1. **(Linear independence)** Let $v_1, \ldots, v_k \in \mathbb{R}^n$ be non-zero vectors orthogonal to each other. Show that $v_1, \ldots, v_k$ are linearly independent. **(5 points)**

Solution: Consider the linear combination

$$c_1 v_1 + \cdots + c_k v_k = 0.$$

By orthogonality, multiplying the equation above by $v_i^\top$ on the left yields

$$c_i \|v_i\|^2 = 0$$

for $i \in \{1, 2, \ldots, k\}$ whereby

$$c_i = 0.$$

2. **(Fundamental subspaces)** Given the matrix

$$A = \begin{bmatrix} 0 & 3 & -1 & 2 \\ 0 & -1 & 8 & 7 \end{bmatrix},$$

provide: (*no need to show calculations*)

(a) a basis for the **row space** of A **(3 points)**

$$\left\{ \begin{bmatrix} 0 & 3 & -1 & 2 \end{bmatrix}^\top, \begin{bmatrix} 0 & -1 & 8 & 7 \end{bmatrix}^\top \right\}$$

(b) a basis for the **null space** of A **(3 points)**

$$\left\{ \begin{bmatrix} 0 & 1 & 1 & -1 \end{bmatrix}^\top, \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}^\top \right\}$$

(c) a basis for the **column space** of A **(3 points)**

$$\left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 8 \end{bmatrix} \right\}$$

(d) rank $A = \underline{\ 2\ }$. **(1 point)**

3. **(Euclidean norm)** Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let $x_1 = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix}^\top$, $x_2 = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^\top$, and $x_3 = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}^\top$.

   i. Verify that x_1, x_2, and x_3 are all solutions to

$$Ax = b.$$

   Which among x_1, x_2, and x_3 has the smallest Euclidean norm? **(5 points)**

   Solution: It is easy to check that all three vectors satisfy $Ax = b$. We calculate the norms.

$$\begin{aligned} \|x_1\| &= \sqrt{1+1} = \sqrt{2} \\ \|x_2\| &= \sqrt{1+1} = \sqrt{2} \\ \|x_3\| &= \sqrt{1/4+1+1/4} = \sqrt{3/2} \end{aligned}$$

   Therefore, x_3 has the smallest Euclidean norm among x_1, x_2, and x_3.

   ii. Can you construct a solution to $Ax = b$, call it x_4, such that

$$\|x_4\| < \min\{\|x_1\|, \|x_2\|, \|x_3\|\}?$$

   (5 points)

   Solution: The solution to $Ax = b$ with the smallest Euclidean norm is given by

$$x^* = A^T(AA^T)^{-1}b = \begin{bmatrix} \frac{1}{2} & 1 & \frac{1}{2} \end{bmatrix}^T = x_3.$$

   Therefore, we cannot construct x_4 such that $\|x_4\| < \|x_3\|$.

   iii. Can you construct a solution to $Ax = b$, call it x_5, such that x_5 is different from x_1, x_2, and x_3, and

$$\|x_5\| = \min\{\|x_1\|, \|x_2\|, \|x_3\|\}?$$

   (5 points)

   Solution: The vector x satisfying $Ax = b$ and having the smallest Euclidean norm is unique. Therefore, we cannot construct x_5 different from x_1, x_2, and x_3 such that $\|x_5\| = \|x_3\|$.

4. **(Linear regression)** We are performing an experiment to calculate the gravitational constant g as follows. We drop a ball from a certain height and measure its distance from the original point at certain time instants. The results of the experiment are shown in the following table.

Time (seconds)	1.00	2.00	3.00
Distance (metres)	5.00	19.50	44.00

The equation relating the distance s and the time t at which s is measured is given by

$$s = \frac{1}{2} g t^2.$$

i. Find the least-squares estimate of g (in metre per second squared) using the experimental results from the table above. Give your answer with precision up to two decimal points. **(10 points)**

Hint: For ease of calculation,

(a) construct the matrix A with entries corresponding to t^2 for the values of t given in the table, and

(b) construct the vector b with entries corresponding to $2s$ for the values of s given in the table.

Solution: We represent the data using the equation $2s = gt^2$ as $Ax = b$ as follows:

$$\begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} [g] = \begin{bmatrix} 10 \\ 39 \\ 88 \end{bmatrix}$$

The least-squares estimate is given by

$$g^* = (A^\top A)^{-1} A^\top b$$

where

$$A^\top A = \begin{bmatrix} 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix} = 98$$

and

$$A^\top b = \begin{bmatrix} 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} 10 \\ 39 \\ 88 \end{bmatrix} = 958$$

whereby

$$g^* = \frac{1}{98} \cdot 958 \approx 9.78.$$

ii. Suppose that we take an additional measurement at time 4.00 seconds, and obtain a distance of 78.5 metres. Use the recursive least-squares algorithm to calculate an updated least-squares estimate of g. **(10 points)**

Solution: We have $A_1 = 16$,

$$G_0 = A_0^\top A_0 = 98$$

and

$$\begin{aligned} G_1 &= G_0 + A_1^\top A_1 \\ &= 98 + (16 \times 16) = 354. \end{aligned}$$

The solution is given by

$$\begin{aligned} x^{(1)} &= x^{(0)} + G_1^{-1} A_1^\top \left(b^{(1)} - A_1 x^{(0)}\right) \\ &= 9.78 + \frac{1}{354} \cdot 16 \, (157 - 16 \cdot 9.78) \\ &= 9.78 + \frac{16}{354} \cdot 0.52 \\ &= 9.78 + 0.024 \\ &= 9.80. \end{aligned}$$