CHAPTER 27
More on Models and Numerical Procedures

Practice Questions

Problem 27.1.
Confirm that the CEV model formulas satisfy put–call parity.

It follows immediately from the equations in Section 27.1 that

$$p - c = Ke^{-rT} - S_0 e^{-qT}$$

in all cases.

Problem 27.2.
Use Monte Carlo simulation to show that Merton's value for a European option is correct when r=0.05, q=0, λ=0.3, k=0.5, σ =0.25, S_0 =30, K=30, s=0.5, and T=1. Use DerivaGem to check your price.

In this case λ'= 0.3×1.5=0.45. The variable f_n is the Black-Scholes-Merton price when the volatility is $0.25^2+ns^2/T$ and the risk-free rate is $-0.1+n\times\ln(1.5)/T$. A spreadsheet can be constructed to value the option using the first (say) 20 terms in the Merton expansion.

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} f_n$$

where λ' =0.3×1.5=0.45, T=1, and f_n is the Black-Scholes-Merton price of a European call option when S_0=30, K=30, the risk-free rate is $0.05-0.3\times0.5+n\times\ln(1.5)/1= -0.1+0.4055n$, T=1, and the variance rate is $0.25^2+n\times0.5^2/1= 0.0625+0.25n$.

There are a number of alternative approaches to valuing the option using Monte Carlo simulation. One is as follows. Sample to determine the number of jumps in time T as described in the text. The probability of N jumps is

$$\frac{e^{-\lambda T}(\lambda T)^N}{N!}$$

The proportional increase in the stock price arising from jumps is the product of N random samples from lognormal distributions. Each random sample is exp(X) where X is a random sample from a normal distribution with mean $\ln(1.5)-s^2/2$ and standard deviation s. (Note that it is one plus the percentage jump that is lognormal.) The proportional increase in the stock price arising from the diffusion component of the process is

$$\exp((0.05 - 0.3\times 0.5 - 0.25^2/2)T + 0.25\varepsilon\sqrt{T})$$

where ε is a random sample from a standard normal distribution. The final stock price is 30 times the product of the increase arising from jumps and the increase from the diffusion component. The payoff from the option can be calculated from this and the present value of the average payoff over many simulations is the estimate of the value of the option.

I find that the two approaches give similar answers. The option value given is about 5.47. This value is confirmed by DerivaGem 3.00. (The diffusion volatility is 25%, the average number of jumps per year is 0.3, the average jump size is 50%, and the jump standard deviation is 50%)

Problem 27.3.

Confirm that Merton's jump diffusion model satisfies put–call parity when the jump size is lognormal.

With the notation in the text the value of a call option, c is

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} c_n$$

where c_n is the Black-Scholes-Merton price of a call option where the variance rate is

$$\sigma^2 + \frac{ns^2}{T}$$

and the risk-free rate is

$$r - \lambda k + \frac{n\gamma}{T}$$

where $\gamma = \ln(1+k)$. Similarly the value of a put option p is

$$\sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} p_n$$

where p_n is the Black-Scholes-Merton price of a put option with this variance rate and risk-free rate. It follows that

$$p - c = \sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} (p_n - c_n)$$

From put–call parity

$$p_n - c_n = Ke^{(-r+\lambda k)T} e^{-n\gamma} - S_0 e^{-qT}$$

Because

$$e^{-n\gamma} = (1+k)^{-n}$$

it follows that

$$p - c = \sum_{n=0}^{\infty} \frac{e^{-\lambda' T + \lambda k T}(\lambda' T/(1+k))^n}{n!} Ke^{-rT} - \sum_{n=0}^{\infty} \frac{e^{-\lambda' T}(\lambda' T)^n}{n!} S_0 e^{-qT}$$

Using $\lambda' = \lambda(1+k)$ this becomes

$$\frac{1}{e^{\lambda T}} \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!} Ke^{-rT} - \frac{1}{e^{\lambda' T}} \sum_{n=0}^{\infty} \frac{(\lambda' T)^n}{n!} S_0 e^{-qT}$$

From the expansion of the exponential function we get

$$e^{\lambda T} = \sum_{n=0}^{\infty} \frac{(\lambda T)^n}{n!}$$

$$e^{\lambda' T} = \sum_{n=0}^{\infty} \frac{(\lambda' T)^n}{n!}$$

Hence

$$p - c = Ke^{-rT} - S_0 e^{-qT}$$

showing that put–call parity holds.

Problem 27.4.

Suppose that the volatility of an asset will be 20% from month 0 to month 6, 22% from month 6 to month 12, and 24% from month 12 to month 24. What volatility should be used in Black-Scholes-Merton to value a two-year option?

The average variance rate is

$$\frac{6 \times 0.2^2 + 6 \times 0.22^2 + 12 \times 0.24^2}{24} = 0.0509$$

The volatility used should be $\sqrt{0.0509} = 0.2256$ or 22.56%.

Problem 27.5.

