The Basel Problem

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Abstract

In this paper We will solve the basel problem with a new elementary proof

1 Introduction

The Basel problem is a problem in mathematical analysis with relevance to number theory, concerning an infinite sum of inverse squares. It was first posed by Pietro Mengoli in 1650 and solved by Leonhard Euler in 1734,[1] and read on 5 December 1735 in The Saint Petersburg Academy of Sciences. Since the problem had withstood the attacks of the leading mathematicians of the day, Euler's solution brought him immediate fame when he was twenty-eight. Euler generalised the problem considerably, and his ideas were taken up years later by Bernhard Riemann in his seminal 1859 paper "On the Number of Primes Less Than a Given Magnitude", in which he defined his zeta function and proved its basic properties. The problem is named after Basel, hometown of Euler as well as of the Bernoulli family who unsuccessfully attacked the problem.
The Basel problem asks for the precise a summation that

equal to

$$\zeta(2) = \sum_{n=0}^{\infty} \frac{1}{n^2}. \tag{1}$$

Lemma 2.1.

$$\zeta(2) = \sum_{n=0}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{2}$$

Proof.

Firstly We will define tow simple integrals and a simple geometric sum.

$$\forall n \in \mathbb{N} : I = \int_0^{\pi} xcos(nx) \tag{3}$$

and

$$J = \int_0^{\pi} x^2 cos(nx). \tag{4}$$

using integration by part technique We will get that :

$$I = \frac{cos(\pi n) - 1}{n^2}; J = \frac{2\pi cos(n\pi)}{n^2} \tag{5}$$

We will find a pair $(a, b) \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$: $aI + bJ = \frac{1}{n^2}$

$$\Rightarrow \frac{acos(\pi n) - a + 2b\pi cos(n\pi)}{n^2} = \frac{1}{n^2} \Rightarrow acos(\pi n) - a + 2b\pi cos(n\pi) = 1 \tag{6}$$

$$\Rightarrow cos(n\pi)(a+2b\pi)-(a+1)=0 \tag{7}$$

$$\Leftrightarrow a=-1 \wedge a+2b\pi=0 \tag{8}$$

$$\Leftrightarrow a=-1 \wedge b=\frac{1}{2\pi} \tag{9}$$

$$therefore(a,b)=(-1,\frac{1}{2\pi}) \tag{10}$$

$$\Rightarrow \forall n \in \mathbb{N} : -I+\frac{1}{2\pi}J=\frac{1}{n^2} \tag{11}$$

$\Rightarrow\zeta(2)=\sum_{n=1}^{\infty}-I+\frac{1}{2\pi}J(12)$

$$\Rightarrow \zeta(2)=\lim_{n\to\infty}\sum_{k=1}^{n}\int_0^{\pi}-xcos(kx)+\frac{x^2cos(kx)}{2\pi}dx \tag{13}$$

$$\Rightarrow \zeta(2)=\lim_{n\to\infty}\int_0^{\pi}\sum_{k=1}^{n}cos(kx)(-x+\frac{x^2}{2\pi})dx \tag{14}$$

Using complex analysis we could obviously prove that :

$$\sum_{k=1}^{n} cos(kx) = \frac{-1}{2} + \frac{sin(\frac{2n+1}{2}x)}{2sin(\frac{x}{2})} \tag{15}$$

$$\Rightarrow \zeta(2) = \lim_{n\to\infty} (-\frac{1}{2}\int_0^{\pi} \frac{x^2}{2\pi} - x) + \int_0^{\pi} \frac{\frac{x^2}{2\pi} - x}{2sin(\frac{x}{2})} sin(\frac{2n+1}{2}x)dx \tag{16}$$

and we have that

$$\lim_{n\to\infty} \frac{\frac{2n+1}{2}}{n} = 1 \tag{17}$$

so using asymptotic expansion

$$\frac{2n+1}{2} \sim n \tag{18}$$

when

$$n \to \infty. \tag{19}$$

and we have that integral

$$(-\frac{1}{2}\int_0^{\pi} \frac{x^2}{2\pi} - x)dx = \frac{-1}{2}(\frac{x^3}{6\pi} - \frac{x^2}{2}\Big|_0^{\pi}) = \frac{-1}{2}(\frac{\pi^3}{6\pi} - \frac{\pi^2}{2}) = \frac{-1}{2}(\frac{\pi^2}{6} - \frac{3\pi^2}{6}) = \frac{\pi^2}{6} \tag{20}$$

$$\Rightarrow \zeta(2) = \lim_{n\to\infty} \frac{\pi^2}{6} + \int_0^{\pi} sin(nx).f(x)dx \tag{21}$$

such that :

$$f(x) = \frac{\frac{x^2}{2\pi} - x}{2} sin(\frac{x}{2}) \tag{22}$$

obviously that f(x) is continuos in the interval$[0, \pi]$

so using Riemann–Lebesgue lemma :

$$\lim_{n \to \infty} \int_0^{\pi} sin(nx).f(x)dx = 0 \tag{23}$$

$$\Rightarrow \zeta(2) = \lim_{n \to \infty} \frac{\pi^2}{6} + \int_0^{\pi} sin(nx)f(x)dx = \frac{\pi^2}{6} \tag{24}$$

2 conclusion

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{25}$$

in conclusion: I wish to say that this work was completed with the intellectual assistance of my friend Bilal Jaouad.