CHAPTER 26
Exotic Options

Practice Questions

Problem 26.1.
Explain the difference between a forward start option and a chooser option.

A forward start option is an option that is paid for now but will start at some time in the future. The strike price is usually equal to the price of the asset at the time the option starts. A chooser option is an option where, at some time in the future, the holder chooses whether the option is a call or a put.

Problem 26.2.
Describe the payoff from a portfolio consisting of a floating lookback call and a floating lookback put with the same maturity.

A floating lookback call provides a payoff of $S_T - S_{\min}$. A floating lookback put provides a payoff of $S_{\max} - S_T$. A combination of a floating lookback call and a floating lookback put therefore provides a payoff of $S_{\max} - S_{\min}$.

Problem 26.3.
Consider a chooser option where the holder has the right to choose between a European call and a European put at any time during a two-year period. The maturity dates and strike prices for the calls and puts are the same regardless of when the choice is made. Is it ever optimal to make the choice before the end of the two-year period? Explain your answer.

No, it is never optimal to choose early. The resulting cash flows are the same regardless of when the choice is made. There is no point in the holder making a commitment earlier than necessary. This argument applies when the holder chooses between two American options providing the options cannot be exercised before the 2-year point. If the early exercise period starts as soon as the choice is made, the argument does not hold. For example, if the stock price fell to almost nothing in the first six months, the holder would choose a put option at this time and exercise it immediately.

Problem 26.4.
Suppose that c_1 and p_1 are the prices of a European average price call and a European average price put with strike price K and maturity T, c_2 and p_2 are the prices of a European average strike call and European average strike put with maturity T, and c_3 and p_3 are the prices of a regular European call and a regular European put with strike price K and maturity T. Show that

$$c_1 + c_2 - c_3 = p_1 + p_2 - p_3$$

The payoffs from c_1, c_2, c_3, p_1, p_2, p_3 are, respectively, as follows:

$$\max(\bar{S} - K, 0)$$

$$\max(S_T - \bar{S}, 0)$$
$$\max(S_T - K, 0)$$
$$\max(K - \bar{S}, 0)$$
$$\max(\bar{S} - S_T, 0)$$
$$\max(K - S_T, 0)$$

The payoff from $c_1 - p_1$ is always $\bar{S} - K$; The payoff from $c_2 - p_2$ is always $S_T - \bar{S}$; The payoff from $c_3 - p_3$ is always $S_T - K$; It follows that

$$c_1 - p_1 + c_2 - p_2 = c_3 - p_3$$

or

$$c_1 + c_2 - c_3 = p_1 + p_2 - p_3$$

Problem 26.5.

The text derives a decomposition of a particular type of chooser option into a call maturing at time T_2 and a put maturing at time T_1. Derive an alternative decomposition into a call maturing at time T_1 and a put maturing at time T_2.

Substituting for c, put-call parity gives

$$\max(c, p) = \max\left[p, p + S_1 e^{-q(T_2-T_1)} - Ke^{-r(T_2-T_1)}\right]$$

$$= p + \max\left[0, S_1 e^{-q(T_2-T_1)} - Ke^{-r(T_2-T_1)}\right]$$

This shows that the chooser option can be decomposed into

1. A put option with strike price K and maturity T_2; and
2. $e^{-q(T_2-T_1)}$ call options with strike price $Ke^{-(r-q)(T_2-T_1)}$ and maturity T_1.

Problem 26.6.

Section 26.9 gives two formulas for a down-and-out call. The first applies to the situation where the barrier, H, is less than or equal to the strike price, K. The second applies to the situation where $H \geq K$. Show that the two formulas are the same when $H = K$.

Consider the formula for c_{do} when $H \geq K$

$$c_{\text{do}} = S_0 N(x_1)e^{-qT} - Ke^{-rT} N(x_1 - \sigma\sqrt{T}) - S_0 e^{-qT}(H/S_0)^{2\lambda} N(y_1)$$

$$+Ke^{-rT}(H/S_0)^{2\lambda-2} N(y_1 - \sigma\sqrt{T})$$

Substituting $H = K$ and noting that

$$\lambda = \frac{r - q + \sigma^2/2}{\sigma^2}$$

we obtain $x_1 = d_1$ so that

$$c_{\text{do}} = c - S_0 e^{-qT}(H/S_0)^{2\lambda} N(y_1) + Ke^{-rT}(H/S_0)^{2\lambda-2} N(y_1 - \sigma\sqrt{T})$$