Consider the case of Merton's jump diffusion model where jumps always reduce the asset price to zero. Assume that the average number of jumps per year is λ. Show that the price of a European call option is the same as in a world with no jumps except that the risk-free rate is $r + \lambda$ rather than r. Does the possibility of jumps increase or reduce the value of the call option in this case? (Hint: Value the option assuming no jumps and assuming one or more jumps. The probability of no jumps in time T is $e^{-\lambda T}$).

In a risk-neutral world the process for the asset price exclusive of jumps is

$$\frac{dS}{S} = (r - q - \lambda k)\,dt + \sigma\,dz$$

In this case $k = -1$ so that the process is

$$\frac{dS}{S} = (r - q + \lambda)\,dt + \sigma\,dz$$

The asset behaves like a stock paying a dividend yield of $q - \lambda$. This shows that, conditional on no jumps, call price

$$S_0 e^{-(q-\lambda)T} N(d_1) - Ke^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(S_0 / K) + (r - q + \lambda + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

There is a probability of $e^{-\lambda T}$ that there will be no jumps and a probability of $1 - e^{-\lambda T}$ that there will be one or more jumps so that the final asset price is zero. It follows that there is a probability of $e^{-\lambda T}$ that the value of the call is given by the above equation and $1 - e^{-\lambda T}$ that it will be zero. Because jumps have no systematic risk it follows that the value of the call option is

$$e^{-\lambda T}\left[S_0 e^{-(q-\lambda)T} N(d_1) - Ke^{-rT} N(d_2)\right]$$

or

$$S_0 e^{-qT} N(d_1) - K e^{-(r+\lambda)T} N(d_2)$$

This is the required result. The value of a call option is an increasing function of the risk-free interest rate (see Chapter 11). It follows that the possibility of jumps increases the value of the call option in this case.

Problem 27.6.

At time zero the price of a non-dividend-paying stock is S_0. Suppose that the time interval between 0 and T is divided into two subintervals of length t_1 and t_2. During the first subinterval, the risk-free interest rate and volatility are r_1 and σ_1, respectively. During the second subinterval, they are r_2 and σ_2, respectively. Assume that the world is risk neutral.

(a) *Use the results in Chapter 15 to determine the stock price distribution at time T in terms of r_1, r_2, σ_1, σ_2, t_1, t_2, and S_0.*
(b) *Suppose that $\bar{r}$ is the average interest rate between time zero and T and that $\bar{V}$ is the average variance rate between times zero and T. What is the stock price distribution as a function of T in terms of $\bar{r}$, $\bar{V}$, T, and S_0?*
(c) *What are the results corresponding to (a) and (b) when there are three subintervals with different interest rates and volatilities?*
(d) *Show that if the risk-free rate, r, and the volatility, σ, are known functions of time, the stock price distribution at time T in a risk-neutral world is*

$$\ln S_T \sim \varphi\left[\ln S_0 + \left(\bar{r} - \frac{\bar{V}}{2}\right)T, \bar{V}T\right]$$

where $\bar{r}$ is the average value of r, $\bar{V}$ is equal to the average value of σ^2, and S_0 is the stock price today.

(a) Suppose that S_1 is the stock price at time t_1 and S_T is the stock price at time T. From equation (15.3), it follows that in a risk- neutral world:

$$\ln S_1 - \ln S_0 \sim \varphi\left[\left(r_1 - \frac{\sigma_1^2}{2}\right)t_1, \sigma_1^2 t_1\right]$$

$$\ln S_T - \ln S_1 \sim \varphi\left[\left(r_2 - \frac{\sigma_2^2}{2}\right)t_2, \sigma_2^2 t_2\right]$$

Since the sum of two independent normal distributions is normal with mean equal to the sum of the means and variance equal to the sum of the variances

$$\ln S_T - \ln S_0 = (\ln S_T - \ln S_1) + (\ln S_1 - \ln S_0)$$
$$\sim \varphi\left(r_1 t_1 + r_2 t_2 - \frac{\sigma_1^2 t_1}{2} - \frac{\sigma_2^2 t_2}{2}, \sigma_1^2 t_1 + \sigma_2^2 t_2 \right)$$

(b) Because

$$r_1 t_1 + r_2 t_2 = \bar{r} T$$

and

$$\sigma_1^2 t_1 + \sigma_2^2 t_2 = \bar{V} T$$

it follows that:

$$\ln S_T - \ln S_0 \sim \varphi\left[\left(\bar{r} - \frac{\bar{V}}{2} \right) T, \bar{V} T \right]$$

(c) If σ_i and r_i are the volatility and risk-free interest rate during the i th subinterval $(i = 1, 2, 3)$, an argument similar to that in (a) shows that:

$$\ln S_T - \ln S_0 \sim \varphi\left(r_1 t_1 + r_2 t_2 + r_3 t_3 - \frac{\sigma_1^2 t_1}{2} - \frac{\sigma_2^2 t_2}{2} - \frac{\sigma_3^2 t_3}{2}, \sigma_1^2 t_1 + \sigma_2^2 t_2 + \sigma_3^2 t_3 \right)$$

where t_1, t_2 and t_3 are the lengths of the three subintervals. It follows that the result in (b) is still true.

