# 第5章 区间估计

《统计推断》 第9章

感谢清华大学自动化系江瑞教授提供PPT

# 内容

- 区间估计定义
- 区间估计构造方法
  - 反转检验统计量
  - 枢轴化(Pivotal)
  - 枢轴化累积分布
  - Bayesian区间估计
- 区间估计的评价方法

# Point estimation

- Our knowledge about the parameter before observing the data

$$\theta \in (-\infty, \infty)$$

- After seeing the data, we made a decision

$$\theta = W(\mathbf{x})$$

  Shrank the parameter space from $(-\infty, \infty)$ to a single point
- The highest possible precision

$$L(W(\mathbf{x}) \mid \mathbf{x}) \geq L(\theta \mid \mathbf{x}) \text{ for any } \theta \in \Theta$$

- The lowest confidence

$$P(W(\mathbf{X}) = \theta) = 0$$

# Interval estimation

- Our knowledge about the parameter before observing the data

$$\theta \in (-\infty, \infty)$$

- After seeing the data, we made a decision

$$L(\mathbf{x}) \leq \theta \leq U(\mathbf{x})$$

  Shrank the parameter space from $(-\infty, \infty)$ to an interval

- The moderate precision

$$L(W(\mathbf{x}) \mid \mathbf{x}) \geq L(\theta \mid \mathbf{x}) \text{ for any } \theta \in [L(\mathbf{x}), U(\mathbf{x})]$$

- The gained confidence

$$P(L(\mathbf{X}) \leq \theta \leq U(\mathbf{X})) = P(\theta \geq L(\mathbf{X}) \text{ and } \theta \leq U(\mathbf{X})) = 1 - \alpha$$

相对于点估计，区间估计希望牺牲参数的“精确性”，来换取更大的“可靠性”

# Interval estimator

***Interval estimator***

An **interval estimate** of a real-valued parameter $\theta$ is any pair of functions, $L(x_1,\dots,x_n)$ and $U(x_1,\dots,x_n)$, of a sample that satisfy $L(\mathbf{x}) \le U(\mathbf{x})$ for all $\mathbf{x} \in \mathcal{X}$. If $\mathbf{X} = \mathbf{x}$ is observed, the inference $L(\mathbf{x}) \le \theta \le U(\mathbf{x})$ is made. The **random interval** $[L(\mathbf{X}), U(\mathbf{X})]$ is called an **interval estimator** for $\theta$.

An interval estimator is typically composed of TWO statistics.
An interval estimate is a pair of real numbers.

**Compare them with point estimate and point estimator**

# Various forms of interval estimator

Two-sided:

$$[L(\mathbf{X}), U(\mathbf{X})]$$

One-sided:

$$[L(\mathbf{X}), \infty),\ (-\infty, U(\mathbf{X})]$$

Closed interval:

$$[L(\mathbf{X}), U(\mathbf{X})]$$

Open interval:

$$(L(\mathbf{X}), U(\mathbf{X})),\ (L(\mathbf{X}), \infty),\ (-\infty, U(\mathbf{X}))$$

Half open interval:

$$[L(\mathbf{X}), U(\mathbf{X})),\ (L(\mathbf{X}), U(\mathbf{X})],\ [L(\mathbf{X}), \infty),\ (-\infty, U(\mathbf{X})]$$

# An example

A random sample $X_1,\ldots,X_n$ is obtained from a $N(\mu,\sigma^2)$ population, where $\sigma^2$ is known. Consider the following interval estimator for $\mu$

$$[L(\mathbf{X}),\ \ U(\mathbf{X})] = [\bar{X} - z,\ \ \bar{X} + z]$$

Because the best unbiased point estimator $\bar{X}$ has a $N(\mu,\sigma^2/n)$ distribution,

$$P(\bar{X} = \mu) = 0.$$

But for the interval estimator,

$$\begin{aligned} P(\bar{X} - z \le \mu \le \bar{X} + z) &= P(-z \le \bar{X} - \mu \le z) \\ &= P\left(-\frac{z}{\sigma/\sqrt{n}} \le \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \le \frac{z}{\sigma/\sqrt{n}}\right) \\ &= \Phi\left(\frac{z}{\sigma/\sqrt{n}}\right) - \Phi\left(-\frac{z}{\sigma/\sqrt{n}}\right) \end{aligned}$$

By choosing $z = 3\sigma/\sqrt{n}$, we have

$$P\left(\bar{X} - z \le \mu \le \bar{X} + z\right) = \Phi(3) - \Phi(-3) = 0.9973.$$

The **random interval** has over 99% chance to cover the true parameter.

# Coverage probability

For an interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$ of a parameter $\theta$, the **coverage probability** of $[L(\mathbf{X}), U(\mathbf{X})]$ is the probability that the random interval $[L(\mathbf{X}), U(\mathbf{X})]$ covers the true parameter.
The coverage probability is denoted by $P_\theta(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$.

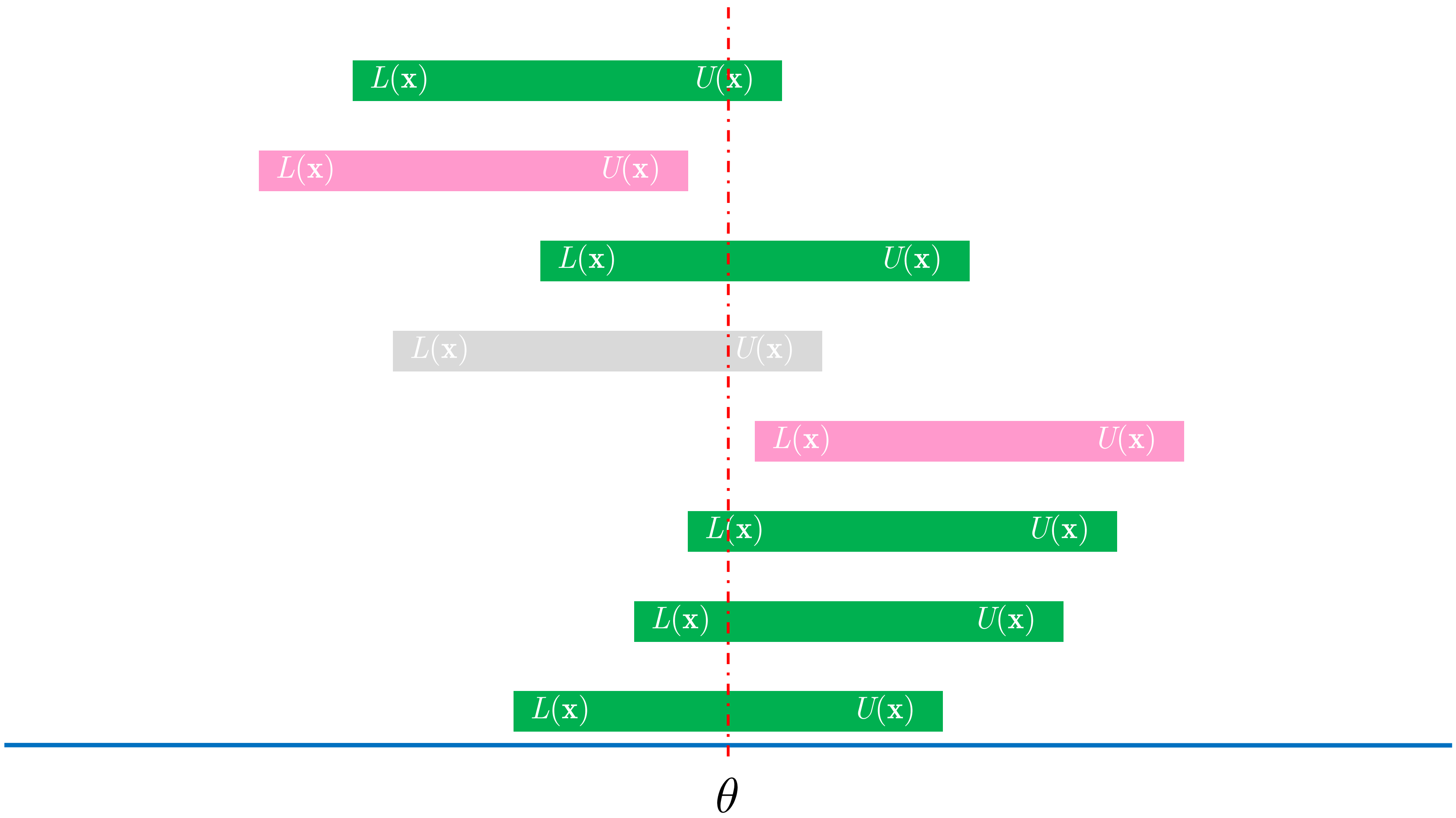
# Confidence Coefficient

The **confidence coefficient** of $[L(\mathbf{X}), U(\mathbf{X})]$ is the infinum of the coverage probability, say,

$$\inf_{\theta \in \Theta} P_\theta(\theta \in [L(\mathbf{X}), U(\mathbf{X})])$$

An interval estimator $[L(\mathbf{X}), U(\mathbf{X})]$, together with its confidence coefficient, is called a **confidence interval**. A confidence interval with the confidence coefficient $1-\alpha$ is called a $1-\alpha$ confidence interval.

# Random interval, fixed parameter



# Inverting test statistics

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*"The random interval [L(X), U(X)] is the interval estimator of the parameter, such that the probability of parameter located in the interval with high probability."*

# Inverting a test statistic

Let $X_1,\ldots,X_n$ be a random sample from a normal population $N(\mu,\sigma^2)$, where $\sigma^2$ is known. Consider the hypothesis testing problem

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0$$

The level $\alpha$ uniformly most powerful (UMP) unbiased test for this problem is

$$R = \left\{ \mathbf{x} : \frac{|\,\bar{x} - \mu_0\,|}{\sigma / \sqrt{n}} > z_{\alpha/2} \right\} \Rightarrow A = R^c = \left\{ \mathbf{x} : \mu_0 - z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \leq \bar{x} \leq \mu_0 + z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \right\}$$

Level $\alpha$ (actually size $\alpha$) means that

$$P(\mathbf{X} \in R \mid \mu = \mu_0) \leq \alpha \;\Rightarrow P(\mathbf{X} \in A \mid \mu = \mu_0) \geq 1 - \alpha \;\Rightarrow$$

$$P(\mu_0 - z_{\alpha/2}\sigma / \sqrt{n} \leq \bar{X} \leq \mu_0 + z_{\alpha/2}\sigma / \sqrt{n} \mid \mu = \mu_0) \geq 1 - \alpha \;\Rightarrow$$

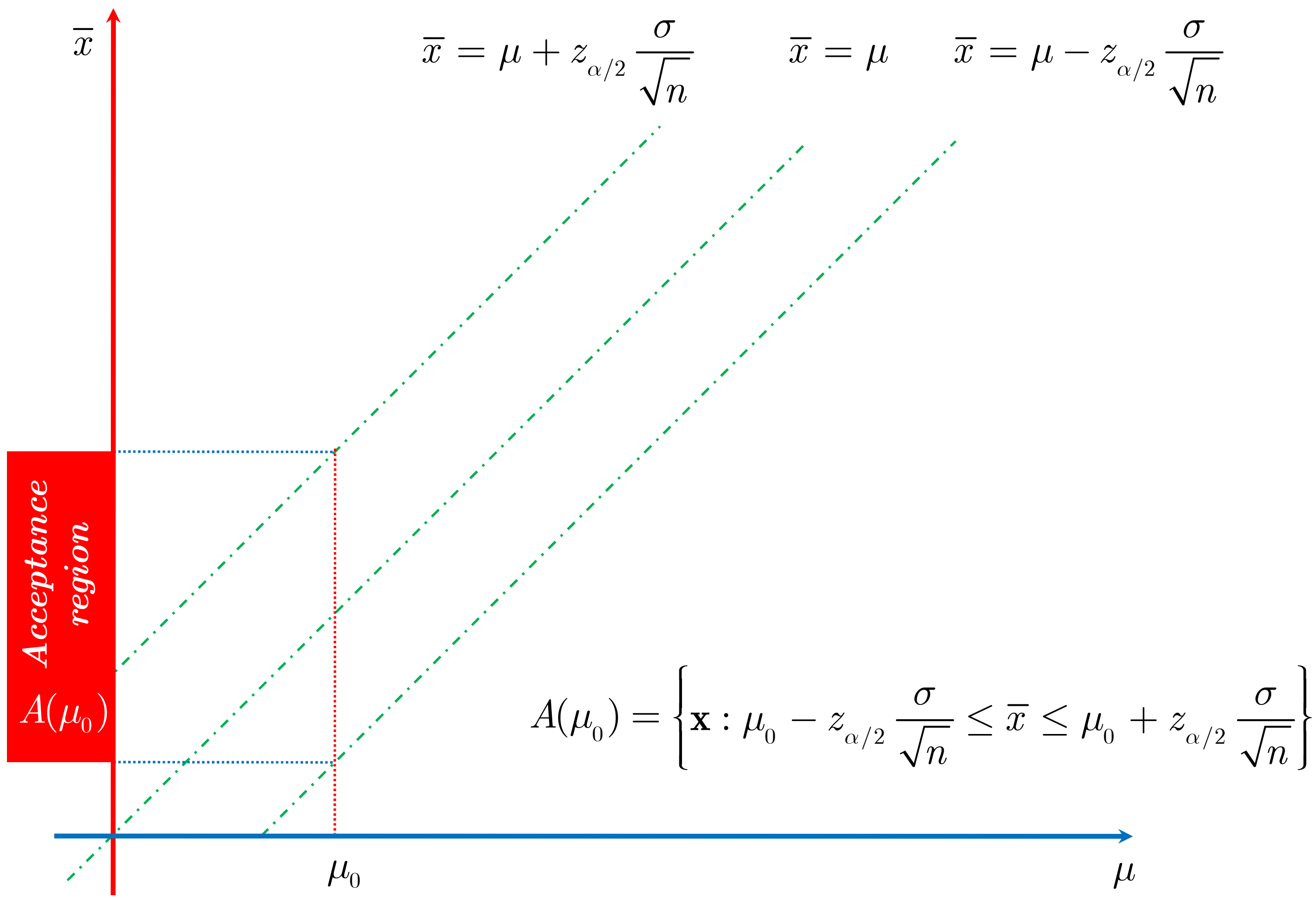
$$P(\bar{X} - z_{\alpha/2}\sigma / \sqrt{n} \leq \mu_0 \leq \bar{X} + z_{\alpha/2}\sigma / \sqrt{n} \mid \mu = \mu_0) \geq 1 - \alpha \;\Rightarrow$$

$$P_\mu(\bar{X} - z_{\alpha/2}\sigma / \sqrt{n} \leq \mu \leq \bar{X} + z_{\alpha/2}\sigma / \sqrt{n}) \geq 1 - \alpha \;\Rightarrow$$

$[\bar{X} - z_{\alpha/2}\sigma / \sqrt{n}, \bar{X} + z_{\alpha/2}\sigma / \sqrt{n}]$ is a $1-\alpha$ confidence interval of $\mu$.