The formula for c_{di} when $H \leq K$ is

$$c_{\text{di}} = S_0 e^{-qT}(H/S_0)^{2\lambda} N(y) - Ke^{-rT}(H/S_0)^{2\lambda-2} N(y - \sigma\sqrt{T})$$

Since $c_{\text{do}} = c - c_{\text{di}}$

$$c_{\text{do}} = c - S_0 e^{-qT}(H/S_0)^{2\lambda} N(y) + Ke^{-rT}(H/S_0)^{2\lambda-2} N(y - \sigma\sqrt{T})$$

From the formulas in the text $y_1 = y$ when $H = K$. The two expression for c_{do} are therefore equivalent when $H = K$

Problem 26.7.

Explain why a down-and-out put is worth zero when the barrier is greater than the strike price.

The option is in the money only when the asset price is less than the strike price. However, in these circumstances the barrier has been hit and the option has ceased to exist.

Problem 26.8.

Suppose that the strike price of an American call option on a non-dividend-paying stock grows at rate g. Show that if g is less than the risk-free rate, r, it is never optimal to exercise the call early.

The argument is similar to that given in Chapter 11 for a regular option on a non-dividend-paying stock. Consider a portfolio consisting of the option and cash equal to the present value of the terminal strike price. The initial cash position is

$$Ke^{gT-rT}$$

By time τ ($0 \le \tau \le T$), the cash grows to

$$Ke^{-r(T-\tau)+gT} = Ke^{g\tau}e^{-(r-g)(T-\tau)}$$

Since $r > g$, this is less than $Ke^{g\tau}$ and therefore is less than the amount required to exercise the option. It follows that, if the option is exercised early, the terminal value of the portfolio is less than S_T. At time T the cash balance is Ke^{gT}. This is exactly what is required to exercise the option. If the early exercise decision is delayed until time T, the terminal value of the portfolio is therefore

$$\max[S_T, Ke^{gT}]$$

This is at least as great as S_T. It follows that early exercise cannot be optimal.

Problem 26.9.

How can the value of a forward start put option on a non-dividend-paying stock be calculated if it is agreed that the strike price will be 10% greater than the stock price at the time the option starts?

When the strike price of an option on a non-dividend-paying stock is defined as 10% greater that the stock price, the value of the option is proportional to the stock price. The same argument as that given in the text for forward start options shows that if t_1 is the time when the option starts and t_2 is the time when it finishes, the option has the same value as an option starting today with a life of $t_2 - t_1$ and a strike price of 1.1 times the current stock price.

Problem 26.10.

If a stock price follows geometric Brownian motion, what process does $A(t)$ follow where $A(t)$ is the arithmetic average stock price between time zero and time t?

Assume that we start calculating averages from time zero. The relationship between $A(t+\Delta t)$ and $A(t)$ is

$$A(t+\Delta t)\times(t+\Delta t)=A(t)\times t+S(t)\times\Delta t$$

where $S(t)$ is the stock price at time t and terms of higher order than Δt are ignored. If we continue to ignore terms of higher order than Δt, it follows that

$$A(t+\Delta t)=A(t)\left[1-\frac{\Delta t}{t}\right]+S(t)\frac{\Delta t}{t}$$

Taking limits as Δt tends to zero

$$dA(t)=\frac{S(t)-A(t)}{t}dt$$

The process for $A(t)$ has a stochastic drift and no dz term. The process makes sense intuitively. Once some time has passed, the change in S in the next small portion of time has only a second order effect on the average. If S equals A the average has no drift; if $S>A$ the average is drifting up; if $S<A$ the average is drifting down.

Problem 26.11.
Explain why delta hedging is easier for Asian options than for regular options.

In an Asian option the payoff becomes more certain as time passes and the delta always approaches zero as the maturity date is approached. This makes delta hedging easy. Barrier options cause problems for delta hedgers when the asset price is close to the barrier because delta is discontinuous.

Problem 26.12.
Calculate the price of a one-year European option to give up 100 ounces of silver in exchange for one ounce of gold. The current prices of gold and silver are $1520 and $16, respectively; the risk-free interest rate is 10% per annum; the volatility of each commodity price is 20%; and the correlation between the two prices is 0.7. Ignore storage costs.