(d) The result in (b) remains true as the time between time zero and time T is divided into more subintervals, each having its own risk-free interest rate and volatility. In the limit, it follows that, if r and σ are known functions of time, the stock price distribution at time T is the same as that for a stock with a constant interest rate and variance rate with the constant interest rate equal to the average interest rate and the constant variance rate equal to the average variance rate.

Problem 27.7.
Write down the equations for simulating the path followed by the asset price in the stochastic volatility model in equation (27.2) and (27.3).

The equations are:

$$S(t + \Delta t) = S(t) \exp[(r - q - V(t)/2)\Delta t + \varepsilon_1 \sqrt{V(t)\Delta t}]$$

$$V(t + \Delta t) - V(t) = a[V_L - V(t)]\Delta t + \xi \varepsilon_2 V(t)^{\alpha} \sqrt{\Delta t}$$

Problem 27.8.
"The IVF model does not necessarily get the evolution of the volatility surface correct." Explain this statement.

The IVF model is designed to match the volatility surface today. There is no guarantee that the volatility surface given by the model at future times will reflect the true evolution of the volatility surface.

Problem 27.9.

"When interest rates are constant the IVF model correctly values any derivative whose payoff depends on the value of the underlying asset at only one time." Explain why.

The IVF model ensures that the risk-neutral probability distribution of the asset price at any future time conditional on its value today is correct (or at least consistent with the market prices of options). When a derivative's payoff depends on the value of the asset at only one time the IVF model therefore calculates the expected payoff from the asset correctly. The value of the derivative is the present value of the expected payoff. When interest rates are constant the IVF model calculates this present value correctly.

Problem 27.10.

Use a three-time-step tree to value an American floating lookback call option on a currency when the initial exchange rate is 1.6, the domestic risk-free rate is 5% per annum, the foreign risk-free interest rate is 8% per annum, the exchange rate volatility is 15%, and the time to maturity is 18 months. Use the approach in Section 27.5.

In this case $S_0 = 1.6$, $r = 0.05$, $r_f = 0.08$, $\sigma = 0.15$, $T = 1.5$, $\Delta t = 0.5$. This means that

$$u = e^{0.15\sqrt{0.5}} = 1.1119$$

$$d = \frac{1}{u} = 0.8994$$

$$a = e^{(0.05-0.08)\times 0.5} = 0.9851$$

$$p = \frac{a-d}{u-d} = 0.4033$$

$$1 - p = 0.5967$$

The option pays off

$$S_T - S_{\min}$$

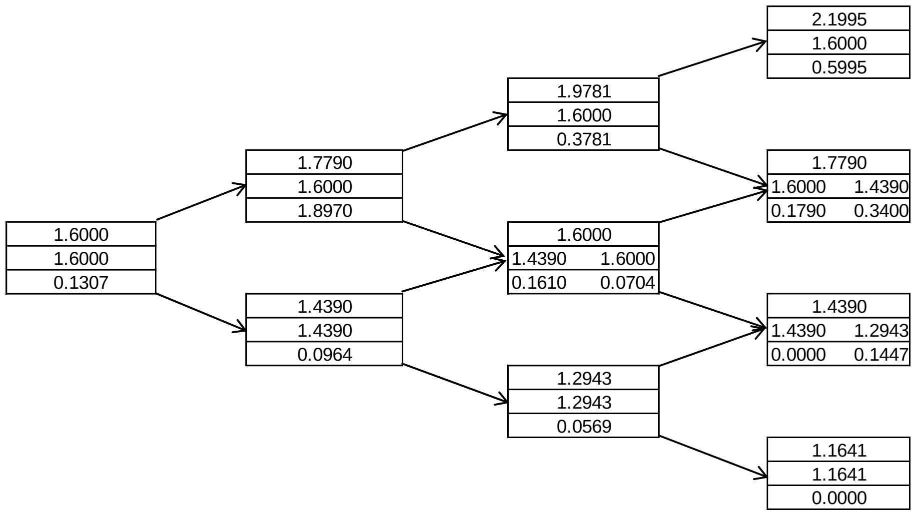


Figure S27.1: Binomial tree for Problem 27.10.

The tree is shown in Figure S27.1. At each node, the upper number is the exchange rate, the middle number(s) are the minimum exchange rate(s) so far, and the lower number(s) are the value(s) of the option. The tree shows that the value of the option today is 0.131.

Problem 27.11.
What happens to the variance-gamma model as the parameter v tends to zero?

As v tends to zero the value of g becomes T with certainty. This can be demonstrated using the GAMMADIST function in Excel. By using a series expansion for the $\ln$ function we see that ω becomes $-\theta T$. In the limit the distribution of $\ln S_T$ therefore has a mean of $\ln S_0 + (r-q)T$ and a standard deviation of $\sigma\sqrt{T}$ so that the model becomes geometric Brownian motion.

Problem 27.12.
Use a three-time-step tree to value an American put option on the geometric average of the price of a non-dividend-paying stock when the stock price is $40, the strike price is $40, the risk-free interest rate is 10% per annum, the volatility is 35% per annum, and the time to maturity is three months. The geometric average is measured from today until the option matures.

