DS5616: Foundations of Statistical Learning **Test 1**

Jan-May 2025 **Date:** 17 **Feb** 2025

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- The test comprises 3 questions for a total of 30 points and is for a duration of 50 minutes.
- Write your fair answer *after* you have tried it out in the rough space. **Only work written in the space below the corresponding questions shall be graded**. If you do not wish something to be graded, strike it out neatly.
- This is a closed-book test. You may not use the textbook, class notes, mobile phone, tablet, laptop, or such devices.
- This is also an opportunity for you to evaluate yourself – in your best interests, be honest.

Question	Points	Score
1.	10	10
2.	10	5
3.	10	1
Total	30	16

1. A coin is tossed independently and repeatedly with the probability of heads equal to p.

(a) What is the probability of getting only heads in the first n tosses? (3 points)

probability of getting heads = p.

p(heads in 'n' tosses) = $(p)^n$.

Since heads occured in 'n' tosses.

(b) What is the probability of obtaining the first tail in the n-th toss? (3 points)

Sol:- Probability of getting heads = p (success)

Probability of getting tail = $1-p$ (failure)

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P(first tail in n^{th} toss) = $(p)^{n-1}(1-p)$.

Since $(n-1)$ tosses heads occured and in $(n-1)^{th}$ toss tail occured (only 1 time).

(c) What is the expected number of tosses required to obtain the first tail? (4 points)

$E[X]$ ⇒ no. of tosses required to obtain first tail in $(n-1)^{th}$ toss

We know, $E[X] = \sum_{x=1}^{\infty} x \cdot P(X=x)$

We know $P(X=x) = (p)^{n-1}(1-p)$

$E[X] = \sum_{x=1}^{\infty} x \cdot (p)^{n-1}(1-p)$

$E[X] = (1-p) \sum_{x=1}^{\infty} x \cdot (p)^{n-1} \rightarrow ①$

We have this equality,

$$\boxed{\sum_{n=0}^{\infty} x^n = \frac{1}{1-x} \text{ for } |x|<1}$$

differentiating both sides:-

$\sum_{n=0}^{\infty} n \cdot x^{n-1} = \frac{-1 \cdot -1}{(1-x)^2}$

$\boxed{\text{Here from eq } x = p \text{ ②}}$

$\Rightarrow \sum_{n=0}^{\infty} n x^{n-1} = \frac{1}{(1-x)^2} \rightarrow ②$

From eq ① and ②

$E[X] = 1-p \cdot \frac{1}{[1-(p)]^2}$

$E[X] = \frac{1-p}{(1-p)^2}$

$$\boxed{E[X] = \frac{1}{1-p}}$$

So, the expected number of tosses to obtain the first tail is $\frac{1}{1-p}$.

$= \frac{1}{\text{failure}}$

2. A fair coin is tossed n times.

(a) Show that the probability of obtaining more than $\frac{1}{\sqrt{2}}$ fraction of heads is at most $\frac{1}{\sqrt{2}}$. **(5 points)**

Using Hoeffding's inequality,
The probability $= \frac{1}{2}$ (fair coin)

$$P(X \geq nt + p) \leq e^{-\frac{t^2}{2n}}$$

~~Given~~ :- $P\left(\frac{X}{n} \geq \sqrt{\frac{1}{2}}\right) = P\left(X \geq n\sqrt{\frac{1}{2}}\right)$

We know from the eqn,

$$P\left(X \geq n\left(\sqrt{\tfrac{1}{2}} - \tfrac{1}{2}\right)\right)$$

$$\boxed{t = \frac{1}{\sqrt{2}} - \frac{1}{2}}$$

$$\leq e^{-\frac{\left(\frac{1}{\sqrt{2}} - \frac{1}{2}\right)^2}{2(n)}}$$

$$\leq e^{-\frac{(0.707-0.5)^2}{2n}}$$

$$\leq e^{-\frac{(0.207)^2}{2n}}$$

$$\leq e^{-\frac{0.021428}{n}}$$

$$\leq e^{-\frac{0.02142}{n}} \text{ as } n \to \text{large}$$

$$\leq \frac{1}{\sqrt{2}}.$$

$$\boxed{P\left(X \geq \frac{1}{\sqrt{2}}\right) \leq \frac{1}{\sqrt{2}}}$$

So, the probability of obtaining more than $\frac{1}{\sqrt{2}}$ fraction of heads is atmost $\frac{1}{\sqrt{2}}$. Hence proved.