The value of the option is given by the formula in the text

$$V_0e^{-q_2T}N(d_1)-U_0e^{-q_1T}N(d_2)$$

where

$$d_1=\frac{\ln(V_0/U_0)+(q_1-q_2+\sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2=d_1-\sigma\sqrt{T}$$

and

$$\sigma=\sqrt{\sigma_1^2+\sigma_2^2-2\rho\sigma_1\sigma_2}$$

In this case, $V_0=1{,}520$, $U_0=1600$, $q_1=0$, $q_2=0$, $T=1$, and

$$\sigma=\sqrt{0.2^2+0.2^2-2\times0.7\times0.2\times0.2}=0.1549$$

Because $d_1=-0.2537$ and $d_2=-0.4086$, the option price is

$$1520N(-0.2537)-1600N(-0.4086)=61.52$$

or $61.52.

Problem 26.13.

Is a European down-and-out option on an asset worth the same as a European down-and-out option on the asset's futures price for a futures contract maturing at the same time as the option?

No. If the future's price is above the spot price during the life of the option, it is possible that the spot price will hit the barrier when the futures price does not.

Problem 26.14.

Answer the following questions about compound options

(a) *What put–call parity relationship exists between the price of a European call on a call and a European put on a call? Show that the formulas given in the text satisfy the relationship.*

(b) *What put–call parity relationship exists between the price of a European call on a put and a European put on a put? Show that the formulas given in the text satisfy the relationship.*

(a) The put–call relationship is

$$cc + K_1 e^{-rT_1} = pc + c$$

where cc is the price of the call on the call, pc is the price of the put on the call, c is the price today of the call into which the options can be exercised at time T_1, and K_1 is the exercise price for cc and pc. The proof is similar to that in Chapter 11 for the usual put–call parity relationship. Both sides of the equation represent the values of portfolios that will be worth $\max(c, K_1)$ at time T_1. Because

$$M(a,b;\rho) = N(a) - M(a,-b;-\rho) = N(b) - M(-a,b,;-\rho)$$

and

$$N(x) = 1 - N(-x)$$

we obtain

$$cc - pc = Se^{-qT_2}N(b_1) - K_2 e^{-rT_2}N(b_2) - K_1 e^{-rT_1}$$

Since

$$c = Se^{-qT_2}N(b_1) - K_2 e^{-rT_2}N(b_2)$$

put–call parity is consistent with the formulas

(b) The put–call relationship is

$$cp + K_1 e^{-rT_1} = pp + p$$

where cp is the price of the call on the put, pp is the price of the put on the put, p is the price today of the put into which the options can be exercised at time T_1, and K_1 is the exercise price for cp and pp. The proof is similar to that in Chapter 11 for the usual put–call parity relationship. Both sides of the equation represent the values of portfolios that will be worth $\max(p, K_1)$ at time T_1. Because

$$M(a,b;\rho) = N(a) - M(a,-b;-\rho) = N(b) - M(-a,b,;-\rho)$$

and

$$N(x) = 1 - N(-x)$$

it follows that

$$cp - pp = -Se^{-qT_2}N(-b_1) + K_2 e^{-rT_2}N(-b_2) - K_1 e^{-rT_1}$$

Because

$$p = -Se^{-qT_2}N(-b_1) + K_2e^{-rT_2}N(-b_2)$$

put–call parity is consistent with the formulas.

Problem 26.15.

Does a floating lookback call become more valuable or less valuable as we increase the frequency with which we observe the asset price in calculating the minimum?

As we increase the frequency we observe a more extreme minimum which increases the value of a floating lookback call.

Problem 26.16.

Does a down-and-out call become more valuable or less valuable as we increase the frequency with which we observe the asset price in determining whether the barrier has been crossed? What is the answer to the same question for a down-and-in call?

As we increase the frequency with which the asset price is observed, the asset price becomes more likely to hit the barrier and the value of a down-and-out call goes down. For a similar reason the value of a down-and-in call goes up. The adjustment mentioned in the text, suggested by Broadie, Glasserman, and Kou, moves the barrier further out as the asset price is observed less frequently. This increases the price of a down-and-out option and reduces the price of a down-and-in option.

Problem 26.17.

Explain why a regular European call option is the sum of a down-and-out European call and a down-and-in European call. Is the same true for American call options?

