \neg \boldsymbol{p} \vee \boldsymbol{q} \vdash \boldsymbol{p} \vdash \boldsymbol{q}$. .... THEOREM *of Material Entailment 2* **106**

¶ .... DEM

106.1 $\Box\neg p\vee q\vdash p\vdash q$. .... LEM

$\Box$¶ .... DEM

106.2 $\Box\Box\neg p\vee q$. .... **[.1]** ASM

106.3 $\Box\Box p\vdash q$. .... LEM

$\Box\Box$¶ .... DEM

106.4 $\Box\Box\Box p$. .... **[.3]** ASM

106.5 $\Box\Box\Box q$. .... **[.2]** **[.4]** TAUT

$\Box\Box\blacksquare$. .... QED

$\Box\blacksquare$. .... QED

106.6 $\Box\forall p\forall q\neg p\vee q\vdash p\vdash q$. .... **[.1]** UGEN$_2$

$\blacksquare$. .... QED

Uses Axioms: **1, 4**

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107 $\forall p\forall q\forall r p\wedge(q\vee r)\Leftrightarrow p\wedge q\vee p\wedge r$. .... THEOREM *of Distributivity 1*

¶ .... DEM

107.1 $\Box p\wedge(q\vee r)\vdash p\wedge q\vee p\wedge r$. .... LEM

$\Box$¶ .... DEM

107.2 $\Box\Box p\wedge(q\vee r)$. .... **[.1]** ASM

107.3 $\Box\Box q\vdash p\wedge q\vee p\wedge r$. .... LEM

$\Box\Box$¶ .... DEM

107.4 $\Box\Box\Box q$. .... **[.3]** ASM

107.5 $\Box\Box\Box p\wedge q\vee p\wedge r$. .... **[.2]** **[.4]** TAUT

$\Box\Box\blacksquare$. .... QED

107.6 $\Box\Box r\vdash p\wedge q\vee p\wedge r$. .... LEM

$\Box\Box$¶ .... DEM

107.7 $\Box\Box\Box r$. .... **[.6]** ASM

107.8 $\Box\Box\Box p\wedge q\vee p\wedge r$. .... **[.2]** **[.7]** TAUT

$\Box\Box\blacksquare$. .... QED

107.9 $\Box\Box p\wedge q\vee p\wedge r$. .... **[.2]** **[.3]** **[.6]** TAUT

$\Box\blacksquare$. .... QED

107.10 $\Box p\wedge q\vee p\wedge r\vdash p\wedge(q\vee r)$. .... LEM

$\Box$¶ .... DEM

107.11 $\Box\Box p\wedge q\vee p\wedge r$. .... **[.10]** ASM

107.12 $\Box\Box p\wedge q\vdash p\wedge(q\vee r)$. .... LEM

$\Box\Box$¶ .... DEM

107.13 $\Box\Box\Box p\wedge q$. .... **[.12]** ASM

107.14 $\Box\Box\Box p\wedge(q\vee r)$. .... **[.13]** TAUT

$\Box\Box\blacksquare$. .... QED

107.15 $\Box\Box p\wedge r\vdash p\wedge(q\vee r)$. .... LEM

$\Box\Box$¶ .... DEM

107.16 $\Box\Box\Box p\wedge r$. .... **[.15]** ASM

107.17 $\Box\Box\Box p\wedge(q\vee r)$. .... **[.16]** TAUT

$\Box\Box\blacksquare$. .... QED

$\Box\Box \boldsymbol{p} \wedge (\boldsymbol{q} \vee \boldsymbol{r})$. .... **[.11] [.12] [.15]** TAUT **107.18**
$\Box\blacksquare$. .... QED
$\Box \boldsymbol{p} \wedge (\boldsymbol{q} \vee \boldsymbol{r}) \Leftrightarrow \boldsymbol{p} \wedge \boldsymbol{q} \vee \boldsymbol{p} \wedge \boldsymbol{r}$. .... **[.1] [.10]** TAUT **107.19**
$\Box \forall \boldsymbol{p} \forall \boldsymbol{q} \forall \boldsymbol{r} \boldsymbol{p} \wedge (\boldsymbol{q} \vee \boldsymbol{r}) \Leftrightarrow \boldsymbol{p} \wedge \boldsymbol{q} \vee \boldsymbol{p} \wedge \boldsymbol{r}$. .... **[.19]** UGEN_3 **107.20**
$\blacksquare$. .... QED

Uses Axioms: **3, 4, 5**

$\forall \boldsymbol{p} \forall \boldsymbol{q} \forall \boldsymbol{r} (\boldsymbol{p} \vee \boldsymbol{q}) \wedge \boldsymbol{r} \Leftrightarrow \boldsymbol{p} \wedge \boldsymbol{r} \vee \boldsymbol{q} \wedge \boldsymbol{r}$. .... THEOREM *of Distributivity 2* **108**
¶ .... DEM
$\Box (\boldsymbol{p} \vee \boldsymbol{q}) \wedge \boldsymbol{r} \vdash \boldsymbol{p} \wedge \boldsymbol{r} \vee \boldsymbol{q} \wedge \boldsymbol{r}$. .... LEM **108.1**
$\Box$¶ .... DEM
$\Box\Box (\boldsymbol{p} \vee \boldsymbol{q}) \wedge \boldsymbol{r}$. .... **[.1]** ASM **108.2**
$\Box\Box \boldsymbol{p} \wedge \boldsymbol{r} \vee \boldsymbol{q} \wedge \boldsymbol{r}$. .... **[.2]** TAUT **108.3**
$\Box\blacksquare$. .... QED
$\Box \boldsymbol{p} \wedge \boldsymbol{r} \vee \boldsymbol{q} \wedge \boldsymbol{r} \vdash (\boldsymbol{p} \vee \boldsymbol{q}) \wedge \boldsymbol{r}$. .... LEM **108.4**
$\Box$¶ .... DEM
$\Box\Box \boldsymbol{p} \wedge \boldsymbol{r} \vee \boldsymbol{q} \wedge \boldsymbol{r}$. .... **[.4]** ASM **108.5**
$\Box\Box (\boldsymbol{p} \vee \boldsymbol{q}) \wedge \boldsymbol{r}$. .... **[.5]** TAUT **108.6**
$\Box\blacksquare$. .... QED
$\Box (\boldsymbol{p} \vee \boldsymbol{q}) \wedge \boldsymbol{r} \Leftrightarrow \boldsymbol{p} \wedge \boldsymbol{r} \vee \boldsymbol{q} \wedge \boldsymbol{r}$. .... **[.1] [.4]** TAUT **108.7**
$\Box \forall \boldsymbol{p} \forall \boldsymbol{q} \forall \boldsymbol{r} (\boldsymbol{p} \vee \boldsymbol{q}) \wedge \boldsymbol{r} \Leftrightarrow \boldsymbol{p} \wedge \boldsymbol{r} \vee \boldsymbol{q} \wedge \boldsymbol{r}$. .... **[.7]** UGEN_3 **108.8**
$\blacksquare$. .... QED

Uses Axioms: **3, 4, 5**

$\forall \boldsymbol{p} \forall \boldsymbol{q} \forall \boldsymbol{r} \boldsymbol{p} \vee (\boldsymbol{q} \wedge \boldsymbol{r}) \Leftrightarrow (\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\boldsymbol{p} \vee \boldsymbol{r})$. .... THEOREM *of Distributivity 3* **109**
¶ .... DEM
$\Box \boldsymbol{p} \vee (\boldsymbol{q} \wedge \boldsymbol{r}) \vdash (\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\boldsymbol{p} \vee \boldsymbol{r})$. .... LEM **109.1**
$\Box$¶ .... DEM
$\Box\Box \boldsymbol{p} \vee (\boldsymbol{q} \wedge \boldsymbol{r})$. .... **[.1]** ASM **109.2**
$\Box\Box (\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\boldsymbol{p} \vee \boldsymbol{r})$. .... **[.2]** TAUT **109.3**
$\Box\blacksquare$. .... QED
$\Box (\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\boldsymbol{p} \vee \boldsymbol{r}) \vdash \boldsymbol{p} \vee (\boldsymbol{q} \wedge \boldsymbol{r})$. .... LEM **109.4**
$\Box$¶ .... DEM
$\Box\Box (\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\boldsymbol{p} \vee \boldsymbol{r})$. .... **[.4]** ASM **109.5**
$\Box\Box \boldsymbol{p} \vee (\boldsymbol{q} \wedge \boldsymbol{r})$. .... **[.5]** TAUT **109.6**
$\Box\blacksquare$. .... QED
$\Box \boldsymbol{p} \vee (\boldsymbol{q} \wedge \boldsymbol{r}) \Leftrightarrow (\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\boldsymbol{p} \vee \boldsymbol{r})$. .... **[.1] [.4]** TAUT **109.7**
$\Box \forall \boldsymbol{p} \forall \boldsymbol{q} \forall \boldsymbol{r} \boldsymbol{p} \vee (\boldsymbol{q} \wedge \boldsymbol{r}) \Leftrightarrow (\boldsymbol{p} \vee \boldsymbol{q}) \wedge (\boldsymbol{p} \vee \boldsymbol{r})$. .... **[.7]** UGEN_3 **109.8**
$\blacksquare$. .... QED

Uses Axioms: **3, 4, 5**

$\forall \boldsymbol{p} \forall \boldsymbol{q} \forall \boldsymbol{r} (\boldsymbol{p} \wedge \boldsymbol{q}) \vee \boldsymbol{r} \Leftrightarrow (\boldsymbol{p} \vee \boldsymbol{r}) \wedge (\boldsymbol{q} \vee \boldsymbol{r})$. .... THEOREM *of Distributivity 4* **110**

¶ .... DEM
110.1 □$(\boldsymbol{p}\wedge\boldsymbol{q})\vee\boldsymbol{r}\vdash(\boldsymbol{p}\vee\boldsymbol{r})\wedge(\boldsymbol{q}\vee\boldsymbol{r})$. .... LEM
□¶ .... DEM
110.2 □□$(\boldsymbol{p}\wedge\boldsymbol{q})\vee\boldsymbol{r}$. .... **[.1]** ASM
110.3 □□$(\boldsymbol{p}\vee\boldsymbol{r})\wedge(\boldsymbol{q}\vee\boldsymbol{r})$. .... **[.2]** TAUT
□■. .... QED
110.4 □$(\boldsymbol{p}\vee\boldsymbol{r})\wedge(\boldsymbol{q}\vee\boldsymbol{r})\vdash(\boldsymbol{p}\wedge\boldsymbol{q})\vee\boldsymbol{r}$. .... LEM
□¶ .... DEM
110.5 □□$(\boldsymbol{p}\vee\boldsymbol{r})\wedge(\boldsymbol{q}\vee\boldsymbol{r})$. .... **[.4]** ASM
110.6 □□$(\boldsymbol{p}\wedge\boldsymbol{q})\vee\boldsymbol{r}$. .... **[.5]** TAUT
□■. .... QED
110.7 □$(\boldsymbol{p}\wedge\boldsymbol{q})\vee\boldsymbol{r}\Leftrightarrow(\boldsymbol{p}\vee\boldsymbol{r})\wedge(\boldsymbol{q}\vee\boldsymbol{r})$. .... **[.1]** **[.4]** TAUT
110.8 □$\forall\boldsymbol{p}\forall\boldsymbol{q}\forall\boldsymbol{r}(\boldsymbol{p}\wedge\boldsymbol{q})\vee\boldsymbol{r}\Leftrightarrow(\boldsymbol{p}\vee\boldsymbol{r})\wedge(\boldsymbol{q}\vee\boldsymbol{r})$. .... **[.7]** UGEN$_3$
■. .... QED

Uses Axioms: **3, 4, 5**

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111 $\forall\boldsymbol{p}\forall\boldsymbol{q}\neg(\boldsymbol{p}\vee\boldsymbol{q})\Leftrightarrow\neg\boldsymbol{p}\wedge\neg\boldsymbol{q}$. .... THEOREM *of De Morgan 1*
¶ .... DEM
111.1 □$\neg(\boldsymbol{p}\vee\boldsymbol{q})\vdash\neg\boldsymbol{p}\wedge\neg\boldsymbol{q}$. .... LEM
□¶ .... DEM
111.2 □□$\neg(\boldsymbol{p}\vee\boldsymbol{q})$. .... **[.1]** ASM
111.3 □□$\boldsymbol{p}\vdash\forall\boldsymbol{x}\boldsymbol{x}$. .... LEM
□□¶ .... DEM
111.4 □□□$\boldsymbol{p}$. .... **[.3]** ASM
111.5 □□□$\forall\boldsymbol{x}\boldsymbol{x}$. .... **[.2]** **[.4]** TAUT
□□■. .... QED
111.6 □□$\neg\boldsymbol{p}$. .... **[.3]** TAUT
111.7 □□$\boldsymbol{q}\vdash\forall\boldsymbol{x}\boldsymbol{x}$. .... LEM
□□¶ .... DEM
111.8 □□□$\boldsymbol{q}$. .... **[.7]** ASM
111.9 □□□$\forall\boldsymbol{x}\boldsymbol{x}$. .... **[.2]** **[.8]** TAUT
□□■. .... QED
111.10 □□$\neg\boldsymbol{p}\wedge\neg\boldsymbol{q}$. .... **[.6]** **[.7]** TAUT
□■. .... QED
111.11 □$\neg\boldsymbol{p}\wedge\neg\boldsymbol{q}\vdash\neg(\boldsymbol{p}\vee\boldsymbol{q})$. .... LEM
□¶ .... DEM
111.12 □□$\neg\boldsymbol{p}\wedge\neg\boldsymbol{q}$. .... **[.11]** ASM
111.13 □□$\boldsymbol{p}\vee\boldsymbol{q}\vdash\forall\boldsymbol{x}\boldsymbol{x}$. .... LEM
□□¶ .... DEM
111.14 □□□$\boldsymbol{p}\vee\boldsymbol{q}$. .... **[.13]** ASM
111.15 □□□$\forall\boldsymbol{x}\boldsymbol{x}$. .... **[.12]** **[.13]** TAUT
□□■. .... QED
111.16 □□$\neg(\boldsymbol{p}\vee\boldsymbol{q})$. .... **[.13]** TAUT
□■. .... QED

$\square\neg(\boldsymbol{p}\lor\boldsymbol{q})\Leftrightarrow\neg\boldsymbol{p}\land\neg\boldsymbol{q}$. …… **[.1] [.11]** TAUT **111.17**
$\square\forall\boldsymbol{p}\forall\boldsymbol{q}\neg(\boldsymbol{p}\lor\boldsymbol{q})\Leftrightarrow\neg\boldsymbol{p}\land\neg\boldsymbol{q}$. …… **[.17]** UGEN$_2$ **111.18**
■. …… QED
Uses Axioms: **1, 3, 4, 5**

$\forall\boldsymbol{p}\forall\boldsymbol{q}\neg\boldsymbol{p}\lor\neg\boldsymbol{q}\vdash\neg(\boldsymbol{p}\land\boldsymbol{q})$. …… THEOREM *of De Morgan 2* **112**
¶ …… DEM
$\square\neg\boldsymbol{p}\lor\neg\boldsymbol{q}\vdash\neg(\boldsymbol{p}\land\boldsymbol{q})$. …… LEM **112.1**
□¶ …… DEM
$\square\square\neg\boldsymbol{p}\lor\neg\boldsymbol{q}$. …… **[.1]** ASM **112.2**
$\square\square\boldsymbol{p}\land\boldsymbol{q}\vdash\forall\boldsymbol{x}\boldsymbol{x}$. …… LEM **112.3**
□□¶ …… DEM
$\square\square\square\boldsymbol{p}\land\boldsymbol{q}$. …… **[.3]** ASM **112.4**
$\square\square\square\neg\boldsymbol{p}\vdash\forall\boldsymbol{x}\boldsymbol{x}$. …… LEM **112.5**
□□□¶ …… DEM
$\square\square\square\square\neg\boldsymbol{p}$. …… **[.5]** ASM **112.6**
$\square\square\square\square\forall\boldsymbol{x}\boldsymbol{x}$. …… **[.4] [.6]** TAUT **112.7**
□□□■. …… QED
$\square\square\square\neg\boldsymbol{q}\vdash\forall\boldsymbol{x}\boldsymbol{x}$. …… LEM **112.8**
□□□¶ …… DEM
$\square\square\square\square\neg\boldsymbol{q}$. …… **[.8]** ASM **112.9**
$\square\square\square\square\forall\boldsymbol{x}\boldsymbol{x}$. …… **[.4] [.9]** TAUT **112.10**
□□□■. …… QED
$\square\square\square\forall\boldsymbol{x}\boldsymbol{x}$. …… **[.2] [.5] [.8]** TAUT **112.11**
□□■. …… QED
$\square\square\neg(\boldsymbol{p}\land\boldsymbol{q})$. …… **[.3]** TAUT **112.12**
□■. …… QED
$\square\forall\boldsymbol{p}\forall\boldsymbol{q}\neg\boldsymbol{p}\lor\neg\boldsymbol{q}\vdash\neg(\boldsymbol{p}\land\boldsymbol{q})$. …… **[.1]** UGEN$_2$ **112.13**
■. …… QED
Uses Axioms: **1, 3, 4**

$\forall\boldsymbol{p}\forall\boldsymbol{q}\neg(\boldsymbol{p}\land\boldsymbol{q})\vdash\neg\boldsymbol{p}\lor\neg\boldsymbol{q}$. …… THEOREM *of De Morgan 3* **113**
¶ …… DEM
$\square\neg(\boldsymbol{p}\land\boldsymbol{q})\vdash\neg\boldsymbol{p}\lor\neg\boldsymbol{q}$. …… LEM **113.1**
□¶ …… DEM
$\square\square\neg(\boldsymbol{p}\land\boldsymbol{q})$. …… **[.1]** ASM **113.2**
$\square\square\neg(\neg\boldsymbol{p}\lor\neg\boldsymbol{q})\vdash\forall\boldsymbol{x}\boldsymbol{x}$. …… LEM **113.3**
□□¶ …… DEM
$\square\square\square\neg(\neg\boldsymbol{p}\lor\neg\boldsymbol{q})$. …… **[.3]** ASM **113.4**
$\square\square\square\neg\neg\boldsymbol{p}\land\neg\neg\boldsymbol{q}$. …… **[.4]** TAUT **113.5**
$\square\square\square\forall\boldsymbol{p}\neg\neg\boldsymbol{p}\vdash\boldsymbol{p}$. …… THM **113.6**
$\square\square\square\forall\boldsymbol{x}\boldsymbol{x}$. …… **[.6] [.5] [.2]** TAUT **113.7**
□□■. …… QED

113.8 $\Box\Box\neg\neg(\neg \boldsymbol{p} \vee \neg \boldsymbol{q})$. .... **[.3]** TAUT
113.9 $\Box\Box\forall \boldsymbol{p}\neg\neg \boldsymbol{p} \vdash \boldsymbol{p}$. .... THM
113.10 $\Box\Box\neg \boldsymbol{p} \vee \neg \boldsymbol{q}$. .... **[.9] [.8]** TAUT
$\Box\blacksquare$. .... QED
113.11 $\Box\forall \boldsymbol{p}\forall \boldsymbol{q}\neg(\boldsymbol{p} \wedge \boldsymbol{q}) \vdash \neg \boldsymbol{p} \vee \neg \boldsymbol{q}$. .... **[.1]** UGEN$_2$
$\blacksquare$. .... QED
Uses Axioms: **1, 3, 4, 22**

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114 $\forall \boldsymbol{p}\forall \boldsymbol{q}\boldsymbol{p} \wedge \boldsymbol{q} \vee \neg \boldsymbol{p} \wedge \boldsymbol{q} \vee \boldsymbol{p} \wedge \neg \boldsymbol{q} \vee \neg \boldsymbol{p} \wedge \neg \boldsymbol{q}$. .... THEOREM *of Propositional Variety 1*
¶ .... DEM
114.1 $\Box\forall \boldsymbol{p}\boldsymbol{p} \vee \neg \boldsymbol{p}$. .... LAW
114.2 $\Box \boldsymbol{p} \wedge \boldsymbol{q} \vee \neg(\boldsymbol{p} \wedge \boldsymbol{q})$. .... **[.1]** $\forall_1$INST
114.3 $\Box \boldsymbol{p} \wedge \boldsymbol{q} \vee \neg \boldsymbol{p} \vee \neg \boldsymbol{q}$. .... **[.2]** TAUT
114.4 $\Box \boldsymbol{q} \vee \neg \boldsymbol{q}$. .... **[.1]** $\forall_1$INST
114.5 $\Box \boldsymbol{p} \wedge \boldsymbol{q} \vee \neg \boldsymbol{p} \wedge (\boldsymbol{q} \vee \neg \boldsymbol{q}) \vee \neg \boldsymbol{q}$. .... **[.4] [.3]** TAUT
114.6 $\Box \boldsymbol{p} \vee \neg \boldsymbol{p}$. .... **[.1]** $\forall_1$INST
114.7 $\Box \boldsymbol{p} \wedge \boldsymbol{q} \vee \neg \boldsymbol{p} \wedge \boldsymbol{q} \vee \boldsymbol{p} \wedge \neg \boldsymbol{q} \vee \neg \boldsymbol{p} \wedge \neg \boldsymbol{q}$. .... **[.6] [.5]** TAUT
114.8 $\Box\forall \boldsymbol{p}\forall \boldsymbol{q}\boldsymbol{p} \wedge \boldsymbol{q} \vee \neg \boldsymbol{p} \wedge \boldsymbol{q} \vee \boldsymbol{p} \wedge \neg \boldsymbol{q} \vee \neg \boldsymbol{p} \wedge \neg \boldsymbol{q}$. .... **[.1]** UGEN$_2$
$\blacksquare$. .... QED
Uses Axioms: **1, 3, 4, 22**

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115 $\forall \boldsymbol{p}\forall \boldsymbol{q}\forall \boldsymbol{r}\boldsymbol{p} \wedge \boldsymbol{q} \wedge \boldsymbol{r} \vee \neg \boldsymbol{p} \wedge \boldsymbol{q} \wedge \boldsymbol{r} \vee \boldsymbol{p} \wedge \neg \boldsymbol{q} \wedge \boldsymbol{r} \vee \boldsymbol{p} \wedge \boldsymbol{q} \wedge \neg \boldsymbol{r} \vee \neg \boldsymbol{p} \wedge \neg \boldsymbol{q} \wedge \boldsymbol{r} \vee \neg \boldsymbol{p} \wedge \boldsymbol{q} \wedge \neg \boldsymbol{r} \vee \boldsymbol{p} \wedge \neg \boldsymbol{q} \wedge \neg \boldsymbol{r} \vee \neg \boldsymbol{p}$
$\wedge \neg \boldsymbol{q} \wedge \neg \boldsymbol{r}$. .... THEOREM *of Propositional Variety 2*
¶ .... DEM
115.1 $\Box\forall \boldsymbol{p}\boldsymbol{p} \vee \neg \boldsymbol{p}$. .... LAW
115.2 $\Box \boldsymbol{r} \vee \neg \boldsymbol{r}$. .... **[.1]** $\forall_1$INST
115.3 $\Box \boldsymbol{p} \wedge \boldsymbol{q} \vee \neg \boldsymbol{p} \wedge \boldsymbol{q} \vee \boldsymbol{p} \wedge \neg \boldsymbol{q} \vee \neg \boldsymbol{p} \wedge \neg \boldsymbol{q}$. .... TAUT
115.4 $\Box \boldsymbol{p} \wedge \boldsymbol{q} \vdash \boldsymbol{p} \wedge \boldsymbol{q} \wedge \boldsymbol{r} \vee \boldsymbol{p} \wedge \boldsymbol{q} \wedge \neg \boldsymbol{r}$. .... LEM
$\Box$¶ .... DEM
115.5 $\Box\Box \boldsymbol{p} \wedge \boldsymbol{q}$. .... **[.4]** ASM
115.6 $\Box\Box \boldsymbol{p} \wedge \boldsymbol{q} \wedge \boldsymbol{r} \vee \boldsymbol{p} \wedge \boldsymbol{q} \wedge \neg \boldsymbol{r}$. .... **[.5] [.2]** TAUT
$\Box\blacksquare$. .... QED
115.7 $\Box\neg \boldsymbol{p} \wedge \boldsymbol{q} \vdash \neg \boldsymbol{p} \wedge \boldsymbol{q} \wedge \boldsymbol{r} \vee \neg \boldsymbol{p} \wedge \boldsymbol{q} \wedge \neg \boldsymbol{r}$. .... LEM
$\Box$¶ .... DEM
115.8 $\Box\Box\neg \boldsymbol{p} \wedge \boldsymbol{q}$. .... **[.7]** ASM
115.9 $\Box\Box\neg \boldsymbol{p} \wedge \boldsymbol{q} \wedge \boldsymbol{r} \vee \neg \boldsymbol{p} \wedge \boldsymbol{q} \wedge \neg \boldsymbol{r}$. .... **[.8] [.2]** TAUT
$\Box\blacksquare$. .... QED
115.10 $\Box \boldsymbol{p} \wedge \neg \boldsymbol{q} \vdash \boldsymbol{p} \wedge \neg \boldsymbol{q} \wedge \boldsymbol{r} \vee \boldsymbol{p} \wedge \neg \boldsymbol{q} \wedge \neg \boldsymbol{r}$. .... LEM
$\Box$¶ .... DEM
115.11 $\Box\Box \boldsymbol{p} \wedge \neg \boldsymbol{q}$. .... **[.10]** ASM
115.12 $\Box\Box \boldsymbol{p} \wedge \neg \boldsymbol{q} \wedge \boldsymbol{r} \vee \boldsymbol{p} \wedge \neg \boldsymbol{q} \wedge \neg \boldsymbol{r}$. .... **[.11] [.2]** TAUT
$\Box\blacksquare$. .... QED
115.13 $\Box\neg \boldsymbol{p} \wedge \neg \boldsymbol{q} \vdash \neg \boldsymbol{p} \wedge \neg \boldsymbol{q} \wedge \boldsymbol{r} \vee \neg \boldsymbol{p} \wedge \neg \boldsymbol{q} \wedge \neg \boldsymbol{r}$. .... LEM

□¶ .... DEM

□□$\neg p \wedge \neg q$. .... [.13] ASM **115.14**

□□$\neg p \wedge \neg q \wedge r \vee \neg p \wedge \neg q \wedge \neg r$. .... [.14] [.2] TAUT **115.15**

□■. .... QED

□$p \wedge q \wedge r \vee \neg p \wedge q \wedge r \vee p \wedge \neg q \wedge r \vee p \wedge q \wedge \neg r \vee \neg p \wedge \neg q \wedge r \vee \neg p \wedge q \wedge \neg r \vee p \wedge \neg q \wedge \neg r \vee \neg p \wedge \neg q \wedge$ **115.16**
$\neg r$. .... [.3] [.4] [.7] [.10] [.13] TAUT

□$\forall p \forall q \forall r p \wedge q \wedge r \vee \neg p \wedge q \wedge r \vee p \wedge \neg q \wedge r \vee p \wedge q \wedge \neg r \vee \neg p \wedge \neg q \wedge r \vee \neg p \wedge q \wedge \neg r \vee p \wedge \neg q \wedge \neg r \vee$ **115.17**
$\neg p \wedge \neg q \wedge \neg r$. .... [.16] UGEN$_3$

■. .... QED

Uses Axioms: **1, 3, 4, 22**

$\forall p \forall q \forall r p \Leftrightarrow q \vee q \Leftrightarrow r \vee r \Leftrightarrow p$. .... THEOREM *of Classical Propositional Luck* **116**

¶ .... DEM

□$p \wedge q \wedge r \vdash p \Leftrightarrow q$. .... LEM **116.1**

□¶ .... DEM

□□$p \wedge q \wedge r$. .... [.1] ASM **116.2**

□□$p \Leftrightarrow q$. .... [.2] TAUT **116.3**

□■. .... QED

□$\neg p \wedge q \wedge r \vdash q \Leftrightarrow r$. .... LEM **116.4**

□¶ .... DEM

□□$\neg p \wedge q \wedge r$. .... [.4] ASM **116.5**

□□$q \Leftrightarrow r$. .... [.5] TAUT **116.6**

□■. .... QED

□$p \wedge \neg q \wedge r \vdash p \Leftrightarrow r$. .... LEM **116.7**

□¶ .... DEM

□□$p \wedge \neg q \wedge r$. .... [.7] ASM **116.8**

□□$p \Leftrightarrow r$. .... [.8] TAUT **116.9**

□■. .... QED

□$p \wedge q \wedge \neg r \vdash p \Leftrightarrow q$. .... LEM **116.10**

□¶ .... DEM

□□$p \wedge q \wedge \neg r$. .... [.10] ASM **116.11**

□□$p \Leftrightarrow q$. .... [.11] TAUT **116.12**

□■. .... QED

□$\neg p \wedge \neg q \wedge r \vdash p \Leftrightarrow q$. .... LEM **116.13**

□¶ .... DEM

□□$\neg p \wedge \neg q \wedge r$. .... [.13] ASM **116.14**

□□$p \Leftrightarrow q$. .... [.14] TAUT **116.15**

□■. .... QED

□$\neg p \wedge q \wedge \neg r \vdash p \Leftrightarrow r$. .... LEM **116.16**

□¶ .... DEM

□□$\neg p \wedge q \wedge \neg r$. .... [.16] ASM **116.17**

□□$p \Leftrightarrow r$. .... [.17] TAUT **116.18**

□■. .... QED

□$p \wedge \neg q \wedge \neg r \vdash q \Leftrightarrow r$. .... LEM **116.19**

$\square\P$ .... DEM

116.20 $\square\square \boldsymbol{p} \wedge \neg \boldsymbol{q} \wedge \neg \boldsymbol{r}$. .... [.19] ASM

116.21 $\square\square \boldsymbol{q} \Leftrightarrow \boldsymbol{r}$. .... [.20] TAUT

$\square\blacksquare$. .... QED

116.22 $\square \neg \boldsymbol{p} \wedge \neg \boldsymbol{q} \wedge \neg \boldsymbol{r} \vdash \boldsymbol{p} \Leftrightarrow \boldsymbol{q}$. .... LEM

$\square\P$ .... DEM

116.23 $\square\square \neg \boldsymbol{p} \wedge \neg \boldsymbol{q} \wedge \neg \boldsymbol{r}$. .... [.22] ASM

116.24 $\square\square \boldsymbol{p} \Leftrightarrow \boldsymbol{q}$. .... [.23] TAUT

$\square\blacksquare$. .... QED

116.25 $\square \boldsymbol{p} \Leftrightarrow \boldsymbol{q} \vee \boldsymbol{q} \Leftrightarrow \boldsymbol{r} \vee \boldsymbol{r} \Leftrightarrow \boldsymbol{p}$. .... [.1] [.4] [.7] [.10] [.13] [.16] [.19] [.22] TAUT

116.26 $\square \forall \boldsymbol{p} \forall \boldsymbol{q} \forall \boldsymbol{r} \boldsymbol{p} \Leftrightarrow \boldsymbol{q} \vee \boldsymbol{q} \Leftrightarrow \boldsymbol{r} \vee \boldsymbol{r} \Leftrightarrow \boldsymbol{p}$. .... [.25] UGEN$_2$

$\blacksquare$. .... QED

Uses Axioms: **1, 3, 4, 5, 22**

The negation of the universal contour of $\boldsymbol{\psi x}$ gives the existential contour of the negation of $\boldsymbol{\psi x}$. PROOF: Assume $\neg\forall \boldsymbol{x \psi x}$ and suppose $\neg\exists \boldsymbol{x} \neg \boldsymbol{\psi x}$. But this gives $\forall \boldsymbol{x} \neg\neg \boldsymbol{\psi x}$ which *classically* implies $\forall \boldsymbol{x \psi x}$, thus giving a contradiction. Hence, by *excluded middle*, $\exists \boldsymbol{x} \neg \boldsymbol{\psi x}$. QED.

117 $\forall \boldsymbol{\psi} (\neg \forall \boldsymbol{x \psi x}) \vdash \exists \boldsymbol{x} \neg \boldsymbol{\psi x}$. .... THEOREM *of the Negative Contour 4*

$\P$ .... DEM

117.1 $\square (\neg \forall \boldsymbol{x \psi x}) \vdash \exists \boldsymbol{x} \neg \boldsymbol{\psi x}$. .... LEM

$\square\P$ .... DEM

117.2 $\square\square \neg \forall \boldsymbol{x \psi x}$. .... [.1] ASM

117.3 $\square\square \neg \exists \boldsymbol{x} \neg \boldsymbol{\psi x} \vdash \forall \boldsymbol{xx}$. .... LEM

$\square\square\P$ .... DEM

117.4 $\square\square\square \neg \exists \boldsymbol{x} \neg \boldsymbol{\psi x}$. .... [.3] ASM

117.5 $\square\square\square \forall \boldsymbol{\psi} (\neg \exists \boldsymbol{x \psi x}) \vdash \forall \boldsymbol{x} \neg \boldsymbol{\psi x}$. .... THM

117.6 $\square\square\square (\neg \exists \boldsymbol{x} \neg \boldsymbol{\psi x}) \vdash \forall \boldsymbol{x} \neg\neg \boldsymbol{\psi x}$. .... [.5] $\forall_2$INST

117.7 $\square\square\square \forall \boldsymbol{x} \neg\neg \boldsymbol{\psi x}$. .... [.6] [.4] MP

117.8 $\square\square\square \boldsymbol{\psi x}$. .... [.7] TAUT

117.9 $\square\square\square \forall \boldsymbol{x \psi x}$. .... [.8] $\forall_1$GEN

117.10 $\square\square\square \forall \boldsymbol{xx}$. .... [.2] [.9] TAUT

$\square\square\blacksquare$. .... QED

117.11 $\square\square \exists \boldsymbol{x} \neg \boldsymbol{\psi x}$. .... [.3] TAUT

$\square\blacksquare$. .... QED

117.12 $\square \forall \boldsymbol{\psi} (\neg \forall \boldsymbol{x \psi x}) \vdash \exists \boldsymbol{x} \neg \boldsymbol{\psi x}$. .... [.1] $\forall_2$GEN

$\blacksquare$. .... QED

Uses Axioms: **1, 4, 22**

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118 $\forall \boldsymbol{p} \forall \boldsymbol{q} ((\boldsymbol{p} \vdash \boldsymbol{q}) \vdash \boldsymbol{p}) \vdash \boldsymbol{p}$. .... THEOREM *of Pierce's Law*

$\P$ .... DEM

118.1 $\square ((\boldsymbol{p} \vdash \boldsymbol{q}) \vdash \boldsymbol{p}) \vdash \boldsymbol{p}$. .... LEM

$\square\P$ .... DEM

$\square\square(\boldsymbol{p}\vdash\boldsymbol{q})\vdash\boldsymbol{p}$. .... [.1] ASM **118.2**

$\square\square\neg\boldsymbol{p}\vee\boldsymbol{q}\vdash\boldsymbol{p}$. .... [.2] TAUT **118.3**

$\square\square\forall\boldsymbol{p}\boldsymbol{p}\vee\neg\boldsymbol{p}$. .... LAW **118.4**

$\square\square\boldsymbol{p}\vee\neg\boldsymbol{p}$. .... [.4] $\forall_1$INST **118.5**

$\square\square\boldsymbol{p}$. .... [.5] [.3] TAUT **118.6**

$\square\blacksquare$. .... QED

$\square\forall\boldsymbol{p}\forall\boldsymbol{q}((\boldsymbol{p}\vdash\boldsymbol{q})\vdash\boldsymbol{p})\vdash\boldsymbol{p}$. .... [.1] UGEN$_2$ **118.7**

$\blacksquare$. .... QED

Uses Axioms: **1, 4, 22**

$\forall\boldsymbol{p}\forall\boldsymbol{q}(\neg\boldsymbol{q}\vdash\neg\boldsymbol{p})\vdash\boldsymbol{p}\vdash\boldsymbol{q}$. .... THEOREM *of Contraposition* **119**

¶ .... DEM

$\square(\neg\boldsymbol{q}\vdash\neg\boldsymbol{p})\vdash\boldsymbol{p}\vdash\boldsymbol{q}$. .... LEM **119.1**

$\square$¶ .... DEM

$\square\square\neg\boldsymbol{q}\vdash\neg\boldsymbol{p}$. .... [.1] ASM **119.2**

$\square\square\boldsymbol{p}\vdash\boldsymbol{q}$. .... LEM **119.3**

$\square\square$¶ .... DEM

$\square\square\square\boldsymbol{p}$. .... [.3] ASM **119.4**

$\square\square\square\neg\boldsymbol{q}\vdash\forall\boldsymbol{x}\boldsymbol{x}$. .... LEM **119.5**

$\square\square\square$¶ .... DEM

$\square\square\square\square\neg\boldsymbol{q}$. .... [.5] ASM **119.6**

$\square\square\square\square\neg\boldsymbol{p}$. .... [.2] [.6] MP **119.7**

$\square\square\square\square\forall\boldsymbol{x}\boldsymbol{x}$. .... [.4] [.7] TAUT **119.8**

$\square\square\square\blacksquare$. .... QED

$\square\square\square\boldsymbol{q}$. .... [.5] TAUT **119.9**

$\square\square\blacksquare$. .... QED

$\square\blacksquare$. .... QED

$\square\forall\boldsymbol{p}\forall\boldsymbol{q}(\neg\boldsymbol{q}\vdash\neg\boldsymbol{p})\vdash\boldsymbol{p}\vdash\boldsymbol{q}$. .... [.1] UGEN$_2$ **119.10**

$\blacksquare$. .... QED

Uses Axioms: **1, 4, 22**

Operational Theory of Sets

The chief aim of type theory is to objectify not only sets, but their operations. One approach is to objectify operations as functions of type $\mathscr{H}\to\mathscr{H}$. If we were to stop here, however, we resign ourselves to Polish notation. Eg. $(\boldsymbol{x}\setminus\boldsymbol{y})\cup(\boldsymbol{y}\setminus\boldsymbol{x})$ becomes instead $(\cup)((\setminus)(\boldsymbol{x})(\boldsymbol{y}))((\setminus)(\boldsymbol{y})(\boldsymbol{x}))$.

