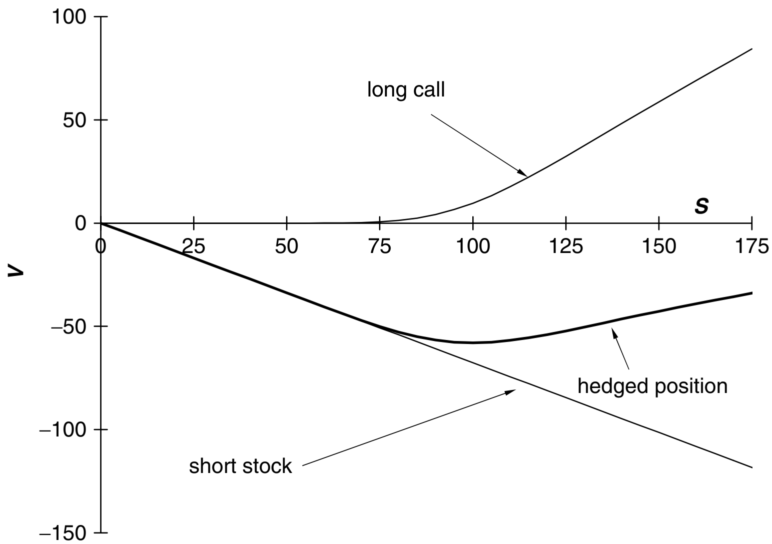


Figure 58.2 The delta-hedged long call portfolio according to Black–Scholes.

signs to consider a short call position we find that a crash is bad, but how do we find the worst case? If there is going to be one crash of 10% when is the worst time for this to happen? This is the motivation for the model below. Note first that, generally speaking, a positive gamma position benefits from a crash, while a negative gamma position loses.

58.4 A MATHEMATICAL MODEL FOR A CRASH

The main idea in the following model is simple. We assume that the worst will happen. We value all contracts assuming this, and then, unless we are very unlucky and the worst does happen, we will be pleasantly surprised. In this context, 'pleasantly surprised' means that we make more money than we expected. We can draw an important distinction between this model and the models of Chapter 57, the jump diffusion models. In the latter we make bold statements about the frequency and distribution of jumps and finally take expectations to arrive at a value for a derivative. Here *we make no statements about the distribution of either the jump size or when it will happen*. At most, the number of jumps is limited. Finally, we examine the worst-case scenario so that no expectations are taken.

I will model the underlying asset price behavior as the classical binomial tree, but with the addition of a third state, corresponding to a large movement in the asset. So, really, we have a trinomial walk but with the lowest branch being to a significantly more distant asset value. The up and down diffusive branches are modeled in the usual binomial fashion. For simplicity, assume that the crash, when it happens, is from S to $(1-k)S$ with k given; this assumption will later be dropped to allow k to cover a range of values, or even to allow a dramatic rise in the value of the underlying. Introduce the subscript 1 to denote values of the option before the crash, i.e. with one crash allowed, and 0 to denote values after. Thus V_0 is the value of the option position after the crash. This is a function of S and t and, since I am only permitting one crash, V_0 must be exactly the Black–Scholes option value.

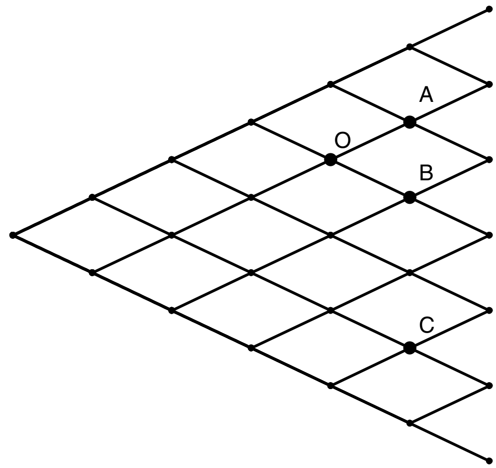


Figure 58.3 The tree structure.