In this case $S_0 = 40$, $K = 40$, $r = 0.1$, $\sigma = 0.35$, $T = 0.25$, $\Delta t = 0.08333$. This means that

$$u = e^{0.35\sqrt{0.08333}} = 1.1063$$

$$d = \frac{1}{u} = 0.9039$$

$$a = e^{0.1\times 0.08333} = 1.008368$$

$$p = \frac{a-d}{u-d} = 0.5161$$

$$1 - p = 0.4839$$

The option pays off

$$40 - \bar{S}$$

where $\bar{S}$ denotes the geometric average. The tree is shown in Figure S27.2. At each node, the upper number is the stock price, the middle number(s) are the geometric average(s), and the lower number(s) are the value(s) of the option. The geometric averages are calculated using the first, the last and all intermediate stock prices on the path. The tree shows that the value of the option today is $1.40.

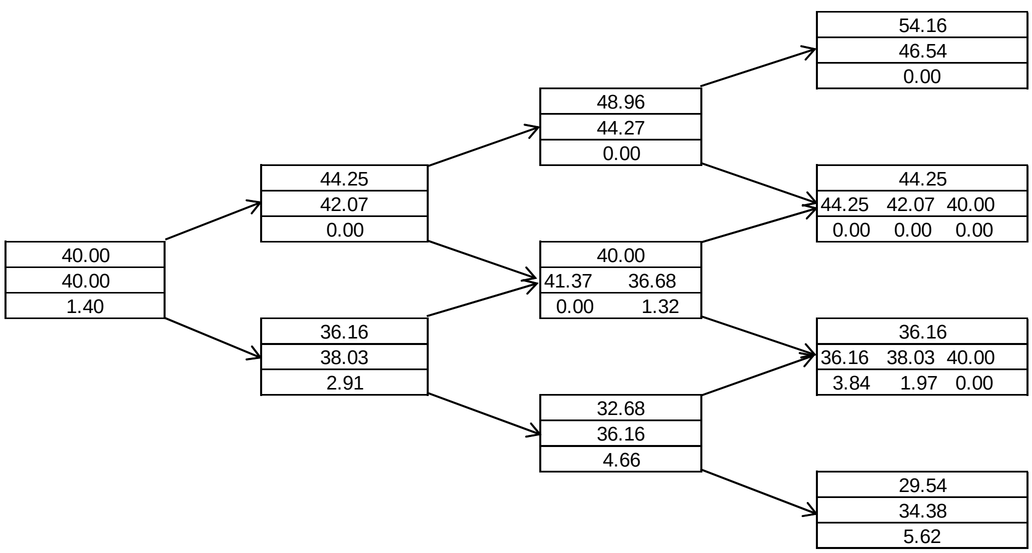


Figure S27.2: Binomial tree for Problem 27.12.

Problem 27.13.

Can the approach for valuing path dependent options in Section 27.5 be used for a two-year American-style option that provides a payoff equal to $\max(S_{\text{ave}} - K, 0)$ *where* S_{ave} *is the average asset price over the three months preceding exercise? Explain your answer.*

As mentioned in Section 27.5, for the procedure to work it must be possible to calculate the value of the path function at time $\tau + \Delta t$ from the value of the path function at time τ and the value of the underlying asset at time $\tau + \Delta t$. When S_{ave} is calculated from time zero until the end of the life of the option (as in the example considered in Section 27.5) this condition is satisfied. When it is calculated over the last three months it is not satisfied. This is because, in order to update the average with a new observation on S, it is necessary to know the observation on S from three months ago that is now no longer part of the average calculation.

Problem 27.14.

Verify that the 6.492 number in Figure 27.4 is correct.

We consider the situation where the average at node X is 53.83. If there is an up movement to node Y the new average becomes:

$$\frac{53.83\times 5+54.68}{6}=53.97$$

Interpolating, the value of the option at node Y when the average is 53.97 is

$$\frac{(53.97-51.12)\times 8.635+(54.26-53.97)\times 8.101}{54.26-51.12}=8.586$$

Similarly if there is a down movement the new average will be

$$\frac{53.83\times 5+45.72}{6}=52.48$$

In this case the option price is 4.416. The option price at node X when the average is 53.83 is therefore:

$$8.586\times 0.5056+4.416\times 0.4944)e^{-0.1\times 0.05}=6.492$$

Problem 27.15.

Examine the early exercise policy for the eight paths considered in the example in Section 27.8. What is the difference between the early exercise policy given by the least squares approach and the exercise boundary parameterization approach? Which gives a higher option price for the paths sampled?

Under the least squares approach we exercise at time $t=1$ in paths 4, 6, 7, and 8. We exercise at time $t=2$ for none of the paths. We exercise at time $t=3$ for path 3. Under the exercise boundary parameterization approach we exercise at time $t=1$ for paths 6 and 8. We exercise at time $t=2$ for path 7. We exercise at time $t=3$ for paths 3 and 4. For the paths sampled the exercise boundary parameterization approach gives a higher value for the option. However, it may be biased upward. As mentioned in the text, once the early exercise boundary has been determined in the exercise boundary parameterization approach a new Monte Carlo simulation should be carried out.