We instead introduce second order variables which *realize* such functions. For second order citizens in TOB are precisely suited for notating functions and operations; the rule of inference of *variation* identifies objects such as $\setminus(\boldsymbol{x})(\boldsymbol{y})$ with $(\boldsymbol{x})\setminus(\boldsymbol{y})$.

In this chapter, we develop *realizers* for some fundamental set operations† and work out their various properties.

† There are some important exceptions to operational realizability, which arise due to TOB's incapability of expressing *third order statements*. Eg. Consider the difficulty of capturing *specification* via some realizer **REP**. For all sets $\boldsymbol{x}$ and formulas $\boldsymbol{\psi}$, we would have liked $\mathbf{REP}(\boldsymbol{x},\boldsymbol{\psi})$ to be the (unique) set $\boldsymbol{y}$ st. $\forall \boldsymbol{y_1 y_1}{:}\boldsymbol{y}\Leftrightarrow \boldsymbol{y_1}{:}\boldsymbol{x}\wedge\boldsymbol{\psi y_1}$. Notice that $\mathbf{REP}(\boldsymbol{x},\boldsymbol{\psi})$ fails to be an object– $\boldsymbol{\psi}$ is a second order variable and therefore cannot be the argument of a realizer.

There are some natural workarounds, however. Supposing there is a notion of *classes*, typed as $\mathscr{C}$. We imagine adding mirrored axioms for class construction, as well as an additional axiom $\mathscr{H}{:}\mathscr{C}$. In particular, it is possible to show that for all formulas $\boldsymbol{\psi}$, their exists a (unique) class $\boldsymbol{A}$ st. $\forall \boldsymbol{aa}{:}\boldsymbol{A}\Leftrightarrow\boldsymbol{a}{:}\mathscr{H}\wedge\boldsymbol{\psi a}$. Formulas of sets can equivalently be seen by their *extension*, ie. the class which captures it. Thus, second order formulas of sets can be captured by first order objects of classes.

§2.1 *Sets and Operations: First Principles*

If for all $\boldsymbol{x}{:}\boldsymbol{a}$ one can exhibit a unique $\boldsymbol{y}{:}\boldsymbol{b}$ which satisfies $\boldsymbol{\psi xy}$, then their exists a unique (type) function $\boldsymbol{f}{:}\boldsymbol{a}\to\boldsymbol{b}$ st. $\boldsymbol{fx}=\boldsymbol{y}$ precisely whenever $\boldsymbol{x}{:}\boldsymbol{a}$, $\boldsymbol{y}{:}\boldsymbol{b}$, satisfy $\boldsymbol{\psi xy}$. PROOF: *Existence.* Guaranteed by *creation*. *Uniqueness.* Guaranteed by *functional extension*. QED.

120 $\forall\boldsymbol{a}\forall\boldsymbol{b}\forall\boldsymbol{\psi}(\forall\boldsymbol{xx}{:}\boldsymbol{a}\vdash\exists!\boldsymbol{yy}{:}\boldsymbol{b}\wedge\boldsymbol{\psi xy})\vdash\exists!\boldsymbol{ff}{:}\boldsymbol{a}\to\boldsymbol{b}\wedge\forall\boldsymbol{x}\forall\boldsymbol{yx}{:}\boldsymbol{a}\vdash\boldsymbol{y}{:}\boldsymbol{b}\vdash\boldsymbol{\psi xy}\Leftrightarrow\boldsymbol{fx}=\boldsymbol{y}$. .... THEOREM *of Unary Creation*

¶ .... DEM

120.1 $\square(\forall\boldsymbol{xx}{:}\boldsymbol{a}\vdash\exists!\boldsymbol{yy}{:}\boldsymbol{b}\wedge\boldsymbol{\psi xy})\vdash\exists!\boldsymbol{ff}{:}\boldsymbol{a}\to\boldsymbol{b}\wedge\forall\boldsymbol{x}\forall\boldsymbol{yx}{:}\boldsymbol{a}\vdash\boldsymbol{y}{:}\boldsymbol{b}\vdash\boldsymbol{\psi xy}\Leftrightarrow\boldsymbol{fx}=\boldsymbol{y}$. .... LEM

$\square$¶ .... DEM

120.2 $\square\square\forall\boldsymbol{xx}{:}\boldsymbol{a}\vdash\exists!\boldsymbol{yy}{:}\boldsymbol{b}\wedge\boldsymbol{\psi xy}$. .... [.1] ASM

120.3 $\square\square\forall\boldsymbol{a}\forall\boldsymbol{b}\forall\boldsymbol{\psi}(\forall\boldsymbol{xx}{:}\boldsymbol{a}\vdash\exists!\boldsymbol{yy}{:}\boldsymbol{b}\wedge\boldsymbol{\psi xy})\vdash\exists\boldsymbol{ff}{:}\boldsymbol{a}\to\boldsymbol{b}\wedge\forall\boldsymbol{x}\forall\boldsymbol{yx}{:}\boldsymbol{a}\vdash\boldsymbol{y}{:}\boldsymbol{b}\vdash\boldsymbol{\psi xy}\Leftrightarrow\boldsymbol{fx}=\boldsymbol{y}$. .... AXM

120.4 $\square\square\forall\boldsymbol{\psi}(\forall\boldsymbol{xx}{:}\boldsymbol{a}\vdash\exists!\boldsymbol{yy}{:}\boldsymbol{b}\wedge\boldsymbol{\psi xy})\vdash\exists\boldsymbol{ff}{:}\boldsymbol{a}\to\boldsymbol{b}\wedge\forall\boldsymbol{x}\forall\boldsymbol{yx}{:}\boldsymbol{a}\vdash\boldsymbol{y}{:}\boldsymbol{b}\vdash\boldsymbol{\psi xy}\Leftrightarrow\boldsymbol{fx}=\boldsymbol{y}$. .... [.3] ${}_0$MINT${}_2$

120.5 $\square\square(\forall\boldsymbol{xx}{:}\boldsymbol{a}\vdash\exists!\boldsymbol{yy}{:}\boldsymbol{b}\wedge\boldsymbol{\psi xy})\vdash\exists\boldsymbol{ff}{:}\boldsymbol{a}\to\boldsymbol{b}\wedge\forall\boldsymbol{x}\forall\boldsymbol{yx}{:}\boldsymbol{a}\vdash\boldsymbol{y}{:}\boldsymbol{b}\vdash\boldsymbol{\psi xy}\Leftrightarrow\boldsymbol{fx}=\boldsymbol{y}$. .... [.4] $\forall_2$INST

120.6 $\square\square\exists\boldsymbol{ff}{:}\boldsymbol{a}\to\boldsymbol{b}\wedge\forall\boldsymbol{x}\forall\boldsymbol{yx}{:}\boldsymbol{a}\vdash\boldsymbol{y}{:}\boldsymbol{b}\vdash\boldsymbol{\psi xy}\Leftrightarrow\boldsymbol{fx}=\boldsymbol{y}$. .... [.5] [.2] MP

120.7 $\square\square(\boldsymbol{f_1}{:}\boldsymbol{a}\to\boldsymbol{b}\wedge\forall\boldsymbol{x}\forall\boldsymbol{yx}{:}\boldsymbol{a}\vdash\boldsymbol{y}{:}\boldsymbol{b}\vdash\boldsymbol{\psi xy}\Leftrightarrow\boldsymbol{f_1x}=\boldsymbol{y})\vdash(\boldsymbol{f_2}{:}\boldsymbol{a}\to\boldsymbol{b}\wedge\forall\boldsymbol{x}\forall\boldsymbol{yx}{:}\boldsymbol{a}\vdash\boldsymbol{y}{:}\boldsymbol{b}\vdash\boldsymbol{\psi xy}\Leftrightarrow\boldsymbol{f_2x}=\boldsymbol{y})\vdash\boldsymbol{f_1}=\boldsymbol{f_2}$. .... LEM

□□¶ .... DEM

□□□$\boldsymbol{f_1{:}a\to b\wedge\forall x\forall yx{:}a\vdash y{:}b\vdash\psi xy\Leftrightarrow f_1x=y}$. .... **[.7]** ASM **120.8**

□□□$\boldsymbol{(f_2{:}a\to b\wedge\forall x\forall yx{:}a\vdash y{:}b\vdash\psi xy\Leftrightarrow f_2x=y)\vdash f_1=f_2}$. .... LEM **120.9**

□□□¶ .... DEM

□□□□$\boldsymbol{f_2{:}a\to b\wedge\forall x\forall yx{:}a\vdash y{:}b\vdash\psi xy\Leftrightarrow f_2x=y}$. .... **[.9]** ASM **120.10**

□□□□$\boldsymbol{\forall a\forall b\forall f\forall gf{:}a\to b\vdash g{:}a\to b\vdash(\forall xx{:}a\vdash fx=gx)\vdash f=g}$. .... AXM **120.11**

□□□□$\boldsymbol{f_1{:}a\to b\vdash f_2{:}a\to b\vdash(\forall xx{:}a\vdash f_1x=f_2x)\vdash f_1=f_2}$. .... **[.11]** $_0$MINT$_4$ **120.12**

□□□□$\boldsymbol{x{:}a\vdash f_1x=f_2x}$. .... LEM **120.13**

□□□□¶ .... DEM

□□□□□$\boldsymbol{x{:}a}$. .... **[.13]** ASM **120.14**

□□□□□$\boldsymbol{\exists!yy{:}b\wedge\psi xy}$. .... **[.2] [.14]** $_1$MINT$_1$ **120.15**

□□□□□$\boldsymbol{\exists yy{:}b\wedge\psi xy}$. .... **[.15]** ∃!EXISTS **120.16**

□□□□□$\boldsymbol{y{:}b\wedge\psi xy}$. .... **[.16]** $\exists_1$INST **120.17**

□□□□□$\boldsymbol{\forall x\forall yx{:}a\vdash y{:}b\vdash\psi xy\Leftrightarrow f_1x=y}$. .... **[.8]** TAUT **120.18**

□□□□□$\boldsymbol{x{:}a\vdash y{:}b\vdash\psi xy\Leftrightarrow f_1x=y}$. .... **[.18]** $_0$MINT$_2$ **120.19**

□□□□□$\boldsymbol{f_1x=y}$. .... **[.19] [.14] [.17]** TAUT **120.20**

□□□□□$\boldsymbol{\forall x\forall yx{:}a\vdash y{:}b\vdash\psi xy\Leftrightarrow f_2x=y}$. .... **[.10]** TAUT **120.21**

□□□□□$\boldsymbol{x{:}a\vdash y{:}b\vdash\psi xy\Leftrightarrow f_2x=y}$. .... **[.21]** $_0$MINT$_2$ **120.22**

□□□□□$\boldsymbol{f_2x=y}$. .... **[.22] [.14] [.17]** TAUT **120.23**

□□□□□$\boldsymbol{f_1x=f_2x}$. .... **[.23] [.20]** SUB **120.24**

□□□□■ .... QED

□□□□$\boldsymbol{\forall xx{:}a\vdash f_1x=f_2x}$. .... **[.13]** $\forall_1$GEN **120.25**

□□□□$\boldsymbol{f_1=f_2}$. .... **[.12] [.8] [.10] [.25]** TAUT **120.26**

□□□■. .... QED

□□■. .... QED

□□$\boldsymbol{\forall f_1\forall f_2(f_1{:}a\to b\wedge\forall x\forall yx{:}a\vdash y{:}b\vdash\psi xy\Leftrightarrow f_1x=y)\vdash(f_2{:}a\to b\wedge\forall x\forall yx{:}a\vdash y{:}b\vdash\psi xy\Leftrightarrow f_2x=y)\vdash f_1=f_2}$. .... **[.7]** UGEN$_2$ **120.27**

□□$\boldsymbol{\exists!ff{:}a\to b\wedge\forall x\forall yx{:}a\vdash y{:}b\vdash\psi xy\Leftrightarrow fx=y}$. .... **[.6] [.27]** ∃!INTROS **120.28**

□■. .... QED

□$\boldsymbol{\forall\psi(\forall xx{:}a\vdash\exists!yy{:}b\wedge\psi xy)\vdash\exists!ff{:}a\to b\wedge\forall x\forall yx{:}a\vdash y{:}b\vdash\psi xy\Leftrightarrow fx=y}$. .... **[.1]** $\forall_2$GEN **120.29**

■. .... QED

Uses Axioms: **3, 5, 6, 8**

$\boldsymbol{\forall\psi(\forall xx{:}\mathscr{H}\vdash\exists!yy{:}\mathscr{H}\wedge\psi xy)\vdash\exists!ff{:}\mathscr{H}\to\mathscr{H}\wedge\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\psi xy\Leftrightarrow fx=y}$. **121**

.... THEOREM *of Unary Operation*

¶ .... DEM

□$\boldsymbol{\forall a\forall b\forall\psi(\forall xx{:}a\vdash\exists!yy{:}b\wedge\psi xy)\vdash\exists!ff{:}a\to b\wedge\forall x\forall yx{:}a\vdash y{:}b\vdash\psi xy\Leftrightarrow fx=y}$. .... THM **121.1**

□$\boldsymbol{\forall\psi(\forall xx{:}\mathscr{H}\vdash\exists!yy{:}\mathscr{H}\wedge\psi xy)\vdash\exists!ff{:}\mathscr{H}\to\mathscr{H}\wedge\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\psi xy\Leftrightarrow fx=y}$. . **[.1]** $_0$MINT$_2$ **121.2**

■. .... QED

Uses Axioms: **3, 5, 6, 8**

$\boldsymbol{\forall\psi(\forall xx{:}\mathscr{H}\vdash\exists!yy{:}\mathscr{H}\wedge\psi xy)\vdash\exists\phi\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\psi xy\Leftrightarrow\phi x=y}$. **122**

.... THEOREM *of Unary Realization*

¶ . . . . . . DEM

122.1 □$(\forall x x{:}\mathscr{H} \vdash \exists! y y{:}\mathscr{H} \wedge \psi xy) \vdash \exists\phi \forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow \phi x = y$. . . . . . . LEM

□¶ . . . . . . DEM

122.2 □□$\forall x x{:}\mathscr{H} \vdash \exists! y y{:}\mathscr{H} \wedge \psi xy$. . . . . . . [.1] ASM

122.3 □□$\forall \psi (\forall x x{:}\mathscr{H} \vdash \exists! y y{:}\mathscr{H} \wedge \psi xy) \vdash \exists! f f{:}\mathscr{H} \to \mathscr{H} \wedge \forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow f x = y$. . . . . . . THM

122.4 □□$(\forall x x{:}\mathscr{H} \vdash \exists! y y{:}\mathscr{H} \wedge \psi xy) \vdash \exists! f f{:}\mathscr{H} \to \mathscr{H} \wedge \forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow f x = y$. . . . [.3] $\forall_2$INST

122.5 □□$\exists! f f{:}\mathscr{H} \to \mathscr{H} \wedge \forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow f x = y$. . . . . . . [.4] [.2] MP

122.6 □□$\exists f f{:}\mathscr{H} \to \mathscr{H} \wedge \forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow f x = y$. . . . . . . [.5] ∃!EXISTS

122.7 □□$f{:}\mathscr{H} \to \mathscr{H} \wedge \forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow f x = y$. . . . . . . [.6] $\exists_1$INST

122.8 □□$\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow f x = y$. . . . . . . [.7] TAUT

122.9 □□$\exists \phi \forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow \phi x = y$. . . . . . . [.8] $\exists_2$GEN

□■. . . . . . . QED

122.10 □$\forall \psi (\forall x x{:}\mathscr{H} \vdash \exists! y y{:}\mathscr{H} \wedge \psi xy) \vdash \exists \phi \forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow \phi x = y$. . . . . . . [.1] $\forall_2$GEN

■. . . . . . . QED

Uses Axioms: **3, 5, 6, 8**

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123 $\forall \psi (\forall x x{:}\mathscr{H} \vdash \exists! y y{:}\mathscr{H} \wedge \psi xy) \vdash \forall \phi (\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow \phi x = y) \vdash \forall x x{:}\mathscr{H} \vdash \phi x{:}\mathscr{H} \wedge \psi x(\phi x)$. . . . . . . THEOREM *of Unary Realization Strength*

¶ . . . . . . DEM

123.1 □$(\forall x x{:}\mathscr{H} \vdash \exists! y y{:}\mathscr{H} \wedge \psi xy) \vdash \forall \phi (\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow \phi x = y) \vdash \forall x x{:}\mathscr{H} \vdash \phi x{:}\mathscr{H} \wedge \psi x(\phi x)$. . . . . . . LEM

□¶ . . . . . . DEM

123.2 □□$\forall x x{:}\mathscr{H} \vdash \exists! y y{:}\mathscr{H} \wedge \psi xy$. . . . . . . [.1] ASM

123.3 □□$(\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow \phi x = y) \vdash \forall x x{:}\mathscr{H} \vdash \phi x{:}\mathscr{H} \wedge \psi x(\phi x)$. . . . . . . LEM

□□¶ . . . . . . DEM

123.4 □□□$\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow \phi x = y$. . . . . . . [.3] ASM

123.5 □□□$\forall \psi (\forall x x{:}\mathscr{H} \vdash \exists! y y{:}\mathscr{H} \wedge \psi xy) \vdash \exists! f f{:}\mathscr{H} \to \mathscr{H} \wedge \forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow f x = y$. . . . . THM

123.6 □□□$(\forall x x{:}\mathscr{H} \vdash \exists! y y{:}\mathscr{H} \wedge \psi xy) \vdash \exists! f f{:}\mathscr{H} \to \mathscr{H} \wedge \forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow f x = y$. . [.5] $\forall_2$INST

123.7 □□□$\exists! f f{:}\mathscr{H} \to \mathscr{H} \wedge \forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow f x = y$. . . . . . . [.6] [.2] MP

123.8 □□□$\exists f f{:}\mathscr{H} \to \mathscr{H} \wedge \forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow f x = y$. . . . . . . [.7] ∃!EXISTS

123.9 □□□$f{:}\mathscr{H} \to \mathscr{H} \wedge \forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow f x = y$. . . . . . . [.8] $\exists_1$INST

123.10 □□□$x{:}\mathscr{H} \vdash \phi x{:}\mathscr{H} \wedge \psi x(\phi x)$. . . . . . . LEM

□□□¶ . . . . . . DEM

123.11 □□□□$x{:}\mathscr{H}$. . . . . . . [.10] ASM

123.12 □□□□$f{:}\mathscr{H} \to \mathscr{H}$. . . . . . . [.9] TAUT

123.13 □□□□$\forall a \forall b \forall f f{:}a \to b \vdash \forall x x{:}a \vdash f x{:}b$. . . . . . . AXM

123.14 □□□□$\forall x x{:}\mathscr{H} \vdash f x{:}\mathscr{H}$. . . . . . . [.13] [.12] ${}_1$MINT$_3$

123.15 □□□□$f x{:}\mathscr{H}$. . . . . . . [.14] [.11] ${}_1$MINT$_1$

123.16 □□□□$\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow f x = y$. . . . . . . [.9] TAUT

123.17 □□□□$\psi x(f x) \Leftrightarrow f x = f x$. . . . . . . [.16] [.11] [.15] ${}_2$MINT$_2$

123.18 □□□□$f x = f x$. . . . . . . EQ

123.19 □□□□$\psi x(f x)$. . . . . . . [.17] [.18] TAUT

123.20 □□□□$\psi x(f x) \Leftrightarrow \phi x = f x$. . . . . . . [.4] [.11] [.15] ${}_2$MINT$_2$

123.21 □□□□$\phi x = f x$. . . . . . . [.20] [.19] TAUT

$\square\square\square\square\boldsymbol{\phi x{:}\mathscr{H}}$. .... [.21] [.15] SUB **123.22**

$\square\square\square\square\boldsymbol{\psi x(\phi x)}$. .... [.21] [.19] SUB **123.23**

$\square\square\square\square\boldsymbol{\phi x{:}\mathscr{H}\wedge\psi x(\phi x)}$. .... [.22] [.23] TAUT **123.24**

$\square\square\square\blacksquare$. .... QED

$\square\square\square\forall\boldsymbol{xx{:}\mathscr{H}\vdash\phi x{:}\mathscr{H}\wedge\psi x(\phi x)}$. .... [.10] $\forall_1$GEN **123.25**

$\square\square\blacksquare$. .... QED

$\square\square\forall\boldsymbol{\phi}(\forall\boldsymbol{x}\forall\boldsymbol{yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\psi xy\Leftrightarrow\phi x=y})\vdash\forall\boldsymbol{xx{:}\mathscr{H}\vdash\phi x{:}\mathscr{H}\wedge\psi x(\phi x)}$. .... [.3] $\forall_2$GEN **123.26**

$\square\blacksquare$. .... QED

$\square\forall\boldsymbol{\psi}(\forall\boldsymbol{xx{:}\mathscr{H}}\vdash\exists!\boldsymbol{yy{:}\mathscr{H}\wedge\psi xy})\vdash\forall\boldsymbol{\phi}(\forall\boldsymbol{x}\forall\boldsymbol{yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\psi xy\Leftrightarrow\phi x=y})\vdash\forall\boldsymbol{xx{:}\mathscr{H}\vdash\phi x{:}\mathscr{H}}\wedge$ **123.27**
$\boldsymbol{\psi x(\phi x)}$. .... [.1] $\forall_2$GEN

$\blacksquare$. .... QED

Uses Axioms: **3, 5, 6, 7, 8**

If for all $\boldsymbol{x{:}a}$, $\boldsymbol{y{:}b}$, one can exhibit a unique $\boldsymbol{z{:}c}$ which satisfies $\boldsymbol{\psi xyz}$, then their exists a unique (type) function $\boldsymbol{f{:}a\to b\to c}$ st. $\boldsymbol{(fx)y=z}$ iff $\boldsymbol{\psi xyz}$ for each $\boldsymbol{x{:}a}$, $\boldsymbol{y{:}b}$, $\boldsymbol{z{:}c}$. PROOF: *Existence.* Take any $\boldsymbol{x{:}a}$. By *unary creation*, their must exist a unique function $\boldsymbol{g_1{:}b\to c}$ st. $\boldsymbol{g_1 y=z}$ iff $\boldsymbol{\psi xyz}$ for each $\boldsymbol{y{:}b}$, $\boldsymbol{z{:}c}$. *Creation* gives a function $\boldsymbol{f{:}a\to b\to c}$ which satisfies precisely the desired property. *Uniqueness.* Suppose $\boldsymbol{f_1}$, $\boldsymbol{f_2}$, are two such functions. For every $\boldsymbol{x{:}a}$, functions $\boldsymbol{f_1x, f_2x{:}b\to c}$ are equal. For take any $\boldsymbol{y{:}b}$; since $\boldsymbol{(f_2x)y{:}c}$ and $\boldsymbol{(f_2x)y=(f_2x)y}$ it follows $\boldsymbol{\psi xy((f_2x)y)}$. Since $\boldsymbol{(f_1x)y=z\Leftrightarrow\psi xyz}$, we conclude $\boldsymbol{(f_1x)y=(f_2x)y}$; thus $\boldsymbol{f_1x=f_2x}$. Therefore $\boldsymbol{f_1=f_2}$. QED.

$\forall\boldsymbol{a}\forall\boldsymbol{b}\forall\boldsymbol{c}\forall\boldsymbol{\psi}(\forall\boldsymbol{x}\forall\boldsymbol{yx{:}a\vdash y{:}b}\vdash\exists!\boldsymbol{zz{:}c\wedge\psi xyz})\vdash\exists!\boldsymbol{ff{:}a\to b\to c}\wedge\forall\boldsymbol{x}\forall\boldsymbol{y}\forall\boldsymbol{zx{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz}$ **124**
$\Leftrightarrow\boldsymbol{(fx)y=z}$. .... THEOREM *of Binary Creation*

¶ .... DEM

$\square(\forall\boldsymbol{x}\forall\boldsymbol{yx{:}a\vdash y{:}b}\vdash\exists!\boldsymbol{zz{:}c\wedge\psi xyz})\vdash\exists!\boldsymbol{ff{:}a\to b\to c}\wedge\forall\boldsymbol{x}\forall\boldsymbol{y}\forall\boldsymbol{zx{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(fx)y=z}$. **124.1**
.... LEM

$\square$¶ .... DEM

$\square\square\forall\boldsymbol{x}\forall\boldsymbol{yx{:}a\vdash y{:}b}\vdash\exists!\boldsymbol{zz{:}c\wedge\psi xyz}$. .... [.1] ASM **124.2**

$\square\square\forall\boldsymbol{b}\forall\boldsymbol{c}\forall\boldsymbol{\psi}(\forall\boldsymbol{xx{:}b}\vdash\exists!\boldsymbol{yy{:}c\wedge\psi xy})\vdash\exists!\boldsymbol{ff{:}b\to c}\wedge\forall\boldsymbol{x}\forall\boldsymbol{yx{:}b\vdash y{:}c\vdash\psi xy\Leftrightarrow fx=y}$. .... THM **124.3**

$\square\square\forall\boldsymbol{\psi}(\forall\boldsymbol{xx{:}b}\vdash\exists!\boldsymbol{yy{:}c\wedge\psi xy})\vdash\exists!\boldsymbol{ff{:}b\to c}\wedge\forall\boldsymbol{x}\forall\boldsymbol{yx{:}b\vdash y{:}c\vdash\psi xy\Leftrightarrow fx=y}$. .... [.3] ${}_0$MINT$_2$ **124.4**

$\square\square\forall\boldsymbol{\psi}(\forall\boldsymbol{yy{:}b}\vdash\exists!\boldsymbol{zz{:}c\wedge\psi yz})\vdash\exists!\boldsymbol{ff{:}b\to c}\wedge\forall\boldsymbol{y}\forall\boldsymbol{zy{:}b\vdash z{:}c\vdash\psi yz\Leftrightarrow fy=z}$. .... [.4] QNT **124.5**

$\square\square(\forall\boldsymbol{yy{:}b}\vdash\exists!\boldsymbol{zz{:}c\wedge\psi xyz})\vdash\exists!\boldsymbol{ff{:}b\to c}\wedge\forall\boldsymbol{y}\forall\boldsymbol{zy{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow fy=z}$. .... [.5] $\forall_2$INST **124.6**

$\square\square\boldsymbol{x_1{:}a}\vdash\exists!\boldsymbol{ff{:}b\to c}\wedge\forall\boldsymbol{y}\forall\boldsymbol{zy{:}b\vdash z{:}c\vdash\psi x_1yz\Leftrightarrow fy=z}$. .... LEM **124.7**

$\square\square$¶ .... DEM

$\square\square\square\boldsymbol{x_1{:}a}$. .... [.7] ASM **124.8**

$\square\square\square\boldsymbol{y{:}b}\vdash\exists!\boldsymbol{zz{:}c\wedge\psi x_1yz}$. .... LEM **124.9**

$\square\square\square$¶ .... DEM

$\square\square\square\square\boldsymbol{y{:}b}$. .... [.9] ASM **124.10**

$\square\square\square\square\exists!\boldsymbol{zz{:}c\wedge\psi x_1yz}$. .... [.2] [.8] [.10] ${}_2$MINT$_2$ **124.11**

$\square\square\square\blacksquare$. .... QED

$\square\square\square\forall\boldsymbol{yy{:}b}\vdash\exists!\boldsymbol{zz{:}c\wedge\psi x_1yz}$. .... [.9] $\forall_1$GEN **124.12**

$\square\square\square\exists!\boldsymbol{ff{:}b\to c}\wedge\forall\boldsymbol{y}\forall\boldsymbol{zy{:}b\vdash z{:}c\vdash\psi x_1yz\Leftrightarrow fy=z}$. .... [.6] [.12] MP **124.13**

$\square\square\blacksquare$. .... QED

$\square\square\forall\boldsymbol{xx{:}a}\vdash\exists!\boldsymbol{ff{:}b\to c}\wedge\forall\boldsymbol{y}\forall\boldsymbol{zy{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow fy=z}$. .... [.7] $\forall_1$GEN **124.14**

124.15 $\Box\Box\forall\psi(\forall xx{:}a\vdash\exists!yy{:}b\to c\wedge\psi xy)\vdash\exists!ff{:}a\to b\to c\wedge\forall x\forall yx{:}a\vdash y{:}b\to c\vdash\psi xy\Leftrightarrow fx=y.$ …… **[.3]** $_0$MINT$_2$

124.16 $\Box\Box\forall\psi(\forall xx{:}a\vdash\exists!ff{:}b\to c\wedge\psi xf)\vdash\exists!gg{:}a\to b\to c\wedge\forall x\forall fx{:}a\vdash f{:}b\to c\vdash\psi xf\Leftrightarrow gx=f.$ …… **[.15]** QNT

124.17 $\Box\Box(\forall xx{:}a\vdash\exists!ff{:}b\to c\wedge\forall y\forall zy{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow fy=z)\vdash\exists!gg{:}a\to b\to c\wedge\forall x\forall fx{:}a\vdash f{:}b\to c$ $\vdash(\forall y\forall zy{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow fy=z)\Leftrightarrow gx=f.$ …… **[.26]** $\forall_2$INST

124.18 $\Box\Box\exists!gg{:}a\to b\to c\wedge\forall x\forall fx{:}a\vdash f{:}b\to c\vdash(\forall y\forall zy{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow fy=z)\Leftrightarrow gx=f.$ …… **[.17] [.14]** MP

124.19 $\Box\Box\exists gg{:}a\to b\to c\wedge\forall x\forall fx{:}a\vdash f{:}b\to c\vdash(\forall y\forall zy{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow fy=z)\Leftrightarrow gx=f.$ …… **[.18]** $\exists!$EXISTS

124.20 $\Box\Box g{:}a\to b\to c\wedge\forall x\forall fx{:}a\vdash f{:}b\to c\vdash(\forall y\forall zy{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow fy=z)\Leftrightarrow gx=f.$ …… **[.19]** $\exists_1$INST

124.21 $\Box\Box x{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(gx)y=z.$ …… LEM

$\Box\Box$¶ …… DEM

124.22 $\Box\Box\Box x{:}a.$ …… **[.21]** ASM

124.23 $\Box\Box\Box y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(gx)y=z.$ …… LEM

$\Box\Box\Box$¶ …… DEM

124.24 $\Box\Box\Box\Box y{:}b.$ …… **[.23]** ASM

124.25 $\Box\Box\Box\Box z{:}c\vdash\psi xyz\Leftrightarrow(gx)y=z.$ …… LEM

$\Box\Box\Box\Box$¶ …… DEM

124.26 $\Box\Box\Box\Box\Box z{:}c.$ …… **[.25]** ASM

124.27 $\Box\Box\Box\Box\Box\forall a\forall b\forall ff{:}a\to b\vdash\forall xx{:}a\vdash fx{:}b.$ …… AXM

124.28 $\Box\Box\Box\Box\Box g{:}a\to b\to c.$ …… **[.20]** TAUT

124.29 $\Box\Box\Box\Box\Box\forall xx{:}a\vdash gx{:}b\to c.$ …… **[.27] [.28]** $_1$MINT$_3$

124.30 $\Box\Box\Box\Box\Box gx{:}b\to c.$ …… **[.29] [.22]** $_1$MINT$_1$

124.31 $\Box\Box\Box\Box\Box\forall x\forall fx{:}a\vdash f{:}b\to c\vdash(\forall y\forall zy{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow fy=z)\Leftrightarrow gx=f.$ …… **[.20]** TAUT

124.32 $\Box\Box\Box\Box\Box(\forall y\forall zy{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(gx)y=z)\Leftrightarrow gx=gx.$ …… **[.31] [.22] [.30]** $_2$MINT$_2$

124.33 $\Box\Box\Box\Box\Box gx=gx.$ …… EQ

124.34 $\Box\Box\Box\Box\Box\forall y\forall zy{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(gx)y=z.$ …… **[.32] [.33]** TAUT

124.35 $\Box\Box\Box\Box\Box\psi xyz\Leftrightarrow(gx)y=z.$ …… **[.34] [.24] [.26]** $_2$MINT$_2$

$\Box\Box\Box\Box\blacksquare.$ …… QED

$\Box\Box\Box\blacksquare.$ …… QED

$\Box\Box\blacksquare.$ …… QED

124.36 $\Box\Box\forall x\forall y\forall zx{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(gx)y=z.$ …… **[.21]** UGEN$_3$

124.37 $\Box\Box g{:}a\to b\to c\wedge\forall x\forall y\forall zx{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(gx)y=z.$ …… **[.20] [.36]** TAUT

124.38 $\Box\Box\exists ff{:}a\to b\to c\wedge\forall x\forall y\forall zx{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(fx)y=z.$ …… **[.37]** $\exists_1$GEN

124.39 $\Box\Box(f_1{:}a\to b\to c\wedge\forall x\forall y\forall zx{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(f_1x)y=z)\vdash(f_2{:}a\to b\to c\wedge\forall x\forall y\forall zx{:}a\vdash$ $y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(f_2x)y=z)\vdash f_1=f_2.$ …… LEM

$\Box\Box$¶ …… DEM

124.40 $\Box\Box\Box f_1{:}a\to b\to c\wedge\forall x\forall y\forall zx{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(f_1x)y=z.$ …… **[.39]** ASM

124.41 $\Box\Box\Box(f_2{:}a\to b\to c\wedge\forall x\forall y\forall zx{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(f_2x)y=z)\vdash f_1=f_2.$ …… LEM

$\Box\Box\Box$¶ …… DEM

124.42 $\Box\Box\Box\Box f_2{:}a\to b\to c\wedge\forall x\forall y\forall zx{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(f_2x)y=z.$ …… **[.41]** ASM

124.43 $\Box\Box\Box\Box\forall a\forall b\forall f\forall gf{:}a\to b\vdash g{:}a\to b\vdash(\forall xx{:}a\vdash fx=gx)\vdash f=g.$ …… AXM

