1. A fair six-sided die is rolled independently and repeatedly. Recall that the outcome of a roll of a fair six-sided die is an element of $\{1, 2, 3, 4, 5, 6\}$, each occurring with probability $1/6$.

   (a) What is the probability that none of the first n rolls results in a 6? **(3 points)**
   Solution: Let X_i be the outcome of the i-th roll. Then,

   $$\mathbb{P}[X_i \neq 6] = \frac{5}{6}.$$

   Since the rolls are independent, the probability that all of the first n rolls are not 6 is

   $$\left(\frac{5}{6}\right)^n.$$

   (b) What is the probability that a 6 appears for the first time on the n-th roll? **(3 points)**
   Solution: $\left(\frac{5}{6}\right)^{n-1} \frac{1}{6}$

   (c) What is the expected number of rolls required to obtain the first 6? **(4 points)**
   Solution: This follows a geometric distribution with success probability

   $$p = \frac{1}{6}.$$

   Hence, the expected number of rolls is $\frac{1}{p} = 6$.

2. A fair six-sided die is rolled n times.

   (a) Let Y denote the number of times a 6 appears. Compute $\mathbb{E}Y^2$. **(2 points)**
   Hint: For $Z \sim \text{Ber}(p)$, we have $\text{Var}(Z) = p(1-p)$.

   Solution: First, we compute $\text{Var}(Y)$ by considering Y as a sum of Bernoulli random variables each with bias $p = 1/6$. Therefore,

   $$\text{Var}(Y) = np(1-p) = \frac{5n}{36}.$$

   Thus,

   $$\mathbb{E}Y^2 = \text{Var}(Y) + (\mathbb{E}Y)^2 = \frac{5n + n^2}{36}.$$

   (b) Show that the probability of obtaining more than $1/\sqrt{6}$ fraction of 6s is at most $1/\sqrt{6}$. **(3 points)**

   Solution: We have $\mathbb{E}Y = n/6$. Then, by Markov's inequality,

   $$\begin{aligned}\mathbb{P}\left[Y \geq \frac{n}{\sqrt{6}}\right] &\leq \frac{\mathbb{E}Y}{n/\sqrt{6}} \\ &= \frac{n/6}{n/\sqrt{6}} \\ &= \frac{1}{\sqrt{6}}.\end{aligned}$$

(c) Show that the probability of obtaining more than $1/7$ fraction of 6s is at least $1/49$ as $n \to \infty$. **(5 points)**

Hint: Use the Paley-Zygmund inequality: for $Y \geq 0$ and $0 < \alpha < 1$,

$$\mathbb{P}[Y \geq \alpha \mathbb{E} Y] \geq (1-\alpha)^2 \frac{(\mathbb{E}Y)^2}{\mathbb{E}Y^2}.$$

Solution: We have $\mathbb{E}Y = n/6$ and $\mathbb{E}Y^2 = \frac{n^2+5n}{36}$.

$$\begin{aligned} \mathbb{P}\left[Y \geq \frac{n}{7}\right] &= \mathbb{P}\left[Y \geq \frac{6}{7}\mathbb{E}Y\right] \\ &\geq \left(\frac{1}{7}\right)^2 \frac{(n/6)^2}{\frac{n^2}{36} + \frac{5n}{36}} \\ &= \frac{1}{49} \cdot \frac{n}{n+5} \end{aligned}$$

which goes to $1/49$ as $n \to \infty$.

3. Let $X_1, \ldots, X_n$ be independent random variables with

$$\mathbb{P}[X_i = 1] = p_i = 1 - \mathbb{P}[X_i = -1]$$

where $p_i \in (0,1), i = 1, \ldots, n$. Show that, for any $t \geq 0$, we have

$$\mathbb{P}\left[\sum_{i=1}^n (X_i - \mathbb{E}X_i) \geq t\right] \leq \exp\left\{-\frac{t^2}{2n}\right\}.$$

(10 points)

Hint: Use Chernoff bound. You may assume the inequality: for all $\lambda > 0$ and $0 < p < 1$,

$$pe^{2(1-p)\lambda} + (1-p)e^{-2p\lambda} \leq e^{\lambda^2/2}.$$

Solution: We have

$$\mathbb{E}X_i = p_i - (1-p_i) = 2p_i - 1$$

for $i \in [n]$. The Chernoff bound is given by

$$\mathbb{P}\left[\sum_{i=1}^n (X_i - \mathbb{E}X_i) \geq t\right] \leq e^{-\lambda t} \prod_{i=1}^n M_{(X_i - \mathbb{E}X_i)}(\lambda)$$

where

$$\begin{aligned} M_{(X_i - \mathbb{E}X_i)}(\lambda) &= \mathbb{E}\left[e^{\lambda(X_i - (2p_i - 1))}\right] \\ &= p_i e^{2(1-p_i)\lambda} + (1-p_i)e^{-2p_i\lambda} \\ &\leq e^{\lambda^2/2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbb{P}\left[\sum_{i=1}^n (X_i - \mathbb{E}X_i) \geq t\right] &\leq \min_{\lambda > 0} e^{-\lambda t} e^{n\lambda^2/2} \\ &= \exp\left\{-\frac{t^2}{2n}\right\}. \end{aligned}$$