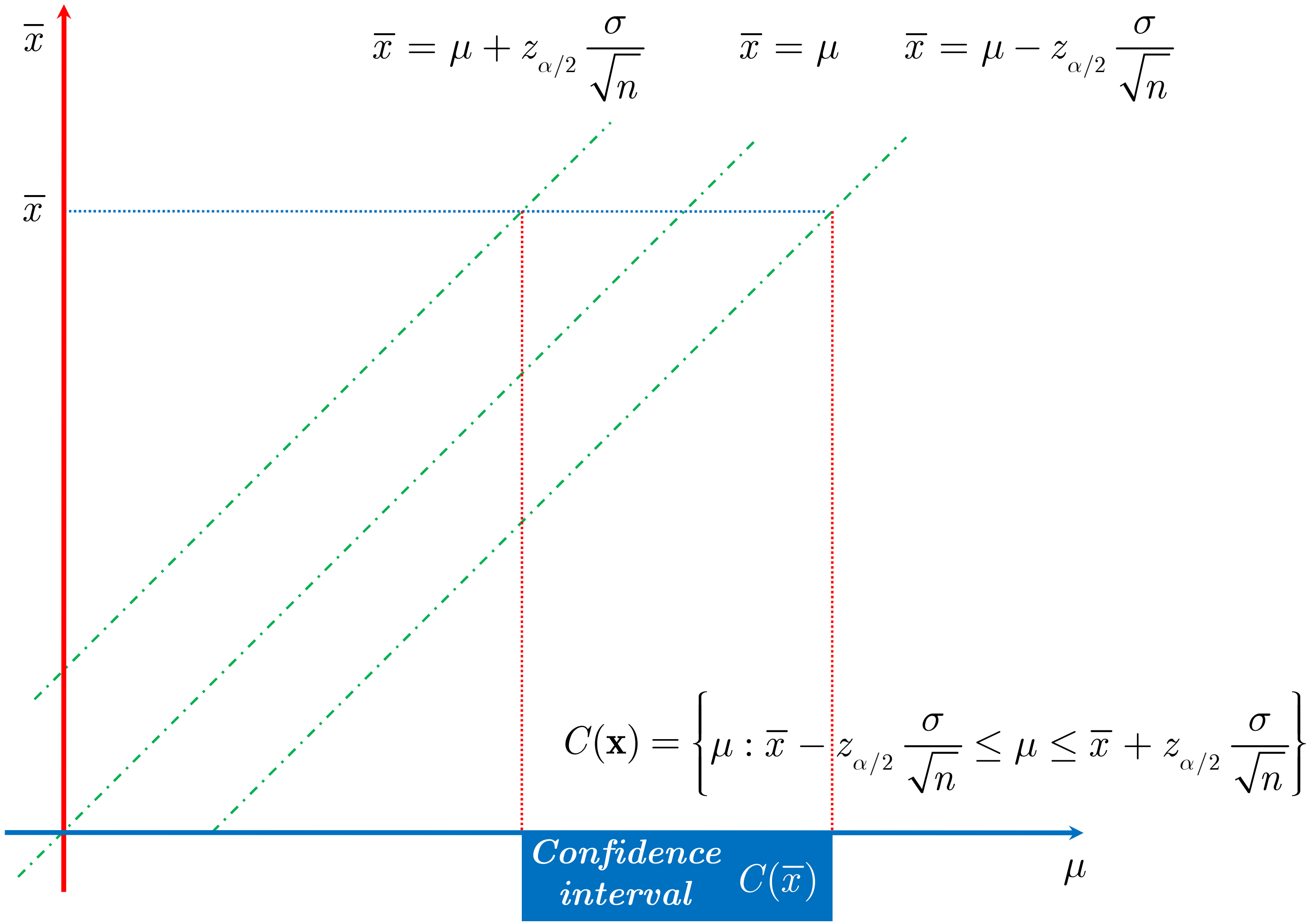
# Hypothesis testing

**Which sample values are consistent with the fixed parameter?**



# Interval estimation

**Which parameter values make the observed sample value most plausible?**

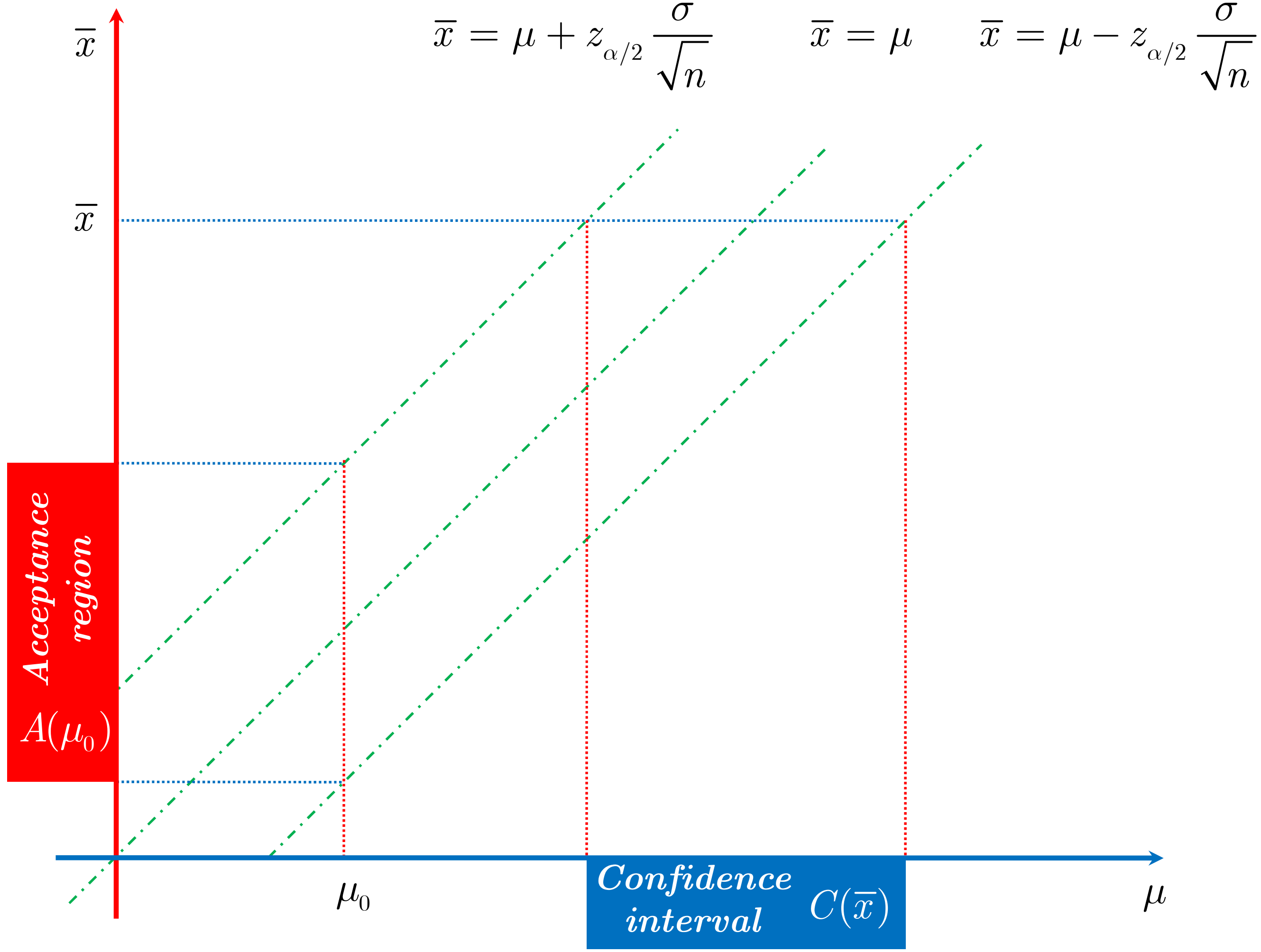


# Hypothesis testing *vs.* interval estimation

$A(\mu_0) \Rightarrow C(\bar{x}):$ define $C(\bar{x}) = \{\mu_0 : \bar{x} \in A(\mu_0)\}$

$C(\bar{x}) \Rightarrow A(\mu_0):$ define $A(\mu_0) = \{\bar{x} : \mu_0 \in C(\bar{x})\}$

$\bar{x} \in A(\mu_0) \Leftrightarrow \mu_0 \in C(\bar{x})$



# In general

***Inverting a test statistic***

For each $\theta_0 \in \Theta$, let $A(\theta_0)$ be the acceptance region of a level $\alpha$ test of $H_0 : \theta = \theta_0$. For each $\mathbf{x} \in \mathcal{X}$, define a set $C(\mathbf{x})$ in the parameter space by

$$C(\mathbf{x}) = \{\theta_0 : \mathbf{x} \in A(\theta_0)\}$$

Then the random set $C(\mathbf{X})$ is a $1-\alpha$ confidence set.

Conversely, let $C(\mathbf{x})$ be a $1-\alpha$ confidence set. For each $\theta_0 \in \Theta$, define a set in the sample space by

$$A(\theta_0) = \{\mathbf{x} : \theta_0 \in C(\mathbf{x})\}$$

Then $A(\theta_0)$ is the acceptance region of a level $\alpha$ test of $H_0 : \theta = \theta_0$.

# Proof

Since $A(\theta_0)$ is the acceptance region of a level $\alpha$ test,

$$P_{\theta_0}(\mathbf{X} \notin A(\theta_0)) \leq \alpha \text{ and hence } P_{\theta_0}(\mathbf{X} \in A(\theta_0)) \geq 1-\alpha.$$

We can write $\theta$ instead of $\theta_0$ because $\theta_0$ is arbitrary. The above inequality, together with the given condition $C(\mathbf{x}) = \{\theta_0 : \mathbf{x} \in A(\theta_0)\}$, shows that the coverage probability of the set $C(\mathbf{X})$ is given by

$$P_\theta(\theta \in C(\mathbf{X})) = P_\theta(\mathbf{X} \in A(\theta)) \geq 1-\alpha,$$

showing that $C(\mathbf{X})$ is a $1-\alpha$ confidence set.

The Type I error probability for the test of $H_0 : \theta = \theta_0$ with acceptance region $A(\theta_0)$ is

$$P_{\theta_0}(\mathbf{X} \notin A(\theta_0)) = P_{\theta_0}(\theta_0 \notin C(\mathbf{X})) \leq \alpha,$$

showing this is a level $\alpha$ test.