124.44 $\Box\Box\Box\Box f_1{:}a\to b\to c.$ …… **[.40]** TAUT

□□□□$f_2{:}a\to b\to c$. … [.42] TAUT **124.45**
□□□□$x{:}a\vdash f_1x=f_2x$. … LEM **124.46**
□□□□¶ … DEM
□□□□□$x{:}a$. … [.46] ASM **124.47**
□□□□□$\forall a\forall b\forall f\,f{:}a\to b\vdash\forall xx{:}a\vdash fx{:}b$. … AXM **124.48**
□□□□□$\forall xx{:}a\vdash f_1x{:}b\to c$. … [.48] [.44] ${}_1$MINT${}_3$ **124.49**
□□□□□$f_1x{:}b\to c$. … [.49] [.47] ${}_1$MINT${}_1$ **124.50**
□□□□□$\forall xx{:}a\vdash f_2x{:}b\to c$. … [.48] [.45] ${}_1$MINT${}_3$ **124.51**
□□□□□$f_2x{:}b\to c$. … [.51] [.47] ${}_1$MINT${}_1$ **124.52**
□□□□□$y{:}b\vdash(f_1x)y=(f_2x)y$. … LEM **124.53**
□□□□□¶ … DEM
□□□□□□$y{:}b$. … [.53] ASM **124.54**
□□□□□□$\forall b\forall c\forall f\,f{:}b\to c\vdash\forall yy{:}b\vdash fy{:}c$. … [.48] QNT **124.55**
□□□□□□$\forall yy{:}b\vdash(f_1x)y{:}c$. … [.55] [.50] ${}_1$MINT${}_3$ **124.56**
□□□□□□$(f_1x)y{:}c$. … [.56] [.54] ${}_1$MINT${}_1$ **124.57**
□□□□□□$\forall yy{:}b\vdash(f_2x)y{:}c$. … [.55] [.52] ${}_1$MINT${}_3$ **124.58**
□□□□□□$(f_2x)y{:}c$. … [.58] [.54] ${}_1$MINT${}_1$ **124.59**
□□□□□□$\forall x\forall y\forall zx{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(f_2x)y=z$. … [.42] TAUT **124.60**
□□□□□□$\psi xy((f_2x)y)\Leftrightarrow(f_2x)y=(f_2x)y$. … [.60] [.47] [.54] [.59] ${}_3$MINT${}_3$ **124.61**
□□□□□□$(f_2x)y=(f_2x)y$. … EQ **124.62**
□□□□□□$\psi xy((f_2x)y)$. … [.62] [.61] TAUT **124.63**
□□□□□□$\forall x\forall y\forall zx{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(f_1x)y=z$. … [.40] TAUT **124.64**
□□□□□□$\psi xy((f_2x)y)\Leftrightarrow(f_1x)y=(f_2x)y$. … [.64] [.47] [.54] [.59] ${}_3$MINT${}_3$ **124.65**
□□□□□□$(f_1x)y=(f_2x)y$. … [.65] [.63] TAUT **124.66**
□□□□□■. … QED
□□□□□$\forall yy{:}b\vdash(f_1x)y=(f_2x)y$. … [.53] $\forall_1$GEN **124.67**
□□□□□$\forall b\forall c\forall f\forall gf{:}b\to c\vdash g{:}b\to c\vdash(\forall yy{:}b\vdash fy=gy)\vdash f=g$. … [.43] QNT **124.68**
□□□□□$f_1x=f_2x$. … [.68] [.50] [.52] [.67] ${}_3$MINT${}_4$ **124.69**
□□□□■. … QED
□□□□$\forall xx{:}a\vdash f_1x=f_2x$. … [.46] $\forall_1$GEN **124.70**
□□□□$f_1=f_2$. … [.43] [.44] [.45] [.70] ${}_3$MINT${}_4$ **124.71**
□□□■. … QED
□□■. … QED
□□$\forall f_1\forall f_2(f_1{:}a\to b\to c\wedge\forall x\forall y\forall zx{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(f_1x)y=z)\vdash(f_2{:}a\to b\to c\wedge\forall x\forall y\forall z$ **124.72**
$x{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(f_2x)y=z)\vdash f_1=f_2$. … [.39] UGEN${}_2$
□□$\exists!f\,f{:}a\to b\to c\wedge\forall x\forall y\forall zx{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(fx)y=z$. … [.38] [.72] $\exists!$INTROS **124.73**
□■. … QED
□$\forall\psi(\forall x\forall yx{:}a\vdash y{:}b\vdash\exists!zz{:}c\wedge\psi xyz)\vdash\exists!f\,f{:}a\to b\to c\wedge\forall x\forall y\forall zx{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz\Leftrightarrow(fx)y$ **124.74**
$=z$. … [.1] $\forall_2$GEN
□$\forall a\forall b\forall c\forall\psi(\forall x\forall yx{:}a\vdash y{:}b\vdash\exists!zz{:}c\wedge\psi xyz)\vdash\exists!f\,f{:}a\to b\to c\wedge\forall x\forall y\forall zx{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz$ **124.75**
$\Leftrightarrow(fx)y=z$. … [.74] UGEN${}_3$
■. … QED

Uses Axioms: **3, 5, 6, 7, 8**

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125 $\forall\psi(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\psi xyz)\vdash\exists!ff{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}\wedge\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash$
$\psi xyz\Leftrightarrow(fx)y=z$. . . . . THEOREM *of Binary Operation*

¶ . . . . DEM

125.1 $\Box\forall a\forall b\forall c\forall\psi(\forall x\forall yx{:}a\vdash y{:}b\vdash\exists!zz{:}c\wedge\psi xyz)\vdash\exists!ff{:}a\to b\to c\wedge\forall x\forall y\forall zx{:}a\vdash y{:}b\vdash z{:}c\vdash\psi xyz$
$\Leftrightarrow(fx)y=z$. . . . . THM

125.2 $\Box\forall\psi(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\psi xyz)\vdash\exists!ff{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}\wedge\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash$
$\psi xyz\Leftrightarrow(fx)y=z$. . . . . **[.1]** $_0$MINT$_3$

■. . . . . QED

Uses Axioms: **3, 5, 6, 7, 8**

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126 $\forall\psi(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\psi xyz)\vdash\exists\phi\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=z$. . . . .
. . . . THEOREM *of Binary Realization*

¶ . . . . DEM

126.1 $\Box(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\psi xyz)\vdash\exists\phi\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=z$. . . LEM

□¶ . . . . DEM

126.2 $\Box\Box\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\psi xyz$. . . . . **[.1]** ASM

126.3 $\Box\Box\forall\psi(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\psi xyz)\vdash\exists!ff{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}\wedge\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}$
$\vdash\psi xyz\Leftrightarrow(fx)y=z$. . . . . THM

126.4 $\Box\Box(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\psi xyz)\vdash\exists!ff{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}\wedge\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash$
$\psi xyz\Leftrightarrow(fx)y=z$. . . . . **[.3]** $\forall_2$INST

126.5 $\Box\Box\exists!ff{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}\wedge\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow(fx)y=z$. . . . . **[.4] [.2]** MP

126.6 $\Box\Box\exists ff{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}\wedge\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow(fx)y=z$. . . . . **[.5]** ∃!EXISTS

126.7 $\Box\Box f{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}\wedge\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow(fx)y=z$. . . . . **[.6]** $\exists_1$INST

126.8 $\Box\Box\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow(fx)y=z$. . . . . **[.7]** TAUT

126.9 $\Box\Box\exists\phi\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=z$. . . . . **[.8]** $\exists_2$GEN

□■. . . . . QED

126.10 $\Box\forall\psi(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\psi xyz)\vdash\exists\phi\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=z$. . . .
. . . . **[.1]** $\forall_2$GEN

■. . . . . QED

Uses Axioms: **3, 5, 6, 7, 8**

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127 $\forall\psi(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\psi xyz)\vdash\forall\phi(\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=z)\vdash$
$\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\psi xy(x\phi y)$. . . . . THEOREM *of Binary Realization Strength*

¶ . . . . DEM

127.1 $\Box(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\psi xyz)\vdash\forall\phi(\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=z)\vdash$
$\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\psi xy(x\phi y)$. . . . . LEM

□¶ . . . . DEM

127.2 $\Box\Box\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\psi xyz$. . . . . **[.1]** ASM

127.3 $\Box\Box(\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=z)\vdash\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\psi xy(x\phi y)$. . . .
. . . . LEM

□□¶ . . . . DEM

$\Box\Box\Box\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash \psi xyz\Leftrightarrow x\phi y=z.$ …… [.3] ASM **127.4**

$\Box\Box\Box\forall\psi(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\psi xyz)\vdash\exists! f f{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}\wedge\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash$ **127.5**
$z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow(fx)y=z.$ …… THM

$\Box\Box\Box(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\psi xyz)\vdash\exists! f f{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}\wedge\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash$ **127.6**
$\psi xyz\Leftrightarrow(fx)y=z.$ …… [.5] $\forall_2$INST

$\Box\Box\Box\exists! f f{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}\wedge\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow(fx)y=z.$ …… [.6] [.2] MP **127.7**

$\Box\Box\Box\exists f f{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}\wedge\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow(fx)y=z.$ …… [.7] $\exists!$EXISTS **127.8**

$\Box\Box\Box f{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}\wedge\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow(fx)y=z.$ …… [.8] $\exists_1$INST **127.9**

$\Box\Box\Box x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\psi xy(x\phi y).$ …… LEM **127.10**

$\Box\Box\Box$¶ …… DEM

$\Box\Box\Box\Box x{:}\mathscr{H}.$ …… [.10] ASM **127.11**

$\Box\Box\Box\Box y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\psi xy(x\phi y).$ …… LEM **127.12**

$\Box\Box\Box\Box$¶ …… DEM

$\Box\Box\Box\Box\Box y{:}\mathscr{H}.$ …… [.12] ASM **127.13**

$\Box\Box\Box\Box\Box f{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}.$ …… [.9] TAUT **127.14**

$\Box\Box\Box\Box\Box\forall a\forall b\forall f f{:}a\to b\vdash\forall x x{:}a\vdash fx{:}b.$ …… AXM **127.15**

$\Box\Box\Box\Box\Box\forall x x{:}\mathscr{H}\vdash fx{:}\mathscr{H}\to\mathscr{H}.$ …… [.15] [.14] ${}_1$MINT$_3$ **127.16**

$\Box\Box\Box\Box\Box fx{:}\mathscr{H}\to\mathscr{H}.$ …… [.16] [.11] ${}_1$MINT$_1$ **127.17**

$\Box\Box\Box\Box\Box\forall a\forall b\forall f f{:}a\to b\vdash\forall y y{:}a\vdash fy{:}b.$ …… [.15] QNT **127.18**

$\Box\Box\Box\Box\Box\forall y y{:}\mathscr{H}\vdash(fx)y{:}\mathscr{H}.$ …… [.18] [.17] ${}_1$MINT$_3$ **127.19**

$\Box\Box\Box\Box\Box(fx)y{:}\mathscr{H}.$ …… [.19] [.13] ${}_1$MINT$_1$ **127.20**

$\Box\Box\Box\Box\Box\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow(fx)y=z.$ …… [.9] TAUT **127.21**

$\Box\Box\Box\Box\Box\psi xy((fx)y)\Leftrightarrow(fx)y=(fx)y.$ …… [.21] [.11] [.13] [.20] ${}_3$MINT$_3$ **127.22**

$\Box\Box\Box\Box\Box(fx)y=(fx)y.$ …… EQ **127.23**

$\Box\Box\Box\Box\Box\psi xy((fx)y).$ …… [.22] [.23] TAUT **127.24**

$\Box\Box\Box\Box\Box\psi xy((fx)y)\Leftrightarrow x\phi y=(fx)y.$ …… [.4] [.11] [.13] [.20] ${}_3$MINT$_3$ **127.25**

$\Box\Box\Box\Box\Box x\phi y=(fx)y.$ …… [.25] [.24] TAUT **127.26**

$\Box\Box\Box\Box\Box x\phi y{:}\mathscr{H}.$ …… [.26] [.20] SUB **127.27**

$\Box\Box\Box\Box\Box\psi xy(x\phi y).$ …… [.26] [.24] SUB **127.28**

$\Box\Box\Box\Box\Box x\phi y{:}\mathscr{H}\wedge\psi xy(x\phi y).$ …… [.27] [.28] TAUT **127.29**

$\Box\Box\Box\Box\blacksquare.$ …… QED

$\Box\Box\Box\blacksquare.$ …… QED

$\Box\Box\Box\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\psi xy(x\phi y).$ …… [.10] UGEN$_2$ **127.30**

$\Box\Box\blacksquare.$ …… QED

$\Box\Box\forall\phi(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\psi xy(x\phi y).$ **127.31**
…… [.3] $\forall_2$GEN

$\Box\blacksquare.$ …… QED

$\Box\forall\psi(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\psi xyz)\vdash\forall\phi(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=z)\vdash$ **127.32**
$\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\psi xy(x\phi y).$ …… [.1] $\forall_2$GEN

$\blacksquare.$ …… QED

Uses Axioms: **3, 5, 6, 7, 8**

For any $f,g{:}a\to b\to c$, if $(fx)y=(gx)y$ for every $x{:}a$, $y{:}b$, then $f=g$. PROOF: Consider two such functions $f,g{:}a\to b\to c$. By *application*, for every $x{:}a$, objects fx, gx, are functions from $b\to c$.

Since for every $\boldsymbol{y{:}b}$, $\boldsymbol{(fx)y=(gx)y}$, it follows by *functional extension*, $\boldsymbol{fx=gx}$ and $\boldsymbol{f=g}$. QED.

128 $\forall a\forall b\forall c\forall f\forall g f{:}a\to b\to c\vdash g{:}a\to b\to c\vdash(\forall x\forall y x{:}a\vdash y{:}b\vdash(fx)y=(gx)y)\vdash f=g$. .... THEOREM *of Binary Extensionality of Functions*

¶ .... DEM

128.1 □$f{:}a\to b\to c\vdash g{:}a\to b\to c\vdash(\forall x\forall y x{:}a\vdash y{:}b\vdash(fx)y=(gx)y)\vdash f=g$. .... LEM

□¶ .... DEM

128.2 □□$f{:}a\to b\to c$. .... **[.1]** ASM

128.3 □□$g{:}a\to b\to c\vdash(\forall x\forall y x{:}a\vdash y{:}b\vdash(fx)y=(gx)y)\vdash f=g$. .... LEM

□□¶ .... DEM

128.4 □□□$g{:}a\to b\to c$. .... **[.3]** ASM

128.5 □□□$(\forall x\forall y x{:}a\vdash y{:}b\vdash(fx)y=(gx)y)\vdash f=g$. .... LEM

□□□¶ .... DEM

128.6 □□□□$\forall x\forall y x{:}a\vdash y{:}b\vdash(fx)y=(gx)y$. .... **[.5]** ASM

128.7 □□□□$\forall a\forall b\forall f\forall g f{:}a\to b\vdash g{:}a\to b\vdash(\forall x x{:}a\vdash fx=gx)\vdash f=g$. .... AXM

128.8 □□□□$(\forall x x{:}a\vdash fx=gx)\vdash f=g$. .... **[.7] [.2] [.4]** $_2$MINT$_4$

128.9 □□□□$x{:}a\vdash fx=gx$. .... LEM

□□□□¶ .... DEM

128.10 □□□□□$x{:}a$. .... **[.9]** ASM

128.11 □□□□□$\forall b\forall c\forall f\forall g f{:}b\to c\vdash g{:}b\to c\vdash(\forall y y{:}b\vdash fy=gy)\vdash f=g$. .... **[.7]** QNT

128.12 □□□□□$\forall a\forall b\forall f f{:}a\to b\vdash\forall x x{:}a\vdash fx{:}b$. .... AXM

128.13 □□□□□$\forall x x{:}a\vdash fx{:}b\to c$. .... **[.12] [.2]** $_1$MINT$_3$

128.14 □□□□□$fx{:}b\to c$. .... **[.13] [.10]** $_1$MINT$_1$

128.15 □□□□□$\forall x x{:}a\vdash gx{:}b\to c$. .... **[.12] [.4]** $_1$MINT$_3$

128.16 □□□□□$gx{:}b\to c$. .... **[.15] [.10]** $_1$MINT$_1$

128.17 □□□□□$(\forall y y{:}b\vdash(fx)y=(gx)y)\vdash fx=gx$. .... **[.11] [.14] [.16]** $_2$MINT$_4$

128.18 □□□□□$y{:}b\vdash(fx)y=(gx)y$. .... LEM

□□□□□¶ .... DEM

128.19 □□□□□□$y{:}b$. .... **[.18]** ASM

128.20 □□□□□□$(fx)y=(gx)y$. .... **[.6] [.10] [.19]** $_2$MINT$_2$

□□□□□■. .... QED

128.21 □□□□□$\forall y y{:}b\vdash(fx)y=(gx)y$. .... **[.18]** $\forall_1$GEN

128.22 □□□□□$fx=gx$. .... **[.17] [.21]** MP

□□□□■. .... QED

128.23 □□□□$\forall x x{:}a\vdash fx=gx$. .... **[.9]** $\forall_1$GEN

128.24 □□□□$f=g$. .... **[.8] [.23]** MP

□□□■. .... QED

□□■. .... QED

□■. .... QED

128.25 □$\forall a\forall b\forall c\forall f\forall g f{:}a\to b\to c\vdash g{:}a\to b\to c\vdash(\forall x\forall y x{:}a\vdash y{:}b\vdash(fx)y=(gx)y)\vdash f=g$. .. **[.1]** UGEN$_5$

■. .... QED

Uses Axioms: **7, 8**

For any $\boldsymbol{f{:}a\to b}$, $\boldsymbol{g{:}b\to c}$, their exists a unique $\boldsymbol{h{:}a\to c}$ which is the “composition” of $\boldsymbol{g}$ with $\boldsymbol{f}$.

PROOF: *Existence.* For every $x{:}a$, their exists a unique $y{:}c$ st. $g(fx)=y$. *Creation* gives a desired function. *Uniqueness.* Trivial. QED.

$\forall a\forall b\forall c\forall f\forall g f{:}a\to b\vdash g{:}b\to c\vdash \exists! hh{:}a\to c\land\forall xx{:}a\vdash g(fx)=hx$. **129**
.... THEOREM *of Implicit Composition*
¶ .... DEM
□$f{:}a\to b\vdash g{:}b\to c\vdash \exists! hh{:}a\to c\land\forall xx{:}a\vdash g(fx)=hx$. .... LEM **129.1**
□¶ .... DEM
□□$f{:}a\to b$. .... [.1] ASM **129.2**
□□$g{:}b\to c\vdash \exists! hh{:}a\to c\land\forall xx{:}a\vdash g(fx)=hx$. .... LEM **129.3**
□□¶ .... DEM
□□□$g{:}b\to c$. .... [.3] ASM **129.4**
□□□$x{:}a\vdash \exists! yy{:}c\land g(fx)=y$. .... LEM **129.5**
□□□¶ .... DEM
□□□□$x{:}a$. .... [.5] ASM **129.6**
□□□□$\forall a\forall b\forall f f{:}a\to b\vdash\forall xx{:}a\vdash fx{:}b$. .... AXM **129.7**
□□□□$\forall xx{:}a\vdash fx{:}b$. .... [.7] [.2] $_{1}\text{MINT}_{3}$ **129.8**
□□□□$fx{:}b$. .... [.8] [.6] $_{1}\text{MINT}_{1}$ **129.9**
□□□□$\forall b\forall c\forall f f{:}b\to c\vdash\forall xx{:}b\vdash fx{:}c$. .... [.7] QNT **129.10**
□□□□$\forall xx{:}b\vdash gx{:}c$. .... [.10] [.4] $_{1}\text{MINT}_{3}$ **129.11**
□□□□$g(fx){:}c$. .... [.11] [.9] $_{1}\text{MINT}_{1}$ **129.12**
□□□□$g(fx)=g(fx)$. .... EQ **129.13**
□□□□$g(fx){:}c\land g(fx)=g(fx)$. .... [.12] [.13] TAUT **129.14**
□□□□$\exists yy{:}c\land g(fx)=y$. .... [.14] $\exists_1$GEN **129.15**
□□□□$y_1{:}c\land g(fx)=y_1\vdash y_2{:}c\land g(fx)=y_2\vdash y_1=y_2$. .... LEM **129.16**
□□□□¶ .... DEM
□□□□□$y_1{:}c\land g(fx)=y_1$. .... [.16] ASM **129.17**
□□□□□$y_2{:}c\land g(fx)=y_2\vdash y_1=y_2$. .... LEM **129.18**
□□□□□¶ .... DEM
□□□□□□$y_2{:}c\land g(fx)=y_2$. .... [.18] ASM **129.19**
□□□□□□$g(fx)=y_1$. .... [.17] TAUT **129.20**
□□□□□□$g(fx)=y_2$. .... [.19] TAUT **129.21**
□□□□□□$y_1=y_2$. .... [.20] [.21] SUB **129.22**
□□□□□■. .... QED
□□□□■. .... QED
□□□□$\forall y_1\forall y_2 y_1{:}c\land g(fx)=y_1\vdash y_2{:}c\land g(fx)=y_2\vdash y_1=y_2$. .... [.16] UGEN_2 **129.23**
□□□□$\exists! yy{:}c\land g(fx)=y$. .... [.15] [.23] $\exists!$INTROS **129.24**
□□□■. .... QED
□□□$\forall xx{:}a\vdash\exists! yy{:}c\land g(fx)=y$. .... [.5] $\forall_1$GEN **129.25**
□□□$\forall a\forall b\forall\psi(\forall xx{:}a\vdash\exists! yy{:}b\land\psi xy)\vdash\exists f f{:}a\to b\land\forall x\forall yx{:}a\vdash y{:}b\vdash\psi xy\Leftrightarrow fx=y$. .... AXM **129.26**
□□□$\forall a\forall b\forall\psi(\forall xx{:}a\vdash\exists! yy{:}b\land\psi xy)\vdash\exists hh{:}a\to b\land\forall x\forall yx{:}a\vdash y{:}b\vdash\psi xy\Leftrightarrow hx=y$. ... [.26] QNT **129.27**
□□□$\forall\psi(\forall xx{:}a\vdash\exists! yy{:}c\land\psi xy)\vdash\exists hh{:}a\to c\land\forall x\forall yx{:}a\vdash y{:}c\vdash\psi xy\Leftrightarrow hx=y$. ...... [.27] $_{0}\text{MINT}_{2}$ **129.28**
□□□$(\forall xx{:}a\vdash\exists! yy{:}c\land g(fx)=y)\vdash\exists hh{:}a\to c\land\forall x\forall yx{:}a\vdash y{:}c\vdash g(fx)=y\Leftrightarrow hx=y$. [.28] $\forall_2$INST **129.29**
□□□$\exists hh{:}a\to c\land\forall x\forall yx{:}a\vdash y{:}c\vdash g(fx)=y\Leftrightarrow hx=y$. .... [.29] [.25] MP **129.30**

129.31 □□□$h{:}a\to c\wedge\forall x\forall y x{:}a\vdash y{:}c\vdash g(fx)=y\Leftrightarrow hx=y$. .... [.30] $\exists_1$INST
129.32 □□□$x{:}a\vdash g(fx)=hx$. .... LEM
□□□¶ .... DEM
129.33 □□□□$x{:}a$. .... [.32] ASM
129.34 □□□□$\forall a\forall b\forall f f{:}a\to b\vdash\forall x x{:}a\vdash fx{:}b$. .... AXM
129.35 □□□□$\forall x x{:}a\vdash fx{:}b$. .... [.34] [.2] $_1$MINT$_3$
129.36 □□□□$fx{:}b$. .... [.35] [.33] $_1$MINT$_1$
129.37 □□□□$\forall b\forall c\forall f f{:}b\to c\vdash\forall x x{:}b\vdash fx{:}c$. .... [.34] QNT
129.38 □□□□$\forall x x{:}b\vdash gx{:}c$. .... [.37] [.4] $_1$MINT$_3$
129.39 □□□□$g(fx){:}c$. .... [.38] [.36] $_1$MINT$_1$
129.40 □□□□$\forall x\forall y x{:}a\vdash y{:}c\vdash g(fx)=y\Leftrightarrow hx=y$. .... [.31] TAUT
129.41 □□□□$g(fx)=g(fx)\Leftrightarrow hx=g(fx)$. .... [.40] [.33] [.39] $_2$MINT$_2$
129.42 □□□□$g(fx)=g(fx)$. .... EQ
129.43 □□□□$g(fx)=hx$. .... [.41] [.42] TAUT
□□□■. .... QED
129.44 □□□$\forall x x{:}a\vdash g(fx)=hx$. .... [.32] $\forall_1$GEN
129.45 □□□$h{:}a\to c\wedge\forall x x{:}a\vdash g(fx)=hx$. .... [.31] [.44] TAUT
129.46 □□□$\exists h h{:}a\to c\wedge\forall x x{:}a\vdash g(fx)=hx$. .... [.45] $\exists_1$GEN
129.47 □□□$(h_1{:}a\to c\wedge\forall x x{:}a\vdash g(fx)=h_1x)\vdash(h_2{:}a\to c\wedge\forall x x{:}a\vdash g(fx)=h_2x)\vdash h_1=h_2$. .... LEM
□□□¶ .... DEM
129.48 □□□□$h_1{:}a\to c\wedge\forall x x{:}a\vdash g(fx)=h_1x$. .... [.47] ASM
129.49 □□□□$(h_2{:}a\to c\wedge\forall x x{:}a\vdash g(fx)=h_2x)\vdash h_1=h_2$. .... LEM
□□□□¶ .... DEM
129.50 □□□□□$h_2{:}a\to c\wedge\forall x x{:}a\vdash g(fx)=h_2x$. .... [.49] ASM
129.51 □□□□□$\forall a\forall b\forall f\forall g f{:}a\to b\vdash g{:}a\to b\vdash(\forall x x{:}a\vdash fx=gx)\vdash f=g$. .... AXM
129.52 □□□□□$h_1{:}a\to c\vdash h_2{:}a\to c\vdash(\forall x x{:}a\vdash h_1x=h_2x)\vdash h_1=h_2$. .... [.51] $_0$MINT$_4$
129.53 □□□□□$(\forall x x{:}a\vdash h_1x=h_2x)\vdash h_1=h_2$. .... [.52] [.48] [.50] TAUT
129.54 □□□□□$x_1{:}a\vdash h_1x_1=h_2x_1$. .... LEM
□□□□□¶ .... DEM
129.55 □□□□□□$x_1{:}a$. .... [.54] ASM
129.56 □□□□□□$\forall x x{:}a\vdash g(fx)=h_1x$. .... [.48] TAUT
129.57 □□□□□□$g(fx_1)=h_1x_1$. .... [.56] [.55] $_1$MINT$_1$
129.58 □□□□□□$\forall x x{:}a\vdash g(fx)=h_2x$. .... [.50] TAUT
129.59 □□□□□□$g(fx_1)=h_2x_1$. .... [.58] [.55] $_1$MINT$_1$
129.60 □□□□□□$h_1x_1=h_2x_1$. .... [.57] [.59] SUB
□□□□□■. .... QED
129.61 □□□□□$\forall x x{:}a\vdash h_1x=h_2x$. .... [.54] $\forall_1$GEN
129.62 □□□□□$h_1=h_2$. .... [.53] [.61] MP
□□□□■. .... QED
□□□■. .... QED
129.63 □□□$\forall h_1\forall h_2(h_1{:}a\to c\wedge\forall x x{:}a\vdash g(fx)=h_1x)\vdash(h_2{:}a\to c\wedge\forall x x{:}a\vdash g(fx)=h_2x)\vdash h_1=h_2$. ....
.... [.47] UGEN$_2$
129.64 □□□$\exists! h h{:}a\to c\wedge\forall x x{:}a\vdash g(fx)=hx$. .... [.46] [.63] $\exists!$INTROS
□□■. .... QED

$\square\blacksquare$ ... QED

$\square\forall a\forall b\forall c\forall f\forall g f{:}a\to b\vdash g{:}b\to c\vdash\exists! hh{:}a\to c\wedge\forall xx{:}a\vdash g(fx)=hx.$ ... [.1] UGEN$_5$ **129.65**

$\blacksquare.$ ... QED

Uses Axioms: **3, 5, 6, 7, 8**

The set created by *specification* is unique. PROOF: Trivial. QED.

$\forall x\forall\psi x{:}\mathscr{H}\vdash\exists! yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}x\wedge\psi y_1.$ ... **130**

... THEOREM *of Unique Specification*

¶ ... DEM

$\square x{:}\mathscr{H}\vdash\exists! yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}x\wedge\psi y_1.$ ... LEM **130.1**

$\square$¶ ... DEM

$\square\square x{:}\mathscr{H}.$ ... [.1] ASM **130.2**

$\square\square\forall x\forall\psi x{:}\mathscr{H}\vdash\exists yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}x\wedge\psi y_1.$ ... AXM **130.3**

$\square\square\forall\psi x{:}\mathscr{H}\vdash\exists yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}x\wedge\psi y_1.$ ... [.3] $\forall_1$INST **130.4**

$\square\square x{:}\mathscr{H}\vdash\exists yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}x\wedge\psi y_1.$ ... [.4] $\forall_2$INST **130.5**

$\square\square\exists yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}x\wedge\psi y_1.$ ... [.5] [.2] MP **130.6**

$\square\square(z_1{:}\mathscr{H}\wedge\forall y_1y_1{:}z_1\Leftrightarrow y_1{:}x\wedge\psi y_1)\vdash(z_2{:}\mathscr{H}\wedge\forall y_1y_1{:}z_2\Leftrightarrow y_1{:}x\wedge\psi y_1)\vdash z_1=z_2.$ ... LEM **130.7**

$\square\square$¶ ... DEM

$\square\square\square z_1{:}\mathscr{H}\wedge\forall y_1y_1{:}z_1\Leftrightarrow y_1{:}x\wedge\psi y_1.$ ... [.7] ASM **130.8**

$\square\square\square(z_2{:}\mathscr{H}\wedge\forall y_1y_1{:}z_2\Leftrightarrow y_1{:}x\wedge\psi y_1)\vdash z_1=z_2.$ ... LEM **130.9**

$\square\square\square$¶ ... DEM

$\square\square\square\square z_2{:}\mathscr{H}\wedge\forall y_1y_1{:}z_2\Leftrightarrow y_1{:}x\wedge\psi y_1.$ ... [.9] ASM **130.10**

$\square\square\square\square\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash(\forall tt{:}x\Leftrightarrow t{:}y)\vdash x=y.$ ... AXM **130.11**

$\square\square\square\square z_1{:}\mathscr{H}\vdash z_2{:}\mathscr{H}\vdash(\forall tt{:}z_1\Leftrightarrow t{:}z_2)\vdash z_1=z_2.$ ... [.11] ${}_0$MINT$_2$ **130.12**

$\square\square\square\square\forall y_1y_1{:}z_1\Leftrightarrow y_1{:}x\wedge\psi y_1.$ ... [.8] TAUT **130.13**

$\square\square\square\square\forall y_1y_1{:}z_2\Leftrightarrow y_1{:}x\wedge\psi y_1.$ ... [.10] TAUT **130.14**

$\square\square\square\square t{:}z_1\Leftrightarrow t{:}x\wedge\psi t.$ ... [.13] $\forall_1$INST **130.15**

$\square\square\square\square t{:}z_2\Leftrightarrow t{:}x\wedge\psi t.$ ... [.14] $\forall_1$INST **130.16**

$\square\square\square\square t{:}z_1\Leftrightarrow t{:}z_2.$ ... [.15] [.16] TAUT **130.17**

$\square\square\square\square\forall tt{:}z_1\Leftrightarrow t{:}z_2.$ ... [.17] $\forall_1$GEN **130.18**

$\square\square\square\square z_1=z_2.$ ... [.12] [.8] [.10] [.18] TAUT **130.19**

$\square\square\square\blacksquare.$ ... QED

$\square\square\blacksquare.$ ... QED

$\square\square\forall z_1\forall z_2(z_1{:}\mathscr{H}\wedge\forall y_1y_1{:}z_1\Leftrightarrow y_1{:}x\wedge\psi y_1)\vdash(z_2{:}\mathscr{H}\wedge\forall y_1y_1{:}z_2\Leftrightarrow y_1{:}x\wedge\psi y_1)\vdash z_1=z_2.$ ... **130.20**

... [.7] UGEN$_2$

$\square\square\exists! yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}x\wedge\psi y_1.$ ... [.6] [.20] $\exists!$INTROS **130.21**

$\square\blacksquare.$ ... QED

$\square\forall\psi x{:}\mathscr{H}\vdash\exists! yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}x\wedge\psi y_1.$ ... [.1] $\forall_2$GEN **130.22**

$\square\forall x\forall\psi x{:}\mathscr{H}\vdash\exists! yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}x\wedge\psi y_1.$ ... [.22] $\forall_1$GEN **130.23**

$\blacksquare.$ ... QED

Uses Axioms: **3, 5, 9, 15**

The set created by *replacement* is unique. PROOF: Trivial. QED.

131 $\forall x \forall f\, x{:}\mathscr{H} \vdash f{:}x \to \mathscr{H} \vdash \exists! y\, y{:}\mathscr{H} \wedge \forall y_1\, y_1{:}y \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1$. .... THEOREM *of Unique Replacement*

¶ .... DEM

131.1 $\square x{:}\mathscr{H} \vdash f{:}x \to \mathscr{H} \vdash \exists! y\, y{:}\mathscr{H} \wedge \forall y_1\, y_1{:}y \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1$. .... LEM

$\square$¶ .... DEM

131.2 $\square\square x{:}\mathscr{H}$. .... **[.1]** ASM

131.3 $\square\square f{:}x \to \mathscr{H} \vdash \exists! y\, y{:}\mathscr{H} \wedge \forall y_1\, y_1{:}y \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1$. .... LEM

$\square\square$¶ .... DEM

131.4 $\square\square\square f{:}x \to \mathscr{H}$. .... **[.3]** ASM

131.5 $\square\square\square \forall x \forall f\, x{:}\mathscr{H} \vdash f{:}x \to \mathscr{H} \vdash \exists y\, y{:}\mathscr{H} \wedge \forall y_1\, y_1{:}y \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1$. .... AXM

131.6 $\square\square\square \exists y\, y{:}\mathscr{H} \wedge \forall y_1\, y_1{:}y \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1$. .... **[.5] [.2] [.4]** ${}_2$MINT$_2$

131.7 $\square\square\square (z_1{:}\mathscr{H} \wedge \forall y_1\, y_1{:}z_1 \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1) \vdash (z_2{:}\mathscr{H} \wedge \forall y_1\, y_1{:}z_2 \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1) \vdash z_1 = z_2$. .... LEM

$\square\square\square$¶ .... DEM

131.8 $\square\square\square\square z_1{:}\mathscr{H} \wedge \forall y_1\, y_1{:}z_1 \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1$. .... **[.7]** ASM

131.9 $\square\square\square\square (z_2{:}\mathscr{H} \wedge \forall y_1\, y_1{:}z_2 \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1) \vdash z_1 = z_2$. .... LEM

$\square\square\square\square$¶ .... DEM

131.10 $\square\square\square\square\square z_2{:}\mathscr{H} \wedge \forall y_1\, y_1{:}z_2 \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1$. .... **[.9]** ASM

131.11 $\square\square\square\square\square \forall x \forall y\, x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash (\forall t\, t{:}x \Leftrightarrow t{:}y) \vdash x = y$. .... AXM

131.12 $\square\square\square\square\square z_1{:}\mathscr{H} \vdash z_2{:}\mathscr{H} \vdash (\forall t\, t{:}z_1 \Leftrightarrow t{:}z_2) \vdash z_1 = z_2$. .... **[.11]** ${}_0$MINT$_2$

131.13 $\square\square\square\square\square \forall y_1\, y_1{:}z_1 \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1$. .... **[.8]** TAUT

131.14 $\square\square\square\square\square \forall y_1\, y_1{:}z_2 \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1$. .... **[.10]** TAUT

131.15 $\square\square\square\square\square t{:}z_1 \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1$. .... **[.13]** $\forall_1$INST

131.16 $\square\square\square\square\square t{:}z_2 \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1$. .... **[.14]** $\forall_1$INST

131.17 $\square\square\square\square\square t{:}z_1 \Leftrightarrow t{:}z_2$. .... **[.15] [.16]** TAUT

131.18 $\square\square\square\square\square \forall t\, t{:}z_1 \Leftrightarrow t{:}z_2$. .... **[.17]** $\forall_1$GEN

131.19 $\square\square\square\square\square z_1 = z_2$. .... **[.12] [.8] [.10] [.18]** TAUT

$\square\square\square\square\blacksquare$. .... QED

$\square\square\square\blacksquare$. .... QED

131.20 $\square\square\square \forall z_1 \forall z_2 (z_1{:}\mathscr{H} \wedge \forall y_1\, y_1{:}z_1 \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1) \vdash (z_2{:}\mathscr{H} \wedge \forall y_1\, y_1{:}z_2 \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1) \vdash z_1 = z_2$. .... **[.7]** UGEN$_2$

131.21 $\square\square\square \exists! y\, y{:}\mathscr{H} \wedge \forall y_1\, y_1{:}y \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1$. .... **[.6] [.20]** $\exists!$INTROS

$\square\square\blacksquare$. .... QED

$\square\blacksquare$. .... QED

131.22 $\square \forall x \forall f\, x{:}\mathscr{H} \vdash f{:}x \to \mathscr{H} \vdash \exists! y\, y{:}\mathscr{H} \wedge \forall y_1\, y_1{:}y \Leftrightarrow \exists x_1\, x_1{:}x \wedge f x_1 = y_1$. .... **[.1]** UGEN$_2$

$\blacksquare$. .... QED

Uses Axioms: **3, 5, 9, 16**

§2.2 *Sets and Operations: Pairing and Union*

For all sets $\boldsymbol{x},\boldsymbol{y}$, one can exhibit a unique set $\boldsymbol{z}$ which satisfies $\forall \boldsymbol{tt}{:}\boldsymbol{z}\Leftrightarrow \boldsymbol{t}=\boldsymbol{x}\vee \boldsymbol{t}=\boldsymbol{y}$. PROOF: *Existence.* Immediate by *pairing*. *Uniqueness.* Take any two such sets $\boldsymbol{z_1},\boldsymbol{z_2}$. $\boldsymbol{t}{:}\boldsymbol{z_1}$ iff. $\boldsymbol{t}=\boldsymbol{x}\vee\boldsymbol{t}=\boldsymbol{y}$; $\boldsymbol{t}{:}\boldsymbol{z_2}$ iff. $\boldsymbol{t}=\boldsymbol{x}\vee\boldsymbol{t}=\boldsymbol{y}$. Thus $\boldsymbol{t}{:}\boldsymbol{z_1}$ iff. $\boldsymbol{t}{:}\boldsymbol{z_2}$, for each object $\boldsymbol{t}$. Hence, by *extensionality*, they must be equal. QED.
Imagine forming a new box by packing exactly two old ones. We name this realizer **PAR**.