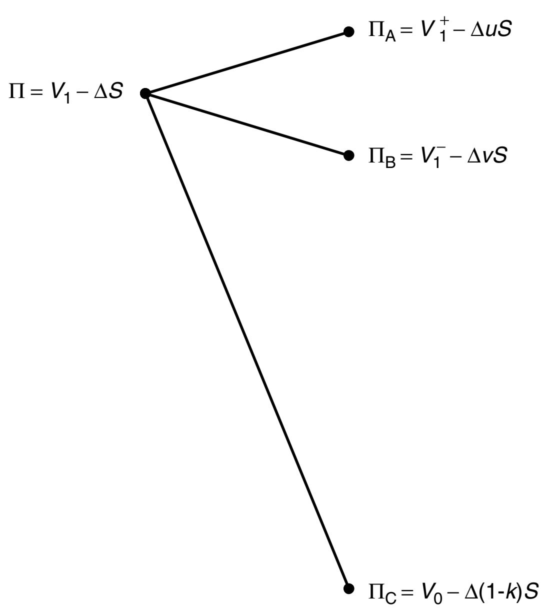


Figure 58.4 The tree and portfolio values.

As shown in Figure 58.3, if the underlying asset starts at value S (point O) it can go to one of three values: uS, if the asset rises; vS, if the asset falls; $(1-k)S$, if there is a crash. These three points are denoted by A, B and C respectively. The values for uS and vS are chosen in the usual manner for the traditional binomial model (see Chapter 15).

Before the asset price moves, we set up a 'hedged' portfolio, consisting of our option position and $-\Delta$ of the underlying asset. At this time our option has value V_1. We must find both an optimal Δ and then V_1. The hedged portfolio is shown in Figure 58.4.

A time δt later the asset value has moved to one of the three states, A, B or C and at the same time the option value becomes either V_1^+ (for state A), V_1^- (for state B) or the Black–Scholes value V_0 (for state C).

The change in the value of the portfolio, between times t and $t + \delta t$ (denoted by $\delta\Pi$) is given by the following expressions for the three possible states:

$$\delta\Pi_A = V_1^+ - \Delta uS + \Delta S - V_1 \text{ (diffusive rise)}$$

$$\delta\Pi_B = V_1^- \Delta vS + \Delta S - V_1 \text{ (diffusive fall)}$$

$$\delta\Pi_C = V_0 + \Delta kS - V_1 \text{ (crash).}$$

These three functions are plotted against Δ in Figures 58.5 and 58.6. I will explain the difference between the two figures very shortly. My aim in what follows is to choose the hedge ratio Δ so as to minimize the pessimistic, worst outcome among the three possible.

There are two cases to consider, shown in Figures 58.5 and 58.6. The former, Case I, is when the worst-case scenario is not the crash but the simple diffusive movement of S. In this case V_0 is sufficiently large for a crash to be beneficial:

$$V_0 \geq V_1^+ + (S - uS - kS)\frac{V_1^+ - V_1^-}{uS - vS}. \tag{58.1}$$

If V_0 is smaller than this, then the worst scenario is a crash; this is Case II.

58.4.1 Case I: Black–Scholes Hedging

Refer to Figure 58.5. In this figure we see the three lines representing $\delta\Pi$ for each of the moves to A, B and C. Pick a value for the hedge ratio Δ (for example, see the dashed vertical line in Figure 58.5), and determine on which of the three lines lies the worst possible value for $\delta\Pi$ (in the example in the figure, the point is marked by a cross and lies on the A line). Change

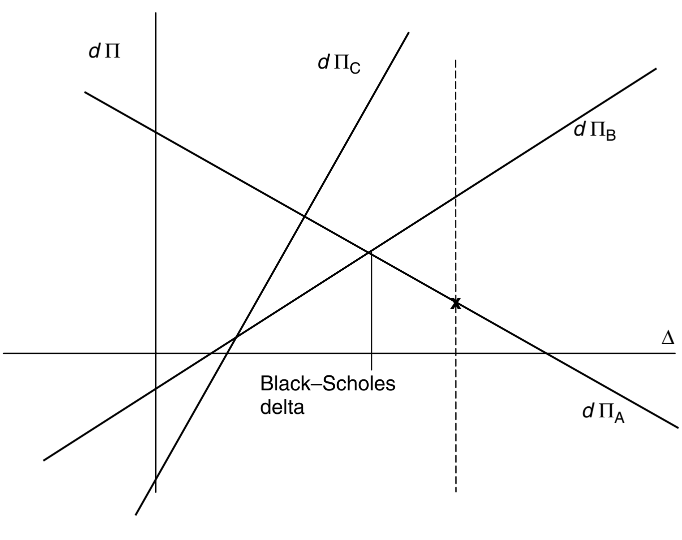


Figure 58.5 Case I: worst case is diffusive motion.