# Inverting the test of normal mean

Let $X_1,\ldots,X_n$ be a random sample from a normal population $N(\mu,\sigma^2)$, where $\sigma^2$ is known. Consider the hypothesis testing problem

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0$$

For the level $\alpha$ uniformly most powerful unbiased test

$$A(\mu_0) = \left\{ \mathbf{x} : \frac{|\,\overline{x} - \mu_0\,|}{\sigma / \sqrt{n}} \leq z_{\alpha/2} \right\} = \left\{ \mathbf{x} : \mu_0 - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \overline{x} \leq \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}. \right\}$$

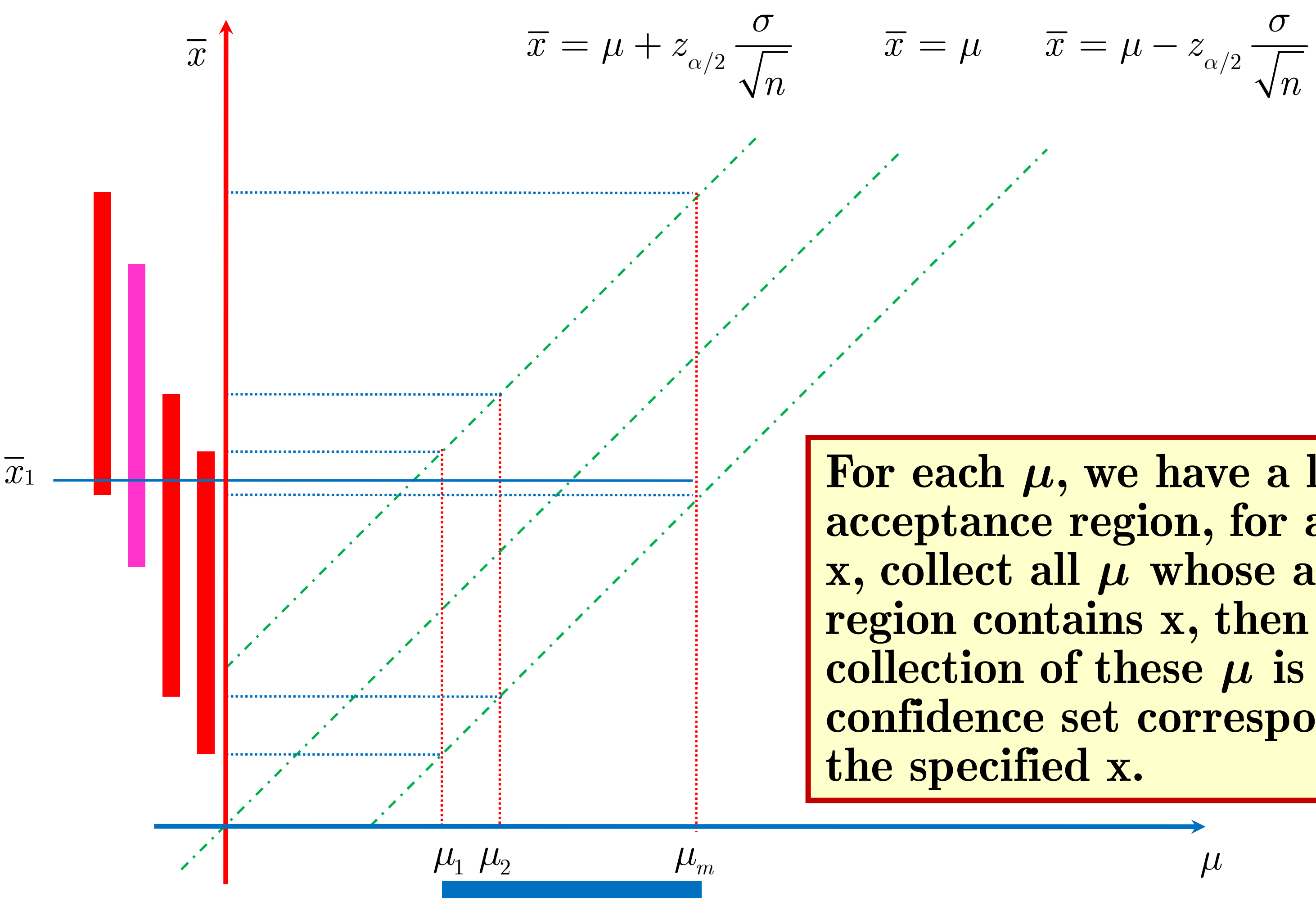
Now invert $A(\mu_0)$ to obtaine $C(\mathbf{x})$

$$C(\mathbf{x}) = \left\{ \mu_0 : \mathbf{x} \in A(\mu_0) \right\} = \left\{ \mu_0 : \overline{x} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \leq \mu_0 \leq \overline{x} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \right\}$$

Substitute $\mathbf{x}$ by $\mathbf{X}$, we obtain that

$$[\bar{X} - z_{\alpha/2}\sigma / \sqrt{n}, \bar{X} + z_{\alpha/2}\sigma / \sqrt{n}] \text{ is a } 1-\alpha \text{ confidence interval for } \mu.$$

# Inverting a family of tests




**For each $\mu$, we have a level $\alpha$ acceptance region, for a certain x, collect all $\mu$ whose acceptance region contains x, then the collection of these $\mu$ is the 1-$\alpha$ confidence set corresponding to the specified x.**

# Inverting the test of the normal mean

Let $X_1,\ldots,X_n$ be a random sample from a normal population $N(\mu,\sigma^2)$, where $\sigma^2$ is **unknown**.

Can we obtain a half-open interval $[L(X),+\infty)$ for $\mu$?

# One-sample normal mean LRT, one-sided, variance unknown

**$H_0$: $\mu = \mu_0$ versus $H_1$: $\mu > \mu_0$**

The parameter spaces are

$$\ldots$$

The restricted MLE yields

$$\ldots$$

The unrestricted MLE yields

$$\ldots$$

Therefore,

$$\lambda(\mathbf{x}) = \ldots$$

Therefore

$$R = \left\{ \mathbf{x} : \frac{\bar{x} - \mu_0}{s / \sqrt{n}} > t \right\}, t \geq 0.$$

# Inverting the test of normal mean

Let $X_1, \ldots, X_n$ be a random sample from a normal population $N(\mu, \sigma^2)$, where $\sigma^2$ is unknown. Consider the hypothesis testing problem

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu > \mu_0$$

For the level $\alpha$ $t$ test

$$A(\mu_0) = \left\{ \mathbf{x} : \frac{\bar{x} - \mu_0}{s / \sqrt{n}} \le t_{n-1,\alpha} \right\} = \left\{ \mathbf{x} : \bar{x} \le \mu_0 + t_{n-1,\alpha} \frac{s}{\sqrt{n}} \right\}$$

Now invert $A(\mu_0)$ to obtaine $C(\mathbf{x})$

$$C(\mathbf{x}) = \{ \mu_0 : \mathbf{x} \in A(\mu_0) \} = \left\{ \mu_0 : \mu_0 \ge \bar{x} - t_{n-1,\alpha} \frac{s}{\sqrt{n}} \right\}$$

Substitute $\mathbf{x}$ by $\mathbf{X}$, we obtain that

$[\bar{X} - t_{n-1,\alpha} s / \sqrt{n}, \infty)$ is a $1 - \alpha$ confidence interval for $\mu$.

# Location exponential model

Let $Z$ be an exponential(1) random variable. In other words,

$$f(z) = e^{-z} I_{[0,\infty)}(z)$$

The transformation $X = Z + \mu$ yields a location exponential random variable with pdf

$$f(x) = e^{-(x-\mu)} I_{[\mu,\infty)}(x)$$

Now, consider $X_1, \ldots, X_n$, a ramdom sample of size $n$, from this location exponential population. **How do we obtian an interval estimator of the location parameter $\mu$?**

# Minimal sufficient statistic

The pdf of the sample is

$$\begin{aligned} f(\mathbf{x} \mid \mu) &= \prod_{i=1}^{n} e^{-(x_i-\mu)} I_{[\mu,\infty)}(x_i) = e^{-(\sum x_i - n\mu)} \prod_{i=1}^{n} I_{[\mu,\infty)}(x_i) \\ &= e^{-(\sum x_i - n\mu)} I_{[\mu,\infty)}(\min x_i) \end{aligned}$$

Now, consider two samples $\mathbf{X}$ and $\mathbf{Y}$ of the samle size,

$$\frac{f(\mathbf{x} \mid \mu)}{f(\mathbf{y} \mid \mu)} = \frac{e^{-(\sum x_i - n\mu)} I_{[\mu,\infty)}(\min x_i)}{e^{-(\sum y_i - n\mu)} I_{[\mu,\infty)}(\min y_i)} = \frac{e^{-\sum x_i} I_{[\mu,\infty)}(\min x_i)}{e^{-\sum y_i} I_{[\mu,\infty)}(\min y_i)}$$

To make this ratio independent of $\mu$, we need the ratio of indicator functions independent of $\mu$. This will be the case if and only if

$$\min x_i = \min y_i$$

Therefore, $T(X) = X_{(1)} = \min X_i$ is a minimal sufficient statistic of this location exponential population $\mu$.

# Sampling distribution

Let's look at the CDF of $Y = \min X_i$

$$\begin{aligned} F(y \mid \mu) &= P(\min X_i \le y \mid \mu) \\ &= 1 - P(\min X_i > y \mid \mu) \\ &= 1 - P(X_1 > y \text{ AND } \cdots \text{ AND } X_n > y \mid \mu) \\ &= 1 - \prod_{i=1}^{n} P(X_i > y \mid \mu) \\ &= 1 - \prod_{i=1}^{n} [1 - P(X_i \le y \mid \mu)] \end{aligned}$$

Because

$$P(X_i \le y \mid \mu) = \int_{\mu}^{y} e^{-(t-\mu)} dt = 1 - e^{-(y-\mu)} \quad (\mu \le y)$$

$$F(y \mid \mu) = 1 - \prod_{i=1}^{n} e^{-(y-\mu)} = [1 - e^{-n(y-\mu)}] \quad (\mu \le y)$$

Therefore

$$f(y \mid \mu) = \frac{d}{dy} F(y \mid \mu) = n e^{-n(y-\mu)} I_{[\mu,\infty)}(y)$$

# MLE

To obtain a point estimate of the location parameter $\mu$,
we conditional on the minimal sufficient statistic $Y = \min X_i$.

The likelihood function is

$$L(\mu \mid y) = f(y \mid \mu) = ne^{-n(y-\mu)} I_{[\mu,\infty)}(y)$$

Obsioiusly, when $y$ is fixed, $L(\mu \mid y)$ is a strict increasing function of $\mu$, the maximum of $L(\mu \mid y)$ is attained at the maximum of $\mu$, which is $y$.

Therefore, the maximum likelihood estimate of $\mu$ is

$$\hat{\mu} = y = \min x_i$$

# LRT

Consider the hypothesis testing problem

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0$$

Conditional on the minimal sufficient statistic $Y = \min X$, the restricted MLE yields $\hat{\mu}_0 = \mu_0$, while the unrestricted MLE yields $\hat{\mu} = y$.

The likelihood ratio test statistic is therefore

$$\lambda(y) = \frac{\sup_{\mu_0} L(\mu \mid y)}{\sup_{(-\infty,\infty)} L(\mu \mid y)} = \frac{ne^{-n(y-\mu_0)} I_{[\mu_0,\infty)}(y)}{ne^{-n(y-y)} I_{[y,\infty)}(y)} = e^{-n(y-\mu_0)} I_{[\mu_0,\infty)}(y)$$

LRT suggests to reject $H_0$ if $\lambda(y) < c$. To ensure a level $\alpha$ test, consider

$$\begin{aligned} P(\text{reject } H_0 \mid \mu = \mu_0) &= P(e^{-n(Y-\mu_0)} < c \text{ OR } Y < \mu_0 \mid \mu = \mu_0) \\ &= P(Y > \mu_0 - (1/n)\log c \mid \mu = \mu_0) \\ &= \int_{\mu_0 - (1/n)\log c}^{\infty} e^{-n(y-\mu_0)} dy \\ &= c \end{aligned}$$

Therefore, $c = \alpha$, and the acceptance region is

$$A(\mu_0) = \{ y : \mu_0 \leq y \leq \mu_0 - (1/n)\log \alpha \}$$

# Inverting the LRT

The level $\alpha$ acceptance region is

$$A(\mu_0) = \{y : \mu_0 \le y \le \mu_0 - (1/n)\log\alpha\}$$

with an associated interval

$$C(y) = \{\mu : y + (1/n)\log\alpha \le \mu \le y\}$$

Therefore, by inverting the LRT, we obtain a $1-\alpha$ confidence interval

$$Y + (1/n)\log\alpha \le \mu \le Y$$

# Pivotal Quantities

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*"The random interval [L(X), U(X)] is the interval estimator of the parameter, such that the probability of parameter located in the interval with high probability."*

# Parameter free distributions

We have seen that for a random sample $X_1, \ldots, X_n$ from a normal population $N(\mu, \sigma^2)$.

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1), \;\; \sigma^2 \text{ known.}$$

$$T = \frac{\bar{X} - \mu}{S / \sqrt{n}} \sim T_{n-1}.$$

$$K = \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} \right)^2 \sim \chi_n^2, \;\; \mu \text{ known.}$$

$$K = \frac{(n-1)S^2}{\sigma^2} = \sum_{i=1}^{n} \left( \frac{X_i - \bar{X}}{\sigma} \right)^2 \sim \chi_{n-1}^2.$$

What's the common characteristics of these random variables?

**Parameter free!**

# Pivotal quantities

***Pivotal quantities***

A random variable $Q(\mathbf{X},\theta) = Q(X_1,\ldots,X_n,\theta)$ is a **pivotal quantity** (**pivot**) if the distribution of $Q(\mathbf{X},\theta)$ is independent of all parameters. That is, if $\mathbf{X} \sim F(\mathbf{x} \mid \theta)$, then $Q(\mathbf{X},\theta)$ has the same distribution for all values of $\theta$.