$\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow t=x\vee t=y$. . . . . THEOREM *of Implicit Pairing* **132**
¶ . . . . . DEM
□$x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow t=x\vee t=y$. . . . . LEM **132.1**
□¶ . . . . . DEM
□□$x{:}\mathscr{H}$. . . . . **[.1]** ASM **132.2**
□□$y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow t=x\vee t=y$. . . . . LEM **132.3**
□□¶ . . . . . DEM
□□□$y{:}\mathscr{H}$. . . . . **[.3]** ASM **132.4**
□□□$\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists zz{:}\mathscr{H}\wedge\forall z_1z_1{:}z\Leftrightarrow z_1=x\vee z_1=y$. . . . . AXM **132.5**
□□□$\exists zz{:}\mathscr{H}\wedge\forall z_1z_1{:}z\Leftrightarrow z_1=x\vee z_1=y$. . . . . **[.5] [.2] [.4]** ${}_2$MINT${}_2$ **132.6**
□□□$\exists zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow t=x\vee t=y$. . . . . **[.6]** QNT **132.7**
□□□$(z_1{:}\mathscr{H}\wedge\forall tt{:}z_1\Leftrightarrow t=x\vee t=y)\vdash(z_2{:}\mathscr{H}\wedge\forall tt{:}z_2\Leftrightarrow t=x\vee t=y)\vdash z_1=z_2$. . . . . LEM **132.8**
□□□¶ . . . . . DEM
□□□□$z_1{:}\mathscr{H}\wedge\forall tt{:}z_1\Leftrightarrow t=x\vee t=y$. . . . . **[.8]** ASM **132.9**
□□□□$(z_2{:}\mathscr{H}\wedge\forall tt{:}z_2\Leftrightarrow t=x\vee t=y)\vdash z_1=z_2$. . . . . LEM **132.10**
□□□□¶ . . . . . DEM
□□□□□$z_2{:}\mathscr{H}\wedge\forall tt{:}z_2\Leftrightarrow t=x\vee t=y$. . . . . **[.10]** ASM **132.11**
□□□□□$\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash(\forall tt{:}x\Leftrightarrow t{:}y)\vdash x=y$. . . . . AXM **132.12**
□□□□□$z_1{:}\mathscr{H}\vdash z_2{:}\mathscr{H}\vdash(\forall tt{:}z_1\Leftrightarrow t{:}z_2)\vdash z_1=z_2$. . . . . **[.12]** ${}_0$MINT${}_2$ **132.13**
□□□□□$\forall tt{:}z_1\Leftrightarrow t=x\vee t=y$. . . . . **[.9]** TAUT **132.14**
□□□□□$t{:}z_1\Leftrightarrow t=x\vee t=y$. . . . . **[.14]** $\forall_1$INST **132.15**
□□□□□$\forall tt{:}z_2\Leftrightarrow t=x\vee t=y$. . . . . **[.11]** TAUT **132.16**
□□□□□$t{:}z_2\Leftrightarrow t=x\vee t=y$. . . . . **[.16]** $\forall_1$INST **132.17**
□□□□□$t{:}z_1\Leftrightarrow t{:}z_2$. . . . . **[.15] [.17]** TAUT **132.18**
□□□□□$\forall tt{:}z_1\Leftrightarrow t{:}z_2$. . . . . **[.18]** $\forall_1$GEN **132.19**
□□□□□$z_1=z_2$. . . . . **[.13] [.9] [.11] [.19]** TAUT **132.20**
□□□□■. . . . . QED
□□□■. . . . . QED
□□□$\forall z_1\forall z_2(z_1{:}\mathscr{H}\wedge\forall tt{:}z_1\Leftrightarrow t=x\vee t=y)\vdash(z_2{:}\mathscr{H}\wedge\forall tt{:}z_2\Leftrightarrow t=x\vee t=y)\vdash z_1=z_2$. . . **[.8]** UGEN${}_2$ **132.21**
□□□$\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow t=x\vee t=y$. . . . . **[.7] [.21]** $\exists!$INTROS **132.22**
□□■. . . . . QED
□■. . . . . QED
□$\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow t=x\vee t=y$. . . . . **[.1]** UGEN${}_2$ **132.23**
■. . . . . QED

Uses Axioms: **3, 5, 9, 12**

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133 $\exists! f f{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}\wedge\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t=x\vee t=y)\Leftrightarrow(f x)y=z.$ .... THEOREM *of a Natural Pairing Operator*

¶ .... DEM

133.1 $\Box\forall\psi(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\psi x y z)\vdash\exists! f f{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}\wedge\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash \psi x y z\Leftrightarrow(f x)y=z.$ .... THM

133.2 $\Box(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t=x\vee t=y)\vdash\exists! f f{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}\wedge\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t=x\vee t=y)\Leftrightarrow(f x)y=z.$ .... **[.1]** $\forall_2$INST

133.3 $\Box\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t=x\vee t=y.$ .... THM

133.4 $\Box\exists! f f{:}\mathscr{H}\to\mathscr{H}\to\mathscr{H}\wedge\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t=x\vee t=y)\Leftrightarrow(f x)y=z.$. **[.2] [.3]** MP

■. .... QED

Uses Axioms: **3, 5, 6, 7, 8, 9, 12**

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134 $\exists\phi\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t=x\vee t=y)\Leftrightarrow x\phi y=z.$ .... THEOREM *of the Existence of a Pairing Realizer*

¶ .... DEM

134.1 $\Box\forall\psi(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\psi x y z)\vdash\exists\phi\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi x y z\Leftrightarrow x\phi y=z.$ .... THM

134.2 $\Box(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t=x\vee t=y)\vdash\exists\phi\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t=x\vee t=y)\Leftrightarrow x\phi y=z.$ .... **[.1]** $\forall_2$INST

134.3 $\Box\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t=x\vee t=y.$ .... THM

134.4 $\Box\exists\phi\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t=x\vee t=y)\Leftrightarrow x\phi y=z.$ .... **[.2] [.3]** MP

■. .... QED

Uses Axioms: **3, 5, 6, 7, 8, 9, 12**

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135 $\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t=x\vee t=y)\Leftrightarrow x\,\mathrm{PAR}\,y=z.$ .... DEFINITION *of a Pairing Realizer*

¶ .... DEM

135.1 $\Box\exists\phi\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t=x\vee t=y)\Leftrightarrow x\phi y=z.$ .... THM

135.2 $\Box\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t=x\vee t=y)\Leftrightarrow x\,\mathrm{PAR}\,y=z.$ .... **[.1]** $_2$NAME

■. .... QED

Uses Axioms: **3, 5, 6, 7, 8, 9, 12**

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136 $\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y{:}\mathscr{H}\wedge\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y.$ .... THEOREM *of Pairing 1*

¶ .... DEM

136.1 $\Box x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y{:}\mathscr{H}.$ .... LEM

□¶ .... DEM

136.2 $\Box\Box x{:}\mathscr{H}.$ .... **[.1]** ASM

136.3 $\Box\Box y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y{:}\mathscr{H}\wedge\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y.$ .... LEM

□□¶ .... DEM

$\square\square\square y{:}\mathscr{H}$ . . . . . . [.3] ASM **136.4**

$\square\square\square\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t=x\vee t=y$ . . . . . . THM **136.5**

$\square\square\square\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t=x\vee t=y)\Leftrightarrow x\,\mathrm{PAR}\,y=z$ . . . . . . DEF **136.6**

$\square\square\square\forall\psi(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\psi xyz)\vdash\forall\phi(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=$ **136.7**
$z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\psi xy(x\phi y)$ . . . . . . THM

$\square\square\square(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t=x\vee t=y)\vdash\forall\phi(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}$ **136.8**
$z\Leftrightarrow t=x\vee t=y)\Leftrightarrow x\phi y=z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\forall t t{:}x\phi y\Leftrightarrow t=x\vee t=y$ . . . . . . [.7] $\forall_2$INST

$\square\square\square\forall\phi(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t=x\vee t=y)\Leftrightarrow x\phi y=z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}$ **136.9**
$\mathscr{H}\wedge\forall t t{:}x\phi y\Leftrightarrow t=x\vee t=y$ . . . . . . [.8] [.5] MP

$\square\square\square(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t=x\vee t=y)\Leftrightarrow x\,\mathrm{PAR}\,y=z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash$ **136.10**
$x\,\mathrm{PAR}\,y{:}\mathscr{H}\wedge\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$ . . . . . . [.9] $\forall_2$INST

$\square\square\square\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y{:}\mathscr{H}\wedge\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y f stop$ . . . . . . [.10] [.6] MP **136.11**

$\square\square\square x\,\mathrm{PAR}\,y{:}\mathscr{H}\wedge\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$ . . . . . . [.11] [.2] [.4] $_2$MINT$_2$ **136.12**

$\square\square\blacksquare$ . . . . . . QED

$\square\blacksquare$ . . . . . . QED

$\square\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y{:}\mathscr{H}\wedge\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$ . . . . . . [.1] UGEN$_2$ **136.13**

$\blacksquare$ . . . . . . QED

Uses Axioms: **3, 5, 6, 7, 8, 9, 12**

$\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y{:}\mathscr{H}$ . . . . . . THEOREM *of Pairing 2* **137**

¶ . . . . . . DEM

$\square x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y{:}\mathscr{H}$ . . . . . . LEM **137.1**

$\square$¶ . . . . . . DEM

$\square\square x{:}\mathscr{H}$ . . . . . . [.1] ASM **137.2**

$\square\square y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y{:}\mathscr{H}$ . . . . . . LEM **137.3**

$\square\square$¶ . . . . . . DEM

$\square\square\square y{:}\mathscr{H}$ . . . . . . [.3] ASM **137.4**

$\square\square\square\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y{:}\mathscr{H}\wedge\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$ . . . . . . THM **137.5**

$\square\square\square x\,\mathrm{PAR}\,y{:}\mathscr{H}\wedge\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$ . . . . . . [.5] [.2] [.4] $_2$MINT$_2$ **137.6**

$\square\square\square x\,\mathrm{PAR}\,y{:}\mathscr{H}$ . . . . . . [.6] TAUT **137.7**

$\square\square\blacksquare$ . . . . . . QED

$\square\blacksquare$ . . . . . . QED

$\square\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y{:}\mathscr{H}$ . . . . . . [.1] UGEN$_2$ **137.8**

$\blacksquare$ . . . . . . QED

Uses Axioms: **3, 5, 6, 7, 8, 9, 12**

$\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$ . . . . . . THEOREM *of Pairing 3* **138**

¶ . . . . . . DEM

$\square x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$ . . . . . . LEM **138.1**

$\square$¶ . . . . . . DEM

$\square\square x{:}\mathscr{H}$ . . . . . . [.1] ASM **138.2**

$\square\square y{:}\mathscr{H}\vdash\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$ . . . . . . LEM **138.3**

$\square\square$¶ . . . . . . DEM

138.4 $\square\square\square y{:}\mathscr{H}$. .... [.3] ASM
138.5 $\square\square\square\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y{:}\mathscr{H}\wedge\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$. .... THM
138.6 $\square\square\square x\,\mathrm{PAR}\,y{:}\mathscr{H}\wedge\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$. .... [.5] [.2] [.4] $_2$MINT$_2$
138.7 $\square\square\square\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$. .... [.6] TAUT
$\square\square\blacksquare$. .... QED
$\square\blacksquare$. .... QED
138.8 $\square\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$. .... [.1] UGEN$_2$
$\blacksquare$. .... QED
Uses Axioms: **3, 5, 6, 7, 8, 9, 12**

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139 $\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x{:}x\,\mathrm{PAR}\,y\wedge y{:}x\,\mathrm{PAR}\,y$. .... THEOREM *of Pairing 4*
¶ .... DEM
139.1 $\square x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y{:}\mathscr{H}$. .... LEM
$\square$¶ .... DEM
139.2 $\square\square x{:}\mathscr{H}$. .... [.1] ASM
139.3 $\square\square y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y{:}\mathscr{H}$. .... LEM
$\square\square$¶ .... DEM
139.4 $\square\square\square y{:}\mathscr{H}$. .... [.3] ASM
139.5 $\square\square\square\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$. .... THM
139.6 $\square\square\square\forall t t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$. .... [.5] [.2] [.4] $_2$MINT$_2$
139.7 $\square\square\square x{:}x\,\mathrm{PAR}\,y\Leftrightarrow x=x\vee x=y$. .... [.6] $\forall_1$INST
139.8 $\square\square\square x=x$. .... EQ
139.9 $\square\square\square x{:}x\,\mathrm{PAR}\,y$. .... [.7] [.8] TAUT
139.10 $\square\square\square y{:}x\,\mathrm{PAR}\,y\Leftrightarrow y=x\vee y=y$. .... [.6] $\forall_1$INST
139.11 $\square\square\square y=y$. .... EQ
139.12 $\square\square\square x{:}x\,\mathrm{PAR}\,y\wedge y{:}x\,\mathrm{PAR}\,y$. .... [.9] [.10] [.11] TAUT
$\square\square\blacksquare$. .... QED
$\square\blacksquare$. .... QED
139.13 $\square\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x{:}x\,\mathrm{PAR}\,y\wedge y{:}x\,\mathrm{PAR}\,y$. .... [.1] UGEN$_2$
$\blacksquare$. .... QED
Uses Axioms: **3, 4, 5, 6, 7, 8, 9, 12**

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140 $\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y=y\,\mathrm{PAR}\,x$. .... THEOREM *of the Commutative Pairing*
¶ .... DEM
140.1 $\square x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y=y\,\mathrm{PAR}\,x$. .... LEM
$\square$¶ .... DEM
140.2 $\square\square x{:}\mathscr{H}$. .... [.1] ASM
140.3 $\square\square y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y=y\,\mathrm{PAR}\,x$. .... LEM
$\square\square$¶ .... DEM
140.4 $\square\square\square y{:}\mathscr{H}$. .... [.3] ASM
140.5 $\square\square\square\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash(\forall t t{:}x\Leftrightarrow t{:}y)\vdash x=y$. .... AXM
140.6 $\square\square\square\forall a\forall b a{:}\mathscr{H}\vdash b{:}\mathscr{H}\vdash(\forall t t{:}a\Leftrightarrow t{:}b)\vdash a=b$. .... [.5] QNT
140.7 $\square\square\square\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y{:}\mathscr{H}$. .... THM

$\square\square\square\forall a\forall b\,a{:}\mathscr{H}\vdash b{:}\mathscr{H}\vdash a\,\mathrm{PAR}\,b{:}\mathscr{H}$ . . . . [.7] QNT **140.8**

$\square\square\square x\,\mathrm{PAR}\,y{:}\mathscr{H}$ . . . . [.8] [.2] [.4] $_2\mathrm{MINT}_2$ **140.9**

$\square\square\square y\,\mathrm{PAR}\,x{:}\mathscr{H}$ . . . . [.8] [.4] [.2] $_2\mathrm{MINT}_2$ **140.10**

$\square\square\square(\forall t\,t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t{:}y\,\mathrm{PAR}\,x)\vdash x\,\mathrm{PAR}\,y=y\,\mathrm{PAR}\,x$ . . . . [.6] [.9] [.10] $_2\mathrm{MINT}_2$ **140.11**

$\square\square\square\forall x\forall y\,x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\forall t\,t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$ . . . . THM **140.12**

$\square\square\square\forall a\forall b\,a{:}\mathscr{H}\vdash b{:}\mathscr{H}\vdash\forall t\,t{:}a\,\mathrm{PAR}\,b\Leftrightarrow t=a\vee t=b$ . . . . [.12] QNT **140.13**

$\square\square\square\forall t\,t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$ . . . . [.13] [.2] [.4] $_2\mathrm{MINT}_2$ **140.14**

$\square\square\square t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$ . . . . [.14] $\forall_1$INST **140.15**

$\square\square\square\forall t\,t{:}y\,\mathrm{PAR}\,x\Leftrightarrow t=y\vee t=x$ . . . . [.13] [.4] [.2] $_2\mathrm{MINT}_2$ **140.16**

$\square\square\square t{:}y\,\mathrm{PAR}\,x\Leftrightarrow t=y\vee t=x$ . . . . [.16] $\forall_1$INST **140.17**

$\square\square\square t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t{:}y\,\mathrm{PAR}\,x$ . . . . [.15] [.17] TAUT **140.18**

$\square\square\square\forall t\,t{:}x\,\mathrm{PAR}\,y\Leftrightarrow t=x\vee t=y$ . . . . [.18] $\forall_1$GEN **140.19**

$\square\square\square x\,\mathrm{PAR}\,y=y\,\mathrm{PAR}\,x$ . . . . [.11] [.19] MP **140.20**

$\square\square\blacksquare$ . . . . QED

$\square\blacksquare$ . . . . QED

$\square\forall x\forall y\,x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathrm{PAR}\,y=y\,\mathrm{PAR}\,x$ . . . . [.1] UGEN_2 **140.21**

$\blacksquare$ . . . . QED

Uses Axioms: **3, 4, 5, 6, 7, 8, 9, 12**

For all sets x, one can exhibit a unique set y which satisfies $\forall t\,t{:}y\Leftrightarrow\exists x_1\,x_1{:}x\wedge t{:}x_1$. PROOF: *Existence.* Guaranteed by the *union* axiom. *Uniqueness.* Straightforward. QED.
Imagine unpacking all elements of a box, and forming a new box containing all elements of their elements. We name this realizer $\bigcup$. Together with *pairing*, we are able to construct all the finite sets of our intuition.

$\forall x\,x{:}\mathscr{H}\vdash\exists!y\,y{:}\mathscr{H}\wedge\forall t\,t{:}y\Leftrightarrow\exists x_1\,x_1{:}x\wedge t{:}x_1$ . . . . THEOREM *of Implicit Upper Union* **141**

¶ . . . . DEM

$\square x{:}\mathscr{H}\vdash\exists!y\,y{:}\mathscr{H}\wedge\forall t\,t{:}y\Leftrightarrow\exists x_1\,x_1{:}x\wedge t{:}x_1$ . . . . LEM **141.1**

$\square$¶ . . . . DEM

$\square\square x{:}\mathscr{H}$ . . . . [.1] ASM **141.2**

$\square\square\forall x\,x{:}\mathscr{H}\vdash\exists y\,y{:}\mathscr{H}\wedge\forall y_1\,y_1{:}y\Leftrightarrow\exists x_1\,x_1{:}x\wedge y_1{:}x_1$ . . . . AXM **141.3**

$\square\square\exists y\,y{:}\mathscr{H}\wedge\forall y_1\,y_1{:}y\Leftrightarrow\exists x_1\,x_1{:}x\wedge y_1{:}x_1$ . . . . [.3] [.2] $_1\mathrm{MINT}_1$ **141.4**

$\square\square\exists y\,y{:}\mathscr{H}\wedge\forall t\,t{:}y\Leftrightarrow\exists x_1\,x_1{:}x\wedge t{:}x_1$ . . . . [.4] QNT **141.5**

$\square\square(y_1{:}\mathscr{H}\wedge\forall t\,t{:}y_1\Leftrightarrow\exists x_1\,x_1{:}x\wedge t{:}x_1)\vdash(y_2{:}\mathscr{H}\wedge\forall t\,t{:}y_2\Leftrightarrow\exists x_1\,x_1{:}x\wedge t{:}x_1)\vdash y_1=y_2$ . . . . LEM **141.6**

$\square\square$¶ . . . . DEM

$\square\square\square y_1{:}\mathscr{H}\wedge\forall t\,t{:}y_1\Leftrightarrow\exists x_1\,x_1{:}x\wedge t{:}x_1$ . . . . [.6] ASM **141.7**

$\square\square\square(y_2{:}\mathscr{H}\wedge\forall t\,t{:}y_2\Leftrightarrow\exists x_1\,x_1{:}x\wedge t{:}x_1)\vdash y_1=y_2$ . . . . LEM **141.8**

$\square\square\square$¶ . . . . DEM

$\square\square\square\square y_2{:}\mathscr{H}\wedge\forall t\,t{:}y_2\Leftrightarrow\exists x_1\,x_1{:}x\wedge t{:}x_1$ . . . . [.8] ASM **141.9**

$\square\square\square\square\forall x\forall y\,x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash(\forall t\,t{:}x\Leftrightarrow t{:}y)\vdash x=y$ . . . . AXM **141.10**

$\square\square\square\square y_1{:}\mathscr{H}\vdash y_2{:}\mathscr{H}\vdash(\forall t\,t{:}y_1\Leftrightarrow t{:}y_2)\vdash y_1=y_2$ . . . . [.10] $_0\mathrm{MINT}_2$ **141.11**

$\square\square\square\square\forall t\,t{:}y_1\Leftrightarrow\exists x_1\,x_1{:}x\wedge t{:}x_1$ . . . . [.7] TAUT **141.12**

$\square\square\square\square t{:}y_1\Leftrightarrow\exists x_1\,x_1{:}x\wedge t{:}x_1$ . . . . [.12] $\forall_1$INST **141.13**

$\square\square\square\square\forall t\,t{:}y_2\Leftrightarrow\exists x_1\,x_1{:}x\wedge t{:}x_1$ . . . . [.9] TAUT **141.14**

141.15 $\square\square\square\square t{:}y_2 \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1$. .... [.14] $\forall_1$INST
141.16 $\square\square\square\square t{:}y_1 \Leftrightarrow t{:}y_2$. .... [.13] [.15] TAUT
141.17 $\square\square\square\square \forall tt{:}y_1 \Leftrightarrow t{:}y_2$. .... [.16] $\forall_1$GEN
141.18 $\square\square\square\square y_1 = y_2$. .... [.11] [.7] [.9] [.17] TAUT
$\square\square\square\blacksquare$. .... QED
$\square\square\blacksquare$. .... QED
141.19 $\square\square \forall y_1 \forall y_2 (y_1{:}\mathscr{H} \wedge \forall tt{:}y_1 \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1) \vdash (y_2{:}\mathscr{H} \wedge \forall tt{:}y_2 \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1) \vdash y_1 = y_2$. .... [.6] UGEN$_2$
141.20 $\square\square \exists! yy{:}\mathscr{H} \wedge \forall tt{:}y \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1$. .... [.5] [.19] $\exists!$INTROS
$\square\blacksquare$. .... QED
141.21 $\square \forall xx{:}\mathscr{H} \vdash \exists! yy{:}\mathscr{H} \wedge \forall tt{:}y \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1$. .... [.1] $\forall_1$GEN
$\blacksquare$. .... QED
Uses Axioms: **3, 5, 9, 13**

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142 $\forall xx{:}\mathscr{H} \vdash \bigcup x{:}\mathscr{H} \wedge \forall tt{:}\bigcup x \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1$. .... DEFINITION *of Upper Union*
¶ .... DEM
142.1 $\square x{:}\mathscr{H} \vdash \bigcup x{:}\mathscr{H} \wedge \forall tt{:}\bigcup x \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1$. .... LEM
$\square$¶ .... DEM
142.2 $\square\square x{:}\mathscr{H}$. .... [.1] ASM
142.3 $\square\square \forall xx{:}\mathscr{H} \vdash \exists! yy{:}\mathscr{H} \wedge \forall tt{:}y \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1$. .... THM
142.4 $\square\square \forall \psi (\forall xx{:}\mathscr{H} \vdash \exists! yy{:}\mathscr{H} \wedge \psi xy) \vdash \exists \phi \forall x \forall yx{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow \phi x = y$. .... THM
142.5 $\square\square (\forall xx{:}\mathscr{H} \vdash \exists! yy{:}\mathscr{H} \wedge \forall tt{:}y \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1) \vdash \exists \phi \forall x \forall yx{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash (\forall tt{:}y \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1) \Leftrightarrow \phi x = y$. .... [.4] $\forall_2$INST
142.6 $\square\square \exists \phi \forall x \forall yx{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash (\forall tt{:}y \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1) \Leftrightarrow \phi x = y$. .... [.5] [.3] MP
142.7 $\square\square \forall x \forall yx{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash (\forall tt{:}y \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1) \Leftrightarrow \bigcup x = y$. .... [.6] ${}_2$NAME
142.8 $\square\square \forall \psi (\forall xx{:}\mathscr{H} \vdash \exists! yy{:}\mathscr{H} \wedge \psi xy) \vdash \forall \phi (\forall x \forall yx{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \psi xy \Leftrightarrow \phi x = y) \vdash \forall xx{:}\mathscr{H} \vdash \phi x{:}\mathscr{H} \wedge \psi x(\phi x)$. .... THM
142.9 $\square\square (\forall xx{:}\mathscr{H} \vdash \exists! yy{:}\mathscr{H} \wedge \forall tt{:}y \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1) \vdash \forall \phi (\forall x \forall yx{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash (\forall tt{:}y \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1) \Leftrightarrow \phi x = y) \vdash \forall xx{:}\mathscr{H} \vdash \phi x{:}\mathscr{H} \wedge \forall tt{:}\phi x \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1$. .... [.8] $\forall_2$INST
142.10 $\square\square \forall \phi (\forall x \forall yx{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash (\forall tt{:}y \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1) \Leftrightarrow \phi x = y) \vdash \forall xx{:}\mathscr{H} \vdash \phi x{:}\mathscr{H} \wedge \forall tt{:}\phi x \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1$. .... [.9] [.3] MP
142.11 $\square\square (\forall x \forall yx{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash (\forall tt{:}y \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1) \Leftrightarrow \bigcup x = y) \vdash \forall xx{:}\mathscr{H} \vdash \bigcup x{:}\mathscr{H} \wedge \forall tt{:}\bigcup x \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1$. .... [.10] $\forall_2$INST
142.12 $\square\square \forall xx{:}\mathscr{H} \vdash \bigcup x{:}\mathscr{H} \wedge \forall tt{:}\bigcup x \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1$. .... [.11] [.7] MP
142.13 $\square\square \bigcup x{:}\mathscr{H} \wedge \forall tt{:}\bigcup x \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1$. .... [.12] [.2] ${}_1$MINT$_1$
$\square\blacksquare$. .... QED
142.14 $\square \forall xx{:}\mathscr{H} \vdash \bigcup x{:}\mathscr{H} \wedge \forall tt{:}\bigcup x \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1$. .... [.1] $\forall_1$GEN
$\blacksquare$. .... QED
Uses Axioms: **3, 5, 6, 7, 8, 9, 13**

For all sets x,y, one can exhibit a unique set z which satisfies $\forall tt{:}z \Leftrightarrow t{:}x \vee t{:}y$. PROOF: *Existence.* The set formed by $\bigcup x\,\mathbf{PAR}\,y$ has the desired property. *Uniqueness.* Straightforward. QED. Imagine forming a new box by packing all contents of two old ones. We name this realizer $\cup$.

$\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \exists! z z{:}\mathscr{H} \wedge \forall t t{:}z \Leftrightarrow t{:}x \vee t{:}y.$	THEOREM *of Implicit Lower Union*	**143**
¶	DEM	
□$x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \exists! z z{:}\mathscr{H} \wedge \forall t t{:}z \Leftrightarrow t{:}x \vee t{:}y.$	LEM	**143.1**
□¶	DEM	
□□$x{:}\mathscr{H}.$	[.1] ASM	**143.2**
□□$y{:}\mathscr{H} \vdash \exists! z z{:}\mathscr{H} \wedge \forall t t{:}z \Leftrightarrow t{:}x \vee t{:}y.$	LEM	**143.3**
□□¶	DEM	
□□□$y{:}\mathscr{H}.$	[.3] ASM	**143.4**
□□□$\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash x \,\mathrm{PAR}\, y{:}\mathscr{H}.$	THM	**143.5**
□□□$x \,\mathrm{PAR}\, y{:}\mathscr{H}.$	[.5] [.2] [.4] $_2$MINT$_2$	**143.6**
□□□$\forall x x{:}\mathscr{H} \vdash \bigcup x{:}\mathscr{H} \wedge \forall t t{:}\bigcup x \Leftrightarrow \exists x_1 x_1{:}x \wedge t{:}x_1.$	DEF	**143.7**
□□□$\bigcup x \,\mathrm{PAR}\, y{:}\mathscr{H} \wedge \forall t t{:}\bigcup x \,\mathrm{PAR}\, y \Leftrightarrow \exists x_1 x_1{:}x \,\mathrm{PAR}\, y \wedge t{:}x_1.$	[.7] [.6] $_1$MINT$_1$	**143.8**
□□□$\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \forall t t{:}x \,\mathrm{PAR}\, y \Leftrightarrow t = x \vee t = y.$	THM	**143.9**
□□□$\forall t t{:}x \,\mathrm{PAR}\, y \Leftrightarrow t = x \vee t = y.$	[.9] [.2] [.4] $_2$MINT$_2$	**143.10**
□□□$t{:}\bigcup x \,\mathrm{PAR}\, y \vdash t{:}x \vee t{:}y.$	LEM	**143.11**
□□□¶	DEM	
□□□□$t{:}\bigcup x \,\mathrm{PAR}\, y.$	[.11] ASM	**143.12**
□□□□$\forall t t{:}\bigcup x \,\mathrm{PAR}\, y \Leftrightarrow \exists x_1 x_1{:}x \,\mathrm{PAR}\, y \wedge t{:}x_1.$	[.8] TAUT	**143.13**
□□□□$t{:}\bigcup x \,\mathrm{PAR}\, y \Leftrightarrow \exists x_1 x_1{:}x \,\mathrm{PAR}\, y \wedge t{:}x_1.$	[.13] $\forall_1$INST	**143.14**
□□□□$\exists x_1 x_1{:}x \,\mathrm{PAR}\, y \wedge t{:}x_1.$	[.14] [.12] TAUT	**143.15**
□□□□$x_1{:}x \,\mathrm{PAR}\, y \wedge t{:}x_1.$	[.15] $\exists_1$INST	**143.16**
□□□□$x_1{:}x \,\mathrm{PAR}\, y \Leftrightarrow x_1 = x \vee x_1 = y.$	[.10] $\forall_1$INST	**143.17**
□□□□$x_1 = x \vee x_1 = y.$	[.17] [.16] TAUT	**143.18**
□□□□$x_1 = x \vdash t{:}x \vee t{:}y.$	LEM	**143.19**
□□□□¶	DEM	
□□□□□$x_1 = x.$	[.19] ASM	**143.20**
□□□□□$t{:}x_1 \vee t{:}y.$	[.16] TAUT	**143.21**
□□□□□$t{:}x \vee t{:}y.$	[.20] [.21] SUB	**143.22**
□□□□■.	QED	
□□□□$x_1 = y \vdash t{:}x \vee t{:}y.$	LEM	**143.23**
□□□□¶	DEM	
□□□□□$x_1 = y.$	[.23] ASM	**143.24**
□□□□□$t{:}x \vee t{:}x_1.$	[.16] TAUT	**143.25**
□□□□□$t{:}x \vee t{:}y.$	[.24] [.25] SUB	**143.26**
□□□□■.	QED	
□□□□$t{:}x \vee t{:}y.$	[.18] [.19] [.23] TAUT	**143.27**
□□□■.	QED	
□□□$t{:}x \vee t{:}y \vdash t{:}\bigcup x \,\mathrm{PAR}\, y.$	LEM	**143.28**
□□□¶	DEM	
□□□□$t{:}x \vee t{:}y.$	[.28] ASM	**143.29**
□□□□$\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash x{:}x \,\mathrm{PAR}\, y \wedge y{:}x \,\mathrm{PAR}\, y.$	THM	**143.30**
□□□□$x{:}x \,\mathrm{PAR}\, y \wedge y{:}x \,\mathrm{PAR}\, y.$	[.30] [.2] [.4] $_2$MINT$_2$	**143.31**
□□□□$t{:}x \vdash t{:}\bigcup x \,\mathrm{PAR}\, y.$	LEM	**143.32**

□□□□¶ . . . DEM
143.33 □□□□□$t{:}x$. . . . [.32] ASM
143.34 □□□□□$x{:}x\,\mathrm{PAR}\,y \wedge t{:}x$. . . . [.31] [.33] TAUT
143.35 □□□□□$\exists x_1 x_1{:}x\,\mathrm{PAR}\,y \wedge t{:}x_1$. . . . [.34] $\exists_1$GEN
143.36 □□□□□$t{:}\bigcup x\,\mathrm{PAR}\,y$. . . . [.14] [.35] TAUT
□□□□■. . . . QED
143.37 □□□□$t{:}y \vdash t{:}\bigcup x\,\mathrm{PAR}\,y$. . . . LEM
□□□□¶ . . . DEM
143.38 □□□□□$t{:}y$. . . . [.37] ASM
143.39 □□□□□$y{:}x\,\mathrm{PAR}\,y \wedge t{:}y$. . . . [.31] [.38] TAUT
143.40 □□□□□$\exists x_1 x_1{:}x\,\mathrm{PAR}\,y \wedge t{:}x_1$. . . . [.39] $\exists_1$GEN
143.41 □□□□□$t{:}\bigcup x\,\mathrm{PAR}\,y$. . . . [.14] [.40] TAUT
□□□□■. . . . QED
143.42 □□□□$t{:}\bigcup x\,\mathrm{PAR}\,y$. . . . [.29] [.32] [.37] TAUT
□□□■. . . . QED
143.43 □□□$t{:}\bigcup x\,\mathrm{PAR}\,y \Leftrightarrow t{:}x \vee t{:}y$. . . . [.11] [.28] TAUT
143.44 □□□$\forall t t{:}\bigcup x\,\mathrm{PAR}\,y \Leftrightarrow t{:}x \vee t{:}y$. . . . [.43] $\forall_1$GEN
143.45 □□□$\bigcup x\,\mathrm{PAR}\,y{:}\mathscr{H} \wedge \forall t t{:}\bigcup x\,\mathrm{PAR}\,y \Leftrightarrow t{:}x \vee t{:}y$. . . . [.8] [.44] TAUT
143.46 □□□$\exists z z{:}\mathscr{H} \wedge \forall t t{:}z \Leftrightarrow t{:}x \vee t{:}y$. . . . [.45] $\exists_1$GEN
143.47 □□□$(z_1{:}\mathscr{H} \wedge \forall t t{:}z_1 \Leftrightarrow t{:}x \vee t{:}y) \vdash (z_2{:}\mathscr{H} \wedge \forall t t{:}z_2 \Leftrightarrow t{:}x \vee t{:}y) \vdash z_1 = z_2$. . . . LEM
□□□¶ . . . DEM
143.48 □□□□$z_1{:}\mathscr{H} \wedge \forall t t{:}z_1 \Leftrightarrow t{:}x \vee t{:}y$. . . . [.47] ASM
143.49 □□□□$(z_2{:}\mathscr{H} \wedge \forall t t{:}z_2 \Leftrightarrow t{:}x \vee t{:}y) \vdash z_1 = z_2$. . . . LEM
□□□□¶ . . . DEM
143.50 □□□□□$z_2{:}\mathscr{H} \wedge \forall t t{:}z_2 \Leftrightarrow t{:}x \vee t{:}y$. . . . [.49] ASM
143.51 □□□□□$\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash (\forall t t{:}x \Leftrightarrow t{:}y) \vdash x = y$. . . . AXM
143.52 □□□□□$z_1{:}\mathscr{H} \vdash z_2{:}\mathscr{H} \vdash (\forall t t{:}z_1 \Leftrightarrow t{:}z_2) \vdash z_1 = z_2$. . . . [.51] $_0$MINT$_2$
143.53 □□□□□$\forall t t{:}z_1 \Leftrightarrow t{:}x \vee t{:}y$. . . . [.48] TAUT
143.54 □□□□□$\forall t t{:}z_2 \Leftrightarrow t{:}x \vee t{:}y$. . . . [.50] TAUT
143.55 □□□□□$t{:}z_1 \Leftrightarrow t{:}x \vee t{:}y$. . . . [.53] $\forall_1$INST
143.56 □□□□□$t{:}z_2 \Leftrightarrow t{:}x \vee t{:}y$. . . . [.54] $\forall_1$INST
143.57 □□□□□$t{:}z_1 \Leftrightarrow t{:}z_2$. . . . [.55] [.56] TAUT
143.58 □□□□□$\forall t t{:}z_1 \Leftrightarrow t{:}z_2$. . . . [.57] $\forall_1$GEN
143.59 □□□□□$z_1 = z_2$. . . . [.52] [.48] [.50] [.58] TAUT
□□□□■. . . . QED
□□□■. . . . QED
143.60 □□□$\forall z_1 \forall z_2 (z_1{:}\mathscr{H} \wedge \forall t t{:}z_1 \Leftrightarrow t{:}x \vee t{:}y) \vdash (z_2{:}\mathscr{H} \wedge \forall t t{:}z_2 \Leftrightarrow t{:}x \vee t{:}y) \vdash z_1 = z_2$. . . . [.47] UGEN$_2$
143.61 □□□$\exists ! z z{:}\mathscr{H} \wedge \forall t t{:}z \Leftrightarrow t{:}x \vee t{:}y$. . . . [.46] [.60] $\exists!$INTROS
□□■. . . . QED
□■. . . . QED
143.62 □$\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \exists ! z z{:}\mathscr{H} \wedge \forall t t{:}z \Leftrightarrow t{:}x \vee t{:}y$. . . . [.1] UGEN$_2$
■. . . . QED