If the barrier is reached the down-and-out option is worth nothing while the down-and-in option has the same value as a regular option. If the barrier is not reached the down-and-in option is worth nothing while the down-and-out option has the same value as a regular option. This is why a down-and-out call option plus a down-and-in call option is worth the same as a regular option. A similar argument cannot be used for American options.

Problem 26.18.

What is the value of a derivative that pays off $100 in six months if the S&P 500 index is greater than 1,000 and zero otherwise? Assume that the current level of the index is 960, the risk-free rate is 8% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 20%.

This is a cash-or-nothing call. The value is $100N(d_2)e^{-0.08\times 0.5}$ where

$$d_2 = \frac{\ln(960/1000) + (0.08 - 0.03 - 0.2^2/2)\times 0.5}{0.2\times\sqrt{0.5}} = -0.1826$$

Since $N(d_2) = 0.4276$ the value of the derivative is $41.08.

Problem 26.19.

In a three-month down-and-out call option on silver futures the strike price is $20 per ounce and the barrier is $18. The current futures price is $19, the risk-free interest rate is 5%, and the volatility of silver futures is 40% per annum. Explain how the option works and calculate its value. What is the value of a regular call option on silver futures with the same terms? What is the value of a down-and-in call option on silver futures with the same terms?

This is a regular call with a strike price of $20 that ceases to exist if the futures price hits $18. With the notation in the text $H = 18$, $K = 20$, $S = 19$, $r = 0.05$, $\sigma = 0.4$, $q = 0.05$, $T = 0.25$. From this $\lambda = 0.5$ and

$$y = \frac{\ln[18^2 / (19 \times 20)]}{0.4\sqrt{0.25}} + 0.5 \times 0.4\sqrt{0.25} = -0.69714$$

The value of a down-and-out call plus a down-and-in call equals the value of a regular call. Substituting into the formula given when $H < K$ we get $c_{\text{di}} = 0.4638$. The regular Black-Scholes-Merton formula gives $c = 1.0902$. Hence $c_{\text{do}} = 0.6264$. (These answers can be checked with DerivaGem.)

Problem 26.20.

A new European-style floating lookback call option on a stock index has a maturity of nine months. The current level of the index is 400, the risk-free rate is 6% per annum, the dividend yield on the index is 4% per annum, and the volatility of the index is 20%. Use DerivaGem to value the option.

DerivaGem shows that the value is 53.38. Note that the Minimum to date and Maximum to date should be set equal to the current value of the index for a new deal. (See material on DerivaGem at the end of the book.)

Problem 26.21.

Estimate the value of a new six-month European-style average price call option on a non-dividend-paying stock. The initial stock price is $30, the strike price is $30, the risk-free interest rate is 5%, and the stock price volatility is 30%.

We can use the analytic approximation given in the text.

$$M_1 = \frac{(e^{0.05\times 0.5} - 1) \times 30}{0.05 \times 0.5} = 30.378$$

Also $M_2 = 936.9$ so that $\sigma = 17.41\%$. The option can be valued as a futures option with $F_0 = 30.378$, $K = 30$, $r = 5\%$, $\sigma = 17.41\%$, and $t = 0.5$. The price is 1.637.

Problem 26.22.

Use DerivaGem to calculate the value of:

(a) *A regular European call option on a non-dividend-paying stock where the stock price is $50, the strike price is $50, the risk-free rate is 5% per annum, the volatility is 30%, and the time to maturity is one year.*

(b) *A down-and-out European call which is as in (a) with the barrier at $45.*

(c) *A down-and-in European call which is as in (a) with the barrier at $45.*

Show that the option in (a) is worth the sum of the values of the options in (b) and (c).

The price of a regular European call option is 7.116. The price of the down-and-out call option is 4.696. The price of the down-and-in call option is 2.419.
The price of a regular European call is the sum of the prices of down-and-out and down-and-in options.

Problem 26.23.