# Hypothesis testing

Let $X_1,\ldots,X_n$ be a random sample from a normal population $N(\mu,\sigma^2)$, where $\sigma^2$ is known. Consider the hypothesis testing problem

$$H_0 : \mu = \mu_0 \quad \text{vs} \quad H_1 : \mu \neq \mu_0$$

# Hypothesis testing

We consider the acceptance region with the form

$$A(\mu_0) = \left\{ \mathbf{x} : -a \leq \frac{\bar{x} - \mu_0}{\sigma^2 / \sqrt{n}} \leq b \right\}$$

In order to ensure a level $\alpha$ test, we require

$$P(\mathbf{X} \notin A(\mu_0) \mid \mu = \mu_0) \leq \alpha$$

and hence

$$P(\mathbf{X} \in A(\mu_0) \mid \mu = \mu_0) \geq 1 - \alpha$$

Since $\dfrac{\bar{X} - \mu_0}{\sigma^2 / \sqrt{n}}$ has a standard normal distribution when $\mu_0$ is the true parameter (under the null hypothesis $\mu = \mu_0$),

$$P(\mathbf{X} \in A(\mu_0) \mid \mu = \mu_0) = P(-a \leq Z \leq b),$$

where $Z$ is a standard normal random variable. Assume $a = b$, we have $a = b = z_{\alpha/2}$. In order to ensure a level $\alpha$ test, the acceptance region is

$$A(\mu_0) = \left\{ \mathbf{x} : -z_{\alpha/2} \leq \frac{\bar{x} - \mu_0}{\sigma^2 / \sqrt{n}} \leq z_{\alpha/2} \right\}$$

# Test inversion

Since

$$\mathbf{x} \in A(\mu_0) \Leftrightarrow \mu_0 \in C(\mathbf{x})$$

The $1-\alpha$ confidence set can be obtained by inverting the previous test statistic.

Inverting

$$A(\mu_0) = \left\{ \mathbf{x} : -z_{\alpha/2} \leq \frac{\overline{x} - \mu_0}{\sigma^2 / \sqrt{n}} \leq z_{\alpha/2} \right\}$$

yields

$$C(\mathbf{x}) = \left\{ \mu : -z_{\alpha/2} \leq \frac{\overline{x} - \mu}{\sigma^2 / \sqrt{n}} \leq z_{\alpha/2} \right\}$$

which is equivalent to

$$C(\mathbf{x}) = \left\{ \mu : \overline{x} - z_{\alpha/2} \frac{\sigma^2}{\sqrt{n}} \leq \mu \leq \overline{x} + z_{\alpha/2} \frac{\sigma^2}{\sqrt{n}} \right\}$$

# Make use of pivotal quantities

Let $X_1,\ldots,X_n$ be a random sample from a normal population $N(\mu,\sigma^2)$, where $\sigma^2$ is known. Since

$$Z=\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$$

is a pivotal quantity, we can consider the following inequality

$$a\le\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\le b,$$

Because this can immediately give us an interval estimator

$$\bar{X}-b\frac{\sigma}{\sqrt{n}}\le\mu\le\bar{X}+a\frac{\sigma}{\sqrt{n}}$$

In order to insure a $1-\alpha$ confidence coefficient, we need to have

$$P\left(a\le\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\le b\right)=P(a\le Z\le b)=\Phi(b)-\Phi(a)=1-\alpha$$

With the equal tail constraint that $P(Z\le a)=P(Z\ge b)$, it is easy to obtain $b=-a=z_{\alpha/2}$.

# Make use of pivotal quantities

Let $X_1,\ldots,X_n$ be a random sample from a normal population $N(\mu,\sigma^2)$. Since

$$\frac{(n-1)S^2}{\sigma^2}=\sum_{i=1}^{n}\left(\frac{X_i-\bar{X}}{\sigma}\right)^2\sim\chi^2_{n-1}$$

is a pivotal quantity, we can consider the following inequality

$$a\le\frac{(n-1)S^2}{\sigma^2}\le b,$$

Because this can immediately give us an interval estimator

$$\frac{(n-1)S^2}{b}\le\sigma^2\le\frac{(n-1)S^2}{a}$$

In order to insure a $1-\alpha$ confidence coefficient, we need to have

$$P\left(a\le\frac{(n-1)S^2}{\sigma^2}\le b\right)=P(a\le\chi^2_{n-1}\le b)=1-\alpha$$

With the equal tail constraint that $P(\chi^2_{n-1}\le a)=P(\chi^2_{n-1}\ge b)$, it is easy to obtain $a=\chi^2_{n-1,1-\alpha/2}$ and $b=\chi^2_{n-1,\alpha/2}$. Therefore,

$$\left[(n-1)S^2/\chi^2_{n-1,\alpha/2},(n-1)S^2/\chi^2_{n-1,1-\alpha/2}\right]$$

is a $1-\alpha$ confidence interval for the nomal variance.

# Simultaneous inference

Let $X_1,\ldots,X_n$ be a random sample from a normal population $N(\mu,\sigma^2)$, where both $\mu$ and $\sigma^2$ are unknown. How to obtain an interval estimator for both parameters?

For $\mu$ only, the confidence interval has the form

$$C_\mu(\mathbf{X}) = \{\mu : \bar{X} - aS / \sqrt{n} \le \mu \le \bar{X} + bS / \sqrt{n}\}$$

For $\sigma^2$ only, the confidence interval has the form

$$C_{\sigma^2}(\mathbf{X}) = \{\sigma^2 : (n-1)S^2 / c \le \sigma^2 \le (n-1)S^2 / d\}$$

Therefore, it is reasonable to combine $C_\mu(\mathbf{X})$ and $C_{\sigma^2}(\mathbf{X})$ as

$$C_a(\mathbf{X}) = \{(\mu,\sigma^2) : \ \mu \in C_\mu(\mathbf{X}) \text{ and } \sigma^2 \in C_{\sigma^2}(\mathbf{X})\}$$

# Simultaneous inference

Remember the Bonferroni Inequality

$$P(A \cap B) \geq P(A) + P(B) - 1$$

In order to ensure $C_a(\mathbf{X})$ to be a $1-\alpha$ confidence set, we need to have

$$\begin{aligned} P((\mu, \sigma^2) \in C_a(\mathbf{X})) &= P(\mu \in C_\mu(\mathbf{X}) \text{ AND } \sigma^2 \in C_{\sigma^2}(\mathbf{X})) \\ &\geq P(\mu \in C_\mu(\mathbf{X})) + P(\sigma^2 \in C_{\sigma^2}(\mathbf{X})) - 1 \\ &\geq 1 - \alpha \end{aligned}$$

A convinient choice is

$$P(\mu \in C_\mu(\mathbf{X})) = P(\sigma^2 \in C_{\sigma^2}(\mathbf{X})) = \frac{2-\alpha}{2} = 1 - \frac{\alpha}{2}$$

With the equal tail assumption, we can choose

$$C_\mu(\mathbf{X}) = \{\mu : \bar{X} - t_{n-1,\alpha/4} S / \sqrt{n} \leq \mu \leq \bar{X} + t_{n-1,\alpha/4} S / \sqrt{n}\}$$

and

$$C_{\sigma^2}(\mathbf{X}) = \{\sigma^2 : (n-1)S^2 / \chi^2_{n-1,\alpha/4} \leq \sigma^2 \leq (n-1)S^2 / \chi^2_{n-1,1-\alpha/4}\}$$

# Location exponential model

Let $X_1, \ldots, X_n$, a ramdom sample of size $n$, from the location exponential family

$$f(x) = e^{-(x-\mu)} I_{[\mu,\infty)}(x)$$

Can we obtian an interval estimator of the location parameter $\mu$ by using a pivotal quantity?

$Y = X_{(1)} = \min X_i$ is a minimal sufficient statistic of the location parameter $\mu$. The pdf of $Y$ is

$$f(y \mid \mu) = \frac{d}{dy} F(y \mid \mu) = n e^{-n(y-\mu)} I_{[\mu,\infty)}(y)$$

# Pivotal quantity

Note that the minimal sufficient statistic $Y$ is a location family

$$f(y \mid \mu) = ne^{-n(y-\mu)} I_{[\mu,\infty)}(y)$$

Transformation $W = Y - \mu$ yields a pivotal quantity

$$f(w) = ne^{-nw} I_{[0,\infty)}(w)$$

which is independent of the location parameter $\mu$. Because

$$\begin{aligned} P(a \le W \le b) &= \int_a^b ne^{-nw} I_{[0,\infty)}(w)dw \\ &= \int_{-\infty}^b ne^{-nw} I_{[0,\infty)}(w)dw - \int_{-\infty}^a ne^{-nw} I_{[0,\infty)}(w)dw \\ &= 1 - \int_b^\infty ne^{-nw}dw - \int_0^a ne^{-nw}dw \end{aligned}$$

Let this probability equal to $1-\alpha$ and assume equal tail probabilities

$$\int_0^a ne^{-nw}dw = \frac{\alpha}{2} \Rightarrow 1 - e^{-n\alpha} = \frac{\alpha}{2} \Rightarrow a = -(1/n)\log(1-\alpha/2)$$

$$\int_b^\infty ne^{-nw}dw = \frac{\alpha}{2} \Rightarrow e^{-nb} = \frac{\alpha}{2} \Rightarrow b = -(1/n)\log(\alpha/2)$$

Therefore, using the pivital quanty, we obtain a $1-\alpha$ confidence interval

$$C(Y) = \{\mu : Y + (1/n)\log(\alpha/2) \le \mu \le Y + (1/n)\log(1-\alpha/2)\}$$

# Pivoting CDFs



*“The random interval [L(X), U(X)] is the interval estimator of the parameter, such that the probability of parameter located in the interval with high probability.”*

# Probability integral transformation

***Probability integral transformation***

Let $X$ have continuous cdf $F_X(x)$ and define a random variable $Y$ as $Y = F_X(X)$. Then $Y$ is uniformly distributed on $(0,1)$, that is,

$$P(Y \le y) = y,\ 0<\mathrm{y}<1.$$

$$\begin{aligned}
P(Y \le y) &= P(F_X(X) \le y) \\
&= P(F_X^{-1}[F_X(X)] \le F_X^{-1}(y)) \\
&= P(X \le F_X^{-1}(y)) \\
&= F_X(F_X^{-1}(y)) \\
&= y
\end{aligned}$$

# CDF is a pivotal quantity

For any statistic $T$, let its cdf be $F_T(t \mid \theta)$, then $F_T(T \mid \theta)$ has a $Uniform(0,1)$ distribution.

Therefore, $F_T(T \mid \theta)$ is a pivital quantity.

# Hypothesis testing

Consider the hypothesis testing problem

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0$$

We consider the acceptance region with the form

$$A(\theta_0) = \left\{ t : a \le F_T(t \mid \theta_0) \le 1 - b \right\}$$

In order to ensure a level $\alpha$ test, we require

$$P(T \notin A(\theta_0) \mid \theta = \theta_0) \le \alpha$$

and hence

$$P(T \in A(\theta_0) \mid \theta = \theta_0) \ge 1 - \alpha$$

Since $F_T(T \mid \theta_0)$ has a $Uniform(0,1)$ distribution when $\theta_0$ is the true parameter (under the null hypothesis $\theta = \theta_0$),

$$P(T \in A(\theta_0) \mid \theta = \theta_0) = P(a \le U \le 1 - b) = 1 - b - a,$$

where $U$ is a $uniform(0,1)$ random variable. Assume $a = b$, we have $a = b = \alpha / 2$. In order to ensure a level $\alpha$ test, the acceptance region is

$$A(\theta_0) = \left\{ t : \alpha / 2 \le F_T(t \mid \theta_0) \le 1 - \alpha / 2 \right\}$$

# Test inversion

Since

$$t \in A(\theta_0) = \{t : \alpha / 2 \leq F_T(t \mid \theta_0) \leq 1 - \alpha / 2\}$$

$$\Leftrightarrow \qquad \theta_0 \in C(t) = \{\theta : \alpha / 2 \leq F_T(t \mid \theta) \leq 1 - \alpha / 2\}$$

The $1 - \alpha$ confidence set can be obtained by inverting the previous test statistic.

Inverting

$$A(\theta_0) = \{t : \alpha / 2 \leq F_T(t \mid \theta_0) \leq 1 - \alpha / 2\}$$

yields

$$C(t) = \{\theta : \alpha / 2 \leq F_T(t \mid \theta) \leq 1 - \alpha / 2\}$$

This is a $1 - \alpha$ confidence set.

# Normal model

Let $X \sim N(\mu, \sigma^2)$ be a normal random variable, where $\sigma^2$ is known.

As we known, $\bar{X}$ has a $N(\mu, \sigma^2 / n)$ distribution. In other words,

$$Z = \frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

has a standard normal distribution, or $Z \sim N(0,1)$.

Now, consider cdfs.

$$\begin{aligned} F_{\bar{X}}(\bar{x} \mid \mu) &= P(\bar{X} \le \bar{x} \mid \mu) = P(\mu + Z\sigma / \sqrt{n} \le \bar{x} \mid \mu) \\ &= P(Z \le (\bar{x} - \mu) / (\sigma / \sqrt{n}) \mid \mu) \\ &= \Phi((\bar{x} - \mu) / (\sigma / \sqrt{n})) \end{aligned}$$

For two values of $\mu, \mu_1$ and $\mu_2$, where $\mu_1 < \mu_2$, we have $\bar{x} - \mu_1 > \bar{x} - \mu_2$

$$F_{\bar{X}}(\bar{x} \mid \mu_1) = \Phi((\bar{x} - \mu_1) / (\sigma / \sqrt{n})) > \Phi((\bar{x} - \mu_2) / (\sigma / \sqrt{n})) = F_{\bar{X}}(\bar{x} \mid \mu_2),$$

since standard normal cdf is strict increasing function. Therefore,

$F_{\bar{X}}(\bar{x} \mid \mu)$ is a strict decreasing function of $\mu$.