Uses Axioms: **3, 4, 5, 6, 7, 8, 9, 12, 13**

$\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\cup y{:}\mathscr{H}\wedge\forall t t{:}x\cup y\Leftrightarrow t{:}x\vee t{:}y.$ .... DEFINITION *of Lower Union* **144**

¶ .... DEM

$\Box x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\cup y{:}\mathscr{H}\wedge\forall t t{:}x\cup y\Leftrightarrow t{:}x\vee t{:}y.$ .... LEM **144.1**

$\Box$¶ .... DEM

$\Box\Box x{:}\mathscr{H}.$ .... **[.1]** ASM **144.2**

$\Box\Box y{:}\mathscr{H}\vdash x\cup y{:}\mathscr{H}\wedge\forall t t{:}x\cup y\Leftrightarrow t{:}x\vee t{:}y.$ .... LEM **144.3**

$\Box\Box$¶ .... DEM

$\Box\Box\Box y{:}\mathscr{H}.$ .... **[.3]** ASM **144.4**

$\Box\Box\Box\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\vee t{:}y.$ .... THM **144.5**

$\Box\Box\Box\forall\psi(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\psi xyz)\vdash\exists\phi\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=z.$ .... THM **144.6**

$\Box\Box\Box(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\vee t{:}y)\vdash\exists\phi\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t{:}x\vee t{:}y)\Leftrightarrow x\phi y=z.$ .... **[.6]** $\forall_2$INST **144.7**

$\Box\Box\Box\exists\phi\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t{:}x\vee t{:}y)\Leftrightarrow x\phi y=z.$ .... **[.7] [.5]** MP **144.8**

$\Box\Box\Box\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t{:}x\vee t{:}y)\Leftrightarrow x\cup y=z.$ .... **[.8]** ${}_2$NAME **144.9**

$\Box\Box\Box\forall\psi(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\psi xyz)\vdash\forall\phi(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\psi xy(x\phi y).$ .... THM **144.10**

$\Box\Box\Box(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\vee t{:}y)\vdash\forall\phi(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t{:}x\vee t{:}y)\Leftrightarrow x\phi y=z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\forall t t{:}x\phi y\Leftrightarrow t{:}x\vee t{:}y.$ .... **[.10]** $\forall_2$INST **144.11**

$\Box\Box\Box\forall\phi(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t{:}x\vee t{:}y)\Leftrightarrow x\phi y=z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\forall t t{:}x\phi y\Leftrightarrow t{:}x\vee t{:}y.$ .... **[.11] [.5]** MP **144.12**

$\Box\Box\Box(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t{:}x\vee t{:}y)\Leftrightarrow x\cup y=z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\cup y{:}\mathscr{H}\wedge\forall t t{:}x\cup y\Leftrightarrow t{:}x\vee t{:}y.$ .... **[.12]** $\forall_2$INST **144.13**

$\Box\Box\Box\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\cup y{:}\mathscr{H}\wedge\forall t t{:}x\cup y\Leftrightarrow t{:}x\vee t{:}y.$ .... **[.13] [.9]** MP **144.14**

$\Box\Box\Box x\cup y{:}\mathscr{H}\wedge\forall t t{:}x\cup y\Leftrightarrow t{:}x\vee t{:}y.$ .... **[.14] [.2] [.4]** ${}_2$MINT$_2$ **144.15**

$\Box\Box\blacksquare.$ .... QED

$\Box\blacksquare.$ .... QED

$\Box\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\cup y{:}\mathscr{H}\wedge\forall t t{:}x\cup y\Leftrightarrow t{:}x\vee t{:}y.$ .... **[.1]** UGEN$_2$ **144.16**

$\blacksquare.$ .... QED

Uses Axioms: **3, 4, 5, 6, 7, 8, 9, 12, 13**

§2.3 *Sets and Operations: Power Set and Intersection*

$\boldsymbol{x}{\subset}\boldsymbol{y}$ iff. every inhabitant of $\boldsymbol{x}$ inhabits $\boldsymbol{y}$. PROOF: There exists, trivially, a formula $\boldsymbol{\psi}$ whose image under $\boldsymbol{x},\boldsymbol{y}$ is logically equivalent to $\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{x}\vdash \boldsymbol{t}{:}\boldsymbol{y}$. By virtue of naming, it may be so defined. QED.

145 $\forall \boldsymbol{a}\forall \boldsymbol{b}\boldsymbol{a}{\subset}\boldsymbol{b}\Leftrightarrow\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{a}\vdash \boldsymbol{t}{:}\boldsymbol{b}$. .... DEFINITION *of Subtype*
¶ .... DEM
145.1 □$(\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{x}\vdash \boldsymbol{t}{:}\boldsymbol{y})\Leftrightarrow\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{x}\vdash \boldsymbol{t}{:}\boldsymbol{y}$. .... TAUT
145.2 □$\exists\boldsymbol{\psi}\boldsymbol{\psi}\boldsymbol{x}\boldsymbol{y}\Leftrightarrow\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{x}\vdash \boldsymbol{t}{:}\boldsymbol{y}$. .... [.1] $\exists_2$GEN
145.3 □$\boldsymbol{x}{\subset}\boldsymbol{y}\Leftrightarrow\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{x}\vdash \boldsymbol{t}{:}\boldsymbol{y}$. .... [.2] $_2$NAME
145.4 □$\forall \boldsymbol{a}\forall \boldsymbol{b}\boldsymbol{a}{\subset}\boldsymbol{b}\Leftrightarrow\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{a}\vdash \boldsymbol{t}{:}\boldsymbol{b}$. .... [.3] UGEN$_2$
■. .... QED
Uses Axioms: **5**

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146 $\forall \boldsymbol{x}\forall \boldsymbol{y}\boldsymbol{x}{:}\mathscr{H}\vdash \boldsymbol{y}{:}\mathscr{H}\vdash \boldsymbol{x}{\subset}\boldsymbol{y}\vdash \boldsymbol{y}{\subset}\boldsymbol{x}\vdash \boldsymbol{x}=\boldsymbol{y}$. .... THEOREM *of Subset Equality*
¶ .... DEM
146.1 □$\boldsymbol{x}{:}\mathscr{H}\vdash \boldsymbol{y}{:}\mathscr{H}\vdash \boldsymbol{x}{\subset}\boldsymbol{y}\vdash \boldsymbol{y}{\subset}\boldsymbol{x}\vdash \boldsymbol{x}=\boldsymbol{y}$. .... LEM
□¶ .... DEM
146.2 □□$\boldsymbol{x}{:}\mathscr{H}$. .... [.1] ASM
146.3 □□$\boldsymbol{y}{:}\mathscr{H}\vdash \boldsymbol{x}{\subset}\boldsymbol{y}\vdash \boldsymbol{y}{\subset}\boldsymbol{x}\vdash \boldsymbol{x}=\boldsymbol{y}$. .... LEM
□□¶ .... DEM
146.4 □□□$\boldsymbol{y}{:}\mathscr{H}$. .... [.3] ASM
146.5 □□□$\boldsymbol{x}{\subset}\boldsymbol{y}\vdash \boldsymbol{y}{\subset}\boldsymbol{x}\vdash \boldsymbol{x}=\boldsymbol{y}$. .... LEM
□□□¶ .... DEM
146.6 □□□□$\boldsymbol{x}{\subset}\boldsymbol{y}$. .... [.5] ASM
146.7 □□□□$\boldsymbol{y}{\subset}\boldsymbol{x}\vdash \boldsymbol{x}=\boldsymbol{y}$. .... LEM
□□□□¶ .... DEM
146.8 □□□□□$\boldsymbol{y}{\subset}\boldsymbol{x}$. .... [.7] ASM
146.9 □□□□□$\forall \boldsymbol{a}\forall \boldsymbol{b}\boldsymbol{a}{\subset}\boldsymbol{b}\Leftrightarrow\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{a}\vdash \boldsymbol{t}{:}\boldsymbol{b}$. .... DEF
146.10 □□□□□$\boldsymbol{x}{\subset}\boldsymbol{y}\Leftrightarrow\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{x}\vdash \boldsymbol{t}{:}\boldsymbol{y}$. .... [.9] $_0$MINT$_2$
146.11 □□□□□$\boldsymbol{y}{\subset}\boldsymbol{x}\Leftrightarrow\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{y}\vdash \boldsymbol{t}{:}\boldsymbol{x}$. .... [.9] $_0$MINT$_2$
146.12 □□□□□$\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{x}\vdash \boldsymbol{t}{:}\boldsymbol{y}$. .... [.10] TAUT
146.13 □□□□□$\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{y}\vdash \boldsymbol{t}{:}\boldsymbol{x}$. .... [.11] TAUT
146.14 □□□□□$\boldsymbol{t}{:}\boldsymbol{x}\vdash \boldsymbol{t}{:}\boldsymbol{y}$. .... [.12] $\forall_1$INST
146.15 □□□□□$\boldsymbol{t}{:}\boldsymbol{y}\vdash \boldsymbol{t}{:}\boldsymbol{x}$. .... [.13] $\forall_1$INST
146.16 □□□□□$\boldsymbol{t}{:}\boldsymbol{x}\Leftrightarrow \boldsymbol{t}{:}\boldsymbol{y}$. .... [.14] [.15] TAUT
146.17 □□□□□$\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{x}\Leftrightarrow \boldsymbol{t}{:}\boldsymbol{y}$. .... [.16] $\forall_1$GEN
146.18 □□□□□$\forall \boldsymbol{x}\forall \boldsymbol{y}\boldsymbol{x}{:}\mathscr{H}\vdash \boldsymbol{y}{:}\mathscr{H}\vdash(\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{x}\Leftrightarrow \boldsymbol{t}{:}\boldsymbol{y})\vdash \boldsymbol{x}=\boldsymbol{y}$. .... AXM
146.19 □□□□□$\boldsymbol{x}=\boldsymbol{y}$. .... [.18] [.2] [.4] [.17] $_3$MINT$_2$
□□□□■. .... QED
□□□■. .... QED

$\square\square\blacksquare$. . . . . QED

$\square\blacksquare$. . . . . QED

$\square\forall \boldsymbol{x}\forall \boldsymbol{y}\boldsymbol{x}{:}\mathscr{H}\vdash \boldsymbol{y}{:}\mathscr{H}\vdash \boldsymbol{x}\subset \boldsymbol{y}\vdash \boldsymbol{y}\subset \boldsymbol{x}\vdash \boldsymbol{x}=\boldsymbol{y}.$ . . . . [.1] UGEN$_2$ **146.20**

$\blacksquare$. . . . . QED

Uses Axioms: **5, 9**

$\forall \boldsymbol{x}\boldsymbol{x}{:}\mathscr{H}\vdash \mathbf{0}\subset \boldsymbol{x}.$ . . . . THEOREM *of a Natural Subset Relation 1* **147**

¶ . . . . DEM

$\square \boldsymbol{x}{:}\mathscr{H}\vdash \mathbf{0}\subset \boldsymbol{x}.$ . . . . LEM **147.1**

$\square$¶ . . . . DEM

$\square\square \boldsymbol{x}{:}\mathscr{H}.$ . . . . [.1] ASM **147.2**

$\square\square\forall \boldsymbol{a}\forall \boldsymbol{b}\boldsymbol{a}\subset \boldsymbol{b}\Leftrightarrow\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{a}\vdash \boldsymbol{t}{:}\boldsymbol{b}.$ . . . . DEF **147.3**

$\square\square\mathbf{0}\subset \boldsymbol{x}\Leftrightarrow\forall \boldsymbol{t}\boldsymbol{t}{:}\mathbf{0}\vdash \boldsymbol{t}{:}\boldsymbol{x}.$ . . . . [.3] $_0$MINT$_2$ **147.4**

$\square\square\mathbf{0}{:}\mathscr{H}\wedge\forall \boldsymbol{x}\neg \boldsymbol{x}{:}\mathbf{0}.$ . . . . DEF **147.5**

$\square\square\forall \boldsymbol{x}\neg \boldsymbol{x}{:}\mathbf{0}.$ . . . . [.5] TAUT **147.6**

$\square\square\neg \boldsymbol{t}{:}\mathbf{0}.$ . . . . [.6] $\forall_1$INST **147.7**

$\square\square \boldsymbol{t}{:}\mathbf{0}\vdash \boldsymbol{t}{:}\boldsymbol{x}.$ . . . . [.7] TAUT **147.8**

$\square\square\forall \boldsymbol{t}\boldsymbol{t}{:}\mathbf{0}\vdash \boldsymbol{t}{:}\boldsymbol{x}.$ . . . . [.8] $\forall_1$GEN **147.9**

$\square\square\mathbf{0}\subset \boldsymbol{x}.$ . . . . [.4] [.9] TAUT **147.10**

$\square\blacksquare$. . . . . QED

$\square\forall \boldsymbol{x}\boldsymbol{x}{:}\mathscr{H}\vdash \mathbf{0}\subset \boldsymbol{x}.$ . . . . [.1] $\forall_1$GEN **147.11**

$\blacksquare$. . . . . QED

Uses Axioms: **1, 3, 5, 11**

$\forall \boldsymbol{x}\boldsymbol{x}{:}\mathscr{H}\vdash \boldsymbol{x}\subset \boldsymbol{x}.$ . . . . THEOREM *of a Natural Subset Relation 2* **148**

¶ . . . . DEM

$\square \boldsymbol{x}{:}\mathscr{H}\vdash \boldsymbol{x}\subset \boldsymbol{x}.$ . . . . LEM **148.1**

$\square$¶ . . . . DEM

$\square\square \boldsymbol{x}{:}\mathscr{H}.$ . . . . [.1] ASM **148.2**

$\square\square\forall \boldsymbol{a}\forall \boldsymbol{b}\boldsymbol{a}\subset \boldsymbol{b}\Leftrightarrow\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{a}\vdash \boldsymbol{t}{:}\boldsymbol{b}.$ . . . . DEF **148.3**

$\square\square \boldsymbol{x}\subset \boldsymbol{x}\Leftrightarrow\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{x}\vdash \boldsymbol{t}{:}\boldsymbol{x}.$ . . . . [.3] $_0$MINT$_2$ **148.4**

$\square\square \boldsymbol{t}{:}\boldsymbol{x}\vdash \boldsymbol{t}{:}\boldsymbol{x}.$ . . . . TAUT **148.5**

$\square\square\forall \boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{x}\vdash \boldsymbol{t}{:}\boldsymbol{x}.$ . . . . [.5] $\forall_1$GEN **148.6**

$\square\square \boldsymbol{x}\subset \boldsymbol{x}.$ . . . . [.4] [.6] TAUT **148.7**

$\square\blacksquare$. . . . . QED

$\square\forall \boldsymbol{x}\boldsymbol{x}{:}\mathscr{H}\vdash \boldsymbol{x}\subset \boldsymbol{x}.$ . . . . [.1] $\forall_1$GEN **148.8**

$\blacksquare$. . . . . QED

Uses Axioms: **5**

For all sets $\boldsymbol{x}$, one can exhibit a unique set $\boldsymbol{y}$ which contains precisely all the subsets of $\boldsymbol{x}$. PROOF: *Existence.* Guaranteed by the *power set* axiom. *Uniqueness.* Straightforward. QED.
Imagine packing all sub-boxes into a new box. We name this realizer $\mathscr{P}$. Together with *infinite*

application, we are able to construct the elementary cardinals of our intuition.

149 $\forall x x{:}\mathscr{H} \vdash \exists! y y{:}\mathscr{H} \wedge \forall t t{:}y \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x.$ … THEOREM *of Implicit Power Set*

¶ … DEM

149.1 $\Box x{:}\mathscr{H} \vdash \exists! y y{:}\mathscr{H} \wedge \forall t t{:}y \Leftrightarrow t \subset x.$ … LEM

$\Box$¶ … DEM

149.2 $\Box\Box x{:}\mathscr{H}.$ … **[.1]** ASM

149.3 $\Box\Box \forall x x{:}\mathscr{H} \vdash \exists y y{:}\mathscr{H} \wedge \forall y_1 y_1{:}y \Leftrightarrow y_1{:}\mathscr{H} \wedge \forall y_2 y_2{:}y_1 \vdash y_2{:}x.$ … AXM

149.4 $\Box\Box \exists y y{:}\mathscr{H} \wedge \forall y_1 y_1{:}y \Leftrightarrow y_1{:}\mathscr{H} \wedge \forall y_2 y_2{:}y_1 \vdash y_2{:}x.$ … **[.3] [.2]** $_1\mathrm{MINT}_1$

149.5 $\Box\Box y{:}\mathscr{H} \wedge \forall y_1 y_1{:}y \Leftrightarrow y_1{:}\mathscr{H} \wedge \forall y_2 y_2{:}y_1 \vdash y_2{:}x.$ … **[.4]** $\exists_1$INST

149.6 $\Box\Box \forall y_1 y_1{:}y \Leftrightarrow y_1{:}\mathscr{H} \wedge \forall y_2 y_2{:}y_1 \vdash y_2{:}x.$ … **[.5]** TAUT

149.7 $\Box\Box t{:}y \Leftrightarrow t{:}\mathscr{H} \wedge \forall y_2 y_2{:}t \vdash y_2{:}x.$ … **[.6]** $\forall_1$INST

149.8 $\Box\Box \forall a \forall b a \subset b \Leftrightarrow \forall t t{:}a \vdash t{:}b.$ … DEF

149.9 $\Box\Box \forall a \forall b a \subset b \Leftrightarrow \forall y_2 y_2{:}a \vdash y_2{:}b.$ … **[.8]** QNT

149.10 $\Box\Box t \subset x \Leftrightarrow \forall y_2 y_2{:}t \vdash y_2{:}x.$ … **[.9]** $_0\mathrm{MINT}_2$

149.11 $\Box\Box t{:}y \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x.$ … **[.7] [.10]** TAUT

149.12 $\Box\Box \forall t t{:}y \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x.$ … **[.11]** $\forall_1$GEN

149.13 $\Box\Box y{:}\mathscr{H} \wedge \forall t t{:}y \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x.$ … **[.5] [.12]** TAUT

149.14 $\Box\Box \exists y y{:}\mathscr{H} \wedge \forall t t{:}y \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x.$ … **[.13]** $\exists_1$GEN

149.15 $\Box\Box (y_1{:}\mathscr{H} \wedge \forall t t{:}y_1 \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x) \vdash (y_2{:}\mathscr{H} \wedge \forall t t{:}y_2 \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x) \vdash y_1 = y_2.$ … LEM

$\Box\Box$¶ … DEM

149.16 $\Box\Box\Box y_1{:}\mathscr{H} \wedge \forall t t{:}y_1 \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x.$ … **[.15]** ASM

149.17 $\Box\Box\Box (y_2{:}\mathscr{H} \wedge \forall t t{:}y_2 \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x) \vdash y_1 = y_2.$ … LEM

$\Box\Box\Box$¶ … DEM

149.18 $\Box\Box\Box\Box y_2{:}\mathscr{H} \wedge \forall t t{:}y_2 \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x.$ … **[.17]** ASM

149.19 $\Box\Box\Box\Box \forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash (\forall t t{:}x \Leftrightarrow t{:}y) \vdash x = y.$ … AXM

149.20 $\Box\Box\Box\Box y_1{:}\mathscr{H} \vdash y_2{:}\mathscr{H} \vdash (\forall t t{:}y_1 \Leftrightarrow t{:}y_2) \vdash y_1 = y_2.$ … **[.19]** $_0\mathrm{MINT}_2$

149.21 $\Box\Box\Box\Box \forall t t{:}y_1 \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x.$ … **[.16]** TAUT

149.22 $\Box\Box\Box\Box t{:}y_1 \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x.$ … **[.21]** $\forall_1$INST

149.23 $\Box\Box\Box\Box \forall t t{:}y_2 \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x.$ … **[.18]** TAUT

149.24 $\Box\Box\Box\Box t{:}y_2 \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x.$ … **[.23]** $\forall_1$INST

149.25 $\Box\Box\Box\Box t{:}y_1 \Leftrightarrow t{:}y_2.$ … **[.22] [.24]** TAUT

149.26 $\Box\Box\Box\Box \forall t t{:}y_1 \Leftrightarrow t{:}y_2.$ … **[.25]** $\forall_1$GEN

149.27 $\Box\Box\Box\Box y_1 = y_2.$ … **[.19] [.16] [.18] [.26]** TAUT

$\Box\Box\Box\blacksquare.$ … QED

$\Box\Box\blacksquare.$ … QED

149.28 $\Box\Box \forall y_1 \forall y_2 (y_1{:}\mathscr{H} \wedge \forall t t{:}y_1 \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x) \vdash (y_2{:}\mathscr{H} \wedge \forall t t{:}y_2 \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x) \vdash y_1 = y_2.$ … **[.15]** UGEN_2

149.29 $\Box\Box \exists! y y{:}\mathscr{H} \wedge \forall t t{:}y \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x.$ … **[.14] [.28]** $\exists!$INTROS

$\Box\blacksquare.$ … QED

149.30 $\Box \forall x x{:}\mathscr{H} \vdash \exists! y y{:}\mathscr{H} \wedge \forall t t{:}y \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x.$ … **[.1]** $\forall_1$GEN

$\blacksquare.$ … QED

Uses Axioms: **3, 5, 9, 17**

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150 $\forall x x{:}\mathscr{H} \vdash \mathscr{P}x{:}\mathscr{H} \wedge \forall t t{:}\mathscr{P}x \Leftrightarrow t{:}\mathscr{H} \wedge t \subset x.$ … DEFINITION *of the Power Set*

¶ …… DEM

$\Box \boldsymbol{x}{:}\mathscr{H} \vdash \mathscr{P}\boldsymbol{x}{:}\mathscr{H} \wedge \forall \boldsymbol{tt}{:}\mathscr{P}\boldsymbol{x} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}.$ …… LEM **150.1**

$\Box$¶ …… DEM

$\Box\Box \boldsymbol{x}{:}\mathscr{H}.$ …… **[.1]** ASM **150.2**

$\Box\Box \forall \boldsymbol{xx}{:}\mathscr{H} \vdash \exists! \boldsymbol{yy}{:}\mathscr{H} \wedge \forall \boldsymbol{tt}{:}\boldsymbol{y} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}.$ …… THM **150.3**

$\Box\Box \forall \boldsymbol{\psi}(\forall \boldsymbol{xx}{:}\mathscr{H} \vdash \exists! \boldsymbol{yy}{:}\mathscr{H} \wedge \boldsymbol{\psi xy}) \vdash \exists \boldsymbol{\phi} \forall \boldsymbol{x} \forall \boldsymbol{yx}{:}\mathscr{H} \vdash \boldsymbol{y}{:}\mathscr{H} \vdash \boldsymbol{\psi xy} \Leftrightarrow \boldsymbol{\phi x} = \boldsymbol{y}.$ …… THM **150.4**

$\Box\Box (\forall \boldsymbol{xx}{:}\mathscr{H} \vdash \exists! \boldsymbol{yy}{:}\mathscr{H} \wedge \forall \boldsymbol{tt}{:}\boldsymbol{y} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}) \vdash \exists \boldsymbol{\phi} \forall \boldsymbol{x} \forall \boldsymbol{yx}{:}\mathscr{H} \vdash \boldsymbol{y}{:}\mathscr{H} \vdash (\forall \boldsymbol{tt}{:}\boldsymbol{y} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}) \Leftrightarrow \boldsymbol{\phi x} = \boldsymbol{y}.$ …… **[.4]** $\forall_2$INST **150.5**

$\Box\Box \exists \boldsymbol{\phi} \forall \boldsymbol{x} \forall \boldsymbol{yx}{:}\mathscr{H} \vdash \boldsymbol{y}{:}\mathscr{H} \vdash (\forall \boldsymbol{tt}{:}\boldsymbol{y} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}) \Leftrightarrow \boldsymbol{\phi x} = \boldsymbol{y}.$ …… **[.5] [.3]** MP **150.6**

$\Box\Box \forall \boldsymbol{x} \forall \boldsymbol{yx}{:}\mathscr{H} \vdash \boldsymbol{y}{:}\mathscr{H} \vdash (\forall \boldsymbol{tt}{:}\boldsymbol{y} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}) \Leftrightarrow \mathscr{P}\boldsymbol{x} = \boldsymbol{y}.$ …… **[.6]** ${}_2$NAME **150.7**

$\Box\Box \forall \boldsymbol{\psi}(\forall \boldsymbol{xx}{:}\mathscr{H} \vdash \exists! \boldsymbol{yy}{:}\mathscr{H} \wedge \boldsymbol{\psi xy}) \vdash \forall \boldsymbol{\phi}(\forall \boldsymbol{x} \forall \boldsymbol{yx}{:}\mathscr{H} \vdash \boldsymbol{y}{:}\mathscr{H} \vdash \boldsymbol{\psi xy} \Leftrightarrow \boldsymbol{\phi x} = \boldsymbol{y}) \vdash \forall \boldsymbol{xx}{:}\mathscr{H} \vdash \boldsymbol{\phi x}{:}\mathscr{H} \wedge \boldsymbol{\psi x}(\boldsymbol{\phi x}).$ …… THM **150.8**

$\Box\Box (\forall \boldsymbol{xx}{:}\mathscr{H} \vdash \exists! \boldsymbol{yy}{:}\mathscr{H} \wedge \forall \boldsymbol{tt}{:}\boldsymbol{y} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}) \vdash \forall \boldsymbol{\phi}(\forall \boldsymbol{x} \forall \boldsymbol{yx}{:}\mathscr{H} \vdash \boldsymbol{y}{:}\mathscr{H} \vdash (\forall \boldsymbol{tt}{:}\boldsymbol{y} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}) \Leftrightarrow \boldsymbol{\phi x} = \boldsymbol{y}) \vdash \forall \boldsymbol{xx}{:}\mathscr{H} \vdash \boldsymbol{\phi x}{:}\mathscr{H} \wedge \forall \boldsymbol{tt}{:}\boldsymbol{\phi x} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}.$ …… **[.8]** $\forall_2$INST **150.9**

$\Box\Box \forall \boldsymbol{\phi}(\forall \boldsymbol{x} \forall \boldsymbol{yx}{:}\mathscr{H} \vdash \boldsymbol{y}{:}\mathscr{H} \vdash (\forall \boldsymbol{tt}{:}\boldsymbol{y} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}) \Leftrightarrow \boldsymbol{\phi x} = \boldsymbol{y}) \vdash \forall \boldsymbol{xx}{:}\mathscr{H} \vdash \boldsymbol{\phi x}{:}\mathscr{H} \wedge \forall \boldsymbol{tt}{:}\boldsymbol{\phi x} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}.$ …… **[.9] [.3]** MP **150.10**

$\Box\Box (\forall \boldsymbol{x} \forall \boldsymbol{yx}{:}\mathscr{H} \vdash \boldsymbol{y}{:}\mathscr{H} \vdash (\forall \boldsymbol{tt}{:}\boldsymbol{y} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}) \Leftrightarrow \mathscr{P}\boldsymbol{x} = \boldsymbol{y}) \vdash \forall \boldsymbol{xx}{:}\mathscr{H} \vdash \mathscr{P}\boldsymbol{x}{:}\mathscr{H} \wedge \forall \boldsymbol{tt}{:}\mathscr{P}\boldsymbol{x} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}.$ …… **[.10]** $\forall_2$INST **150.11**

$\Box\Box \forall \boldsymbol{xx}{:}\mathscr{H} \vdash \mathscr{P}\boldsymbol{x}{:}\mathscr{H} \wedge \forall \boldsymbol{tt}{:}\mathscr{P}\boldsymbol{x} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}.$ …… **[.11] [.7]** MP **150.12**

$\Box\Box \mathscr{P}\boldsymbol{x}{:}\mathscr{H} \wedge \forall \boldsymbol{tt}{:}\mathscr{P}\boldsymbol{x} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}.$ …… **[.12] [.2]** ${}_1$MINT${}_1$ **150.13**

$\Box\blacksquare.$ …… QED

$\Box \forall \boldsymbol{xx}{:}\mathscr{H} \vdash \mathscr{P}\boldsymbol{x}{:}\mathscr{H} \wedge \forall \boldsymbol{tt}{:}\mathscr{P}\boldsymbol{x} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}.$ …… **[.1]** $\forall_1$GEN **150.14**

$\blacksquare.$ …… QED

Uses Axioms: **3, 5, 6, 7, 8, 9, 17**

$\forall \boldsymbol{xx}{:}\mathscr{H} \vdash \boldsymbol{0}{:}\mathscr{P}\boldsymbol{x}.$ …… THEOREM *of a Natural Powerset Element 1* **151**

¶ …… DEM

$\Box \boldsymbol{x}{:}\mathscr{H} \vdash \boldsymbol{0}{:}\mathscr{P}\boldsymbol{x}.$ …… LEM **151.1**

$\Box$¶ …… DEM

$\Box\Box \boldsymbol{x}{:}\mathscr{H}.$ …… **[.1]** ASM **151.2**

$\Box\Box \forall \boldsymbol{xx}{:}\mathscr{H} \vdash \mathscr{P}\boldsymbol{x}{:}\mathscr{H} \wedge \forall \boldsymbol{tt}{:}\mathscr{P}\boldsymbol{x} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}.$ …… DEF **151.3**

$\Box\Box \mathscr{P}\boldsymbol{x}{:}\mathscr{H} \wedge \forall \boldsymbol{tt}{:}\mathscr{P}\boldsymbol{x} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}.$ …… **[.3] [.2]** ${}_1$MINT${}_1$ **151.4**

$\Box\Box \forall \boldsymbol{tt}{:}\mathscr{P}\boldsymbol{x} \Leftrightarrow \boldsymbol{t}{:}\mathscr{H} \wedge \boldsymbol{t} \subset \boldsymbol{x}.$ …… **[.4]** TAUT **151.5**

$\Box\Box \boldsymbol{0}{:}\mathscr{P}\boldsymbol{x} \Leftrightarrow \boldsymbol{0}{:}\mathscr{H} \wedge \boldsymbol{0} \subset \boldsymbol{x}.$ …… **[.5]** $\forall_1$INST **151.6**

$\Box\Box \boldsymbol{0} \subset \boldsymbol{x}.$ …… THM **151.7**

$\Box\Box \boldsymbol{0}{:}\mathscr{H} \wedge \forall \boldsymbol{x} \neg \boldsymbol{x}{:}\boldsymbol{0}.$ …… DEF **151.8**

$\Box\Box \boldsymbol{0}{:}\mathscr{P}\boldsymbol{x}.$ …… **[.6] [.7] [.8]** TAUT **151.9**

$\Box\blacksquare.$ …… QED

$\Box \forall \boldsymbol{xx}{:}\mathscr{H} \vdash \boldsymbol{0}{:}\mathscr{P}\boldsymbol{x}.$ …… **[.1]** $\forall_1$GEN **151.10**

$\blacksquare.$ …… QED

Uses Axioms: **1, 3, 5, 6, 7, 8, 9, 11, 17**

$\forall \boldsymbol{xx}{:}\mathscr{H} \vdash \boldsymbol{x}{:}\mathscr{P}\boldsymbol{x}.$ …… THEOREM *of a Natural Powerset Element 2* **152**

¶ . . . . DEM

152.1 $\square x{:}\mathscr{H}\vdash x{:}\mathscr{P}x$. . . . . LEM

$\square$¶ . . . . DEM

152.2 $\square\square x{:}\mathscr{H}$. . . . . [.1] ASM

152.3 $\square\square\forall xx{:}\mathscr{H}\vdash\mathscr{P}x{:}\mathscr{H}\wedge\forall tt{:}\mathscr{P}x\Leftrightarrow t{:}\mathscr{H}\wedge t\subset x$. . . . . DEF

152.4 $\square\square\mathscr{P}x{:}\mathscr{H}\wedge\forall tt{:}\mathscr{P}x\Leftrightarrow t{:}\mathscr{H}\wedge t\subset x$. . . . . [.3] [.2] $_1$MINT$_1$

152.5 $\square\square\forall tt{:}\mathscr{P}x\Leftrightarrow t{:}\mathscr{H}\wedge t\subset x$. . . . . [.4] TAUT

152.6 $\square\square x{:}\mathscr{P}x\Leftrightarrow x{:}\mathscr{H}\wedge x\subset x$. . . . . [.5] $\forall_1$INST

152.7 $\square\square x\subset x$. . . . . THM

152.8 $\square\square x{:}\mathscr{P}x$. . . . . [.6] [.2] [.7] TAUT

$\square\blacksquare$. . . . . QED

152.9 $\square\forall xx{:}\mathscr{H}\vdash x{:}\mathscr{P}x$. . . . . [.1] $\forall_1$GEN

$\blacksquare$. . . . . QED

Uses Axioms: **3, 5, 6, 7, 8, 9, 17**

For all sets x, one can exhibit a unique set y which satisfies $\forall tt{:}y\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1$. PROOF: *Existence.* We may specify by admitting only those elements $t{:}\bigcup x$ for which $\forall x_1x_1{:}x\vdash t{:}x_1$. *Uniqueness.* Straightforward. QED.
Imagine unpacking all elements of a box, and forming a new one containing all common elements of their elements. We name this realizer $\bigcap$.

