5.4.2 Multidimensional Market Model

We assume there are m stocks, each with stochastic differential

$$dS_i(t) = \alpha_i(t)S_i(t)\,dt + S_i(t)\sum_{j=1}^{d}\sigma_{ij}(t)\,dW_j(t), \quad i = 1,\ldots,m. \qquad (5.4.6)$$

We assume that the mean rate of return vector $(\alpha_i(t))_{i=1,\ldots,m}$ and the volatility matrix $(\sigma_{ij}(t))_{i=1,\ldots,m;j=1,\ldots,d}$ are adapted processes. These stocks are typically correlated. To see the nature of this correlation, we set $\sigma_i(t) = \sqrt{\sum_{j=1}^{d}\sigma_{ij}^2(t)}$, which we assume is never zero, and we define processes

$$B_i(t) = \sum_{j=1}^{d}\int_0^t \frac{\sigma_{ij}(u)}{\sigma_i(u)}\,dW_j(u), \quad i = 1,\ldots,m. \qquad (5.4.7)$$

Being a sum of stochastic integrals, each $B_i(t)$ is a continuous martingale. Furthermore,

$$dB_i(t)\,dB_i(t) = \sum_{j=1}^{d}\frac{\sigma_{ij}^2(t)}{\sigma_i^2(t)}\,dt = dt.$$

According to Lévy's Theorem, Theorem 4.6.4, $B_i(t)$ is a Brownian motion. We may rewrite (5.4.6) in terms of the Brownian motion $B_i(t)$ as

$$dS_i(t) = \alpha_i(t)S_i(t)\,dt + \sigma_i(t)S_i(t)\,dB_i(t). \qquad (5.4.8)$$

From this formula, we see that $\sigma_i(t)$ is the volatility of $S_i(t)$.

For $i \neq k$, the Brownian motions $B_i(t)$ and $B_k(t)$ are typically not independent. To see this, we first note that

$$dB_i(t)\,dB_k(t) = \sum_{j=1}^{d}\frac{\sigma_{ij}(t)\sigma_{kj}(t)}{\sigma_i(t)\sigma_k(t)}\,dt = \rho_{ik}(t)\,dt, \qquad (5.4.9)$$

where

$$\rho_{ik}(t) = \frac{1}{\sigma_i(t)\sigma_k(t)}\sum_{j=1}^{d}\sigma_{ij}(t)\sigma_{kj}(t). \qquad (5.4.10)$$

Itô's product rule implies

$$d\big(B_i(t)B_k(t)\big) = B_i(t)\,dB_k(t) + B_k(t)\,dB_i(t) + dB_i(t)\,dB_k(t),$$

and integration of this equation yields

$$B_i(t)B_k(t) = \int_0^t B_i(u)\,dB_k(u) + \int_0^t B_k(u)\,dB_i(u) + \int_0^t \rho_{ik}(u)\,du. \qquad (5.4.11)$$

Taking expectations and using the fact that the expectation of an Itô integral is zero, we obtain the covariance formula

$$\operatorname{Cov}\big[B_i(t), B_k(t)\big] = \mathbb{E}\int_0^t \rho_{ik}(u)\,du. \tag{5.4.12}$$

If the processes $\sigma_{ij}(t)$ and $\sigma_{kj}(t)$ are constant (i.e., independent of t and not random), then so are $\sigma_i(t)$, $\sigma_k(t)$, and $\rho_{ik}(t)$. In this case, (5.4.12) reduces to $\operatorname{Cov}\big[B_i(t), B_k(t)\big] = \rho_{ik}t$. Because both $B_i(t)$ and $B_k(t)$ have standard deviation $\sqrt{t}$, the correlation between $B_i(t)$ and $B_j(t)$ is simply ρ_{ik}. When $\sigma_{ij}(t)$ and $\sigma_{kj}(t)$ are themselves random processes, we call $\rho_{ik}(t)$ the *instantaneous correlation* between $B_i(t)$ and $B_k(t)$. At a fixed time t along a particular path, $\rho_{ik}(t)$ is the conditional correlation between the next increments of B_i and B_k over a "small" time interval following time t (see Exercise 4.17 of Chapter 4 with $\Theta_1 = \Theta_2 = 0$, $\sigma_1 = \sigma_2 = 1$).

Finally, we note from (5.4.8) and (5.4.9) that

$$\begin{aligned} dS_i(t)\,dS_k(t) &= \sigma_i(t)\sigma_k(t)S_i(t)S_k(t)\,dB_i(t)\,dB_k(t) \\ &= \rho_{ik}(t)\sigma_i(t)\sigma_k(t)S_i(t)S_k(t)\,dt. \end{aligned} \tag{5.4.13}$$

Rewriting (5.4.13) in terms of "relative differentials," we obtain

$$\frac{dS_i(t)}{S_i(t)} \cdot \frac{dS_k(t)}{S_k(t)} = \rho_{ik}(t)\sigma_i(t)\sigma_k(t)\,dt.$$

The volatility processes $\sigma_i(t)$ and $\sigma_k(t)$ are the respective *instantaneous standard deviations* of the relative changes in S_i and S_k at time t, and the process $\rho_{ik}(t)$ is the *instantaneous correlation* between these relative changes.

Mean rates of return are affected by the change to a risk-neutral measure in the next subsection. Instantaneous standard deviations and correlations are unaffected (Exercise 5.12(ii) and (iii)). If the instantaneous standard deviations and correlations are not random, then (noninstantaneous) standard deviations and correlations are unaffected by the change of measure (see Exercise 5.12(iv) for the case of correlations). However, (noninstantaneous) standard deviations and correlations can be affected by a change of measure when the instantaneous standard deviations and correlations are random (see Exercises 5.12(v) and 5.13 for the case of correlations).

We define a *discount process*

$$D(t) = e^{-\int_0^t R(u)\,du}. \tag{5.4.14}$$

We assume that the interest rate process $R(t)$ is adapted. In addition to stock prices, we shall often work with discounted stock prices. Their differentials are