CS2020A Discrete Mathematics

Aug-Dec 2025 | Final Test | 50 points

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1. Truth tables of two propositional formulae α and β on propositional variables p, q, r are given below

p	q	r	α	β
T	T	T	T	T
T	T	F	T	T
T	F	T	T	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	F
F	F	F	F	F

Write down a propositional formula for α and β

a) in Disjunctive Normal Form (directly from the truth table) (2 points)
b) in Conjunctive Normal Form (directly from the truth table) (2 points)
c) in a simplified form in which no variable repeats (2 points)

Ans.

DNFs (2 pt)

$$\alpha = (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r) \vee (\neg p \wedge q \wedge \neg r) \vee (\neg p \wedge \neg q \wedge r)$$
$$\beta = (p \wedge q \wedge r) \vee (p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (p \wedge \neg q \wedge \neg r) \vee (\neg p \wedge q \wedge r)$$

CNFs (2 pt)

$$\alpha = (\neg p \vee q \vee r) \wedge (p \vee q \vee r)$$
$$\beta = (p \vee \neg q \vee r) \wedge (p \vee q \vee \neg r) \wedge (p \vee q \vee r)$$

Simplified (2 pt)

$$\alpha = q \vee r$$
$$\beta = p \vee (q \wedge r)$$

2. Prove (rigorously) that for every natural number n,

$$\sum_{i=0}^{n} f_i = f_{n+2} - 1,$$

where $f_0, f_1, \ldots$ is the Fibonacci Sequence defined as $f_0 = 0, f_1 = 1$ and for every $k \geq 2$, $f_k = f_{k-1} + f_{k-2}$. (4 points)

Ans. Proof by induction on n:

- Base case is when $n = 0$. LHS is $\sum_{i=0}^{0} f_i = f_0 = 0$. RHS is $f_2 - 1 = 1 - 1 = 0$. Hence the base case is true. (1 pt)
- Let the claim be true for $n = k$. When $n = k + 1$,

$$\begin{aligned}
\sum_{i=0}^{k+1} f_i &= \sum_{i=0}^{k} f_i + f_{k+1} \\
&= f_{k+2} - 1 + f_{k+1} && \text{(Induction Hypo)} \\
&= f_{k+3} - 1 && (f_{k+2} + f_{k+1} = f_{k+3})
\end{aligned}$$

  Hence the claim is true for $n = k + 1$ when it is true for $n = k$. (2 pt)
- By principle of mathematical induction, the claim is true for all natural numbers n. (1 pt)

3. The *complement* of a simple undirected graph $G = (V, E)$ is the graph $\overline{G} = (V, \overline{E})$ where $\overline{E} = \binom{V}{2} \setminus E$. A graph G is called *self-complementary* if it is isomorphic to its complement.

   (a) When do we call two simple undirected graphs *isomorphic*? (1 point)

   (b) Give an example of a graph G_1 on more than one vertex which is self-complementary. Write down the isomorphism from G_1 to $\overline{G}_1$. (2 points)

   (c) Give another example of a graph G_2 on more than one vertex which is self-complementary. G_2 should be non-isomorphic to G_1. Write down the isomorphism from G_2 to $\overline{G}_2$. Also, justify why G_2 and G_1 are non-isomorphic. (3 points)

Ans:

(a) Two simple undirected graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there exists a bijection f from V_1 to V_2 such that (1 pt)

$$\{x, y\} \in E_1 \iff \{f(x), f(y)\} \in E_2$$

(b) Let G_1 be a path on 4 vertices, say a-b-c-d. Then, it's complement $\overline{G_1}$ is also a path b-d-a-c. (1 pt)
These two are isomorphic and the following is an isomorphism (1 pt)

$$f(x) = \begin{cases} b, & x = a \\ d, & x = b \\ a, & x = c \\ c, & x = d \end{cases}$$

(c) Let G_2 be a cycle on 5 vertices: a-b-c-d-e-a. It's complement $\overline{G_2}$ is also a cycle a-c-e-b-d-a. (1 pt)
These two are isomorphic and the following is an isomorphism (1 pt)

$$f(x) = \begin{cases} a, & x = a \\ c, & x = b \\ e, & x = c \\ b, & x = d \\ d, & x = e \end{cases}$$

G_1 and G_2 are not isomorphic since they have different number of vertices. (1 pt)

4. Maximum and Maximal Matching.

   (a) Define *matching*, *maximal matching* and *maximum matching*. (3 points)

   (b) Show that if L is a maximal matching and M is a maximum matching in a graph G, then $|L| \leq |M| \leq 2|L|$. (5 points $\star$)

   Ans.

