1. Consider two events A and B. Let $P(A) = 0.5$, $P(B) = 0.5$, $P(A \cup B) = 0.75$. Check if A and B are independent. **(2 points)**

$\rightarrow$ given data

2. (Approximating a matrix) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Obtain the rank-1 matrix X^* which minimizes $|||A - X|||_F$, where $|||\cdot|||_F$ denotes the Frobenius norm. **(2 points)**

3. Let random variable X take values in $\{1, 2, 3\}$. Given that $p_X(1) = 0.5$, $p_X(2) = 0.4$. Compu **(2 points)**
$E(X)$ and $Var(X)$.

4. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the (possibly repeated) eigenvalues of $A \in \mathbb{R}^{n \times n}$. Show that:

$$\det(A) = \lambda_1 \lambda_2 \cdots \lambda_n.$$

(3 points)

are given $\lambda_1\ \lambda_2 \ldots \lambda_n$ are

5. We are given three coins: the first one has heads on both sides, the second has tails on both sides, and the third has a head on one side and a tail on the other. We choose a coin uniformly at random. We will then toss it (it can land on either side with equal probability) and the result is a head. What is the probability that the other side is also a head?

Hint: You may construct the sample space as follows - $\Omega = \{(1, H), (2, T), (3, H), (3, T)\}$. Here, for each element $\omega \in \Omega$, the first entry (1, 2 or 3) indicates the coin chosen and the second entry (H or T) denotes the toss result we observe. To assign probabilities, consider the events:

$$C_i : \{\text{coin-}i \text{ was chosen}\}, \quad A : \{\text{result is } H\}.$$

We now have, for example, $P(\{(2, T)\}) = P(C_2 \cap A^c)$.

(3 points)

6. An electrical system consists of six identical components (labelled $1, 2, \ldots, 6$) as shown in the figure below. Each component is operational with probability p, independent of the other components. There are three possible "paths" that start at point A and end at point B namely, $A-1-2-6-B$, $A-1-3-6-B$ and $A-1-4-5-6-B$. The system is operational if and only if there is a path such that all components along the path are operational. For instance, if components 1, 3 and 6 are operational, the system is operational. Compute the probability that the system is operational.

Hint: You may try to decompose the system as a sequence of sub-systems, which operate independently.

(3 points)