Suffices to use Markov

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(b) Show that the probability of obtaining more than $\frac{1}{4}$ fraction of heads is at least $\frac{1}{4}$ as $n \to \infty$. **(5 points)**

Hint: Use the Paley-Zygmund inequality: for $X \geq 0$ and $0 < \alpha < 1$,

$$\mathbb{P}[X \geq \alpha \mathbb{E}X] \geq (1-\alpha)^2 \frac{(\mathbb{E}X)^2}{\mathbb{E}X^2}.$$

Sol :- $P\left(X \geq \frac{1}{4}\right) = P\left(\frac{X}{n} \geq \frac{1}{4}\right) \Rightarrow P\left(X \geq \frac{n}{4}\right)$

3. Let $X_1, \ldots, X_n$ be independent random variables with

$$\mathbb{P}[X_1 = 1] = p_i = 1 - \mathbb{P}[X = -1]$$

where $p_i \in (0,1), i = 1, \ldots, n$. Show that, for any $t \geq 0$, we have

$$\mathbb{P}\left[\sum_{i=1}^{n}(X_i - \mathbb{E}X_i) \geq t\right] \leq \exp\left\{-\frac{t^2}{2n}\right\}.$$

(10 points)

Hint: Use Chernoff bound. You may assume the inequality: for all $\lambda > 0$ and $0 < p < 1$,

$$pe^{2(1-p)\lambda} + (1-p)e^{-2p\lambda} \leq e^{\lambda^2/2}.$$

chernoff bound:-

$$P[X \geq (1+\delta)E[X]] \leq e^{-\frac{\delta^2 E[X]}{2+\delta}}$$

$X = X_1 + X_2 + \cdots X_i$'s

$$\Rightarrow \quad P\left[\sum_{i=1}^{n}(X_i) - \sum_{i=1}^{n} E[X_i]) \geq t\right]$$

$$\Rightarrow \quad P[(X - E[X]) \geq t\,E[X]]$$ Multiplying with $E[X]$

$$\Rightarrow \quad P[(X \geq t + E[X]] \Rightarrow P\left[\left(\frac{X}{E[X]} \geq \frac{tE[X] + E[X]}{E[X]}\right)\right]$$

let $t = 1+\delta$

(1) then $P\left[\left(X \geq \frac{\cancel{t E[X]} E[X]}{\cancel{E[X]}}\right)\right] \leq \cancel{e}$

$$P\left[\left(X \geq \frac{(t+1)E[X]}{\cancel{E[X]}}\right)\right] \leq e^{-\frac{\delta^2 E[X]}{2+\delta}}$$

$$\leq e^{-\frac{t^2 E[X]}{2+\delta}}$$

$\sum_{i} \mu P(x, \theta) \; \underline{f}$ ... Bayesbooth ... Do Problem Set

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1. A box has r red and b blue balls. A ball is chosen at random from the box (so that each ball is equally likely to be chosen), and then a second ball is drawn at random from the remaining balls in the box. Find the probability that

   (a) Both balls are of the same colour.

   (b) Both balls are of different colour.

   Do they add up to 1? **(6 points)**

Sol:- Total no. of balls = $r+b$

(a.) Both balls are of same colour.

i, let us think the colour is red.

$$P(\text{red}) = \frac{r}{r+b} \times \frac{r-1}{r+b-1}$$

ii, The colour is blue

$$P(\text{blue}) = \frac{b}{r+b} \times \frac{b-1}{r+b-1}$$

$$\text{Total probability} = \frac{r(r-1)+b(b-1)}{(r+b)(r+b-1)}$$

(b) Both balls are different colour

i, red and then next ball is blue

~~P(1st ball)~~ =

$$P(\text{red}, \text{blue}) = \frac{r}{r+b} \times \frac{b}{r+b-1}$$

ii, blue and then red

$$P(\text{blue}, \text{red}) = \frac{b}{b+r} \times \frac{r}{b+r-1}$$

$$\text{Total probability} = \frac{2rb}{(b+r)(b+r-1)}$$

Yes they add up to 1. $\frac{r(r-1)+b(b-1)+2rb}{(r+b)(r+b-1)} = \frac{r^2 \cancel{-r} + b^2 \cancel{-b} + \cancel{2}rb}{(r+b)(r+b-1)} = 1$

$r^2+b^2-r-2rb$

2. Let $X = 1$ if the second ball drawn is red; 0 otherwise. What is $\mathbb{E}[X]$? **(4 points)**

sol:-

$$\boxed{E[X] = \sum_{n=0} n \cdot P(X = n)}$$

$$E[X] = \sum_{i=0} 1 \cdot P(X=1) \text{ (second ball drawn is red)}$$

[Because it may be different (or) equal]

$$E[X] = \frac{rb}{(b+r)(b+r-1)} + \frac{r(r-1)}{(r+b)(b+r-1)}$$

$$= \frac{rb + r(r-1)}{(r+b)(b+r-1)} = \frac{rb + r^2 - r}{(r+b)(b+r-1)}$$

$$= \frac{r(b+\cancel{r-1})}{(r+b)(\cancel{b+r-1})} = \frac{r}{r+b}$$