Mathematics for Data Science (DS5004)

Quiz-1: Solutions

TAs: Geddam Gowtham (142402006@smail.iitpkd.ac.in),
Suman Pal (142402012@smail.iitpkd.ac.in)
Mehta Family School of Data Science & Artificial Intelligence
IIT Palakkad

Question.1. If W_1, W_2 are subspaces of a vector space V, then

$$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2).$$

Solution: Let W_1, W_2 be subspaces of a vector space V and S be a basis of $W_1 \cap W_2$ (if $W_1 \cap W_2$ is the zero space then $S = \emptyset$). For each $i = 1, 2$, extend S to a basis B_i of W_i.
Let $S = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r\}$, $\quad B_1 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r, \mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_s\}$, $\quad B_2 = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_r, \mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_t\}$.
Then $\dim(W_1) = r + s$, $\dim(W_2) = r + t$. Let $B = \{\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{v}_1, \ldots, \mathbf{v}_s, \mathbf{w}_1, \ldots, \mathbf{w}_t\}$.
It is enough to show that B is a basis of $\mathbf{w}_1 + \mathbf{w}_2$ because then

$$\dim(W_1 + W_2) = r + s + t = (r + s) + (r + t) - r = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$$

To show that B is linearly independent, we need to show that

$$\sum_{i=1}^{r} a_i \mathbf{u}_i + \sum_{j=1}^{s} b_j \mathbf{v}_j + \sum_{k=1}^{t} c_k \mathbf{w}_k = \mathbf{0} \tag{1}$$

implies $a_i = b_j = c_k = 0$ for all i, j, k. Let (1) be re-expressed as:

$$\sum_{i=1}^{r} a_i \mathbf{u}_i + \sum_{j=1}^{s} b_j \mathbf{v}_j = -\sum_{k=1}^{t} c_k \mathbf{w}_k. \tag{2}$$

Let $\mathbf{v} \triangleq -\sum_{k=1}^{t} c_k \mathbf{w}_k$. Now the LHS of (2) is in W_1 and the RHS is in W_2. Thus, $\mathbf{v} \in W_1 \cap W_2$ and $\mathbf{v} = \sum_{i=1}^{r} d_i \mathbf{u}_i$ for some scalars $d_i \in \mathbb{R}$. Since $\mathbf{v} = -\sum_{k=1}^{t} c_k \mathbf{w}_k = \sum_{i=1}^{r} d_i \mathbf{u}_i$, it now follows that:

$$\sum_{i=1}^{r} d_i \mathbf{u}_i + \sum_{k=1}^{t} c_k \mathbf{w}_k = \mathbf{0},$$

which implies $d_i = 0$ and $c_k = 0$ for each i, k (since B_2 is linearly independent). We can now revise (1) as:

$$\sum_{i=1}^{r} a_i \mathbf{u}_i + \sum_{j=1}^{s} b_j \mathbf{v}_j = \mathbf{0},$$

which implies $a_i = b_j = 0$ for each i, j (since B_1 is linearly independent). Thus, B is linearly independent.

Showing that $W_1 + W_2 = \operatorname{span}(B)$ is rather straightforward; nevertheless, we include the argument here for completeness. Note that the vectors in B belong to $W_1 + W_2$. Since $W_1 + W_2$ is a vector space, any linear combination of its elements also belongs to it, and hence $\operatorname{span}(B) \subseteq W_1 + W_2$. To show that $W_1 + W_2 \subseteq \operatorname{span}(B)$, consider any vector $\mathbf{x} \in W_1 + W_2$. By definition of $W_1 + W_2$, we have $\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2$ for some $\mathbf{w}_1 \in W_1$ & $\mathbf{w}_2 \in W_2$. Let $\mathbf{w}_1 = \sum_{i=1}^{r} \alpha_i \mathbf{u}_i + \sum_{j=1}^{s} \beta_j \mathbf{v}_j$ and $\mathbf{w}_2 = \sum_{i=1}^{r} \gamma_i \mathbf{u}_i + \sum_{k=1}^{t} \eta_k \mathbf{w}_k$. Thus:

$$\mathbf{x} = \mathbf{w}_1 + \mathbf{w}_2 = \left(\sum_{i=1}^{r} \alpha_i \mathbf{u}_i + \sum_{j=1}^{s} \beta_j \mathbf{v}_j \right) + \left(\sum_{i=1}^{r} \gamma_i \mathbf{u}_i + \sum_{k=1}^{t} \eta_k \mathbf{w}_k \right)$$

$$\implies \mathbf{x} = \sum_{i=1}^{r} (\alpha_i + \gamma_i) \mathbf{u}_i + \sum_{j=1}^{s} \beta_j \mathbf{v}_j + \sum_{k=1}^{t} \eta_k \mathbf{w}_k$$

So, any vector of $W_1 + W_2$ can be expressed as a linear combination of vectors of B. Hence, B is a basis for $W_1 + W_2$ and this completes the proof.

Question 2. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$. Consider the linear transformation $\varphi(\mathbf{x}) = A\mathbf{x}$.

Obtain the transformation matrix with respect to the basis vectors $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$.

Solution. Given:

$$\mathcal{B} = \left\{ \mathbf{b}_1 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \mathbf{b}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{b}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

$$A\mathbf{b}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ 12 \end{pmatrix}$$

Express in $\mathcal{B}$:

$$\begin{pmatrix} 1 \\ 4 \\ 12 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This gives $c_1 = 1,\ c_2 = 2,\ c_3 = 8$.
So,

$$[A\mathbf{b}_1]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 2 \\ 8 \end{pmatrix}$$

$$A\mathbf{b}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

Express in $\mathcal{B}$:

$$\begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This gives $c_1 = 0,\ c_2 = 2,\ c_3 = 0$. So,

$$[A\mathbf{b}_2]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}$$

$$A\mathbf{b}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

Express in $\mathcal{B}$:

$$\begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

This gives $c_1 = 0,\ c_2 = 0,\ c_3 = 3$. So,

$$[A\mathbf{b}_3]_{\mathcal{B}} = \begin{pmatrix} 0 \\ 0 \\ 3 \end{pmatrix}$$

The transformation matrix with respect to the basis $\mathcal{B}$ is

$$[\varphi]_{\mathcal{B}} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 8 & 0 & 3 \end{pmatrix}.$$