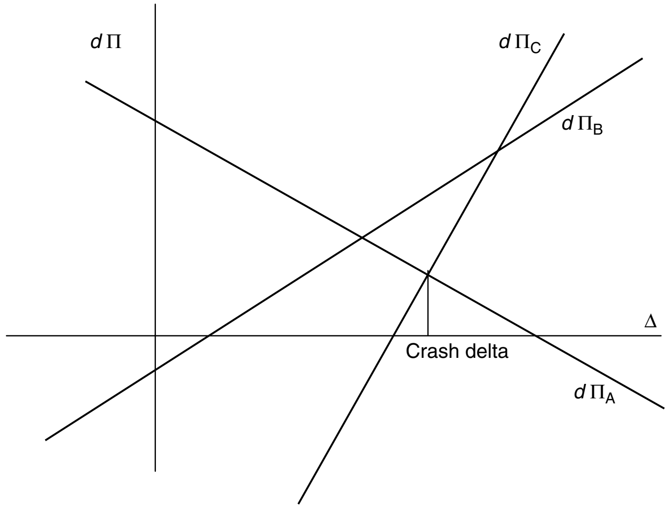


Figure 58.6 Case II: worst case is a crash.

your value of Δ to maximize this worst value. In other words, we want to choose Δ to put us as high up the envelope of the three lines as possible.

In this case the maximal-lowest value for $\delta\Pi$ occurs at the point where

$$\delta\Pi_A = \delta\Pi_B,$$

that is

$$\Delta = \frac{V_1^+ - V_1^-}{uS - vS}. \tag{58.2}$$

This will be recognized as the expression for the hedge ratio in a Black–Scholes world.

Having chosen Δ, we now determine V_1 by setting the return on the portfolio equal to the risk-free interest rate. Thus we set

$$\delta\Pi_A = r\Pi\,\delta t$$

to get

$$V_1 = \frac{1}{1 + r\,\delta t}\left(V_1^+ + (S - uS + rS\,\delta t)\frac{V_1^+ - V_1^-}{uS - vS}\right). \tag{58.3}$$

This is the equation to solve if we are in Case I. Note that it corresponds exactly to the usual binomial version of the Black–Scholes equation; there is no mention of the value of the portfolio at the point C. As δt goes to zero, (58.2) becomes $\partial V/\partial S$ and Equation (58.3) becomes the Black–Scholes partial differential equation.

58.4.2 Case II: Crash Hedging

Refer to Figure 58.6. In this case the value for V_0 is low enough for a crash to give the lowest value for the jump in the portfolio. We therefore choose Δ to maximize this worst case. Again, we want to choose Δ to put us as high up the envelope of the three lines as possible. Thus we choose

$$\delta\Pi_A = \delta\Pi_C,$$

that is,

$$\Delta = \frac{V_0 - V_1^+}{S - uS - kS}. \tag{58.4}$$

Now set

$$\delta\Pi_A = r\Pi\,\delta t$$

to get

$$V_1 = \frac{1}{1 + r\,\delta t}\left(V_0 + S(k + r\,\delta t)\frac{V_0 - V_1^+}{S - uS - kS}\right). \tag{58.5}$$

This is the equation to solve when we are in Case II. Note that this is different from the usual binomial equation, and does not give the Black–Scholes partial differential equation as δt goes to zero (see later in the chapter). Also (58.4) is not the Black–Scholes delta. To appreciate that delta hedging is not necessarily optimal, consider the simple example of the butterfly spread. If the butterfly spread is delta hedged on the right 'wing' of the butterfly, where the delta is negative, a large fall in the underlying will result in a large loss from the hedge, whereas the loss in the butterfly spread will be relatively small. This could result in a negative value for a contract, even though its payoff is everywhere positive.