   (a) A *matching* in an undirected graph is a collection of edges such that no two of them share a common endpoint. (1 pt)

   A *maximum matching* in an undirected graph G is a matching in G such that no other matching in G has a larger size. (1 pt)

   A *maximal matching* in an undirected graph G is a matching in G such that no proper superset of it is a matching in G. (1 pt)

   (b) Since M is a maximum matching, by definition, no other matching has a larger size and hence $|L| \leq |M|$. (1 pt)

   Next we show that $|M| \leq 2|L|$. (4 pt)

   - Consider $M \triangle L$, which is the subgraph of G consisting of edges which are either in M or L but not in both.
   - Since each vertex is incident to at most one edge each from M and L, $M \triangle L$ is a union of paths and even cycles in which the edges alternate between M and L.
   - The edges in even cycles are evenly distributed between M and L.
   - Since L is maximal, every edge e of M has at least one endpoint in common with an edge in L. Otherwise, L is not maximal.
   - Hence every path on $M \triangle L$ has at least one edge of L and hence the number of such paths is at most $|L|$.
   - Each such path can have at most one more edge from M than those from L.
   - Hence the excess number of edges in M (over $|L|$) is at most the number of paths in $M \triangle L$, which is at most L.
   - So $|M| \leq |L| + |L| = 2|L|$.

5. Given two (possibly infinite) sets A and B when do we say that

   (a) A is smaller of equal in size to B

   (b) A is equal in size to B

   (c) A is strictly smaller in size to B (3 points)

   Ans.

   (a) We say that A is smaller of equal in size to B if there exists an injective function (one-to-one function) from A to B. (1 pt)

   (b) We say that A is equal in size to B if there exists a bijective function (one-to-one onto function) from A to B. (1 pt)

   (c) We say that A is strictly smaller in size to B if there exists an injective function from A to B an there does not exist any bijective function from A to B. (1 pt)

6. Show that, for every (possibly uncountable) set A, A is strictly smaller in size to its power set $P(A)$. (4 points)

   Ans.

   $f : A \to P(A)$ defined as $f(x) = \{x\}$ is an example of an injective function from A to $P(A)$. (1 pt)

   We will now show that there is no bijective function from A to $P(A)$ by contradiction. (3 pt)

   - Suppose there exists a bijective function f from A to $P(A)$.
   - Consider the set $B = \{x \in A : x \notin f(x)\}$.
   - Since f is a bijection, there exists a member $b \in A$ such that $f(b) = B$.
   - If $b \in f(b)$, then $b \notin B$ by the definition of B, but then this is a contradiction since $f(b) = B$.
   - If $b \notin f(b)$, then $b \in B$ by the definition of B, but then this is also a contradiction since $f(b) = B$.
   - Since we get a contradiction irrespective of whether b is in $f(b)$ or not, our assumption (existence of a bijective function from A to $P(A)$) is wrong.

   Since there exists an injective function but no bijective function from A to $P(A)$, A is strictly smaller in size to $P(A)$.

7. A and B are two sets with at least two elements each. F_{AB} is the set of all functions from A to B and F_{BA} is the set of all functions from B to A. F_{AB} is countably infinite and F_{BA} is uncountable. What kind of sets are A and B (finite, countably infinite or uncountable)? Justify your answer. You can use without proof any of the results we have already proved in class and tutorials. (6 points $\star$)

   Ans.

   One example: (2 pt)

   - Let $A = \{0, 1\}$ and $B = \mathbb{N}$.
   - Then any function $f \in F_{AB}$ (that is a function from $\{0, 1\}$ to $\mathbb{N}$) can be seen as an ordered pair $(f(0), f(1))$ of natural numbers. This gives a bijection from $\mathbb{N} \times \mathbb{N}$ to F_{AB} and hence F_{AB} is countable.
   - Any function from B to A can be treated as a characteristic vector of a subset of $\mathbb{N}$ (the subset of $\mathbb{N}$ which is mapped to 1, for example). This gives a bijection from F_{BA} to the powerset of A. Since $A = \mathbb{N}$, F_{BA} is uncountable.
   - So we have an example of a finite set A and a countably infinite set B which satisfies the requirements in the quesiton.

   General case. (+2 pt)

   - Let A be a finite set (say $|A| = k$) and B be a countably infinite set.
   - Then F_{AB} is in bijection with B^k (Cartesian product of B with itself k times, which is countably infinite.
   - Let A' be a subset of A with two elements. F_{BA} contains $F_{BA'}$ which is already in bijection with $P(B)$ and hence uncountable.
   - So **any finite** A **and countably infinite** B satisfies the requirements in the question.

   Ruling out other cases (2 pt)

   - If A is infinite, pick any set $B' \subset B$ with two elements. $F_{AB} \supseteq F_{AB'}$ and $F_{AB'}$ has a bijection with the powerset of A which is already uncountable. Hence F_{AB} will be uncountable in this case, violating the first requirement.
   - If A and B are finite, then F_{AB} and F_{BA} are both finite and this violates the requirements in the question.
   - If A is finite and B is uncountable F_{AB} is in bijection with B^k (where $k = |A|$) and hence uncountable. This also violates the requirements.

8. Give formal set theoretic definitions (using sets and functions) of
   (a) Discrete probability space, sample space, probability mass function
   (b) Event , Probability function
   (c) Two mutually exclusive events and two independent events. (6 points)

   Ans.