Explain adjustments that have to be made when $r = q$ for a) the valuation formulas for *lookback call options in Section 26.11 and b) the formulas for M_1 and M_2 in Section 26.13.*

When $r = q$ in the expression for a floating lookback call in Section 26.11 $a_1 = a_3$ and $Y_1 = \ln(S_0 / S_{\min})$ so that the expression for a floating lookback call becomes

$$S_0 e^{-qT} N(a_1) - S_{\min} e^{-rT} N(a_2)$$

As q approaches r in Section 26.13 we get

$$M_1 = S_0$$

$$M_2 = \frac{2e^{\sigma^2 T} S_0^2}{\sigma^4 T^2} - \frac{2S_0^2}{T^2} \frac{1+\sigma^2 T}{\sigma^4}$$

Problem 26.24.
Value the variance swap in Example 26.4 of Section 26.16 assuming that the implied volatilities for options with strike prices 800, 850, 900, 950, 1,000, 1,050, 1,100, 1,150, 1,200 are 20%, 20.5%, 21%, 21.5%, 22%, 22.5%, 23%, 23.5%, 24%, respectively.

In this case, DerivaGem shows that $Q(K_1) = 0.1772$, $Q(K_2) = 1.1857$, $Q(K_3) = 4.9123$, $Q(K_4) = 14.2374$, $Q(K_5) = 45.3738$, $Q(K_6) = 35.9243$, $Q(K_7) = 20.6883$, $Q(K_8) = 11.4135$, $Q(K_9) = 6.1043$. $\hat{E}(\bar{V}) = 0.0502$. The value of the variance swap is \$0.51 million.

Problem 26.25.
Verify that the results in Section 26.2 for the value of a derivative that pays Q when S=H are consistent with those in Section 15.6.

When q=0, $w=r-\sigma^2/2$ so that α_1=1 and $\alpha_2=2r/\sigma^2$. This is consistent with the results for perpetual derivatives in Section 15.6.

Further Questions

Problem 26.26.
What is the value in dollars of a derivative that pays off \$10,000 in one year provided that the dollar–sterling exchange rate is greater than 1.5000 at that time? The current exchange rate is 1.4800. The dollar and sterling interest rates are 4% and 8% per annum respectively. The volatility of the exchange rate is 12% per annum.

It is instructive to consider two different ways of valuing this instrument. From the perspective of a sterling investor it is a cash or nothing put. The variables are $S_0 = 1/1.48 = 0.6757$, $K = 1/1.50 = 0.6667$, $r = 0.08$, $q = 0.04$, $\sigma = 0.12$, and $T = 1$. The derivative pays off if the exchange rate is less than 0.6667. The value of the derivative is $10{,}000 N(-d_2) e^{-0.08\times 1}$ where

$$d_2 = \frac{\ln(0.6757/0.6667) + (0.08 - 0.04 - 0.12^2/2)}{0.12} = 0.3852$$

Since $N(-d_2)$=0.3500, the value of the derivative is $10{,}000 \times 0.3500 \times e^{-0.08}$ or 3,231. In dollars this is $3{,}231 \times 1.48 = \$4782$

From the perspective of a dollar investor the derivative is an asset or nothing call. The variables are $S_0 = 1.48$, $K = 1.50$, $r = 0.04$, $q = 0.08$, $\sigma = 0.12$ and $T = 1$. The value is $10{,}000 N(d_1) e^{-0.08\times 1}$ where

$$d_1 = \frac{\ln(1.48/1.50) + (0.04 - 0.08 + 0.12^2/2)}{0.12} = -0.3852$$

$N(d_1) = 0.3500$ and the value of the derivative is as before

$10{,}000 \times 1.48 \times 0.3500 \times e^{-0.08} = 4782$ or $4,782.

Problem 26.27.

Consider an up-and-out barrier call option on a non-dividend-paying stock when the stock price is 50, the strike price is 50, the volatility is 30%, the risk-free rate is 5%, the time to maturity is one year, and the barrier at $80. Use the DerivaGem software to value the option and graph the relationship between (a) the option price and the stock price, (b) the delta and the stock price, (c) the option price and the time to maturity, and (d) the option price and the volatility. Provide an intuitive explanation for the results you get. Show that the delta, gamma, theta, and vega for an up-and-out barrier call option can be either positive or negative.

The price of the option is 3.528.

(a) The option price is a humped function of the stock price with the maximum option price occurring for a stock price of about $57. If you could choose the stock price there would be a trade off. High stock prices give a high probability that the option will be knocked out. Low stock prices give a low potential payoff. For a stock price less than $57 delta is positive (as it is for a regular call option); for a stock price greater than $57, delta is negative.

(b) Delta increases up to a stock price of about 45 and then decreases. This shows that gamma can be positive or negative.