58.5 AN EXAMPLE

All that remains to be done is to solve equations (58.3) and (58.5) (which one is valid at any asset value and at any point in time depends on whether or not (58.1) is satisfied). This is easily done by working backwards down the tree from expiry in the usual binomial fashion.

As an example, examine the cost of a 15% crash on a portfolio consisting of the call options in Table 58.1.

At the moment the portfolio only contains the first two options. Later I will add some of the third option for static hedging, that is when the bid-ask prices will concern us. The volatility of the underlying is 17.5% and the risk-free interest rate is 6%.

The solution to the problem is shown in Figure 58.7. Observe how the value of the portfolio assuming the worst (21.2 when the spot is 100), is lower than the Black–Scholes value (30.5). This is especially clear where the portfolio's gamma is highly negative. This is because when

Table 58.1 Available contracts.

Strike	Expiry	Bid	Ask	Quantity
100	75 days			–3
80	75 days			2
90	75 days	11.2	12	0

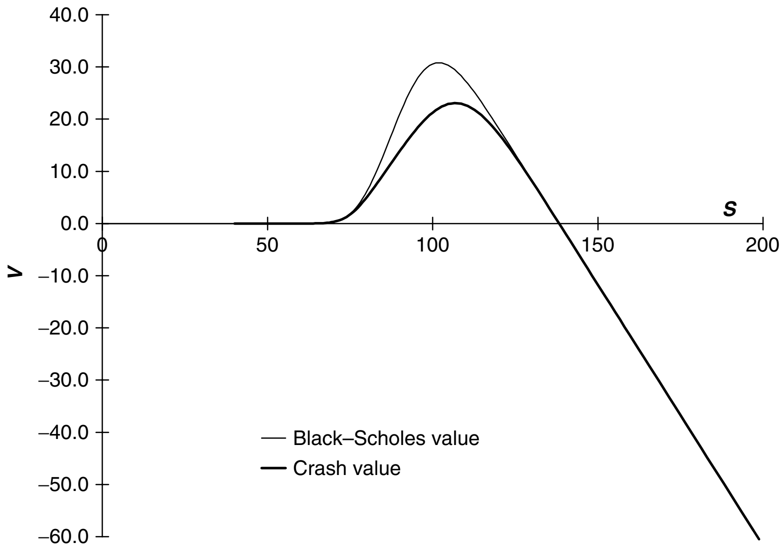
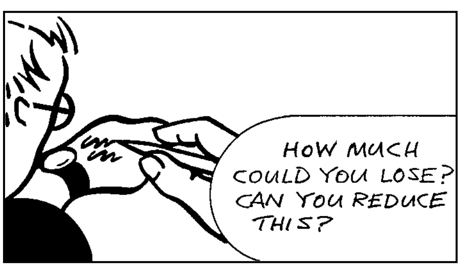


Figure 58.7 Example showing crash value and Black–Scholes value.

the gamma is positive, a crash is beneficial to the portfolio's value. When the gamma is close to zero, the delta hedge is very accurate and the option is insensitive to a crash. If the asset price is currently 100, the difference between the before and after portfolio values is $30.5 - 21.2 = 9.3$. This is the 'Value at Risk' under the worst-case scenario.



58.6 OPTIMAL STATIC HEDGING: VAR REDUCTION

The 9.3 value at risk is due to the negative gamma around the asset price of 100. An obvious hedging strategy that will offset some of this risk is to buy some positive gamma as a 'static' hedge. In other words, we should buy an option or options having a counterbalancing effect on the value at risk. We are willing to pay a premium for such an option. We may even pay more than the Black–Scholes fair value for such a static hedge because of the extra benefit that it gives us in reducing our exposure to a crash. Moreover, if we have a choice of contracts with which to hedge statically we should buy the most 'efficient' one. To see what this means consider the above example in more detail.

Recall that the value of the initial portfolio under the worst-case scenario is 21.2. How many of the 90 calls should we buy (for 12) or sell (for 11.2) to make the best of this scenario? Suppose that we buy λ of these calls. We will now find the optimal value for λ.