153 $\forall xx{:}\mathscr{H}\vdash\exists!yy{:}\mathscr{H}\wedge\forall tt{:}y\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1$. . . . . THEOREM *of Implicit Upper Intersection*

¶ . . . . DEM

153.1 $\square x{:}\mathscr{H}\vdash\exists!yy{:}\mathscr{H}\wedge\forall tt{:}y\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1$. . . . . LEM

$\square$¶ . . . . DEM

153.2 $\square\square x{:}\mathscr{H}$. . . . . [.1] ASM

153.3 $\square\square\forall x\forall\psi x{:}\mathscr{H}\vdash\exists!yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}x\wedge\psi y_1$. . . . . THM

153.4 $\square\square\forall\psi\bigcup x{:}\mathscr{H}\vdash\exists!yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}\bigcup x\wedge\psi y_1$. . . . . [.3] $\forall_1$INST

153.5 $\square\square\bigcup x{:}\mathscr{H}\vdash\exists!yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}\bigcup x\wedge\forall x_1x_1{:}x\vdash y_1{:}x_1$. . . . . [.4] $\forall_2$INST

153.6 $\square\square\forall xx{:}\mathscr{H}\vdash\bigcup x{:}\mathscr{H}\wedge\forall tt{:}\bigcup x\Leftrightarrow\exists x_1x_1{:}x\wedge t{:}x_1$. . . . . DEF

153.7 $\square\square\bigcup x{:}\mathscr{H}\wedge\forall tt{:}\bigcup x\Leftrightarrow\exists x_1x_1{:}x\wedge t{:}x_1$. . . . . [.6] [.2] $_1$MINT$_1$

153.8 $\square\square\exists!yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}\bigcup x\wedge\forall x_1x_1{:}x\vdash y_1{:}x_1$. . . . . [.7] [.5] TAUT

$\square\blacksquare$. . . . . QED

153.9 $\square\forall xx{:}\mathscr{H}\vdash\exists!yy{:}\mathscr{H}\wedge\forall tt{:}y\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1x$. . . . . [.1] $\forall_1$GEN

$\blacksquare$. . . . . QED

Uses Axioms: **3, 5, 6, 7, 8, 9, 13, 15**

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154 $\forall xx{:}\mathscr{H}\vdash\bigcap x{:}\mathscr{H}\wedge\forall tt{:}\bigcap x\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1$. . . . . DEFINITION *of Upper Intersection*

¶ . . . . DEM

154.1 $\square x{:}\mathscr{H}\vdash\bigcap x{:}\mathscr{H}\wedge\forall tt{:}\bigcap x\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1$. . . . . LEM

$\square$¶ . . . . DEM

154.2 $\square\square x{:}\mathscr{H}$. . . . . [.1] ASM

154.3 $\square\square\forall xx{:}\mathscr{H}\vdash\exists!yy{:}\mathscr{H}\wedge\forall tt{:}y\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1$. . . . . THM

$\square\square\forall\psi(\forall xx{:}\mathscr{H}\vdash\exists!yy{:}\mathscr{H}\wedge\psi xy)\vdash\exists\phi\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\psi xy\Leftrightarrow\phi x=y$. .... THM **154.4**

$\square\square(\forall xx{:}\mathscr{H}\vdash\exists!yy{:}\mathscr{H}\wedge\forall tt{:}y\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1)\vdash\exists\phi\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash(\forall tt{:}y\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1)\Leftrightarrow\phi x=y$. .... [.4] $\forall_2$INST **154.5**

$\square\square\exists\phi\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash(\forall tt{:}y\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1)\Leftrightarrow\phi x=y$. .... [.5] [.3] MP **154.6**

$\square\square\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash(\forall tt{:}y\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1)\Leftrightarrow\bigcap x=y$. .... [.6] ${}_2$NAME **154.7**

$\square\square\forall\psi(\forall xx{:}\mathscr{H}\vdash\exists!yy{:}\mathscr{H}\wedge\psi xy)\vdash\forall\phi(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\psi xy\Leftrightarrow\phi x=y)\vdash\forall xx{:}\mathscr{H}\vdash\phi x{:}\mathscr{H}\wedge\psi x(\phi x)$. .... THM **154.8**

$\square\square(\forall xx{:}\mathscr{H}\vdash\exists!yy{:}\mathscr{H}\wedge\forall tt{:}y\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1)\vdash\forall\phi(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash(\forall tt{:}y\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1)\Leftrightarrow\phi x=y)\vdash\forall xx{:}\mathscr{H}\vdash\phi x{:}\mathscr{H}\wedge\forall tt{:}\phi x\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1$. .... [.8] $\forall_2$INST **154.9**

$\square\square\forall\phi(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash(\forall tt{:}y\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1)\Leftrightarrow\phi x=y)\vdash\forall xx{:}\mathscr{H}\vdash\phi x{:}\mathscr{H}\wedge\forall tt{:}\phi x\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1$. .... [.9] [.3] MP **154.10**

$\square\square(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash(\forall tt{:}y\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1)\Leftrightarrow\bigcap x=y)\vdash\forall xx{:}\mathscr{H}\vdash\bigcap x{:}\mathscr{H}\wedge\forall tt{:}y\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1$. .... [.10] $\forall_2$INST **154.11**

$\square\square\forall xx{:}\mathscr{H}\vdash\bigcap x{:}\mathscr{H}\wedge\forall tt{:}\bigcap x\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1$. .... [.11] [.7] MP **154.12**

$\square\square\bigcap x{:}\mathscr{H}\wedge\forall tt{:}\bigcap x\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1$. .... [.12] [.2] ${}_1$MINT$_1$ **154.13**

$\square\blacksquare$. .... QED

$\square\forall xx{:}\mathscr{H}\vdash\bigcap x{:}\mathscr{H}\wedge\forall tt{:}\bigcap x\Leftrightarrow\forall x_1x_1{:}x\vdash t{:}x_1$. .... [.1] $\forall_1$GEN **154.14**

$\blacksquare$. .... QED

Uses Axioms: **3, 5, 6, 7, 8, 9, 13, 15**

For all sets x,y, one can exhibit a unique set z which contains precisely all the common elements. PROOF: *Existence.* Specifying over x in the obvious way gives the desired set. *Uniqueness.* Straightforward. QED.

Imagine forming a new box by packing all common contents of two old ones. We name this realizer $\cap$.

$\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow t{:}x\wedge t{:}y$. .... THEOREM *of Implicit Lower Intersection* **155**

¶ .... DEM

$\square x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow t{:}x\wedge t{:}y$. .... LEM **155.1**

$\square$¶ .... DEM

$\square\square x{:}\mathscr{H}$. .... [.1] ASM **155.2**

$\square\square y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow t{:}x\wedge t{:}y$. .... LEM **155.3**

$\square\square$¶ .... DEM

$\square\square\square y{:}\mathscr{H}$. .... [.3] ASM **155.4**

$\square\square\square\forall x\forall\psi x{:}\mathscr{H}\vdash\exists!yy{:}\mathscr{H}\wedge\forall y_1y_1{:}y\Leftrightarrow y_1{:}x\wedge\psi y_1$. .... THM **155.5**

$\square\square\square\forall x\forall\psi x{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow t{:}x\wedge\psi t$. .... [.5] QNT **155.6**

$\square\square\square\forall\psi x{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow t{:}x\wedge\psi t$. .... [.6] $\forall_1$INST **155.7**

$\square\square\square x{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow t{:}x\wedge t{:}y$. .... [.7] $\forall_2$INST **155.8**

$\square\square\square\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow t{:}x\wedge t{:}y$. .... [.8] [.2] MP **155.9**

$\square\square\blacksquare$. .... QED **155.10**

$\square\blacksquare$. .... QED

$\square\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow t{:}x\wedge t{:}y$. .... [.1] UGEN$_2$ **155.11**

$\blacksquare$. .... QED

Uses Axioms: **3, 5, 9, 15**

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156 $\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\cap y{:}\mathscr{H}\wedge\forall t t{:}x\cap y\Leftrightarrow t{:}x\wedge t{:}y.$ ...... DEFINITION *of Lower Intersection*

¶ ...... DEM

156.1 $\Box x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\cap y{:}\mathscr{H}\wedge\forall t t{:}x\cap y\Leftrightarrow t{:}x\wedge t{:}y.$ ...... LEM

$\Box$¶ ...... DEM

156.2 $\Box\Box x{:}\mathscr{H}.$ ...... [.1] ASM

156.3 $\Box\Box y{:}\mathscr{H}\vdash x\cap y{:}\mathscr{H}\wedge\forall t t{:}x\cap y\Leftrightarrow t{:}x\wedge t{:}y.$ ...... LEM

$\Box\Box$¶ ...... DEM

156.4 $\Box\Box\Box y{:}\mathscr{H}.$ ...... [.3] ASM

156.5 $\Box\Box\Box\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\wedge t{:}y.$ ...... THM

156.6 $\Box\Box\Box\forall\psi(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\psi xyz)\vdash\exists\phi\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=z.$ ...... THM

156.7 $\Box\Box\Box(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\wedge t{:}y)\vdash\exists\phi\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t{:}x\wedge t{:}y)\Leftrightarrow x\phi y=z.$ ...... [.6] $\forall_2$INST

156.8 $\Box\Box\Box\exists\phi\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t{:}x\wedge t{:}y)\Leftrightarrow x\phi y=z.$ ...... [.7] [.5] MP

156.9 $\Box\Box\Box\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t{:}x\wedge t{:}y)\Leftrightarrow x\cap y=z.$ ...... [.8] ${}_2$NAME

156.10 $\Box\Box\Box\forall\psi(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\psi xyz)\vdash\forall\phi(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\psi xy(x\phi y).$ ...... THM

156.11 $\Box\Box\Box(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\wedge t{:}y)\vdash\forall\phi(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t{:}x\wedge t{:}y)\Leftrightarrow x\phi y=z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\forall t t{:}x\phi y\Leftrightarrow t{:}x\wedge t{:}y.$ ...... [.10] $\forall_2$INST

156.12 $\Box\Box\Box\forall\phi(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t{:}x\wedge t{:}y)\Leftrightarrow x\phi y=z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\forall t t{:}x\phi y\Leftrightarrow t{:}x\wedge t{:}y.$ ...... [.11] [.5] MP

156.13 $\Box\Box\Box(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t{:}x\wedge t{:}y)\Leftrightarrow x\cap y=z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\cap y{:}\mathscr{H}\wedge\forall t t{:}x\cap y\Leftrightarrow t{:}x\wedge t{:}y.$ ...... [.12] $\forall_2$INST

156.14 $\Box\Box\Box\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\cap y{:}\mathscr{H}\wedge\forall t t{:}x\cap y\Leftrightarrow t{:}x\wedge t{:}y.$ ...... [.13] [.9] MP

156.15 $\Box\Box\Box x\cap y{:}\mathscr{H}\wedge\forall t t{:}x\cap y\Leftrightarrow t{:}x\wedge t{:}y.$ ...... [.14] [.2] [.4] ${}_2$MINT$_2$

$\Box\Box\blacksquare.$ ...... QED

$\Box\blacksquare.$ ...... QED

156.16 $\Box\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\cap y{:}\mathscr{H}\wedge\forall t t{:}x\cap y\Leftrightarrow t{:}x\wedge t{:}y.$ ...... [.1] UGEN$_2$

$\blacksquare.$ ...... QED

Uses Axioms: **3, 5, 6, 7, 8, 9, 15**

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§2.4 *Sets and Operations: Set Minus and Symmetric Difference*

For all sets x,y, we can exhibit a unique set z containing precisely all elements of x which are not in y. PROOF: *Existence.* Straightforward from *specification*. *Uniqueness.* Straightforward. QED.

We name its realizer \.

$\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\wedge\neg t{:}y.$ ... THEOREM *of Implicit Set Minus* **157**

$\P$ ... DEM

$\square x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\wedge\neg t{:}y.$ ... LEM **157.1**

$\square\P$ ... DEM

$\square\square x{:}\mathscr{H}.$ ... [.1] ASM **157.2**

$\square\square y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\wedge\neg t{:}y.$ ... LEM **157.3**

$\square\square\P$ ... DEM

$\square\square\square y{:}\mathscr{H}.$ ... [.3] ASM **157.4**

$\square\square\square\forall x\forall\psi x{:}\mathscr{H}\vdash\exists! y y{:}\mathscr{H}\wedge\forall y_1 y_1{:}y\Leftrightarrow y_1{:}x\wedge\psi y_1.$ ... THM **157.5**

$\square\square\square\forall x\forall\psi x{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\wedge\psi t.$ ... [.5] QNT **157.6**

$\square\square\square\forall\psi x{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\wedge\psi t.$ ... [.6] $\forall_1$INST **157.7**

$\square\square\square x{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\wedge\neg t{:}y.$ ... [.7] $\forall_2$INST **157.8**

$\square\square\square\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\wedge\neg t{:}y.$ ... [.8] [.2] MP **157.9**

$\square\square\blacksquare.$ ... QED **157.10**

$\square\blacksquare.$ ... QED

$\square\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\wedge\neg t{:}y.$ ... [.1] UGEN$_2$ **157.11**

$\blacksquare.$ ... QED

Uses Axioms: **3, 5, 9, 15**

$\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\setminus y{:}\mathscr{H}\wedge\forall t t{:}x\setminus y\Leftrightarrow t{:}x\wedge\neg t{:}y.$ ... DEFINITION *of Set Minus* **158**

$\P$ ... DEM

$\square x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\setminus y{:}\mathscr{H}\wedge\forall t t{:}x\setminus y\Leftrightarrow t{:}x\wedge\neg t{:}y.$ ... LEM **158.1**

$\square\P$ ... DEM

$\square\square x{:}\mathscr{H}.$ ... [.1] ASM **158.2**

$\square\square y{:}\mathscr{H}\vdash x\setminus y{:}\mathscr{H}\wedge\forall t t{:}x\setminus y\Leftrightarrow t{:}x\wedge\neg t{:}y.$ ... LEM **158.3**

$\square\square\P$ ... DEM

$\square\square\square y{:}\mathscr{H}.$ ... [.3] ASM **158.4**

$\square\square\square\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\wedge\neg t{:}y.$ ... THM **158.5**

$\square\square\square\forall\psi(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\psi xyz)\vdash\exists\phi\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=$ **158.6**
$z.$ ... THM

$\square\square\square(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\wedge\neg t{:}y)\vdash\exists\phi\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z$ **158.7**
$\Leftrightarrow t{:}x\wedge\neg t{:}y)\Leftrightarrow x\phi y=z.$ ... [.6] $\forall_2$INST

$\square\square\square\exists\phi\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t{:}x\wedge\neg t{:}y)\Leftrightarrow x\phi y=z.$ ... [.7] [.5] MP **158.8**

$\square\square\square\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t{:}x\wedge\neg t{:}y)\Leftrightarrow x\setminus y=z.$ ... [.8] $_2$NAME **158.9**

$\square\square\square\forall\psi(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\psi xyz)\vdash\forall\phi(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=$ **158.10**

$z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\psi xy(x\phi y).$ …… THM

158.11 $\Box\Box\Box(\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow t{:}x\wedge\neg t{:}y)\vdash\forall\phi(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z \Leftrightarrow t{:}x\wedge\neg t{:}y)\Leftrightarrow x\phi y=z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\forall t t{:}x\phi y\Leftrightarrow t{:}x\wedge\neg t{:}y.$ …… **[.10]** $\forall_2$INST

158.12 $\Box\Box\Box\forall\phi(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t{:}x\wedge\neg t{:}y)\Leftrightarrow x\phi y=z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\forall t t{:}x\phi y\Leftrightarrow t{:}x\wedge\neg t{:}y.$ …… **[.11] [.5]** MP

158.13 $\Box\Box\Box(\forall x\forall y\forall z x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall t t{:}z\Leftrightarrow t{:}x\wedge\neg t{:}y)\Leftrightarrow x\setminus y=z)\vdash\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\setminus y{:}\mathscr{H}\wedge\forall t t{:}x\setminus y\Leftrightarrow t{:}x\wedge\neg t{:}y.$ …… **[.12]** $\forall_2$INST

158.14 $\Box\Box\Box\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\setminus y{:}\mathscr{H}\wedge\forall t t{:}x\setminus y\Leftrightarrow t{:}x\wedge\neg t{:}y.$ …… **[.13] [.9]** MP

158.15 $\Box\Box\Box x\setminus y{:}\mathscr{H}\wedge\forall t t{:}x\setminus y\Leftrightarrow t{:}x\wedge\neg t{:}y.$ …… **[.14] [.2] [.4]** ${}_2$MINT$_2$

$\Box\Box\blacksquare.$ …… QED

$\Box\blacksquare.$ …… QED

158.16 $\Box\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\setminus y{:}\mathscr{H}\wedge\forall t t{:}x\setminus y\Leftrightarrow t{:}x\wedge\neg t{:}y.$ …… **[.1]** UGEN$_2$

$\blacksquare.$ …… QED

Uses Axioms: **3, 5, 6, 7, 8, 9, 15**

For all sets x,y, we can exhibit a unique set z containing precisely all elements of x which are not in y and all elements of y not in x. PROOF: Immediate via the set $(x\setminus y)\cup y\setminus x$. QED.
We name its realizer $\dot{-}$. Alternatively, we could have *specified* over $x\cup y$ (with no extra axiomatic commitment). The proof, however, would be longer.

159 $\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x).$ …… THEOREM *of Implicit Symmetric Difference*

¶ …… DEM

159.1 $\Box x_1{:}\mathscr{H}\vdash y_1{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1).$ …… LEM

$\Box$¶ …… DEM

159.2 $\Box\Box x_1{:}\mathscr{H}.$ …… **[.1]** ASM

159.3 $\Box\Box y_1{:}\mathscr{H}\vdash\exists! z z{:}\mathscr{H}\wedge\forall t t{:}z\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1).$ …… LEM

$\Box\Box$¶ …… DEM

159.4 $\Box\Box\Box y_1{:}\mathscr{H}.$ …… **[.3]** ASM

159.5 $\Box\Box\Box\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\setminus y{:}\mathscr{H}\wedge\forall t t{:}x\setminus y\Leftrightarrow t{:}x\wedge\neg t{:}y.$ …… DEF

159.6 $\Box\Box\Box\forall x\forall y x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\cup y{:}\mathscr{H}\wedge\forall t t{:}x\cup y\Leftrightarrow t{:}x\vee t{:}y.$ …… DEF

159.7 $\Box\Box\Box x_1\setminus y_1{:}\mathscr{H}\wedge\forall t t{:}x_1\setminus y_1\Leftrightarrow t{:}x_1\wedge\neg t{:}y_1.$ …… **[.5] [.2] [.4]** ${}_2$MINT$_2$

159.8 $\Box\Box\Box x_1\setminus y_1{:}\mathscr{H}.$ …… **[.7]** TAUT

159.9 $\Box\Box\Box y_1\setminus x_1{:}\mathscr{H}\wedge\forall t t{:}y_1\setminus x_1\Leftrightarrow t{:}y_1\wedge\neg t{:}x_1.$ …… **[.5] [.4] [.2]** ${}_2$MINT$_2$

159.10 $\Box\Box\Box y_1\setminus x_1{:}\mathscr{H}.$ …… **[.9]** TAUT

159.11 $\Box\Box\Box(x_1\setminus y_1)\cup(y_1\setminus x_1){:}\mathscr{H}\wedge\forall t t{:}(x_1\setminus y_1)\cup(y_1\setminus x_1)\Leftrightarrow t{:}x_1\setminus y_1\vee t{:}y_1\setminus x_1.$ …… **[.6] [.8] [.10]** ${}_2$MINT$_2$

159.12 $\Box\Box\Box\forall t t{:}(x_1\setminus y_1)\cup(y_1\setminus x_1)\Leftrightarrow t{:}x_1\setminus y_1\vee t{:}y_1\setminus x_1.$ …… **[.11]** TAUT

159.13 $\Box\Box\Box t{:}(x_1\setminus y_1)\cup(y_1\setminus x_1)\Leftrightarrow t{:}x_1\setminus y_1\vee t{:}y_1\setminus x_1.$ …… **[.12]** $\forall_1$INST

159.14 $\Box\Box\Box\forall t t{:}x_1\setminus y_1\Leftrightarrow t{:}x_1\wedge\neg t{:}y_1.$ …… **[.7]** TAUT

159.15 $\Box\Box\Box t{:}x_1\setminus y_1\Leftrightarrow t{:}x_1\wedge\neg t{:}y_1.$ …… **[.14]** $\forall_1$INST

159.16 $\Box\Box\Box\forall t t{:}y_1\setminus x_1\Leftrightarrow t{:}y_1\wedge\neg t{:}x_1.$ …… **[.9]** TAUT

159.17 $\Box\Box\Box t{:}y_1\setminus x_1\Leftrightarrow t{:}y_1\wedge\neg t{:}x_1.$ …… **[.16]** $\forall_1$INST

159.18 $\Box\Box\Box t{:}(x_1\setminus y_1)\cup(y_1\setminus x_1)\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1).$ …… **[.13] [.15] [.17]** TAUT

159.19 $\Box\Box\Box\forall t t{:}(x_1\setminus y_1)\cup(y_1\setminus x_1)\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1).$ …… **[.18]** $\forall_1$GEN

$\Box\Box\Box(x_1\setminus y_1)\cup(y_1\setminus x_1){:}\mathscr{H}\wedge\forall tt{:}(x_1\setminus y_1)\cup(y_1\setminus x_1)\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1)$. **159.20**
.... **[.11] [.19]** TAUT

$\Box\Box\Box\exists zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1)$. .... **[.20]** $\exists_1$GEN **159.21**

$\Box\Box\Box(z_1{:}\mathscr{H}\wedge\forall tt{:}z_1\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1))\vdash(z_2{:}\mathscr{H}\wedge\forall tt{:}z_2\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge$ **159.22**
$\neg t{:}x_1))\vdash z_1=z_2$. .... LEM

$\Box\Box\Box$¶ .... DEM

$\Box\Box\Box\Box z_1{:}\mathscr{H}\wedge\forall tt{:}z_1\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1)$. .... **[.22]** ASM **159.23**

$\Box\Box\Box\Box(z_2{:}\mathscr{H}\wedge\forall tt{:}z_2\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1))\vdash z_1=z_2$. .... LEM **159.24**

$\Box\Box\Box\Box$¶ .... DEM

$\Box\Box\Box\Box\Box z_2{:}\mathscr{H}\wedge\forall tt{:}z_2\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1)$. .... **[.24]** ASM **159.25**

$\Box\Box\Box\Box\Box\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash(\forall tt{:}x\Leftrightarrow t{:}y)\vdash x=y$. .... AXM **159.26**

$\Box\Box\Box\Box\Box z_1{:}\mathscr{H}\vdash z_2{:}\mathscr{H}\vdash(\forall tt{:}z_1\Leftrightarrow t{:}z_2)\vdash z_1=z_2$. .... **[.26]** ${}_0$MINT$_2$ **159.27**

$\Box\Box\Box\Box\Box(\forall tt{:}z_1\Leftrightarrow t{:}z_2)\vdash z_1=z_2$. .... **[.27] [.23] [.25]** TAUT **159.28**

$\Box\Box\Box\Box\Box\forall tt{:}z_1\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1)$. .... **[.23]** TAUT **159.29**

$\Box\Box\Box\Box\Box t{:}z_1\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1)$. .... **[.29]** $\forall_1$INST **159.30**

$\Box\Box\Box\Box\Box\forall tt{:}z_2\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1)$. .... **[.25]** TAUT **159.31**

$\Box\Box\Box\Box\Box t{:}z_2\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1)$. .... **[.31]** $\forall_1$INST **159.32**

$\Box\Box\Box\Box\Box t{:}z_1\Leftrightarrow t{:}z_2$. .... **[.30] [.32]** TAUT **159.33**

$\Box\Box\Box\Box\Box\forall tt{:}z_1\Leftrightarrow t{:}z_2$. .... **[.33]** $\forall_1$GEN **159.34**

$\Box\Box\Box\Box\Box z_1=z_2$. .... **[.28] [.34]** MP **159.35**

$\Box\Box\Box\Box\blacksquare$. .... QED

$\Box\Box\Box\blacksquare$. .... QED

$\Box\Box\Box\forall z_1\forall z_2(z_1{:}\mathscr{H}\wedge\forall tt{:}z_1\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1))\vdash(z_2{:}\mathscr{H}\wedge\forall tt{:}z_2\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)$ **159.36**
$\vee(t{:}y_1\wedge\neg t{:}x_1))\vdash z_1=z_2$. .... **[.22]** UGEN$_2$

$\Box\Box\Box\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1)$. .... **[.21] [.36]** $\exists!$INTROS **159.37**

$\Box\Box\blacksquare$. .... QED

$\Box\blacksquare$. .... QED

$\Box\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x)$. .... **[.1]** UGEN$_2$ **159.38**

$\blacksquare$. .... QED

Uses Axioms: **3, 4, 5, 6, 7, 8, 9, 12, 13, 15**

$\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\dot{-}y{:}\mathscr{H}\wedge\forall tt{:}x\dot{-}y\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x)$. **160**
.... DEFINITION *of Symmetric Difference*

¶ .... DEM

$\Box x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\dot{-}y{:}\mathscr{H}\wedge\forall tt{:}x\dot{-}y\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x)$. .... LEM **160.1**

$\Box$¶ .... DEM

$\Box\Box x{:}\mathscr{H}$. .... **[.1]** ASM **160.2**

$\Box\Box y{:}\mathscr{H}\vdash x\dot{-}y{:}\mathscr{H}\wedge\forall tt{:}x\dot{-}y\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x)$. .... LEM **160.3**

$\Box\Box$¶ .... DEM

$\Box\Box\Box y{:}\mathscr{H}$. .... **[.3]** ASM **160.4**

$\Box\Box\Box\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x)$. .... THM **160.5**

$\Box\Box\Box\forall\psi(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\psi xyz)\vdash\exists\phi\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=$ **160.6**
z. .... THM

$\Box\Box\Box(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x))\vdash\exists\phi\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}$ **160.7**

$\vdash z{:}\mathscr{H}\vdash(\forall tt{:}z\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x))\Leftrightarrow x\phi y=z$. .... [.6] $\forall_2$INST

160.8 $\Box\Box\Box\exists\phi\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall tt{:}z\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x))\Leftrightarrow x\phi y=z$. .. [.7] [.5] MP

160.9 $\Box\Box\Box\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall tt{:}z\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x))\Leftrightarrow x\dot{-}y=z$. ..... [.8] ${}_2$NAME

160.10 $\Box\Box\Box\forall\psi(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\psi xyz)\vdash\forall\phi(\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=z)\vdash\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\psi xy(x\phi y)$. .... THM

160.11 $\Box\Box\Box(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\forall tt{:}z\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x))\vdash\forall\phi(\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall tt{:}z\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x))\Leftrightarrow x\phi y=z)\vdash\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\forall tt{:}x\phi y\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x)$. .... [.10] $\forall_2$INST

160.12 $\Box\Box\Box\forall\phi(\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall tt{:}z\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x))\Leftrightarrow x\phi y=z)\vdash\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\forall tt{:}x\phi y\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x)$. .... [.11] [.5] MP

160.13 $\Box\Box\Box(\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(\forall tt{:}z\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x))\Leftrightarrow x\dot{-}y=z)\vdash\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\dot{-}y{:}\mathscr{H}\wedge\forall tt{:}x\dot{-}y\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x)$. .... [.12] $\forall_2$INST

160.14 $\Box\Box\Box\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\dot{-}y{:}\mathscr{H}\wedge\forall tt{:}x\dot{-}y\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x)$. .... [.13] [.9] MP

160.15 $\Box\Box\Box x\dot{-}y{:}\mathscr{H}\wedge\forall tt{:}x\dot{-}y\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x)$. .... [.14] [.2] [.4] ${}_2$MINT$_2$

$\Box\Box\blacksquare$. .... QED

$\Box\blacksquare$. .... QED

160.16 $\Box\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\dot{-}y{:}\mathscr{H}\wedge\forall tt{:}x\dot{-}y\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x)$. .... [.1] UGEN$_2$

$\blacksquare$. .... QED

Uses Axioms: **3, 4, 5, 6, 7, 8, 9, 12, 13, 15**

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161 $\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\dot{-}y=(x\setminus y)\cup(y\setminus x)$. .... THEOREM *of Symmetric Difference*

¶ .... DEM

161.1 $\Box x_1{:}\mathscr{H}\vdash y_1{:}\mathscr{H}\vdash x_1\dot{-}y_1=(x_1\setminus y_1)\cup(y_1\setminus x_1)$. .... LEM

$\Box$¶ .... DEM

161.2 $\Box\Box x_1{:}\mathscr{H}$. .... [.1] ASM

161.3 $\Box\Box y_1{:}\mathscr{H}\vdash x_1\dot{-}y_1=(x_1\setminus y_1)\cup(y_1\setminus x_1)$. .... LEM

$\Box\Box$¶ .... DEM

161.4 $\Box\Box\Box y_1{:}\mathscr{H}$. .... [.3] ASM

161.5 $\Box\Box\Box\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\setminus y{:}\mathscr{H}\wedge\forall tt{:}x\setminus y\Leftrightarrow t{:}x\wedge\neg t{:}y$. .... DEF

161.6 $\Box\Box\Box\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\cup y{:}\mathscr{H}\wedge\forall tt{:}x\cup y\Leftrightarrow t{:}x\vee t{:}y$. .... DEF

161.7 $\Box\Box\Box x_1\setminus y_1{:}\mathscr{H}\wedge\forall tt{:}x_1\setminus y_1\Leftrightarrow t{:}x_1\wedge\neg t{:}y_1$. .... [.5] [.2] [.4] ${}_2$MINT$_2$

161.8 $\Box\Box\Box x_1\setminus y_1{:}\mathscr{H}$. .... [.7] TAUT

161.9 $\Box\Box\Box y_1\setminus x_1{:}\mathscr{H}\wedge\forall tt{:}y_1\setminus x_1\Leftrightarrow t{:}y_1\wedge\neg t{:}x_1$. .... [.5] [.4] [.2] ${}_2$MINT$_2$

161.10 $\Box\Box\Box y_1\setminus x_1{:}\mathscr{H}$. .... [.9] TAUT

161.11 $\Box\Box\Box(x_1\setminus y_1)\cup(y_1\setminus x_1){:}\mathscr{H}\wedge\forall tt{:}(x_1\setminus y_1)\cup(y_1\setminus x_1)\Leftrightarrow t{:}x_1\setminus y_1\vee t{:}y_1\setminus x_1$. . [.6] [.8] [.10] ${}_2$MINT$_2$

161.12 $\Box\Box\Box\forall tt{:}(x_1\setminus y_1)\cup(y_1\setminus x_1)\Leftrightarrow t{:}x_1\setminus y_1\vee t{:}y_1\setminus x_1$. .... [.11] TAUT

161.13 $\Box\Box\Box t{:}(x_1\setminus y_1)\cup(y_1\setminus x_1)\Leftrightarrow t{:}x_1\setminus y_1\vee t{:}y_1\setminus x_1$. .... [.12] $\forall_1$INST

161.14 $\Box\Box\Box\forall tt{:}x_1\setminus y_1\Leftrightarrow t{:}x_1\wedge\neg t{:}y_1$. .... [.7] TAUT

161.15 $\Box\Box\Box t{:}x_1\setminus y_1\Leftrightarrow t{:}x_1\wedge\neg t{:}y_1$. .... [.14] $\forall_1$INST

161.16 $\Box\Box\Box\forall tt{:}y_1\setminus x_1\Leftrightarrow t{:}y_1\wedge\neg t{:}x_1$. .... [.9] TAUT

161.17 $\Box\Box\Box t{:}y_1\setminus x_1\Leftrightarrow t{:}y_1\wedge\neg t{:}x_1$. .... [.16] $\forall_1$INST

161.18 $\Box\Box\Box t{:}(x_1\setminus y_1)\cup(y_1\setminus x_1)\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1\wedge\neg t{:}x_1)$. .... [.13] [.15] [.17] TAUT

161.19 $\Box\Box\Box\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\dot{-}y{:}\mathscr{H}\wedge\forall tt{:}x\dot{-}y\Leftrightarrow(t{:}x\wedge\neg t{:}y)\vee(t{:}y\wedge\neg t{:}x)$. .... DEF

161.20 $\Box\Box\Box x_1\dot{-}y_1{:}\mathscr{H}\wedge\forall tt{:}x_1\dot{-}y_1\Leftrightarrow(t{:}x_1\wedge\neg t{:}y_1)\vee(t{:}y_1)\wedge\neg t{:}x_1)$. .... [.19] [.2] [.4] ${}_2$MINT$_2$

$\square\square\square\forall t\, t{:}x_1 \dot{-} y_1 \Leftrightarrow (t{:}x_1 \wedge \neg t{:}y_1) \vee (t{:}y_1) \wedge \neg t{:}x_1)$. .... [.20] TAUT **161.21**

$\square\square\square t{:}x_1 \dot{-} y_1 \Leftrightarrow (t{:}x_1 \wedge \neg t{:}y_1) \vee (t{:}y_1) \wedge \neg t{:}x_1)$. .... [.21] $\forall_1$INST **161.22**

$\square\square\square t{:}x_1 \dot{-} y_1 \Leftrightarrow t{:}(x_1 \setminus y_1) \cup (y_1 \setminus x_1)$. .... [.22] [.18] TAUT **161.23**

$\square\square\square\forall t\, t{:}x_1 \dot{-} y_1 \Leftrightarrow t{:}(x_1 \setminus y_1) \cup (y_1 \setminus x_1)$. .... [.23] $\forall_1$GEN **161.24**

$\square\square\square\forall x \forall y\, x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash (\forall t\, t{:}x \Leftrightarrow t{:}y) \vdash x = y$. .... AXM **161.25**

$\square\square\square x_1 \dot{-} y_1{:}\mathscr{H} \vdash (x_1 \setminus y_1) \cup (y_1 \setminus x_1){:}\mathscr{H} \vdash (\forall t\, t{:}x_1 \dot{-} y_1 \Leftrightarrow t{:}(x_1 \setminus y_1) \cup (y_1 \setminus x_1)) \vdash x_1 \dot{-} y_1 = (x_1 \setminus y_1)$ **161.26**
$\cup (y_1 \setminus x_1)$. .... [.25] ${}_0$MINT$_2$

$\square\square\square x_1 \dot{-} y_1 = (x_1 \setminus y_1) \cup (y_1 \setminus x_1)$. .... [.26] [.20] [.11] [.24] TAUT **161.27**

$\square\square\blacksquare$. .... QED

$\square\blacksquare$. .... QED

$\square\forall x \forall y\, x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash x \dot{-} y = (x \setminus y) \cup (y \setminus x)$. .... [.1] UGEN$_2$ **161.28**

$\blacksquare$. .... QED

Uses Axioms: **3, 4, 5, 6, 7, 8, 9, 12, 13, 15**

§2.5 *Sets and Operations: Ordered Pairing and Cartesian Product*

For all sets $\boldsymbol{x},\boldsymbol{y}$, we can exhibit a unique set $\boldsymbol{z}$ containing precisely $\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x}$ and $\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}$. PROOF: Immediate via the set $(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}$. QED.
We name its realizer $\mathbf{ORP}$. While there are other candidates for defining ordered pairs, we select this since it requires little axiomatic purchase to prove $\boldsymbol{x}_1\,\mathbf{ORP}\,\boldsymbol{y}_1=\boldsymbol{x}_2\,\mathbf{ORP}\,\boldsymbol{y}_2\Leftrightarrow\boldsymbol{x}_1=\boldsymbol{x}_2\wedge\boldsymbol{y}_1=\boldsymbol{y}_2$.