# Pivoting the normal cdf

Now, in order to ensure

$$C(\overline{x}) = \{\mu : a \le F_{\overline{X}}(\overline{x} \mid \mu) \le 1 - b\}$$

We can determine the upper bound of $\mu$ by solving the equation

$$F_{\overline{X}}(\overline{x} \mid \mu_U) = \Phi((\overline{x} - \mu_U) / (\sigma / \sqrt{n})) = a,$$

which yields

$$\mu_U = \overline{x} - (\sigma / \sqrt{n})\Phi^{-1}(a)$$

To determine the lower bound of $\mu$, we solve the equation

$$F_{\overline{X}}(\overline{x} \mid \mu_L) = \Phi(\overline{x} - \mu_L) = 1 - b,$$

which yields

$$\mu_L = \overline{x} - (\sigma / \sqrt{n})\Phi^{-1}(1 - b)$$

Let $a = b = \alpha / 2$, we obtain a $1 - \alpha$ confidence interval

$$\overline{x} - (\sigma / \sqrt{n})\Phi^{-1}(1 - \alpha / 2) \le \mu \le \overline{x} - (\sigma / \sqrt{n})\Phi^{-1}(\alpha / 2)$$

which is equivalent to

$$\overline{x} - z_{\alpha/2}\sigma / \sqrt{n} \le \mu \le \overline{x} + z_{\alpha/2}\sigma / \sqrt{n}$$

# Pivoting a continuous cdf

Let $T$ be a statistic with continuous cdf $F_T(t \mid \theta)$, where $F_T(t \mid \theta)$ is a strict **decreasing** function of $\theta$ for each $t$. Let $a + b = \alpha$, with $0 < \alpha < 1$ being fixed values. Suppose that for each $t \in \mathcal{T}$, the domain of $t$, the functions $\theta_L(t)$ and $\theta_U(t)$ can be defined as

$$F_T(t \mid \theta_U(t)) = a$$

and

$$F_T(t \mid \theta_L(t)) = 1 - b$$

Then the random interval $[\theta_L(T), \theta_U(T)]$ is a $1 - \alpha$ confidence interval for $\theta$.

# Pivoting a continuous cdf

Let $T$ be a statistic with continuous cdf $F_T(t \mid \theta)$, where $F_T(t \mid \theta)$ is a strict **increasing** function of $\theta$ for each $t$. Let $a+b=\alpha$, with $0<\alpha<1$ being fixed values. Suppose that for each $t \in \mathcal{T}$, the functions $\theta_L(t)$ and $\theta_U(t)$ can be defined as

$$F_T(t \mid \theta_U(t)) = 1-b$$

and

$$F_T(t \mid \theta_L(t)) = a$$

Then the random interval $[\theta_L(T), \theta_U(T)]$ is a $1-\alpha$ confidence interval for $\theta$.

# Proof

Assume that we have constructed the $1-\alpha$ acceptance region

$$\{t : a \le F_T(t \mid \theta_0) \le 1-b\}$$

Since $F_T(t \mid \theta)$ is a strict decreasing function of $\theta$ for each $t$ and $1-b > a$ (why?), $\theta_L(t) < \theta_U(t)$, and $\theta_L(t)$ and $\theta_U(t)$ are unique. Also,

$$\begin{aligned} F_T(t \mid \theta) < a \qquad & \Leftrightarrow \theta > \theta_U(t), \\ F_T(t \mid \theta) > 1-b \quad & \Leftrightarrow \theta < \theta_L(t), \end{aligned}$$

and hence

$$\{\theta : a \le F_T(t \mid \theta) \le 1-b\}$$

is equivalent to

$$\{\theta : \theta_L(t) \le \theta \le \theta_U(t)\}.$$

# In terms of pdf

We actually need to solve two equations

$$F_T(t \mid \theta_U(t)) = a$$

and

$$F_T(t \mid \theta_L(t)) = 1 - b$$

Let $f_T(t \mid \theta)$ be the pdf of the statistic $T$. Then

$$F_T(t \mid \theta) = \int_{-\infty}^{t} f_T(u \mid \theta) du$$

that is to say, we need to solve

$$\int_{-\infty}^{t} f_T(u \mid \theta_U(t)) du = a$$

and

$$\int_{t}^{\infty} f_T(u \mid \theta_L(t)) du = b$$

# Location exponential model

Let $X_1, \ldots, X_n$, a ramdom sample of size $n$, from the location exponential family

$$f(x) = e^{-(x-\mu)} I_{[\mu,\infty)}(x)$$

Can we obtian an interval estimator of the location parameter $\mu$ by pivoting a CDF?

$Y = X_{(1)} = \min X_i$ is a minimal sufficient statistic of the location parameter $\mu$. The pdf of $Y$ is

$$f(y \mid \mu) = \frac{d}{dy} F(y \mid \mu) = n e^{-n(y-\mu)} I_{[\mu,\infty)}(y)$$

# Pivoting the CDF

Consider the cdf of $Y$,

$$F_Y(y \mid \mu) = P(Y \le y \mid \mu) = P(W + \mu \le y \mid \mu) = P(W \le y - \mu \mid \mu) = F_W(y - \mu),$$

where $W = Y - \mu$ and has the cdf

$$F_W(w) = \int_0^w ne^{-nt} dt = -e^{-nt} \Big|_0^w = 1 - e^{-nw}$$

Obviously, $F_W(w)$ is a strict increasing function of $w$. Hence, for any $\mu_1 < \mu_2$

$$F_Y(y \mid \mu_1) = F_W(y - \mu_1) > F_W(y - \mu_2) = F_Y(y \mid \mu_2),$$

In other words,

$$F_Y(y \mid \mu) \text{ is a strict decreasing function of } \mu.$$

Now, let

$$\int_{-\infty}^{y} f_Y(t \mid \theta_U(y)) dt = \int_{\mu_U(y)}^{y} ne^{-n(t - \mu_U(y))} dt = \frac{\alpha}{2} \Rightarrow \mu_U(y) = y + (1/n)\log(1 - \alpha/2)$$

$$\int_{y}^{\infty} f_Y(t \mid \theta_L(y)) dt = \int_{y}^{\infty} ne^{-n(t - \mu_L(y))} dt = \frac{\alpha}{2} \Rightarrow \mu_L(y) = y + (1/n)\log(\alpha/2)$$

Therefore, by pivoting the cdf, we obtain a $1 - \alpha$ confidence interval

$$C(Y) = \{\mu : Y + (1/n)\log(\alpha/2) \le \mu \le Y + (1/n)\log(1 - \alpha/2)\}$$

# Pivoting a discrete cdf

Let $T$ be a discrete statistic with cdf $F_T(t \mid \theta) = P(T \leq t \mid \theta)$. Let $a + b = \alpha$ with $0 < \alpha < 1$ being fixed values. If $F_T(t \mid \theta)$ is a **decreasing** function of $\theta$ for each $t \in \mathcal{T}$, define the functions $\theta_L(t)$ and $\theta_U(t)$ as

$$P(T \leq t \mid \theta_U(t)) = a$$

and

$$P(T \geq t \mid \theta_L(t)) = b$$

Then the random interval $[\theta_L(T), \theta_U(T)]$ is a $1 - \alpha$ confidence interval for $\theta$.

# Pivoting a discrete cdf

Let $T$ be a discrete statistic with cdf $F_T(t \mid \theta) = P(T \leq t \mid \theta)$. Let $a + b = \alpha$ with $0 < \alpha < 1$ being fixed values. If $F_T(t \mid \theta)$ is an **increasing** function of $\theta$ for each $t \in \mathcal{T}$, define the functions $\theta_L(t)$ and $\theta_U(t)$ as

$$P(T \geq t \mid \theta_U(t)) = a$$

and

$$P(T \leq t \mid \theta_L(t)) = b$$

Then the random interval $[\theta_L(T), \theta_U(T)]$ is a $1 - \alpha$ confidence interval for $\theta$.

# Poisson model

Let $X_1, \ldots, X_n$ be a random sample from a $Poisson(\lambda)$ population. Then the joint pmf of the sample is

$$f(\mathbf{x} \mid \lambda) = \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i\,!} = e^{-n\lambda}\lambda^{\Sigma x_i} \prod_{i=1}^{n} \frac{1}{x_i\,!}$$

Therefore, $Y = \sum_{i=1}^{n} X_i$ is a sufficient statistic of $\lambda$. Furthermore, $Y$ has a $Poisson(n\lambda)$ distribution. In other words

$$f(y \mid \lambda) = \frac{e^{-n\lambda}(n\lambda)^y}{y\,!}$$

How to calculate Poisson cdf, say, if $X \sim Poisson(\lambda)$, how about

$$P(X \le x \mid \lambda) = \sum_{k=0}^{x} \frac{e^{-\lambda}\lambda^{k}}{k\,!}$$

# Poisson-Gamma relation

If $X \sim Gamma(shape = \alpha, scale = \theta)$, then for any $x$

$$P(X \le x) = P(Y \ge \alpha)$$

where $Y \sim Poisson(x \,/\, \theta)$.

$\alpha$ should be integer.

$$\begin{aligned}
P(X \le x) &= \int_0^x \frac{1}{\Gamma(\alpha)\theta^\alpha} t^{\alpha-1} e^{-t/\theta} dt \\
&= \frac{1}{(\alpha-1)!\,\theta^\alpha} \int_0^x t^{\alpha-1} e^{-t/\theta} dt \\
&= \frac{1}{(\alpha-1)!\,\theta^\alpha} \left[ -\theta t^{\alpha-1} e^{-t/\theta} \Big|_0^x + (\alpha-1)\theta \int_0^x t^{\alpha-2} e^{-t/\theta} dt \right] \\
&= \frac{1}{(\alpha-2)!\,\theta^{\alpha-1}} \int_0^x t^{\alpha-2} e^{-t/\theta} dt - \frac{(x\,/\,\theta)^{\alpha-1} e^{-x/\theta}}{(\alpha-1)!} \\
&= \frac{1}{(\alpha-2)!\,\theta^{\alpha-1}} \int_0^x t^{(\alpha-1)-1} e^{-t/\theta} dt - P(Y = \alpha - 1) \\
&= \ldots \\
&= \frac{1}{\theta} \int_0^x e^{-t/\theta} dt - \sum_{k=1}^{\alpha-1} P(Y = k) \\
&= 1 - e^{-x/\theta} - (1 - P(Y=0) - P(Y \ge \alpha)) \\
&= P(Y \ge \alpha)
\end{aligned}$$

# Poisson-Chi-square relation

$$\text{If } Z \sim Poisson(\lambda), \text{ then for any integer } z$$
$$P(Z \geq z) = P(\chi^2_{2z} < 2\lambda)$$
$$P(Z \leq z) = P(\chi^2_{2(z+1)} > 2\lambda)$$

We already know if $X \sim Gamma(shape = \alpha, scale = \theta)$, then for any $x$

$$P(X \leq x) = P(Y \geq \alpha),$$

where $Y \sim Poisson(x / \theta)$. Therefore, if $Z \sim Poisson(\lambda)$, then

$$P(Z \geq z) = P(W \leq \lambda\theta),$$

where $W \sim Gamma(shape = z, scale = \theta)$.

More conviniently, as a special case of the Gamma distribution,

$\chi^2_p$ is $Gamma(shape = p / 2, scale = 2)$. Thus,

$$P(Z \geq z) = P(\chi^2_{2z} \leq 2\lambda).$$

Therefore,

$$P(Z \leq z) = 1 - P(Z \geq z + 1) = 1 - P(\chi^2_{2(z+1)} \leq 2\lambda) = P(\chi^2_{2(z+1)} > 2\lambda)$$

# Pivoting the Poisson cdf

Now, the sufficient statistic $Y = \sum_{i=1}^{n} X_i$ for a $Poisson(\lambda)$ population has a $Poisson(n\lambda)$ distribution. The cdf is

$$P(Y \le y \mid \lambda) = P(\chi^2_{2(y+1)} > 2n\lambda)$$

For fixed $y$, it is a decreasing function of $\lambda$. Assuming $a = b = \alpha / 2$, we have,

$$\alpha / 2 = P(Y \le y \mid \lambda_U) = P(\chi^2_{2(y+1)} > 2n\lambda_U),$$

$$\Rightarrow \qquad \lambda_U = \frac{1}{2n}\chi^2_{2(y+1),\alpha/2}$$

$$\alpha / 2 = P(Y \ge y \mid \lambda_L) = 1 - P(Y \le y-1 \mid \lambda_L) = P(\chi^2_{2y} < 2n\lambda_L),$$

$$\Rightarrow \qquad \lambda_L = \frac{1}{2n}\chi^2_{2y,1-\alpha/2}$$

Putting together, we have a $1-\alpha$ confidence interval

$$C(Y) = \{\lambda : \frac{1}{2n}\chi^2_{2Y,1-\alpha/2} \le \lambda \le \frac{1}{2n}\chi^2_{2(Y+1),\alpha/2}\}$$

# Bayesian Interval Estimator

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*“The random interval [L(X), U(X)] is the interval estimator of the parameter, such that the probability of parameter located in the interval with high probability.”*

# Credible interval

Let $\pi(\theta|\mathbf{x})$ be the posterior distribution of $\theta$ given $\mathbf{X} = \mathbf{x}$.

We have that

$$P(\theta \in A \mid \mathbf{x}) = \int_A \pi(\theta|\mathbf{x}) d\theta$$

$A$ is called a **credible set** for $\theta$.

