1. Let $\alpha \in \mathbb{R}$. Consider the following linear transformation $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by:

$$\phi\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} 3x_1 + 2x_2 \\ x_1 + \alpha x_2 \end{bmatrix}.$$

Obtain the condition(s) on α such that the function is injective (i.e., one-to-one)? No justification is required for your answer.

$f(x_1, x_2) = y$

(2 points)

2. Consider two matrices $A, B \in \mathbb{R}^{3\times 3}$ such that $AB = \mathbf{0}$. Is it always the case that either $A = \mathbf{0}$ or $B = \mathbf{0}$? If yes, prove it; otherwise give a counterexample.

(2 points)

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3. Let $A \in \mathbb{R}^{n\times n}$ be diagonalizable. Define $B \triangleq A^2 + 2A + 3I_n$. Is B always diagonalizable? If yes, prove it; otherwise, provide a counterexample.

Hint: If $A = PDP^{-1}$, what is A^2?

(3 points)

4. Compute the inverse of the following matrix.

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}.$$

Hint: Compare the columns. Do you notice any interesting property? **(3 points)**

5. **This question has three parts.** Consider the vector space of real-valued functions (signals) defined on $[0, 1]$, with the inner product:

$$\langle y_1(t), y_2(t) \rangle = \int_0^1 y_1(t)\, y_2(t)\, dt.$$

(i) Show that the signals $x_1(t)$ and $x_2(t)$ given below are orthogonal. **(1 point)**

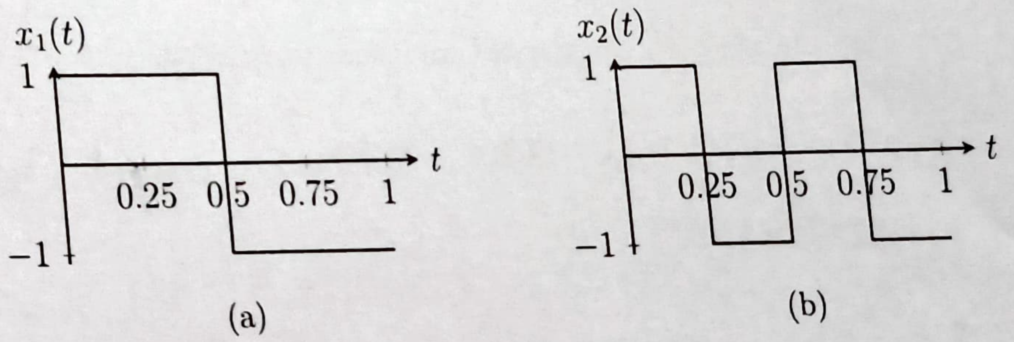

Figure 1: The two signals: (a) $x_1(t)$ and (b) $x_2(t)$.

(ii) A transmitter sends a signal $x(t) = a_1 x_1(t) + a_2 x_2(t)$, where $a_1, a_2 \in \mathbb{R}$ to a receiver. The receiver knows $x_1(t)$ and $x_2(t)$, but not $x(t)$. Instead, it receives the following noisy signal $y(t)$. Compute the orthogonal projection of $y(t)$ onto the subspace spanned by $x_1(t)$ and $x_2(t)$. **(2 points)**

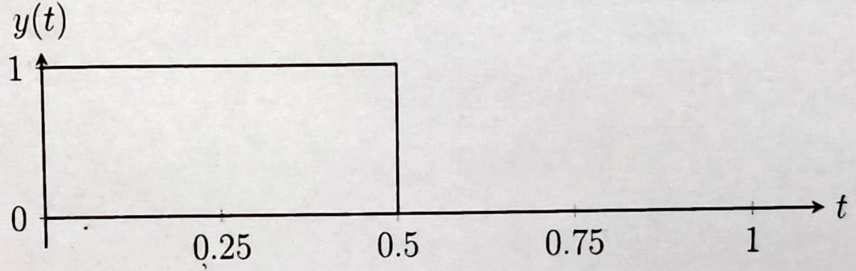

Figure 2: The noisy signal $y(t)$ received by the receiver.

(iii) The receiver wants to estimate a_1 and a_2. Find the best estimates $\tilde{a}_1$ and $\tilde{a}_2$ that minimize the squared error

$$\int_0^1 \left(\tilde{x}(t) - y(t)\right)^2 dt,$$

where $\tilde{x}(t) = \tilde{a}_1 x_1(t) + \tilde{a}_2 x_2(t)$. No justification is required for your answer.

(2 points)