   (a) A *discrete probability space* is a pair (Ω, p) where Ω is a countable set and p is a function from Ω to the closed interval $[0, 1]$ such that $\sum_{x \in \Omega} p(x) = 1$. (1 pt)

   Ω above is called the *sample space* and p is the *probability mass function.* (1 pt)

   (b) An *event* is any subset of Ω in a probability space (Ω, p) (1 pt)

   The *probability function* P in a probability space (Ω, p) is an extention of p to the powerset of Ω defined by $P(A) = \sum_{x \in A} p(x)$ for every $A \subseteq \Omega$. (1 pt)

   (c) Two events A and B are called *mutually exclusive* if $A \cap B = \emptyset$. (1 pt)

   Two events A and B are called *independent* if $P(A \cap B) = P(A)P(B)$ where P is the probability function. (1 pt)

9. Give an example of a probability space and a collection of events which are 3-wise independent but not mutually independent. Justify your answer.

   Defn. A collection of events is 3-wise independent if every subcollection of three or fewer events is independent. (4 points)

   Ans.

   - Consider the experiment of tossing a fair coin independently four times. $\Omega = \{H, T\}^4$ and p is the uniform distribution (each of the 16 possible outcomes have probability 1/16 each.
   - For $1 \leq i \neq j \leq 4$, let A_{ij} denote the event that the i-th and j-toss had the same result (both heads or both tails).
   - The events $A_{12}, A_{23}, A_{34}, A_{41}$ are 3-wise independent but not mutually indpendent. (1 pt)

   Proof (3 pt)

   - $P(A_{12}) = P(\{HH**, TT**\}) = 8/16 = 1/2$.
   - Similarly $P(A_{23}) = P(A_{34}) = P(A_{41}) = 1/2$.
   - $P(A_{12} \cap P_A 34) = P(\{HHHH, HHTT, TTHH, TTTT\}) = 4/16 = 1/4$
   - $P(A_{12} \cap P_A 23) = P(\{HHH*, TTT*\}) = 4/16 = 1/4$
   - Similarly all pairwise intersection have probability 1/4 which is the product of the marginals.
   - If any three of the four events happen, then all the four tosses have to agree and hence the probability of each of the four 3-wise intersections is $2/16 = 1/8$ which is also the product of the marginals.
   - But once all the tosses agree, all the four events happen and hence $P(A_{12} \cap A_{23} \cap A_{34} \cap A_{41}) = 1/8$ which is not equal to the product of the marginals.

10. Let K_5 be the complete graph on 5 vertices $\{a, b, c, d, e\}$. Each edge of this K_5 is colored either red or green based on the toss of a fair coin. That is a toss is done for each edge and the edge is colored red if the coin shows head and green if the coin shows tail. The tosses are mutually independent. For any three distinct vertices x, y, z, let E_{xyz} denote the event that the triangle formed by these three vertices is single-colored. (That is, all the three edges between x, y and z get the same color).

   a) What is the probabiity of E_{abc} (1 point)
   b) Are the events E_{abc} and E_{cde} independent? Justify. (1 point)
   c) Are the events E_{abc} and E_{bcd} independent? Justify (1 point)
   d) Are the three events $E_{abc}, E_{bcd}, E_{abd}$ mutually independent? Justify (2 points)
   e) Are the three events $E_{abc}, E_{bcd}, E_{cde}$ mutually independent? Justify (2 points)

   Note. The *complete graph* K_n is a simple undirected graph on n vertices in which every pair of vertices form an edge.

   Ans.

   (a) $P(E_{abc}) = 1/4$, since among the three relevant tosses (on the edges ab, bc and ca) the probability that they agree is $2/8 = 1/4$. (1 pt)
   (b) $P(E_{abc} \cap E_{cde}) = 1/16$, since among the six relevant tosses (on the edges ab, bc, ca, cd, de and ec) the probability that the first three agree and the last three agree is $1/4 \times 1/4 = 1/16$. Hence the events E_{abc} and E_{cde} are independent. (1 pt)
   (c) $P(E_{abc} \cap E_{bcd}) = 1/16$, since among the five relevant tosses (on the edges ab, bc, ca, cd, and db) the probability that all five agree is $2/32 = 1/16$. Since this also is equal to the product of the marginals, the events E_{abc} and E_{bcd} are independent. (1 pt)
   (d) $P(E_{abc} \cap E_{bcd} \cap E_{abd}) = 1/32$, since among the six relevant tosses (on the edges ab, bc, ca, cd db, and ad) the probability that all six agree is $2/64 = 1/32$. Since this also is not equal to the product of the marginals, these three events are not mutually independent. (2 pt)
   (e) $P(E_{abc} \cap E_{bcd} \cap E_{cde}) = 2/2^7$, since among the seven relevant tosses (on the edges ab, bc, ca, cd db, ce and ed) the probability that all seven agree is $2/2^7 = 1/64$. Since this also is equal to the product of the marginals, and we have verified the pairwise independence of triangles which share a vertex or edge already, these three events are mutually independent. (2 pt)