The cost of this hedge is

$$\lambda C(\lambda)$$

where $C(\lambda)$ is 12 if λ is positive and 11.2 otherwise (see Table 58.1). Now solve Equations (58.3) and (58.5) with the final total payoffs

$$V_0(S,T) = V_1(S,T) = 2\max(S-80,0) - 3\max(S-100,0) + \lambda\max(S-90,0).$$

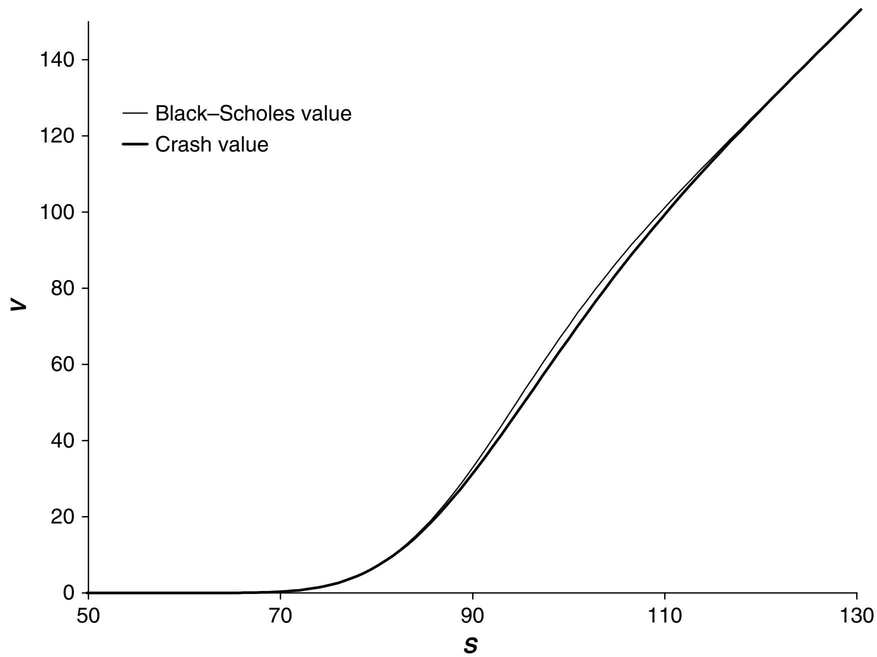


Figure 58.8 Optimally-hedged portfolio, before and after crash.

This is the payoff at time T for the statically hedged portfolio. The *marginal* value of the original portfolio (that is, the portfolio of the 80 and 100 calls) is therefore

$$V_1(100, 0) - \lambda C(\lambda) \qquad (58.6)$$

i.e. the worst-case value for the new portfolio less the cost of the static hedge. The arguments of the before-crash option value are 100 and 0 because they are today's asset value and date. The optimality in this hedge arises when we choose the quantity λ to maximize the value, expression (58.6). With the bid-ask spread in the 90 calls being 11.2–12, we find that buying 3.5 of the calls maximizes expression (58.6). The value of the new portfolio is 70.7 in a Black–Scholes world and 65.0 under our worst-case scenario. The value at risk has been reduced from 9.3 to $70.7 - 65 = 5.7$. The optimal portfolio values before and after the crash are shown in Figure 58.8. The optimal static hedge is known as the **Platinum Hedge**.

The issues of static hedging and optimal static hedging are covered in detail in Chapter 60.

58.7 CONTINUOUS-TIME LIMIT

If we let $\delta t \to 0$ in Equations (58.1), (58.2), (58.3), (58.4) and (58.5) we find that the Black–Scholes equation is still satisfied by $V_1(S, t)$ but we also have the constraint

$$\boxed{V_1(S, t) - kS\frac{\partial V_1}{\partial S}(S, t) \le V_0(S(1 - k), t)}$$

Such a problem is similar in principal to the American option valuation problem, where we also saw a constraint on the derivative's value. Here the constraint is more complicated. To this we must add the condition that the first derivative of V_1 must be continuous for $t < T$.

