Algebraic Number theory : Fermat last theorem

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October 2022

1 1-Introduction :

In number theory, Fermat's Last Theorem (sometimes called Fermat's conjecture, especially in older texts) states that no three positive integers a, b, and c satisfy the equation

$$a^n + b^n = c^n \tag{1}$$

for any integer value of n greater than 2. The cases n = 1 and n = 2 have been known since antiquity to have infinitely many solutions , The proposition was first stated as a theorem by Pierre de Fermat around 1637 in the margin of a copy of Arithmetica. After 358 years of effort by mathematicians, the first successful proof was released in 1994 by Andrew Wiles and formally published in 1995(using Taniyama Shimura conjecture).

2 2-abc conjecture and applications :

Theorem :(Fermat Last Theorem) No three positive integers a, b, and c satisfy the equation

$$a^n + b^n = c^n \tag{2}$$

Before proving it We have to talk about abc conjecture

Conjecture : (abc conjecture) For all triples (a, b, c) of coprime positive integers with a + b = c, rad(abc) is at least c$^{1-o(1)}$

An equivalent formulation is: For every positive real number ϵ, there exists a constant $K\epsilon$ such that for all triples (a, b, c) of coprime positive integers, with a + b = c

We have that

$$: c < k\epsilon(rad(abc))^{1+\epsilon} \tag{3}$$

The conjecture was proved (not officially) by **Shinichi Mochizuki** he proved it using a new theory he called **inter-universal Teichmüller theory (IUTT)**.

3 3-abc application to fermat last theorem

let x,y,z three integers such that

$$x^n + y^n = z^n \tag{4}$$

(obviously that x,y,z are co-primes because we could subtract the gcd(x,y,z))
let

$$a = x^n, b = y^n c = z^n \tag{5}$$

so a+b=c and a,b,c are coprimes .
so for every

$$\epsilon > 0 : c < k\epsilon(rad(abc)^{1+\epsilon}) \tag{6}$$

$\Rightarrow c < \lim_{\epsilon \to 0} k\epsilon.rad(abc)^{1+\epsilon}$
(7)
therefore

$$c < k_{0+} rad(abc) \Rightarrow z^n < (k_{0+} rad((xyz)^n) = k_{0+} rad((xyz)) < k_{0+}$$
(8)

(because

$$rad(h^n) = rad(h) \tag{9}$$

and

$$rad(h) < h \tag{10}$$

for every integer h)
then

$$z^n < k_{0+}xyz \Rightarrow z^n < k_{0+}z^3 \qquad (11)$$

(because $x < z$ and $y < z$)
therefore

$$\mathbf{n} < \log_z(k_{0+}) + 3(12)$$

4-Using Professor S-Mochizuki result

in 2012 Professor **Shinichi Mochizuki** in kyoto university proved that for any three integers a,b,c such that a coprime with b and a+b=c we have

$$c < rad(abc)^2 \qquad (13)$$

$$(\forall n \in \mathbb{Z}) : rad(n) \stackrel{def}{=} \prod_{p|n} p \qquad (14)$$

(such that p is a prime)
let x , y , z , n three positive integers > 0 such that n>2 : and

$$x^n + y^n = z^n \qquad (15)$$

we could suppose that x,y are coprimes because it is enough ,
okay let's suppose that x,y are not coprimes so
$\exists p \in \mathbb{P}$:

p —x $\wedge p|y \rightarrow \exists(q,q') \in \mathbb{N}^{*2}$:
$x = p.q \wedge y = p.q'$
$\rightarrow x^n + y^n = p^n(q^n + q'^n) = z^n \rightarrow p|z^n \rightarrow p|z \rightarrow \exists d \in \mathbb{N}$:
$z = p.d$

$$\rightarrow z^n = p^n d^n \rightarrow p^n(q^n + q'^n) = p^n d^n \rightarrow q^n + q'^n = d^n \tag{16}$$

and so one until we subtract all coprimes
so after **mochizuki** we have
$z^n < rad((xyz)^n)^2 = rad(xyz)^2 \rightarrow z^n < rad(z^3)^2$
$\rightarrow z^n \leq z^6$
$\rightarrow n \leq 6 \rightarrow n \in \{3,4,5,6\}$
-The case n = 3 was proven by Euler in 1770.
-The case n = 5 was proved by Dirichlet and Legendre around 1825.
-Alternative proofs of the case n = 4 were developed later by Frénicle de Bessy,
-Proofs for n = 6 have been published by Kausler,

4 conclusion

as we see in this paper abc is a powerful conjecture .