Problem 27.16.
Consider a European put option on a non-dividend paying stock when the stock price is $100, the strike price is $110, the risk-free rate is 5% per annum, and the time to maturity is one year. Suppose that the average variance rate during the life of an option has a 0.20 probability of being 0.06, a 0.5 probability of being 0.09, and a 0.3 probability of being 0.12. The volatility is uncorrelated with the stock price. Estimate the value of the option. Use DerivaGem.

If the average variance rate is 0.06, the value of the option is given by Black-Scholes with a volatility of $\sqrt{0.06} = 24.495\%$; it is 12.460. If the average variance rate is 0.09, the value of the option is given by Black-Scholes with a volatility of $\sqrt{0.09} = 30.000\%$; it is 14.655. If the average variance rate is 0.12, the value of the option is given by Black-Scholes-Merton with a volatility of $\sqrt{0.12} = 34.641\%$; it is 16.506. The value of the option is the Black-Scholes-Merton price integrated over the probability distribution of the average variance rate. It is

$$0.2\times 12.460 + 0.5\times 14.655 + 0.3\times 16.506 = 14.77$$

Problem 27.17.
When there are two barriers how can a tree be designed so that nodes lie on both barriers?

Suppose that there are two horizontal barriers, H_1 and H_2, with $H_1 < H_2$ and that the underlying stock price follows geometric Brownian motion. In a trinomial tree, there are three possible movements in the asset's price at each node: up by a proportional amount u; stay the same; and down by a proportional amount d where $d = 1/u$. We can always choose u so that nodes lie on both barriers. The condition that must be satisfied by u is

$$H_2 = H_1 u^N$$

or

$$\ln H_2 = \ln H_1 + N \ln u$$

for some integer N.

When discussing trinomial trees in Section 21.4, the value suggested for u was $e^{\sigma\sqrt{3\Delta t}}$ so that $\ln u = \sigma\sqrt{3\Delta t}$. In the situation considered here, a good rule is to choose $\ln u$ as close as possible to this value, consistent with the condition given above. This means that we set

$$\ln u = \frac{\ln H_2 - \ln H_1}{N}$$

where

$$N = \mathrm{int}\left[\frac{\ln H_2 - \ln H_1}{\sigma\sqrt{3\Delta t}} + 0.5\right]$$

and $\mathrm{int}(x)$ is the integral part of x. This means that nodes are at values of the stock price equal to H_1, $H_1 u$, $H_1 u^2$, …, $H_1 u^N = H_2$

Normally the trinomial stock price tree is constructed so that the central node is the initial stock price. In this case, it is unlikely that the current stock price happens to be H_1u^i for some i. To deal with this, the first trinomial movement should be from the initial stock price to H_1u^{i-1}, H_1u^i and H_1u^{i+1} where i is chosen so that H_1u^i is closest to the current stock price. The probabilities on all branches of the tree are chosen, as usual, to match the first two moments of the stochastic process followed by the asset price. The approach works well except when the initial asset price is close to a barrier.

Problem 27.18.

Consider an 18-month zero-coupon bond with a face value of $100 that can be converted into five shares of the company's stock at any time during its life. Suppose that the current share price is $20, no dividends are paid on the stock, the risk-free rate for all maturities is 6% per annum with continuous compounding, and the share price volatility is 25% per annum. Assume that the hazard rate is 3% per year and the recovery rate is 35%. The bond is callable at $110. Use a three-time-step tree to calculate the value of the bond. What is the value of the conversion option (net of the issuer's call option)?

In this case $\Delta t = 0.5$, $\lambda = 0.03$, $\sigma = 0.25$, $r = 0.06$ and $q = 0$ so that $u = 1.1360$, $d = 0.8803$, $a = 1.0305$, $p_u = 0.6386$, $p_d = 0.3465$, and the probability on default branches is 0.0149. This leads to the tree shown in Figure S27.3. The bond is called at nodes B and D and this forces exercise. Without the call the value at node D would be 129.55, the value at node B would be 115.94, and the value at node A would be 105.18. The value of the call option to the bond issuer is therefore $105.18 - 103.72 = 1.46$.