162 $\forall\boldsymbol{x}\forall\boldsymbol{y}\boldsymbol{x}{:}\mathscr{H}\vdash\boldsymbol{y}{:}\mathscr{H}\vdash\exists!\boldsymbol{z}\boldsymbol{z}{:}\mathscr{H}\wedge\boldsymbol{z}=(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}.$ ... THEOREM *of Implicit Ordered Pairing*

$\P$ ... DEM

162.1 $\square\boldsymbol{x}{:}\mathscr{H}\vdash\boldsymbol{y}{:}\mathscr{H}\vdash\exists!\boldsymbol{z}\boldsymbol{z}{:}\mathscr{H}\wedge(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})=\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}.$ ... LEM

$\square\P$ ... DEM

162.2 $\square\square\boldsymbol{x}{:}\mathscr{H}.$ ... [.1] ASM

162.3 $\square\square\boldsymbol{y}{:}\mathscr{H}\vdash\exists!\boldsymbol{z}\boldsymbol{z}{:}\mathscr{H}\wedge\boldsymbol{z}=(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}.$ ... LEM

$\square\square\P$ ... DEM

162.4 $\square\square\square\boldsymbol{y}{:}\mathscr{H}.$ ... [.3] ASM

162.5 $\square\square\square\forall\boldsymbol{x}\forall\boldsymbol{y}\boldsymbol{x}{:}\mathscr{H}\vdash\boldsymbol{y}{:}\mathscr{H}\vdash\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}{:}\mathscr{H}\wedge\forall\boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}\Leftrightarrow\boldsymbol{t}=\boldsymbol{x}\vee\boldsymbol{t}=\boldsymbol{y}.$ ... THM

162.6 $\square\square\square\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x}{:}\mathscr{H}\wedge\forall\boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x}\Leftrightarrow\boldsymbol{t}=\boldsymbol{x}\vee\boldsymbol{t}=\boldsymbol{x}.$ ... [.5] [.2] [.2] $_2$MINT$_2$

162.7 $\square\square\square\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}{:}\mathscr{H}\wedge\forall\boldsymbol{t}\boldsymbol{t}{:}\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}\Leftrightarrow\boldsymbol{t}=\boldsymbol{x}\vee\boldsymbol{t}=\boldsymbol{y}.$ ... [.5] [.2] [.4] $_2$MINT$_2$

162.8 $\square\square\square\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x}{:}\mathscr{H}\vdash\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}{:}\mathscr{H}\vdash(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}{:}\mathscr{H}\wedge\forall\boldsymbol{t}\boldsymbol{t}{:}(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}\Leftrightarrow\boldsymbol{t}=\boldsymbol{x}\,\mathbf{PAR}$
$\boldsymbol{x}\vee\boldsymbol{t}=\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}.$ ... [.5] $_0$MINT$_2$

162.9 $\square\square\square(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}{:}\mathscr{H}\wedge\forall\boldsymbol{t}\boldsymbol{t}{:}(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}\Leftrightarrow\boldsymbol{t}=\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x}\vee\boldsymbol{t}=\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}.$ ...
... [.8] [.6] [.7] TAUT

162.10 $\square\square\square(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}=(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}.$ ... EQ

162.11 $\square\square\square(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}{:}\mathscr{H}\wedge(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}=(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}.$ ... [.9] [.10] TAUT

162.12 $\square\square\square\exists\boldsymbol{z}\boldsymbol{z}{:}\mathscr{H}\wedge\boldsymbol{z}=(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}.$ ... [.11] $\exists_1$GEN

162.13 $\square\square\square\boldsymbol{z}_1{:}\mathscr{H}\wedge\boldsymbol{z}_1=(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}\vdash\boldsymbol{z}_2{:}\mathscr{H}\wedge\boldsymbol{z}_2=(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}\vdash\boldsymbol{z}_1=\boldsymbol{z}_2.$ ... LEM

$\square\square\square\P$ ... DEM

162.14 $\square\square\square\square\boldsymbol{z}_1{:}\mathscr{H}\wedge\boldsymbol{z}_1=(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}.$ ... [.13] ASM

162.15 $\square\square\square\square\boldsymbol{z}_2{:}\mathscr{H}\wedge\boldsymbol{z}_2=(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}\vdash\boldsymbol{z}_1=\boldsymbol{z}_2.$ ... LEM

$\square\square\square\square\P$ ... DEM

162.16 $\square\square\square\square\square\boldsymbol{z}_2{:}\mathscr{H}\wedge\boldsymbol{z}_2=(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}.$ ... [.15] ASM

162.17 $\square\square\square\square\square\boldsymbol{z}_1=\boldsymbol{z}_2.$ ... [.14] [.16] TAUT

162.18 $\square\square\square\square\blacksquare.$ ... QED

$\square\square\square\blacksquare.$ ... QED

162.19 $\square\square\square\forall\boldsymbol{z}_1\forall\boldsymbol{z}_2\boldsymbol{z}_1{:}\mathscr{H}\wedge\boldsymbol{z}_1=(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}\vdash\boldsymbol{z}_2{:}\mathscr{H}\wedge\boldsymbol{z}_2=(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}\vdash\boldsymbol{z}_1=\boldsymbol{z}_2.$ ...
... [.13] UGEN$_2$

162.20 $\square\square\square\exists!\boldsymbol{z}\boldsymbol{z}{:}\mathscr{H}\wedge\boldsymbol{z}=(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}.$ ... [.12] [.19] $\exists!$INTROS

$\square\square\blacksquare.$ ... QED

$\square\blacksquare.$ ... QED

162.21 $\square\forall\boldsymbol{x}\forall\boldsymbol{y}\boldsymbol{x}{:}\mathscr{H}\vdash\boldsymbol{y}{:}\mathscr{H}\vdash\exists!\boldsymbol{z}\boldsymbol{z}{:}\mathscr{H}\wedge\boldsymbol{z}=(\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{x})\,\mathbf{PAR}\,\boldsymbol{x}\,\mathbf{PAR}\,\boldsymbol{y}.$ ... [.1] UGEN$_2$

$\blacksquare.$ ... QED

Uses Axioms: **3, 5, 6, 7, 8, 9, 12**

$\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathbf{ORP}\,y{:}\mathscr{H}\wedge x\,\mathbf{ORP}\,y=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y$. DEFINITION *of Ordered Pairing* **163**

¶ .... DEM

$\Box x{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathbf{ORP}\,y{:}\mathscr{H}\wedge x\,\mathbf{ORP}\,y=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y$. .... LEM **163.1**

$\Box$¶ .... DEM

$\Box\Box x{:}\mathscr{H}$. .... [.1] ASM **163.2**

$\Box\Box y{:}\mathscr{H}\vdash x\,\mathbf{ORP}\,y{:}\mathscr{H}\wedge x\,\mathbf{ORP}\,y=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y$. .... LEM **163.3**

$\Box\Box$¶ .... DEM

$\Box\Box\Box y{:}\mathscr{H}$. .... [.3] ASM **163.4**

$\Box\Box\Box\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge z=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y$. .... THM **163.5**

$\Box\Box\Box\forall\psi(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\psi xyz)\vdash\exists\phi\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=z$. .... THM **163.6**

$\Box\Box\Box(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge z=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y)\vdash\exists\phi\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash(z=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y)\Leftrightarrow x\phi y=z$. .... [.6] $\forall_2$INST **163.7**

$\Box\Box\Box\exists\phi\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash z=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y\Leftrightarrow x\phi y=z$. .... [.7] [.5] MP **163.8**

$\Box\Box\Box\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash z=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y\Leftrightarrow x\,\mathbf{ORP}\,y=z$. .... [.8] $_2$NAME **163.9**

$\Box\Box\Box\forall\psi(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge\psi xyz)\vdash\forall\phi(\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash\psi xyz\Leftrightarrow x\phi y=z)\vdash\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge\psi xy(x\phi y)$. .... THM **163.10**

$\Box\Box\Box(\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash\exists!zz{:}\mathscr{H}\wedge z=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y)\vdash\forall\phi(\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash z=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y\Leftrightarrow x\phi y=z)\vdash\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge x\phi y=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y$. .... [.10] $\forall_2$INST **163.11**

$\Box\Box\Box\forall\phi(\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash z=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y\Leftrightarrow x\phi y=z)\vdash\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\phi y{:}\mathscr{H}\wedge x\phi y=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y$. .... [.11] [.5] MP **163.12**

$\Box\Box\Box(\forall x\forall y\forall zx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash z{:}\mathscr{H}\vdash z=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y\Leftrightarrow x\,\mathbf{ORP}\,y=z)\vdash\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathbf{ORP}\,y{:}\mathscr{H}\wedge x\,\mathbf{ORP}\,y=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y$. .... [.12] $\forall_2$INST **163.13**

$\Box\Box\Box\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathbf{ORP}\,y{:}\mathscr{H}\wedge x\,\mathbf{ORP}\,y=x\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y$. .... [.13] [.9] MP **163.14**

$\Box\Box\Box x\,\mathbf{ORP}\,y{:}\mathscr{H}\wedge x\,\mathbf{ORP}\,y=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y$. .... [.14] [.2] [.4] $_2$MINT$_2$ **163.15**

$\Box\Box\blacksquare$. .... QED

$\Box\blacksquare$. .... QED

$\Box\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash x\,\mathbf{ORP}\,y{:}\mathscr{H}\wedge x\,\mathbf{ORP}\,y=(x\,\mathbf{PAR}\,x)\,\mathbf{PAR}\,x\,\mathbf{PAR}\,y$. .... [.1] UGEN$_2$ **163.16**

$\blacksquare$. .... QED

Uses Axioms: **3, 5, 6, 7, 8, 9, 12**

Two ordered pairs are equal if(f.) their coordinates are equal. PROOF: We need only consider the nontrivial direction. Let $x_1\,\mathbf{ORP}\,y_1=x_2\,\mathbf{ORP}\,y_2$. By definition $(x_1\,\mathbf{PAR}\,x_1)\,\mathbf{PAR}\,x_1\,\mathbf{PAR}\,y_1=(x_2\,\mathbf{PAR}\,x_2)\,\mathbf{PAR}\,x_2\,\mathbf{PAR}\,y_2$. Either $x_1\,\mathbf{PAR}\,x_1=x_2\,\mathbf{PAR}\,x_2$ or $x_1\,\mathbf{PAR}\,x_1=x_2\,\mathbf{PAR}\,y_2$. In either case, $x_1=x_2$. It follows $y_1=y_2$. QED.

$\forall x_1\forall x_2\forall y_1\forall y_2x_1{:}\mathscr{H}\vdash x_2{:}\mathscr{H}\vdash y_1{:}\mathscr{H}\vdash y_2{:}\mathscr{H}\vdash x_1\,\mathbf{ORP}\,y_1=x_2\,\mathbf{ORP}\,y_2\vdash x_1=x_2\wedge y_1=y_2$. .... THEOREM *of Ordered Pairs* **164**

¶ .... DEM

$\Box x_1{:}\mathscr{H}\vdash x_2{:}\mathscr{H}\vdash y_1{:}\mathscr{H}\vdash y_2{:}\mathscr{H}\vdash x_1\,\mathbf{ORP}\,y_1=x_2\,\mathbf{ORP}\,y_2\vdash x_1=x_2\wedge y_1=y_2$. .... LEM **164.1**

$\Box$¶ .... DEM

164.2 □□$x_1{:}\mathscr{H}$. …… [.1] ASM

164.3 □□$x_2{:}\mathscr{H} \vdash y_1{:}\mathscr{H} \vdash y_2{:}\mathscr{H} \vdash x_1 \text{ ORP } y_1 = x_2 \text{ ORP } y_2 \vdash x_1 = x_2 \wedge y_1 = y_2$. …… LEM

□□¶ …… DEM

164.4 □□□$x_2{:}\mathscr{H}$. …… [.3] ASM

164.5 □□□$y_1{:}\mathscr{H} \vdash y_2{:}\mathscr{H} \vdash x_1 \text{ ORP } y_1 = x_2 \text{ ORP } y_2 \vdash x_1 = x_2 \wedge y_1 = y_2$. …… LEM

□□□¶ …… DEM

164.6 □□□□$y_1{:}\mathscr{H}$. …… [.5] ASM

164.7 □□□□$y_2{:}\mathscr{H} \vdash x_1 \text{ ORP } y_1 = x_2 \text{ ORP } y_2 \vdash x_1 = x_2 \wedge y_1 = y_2$. …… LEM

□□□□¶ …… DEM

164.8 □□□□□$y_2{:}\mathscr{H}$. …… [.7] ASM

164.9 □□□□□$x_1 \text{ ORP } y_1 = x_2 \text{ ORP } y_2 \vdash x_1 = x_2 \wedge y_1 = y_2$. …… LEM

□□□□□¶ …… DEM

164.10 □□□□□□$x_1 \text{ ORP } y_1 = x_2 \text{ ORP } y_2$. …… [.9] ASM

164.11 □□□□□□$\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash x \text{ ORP } y{:}\mathscr{H} \wedge x \text{ ORP } y = (x \text{ PAR } x) \text{ PAR } x \text{ PAR } y$. …… DEF

164.12 □□□□□□$x_1 \text{ ORP } y_1{:}\mathscr{H} \wedge x_1 \text{ ORP } y_1 = (x_1 \text{ PAR } x_1) \text{ PAR } x_1 \text{ PAR } y_1$. …… [.11] [.2] [.6] ${}_2$MINT${}_2$

164.13 □□□□□□$x_2 \text{ ORP } y_2{:}\mathscr{H} \wedge x_2 \text{ ORP } y_2 = (x_2 \text{ PAR } x_2) \text{ PAR } x_2 \text{ PAR } y_2$. …… [.11] [.4] [.8] ${}_2$MINT${}_2$

164.14 □□□□□□$(x_1 \text{ PAR } x_1) \text{ PAR } x_1 \text{ PAR } y_1 = (x_2 \text{ PAR } x_2) \text{ PAR } x_2 \text{ PAR } y_2$. …… [.10] [.12] [.13] TAUT

164.15 □□□□□□$\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash x{:}x \text{ PAR } y \wedge y{:}x \text{ PAR } y$. …… THM

164.16 □□□□□□$x_1 \text{ PAR } x_1{:}\mathscr{H} \vdash x_1 \text{ PAR } y_1{:}\mathscr{H} \vdash x_1 \text{ PAR } x_1{:}(x_1 \text{ PAR } x_1) \text{ PAR } x_1 \text{ PAR } y_1 \wedge x_1 \text{ PAR } y_1{:}$
$(x_1 \text{ PAR } x_1) \text{ PAR } x_1 \text{ PAR } y_1$. …… [.15] ${}_0$MINT${}_2$

164.17 □□□□□□$\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash x \text{ PAR } y{:}\mathscr{H}$. …… THM

164.18 □□□□□□$x_1 \text{ PAR } x_1{:}\mathscr{H}$. …… [.17] [.2] [.2] ${}_2$MINT${}_2$

164.19 □□□□□□$x_1 \text{ PAR } y_1{:}\mathscr{H}$. …… [.17] [.2] [.6] ${}_2$MINT${}_2$

164.20 □□□□□□$x_1 \text{ PAR } x_1{:}(x_2 \text{ PAR } x_2) \text{ PAR } x_2 \text{ PAR } y_2$. …… [.16] [.18] [.19] [.14] TAUT

164.21 □□□□□□$x_2 \text{ PAR } x_2{:}\mathscr{H}$. …… [.17] [.4] [.4] ${}_2$MINT${}_2$

164.22 □□□□□□$x_2 \text{ PAR } y_2{:}\mathscr{H}$. …… [.17] [.4] [.8] ${}_2$MINT${}_2$

164.23 □□□□□□$\forall x \forall y x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash x \text{ PAR } y{:}\mathscr{H} \wedge \forall t t{:}x \text{ PAR } y \Leftrightarrow t = x \vee t = y$. …… THM

164.24 □□□□□□$(x_2 \text{ PAR } x_2) \text{ PAR } x_2 \text{ PAR } y_2{:}\mathscr{H} \wedge \forall t t{:}(x_2 \text{ PAR } x_2) \text{ PAR } x_2 \text{ PAR } y_2 \Leftrightarrow t = x_2 \text{ PAR } x_2 \vee t$
$= x_2 \text{ PAR } y_2$. …… [.23] [.21] [.22] ${}_2$MINT${}_2$

164.25 □□□□□□$\forall t t{:}(x_2 \text{ PAR } x_2) \text{ PAR } x_2 \text{ PAR } y_2 \Leftrightarrow t = x_2 \text{ PAR } x_2 \vee t = x_2 \text{ PAR } y_2$. …… [.24] TAUT

164.26 □□□□□□$x_1 \text{ PAR } x_1{:}(x_2 \text{ PAR } x_2) \text{ PAR } x_2 \text{ PAR } y_2 \Leftrightarrow x_1 \text{ PAR } x_1 = x_2 \text{ PAR } x_2 \vee x_1 \text{ PAR } x_1 = x_2 \text{ PAR } y_2$.
…… [.24] $\forall_1$INST

164.27 □□□□□□$x_1 \text{ PAR } x_1 = x_2 \text{ PAR } x_2 \vee x_1 \text{ PAR } x_1 = x_2 \text{ PAR } y_2$. …… [.26] [.20] TAUT

164.28 □□□□□□$x_1 \text{ PAR } x_1 = x_2 \text{ PAR } x_2 \vdash x_1 = x_2$. …… LEM

□□□□□□¶ …… DEM

164.29 □□□□□□□$x_1 \text{ PAR } x_1 = x_2 \text{ PAR } x_2$. …… [.28] ASM

164.30 □□□□□□□$x_1 \text{ PAR } x_1{:}\mathscr{H} \wedge \forall t t{:}x_1 \text{ PAR } x_1 \Leftrightarrow t = x_1 \vee t = x_1$. …… [.23] [.2] [.2] ${}_2$MINT${}_2$

164.31 □□□□□□□$x_2 \text{ PAR } x_2{:}\mathscr{H} \wedge \forall t t{:}x_2 \text{ PAR } x_2 \Leftrightarrow t = x_2 \vee t = x_2$. …… [.23] [.4] [.4] ${}_2$MINT${}_2$

164.32 □□□□□□□$\forall t t{:}x_1 \text{ PAR } x_1 \Leftrightarrow t = x_2 \vee t = x_2$. …… [.31] [.29] TAUT

164.33 □□□□□□□$x_1{:}x_1 \text{ PAR } x_1 \Leftrightarrow x_1 = x_2 \vee x_1 = x_2$. …… [.32] $\forall_1$INST

164.34 □□□□□□□$\forall t t{:}x_1 \text{ PAR } x_1 \Leftrightarrow t = x_1 \vee t = x_1$. …… [.30] TAUT

164.35 □□□□□□□$x_1{:}x_1 \text{ PAR } x_1 \Leftrightarrow x_1 = x_1 \vee x_1 = x_1$. …… [.34] $\forall_1$INST

164.36 □□□□□□□$x_1 = x_1$. …… EQ

164.37 □□□□□□□$x_1 = x_2$. …… [.35] [.36] [.33] TAUT

□□□□□□■. .... QED

□□□□□□$x_1 \,\mathrm{PAR}\, x_1 = x_2 \,\mathrm{PAR}\, y_2 \vdash x_1 = x_2$. .... LEM 164.38

□□□□□□¶ .... DEM

□□□□□□□$x_1 \,\mathrm{PAR}\, x_1 = x_2 \,\mathrm{PAR}\, y_2$. .... [.38] ASM 164.39

□□□□□□□$x_1 \,\mathrm{PAR}\, x_1 : \mathscr{H} \wedge \forall tt : x_1 \,\mathrm{PAR}\, x_1 \Leftrightarrow t = x_1 \vee t = x_1$. .... [.23] [.2] [.2] $_2$MINT$_2$ 164.40

□□□□□□□$x_2 \,\mathrm{PAR}\, y_2 : \mathscr{H} \wedge \forall tt : x_2 \,\mathrm{PAR}\, y_2 \Leftrightarrow t = x_2 \vee t = y_2$. .... [.23] [.4] [.8] $_2$MINT$_2$ 164.41

□□□□□□□$\forall tt : x_1 \,\mathrm{PAR}\, x_1 \Leftrightarrow t = x_2 \vee t = y_2$. .... [.41] [.39] TAUT 164.42

□□□□□□□$x_2 : x_1 \,\mathrm{PAR}\, x_1 \Leftrightarrow x_2 = x_2 \vee x_1 = y_2$. .... [.42] $\forall_1$INST 164.43

□□□□□□□$\forall tt : x_1 \,\mathrm{PAR}\, x_1 \Leftrightarrow t = x_1 \vee t = x_1$. .... [.40] TAUT 164.44

□□□□□□□$x_2 : x_1 \,\mathrm{PAR}\, x_1 \Leftrightarrow x_2 = x_1 \vee x_2 = x_1$. .... [.44] $\forall_1$INST 164.45

□□□□□□□$x_2 = x_2$. .... EQ 164.46

□□□□□□□$x_1 = x_2$. .... [.46] [.43] [.45] TAUT 164.47

□□□□□□□■. .... QED

□□□□□□$x_1 = x_2$. .... [.27] [.28] [.38] TAUT 164.48

□□□□□□$x_1 \,\mathrm{PAR}\, y_1 : (x_1 \,\mathrm{PAR}\, x_1) \,\mathrm{PAR}\, x_1 \,\mathrm{PAR}\, y_2$. .... [.16] [.18] [.19] [.14] [.48] TAUT 164.49

□□□□□□$(x_1 \,\mathrm{PAR}\, x_1) \,\mathrm{PAR}\, x_1 \,\mathrm{PAR}\, y_2 : \mathscr{H} \wedge \forall tt : (x_1 \,\mathrm{PAR}\, x_1) \,\mathrm{PAR}\, x_1 \,\mathrm{PAR}\, y_2 \Leftrightarrow t = x_1 \,\mathrm{PAR}\, x_1 \vee t$ 164.50
$= x_1 \,\mathrm{PAR}\, y_2$. .... [.23] [.18] [.19] $_2$MINT$_2$

□□□□□□$\forall tt : (x_1 \,\mathrm{PAR}\, x_1) \,\mathrm{PAR}\, x_1 \,\mathrm{PAR}\, y_2 \Leftrightarrow t = x_1 \,\mathrm{PAR}\, x_1 \vee t = x_1 \,\mathrm{PAR}\, y_2$. .... [.50] TAUT 164.51

□□□□□□$x_1 \,\mathrm{PAR}\, y_1 : (x_1 \,\mathrm{PAR}\, x_1) \,\mathrm{PAR}\, x_1 \,\mathrm{PAR}\, y_2 \Leftrightarrow x_1 \,\mathrm{PAR}\, y_1 = x_1 \,\mathrm{PAR}\, x_1 \vee x_1 \,\mathrm{PAR}\, y_1 = x_1 \,\mathrm{PAR}\, y_2$. 164.52
.... [.51] $\forall_1$INST

□□□□□□$x_1 \,\mathrm{PAR}\, y_1 = x_1 \,\mathrm{PAR}\, x_1 \vee x_1 \,\mathrm{PAR}\, y_1 = x_1 \,\mathrm{PAR}\, y_2$. .... [.52] [.49] TAUT 164.53

□□□□□□$x_1 \,\mathrm{PAR}\, y_1 = x_1 \,\mathrm{PAR}\, x_1 \vdash y_1 = y_2$. .... LEM 164.54

□□□□□□¶ .... DEM

□□□□□□□$x_1 \,\mathrm{PAR}\, y_1 = x_1 \,\mathrm{PAR}\, x_1$. .... [.54] ASM 164.55

□□□□□□□$x_1 \,\mathrm{PAR}\, y_1 : \mathscr{H} \wedge \forall tt : x_1 \,\mathrm{PAR}\, x_1 \Leftrightarrow t = x_1 \vee t = y_1$. .... [.23] [.2] [.6] $_2$MINT$_2$ 164.56

□□□□□□□$x_1 \,\mathrm{PAR}\, x_1 : \mathscr{H} \wedge \forall tt : x_1 \,\mathrm{PAR}\, x_1 \Leftrightarrow t = x_1 \vee t = x_1$. .... [.23] [.2] [.2] $_2$MINT$_2$ 164.57

□□□□□□□$\forall tt : x_1 \,\mathrm{PAR}\, y_1 \Leftrightarrow t = x_1 \vee t = x_1$. .... [.57] [.55] TAUT 164.58

□□□□□□□$y_1 : x_1 \,\mathrm{PAR}\, y_1 \Leftrightarrow y_1 = x_1 \vee y_1 = x_1$. .... [.58] $\forall_1$INST 164.59

□□□□□□□$\forall tt : x_1 \,\mathrm{PAR}\, y_1 \Leftrightarrow t = x_1 \vee t = y_1$. .... [.56] TAUT 164.60

□□□□□□□$y_1 : x_1 \,\mathrm{PAR}\, y_1 \Leftrightarrow y_1 = x_1 \vee y_1 = y_1$. .... [.60] $\forall_1$INST 164.61

□□□□□□□$y_1 = y_1$. .... EQ 164.62

□□□□□□□$y_1 = x_1$. .... [.62] [.61] [.59] TAUT 164.63

□□□□□□□$x_2 \,\mathrm{PAR}\, x_2 : (x_2 \,\mathrm{PAR}\, x_2) \,\mathrm{PAR}\, x_2 \,\mathrm{PAR}\, y_2 \wedge x_2 \,\mathrm{PAR}\, y_2 : (x_2 \,\mathrm{PAR}\, x_2) \,\mathrm{PAR}\, x_2 \,\mathrm{PAR}\, y_2$. 164.64
.... [.15] [.21] [.22] $_2$MINT$_2$

□□□□□□□$x_1 \,\mathrm{PAR}\, y_2 : (x_1 \,\mathrm{PAR}\, x_1) \,\mathrm{PAR}\, x_1 \,\mathrm{PAR}\, x_1$. .... [.64] [.14] [.48] [.63] TAUT 164.65

□□□□□□□$\forall x \forall y x : \mathscr{H} \vdash y : \mathscr{H} \vdash \forall tt : x \,\mathrm{PAR}\, y \Leftrightarrow t = x \vee t = y$. .... THM 164.66

□□□□□□□$\forall tt : (x_1 \,\mathrm{PAR}\, x_1) \,\mathrm{PAR}\, x_1 \,\mathrm{PAR}\, x_1 \Leftrightarrow t = x_1 \,\mathrm{PAR}\, x_1 \vee t = x_1 \,\mathrm{PAR}\, x_1$. 164.67
.... [.66] [.18] [.18] $_2$MINT$_2$

□□□□□□□$x_1 \,\mathrm{PAR}\, y_2 : (x_1 \,\mathrm{PAR}\, x_1) \,\mathrm{PAR}\, x_1 \,\mathrm{PAR}\, x_1 \Leftrightarrow x_1 \,\mathrm{PAR}\, y_2 = x_1 \,\mathrm{PAR}\, x_1 \vee x_1 \,\mathrm{PAR}\, y_2 = x_1$ PAR 164.68
x_1. .... [.67] $\forall_1$INST

□□□□□□□$x_1 \,\mathrm{PAR}\, y_2 = x_1 \,\mathrm{PAR}\, x_1 \vee x_1 \,\mathrm{PAR}\, y_2 = x_1 \,\mathrm{PAR}\, x_1$. .... [.68] [.65] $\forall_1$INST 164.69

□□□□□□□$x_1 \,\mathrm{PAR}\, y_2 = x_1 \,\mathrm{PAR}\, x_1$. .... [.69] TAUT 164.70

□□□□□□□$\forall tt : x_1 \,\mathrm{PAR}\, y_2 \Leftrightarrow t = x_1 \vee t = y_2$. .... [.66] [.2] [.8] $_2$MINT$_2$ 164.71

□□□□□□□$y_2 : x_1 \,\mathrm{PAR}\, y_2 \Leftrightarrow y_2 = x_1 \vee y_2 = y_2$. .... [.70] $\forall_1$INST 164.72

164.73 □□□□□□□$\forall t\, t{:}x_1 \,\mathrm{PAR}\, x_1 \Leftrightarrow t = x_1 \vee t = x_1$. ... [.66] [.2] [.2] $_2$MINT$_2$
164.74 □□□□□□□$y_2{:}x_1 \,\mathrm{PAR}\, x_1 \Leftrightarrow y_2 = x_1 \vee y_2 = x_1$. ... [.73] $\forall_1$INST
164.75 □□□□□□□$y_2 = y_2$. ... EQ
164.76 □□□□□□□$y_1 = y_2$. ... [.75] [.72] [.74] [.63] TAUT
□□□□□□■. ... QED
164.77 □□□□□□$x_1 \,\mathrm{PAR}\, y_1 = x_1 \,\mathrm{PAR}\, y_2 \vdash y_1 = y_2$. ... LEM
□□□□□□¶ ... DEM
164.78 □□□□□□□$x_1 \,\mathrm{PAR}\, y_1 = x_1 \,\mathrm{PAR}\, y_2$. ... [.77] ASM
164.79 □□□□□□□$\forall x \forall y\, x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \forall t\, t{:}x \,\mathrm{PAR}\, y \Leftrightarrow t = x \vee t = y$. ... THM
164.80 □□□□□□□$\forall t\, t{:}x_1 \,\mathrm{PAR}\, y_1 \Leftrightarrow t = x_1 \vee t = y_1$. ... [.79] [.2] [.6] $_2$MINT$_2$
164.81 □□□□□□□$y_2{:}x_1 \,\mathrm{PAR}\, y_1 \Leftrightarrow y_2 = x_1 \vee y_2 = y_1$. ... [.80] $\forall_1$INST
164.82 □□□□□□□$\forall t\, t{:}x_1 \,\mathrm{PAR}\, y_2 \Leftrightarrow t = x_1 \vee t = y_2$. ... [.79] [.2] [.8] $_2$MINT$_2$
164.83 □□□□□□□$y_1{:}x_1 \,\mathrm{PAR}\, y_2 \Leftrightarrow y_1 = x_1 \vee y_1 = y_2$. ... [.82] $\forall_1$INST
164.84 □□□□□□□$y_2{:}x_1 \,\mathrm{PAR}\, y_2 \Leftrightarrow y_2 = x_1 \vee y_2 = y_2$. ... [.82] $\forall_1$INST
164.85 □□□□□□□$y_2 = y_2$. ... EQ
164.86 □□□□□□□$y_2 = x_1 \vee y_2 = y_1$. ... [.85] [.84] [.78] [.81] TAUT
164.87 □□□□□□□$y_1{:}x_1 \,\mathrm{PAR}\, y_1 \Leftrightarrow y_1 = x_1 \vee y_1 = y_1$. ... [.80] $\forall_1$INST
164.88 □□□□□□□$y_1 = y_1$. ... EQ
164.89 □□□□□□□$y_1 = x_1 \vee y_1 = y_2$. ... [.88] [.87] [.78] [.83] TAUT
164.90 □□□□□□□$y_1 = y_2$. ... [.86] [.89] TAUT
□□□□□□■. ... QED
164.91 □□□□□□$y_1 = y_2$. ... [.53] [.54] [.77] TAUT
164.92 □□□□□□$x_1 = x_2 \wedge y_1 = y_2$. ... [.48] [.91] TAUT
□□□□□■. ... QED
□□□□■. ... QED
□□□■. ... QED
□□■. ... QED
□■. ... QED
164.93 □$\forall x_1 \forall x_2 \forall y_1 \forall y_2\, x_1{:}\mathscr{H} \vdash x_2{:}\mathscr{H} \vdash y_1{:}\mathscr{H} \vdash y_2{:}\mathscr{H} \vdash x_1 \,\mathrm{ORP}\, y_1 = x_2 \,\mathrm{ORP}\, y_2 \vdash x_1 = x_2 \wedge y_1 = y_2$. ... [.1] UGEN$_4$
■. ... QED
Uses Axioms: **3, 4, 5, 6, 7, 8, 9, 12**

For all sets x, y, we can exhibit a unique set z containing precisely all the ordered pairs $x_1 \,\mathrm{ORP}\, y_1$ for each $x_1{:}x$, $y_1{:}y$. PROOF: Fix any $x_1{:}x$. We may objectify a function $f{:}y \to \mathscr{H}$ which assigns each $y_1{:}y$ to $x_1 \,\mathrm{ORP}\, y_1$. By *replacement* there exists a unique set a st. $\forall a_1\, a_1{:}a \Leftrightarrow \exists y_1\, y_1{:}y \wedge a_1 = x_1 \,\mathrm{ORP}\, y_1$. Thus we may lift ourselves from x_1, and objectify a function $g{:}x \to \mathscr{H}$ which assigns each $x_1{:}x$ to the unique set a st. $\forall a_1\, a_1{:}a \Leftrightarrow \exists y_1\, y_1{:}y \wedge a_1 = x_1 \,\mathrm{ORP}\, y_1$. *Replacing* the elements of x by its evaluation under g, gives the set b. Finally, $\bigcup b$ gives the desired set. QED.
We name its realizer $\times$. Another proof uses the *power set* axiom, as opposed to *replacement*, by *specifying* over $\mathscr{P}\mathscr{P}(x \cup y)$. We prefer ours since *replacement* is pathologically less intrusive.

165 $\forall x \forall y\, x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash \exists! z\, z{:}\mathscr{H} \wedge \forall z_1\, z_1{:}z \Leftrightarrow \exists x_1 \exists y_1\, x_1{:}x \wedge y_1{:}y \wedge z_1 = x_1 \,\mathrm{ORP}\, y_1$. ... THEOREM *of Implicit Cartesian Product*

¶ . . . . . . DEM

$\square \boldsymbol{x}{:}\mathscr{H} \vdash \boldsymbol{y}{:}\mathscr{H} \vdash \exists! \boldsymbol{z}\boldsymbol{z}{:}\mathscr{H} \wedge \forall \boldsymbol{z}_1 \boldsymbol{z}_1{:}\boldsymbol{z} \Leftrightarrow \exists \boldsymbol{x}_1 \exists \boldsymbol{y}_1 \boldsymbol{x}_1{:}\boldsymbol{x} \wedge \boldsymbol{y}_1{:}\boldsymbol{y} \wedge \boldsymbol{z}_1 = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1$. . . . . . . LEM **165.1**

$\square$¶ . . . . . . DEM

$\square\square \boldsymbol{x}{:}\mathscr{H}$. . . . . . . **[.1]** ASM **165.2**

$\square\square \boldsymbol{y}{:}\mathscr{H} \vdash \exists! \boldsymbol{z}\boldsymbol{z}{:}\mathscr{H} \wedge \forall \boldsymbol{z}_1 \boldsymbol{z}_1{:}\boldsymbol{z} \Leftrightarrow \exists \boldsymbol{x}_1 \exists \boldsymbol{y}_1 \boldsymbol{x}_1{:}\boldsymbol{x} \wedge \boldsymbol{y}_1{:}\boldsymbol{y} \wedge \boldsymbol{z}_1 = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1$. . . . . . . LEM **165.3**

$\square\square$¶ . . . . . . DEM

$\square\square\square \boldsymbol{y}{:}\mathscr{H}$. . . . . . . **[.3]** ASM **165.4**

$\square\square\square \boldsymbol{x}_1{:}\boldsymbol{x} \vdash \exists! \boldsymbol{a}\boldsymbol{a}{:}\mathscr{H} \wedge \forall \boldsymbol{a}_1 \boldsymbol{a}_1{:}\boldsymbol{a} \Leftrightarrow \exists \boldsymbol{y}_1 \boldsymbol{y}_1{:}\boldsymbol{y} \wedge \boldsymbol{a}_1 = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1$. . . . . . . LEM **165.5**

$\square\square\square$¶ . . . . . . DEM

$\square\square\square\square \boldsymbol{x}_1{:}\boldsymbol{x}$. . . . . . . **[.5]** ASM **165.6**

$\square\square\square\square \forall \boldsymbol{a} \forall \boldsymbol{b} \forall \boldsymbol{\psi} (\forall \boldsymbol{x}\boldsymbol{x}{:}\boldsymbol{a} \vdash \exists! \boldsymbol{y}\boldsymbol{y}{:}\boldsymbol{b} \wedge \boldsymbol{\psi}\boldsymbol{x}\boldsymbol{y}) \vdash \exists \boldsymbol{f}\boldsymbol{f}{:}\boldsymbol{a} \to \boldsymbol{b} \wedge \forall \boldsymbol{x} \forall \boldsymbol{y} \boldsymbol{x}{:}\boldsymbol{a} \vdash \boldsymbol{y}{:}\boldsymbol{b} \vdash \boldsymbol{\psi}\boldsymbol{x}\boldsymbol{y} \Leftrightarrow \boldsymbol{f}\boldsymbol{x} = \boldsymbol{y}$. . . . . . . AXM **165.7**

$\square\square\square\square \forall \boldsymbol{y} \forall \boldsymbol{b} \forall \boldsymbol{\psi} (\forall \boldsymbol{y}_1 \boldsymbol{y}_1{:}\boldsymbol{y} \vdash \exists! \boldsymbol{z}\boldsymbol{z}{:}\boldsymbol{b} \wedge \boldsymbol{\psi}\boldsymbol{y}_1\boldsymbol{z}) \vdash \exists \boldsymbol{f}\boldsymbol{f}{:}\boldsymbol{y} \to \boldsymbol{b} \wedge \forall \boldsymbol{y}_1 \forall \boldsymbol{z} \boldsymbol{y}_1{:}\boldsymbol{y} \vdash \boldsymbol{z}{:}\boldsymbol{b} \vdash \boldsymbol{\psi}\boldsymbol{y}_1\boldsymbol{z} \Leftrightarrow \boldsymbol{f}\boldsymbol{y}_1 = \boldsymbol{z}$. **165.8**
. . . . **[.7]** QNT

$\square\square\square\square \forall \boldsymbol{\psi} (\forall \boldsymbol{y}_1 \boldsymbol{y}_1{:}\boldsymbol{y} \vdash \exists! \boldsymbol{z}\boldsymbol{z}{:}\mathscr{H} \wedge \boldsymbol{\psi}\boldsymbol{y}_1\boldsymbol{z}) \vdash \exists \boldsymbol{f}\boldsymbol{f}{:}\boldsymbol{y} \to \mathscr{H} \wedge \forall \boldsymbol{y}_1 \forall \boldsymbol{z} \boldsymbol{y}_1{:}\boldsymbol{y} \vdash \boldsymbol{z}{:}\mathscr{H} \vdash \boldsymbol{\psi}\boldsymbol{y}_1\boldsymbol{z} \Leftrightarrow \boldsymbol{f}\boldsymbol{y}_1 = \boldsymbol{z}$. **165.9**
. . . . **[.8]** UGEN$_2$

$\square\square\square\square (\forall \boldsymbol{y}_1 \boldsymbol{y}_1{:}\boldsymbol{y} \vdash \exists! \boldsymbol{z}\boldsymbol{z}{:}\mathscr{H} \wedge \boldsymbol{z} = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1) \vdash \exists \boldsymbol{f}\boldsymbol{f}{:}\boldsymbol{y} \to \mathscr{H} \wedge \forall \boldsymbol{y}_1 \forall \boldsymbol{z} \boldsymbol{y}_1{:}\boldsymbol{y} \vdash \boldsymbol{z}{:}\mathscr{H} \vdash \boldsymbol{z} = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1 \Leftrightarrow$ **165.10**
$\boldsymbol{f}\boldsymbol{y}_1 = \boldsymbol{z}$. . . . . . . **[.9]** $\forall_2$INST

$\square\square\square\square \boldsymbol{y}_1{:}\boldsymbol{y} \vdash \exists! \boldsymbol{z}\boldsymbol{z}{:}\mathscr{H} \wedge \boldsymbol{z} = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1$. . . . . . . LEM **165.11**

