Modeling Revenue Projection Via *M/M/1/1 Queuing Process* for Unstructured Business

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The utilization of machine learning models spans a diverse spectrum of applications, ranging from robotic systems to predicting weather patterns, wheat productions and detecting credit card fraudulent. These ML models exhibit potential for analyzing and potentially optimizing intricate systems, exemplified by their capacity to comprehend and navigate the complexities of chaotic systems such as Lorenz 1963 given appropriate informational inputs and adequate data. Nevertheless, it is crucial to acknowledge the inherent challenges associated with employing machine learning methodologies in scenarios where foundational knowledge about the system is limited, and where data availability for pre-training purposes is insufficient.

To illustrate, consider a family-owned small lawn care business endeavoring to forecast its requisite workforce to avoid both understaffed, which may lead to revenue loss, or overstaffing which leads to increased labor cost. This predicament is compounded by the stochastic nature of contract acquisition, rendering the number of contracts a random variable independent and identically distributed. Suppose the owner is interested in knowing the average revenue he could generate with X number of people with $\boldsymbol{\mu}$ completion rate and the potential revenue loss at the same time. At the same time, he wants to know how many of the contracts would need more than some days to complete. Since its summer everyone wants their work done as fast as possible, however he can only work in one contract at a time where each contract is processes sequentially, adhering to a first-in first-out (FIFO) approach. The next contract can get started if and only if current one has been completed or must be relinquished.

Potential contracts come according to the Poisson process with rate $\boldsymbol{\lambda}$ which has stationary independent increment[1] and time to complete each project are independently exponentially distributed with rate $\boldsymbol{\mu}$.

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[1] $X(t)$ is the process at time t, then any times $t_1 < t_2 < \cdots < t_n$ and the random variable $T_2 - T_1, T_3 - T_2, \dots, T_n - T_{n-1}$ are independent and distribution of the increments of the process remain same.

**ξ_n is the time between two contracts. It is the time taken to complete each job.*

Let, the revenue for completing and loss for rejecting the contrasts are iid with mean θ and η respectively. $X(t)$ be the state of the company at time t with transition rates $q_{10}= \boldsymbol{\mu}$ and $q_{01} = \boldsymbol{\lambda}$

i.e. $X(t) = \begin{cases} 1, & inservice \\ 0, & otherwise \end{cases}$

The stationary distribution of this two-state[2] Continuous Time Markov Chain[3] is.

$\mathbb{p}_0 = \frac{\mu}{\lambda+\mu}$, i.e. probability of being in state 0

$\mathbb{P}_1 = 1 - \mathbb{p}_0 = \frac{\lambda}{\lambda+\mu}$ i.e. probability of being Inservice.

From renewal point process we know,

$$N(t) = \sum_{n=1}^{\infty} 1(T_n \leq t\,; t \geq 0)$$

$N(t)$ is the number of completed contract at $T_n \in (0, t]$. Then total revenue at time T_n is.

$$Z(t) = \sum_{n=1}^{N(t)} Y_n \;; t \geq 0$$

Where, Y_n are iid contract revenue independent of the $X(t)$ with mean θ.

Average revenue from the contract's completion is $\lim_{t\to\infty} t^{-1} \sum_{n=0}^{N(t)} Z(t) = \theta \mathbb{p}_0$

Likewise operating under the assumption that it is a small business where only two people carry out the work and when they are engaged in a current contract, it is unable to accept new contracts. The revenue lost in such instances can be modeled using a Compound Poisson Process[4] $M(t)$ with rate λ and the distribution of a single lost project with mean η. The total lost revenue at time $T_n \in (0, t]$.

$Z'(t) \;= M_n(\xi_n)1(X_n = 1)$

where,

[5]$M_n = \sum_{n=1}^{\infty} \delta_{(T_n, Y_n)})$

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[2] *When the chain is ergodic its two-state stationary distribution is given by* $\pi_1 = \frac{b}{(a+b)}$ *and* $\pi_2 = \frac{a}{(a+b)}$

[3] *A Markov process* $\{X(t): t \geq 0\}$ *on a countable set S is a Continuous Time Markov chain if its sample paths are right continuous and piecewise constant with finite lengths. i,e* $X(t) = X_n \; if \; t \in [T_n, T_{n+1}) \; for \; some \; n$

[4] *If a Poisson Process has a real-valued marks at its points, then the cumulative value if the mark in time is a Compound Poisson Process.*

[5] M_n *are i.i.d compound Poisson Processes with* $M_n \stackrel{\text{def}}{=} M$ *are independent of the* (X_n, ξ_n)

And, the average revenue lost due to understaffed is $\lim_{t\to\infty} t^{-1} \sum_{n=0}^{N(t)} Z'(t) = \frac{1}{\lambda\mu} \eta \mathbb{p}_1$

Let, the cost per unit time of having a rate μ is H (μ). Then the total average revenue would be

$$\bar{Z}(t) = \theta \mathbb{p}_0 - \frac{1}{\lambda\mu} \eta \mathbb{p}_1 - \mathrm{H}\,(\mu).$$

We can use the preceding information as optimization to maximize average revenue by varying μ. The service rate μ can be varied by changing the number of workers, hour worked and so on.

Understanding the distribution of the completion time, specifically how many contracts lasts longer than a day is also crucial to make an informed decision and to optimize operations. If a significant number of contracts extend beyond someday (say average day), it may necessitate adjustments in staffing levels, equipment's and so on. The average number of projects that take longer than D days to complete is.

$$\lim_{t\to\infty} t^{-1} \sum_{n=0}^{N(t)} 1(X_n = 1, \xi_n > D) = \mathbb{p}_1 e^{-D\mu}$$

Now we will use above solution to run the simulation aimed to determining the overall average revenue. The provided pseudocode specifically illustrates the calculation of total average revenue change resulting from variations in lambda. However, it's worth noting that we have the flexibility to adjust the parameter μ as well.

```
Define parameters for total_average_revuenue:
Average_revenue = []
lambdas = any arbiter value
for lambda in lambdas:
        Revenue for each contract = range (min_value, max_value)
        Project_completion_time ():
                mu_rate = generate_random_exponential_rate ()
                random no = generate_uniform_random_no()
                mu = -ln(1-random_no)/ mu_rate
           return mu.
        Transition rate_p0= mu / lambda – mu
        Transition rate_p1 = 1 - Transition rate_p0
        revenue from contract = select randomly from Revenue for each contract
        revenue loss from contract = select randomly from Revenue for each contract
        total_average_revenue = revenue from contract* Transition rate_p0 – (1/
                                Lambda*mu) * revenue loss from contract* Transition
                                rate_p1- H (μ).
append total_average_revenue to Average_revenue.
```

<u>Define parameters for average no. of projects that take longer than some D days:</u>

```
Lambda = any arbiter value
Project_completion_time ():
        mu_rate = generate_random_exponential_rate()
        random no = generate_uniform_random_no ()
        mu = -ln(1-random_no)/ mu_rate
  return mu
Transition rate_p0= mu / lambda – mu
Transition rate_p1 = 1 - Transition rate_p0
Days = no of days of interest
Average_no_contract_longerthen_d_days = Transition rate_p1 * exp * (- D*mu)
```

Bibliography:

Serfozo, Richard. *Basics of Applied Stochastic Processes*. Springer Berlin Heidelberg, 2009.