(c) The option price is a humped function of the time to maturity with the maximum option price occurring for a time to maturity of 0.5 years. This is because too long a time to maturity means that the option has a high probability of being knocked out; too short a time to maturity means that the option has a low potential payoff. For a time to maturity less than 0.5 years theta is negative (as it is for a regular call option); for a time to maturity greater than 0.5 years the theta of the option is positive.

(d) The option price is also a humped function of volatility with the maximum option price being obtained for a volatility of about 20%. This is because too high a volatility means that the option has a high probability of being knocked out; too low volatility means that the option has a low potential payoff. For volatilities less than 20% vega is positive (as it is for a regular option); for volatilities above 20% vega is negative.

Problem 26.28.

Sample Application F in the DerivaGem Application Builder Software considers the static options replication example in Section 26.17. It shows the way a hedge can be constructed using four options (as in Section 26.17) and two ways a hedge can be constructed using 16 options.

(a) *Explain the difference between the two ways a hedge can be constructed using 16 options.*

(b) *Explain intuitively why the second method works better.*

(c) *Improve on the four-option hedge by changing Tmat for the third and fourth options.*

(d) *Check how well the 16-option portfolios match the delta, gamma, and vega of the barrier* *option.*

(a) Both approaches use a one call option with a strike price of 50 and a maturity of 0.75. In the first approach the other 15 call options have strike prices of 60 and equally spaced times to maturity. In the second approach the other 15 call options have strike prices of 60, but the spacing between the times to maturity decreases as the maturity of the barrier option is approached. The second approach to setting times to maturity produces a better hedge. This is because the chance of the barrier being hit at time t is an increasing function of t. As t increases it therefore becomes more important to replicate the barrier at time t.

(b) By using either trial and error or the Solver tool we see that we come closest to matching the price of the barrier option when the maturities of the third and fourth options are changed from 0.25 and 0.5 to 0.39 and 0.65.

(c) To calculate delta for the two 16-option hedge strategies it is necessary to change the last argument of EPortfolio from 0 to 1 in cells L42 and X42. To calculate delta for the barrier option it is necessary to change the last argument of BarrierOption in cell F12 from 0 to 1. To calculate gamma and vega the arguments must be changed to 2 and 3, respectively. The delta, gamma, and vega of the barrier option are –0.0221, –0.0035, and –0.0254. The delta, gamma, and vega of the first 16-option portfolio are –0.0262, –0.0045, and –0.1470. The delta, gamma, and vega of the second 16-option portfolio are –0.0199, –0.0037, and –0.1449. The second of the two 16-option portfolios provides Greek letters that are closest to the Greek letters of the barrier option. Interestingly neither of the two portfolios does particularly well on vega.

Problem 26.29.

Consider a down-and-out call option on a foreign currency. The initial exchange rate is 0.90, the time to maturity is two years, the strike price is 1.00, the barrier is 0.80, the domestic risk-free interest rate is 5%, the foreign risk-free interest rate is 6%, and the volatility is 25% per annum. Use DerivaGem to develop a static option replication strategy involving five options.

A natural approach is to attempt to replicate the option with positions in:

(a) A European call option with strike price 1.00 maturing in two years
(b) A European put option with strike price 0.80 maturing in two years
(c) A European put option with strike price 0.80 maturing in 1.5 years
(d) A European put option with strike price 0.80 maturing in 1.0 years
(e) A European put option with strike price 0.80 maturing in 0.5 years

The first option can be used to match the value of the down-and-out-call for $t = 2$ and $S > 1.00$. The others can be used to match it at the following (t, S) points: (1.5, 0.80) (1.0, 0.80), (0.5, 0.80), (0.0, 0.80). Following the procedure in the text, we find that the required positions in the options are as shown in the following table.

Option Type	*Strike Price*	*Maturity (yrs)*	*Position*
Call	1.0	2.00	+1.0000
Put	0.8	2.00	−0.1255
Put	0.8	1.50	−0.1758
Put	0.8	1.00	−0.0956
Put	0.8	0.50	−0.0547

The values of the options at the relevant (*t*,*S*) points are as follows

	Value initially	Value at (1.5,0.8)	Value at (1.0,0.8)	Value at (0.5,0.8)	Value at (0,0.8)
option (a)	0.0735	0.0071	0.0199	0.0313	0.0410
option (b)	0.0736	0.0568	0.0792	0.0953	0.1079
option (c)	0.0603		0.0568	0.0792	0.0953
option (d)	0.0440			0.0568	0.0792
option (e)	0.0231				0.0568

The value of the portfolio initially is 0.0482. This is only a little less than the value of the down-and-out-option which is 0.0488. This example is different from the example in the text in a number of ways. Put options and call options are used in the replicating portfolio. The value of the replicating portfolio converges to the value of the option from below rather than from above. Also, even with relatively few options, the value of the replicating portfolio is close to the value of the down-and-out option.