58.8 A RANGE FOR THE CRASH

In the above model, the crash has been specified as taking a certain value. Only the timing was left to be determined for the worst-case scenario. It is simple to allow the crash to cover a range of values, so that S goes to $(1-k)S$ where

$$k^- \le k \le k^+.$$

A negative k^- corresponds to a rise in the asset.

In the discrete setting the worst-case option value is given by

$$V_1 = \min_{k^- \le k \le k^+} \left(\frac{1}{1 + r\,\delta t} \left(V_0 + S(k + r\,\delta t) \frac{V_0 - V_1^+}{S - uS - kS} \right) \right).$$

This contains the min(·) because we want the worst-case crash. When a crash is beneficial we still have (58.3).

58.9 MULTIPLE CRASHES

The model described above can be extended in many ways, one of the most important is to consider the effect of multiple crashes. I describe two possibilities below. The first puts a constraint on the total number of crashes in a time period; there can be three crashes within the horizon of one year, say. The second puts a limit on the time between crashes; there cannot be another crash if there was a crash in the last six months, say.

58.9.1 Limiting the Total Number of Crashes

We will allow up to N crashes. We make no statement about the time these occur. We will assume that the crash size is given, allowing a fall of $k\%$. This can easily be extended to a range of sizes, as described above. Introduce the functions $V_i(S,t)$ with $i = 0, 1, \ldots, N$, such that V_i is the value of the option with i more crashes still allowed. Thus, as before, V_0 is the Black–Scholes value.

We must now solve N coupled equations of the following form. If

$$V_{i-1} \ge V_i^+ + (S - uS - kS) \frac{V_i^+ - V_i^-}{uS - vS}$$

then we are in Case I, a crash is beneficial and is assumed not to happen. In this case we have

$$V_i = \frac{1}{1 + r\,\delta t} \left(V_i^+ + (S - uS + rS\,\delta t) \frac{V_i^+ - V_i^-}{uS - vS} \right).$$

Otherwise a crash is bad for the hedged option; this is Case II. We then have

$$V_i = \frac{1}{1 + r\,\delta t} \left(V_{i-1} + S(k + r\,\delta t) \frac{V_{i-1} - V_i^+}{S - uS - kS} \right).$$

In continuous time the equations become

$$\frac{\partial V_i}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_i}{\partial S^2} + rS\frac{\partial V_i}{\partial S} - rV_i = 0$$

for $i = 0, \dots, N$, subject to

$$V_i(S,t) - kS\frac{\partial V_i}{\partial S}(S,t) \le V_{i-1}(S(1-k),t)$$

for $i = 1, \dots, N$. Each of the V_i has the same final condition, representing the payoff at expiry.

58.9.2 Limiting the Frequency of Crashes

Finally, we model a situation where the time between crashes is limited; if there was a crash less than a time ω ago another is not allowed.

This is slightly harder than the N-crash model and we have to introduce a new variable τ measuring the time since the last crash. We now have two functions to consider $V_c(S,t)$ and $V_n(S,t,\tau)$. The former is the worst-case option value when a crash is allowed (and therefore we don't need to know how long it has been since the last crash) and the latter is the worst-case option value when a crash is not yet allowed.

The governing equations, which are derived in the same way as the original crash model, are, in continuous time, simply

$$\frac{\partial V_c}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_c}{\partial S^2} + rS\frac{\partial V_c}{\partial S} - rV_c = 0$$

subject to

$$V_c(S,t) - kS\frac{\partial V_c}{\partial S}(S,t) \le V_n(S(1-k),t,0),$$

and for $V_n(S,t,\tau)$,

$$\frac{\partial V_n}{\partial t} + \frac{\partial V_n}{\partial \tau} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_n}{\partial S^2} + rS\frac{\partial V_n}{\partial S} - rV_n = 0$$

with the condition

$$V_n(S,t,\omega) = V_c(S,t).$$

Observe how the time τ and real time t increase together in the equation when a crash is not allowed.