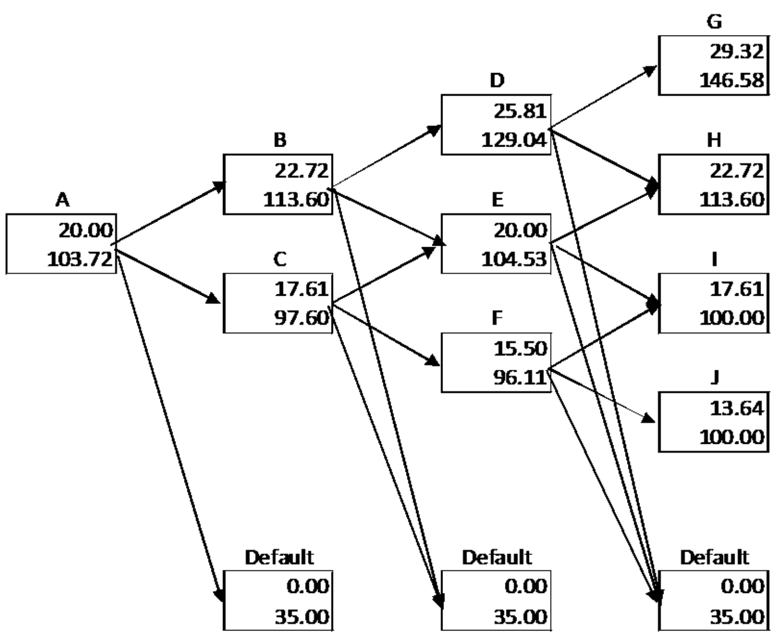


Figure S27.3: Tree for Problem 27.18

Further Questions

Problem 27.19.

A new European-style floating lookback call option on a stock index has a maturity of nine months. The current level of the index is 400, the risk-free rate is 6% per annum, the dividend yield on the index is 4% per annum, and the volatility of the index is 20%. Use the approach in Section 27.5 to value the option and compare your answer to the result given by DerivaGem using the analytic valuation formula.

Using three-month time steps the tree parameters are Δt=0.25, $u = 1.1052$, $d = 0.9048$, $a = 1.0050$, $p = 0.5000$. The tree is shown in Figure S27.4. The value of the floating lookback option is 40.47. (A more efficient procedure for giving the same result is in Technical Note 13.) We construct a tree for $Y(t) = G(t)/S(t)$ where $G(t)$ is the minimum value of the index to date and $S(t)$ is the value of the index at time t. The tree is shown in Figure S27.5. It values the option in units of the stock index. This means that we value an instrument that pays off $1 - Y(t)$. The tree shows that the value of the option is 0.1012 units of the stock index or 400×0.1012 or 40.47 dollars, as given by Figure S27.4. DerivaGem shows that the value given by the analytic formula is 53.38. This is higher than the value given by the tree because the tree assumes that the stock price is observed only three times when the minimum is calculated.

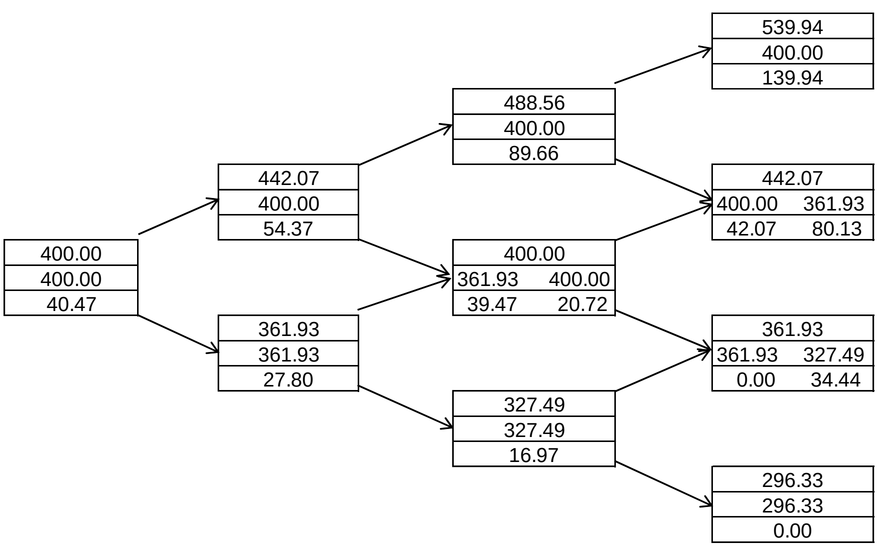


Figure S27.4: Tree for Problem 27.19.

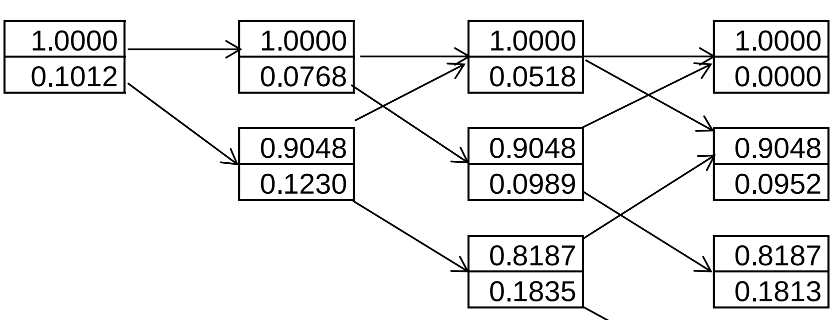


Figure S27.5 Tree for Problem 27.19 Using Results in Technical Note 13.

Problem 27.20.

Suppose that the volatilities used to price a six-month currency option are as in Table 20.2. Assume that the domestic and foreign risk-free rates are 5% per annum and the current exchange rate is 1.00. Consider a bull spread that consists of a long position in a six-month call option with strike price 1.05 and a short position in a six-month call option with a strike price 1.10.