Problem 26.30.

Suppose that a stock index is currently 900. The dividend yield is 2%, the risk-free rate is 5%, and the volatility is 40%. Use the results in the Technical Note 27 to calculate the value of a one-year average price call where the strike price is 900 and the index level is observed at the end of each quarter for the purposes of the averaging. Compare this with the price calculated by DerivaGem for a one-year average price option where the price is observed continuously. Provide an intuitive explanation for any differences between the prices.

In this case

M_1=$(900e^{(0.05-0.03)\times 0.25}+900e^{(0.05-0.03)\times 0.50}+900e^{(0.05-0.03)\times 0.75}+900e^{(0.05-0.03)\times 1})/4=917.07$ and

A more complicated calculation involving 16 terms shows that M_2=907,486.6

so that the option can be valued as an option on futures where the futures price is 917.07 and volatility is $\sqrt{\ln(907{,}486.6/917.07^2}$ or 27.58%. The value of the option is 103.13. DerivaGem gives the price as 86.77 (set option type =Asian). The higher price for the first option arises because the average is calculated from prices at times 0.25, 0.50, 0.75, and 1.00. The mean of these times is 0.625. By contrast the corresponding mean when the price is observed continuously is 0.50. The later a price is observed the more uncertain it is and the more it contributes to the value of the option.

Problem 26.31.

Use the DerivaGem Application Builder software to compare the effectiveness of daily delta hedging for (a) the option considered in Tables 19.2 and 19.3 and (b) an average price call with the same parameters. Use Sample Application C. For the average price option you will find it necessary to change the calculation of the option price in cell C16, the payoffs in cells H15 and H16, and the deltas (cells G46 to G186 and N46 to N186). Carry out 20 Monte Carlo simulation runs for each option by repeatedly pressing F9. On each run record the cost of writing and hedging the option, the volume of trading over the whole 20 weeks and the volume of trading between weeks 11 and 20. Comment on the results.

For the regular option the theoretical price is about $240,000. For the average price option the theoretical price is about 115,000. My 20 simulation runs (40 outcomes because of the antithetic calculations) gave results as shown in the following table.

	Regular Call	*Ave Price Call*
Ave Hedging Cost	247,628	114,837
SD Hedging Cost	17,833	12,123
Ave Trading Vol (20 wks)	412,440	291,237
Ave Trading Vol (last 10 wks)	187,074	51,658

These results show that the standard deviation of using delta hedging for an average price option is lower than that for a regular option. However, using the criterion in Chapter 19 (standard deviation divided by value of option) hedge performance is better for the regular option. Hedging the average price option requires less trading, particularly in the last 10 weeks. This is because we become progressively more certain about the average price as the maturity of the option is approached.

Problem 26.32.
In the DerivaGem Application Builder Software modify Sample Application D to test the effectiveness of delta and gamma hedging for a call on call compound option on a 100,000 units of a foreign currency where the exchange rate is 0.67, the domestic risk-free rate is 5%, the foreign risk-free rate is 6%, the volatility is 12%. The time to maturity of the first option is 20 weeks, and the strike price of the first option is 0.015. The second option matures 40 weeks from today and has a strike price of 0.68. Explain how you modified the cells. Comment on hedge effectiveness.

The value of the option is 1093. It is necessary to change cells F20 and F46 to 0.67. Cells G20 to G39 and G46 to G65 must be changed to calculate delta of the compound option. Cells H20 to H39 and H46 to H65 must be changed to calculate gamma of the compound option. Cells I20 to I40 and I46 to I66 must be changed to calculate the Black–Scholes price of the call option expiring in 40 weeks. Similarly cells J20 to J40 and J46 to J66 must be changed to calculate the delta of this option; cells K20 to K40 and K46 to K66 must be changed to calculate the gamma of the option. The payoffs in cells N9 and N10 must be calculated as MAX(I40-0.015,0)*100000 and MAX(I66-0.015,0)*100000. Delta plus gamma hedging works relatively poorly for the compound option. On 20 simulation runs the cost of writing and hedging the option ranged from 200 to 2500.