58.10 CRASHES IN A MULTI-ASSET WORLD

When we have a portfolio of options with many underlyings we can still examine the worst-case scenario, but we have two choices. Either (a) we allow a crash to happen in any underlyings completely independently of all other underlyings or (b) we assume some relationship between the assets during a crash. Clearly the latter is not as bad a worst case as the former. It is also easier to write down, so we will look at that model only. Assuming that all assets fall simultaneously by the same percentage k we have

$$V_1(S_1, \dots, S_n, t) - k\sum_{i=1}^{n} S_i \frac{\partial V_1}{\partial S_i}(S_1, \dots, S_n, t) \leq V_0((1-k)S_1, \dots, (1-k)S_n, t, t).$$

We examine stock market crashes using CrashMetrics in Chapter 42.

58.11 FIXED AND FLOATING EXCHANGE RATES

Many currencies are linked directly to the currency in another country. Some countries have their currency linked to the US dollar; the Argentine Peso is tied at a rate of one to one to the dollar.

Once an exchange rate is fixed in this way the issue of fluctuating rates becomes a credit risk issue. All being well, the exchange rate will stay constant with all the advantages that stability brings. If economic conditions in the two countries start to diverge then the exchange rate will come under pressure. In Figure 58.9 is a plot of the possible exchange rate, showing a fixed rate for a while, followed by a sudden discontinuous drop and then a random fluctuation. How can we model derivatives of the exchange rate? The models of this chapter are ideally suited to this situation.

I'm going to ignore interest rates in the following. This is because of the complex issues this would otherwise raise. For example, the pressure on the exchange rate and the decoupling of the currencies would be accompanied by changing interest rates. This can be modeled but would distract us from the application of the crash model.

While the exchange rate is fixed, before the 'crash,' the price of an option, $V_1(S, t)$, satisfies

$$\frac{\partial V_1}{\partial t} = 0,$$

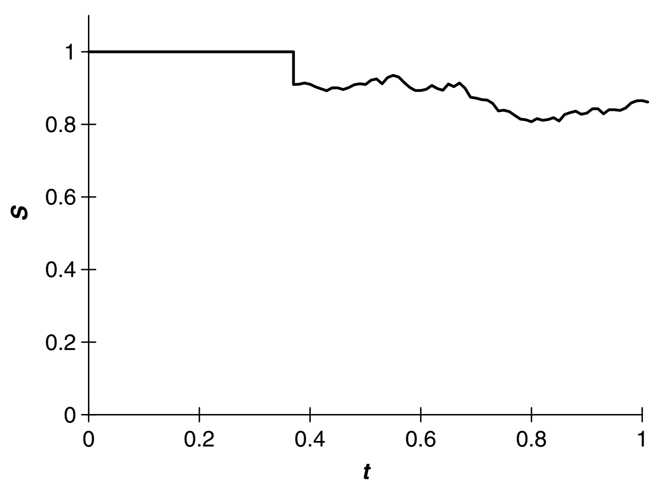


Figure 58.9 Decoupling of an exchange rate.

since I am assuming zero interest rates. Here S is the exchange rate. After the 'crash' we have

$$\frac{\partial V_0}{\partial t} + \tfrac{1}{2}\sigma^2 S^2 \frac{\partial^2 V_0}{\partial S^2} = 0.$$

(But with what volatility?) This is just the Black–Scholes model, with the relevant Black–Scholes value for the particular option payoff.

The worst-case crash model is now almost directly applicable. I will leave the details to the reader.

58.12 SUMMARY

I have presented a model for the effect of an extreme market movement on the value of portfolios of derivative products. This is an alternative way of looking at value at risk. I have shown how to employ static hedging to minimize this risk. In conclusion, note that the above is not a jump-diffusion model since I have deliberately not specified any probability distribution for the size or the timing of the jump: we model the worst-case scenario.

We have examined several possible models of crashes, of increasing complexity. One further thought is that we have not allowed for the rise in volatility that accompanies crashes. This can be done with ease. There is no reason why the after-crash model (V_0 in the simplest case above) cannot have a different volatility from the before-crash model. Of particular interest is the final model where there is a minimum time between crashes; we could easily have the volatility post crash being a decaying function of the time since the crash occurred, τ. This would involve no extra computational effort.

FURTHER READING

- For further details about crash modeling see Hua & Wilmott (1997) and Hua (1997).
- Derman & Zou (1997) describe and model the behavior of implied volatility after a large move in an index.