If $A$ is an interval $[L(\mathbf{x}), U(\mathbf{x})]$ and satisfies

$$P(L(\mathbf{x}) \leq \theta \leq U(\mathbf{x})) \geq 1 - \alpha$$

$A = [L(\mathbf{x}), U(\mathbf{x})]$ is called a $1-\alpha$ **credible interval** for $\theta$.

$$P(L(\mathbf{x}) \leq \theta \leq U(\mathbf{x}) \mid \mathbf{x}) = \int_{L(\mathbf{x})}^{U(\mathbf{x})} \pi(\theta|\mathbf{x}) d\theta$$

# Evaluation of Interval Estimators

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*"The random interval [L(X), U(X)] is the interval estimator of the parameter, such that the probability of parameter located in the interval with high probability."*

# Different interval estimators

For the location exponential family

$$f(x) = e^{-(x-\mu)} I_{[\mu,\infty)}(x)$$

Inverting the LRT suggests

$$Y + (1/n)\log\alpha \le \mu \le Y$$

Using pivotal quantity or pivoting the cdf suggests

$$Y + (1/n)\log(\alpha/2) \le \mu \le Y + (1/n)\log(1-\alpha/2)$$

Although both intervals have confidence coefficient $1-\alpha$

The length of the LRT interval is

$$Y - (Y + (1/n)\log\alpha) = -(1/n)\log\alpha = (1/n)\log(1/\alpha)$$

The length of the pivotal interval is

$$(Y + (1/n)\log(1-\alpha/2)) - (Y + (1/n)\log(\alpha/2)) = (1/n)\log\frac{2-\alpha}{\alpha}$$

Since $\alpha < 1,\ \ 2-\alpha > 1,$ this is to say

The LRT interval is shorter than the pivotal interval.

# Size and coverage probability

We care about two properties of an interval estimator,

Its **coverage probability** and its **Size**

Coverage probability is usually measured using the confidence coefficient, say, the infimum of the coverage probability.

Size is usually mean the length of the confidence set if the set is an interval. If the set is not a interval (e.g., multi-dimensionalset) the size will usually become the volume.

# Optimizing length

Let $X_1,\ldots,X_n$ be a random sample from a normal population $N(\mu,\sigma^2)$, where $\sigma^2$ is known. Then

$$Z = \frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \sim N(0,1)$$

is a pivotal quantity. In order to have

$$P(a \le Z \le b) = 1-\alpha,$$

there could be many choices of $a$ and $b$.

The interval is

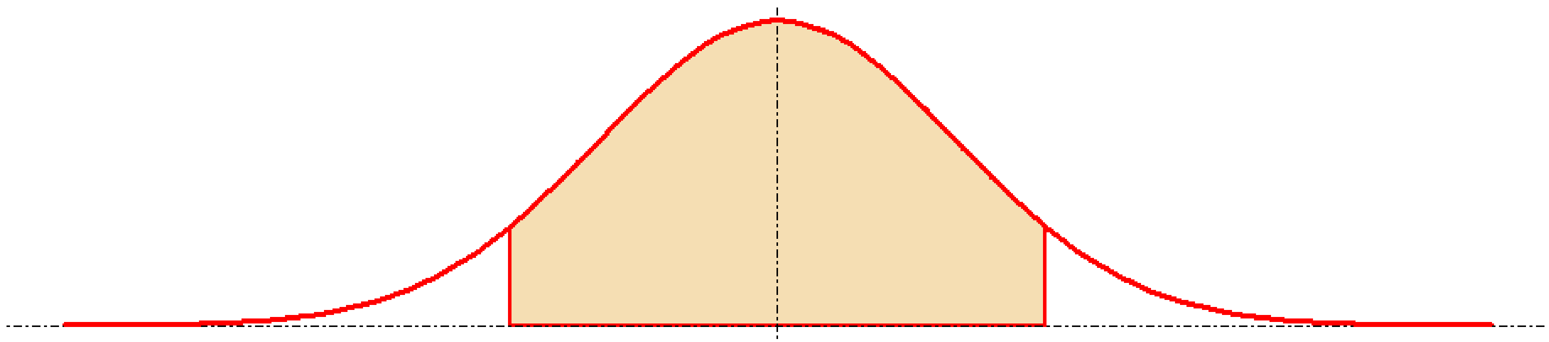
$$\bar{X} - b\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} - a\frac{\sigma}{\sqrt{n}}$$

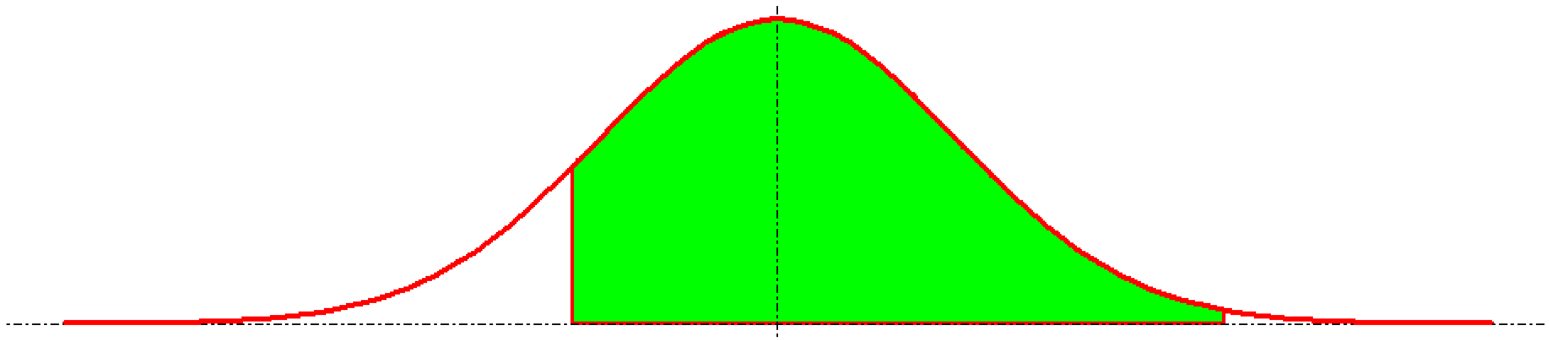
The length of the interval is

$$(b-a)\frac{\sigma}{\sqrt{n}}$$

which will be determined by $b-a$

# Different length





# Minimum length

Let $f(x)$ be a unimodal pdf. If the interval $[a, b]$ satisfies

(1) $\int_a^b f(x)dx = 1 - \alpha$;

(2) $f(a) = f(b) > 0$;

(3) $a \le x^\star \le b$, where $x^\star$ is the mode of the pdf;

Then $[a, b]$ is the shortest among all intervals that satisfy (1).

**Equal height yields optimal length.**

# Proof

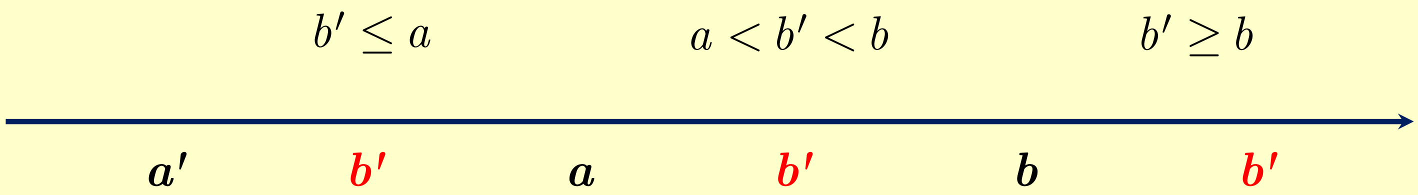
Suppose there exists another interval $[a', b']$ with $b' - a' < b - a$.

We will show

$$\int_{a'}^{b'} f(x)dx < 1 - \alpha.$$

Only proof the case that $a' \leq a$.

Other cases can be proved similarily.



# Proof

If $b' \leq a$, then $a' \leq b' \leq a \leq x^\star \leq b$. Since $x^\star$ is the mode, for any $a' \leq x \leq b'$,

$$f(a') \leq f(x) \leq f(b') \leq f(a) \leq f(x^\star),$$

Therefore

$$\int_{a'}^{b'} f(x)dx \leq \int_{a'}^{b'} f(b')dx = f(b')(b'-a') \leq f(a)(b'-a') < f(a)(b-a).$$

Since $f(a) = f(b)$ and $a \leq x^\star \leq b$, for any $a \leq x \leq b$, $f(a) = f(b) \leq f(x)$

$$f(a)(b-a) = \int_a^b f(a)dx \leq \int_a^b f(x)dx = 1-\alpha.$$

Putting together, we have

$$\int_{a'}^{b'} f(x)dx < 1-\alpha$$

# Proof

If $a < b' < b$, then $a' \leq a < b' < b$. In this case,

$$\begin{aligned}\int_{a'}^{b'} f(x)dx &= \int_{a'}^{a} f(x)dx + \int_{a}^{b} f(x)dx - \int_{b'}^{b} f(x)dx \\ &= 1 - \alpha + \int_{a'}^{a} f(x)dx - \int_{b'}^{b} f(x)dx\end{aligned}$$

Because $a' \leq a \leq x^\star$, for any $a' \leq x \leq a$, $f(a') \leq f(x) \leq f(a)$

$$\int_{a'}^{a} f(x)dx \leq \int_{a'}^{a} f(a)dx = f(a)(a - a')$$

Because $a < b' < b$, for any $b' \leq x \leq b$, $f(x) \geq f(b)$

$$\int_{b'}^{b} f(x)dx \geq \int_{b'}^{b} f(b)dx = f(b)(b - b')$$

Thus

$$\begin{aligned}\int_{a'}^{a} f(x)dx - \int_{b'}^{b} f(x)dx &\leq f(a)(a - a') - f(b)(b - b') \\ &= f(a)[(a - a') - (b - b')] \\ &= f(a)[(b' - a') - (b - a)] \\ &< 0\end{aligned}$$

Therefore $\int_{a'}^{b'} f(x)dx < 1 - \alpha$

$a < b' < b$

$\boldsymbol{a'}$ $\boldsymbol{a}$ $\boldsymbol{b'}$ $\boldsymbol{b}$

# Proof

It is impossible that $b' \geq b$, because this implies $b' - a' > b - a$.

# Equal tail yields minimum length

Let $X \sim f(x),$ where $f$ is a **symmetric unimodal** pdf. For a fixed value of $1-\alpha,$ of all intervals $[a,b]$ that satisfy

$$\int_a^b f(x)dx = 1-\alpha$$

the shortest length inerval is obtained by choosing $a$ and $b$ such that

$$\int_{-\infty}^a f(x)dx = \alpha / 2 \text{ and } \int_b^\infty f(x)dx = \alpha / 2$$

# Proof

Let $a$ and $b$ be the fixed value such that $\int_{-\infty}^{a} f(x)dx = \int_{b}^{\infty} f(x)dx = \alpha / 2$

then $\int_{a}^{b} f(x)dx = 1 - \int_{-\infty}^{a} f(x)dx - \int_{b}^{\infty} f(x)dx = 1 - \alpha.$

Let $\mu$ be the symmetric point of $f(x)$, then for any $x$

$$f(\mu - x) = f(\mu + x) \text{ and } f(2\mu - x) = f(\mu + (\mu - x)) = f(\mu - (\mu - x)) = f(x)$$

Consider

$$\int_{b}^{\infty} f(x)dx \xLeftrightarrow{y=2\mu-x} -\int_{2\mu-b}^{-\infty} f(2\mu - y)dy = \int_{-\infty}^{2\mu-b} f(y)dy = \int_{-\infty}^{a} f(x)dx$$

That is to say, $a = 2\mu - b$, and hence $f(a) = f(2\mu - b) = f(b)$

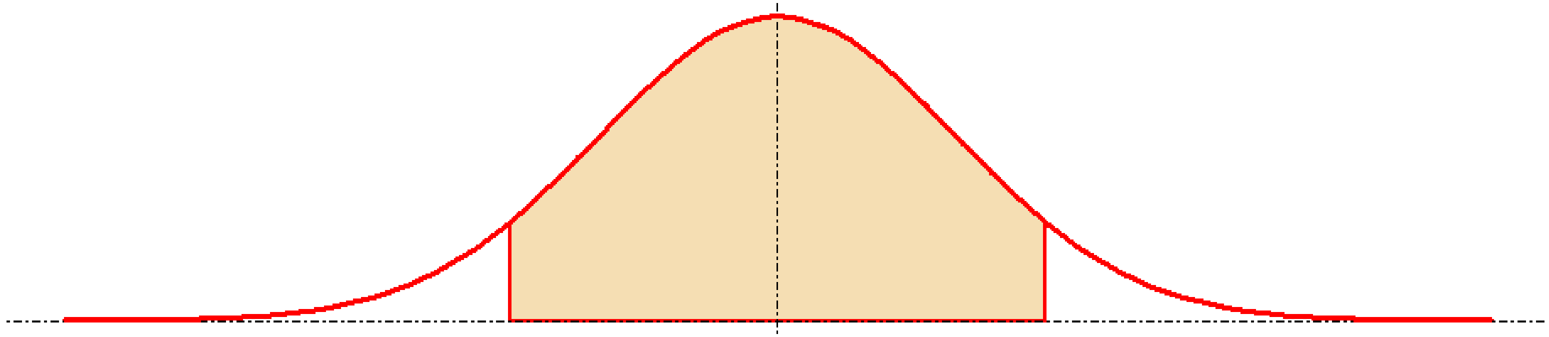
Because $\int_{-\infty}^{a} f(x)dx = \alpha / 2 > 0 \Rightarrow$ for some $x \le a$, $f(x) > 0$,