$\square\square\square\square$¶ . . . . . . DEM

$\square\square\square\square\square \boldsymbol{y}_1{:}\boldsymbol{y}$. . . . . . . **[.11]** ASM **165.12**

$\square\square\square\square\square \forall \boldsymbol{x} \forall \boldsymbol{y} \boldsymbol{x}{:}\boldsymbol{y} \vdash \boldsymbol{y}{:}\mathscr{H} \vdash \boldsymbol{x}{:}\mathscr{H}$. . . . . . . AXM **165.13**

$\square\square\square\square\square \boldsymbol{x}_1{:}\mathscr{H}$. . . . . . . **[.13] [.6] [.2]** $_2$MINT$_2$ **165.14**

$\square\square\square\square\square \boldsymbol{y}_1{:}\mathscr{H}$. . . . . . . **[.13] [.12] [.4]** $_2$MINT$_2$ **165.15**

$\square\square\square\square\square \forall \boldsymbol{x} \forall \boldsymbol{y} \boldsymbol{x}{:}\mathscr{H} \vdash \boldsymbol{y}{:}\mathscr{H} \vdash \boldsymbol{x}\ \mathbf{ORP}\ \boldsymbol{y}{:}\mathscr{H} \wedge \boldsymbol{x}\ \mathbf{ORP}\ \boldsymbol{y} = (\boldsymbol{x}\ \mathbf{PAR}\ \boldsymbol{x})\ \mathbf{PAR}\ \boldsymbol{x}\ \mathbf{PAR}\ \boldsymbol{y}$. . . . . . . DEF **165.16**

$\square\square\square\square\square \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1{:}\mathscr{H} \wedge \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1 = (\boldsymbol{x}_1\ \mathbf{PAR}\ \boldsymbol{x}_1)\ \mathbf{PAR}\ \boldsymbol{x}_1\ \mathbf{PAR}\ \boldsymbol{y}_1$. . . . . . . **[.16] [.14] [.15]** $_2$MINT$_2$ **165.17**

$\square\square\square\square\square \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1 = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1$. . . . . . . EQ **165.18**

$\square\square\square\square\square \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1{:}\mathscr{H} \wedge \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1 = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1$. . . . . . . **[.17] [.18]** TAUT **165.19**

$\square\square\square\square\square \exists \boldsymbol{z}\boldsymbol{z}{:}\mathscr{H} \wedge \boldsymbol{z} = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1$. . . . . . . **[.19]** $\exists_1$GEN **165.20**

$\square\square\square\square\square \boldsymbol{z}_1{:}\mathscr{H} \wedge \boldsymbol{z}_1 = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1 \vdash \boldsymbol{z}_2{:}\mathscr{H} \wedge \boldsymbol{z}_2 = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1 \vdash \boldsymbol{z}_1 = \boldsymbol{z}_2$. . . . . . . LEM **165.21**

$\square\square\square\square\square$¶ . . . . . . DEM

$\square\square\square\square\square\square \boldsymbol{z}_1{:}\mathscr{H} \wedge \boldsymbol{z}_1 = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1$. . . . . . . **[.21]** ASM **165.22**

$\square\square\square\square\square\square \boldsymbol{z}_2{:}\mathscr{H} \wedge \boldsymbol{z}_2 = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1 \vdash \boldsymbol{z}_1 = \boldsymbol{z}_2$. . . . . . . LEM **165.23**

$\square\square\square\square\square\square$¶ . . . . . . DEM

$\square\square\square\square\square\square\square \boldsymbol{z}_2{:}\mathscr{H} \wedge \boldsymbol{z}_2 = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1$. . . . . . . **[.23]** ASM **165.24**

$\square\square\square\square\square\square\square \boldsymbol{z}_1 = \boldsymbol{z}_2$. . . . . . . **[.22] [.24]** TAUT **165.25**

$\square\square\square\square\square\square\blacksquare$. . . . . . . QED

$\square\square\square\square\square\blacksquare$. . . . . . . QED

$\square\square\square\square\square \forall \boldsymbol{z}_1 \forall \boldsymbol{z}_2 \boldsymbol{z}_1{:}\mathscr{H} \wedge \boldsymbol{z}_1 = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1 \vdash \boldsymbol{z}_2{:}\mathscr{H} \wedge \boldsymbol{z}_2 = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1 \vdash \boldsymbol{z}_1 = \boldsymbol{z}_2$. . . . . . . **[.21]** UGEN$_2$ **165.26**

$\square\square\square\square\square \exists! \boldsymbol{z}\boldsymbol{z}{:}\mathscr{H} \wedge \boldsymbol{z} = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1$. . . . . . . **[.20] [.26]** $\exists!$INTROS **165.27**

$\square\square\square\square\blacksquare$. . . . . . . QED

$\square\square\square\square \forall \boldsymbol{y}_1 \boldsymbol{y}_1{:}\boldsymbol{y} \vdash \exists! \boldsymbol{z}\boldsymbol{z}{:}\mathscr{H} \wedge \boldsymbol{z} = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1$. . . . . . . **[.11]** $\forall_1$GEN **165.28**

$\square\square\square\square \exists \boldsymbol{f}\boldsymbol{f}{:}\boldsymbol{y} \to \mathscr{H} \wedge \forall \boldsymbol{y}_1 \forall \boldsymbol{z} \boldsymbol{y}_1{:}\boldsymbol{y} \vdash \boldsymbol{z}{:}\mathscr{H} \vdash \boldsymbol{z} = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1 \Leftrightarrow \boldsymbol{f}\boldsymbol{y}_1 = \boldsymbol{z}$. . . . . . . **[.10] [.28]** MP **165.29**

$\square\square\square\square \boldsymbol{f}{:}\boldsymbol{y} \to \mathscr{H} \wedge \forall \boldsymbol{y}_1 \forall \boldsymbol{z} \boldsymbol{y}_1{:}\boldsymbol{y} \vdash \boldsymbol{z}{:}\mathscr{H} \vdash \boldsymbol{z} = \boldsymbol{x}_1\ \mathbf{ORP}\ \boldsymbol{y}_1 \Leftrightarrow \boldsymbol{f}\boldsymbol{y}_1 = \boldsymbol{z}$. . . . . . . **[.29]** $\exists_1$INST **165.30**

$\square\square\square\square \boldsymbol{f}{:}\boldsymbol{y} \to \mathscr{H}$. . . . . . . **[.30]** TAUT **165.31**

165.32 □□□□$\forall x \forall f\, x{:}\mathscr{H} \vdash f{:}x \to \mathscr{H} \vdash \exists y\, y{:}\mathscr{H} \wedge \forall y_1\, y_1{:}y \Leftrightarrow \exists x_1\, x_1{:}x \wedge f\, x_1 = y_1.$ ...... AXM

165.33 □□□□$\forall x \forall f\, x{:}\mathscr{H} \vdash f{:}x \to \mathscr{H} \vdash \exists a\, a{:}\mathscr{H} \wedge \forall a_1\, a_1{:}a \Leftrightarrow \exists y_1\, y_1{:}x \wedge f\, y_1 = a_1.$ ...... [.32] QNT

165.34 □□□□$\exists a\, a{:}\mathscr{H} \wedge \forall a_1\, a_1{:}a \Leftrightarrow \exists y_1\, y_1{:}y \wedge f\, y_1 = a_1.$ ...... [.33] [.4] [.31] ${}_2$MINT${}_2$

165.35 □□□□$a{:}\mathscr{H} \wedge \forall a_1\, a_1{:}a \Leftrightarrow \exists y_1\, y_1{:}y \wedge f\, y_1 = a_1.$ ...... [.34] $\exists_1$INST

165.36 □□□□$\forall a_1\, a_1{:}a \Leftrightarrow \exists y_1\, y_1{:}y \wedge f\, y_1 = a_1.$ ...... [.35] TAUT

165.37 □□□□$a{:}\mathscr{H}.$ ...... [.35] TAUT

165.38 □□□□$a_1{:}a \vdash \exists y_1\, y_1{:}y \wedge a_1 = x_1 \ \mathrm{ORP}\ y_1.$ ...... LEM

□□□□¶ ...... DEM

165.39 □□□□□$a_1{:}a.$ ...... [.38] ASM

165.40 □□□□□$a_1{:}a \Leftrightarrow \exists y_1\, y_1{:}y \wedge f\, y_1 = a_1.$ ...... [.36] $\forall_1$INST

165.41 □□□□□$\exists y_1\, y_1{:}y \wedge f\, y_1 = a_1.$ ...... [.41] [.39] TAUT

165.42 □□□□□$y_2{:}y \wedge f\, y_2 = a_1.$ ...... [.41] $\exists_1$INST

165.43 □□□□□$\forall y_1 \forall z\, y_1{:}y \vdash z{:}\mathscr{H} \vdash z = x_1 \ \mathrm{ORP}\ y_1 \Leftrightarrow f\, y_1 = z.$ ...... [.30] TAUT

165.44 □□□□□$y_2{:}y \vdash f\, y_2{:}\mathscr{H} \vdash f\, y_2 = x_1 \ \mathrm{ORP}\ y_2 \Leftrightarrow f\, y_2 = f\, y_2.$ ...... [.43] ${}_0$MINT${}_2$

165.45 □□□□□$\forall a \forall b \forall f\, f{:}a \to b \vdash \forall x\, x{:}a \vdash f\, x{:}b.$ ...... AXM

165.46 □□□□□$\forall x\, x{:}y \vdash f\, y{:}\mathscr{H}.$ ...... [.45] [.31] ${}_1$MINT${}_3$

165.47 □□□□□$y_2{:}y \vdash f\, y_2{:}\mathscr{H}.$ ...... [.46] $\forall_1$INST

165.48 □□□□□$f\, y_2 = f\, y_2.$ ...... EQ

165.49 □□□□□$a_1 = x_1 \ \mathrm{ORP}\ y_2.$ ...... [.47] [.42] [.44] [.48] TAUT

165.50 □□□□□$y_2{:}y \wedge a_1 = x_1 \ \mathrm{ORP}\ y_2.$ ...... [.42] [.49] TAUT

165.51 □□□□□$\exists y_1\, y_1{:}y \wedge a_1 = x_1 \ \mathrm{ORP}\ y_1.$ ...... [.50] $\exists_1$GEN

□□□□■. ...... QED

165.52 □□□□$(\exists y_1\, y_1{:}y \wedge a_1 = x_1 \ \mathrm{ORP}\ y_1) \vdash a_1{:}a.$ ...... LEM

□□□□¶ ...... DEM

165.53 □□□□□$\exists y_1\, y_1{:}y \wedge a_1 = x_1 \ \mathrm{ORP}\ y_1.$ ...... [.52] ASM

165.54 □□□□□$y_2{:}y \wedge a_1 = x_1 \ \mathrm{ORP}\ y_2.$ ...... [.52] $\exists_1$INST

165.55 □□□□□$\forall y_1 \forall z\, y_1{:}y \vdash z{:}\mathscr{H} \vdash z = x_1 \ \mathrm{ORP}\ y_1 \Leftrightarrow f\, y_1 = z.$ ...... [.30] TAUT

165.56 □□□□□$y_2{:}y \vdash x_1 \ \mathrm{ORP}\ y_2{:}\mathscr{H} \vdash x_1 \ \mathrm{ORP}\ y_2 = x_1 \ \mathrm{ORP}\ y_2 \Leftrightarrow f\, y_2 = x_1 \ \mathrm{ORP}\ y_2.$ ...... [.55] ${}_0$MINT${}_2$

165.57 □□□□□$\forall x \forall y\, x{:}y \vdash y{:}\mathscr{H} \vdash x{:}\mathscr{H}.$ ...... AXM

165.58 □□□□□$x_1{:}\mathscr{H}.$ ...... [.57] [.6] [.2] ${}_2$MINT${}_2$

165.59 □□□□□$y_2{:}y.$ ...... [.54] TAUT

165.60 □□□□□$y_2{:}\mathscr{H}.$ ...... [.57] [.59] [.4] ${}_2$MINT${}_2$

165.61 □□□□□$\forall x \forall y\, x{:}\mathscr{H} \vdash y{:}\mathscr{H} \vdash x \ \mathrm{ORP}\ y{:}\mathscr{H} \wedge x \ \mathrm{ORP}\ y = (x \ \mathrm{PAR}\ x) \ \mathrm{PAR}\ x \ \mathrm{PAR}\ y.$ ...... DEF

165.62 □□□□□$x_1 \ \mathrm{ORP}\ y_2{:}\mathscr{H} \wedge x_1 \ \mathrm{ORP}\ y_2 = (x_1 \ \mathrm{PAR}\ x_1) \ \mathrm{PAR}\ x_1 \ \mathrm{PAR}\ y_2.$ ...... [.61] [.58] [.60] ${}_2$MINT${}_2$

165.63 □□□□□$x_1 \ \mathrm{ORP}\ y_2 = x_1 \ \mathrm{ORP}\ y_2.$ ...... EQ

165.64 □□□□□$y_2{:}y \wedge f\, y_2 = a_1.$ ...... [.56] [.59] [.62] [.63] [.54] TAUT

165.65 □□□□□$\exists y_1\, y_1{:}y \wedge f\, y_1 = a_1.$ ...... [.65] $\exists_1$GEN

165.66 □□□□□$a_1{:}a \Leftrightarrow \exists y_1\, y_1{:}y \wedge f\, y_1 = a_1.$ ...... [.36] $\forall_1$GEN

165.67 □□□□□$a_1{:}a.$ ...... [.66] [.65] TAUT

□□□□■. ...... QED

165.68 □□□□$a_1{:}a \Leftrightarrow \exists y_1\, y_1{:}y \wedge a_1 = x_1 \ \mathrm{ORP}\ y_1.$ ...... [.38] [.52] TAUT

165.69 □□□□$\forall a_1\, a_1{:}a \Leftrightarrow \exists y_1\, y_1{:}y \wedge a_1 = x_1 \ \mathrm{ORP}\ y_1.$ ...... [.68] $\forall_1$GEN

165.70 □□□□$a{:}\mathscr{H} \wedge \forall a_1\, a_1{:}a \Leftrightarrow \exists y_1\, y_1{:}y \wedge a_1 = x_1 \ \mathrm{ORP}\ y_1.$ ...... [.35] [.69] TAUT

165.71 □□□□$\exists a\, a{:}\mathscr{H} \wedge \forall a_1\, a_1{:}a \Leftrightarrow \exists y_1\, y_1{:}y \wedge a_1 = x_1 \ \mathrm{ORP}\ y_1.$ ...... [.68] $\exists_1$GEN

□□□□$(a_x{:}\mathscr{H}\wedge\forall a_1 a_1{:}a_x\Leftrightarrow\exists y_1 y_1{:}y\wedge a_1=x_1\,\mathrm{ORP}\,y_1)\vdash(a_y{:}\mathscr{H}\wedge\forall a_1 a_1{:}a_y\Leftrightarrow\exists y_1 y_1{:}y\wedge a_1=x_1\,\mathrm{ORP}\,y_1)\vdash a_x=a_y$ .... LEM **165.72**

□□□□¶ .... DEM

□□□□□$a_x{:}\mathscr{H}\wedge\forall a_1 a_1{:}a_x\Leftrightarrow\exists y_1 y_1{:}y\wedge a_1=x_1\,\mathrm{ORP}\,y_1$ .... [.72] ASM **165.73**

□□□□□$(a_y{:}\mathscr{H}\wedge\forall a_1 a_1{:}a_y\Leftrightarrow\exists y_1 y_1{:}y\wedge a_1=x_1\,\mathrm{ORP}\,y_1)\vdash a_x=a_y$ .... LEM **165.74**

□□□□□¶ .... DEM

□□□□□□$a_y{:}\mathscr{H}\wedge\forall a_1 a_1{:}a_y\Leftrightarrow\exists y_1 y_1{:}y\wedge a_1=x_1\,\mathrm{ORP}\,y_1$ .... [.74] ASM **165.75**

□□□□□□$\forall a_1 a_1{:}a_x\Leftrightarrow\exists y_1 y_1{:}y\wedge a_1=x_1\,\mathrm{ORP}\,y_1$ .... [.73] TAUT **165.76**

□□□□□□$t{:}a_x\Leftrightarrow\exists y_1 y_1{:}y\wedge t=x_1\,\mathrm{ORP}\,y_1$ .... [.76] $\forall_1$INST **165.77**

□□□□□□$\forall a_1 a_1{:}a_y\Leftrightarrow\exists y_1 y_1{:}y\wedge a_1=x_1\,\mathrm{ORP}\,y_1$ .... [.75] TAUT **165.78**

□□□□□□$t{:}a_y\Leftrightarrow\exists y_1 y_1{:}y\wedge t=x_1\,\mathrm{ORP}\,y_1$ .... [.78] $\forall_1$INST **165.79**

□□□□□□$t{:}a_x\Leftrightarrow t{:}a_y$ .... [.77] [.79] TAUT **165.80**

□□□□□□$\forall tt{:}a_x\Leftrightarrow t{:}a_y$ .... [.80] $\forall_1$GEN **165.81**

□□□□□□$\forall x\forall yx{:}\mathscr{H}\vdash y{:}\mathscr{H}\vdash(\forall tt{:}x\Leftrightarrow t{:}y)\vdash x=y$ .... AXM **165.82**

□□□□□□$a_x{:}\mathscr{H}\vdash a_y{:}\mathscr{H}\vdash(\forall tt{:}a_x\Leftrightarrow t{:}a_y)\vdash a_x=a_y$ .... [.82] ${}_0\mathrm{MINT}_3$ **165.83**

□□□□□□$a_x=a_y$ .... [.83] [.73] [.75] [.81] TAUT **165.84**

□□□□□■ .... QED

□□□□■ .... QED

□□□□$\forall a_x\forall a_y(a_x{:}\mathscr{H}\wedge\forall a_1 a_1{:}a_x\Leftrightarrow\exists y_1 y_1{:}y\wedge a_1=x_1\,\mathrm{ORP}\,y_1)\vdash(a_y{:}\mathscr{H}\wedge\forall a_1 a_1{:}a_y\Leftrightarrow\exists y_1 y_1{:}y\wedge a_1=x_1\,\mathrm{ORP}\,y_1)\vdash a_x=a_y$ .... [.72] UGEN_2 **165.85**

□□□□$\exists!aa{:}\mathscr{H}\wedge\forall a_1 a_1{:}a\Leftrightarrow\exists y_1 y_1{:}y\wedge a_1=x_1\,\mathrm{ORP}\,y_1$ .... [.71] [.85] $\exists!$INTROS **165.86**

□□□■ .... QED

□□□$\forall x_1 x_1{:}x\vdash\exists!aa{:}\mathscr{H}\wedge\forall a_1 a_1{:}a\Leftrightarrow\exists y_1 y_1{:}y\wedge a_1=x_1\,\mathrm{ORP}\,y_1$ .... [.5] $\forall_1$GEN **165.87**

□□□$\forall a\forall b\forall\psi(\forall xx{:}a\vdash\exists!yy{:}b\wedge\psi xy)\vdash\exists ff{:}a\to b\wedge\forall x\forall yx{:}a\vdash y{:}b\vdash\psi xy\Leftrightarrow fx=y$ .... AXM **165.88**

□□□$\forall x\forall b\forall\psi(\forall x_1 x_1{:}x\vdash\exists!aa{:}b\wedge\psi x_1 a)\vdash\exists ff{:}x\to b\wedge\forall x_1\forall ax_1{:}x\vdash a{:}b\vdash\psi x_1 a\Leftrightarrow fx_1=a$ .... [.88] QNT **165.89**

□□□$\forall\psi(\forall x_1 x_1{:}x\vdash\exists!aa{:}\mathscr{H}\wedge\psi x_1 a)\vdash\exists ff{:}x\to\mathscr{H}\wedge\forall x_1\forall ax_1{:}x\vdash a{:}\mathscr{H}\vdash\psi x_1 a\Leftrightarrow fx_1=a$ .... [.89] UGEN_2 **165.90**

□□□$(\forall x_1 x_1{:}x\vdash\exists!aa{:}\mathscr{H}\wedge\forall a_1 a_1{:}a\Leftrightarrow\exists y_1 y_1{:}y\wedge a_1=x_1\,\mathrm{ORP}\,y_1)\vdash\exists ff{:}x\to\mathscr{H}\wedge\forall x_1\forall ax_1{:}x\vdash a{:}\mathscr{H}\vdash(\forall a_1 a_1{:}a\Leftrightarrow\exists y_1 y_1{:}y\wedge a_1=x_1\,\mathrm{ORP}\,y_1)\Leftrightarrow fx_1=a$ .... [.90] $\forall_2$INST **165.91**

□□□$\exists ff{:}x\to\mathscr{H}\wedge\forall x_1\forall ax_1{:}x\vdash a{:}\mathscr{H}\vdash(\forall a_1 a_1{:}a\Leftrightarrow\exists y_1 y_1{:}y\wedge a_1=x_1\,\mathrm{ORP}\,y_1)\Leftrightarrow fx_1=a$ .... [.91] [.87] MP **165.92**

□□□$f{:}x\to\mathscr{H}\wedge\forall x_1\forall ax_1{:}x\vdash a{:}\mathscr{H}\vdash(\forall a_1 a_1{:}a\Leftrightarrow\exists y_1 y_1{:}y\wedge a_1=x_1\,\mathrm{ORP}\,y_1)\Leftrightarrow fx_1=a$ .... [.92] $\exists_1$INST **165.93**

□□□$f{:}x\to\mathscr{H}$ .... [.93] TAUT **165.94**

□□□$\forall x\forall fx{:}\mathscr{H}\vdash f{:}x\to\mathscr{H}\vdash\exists yy{:}\mathscr{H}\wedge\forall y_1 y_1{:}y\Leftrightarrow\exists x_1 x_1{:}x\wedge fx_1=y_1$ .... AXM **165.95**

□□□$\forall x\forall fx{:}\mathscr{H}\vdash f{:}x\to\mathscr{H}\vdash\exists bb{:}\mathscr{H}\wedge\forall b_1 b_1{:}b\Leftrightarrow\exists x_1 x_1{:}x\wedge fx_1=b_1$ .... [.95] QNT **165.96**

□□□$\exists bb{:}\mathscr{H}\wedge\forall b_1 b_1{:}b\Leftrightarrow\exists x_1 x_1{:}x\wedge fx_1=b_1$ .... [.96] [.2] [.94] ${}_2\mathrm{MINT}_2$ **165.97**

□□□$b{:}\mathscr{H}\wedge\forall b_1 b_1{:}b\Leftrightarrow\exists x_1 x_1{:}x\wedge fx_1=b_1$ .... [.97] $\exists_1$INST **165.98**

□□□$\forall xx{:}\mathscr{H}\vdash\bigcup x{:}\mathscr{H}\wedge\forall tt{:}\bigcup x\Leftrightarrow\exists x_1 x_1{:}x\wedge t{:}x_1$ .... DEF **165.99**

□□□$b{:}\mathscr{H}\vdash\bigcup b{:}\mathscr{H}\wedge\forall tt{:}\bigcup b\Leftrightarrow\exists x_1 x_1{:}b\wedge t{:}x_1$ .... [.99] $\forall_1$INST **165.100**

□□□$\bigcup b{:}\mathscr{H}$ .... [.100] TAUT **165.101**

□□□$\forall tt{:}\bigcup b\Leftrightarrow\exists x_1 x_1{:}b\wedge t{:}x_1$ .... [.100] TAUT **165.102**

□□□$t{:}\bigcup b\vdash\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge t=x_1\,\mathrm{ORP}\,y_1$ .... LEM **165.103**

□□□¶ … DEM

165.104 □□□□$t{:}\bigcup b.$ … [.103] ASM

165.105 □□□□$t{:}\bigcup b \Leftrightarrow \exists x_1 x_1{:}b \wedge t{:}x_1.$ … [.102] $\forall_1$INST

165.106 □□□□$\exists x_1 x_1{:}b \wedge t{:}x_1.$ … [.105] [.104] TAUT

165.107 □□□□$b_1{:}b \wedge t{:}b_1.$ … [.106] $\exists_1$INST

165.108 □□□□$\forall b_1 b_1{:}b \Leftrightarrow \exists x_1 x_1{:}x \wedge f x_1 = b_1.$ … [.98] TAUT

165.109 □□□□$b_1{:}b \Leftrightarrow \exists x_1 x_1{:}x \wedge f x_1 = b_1.$ … [.108] $\forall_1$INST

165.110 □□□□$\exists x_1 x_1{:}x \wedge f x_1 = b_1.$ … [.109] [.107] TAUT

165.111 □□□□$x_1{:}x \wedge f x_1 = b_1.$ … [.110] $\exists_1$INST

165.112 □□□□$\forall x_1 \forall a x_1{:}x \vdash a{:}\mathscr{H} \vdash (\forall a_1 a_1{:}a \Leftrightarrow \exists y_1 y_1{:}y \wedge a_1 = x_1 \,\mathrm{ORP}\, y_1) \Leftrightarrow f x_1 = a.$ … [.93] TAUT

165.113 □□□□$x_1{:}x \vdash f x_1{:}\mathscr{H} \vdash (\forall a_1 a_1{:}f x_1 \Leftrightarrow \exists y_1 y_1{:}y \wedge a_1 = x_1 \,\mathrm{ORP}\, y_1) \Leftrightarrow f x_1 = f x_1.$ … [.112] ${}_0$MINT$_2$

165.114 □□□□$\forall a \forall b \forall f\, f{:}a \to b \vdash \forall x x{:}a \vdash f x{:}b.$ … AXM

165.115 □□□□$\forall a \forall b \forall f\, f{:}a \to b \vdash \forall x_1 x_1{:}a \vdash f x_1{:}b.$ … [.114] QNT

165.116 □□□□$\forall x_1 x_1{:}x \vdash f x_1{:}\mathscr{H}.$ … [.115] [.93] ${}_1$MINT$_3$

165.117 □□□□$x_1{:}x \vdash f x_1{:}\mathscr{H}.$ … [.116] $\forall_1$INST

165.118 □□□□$f x_1 = f x_1.$ … EQ

165.119 □□□□$\forall a_1 a_1{:}b_1 \Leftrightarrow \exists y_1 y_1{:}y \wedge a_1 = x_1 \,\mathrm{ORP}\, y_1.$ … [.117] [.111] [.113] [.118] TAUT

165.120 □□□□$t{:}b_1 \Leftrightarrow \exists y_1 y_1{:}y \wedge t = x_1 \,\mathrm{ORP}\, y_1.$ … [.119] $\forall_1$INST

165.121 □□□□$\exists y_1 y_1{:}y \wedge t = x_1 \,\mathrm{ORP}\, y_1.$ … [.120] [.107] TAUT

165.122 □□□□$y_1{:}y \wedge t = x_1 \,\mathrm{ORP}\, y_1.$ … [.121] $\exists_1$INST

165.123 □□□□$x_1{:}x \wedge y_1{:}y \wedge t = x_1 \,\mathrm{ORP}\, y_1.$ … [.111] [.122] TAUT

165.124 □□□□$\exists y_1 x_1{:}x \wedge y_1{:}y \wedge t = x_1 \,\mathrm{ORP}\, y_1.$ … [.123] $\exists_1$GEN

165.125 □□□□$\exists x_1 \exists y_1 x_1{:}x \wedge y_1{:}y \wedge t = x_1 \,\mathrm{ORP}\, y_1.$ … [.123] $\exists_1$GEN

□□□■. … QED

165.126 □□□$(\exists x_1 \exists y_1 x_1{:}x \wedge y_1{:}y \wedge t = x_1 \,\mathrm{ORP}\, y_1) \vdash t{:}\bigcup b.$ … LEM

□□□¶ … DEM

165.127 □□□□$\exists x_1 \exists y_1 x_1{:}x \wedge y_1{:}y \wedge t = x_1 \,\mathrm{ORP}\, y_1.$ … [.126] ASM

165.128 □□□□$\exists y_1 x_1{:}x \wedge y_1{:}y \wedge t = x_1 \,\mathrm{ORP}\, y_1.$ … [.127] $\exists_1$INST

165.129 □□□□$x_1{:}x \wedge y_1{:}y \wedge t = x_1 \,\mathrm{ORP}\, y_1.$ … [.128] $\exists_1$INST

165.130 □□□□$t{:}\bigcup b \Leftrightarrow \exists x_1 x_1{:}b \wedge t{:}x_1.$ … [.102] $\forall_1$INST

165.131 □□□□$\forall x_1 \forall a x_1{:}x \vdash a{:}\mathscr{H} \vdash (\forall a_1 a_1{:}a \Leftrightarrow \exists y_1 y_1{:}y \wedge a_1 = x_1 \,\mathrm{ORP}\, y_1) \Leftrightarrow f x_1 = a.$ … [.93] TAUT

165.132 □□□□$x_1{:}x \vdash f x_1{:}\mathscr{H} \vdash (\forall a_1 a_1{:}f x_1 \Leftrightarrow \exists y_1 y_1{:}y \wedge a_1 = x_1 \,\mathrm{ORP}\, y_1) \Leftrightarrow f x_1 = f x_1.$ … [.131] ${}_0$MINT$_2$

165.133 □□□□$\forall a \forall b \forall f\, f{:}a \to b \vdash \forall x x{:}a \vdash f x{:}b.$ … AXM

165.134 □□□□$\forall a \forall b \forall f\, f{:}a \to b \vdash \forall x_1 x_1{:}a \vdash f x_1{:}b.$ … [.133] QNT

165.135 □□□□$\forall x_1 x_1{:}x \vdash f x_1{:}\mathscr{H}.$ … [.134] [.93] ${}_1$MINT$_3$

165.136 □□□□$x_1{:}x \vdash f x_1{:}\mathscr{H}.$ … [.135] $\forall_1$INST

165.137 □□□□$f x_1 = f x_1.$ … EQ

165.138 □□□□$\forall a_1 a_1{:}f x_1 \Leftrightarrow \exists y_1 y_1{:}y \wedge a_1 = x_1 \,\mathrm{ORP}\, y_1.$ … [.132] [.129] [.136] [.137] TAUT

165.139 □□□□$t{:}f x_1 \Leftrightarrow \exists y_1 y_1{:}y \wedge t = x_1 \,\mathrm{ORP}\, y_1.$ … [.138] $\forall_1$INST

165.140 □□□□$y_1{:}y \wedge t = x_1 \,\mathrm{ORP}\, y_1.$ … [.129] TAUT

165.141 □□□□$\exists y_1 y_1{:}y \wedge t = x_1 \,\mathrm{ORP}\, y_1.$ … [.140] $\exists_1$GEN

165.142 □□□□$t{:}f x_1.$ … [.139] [.141] TAUT

165.143 □□□□$\forall b_1 b_1{:}b \Leftrightarrow \exists x_1 x_1{:}x \wedge f x_1 = b_1.$ … [.98] TAUT

165.144 □□□□$\forall b_1 b_1{:}b \Leftrightarrow \exists x_2 x_2{:}x \wedge f x_2 = b_1.$ … [.143] QNT

□□□□$fx_1{:}b\Leftrightarrow\exists x_2 x_2{:}x\wedge fx_2=fx_1$. .... [.144] $\forall_1$INST **165.145**

□□□□$x_1{:}x\wedge fx_1=fx_1$. .... [.129] [.137] TAUT **165.146**

□□□□$\exists x_2 x_2{:}x\wedge fx_2=fx_1$. .... [.146] $\exists_1$GEN **165.147**

□□□□$fx_1{:}b\wedge t{:}fx_1$. .... [.145] [.147] [.142] TAUT **165.148**

□□□□$\exists x_1 x_1{:}b\wedge t{:}x_1$. .... [.148] $\exists_1$GEN **165.149**

□□□□$t{:}\bigcup b$. .... [.130] [.149] TAUT **165.150**

□□□■. .... QED

□□□$t{:}\bigcup b\Leftrightarrow\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge t=x_1\,\mathbf{ORP}\,y_1$. .... [.103] [.126] TAUT **165.151**

□□□$\forall z_1 z_1{:}\bigcup b\Leftrightarrow\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge z_1=x_1\,\mathbf{ORP}\,y_1$. .... [.151] $\forall_1$GEN **165.152**

□□□$\bigcup b{:}\mathcal{H}\wedge\forall z_1 z_1{:}\bigcup b\Leftrightarrow\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge z_1=x_1\,\mathbf{ORP}\,y_1$. .... [.101] [.152] TAUT **165.153**

□□□$\exists z z{:}\mathcal{H}\wedge\forall z_1 z_1{:}z\Leftrightarrow\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge z_1=x_1\,\mathbf{ORP}\,y_1$. .... [.153] $\exists_1$GEN **165.154**

□□□$(z_x{:}\mathcal{H}\wedge\forall z_1 z_1{:}z_x\Leftrightarrow\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge z_1=x_1\,\mathbf{ORP}\,y_1)\vdash(z_y{:}\mathcal{H}\wedge\forall z_1 z_1{:}z_y\Leftrightarrow\exists x_1\exists y_1$ **165.155**
$x_1{:}x\wedge y_1{:}y\wedge z_1=x_1\,\mathbf{ORP}\,y_1)\vdash z_x=z_y$. .... LEM

□□□¶ .... DEM

□□□□$z_x{:}\mathcal{H}\wedge\forall z_1 z_1{:}z_x\Leftrightarrow\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge z_1=x_1\,\mathbf{ORP}\,y_1$. .... [.155] ASM **165.156**

□□□□$(z_y{:}\mathcal{H}\wedge\forall z_1 z_1{:}z_y\Leftrightarrow\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge z_1=x_1\,\mathbf{ORP}\,y_1)\vdash z_x=z_y$. .... LEM **165.157**

□□□□¶ .... DEM

□□□□□$z_y{:}\mathcal{H}\wedge\forall z_1 z_1{:}z_y\Leftrightarrow\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge z_1=x_1\,\mathbf{ORP}\,y_1$. .... [.157] ASM **165.158**

□□□□□$\forall x\forall y x{:}\mathcal{H}\vdash y{:}\mathcal{H}\vdash(\forall t t{:}x\Leftrightarrow t{:}y)\vdash x=y$. .... AXM **165.159**

□□□□□$z_x{:}\mathcal{H}\vdash z_y{:}\mathcal{H}\vdash(\forall t t{:}z_x\Leftrightarrow t{:}z_y)\vdash z_x=z_y$. .... [.159] $_0$MINT$_2$ **165.160**

□□□□□$\forall z_1 z_1{:}z_x\Leftrightarrow\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge z_1=x_1\,\mathbf{ORP}\,y_1$. .... [.156] TAUT **165.161**

□□□□□$z_1{:}z_x\Leftrightarrow\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge z_1=x_1\,\mathbf{ORP}\,y_1$. .... [.161] $\forall_1$INST **165.162**

□□□□□$\forall z_1 z_1{:}z_y\Leftrightarrow\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge z_1=x_1\,\mathbf{ORP}\,y_1$. .... [.158] TAUT **165.163**

□□□□□$z_1{:}z_y\Leftrightarrow\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge z_1=x_1\,\mathbf{ORP}\,y_1$. .... [.163] $\forall_1$INST **165.164**

□□□□□$z_1{:}z_x\Leftrightarrow z_1{:}z_y$. .... [.162] [.164] TAUT **165.165**

□□□□□$\forall t t{:}z_x\Leftrightarrow t{:}z_y$. .... [.165] $\forall_1$GEN **165.166**

□□□□□$z_x=z_y$. .... [.160] [.156] [.158] [.166] TAUT **165.167**

□□□□■. .... QED

□□□■. .... QED

□□□$\forall z_x\forall z_y(z_x{:}\mathcal{H}\wedge\forall z_1 z_1{:}z_x\Leftrightarrow\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge z_1=x_1\,\mathbf{ORP}\,y_1)\vdash(z_y{:}\mathcal{H}\wedge\forall z_1 z_1{:}z_y\Leftrightarrow$ **165.168**
$\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge z_1=x_1\,\mathbf{ORP}\,y_1)\vdash z_x=z_y$. .... [.155] UGEN$_2$

□□□$\exists! z z{:}\mathcal{H}\wedge\forall z_1 z_1{:}z\Leftrightarrow\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge z_1=x_1\,\mathbf{ORP}\,y_1$. .... [.154] [.168] $\exists!$INTROS **165.169**

□□■. .... QED

□■. .... QED

□$\forall x\forall y x{:}\mathcal{H}\vdash y{:}\mathcal{H}\vdash\exists! z z{:}\mathcal{H}\wedge\forall z_1 z_1{:}z\Leftrightarrow\exists x_1\exists y_1 x_1{:}x\wedge y_1{:}y\wedge z_1=x_1\,\mathbf{ORP}\,y_1$. .... [.1] UGEN$_2$ **165.170**

■. .... QED

Uses Axioms: **3, 5, 6, 7, 8, 9, 10, 12, 13, 16**