(a) What is the value of the spread?
(b) What single volatility if used for both options gives the correct value of the bull spread? (Use the DerivaGem Application Builder in conjunction with Goal Seek or Solver.)
(c) Does your answer support the assertion at the beginning of the chapter that the correct volatility to use when pricing exotic options can be counterintuitive?
(d) Does the IVF model give the correct price for the bull spread?

(a) The six-month call option with a strike price of 1.05 should be valued with a volatility of 13.4% and is worth 0.01829. The call option with a strike price of 1.10 should be valued with a volatility of 14.3% and is worth 0.00959. The bull spread is therefore worth $0.01829 - 0.00959 = 0.00870$.

(b) We now ask what volatility, if used to value both options, gives this price. Using the DerivaGem Application Builder in conjunction with Goal Seek we find that the answer is 11.42%.

(c) Yes, this does support the contention at the beginning of the chapter that the correct volatility for valuing exotic options can be counterintuitive. We might reasonably expect the volatility to be between 13.4% (the volatility used to value the first option) and 14.3% (the volatility used to value the second option). 11.42% is well outside this range. The reason why the volatility is relatively low is as follows. The option provides the same payoff as a regular option with a 1.05 strike price when the asset price is between 1.05 and 1.10 and a lower payoff when the asset price is over 1.10. The implied probability distribution of the asset price (see Figure 20.2) is less heavy than the lognormal distribution in the 1.05 to 1.10 range and heavier than the lognormal distribution in the >1.10 range. This means that using a volatility of 13.4% (which is the implied volatility of a regular option with a strike price of 1.05) will give a price than is too high.

(d) The bull spread provides a payoff at only one time. It is therefore correctly valued by the IVF model.

Problem 27.21.

Repeat the analysis in Section 27.8 for the put option example on the assumption that the strike price is 1.13. Use both the least squares approach and the exercise boundary parameterization approach.

Consider first the least squares approach. At the two-year point the option is in the money for paths 1, 3, 4, 6, and 7. The five observations on S are 1.08, 1.07, 0.97, 0.77, and 0.84. The five continuation values are 0, $0.10e^{-0.06}$, $0.21e^{-0.06}$, $0.23e^{-0.06}$, $0.12e^{-0.06}$. The best fit continuation value is

$$-1.394+3.795S-2.276S^2$$

The best fit continuation values for the five paths are 0.0495, 0.0605, 0.1454, 0.1785, and 0.1876. These show that the option should be exercised for paths 1, 4, 6, and 7 at the two-year point. There are six paths at the one-year point for which the option is in the money. These are paths 1, 4, 5, 6, 7, and 8. The six observations on S are 1.09, 0.93, 1.11, 0.76, 0.92, and 0.88. The six continuation values are $0.05e^{-0.06}$, $0.16e^{-0.06}$, 0, $0.36e^{-0.06}$, $0.29e^{-0.06}$, and 0. The best fit continuation value is

$$2.055-3.317S+1.341S^2$$

The best fit continuation values for the six paths are 0.0327, 0.1301, 0.0253, 0.3088, 0.1385, and 0.1746. These show that the option should be exercised at the one-year point for paths 1, 4, 6, 7, and 8 The value of the option if not exercised at time zero is therefore

$$\frac{1}{8}(0.04e^{-0.06}+0+0.10e^{-0.18}+0.20e^{-0.06}+0+0.37e^{-0.06}+0.21e^{-0.06}+0.25e^{-0.06})$$

or 0.136. Exercising at time zero would yield 0.13. The option should therefore not be exercised at time zero and its value is 0.136.

Consider next the exercise boundary parameterization approach. At time two years it is optimal to exercise when the stock price is 0.84 or below. At time one year it is optimal to exercise whenever the option is in the money. The value of the option assuming no early exercise at time zero is therefore

$$\frac{1}{8}(0.04e^{-0.06}+0+0.10e^{-0.018}+0.20e^{-0.06}+0.02e^{-0.06}$$
$$+0.37e^{-0.06}+0.21e^{-0.06}+0.25e^{-0.06})$$

or 0.139. Exercising at time zero would yield 0.13. The option should therefore not be exercised at time zero. The value at time zero is 0.139. However, this tends to be high. As explained in the text, we should use one Monte Carlo simulation to determine the early exercise boundary. We should then carry out a new Monte Carlo simulation using the early exercise boundary to value the option.

Problem 27.22.

A European call option on a non-dividend-paying stock has a time to maturity of 6 months and a strike price of $100. The stock price is $100 and the risk-free rate is 5%. Use DerivaGem to answer the following questions.

(a) What is the Black-Scholes-Merton price of the option if the volatility is 30%.

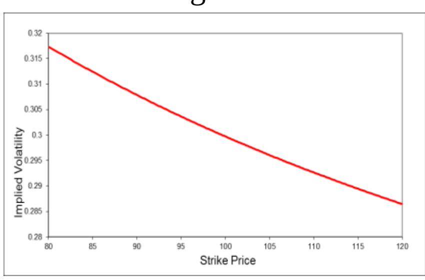
(b) What is the CEV volatility parameter that gives the same price for the option as you calculated in (a) when α=0.5?