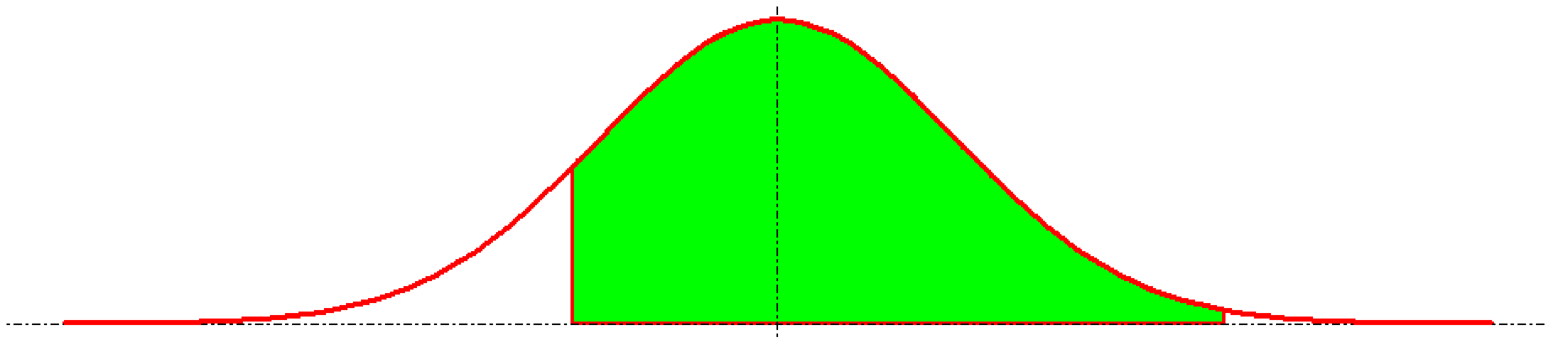
but $f(x) \le f(a)$ for any $x \le a$. Therefore $f(a) = f(b) > 0$

$\int_{-\infty}^{a} f(x)dx = \alpha / 2 \le 1/2 = \int_{-\infty}^{\mu} f(x)dx \Rightarrow a \le \mu$. Hence $b = 2\mu - a \ge 2\mu - \mu = \mu.$

For a unimodal and symmetry pdf $f(x)$, $f(x^\star) = f(2\mu - x^\star)$, the mode $x^\star$ must be $\mu$

Therefore $a \le x^\star \le b$





Symmetric, equal tail is equal height

# Normal optimal interval

Let $X_1,\ldots,X_n$ be a random sample from a normal population $N(\mu,\sigma^2)$, where $\sigma^2$ is known. Then

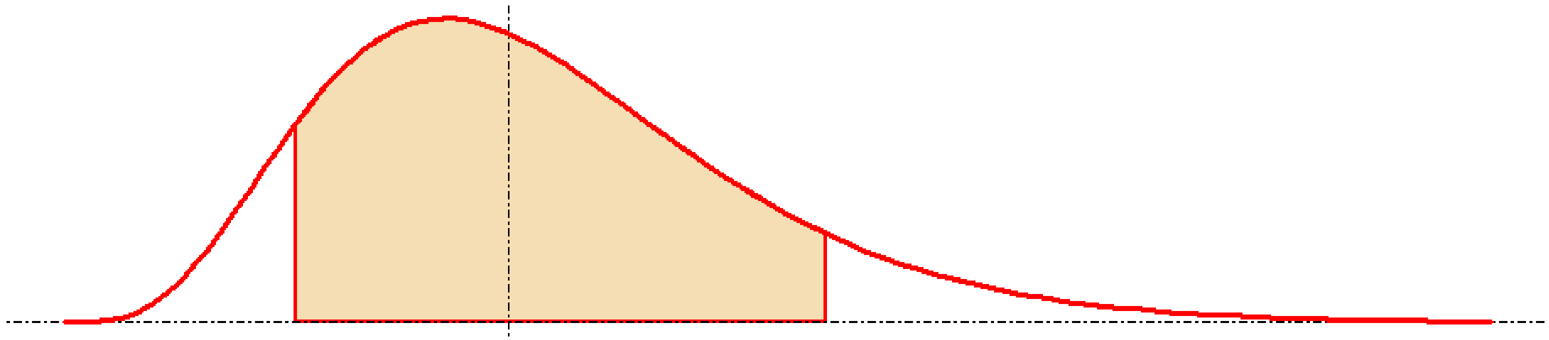
$$Z = \frac{\bar{X}-\mu}{\sigma / \sqrt{n}} \sim N(0,1)$$

is a pivotal quantity. In order to have

$$P(a \le Z \le b) = 1-\alpha,$$

With the use of the equal tail theorem, we choose $a = -z_{\alpha/2}$ and $b = z_{\alpha/2}$, yielding a minimum length interval

$$\bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \le \mu \le \bar{X} + z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$



Non-symmetric, equal tail is obviously not optimal

# Test-related optimality

Since there is a one-to-one correspondence between tests of hypotheses and confidence sets, there is some **correspondence between optimality of tests and optimality of confidence sets.** Usually, test-related optimality properties of confidence sets do not directly related to the size of the sets but rather to the probability of the set covering false values.

# True coverage and false coverage

Consider the general situation, where $\mathbf{X} \sim f(\mathbf{x} \mid \theta)$, and we construct a $1-\alpha$ condifence set for $\theta$, $C(\mathbf{x})$, by inverting an acceptance region $A(\theta)$. The probability that $C(\mathbf{x})$ covers the true parameter $\theta$ is called the **probability of true coverage**, which is a function of $\theta$ given by

$$P_\theta(\theta \in C(\mathbf{X}))$$

Therefore, the infimum of the probability of true coverage is the confidence coefficient $(1-\alpha)$ of the confidence set

The probability that $C(\mathbf{x})$ covers wrong parameters $\theta'$ is called the **probability of false coverage**, which is a function of $\theta$ given by

$$P_\theta(\theta' \in C(\mathbf{X})),\ \theta' \neq \theta,\ \text{if } C(\mathbf{X}) = [L(\mathbf{X}), U(\mathbf{X})]$$

$$P_\theta(\theta' \in C(\mathbf{X})),\ \theta' < \theta,\ \text{if } C(\mathbf{X}) = [L(\mathbf{X}), \infty)$$

$$P_\theta(\theta' \in C(\mathbf{X})),\ \theta' > \theta,\ \text{if } C(\mathbf{X}) = (-\infty, U(\mathbf{X})]$$

# Unbiasedness

A $1-\alpha$ confidence set $C(\mathbf{x})$ is said to be unbiased if

$$P_{\theta}(\theta' \in C(\mathbf{X})) \leq 1-\alpha \text{ for all } \theta' \neq \theta$$

That is, for an unbiased confidence set, the probability of false coverage is never more than the minimum probability of true coverage. This concept is usually related to two-sided intervals.

Recall that for an unbiased test, the power in the alternative is always greater than the power in the null.

# Unbiased interval of normal mean

$H_0$: $\mu = \mu_0$ versus $H_1$: $\mu \neq \mu_0$

The likelihood ratio test is to reject $H_0$ when

$$\bar{x} < \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} \text{ or } \bar{x} > \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}}$$

We have seen this test is unbiased.

The acceptance region of this unbiased test is

$$A(\mu_0) = \left\{ \mathbf{x} : \mu_0 - z_\alpha \frac{\sigma}{\sqrt{n}} \leq \bar{x} \leq \mu_0 + z_\alpha \frac{\sigma}{\sqrt{n}} \right\}$$

By inverting this acceptance region, we obtain the two-sided $1-\alpha$ unbiased confidence interval

$$C(\mathbf{x}) = \left\{ \mu : \bar{x} - z_\alpha \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{x} + z_\alpha \frac{\sigma}{\sqrt{n}} \right\}$$

# Uniformly most accurate (UMA) set

A $1-\alpha$ confidence set $C(\mathbf{x})$ is said to be **uniformly most accurate (UMA)** if it minimizes the probability of false coverage over a class of $1-\alpha$ confidence sets. This concept is usually related to one-sided intervals.

Because the one-to-one correspondence between the acceptance region and the confidence set, by inverting a UMP test, we will obtain a UMA confidence set.

$$P_\theta(\theta' \in C^\star(\mathbf{X})) \leq P_\theta(\theta' \in C(\mathbf{X}))$$

# Uniformly most powerful (UMP) test

Let $\mathcal{C}$ be the class of tests for testing

$$H_0 : \theta \in \Theta_0 \text{ versus } H_1 : \theta \in \Theta_0^c$$

A test in class $\mathcal{C}$, with power function $\beta(\theta)$ is a **uniformly most powerful** (UMP) class $\mathcal{C}$ test if $\beta(\theta) \geq \beta'(\theta)$ for every $\theta \in \Theta_0^c$ and every $\beta'(\theta)$ that is a power function of a test in class $\mathcal{C}$.

For a uniformly most powerful (UMP) test, its power in the alternative is always no less than that of any other test. In other words, 1−power is always no more than that of any other test. That is to say, a UMP test minimizes the probability that a sample belongs to the acceptance region when the alternative is true.

# Inverting a UMP

Let $A^{\star}(\theta_0)$ be the acceptance region of a level $\alpha$ UMP test for

$$H_0 : \theta = \theta_0 \text{ versus } H_1 : \theta > \theta_0$$

Let $\theta'$ be any value less than the true parameter $\theta$. Consider the hypothesis

$$H_0 : \theta = \theta' \text{ versus } H_1 : \theta > \theta'$$

By the definition of UMP,

$$P_\theta(\mathbf{X} \in R^{\star}(\theta')) \geq P_\theta(\mathbf{X} \in R(\theta')), \text{ for any } \theta > \theta'$$

$$\Rightarrow \quad P_\theta(\mathbf{X} \in A^{\star}(\theta')) \leq P_\theta(\mathbf{X} \in A(\theta')), \text{ for any } \theta > \theta'$$

where $A(\theta')$ is the acceptance region of any level $\alpha$ test other than the UMP one. Now, inverting $A^{\star}(\theta')$ to obtain $C^{\star}(\mathbf{x})$, inverting $A(\theta')$ to obtain $C(\mathbf{x})$, both $C^{\star}(\mathbf{x})$ and $C(\mathbf{x})$ are $1-\alpha$ confidence set, but for any $\theta' < \theta$,

$$\begin{aligned} P_\theta(\theta' \in C^{\star}(\mathbf{X})) &= P_\theta(\mathbf{X} \in A^{\star}(\theta')) \\ &\leq P_\theta(\mathbf{X} \in A(\theta')) \\ &= P_\theta(\theta' \in C(\mathbf{X})) \end{aligned}$$

Therefore, $C^{\star}(\mathbf{x})$ is the UMA condidence set.

# UMA interval of normal mean

**$H_0$: $\mu = \mu_0$ versus $H_1$: $\mu > \mu_0$**

The likelihood ratio test suggest to reject $H_0$ if

$$\bar{x} > \mu_0 + z_\alpha \sigma / \sqrt{n}$$

We have proved that $\bar{X}$ is a sufficient statistic of $\mu$ in the case that $\sigma^2$ is known, and the likelihood ratio is monotone (MLR), as we have seen. By Karlin-Rubin Theorem, the LRT, which rejects $H_0$ if and only if

$$\bar{x} > \mu_0 + z_\alpha \sigma / \sqrt{n}$$

is a UMP level $\alpha$ test, where $\alpha = P(Z > z_\alpha)$ defines $z_\alpha$.

The acceptance region of the above UMP test is

$$A(\mu_0) = \{\mathbf{x} : \bar{x} \le \mu_0 + z_\alpha \sigma / \sqrt{n}\}$$

By inverting this acceptance region, we obtain the one-sided $1 - \alpha$ UMA confidence interval

$$C(\mathbf{x}) = \{\mu : \mu \ge \bar{x} - z_\alpha \sigma / \sqrt{n}\}$$

# Practice

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*"The random interval [L(X), U(X)] is the interval estimator of the parameter, such that the probability of parameter located in the interval with high probability."*

# Active group *vs* Placebo group

**1**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 98 | 113 | 103 | 96 | 110 | 96 | 97 | 96 | 94 | 91 |
| 97 | 113 | 110 | | | | | | | |

**2**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 77 | 88 | 100 | 83 | 108 | 64 | 74 | 87 | 104 | 59 |
| 91 | 96 | 99 | 108 | 98 | 83 | 83 | 92 | 110 | 73 |
| 98 | 89 | 88 | 87 | 102 | 89 | 91 | 108 | 119 | 90 |
| 105 | 72 | 88 | 117 | 78 | 96 | 80 | 97 | 96 | 92 |
| 85 | 109 | 108 | 87 | 80 | 88 | 85 | 85 | 82 | 98 |
| 106 | 114 | 90 | 81 | 61 | 84 | 64 | 79 | 77 | 111 |
| 80 | 65 | 97 | 89 | 107 | 92 | 94 | 112 | 70 | 110 |
| 76 | 77 | 64 | 80 | 96 | 93 | 94 | 73 | 87 | 111 |
| 102 | 94 | 78 | 81 | 85 | 75 | 91 | 113 | 87 | 80 |
| 90 | 77 | 119 | 113 | 96 | 94 | 88 | 103 | 89 | 84 |
| 71 | 106 | 77 | 86 | 71 | 104 | 108 | 81 | 117 | 99 |
| 86 | 96 | 91 | 87 | 77 | 100 | 95 | 83 | 93 | 96 |
| 94 | 93 | 91 | 106 | 101 | 95 | 109 | 127 | 95 | 83 |
| 78 | 69 | 82 | 89 | 120 | 109 | 85 | 114 | 94 | 92 |
| 92 | 94 | 100 | 98 | 104 | 109 | 112 | 66 | 83 | 85 |
| 124 | 95 | 75 | 100 | 95 | 92 | 95 | 113 | 100 | 95 |
| 87 | 90 | 92 | 101 | 97 | 82 | 82 | 88 | 90 | 103 |
| 106 | 114 | 93 | 57 | 76 | 77 | 72 | 82 | 101 | 98 |
| 96 | 114 | 109 | 76 | 99 | 90 | 95 | 64 | 67 | 69 |
| 80 | 101 | 103 | | | | | | | |

