Zeta function : Euler Product Applications

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Abstract

In this paper , We will prove the euler product formula and we will see some of it's applications in prime number theorem

1-Zeta function relation with prime numbers :

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p^s}} \tag{1}$$

-Proof :

We have that:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} (= 1 + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \frac{1}{5^s} + ...)$$

let:

$$p \in \mathbb{P} :$$

$$\frac{1}{p^s} . \zeta(s) = \sum_{n=1}^{\infty} \frac{1}{(np)^s} = \sum_{k \in M_p} \frac{1}{k^s}$$

such that:

$$M_p := \{n \in \mathbb{N} : p | n\} = p\mathbb{Z} \cap \mathbb{N}$$

therefore

$$\zeta(s) - \frac{1}{p^s}\zeta(s) = \sum_{n \in \mathbb{N}^* : n \notin M_p} \frac{1}{n^s} = \zeta(s)(1 - \frac{1}{p^s})$$

So using this operation we subtract all terms of multiples of p for example let's take that p = 3 then :

$$(1 - \frac{1}{3^s})\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{4^s} + \frac{1}{5^s} + ..$$

so let's subtract also the multiples of 2 :

$$\rightarrow (1 - \frac{1}{2^s})(1 - \frac{1}{3^s})\zeta(s) = \sum_{n \in \mathbb{N}^* : n \notin M_2 \cup M_3} \frac{1}{n^s}$$

Repeating for the next terms ... : until we get :

$$(\prod_{p \in \mathbb{P}} 1 - \frac{1}{p^s})\zeta(s) = \sum_{n \in \mathbb{N}^* : n \notin \bigcup_{i \in \mathbb{N}^*} M_i} \frac{1}{n^s} \qquad (2)$$

therefore :

$$(\prod_{p \in \mathbb{P}} 1 - \frac{1}{p^s})\zeta(s) = \sum_{n \in \mathbb{N}^* : n \in \bigcap_{i \in \mathbb{N}^*} \bar{M}_i} \frac{1}{n^s} \qquad (3)$$

and we know that :

$$\bigcap_{i \in \mathbb{N}^*} \bar{M}_i = \bigcap_{i \in \mathbb{N}^*} \{n \in \mathbb{N}^* | n \not\equiv 0[i]\} = 1 \qquad (4)$$

because :

$$(\forall n \in \mathbb{N}^* - \{1\})(\exists p \in \mathbb{P}) : p|n$$

$$\therefore (\prod_{p \in \mathbb{P}} 1 - \frac{1}{p^s})\zeta(s) = 1 \Rightarrow \zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p^s}} \quad (5)$$

2-Proof of infinitely many primes using Zeta Euler product

as we proved before that

$$\zeta(s) = \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p^s}} \Rightarrow \zeta(1) = H_\infty = \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p}} = \infty$$

such that

$$(\forall n \in \mathbb{N}^*) : H_n := \sum_{k=1}^{n} \frac{1}{n}$$

(the harmonic sequence)

$$\therefore \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p}} = \infty$$

Suppose there exist a finite number of primes : We know that

$$\forall p \in \mathbb{P} : p > 1$$

so the product:

$$\prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p}}$$

is a finite product because:

$$\forall p \in \mathbb{P} : p > 1 \Rightarrow 0 < 1 - \frac{1}{p} < 1$$

and while that the product :

$$\prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p}}$$

is finite and

$$1 - \frac{1}{p} \nsim 0 \Rightarrow \prod_{p \in \mathbb{P}} \frac{1}{1 - \frac{1}{p}} \nsim \infty$$

Absurd because the Harmonic series is a divergent

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