(c) In Merton's mixed jump diffusion model the average frequency of jumps is one per year, the average percentage jump size is 2%, and the standard deviation of the logarithm of one plus the percentage jump size is 20%. What is the volatility of the diffusion part of the process that gives the same price for the option as you calculated in (a)?

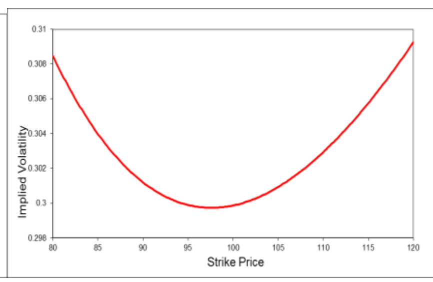
(d) In the variance -gamma model, θ=0 and v=40%. What value of the volatility gives the same price for the option as you calculated in (a)?

(e) For the models you have developed in (b), (c) and (d), calculate the volatility smile by considering European call options with strike prices between 80 and 120. Describe the nature of the probability distributions implied by the smiles.

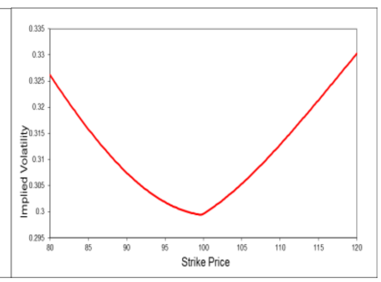
(a) 9.63
(b) 3.0
(c) 23.75%
(d) 33%
(e) The volatility smiles are shown below. The CEV model gives a heavy left tail and thin right tail relative to the lognormal distribution and might be appropriate for equities. The other two models give heavier left and right tails and might be appropriate for foreign exchange.



CEV



Merton's Jump Diffusion



Variance-Gamma

Problem 27.23.

A three-year convertible bond with a face value of $100 has been issued by company ABC. It pays a coupon of $5 at the end of each year. It can be converted into ABC's equity at the end of the first year or at the end of the second year. At the end of the first year, it can be exchanged for 3.6 shares immediately after the coupon date. At the end of the second year it can be exchanged for 3.5 shares immediately after the coupon date. The current stock price is $25 and the stock price volatility is 25%. No dividends are paid on the stock. The risk-free interest rate is 5% with continuous compounding. The yield on bonds issued by ABC is 7% with continuous compounding and the recovery rate is 30%.

(a) Use a three-step tree to calculate the value of the bond

(b) How much is the conversion option worth?

(c) What difference does it make to the value of the bond and the value of the conversion option if the bond is callable any time within the first two years for \$115?
(d) Explain how your analysis would change if there were a dividend payment of \$1 on the equity at the six month, 18-month, and 30-month points. Detailed calculations are not required.
(Hint: Use equation (24.2) to estimate the average hazard rate.)

In this case $\Delta t = 1$, $\lambda = 0.02/0.7 = 0.02857$, $\sigma = 0.25$, $r = 0.05$, $q = 0$, $u = 1.2023$, $d = 0.8318$, $a = 1.0513$, $p_u = 0.6557$, $p_d = 0.3161$, and the probability of a default is 0.0282. The calculations are shown in Figure S27.6. The values at the nodes include the value of the coupon paid just before the node is reached. The value of the convertible is 105.21. The value if there is no conversion is calculated by working out the present value of the coupons and principal at 7%. It is 94.12. The value of the conversion option is therefore 11.09. Calling at node D makes no difference because the bond will be converted at that node anyway. Calling at node B (just before the coupon payment) does make a difference. It reduces the value of the convertible at node B to \$115. The value of the bond at node A is reduced by 2.34. This is a reduction in the value of the conversion option. A dividend payment would affect the way the tree is constructed as described in Chapter 21.

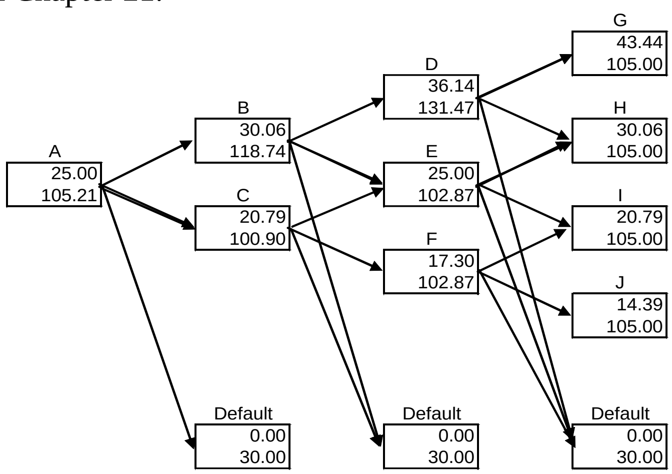


Figure S27.6: Tree for Problem 27.23