Problem 26.33.
Outperformance certificates (also called "sprint certificates", "accelerator certificates", or "speeders") are offered to investors by many European banks as a way of investing in a company's stock. The initial investment equals the company's stock price, S_0. If the stock price goes up between time 0 and time T, the investor gains k times the increase at time T where k is a constant greater than 1.0. However, the stock price used to calculate the gain at time T is capped at some maximum level M. If the stock price goes down the investor's loss is equal to the decrease. The investor does not receive dividends.

(a) *Show that the outperformance certificate is a package.*
(b) *Calculate using DerivaGem the value of a one-year outperformance certificate when the stock price is 50 euros, $k = 1.5$, M = 70 euros, the risk-free rate is 5%, and the stock price volatility is 25%. Dividends equal to 0.5 euros are expected in 2 months, 5 month, 8 months, and 11 months.*

a) The outperformance certificate is equivalent to a package consisting of

(i) A zero coupon bond that pays off S_0 at time T

(ii) A long position in k one-year European call options on the stock with a strike price equal to the current stock price.

(iii) A short position in k one-year European call options on the stock with a strike price equal to M

(iv) A short position in one European one-year put option on the stock with a strike price equal to the current stock price.

b) In this case the present value of the four parts of the package are

(i) $50e^{-0.05\times 1}=47.56$

(ii) 1. 5× 5. 0 0 5 6 .7 [illegible]

(iii) $-1.5\times 0.6339=-0.9509$

(iv) -4.5138

The total of these is 47.56 +7.5084 – 0.9509−4.5138 = 49.6. This is less than the initial investment of 50.

Problem 26.34.

Carry out the analysis in Example 26.4 of Section 26.16 to value the variance swap on the assumption that the life of the swap is 1 month rather than 3 months.

In this case, $F_0=1022.55$ and DerivaGem shows that $Q(K_1)=0.0366$, $Q(K_2)=0.2858$, $Q(K_3)=1.5822$, $Q(K_4)=6.3708$, $Q(K_5)=30.3864$, $Q(K_6)=16.9233$, $Q(K_7)=4.8180$, $Q(K_8)=0.8639$, and $Q_9=0.0863$. $\hat{E}(\bar{V})=0.0661$. The value of the variance swap is \$2.09 million.

Problem 26.35.

What is the relationship between a regular call option, a binary call option, and a gap call option?

With the notation in the text, a regular call option with strike price K_2 plus a binary call option that pays off $K_2 - K_1$ is a gap call option that pays off $S_T - K_1$ when $S_T > K_2$.

Problem 26.36.

Produce a formula for valuing a cliquet option where an amount Q is invested to produce a payoff at the end of n periods. The return each period is the greater of the return on an index (excluding dividends) and zero.

Suppose that there are n periods each of length τ, the risk-free interest rate is r, the dividend yield on the index is q, and the volatility of the index is σ. The value of the investment is

$$e^{-rn\tau}Q\hat{E}\left[\prod_{i=1}^{n}\max(1+R_i,1)\right]$$

where R_i is the return in period i and as usual $\hat{E}$ denotes expected value in a risk-neutral world. Because (assuming efficient markets) the returns in successive periods are independent, this is

$$e^{-rn\tau}Q\prod_{i=1}^{n}\{\hat{E}[\max(1+R_i,1)]\}$$

$$=e^{-rn\tau}Q\prod_{i=1}^{n}\left\{\hat{E}\left[1+\max\left(\frac{S_i-S_{i-1}}{S_{i-1}},\ 0\right)\right]\right\}$$

where S_i is the value of the index at the end of the ith period.

From Black-Scholes-Merton the risk-neutral expectation at time $(i\text{-}1)\tau$ of $\max(S_i-S_{i-1}, 0)$ is

$$e^{(r-q)\tau}S_{i-1}N(d_1)-S_{i-1}N(d_2)$$

where

$$d_1=\frac{(r-q+\sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

$$d_2=\frac{(r-q-\sigma^2/2)\tau}{\sigma\sqrt{\tau}}$$

The value of the investment is therefore

$$e^{-rn\tau}Q\left[1+e^{(r-q)\tau}N(d_1)-N(d_2)\right]^n$$