**3**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 109 | 97 | 112 | 115 | 113 | 110 | 105 | 114 | 115 | 90 |
| 92 | 104 | 109 | 104 | 115 | 90 | | | | |

**4**

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 82 | 103 | 116 | 94 | 87 | 93 | 124 | 126 | 131 | 102 |
| 115 | 103 | 92 | 105 | 105 | 103 | 92 | 103 | 96 | 133 |
| 99 | 85 | 103 | 109 | 101 | 97 | 130 | 98 | 101 | 87 |
| 112 | 92 | 96 | 102 | 89 | 108 | 115 | 83 | 116 | 101 |
| 93 | 96 | 130 | 113 | 135 | 112 | 90 | 92 | 102 | 102 |
| 97 | 107 | 130 | 121 | 99 | 102 | 103 | 109 | 105 | 77 |
| 93 | 97 | 96 | 86 | 110 | 107 | 91 | 113 | 133 | 112 |
| 86 | 77 | 94 | 134 | 108 | 92 | 101 | 104 | 95 | 81 |
| 112 | 98 | 91 | 90 | 100 | 93 | 69 | 110 | 91 | 92 |
| 103 | 103 | 85 | 80 | 93 | 100 | 93 | 91 | 96 | 102 |
| 110 | 124 | 106 | 100 | 133 | 128 | 126 | 92 | 91 | 92 |
| 78 | 104 | 117 | 133 | 111 | 110 | 116 | 92 | 106 | 110 |
| 130 | 116 | 110 | 111 | 94 | 100 | 95 | 94 | 95 | 111 |
| 99 | 110 | 102 | 116 | 99 | 98 | 107 | 67 | 113 | 102 |
| 125 | 137 | 97 | 102 | 107 | 95 | 125 | 95 | 107 | 131 |
| 136 | 90 | 113 | 87 | 105 | 134 | 105 | 110 | 132 | 109 |
| 97 | 100 | 107 | 81 | 90 | 100 | 115 | 75 | 106 | 116 |
| 100 | 93 | 132 | 105 | 103 | 135 | 79 | 105 | 134 | 87 |
| 106 | 102 | 122 | 130 | 105 | 142 | 132 | 136 | 132 | 102 |
| 99 | 106 | 106 | 96 | 102 | 72 | | | | |

# Practice

```
x <- read.table("active.txt");
y <- read.table("placebo.txt");

t.test(x[[1]], mu=90, alternative="l");

One Sample t-test

data:  x[[1]]

t = 22.1619, df = 192, p-value = 1
alternative hypothesis: true mean is less than 90

95 percent confidence interval:
     -Inf 103.0731

sample estimates:
mean of x
 102.1658
```

# Practice

```
x <- read.table("active.txt");
y <- read.table("placebo.txt");

t.test(x[[1]], x[[2]], paired=T);

Paired t-test

data:  x[[1]] and x[[2]]

----------------------- Hypothesis testing ---------------------
t = 12.3842, df = 192, p-value < 2.2e-16
alternative hypothesis: true difference in means is not equal to
0

----------------------- Interval estimation --------------------
95 percent confidence interval:
  8.555443 11.796888

------------------------  Point estimation ---------------------
sample estimates:
mean of the differences
               10.17617
```

# Practice

```
x <- read.table("d:/temp/active.txt");
y <- read.table("d:/temp/placebo.txt");

t.test(x[[2]], y[[2]]);

Welch Two Sample t-test

data:  x[[2]] and y[[2]]

----------------------- Hypothesis testing ---------------------
t = -7.5291, df = 384.883, p-value = 3.666e-13
alternative hypothesis: true difference in means is not equal to
0

----------------------- Interval estimation --------------------
95 percent confidence interval:
 -12.823933  -7.513119

------------------------  Point estimation ---------------------
sample estimates:
mean of x mean of y
 91.98964 102.15816
```

# Other Intervals

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*"The random interval [L(X), U(X)] is the interval estimator of the parameter, such that the probability of parameter located in the interval with high probability."*

# *Wald*-type intervals

Wald statistic:

$$Z_n = \frac{W_n - \theta}{S_n},$$

where $W_n$ is an estimator of $\theta$, $S_n$ is an estimate of the standard deviation **(standard error)** of $W_n$.

Because

$$Z_n = \frac{W_n - \theta}{S_n}$$

has a limiting $N(0,1)$ distribution, a pivotal quantity, the $1-\alpha$ confidence interval is then

$$W_n - z_{\alpha/2} S_n \le \theta \le W_n + z_{\alpha/2} S_n$$

# Prediction interval

$$P(L(\mathbf{X}) \leq Y \leq U(\mathbf{X})) \geq 1-\alpha$$

A prediction interval is a random interval covers a random variable, and the probability that the interval covers the random variable is at least $1-\alpha$.

# Tolerance interval

$$P\Big(F(U(\mathbf{X})\mid\theta)-F(L(\mathbf{X})\mid\theta)\geq p\Big)\geq 1-\alpha$$

A tolerance interval is a random interval covers a proportion ($p$) of a population, and the probability that the interval covers the proportion is at least $1-\alpha$.

# In summary

- Point estimation
  - Which parameter value is most likely to yield the observed sample value?
- Hypothesis testing
  - What sample values are inconsistent with the fixed population parameter?
  - What sample values are consistent with the fixed population parameter?
- **Interval estimation**
  - **What parameter values make the observed sample value most plausible?**

**Thank you very much**

# Distributions of the sample mean

Let $X_1, \ldots, X_n$ be a random sample from a normal population $N(\mu, \sigma^2)$, then

$$\frac{\bar{X} - \mu}{\sigma / \sqrt{n}}$$

has a standard normal distribution.

Let $X_1, \ldots, X_n$ be a random sample from a normal population $N(\mu, \sigma^2)$, then

$$\frac{\bar{X} - \mu}{S / \sqrt{n}}$$

has a Student's $t$ distribution with $n-1$ degrees of freedom.

# Make use of pivotal quantities

Let $X_1, \ldots, X_n$ be a random sample from a normal population $N(\mu, \sigma^2)$, where $\sigma^2$ is known. Since

$$T = \frac{\bar{X} - \mu}{S / \sqrt{n}}$$

is a pivotal quantity, we can consider the following inequality

$$a \le \frac{\bar{X} - \mu}{S / \sqrt{n}} \le b,$$

Because this can immediately give us an interval estimator

$$\bar{X} - bS / \sqrt{n} \le \mu \le \bar{X} + aS / \sqrt{n}$$

In order to insure a $1 - \alpha$ confidence coefficient, we need to have

$$P\left(a \le \frac{\bar{X} - \mu}{S / \sqrt{n}} \le b\right) = P(a \le T_{n-1} \le b)$$

With the equal tail constraint that $P(T_{n-1} \le a) = P(T_{n-1} \ge b)$, it is easy to obtain $b = -a = t_{n-1,\alpha/2}$.

# Distributions of the sample sum

Let $X_1, \ldots, X_n$ be a random sample from a normal population $N(\mu, \sigma^2)$, then

$$\sum_{i=1}^{n}\left(\frac{X_i-\mu}{\sigma}\right)^2$$

has a $\chi^2$ distribution with $n$ degrees of freedom

Let $X_1, \ldots, X_n$ be a random sample from a normal population $N(\mu, \sigma^2)$, then

$$\frac{(n-1)S^2}{\sigma^2}=\sum_{i=1}^{n}\left(\frac{X_i-\overline{X}}{\sigma}\right)^2$$

has a $\chi^2$ distribution with $n-1$ degree of freedom.

# Make use of pivotal quantities

Let $X_1, \dots, X_n$ be a random sample from a normal population $N(\mu, \sigma^2)$, where $\mu$ is known. Since

$$K = \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} \right)^2$$

is a pivotal quantity, we can consider the following inequality

$$a \le \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} \right)^2 \le b,$$

Because this can immediately give us an interval estimator

$$\frac{\sum_{i=1}^{n} (X_i - \mu)^2}{b} \le \sigma^2 \le \frac{\sum_{i=1}^{n} (X_i - \mu)^2}{a}$$

In order to insure a $1-\alpha$ confidence coefficient, we need to have

$$P\left( a \le \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} \right)^2 \le b \right) = P(a \le \chi_n^2 \le b) = 1 - \alpha$$

With the equal tail constraint that $P(\chi_n^2 \le a) = P(\chi_n^2 \ge b)$, it is easy to obtain $a = \chi^2_{n,1-\alpha/2}$ and $b = \chi^2_{n,\alpha/2}$.

# The calculus of probabilities

***For two events***

If $P$ is a probability function and $A$ and B are any two sets in $\mathcal{B}$, then

1. $P(A) = P(A \cap B) + P(A \cap B^c)$;
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$;
3. $P(A \cup B) \leq P(A) + P(B)$;
4. $P(A \cap B) \geq P(A) + P(B) - 1$ (Bonferroni's inequality);
5. If $A \subset B$, then $P(A) \leq P(B)$.

# Proof

Because

$$(A \cap B) \cap (A \cap B^c) = (A \cap A) \cap (B \cap B^c) = A \cap \emptyset = \emptyset$$

and $\quad (A \cap B) \cup (A \cap B^c) = A \cap (B \cup B^c) = A \cap S = A,$

$$P(A) = P((A \cap B) \cup (A \cap B^c)) = P(A \cap B) + P(A \cap B^c)$$

Because

$$(A \cap B^c) \cap B = A \cap (B^c \cap B) = A \cap \emptyset = \emptyset$$

and $\quad (A \cap B^c) \cup B = (A \cup B) \cap (B^c \cup B) = (A \cup B) \cap S = A \cup B,$

$$P(A \cup B) = P(A \cap B^c) + P(B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) \leq P(A) + P(B) - 0 = P(A) + P(B)$$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1$$

If $A \subset B$, then $P(A \cap B) = P(A)$,

$$P(B) = P(B \cap A) + P(B \cap A^c) = P(A) + P(B \cap A^c) \geq P(A) + 0$$

Therefore,

$$P(A) \leq P(B)$$

# Optimizing expected length

Let $X_1,\ldots,X_n$ be a random sample from a normal population $N(\mu,\sigma^2)$, where $\sigma^2$ is **unknown**. Then

$$T = \frac{\bar{X}-\mu}{S / \sqrt{n}} \sim T_{n-1}$$

is a pivotal quantity.

Choose

$$a \le \frac{\bar{X}-\mu}{S / \sqrt{n}} \le b$$

will give us an interval

$$\bar{X} - b\frac{S}{\sqrt{n}} \le \mu \le \bar{X} - a\frac{S}{\sqrt{n}}$$

but the length of the interval

$$Length(S) = 2(b-a)\frac{S}{\sqrt{n}}$$

is a function of sample standard derivation. How to select $a$ and $b$?

# Chi distribution

Suppose $X \sim \chi_p^2$, say,

$$f_X(x) = \frac{1}{\Gamma(p/2)2^{p/2}} x^{p/2-1} e^{-x/2}, \ p > 0, x > 0$$

Consider the transformation $Y = \sqrt{X}$

$$x = y^2, \quad \frac{dx}{dy} = 2y$$

The pdf of $Y$ is then

$$f_Y(y) = 2y \frac{1}{\Gamma(p/2)2^{p/2}} (y^2)^{p/2-1} e^{-y^2/2} = \frac{1}{\Gamma(p/2)2^{p/2-1}} y^{p-1} e^{-y^2/2}, \ p > 0$$

which is called a **chi distribution** with $p$ degrees of freedom $(\chi_p)$.

The mean is calculated as

$$\begin{aligned} \mathrm{E}\,Y &= \int_{-\infty}^{\infty} y \frac{1}{\Gamma(p/2)2^{p/2-1}} y^{p-1} e^{-y^2/2} dy \\ &= \frac{1}{\Gamma(p/2)2^{p/2-1}} \int_{-\infty}^{\infty} y^{(p+1)-1} e^{-y^2/2} dy \\ &= \sqrt{2} \frac{\Gamma((p+1)/2)}{\Gamma(p/2)} \end{aligned}$$

# Normal optimal interval

Because $$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$$

$$\frac{\sqrt{n-1}S}{\sigma} \sim \chi_{n-1}$$

Hence $$\mathrm{E}S = \frac{\sigma}{\sqrt{n-1}}\mathrm{E}\chi_{n-1} = \frac{\sqrt{2}}{\sqrt{n-1}}\frac{\Gamma(n/2)}{\Gamma((n-1)/2)}\sigma = c(n)\sigma$$

The expected length of the interval

$$\bar{X} - b\frac{S}{\sqrt{n}} \le \mu \le \bar{X} - a\frac{S}{\sqrt{n}}$$

is $$\mathrm{E}[Length(S)] = (b-a)\frac{\mathrm{E}S}{\sqrt{n}} = (b-a)\frac{c(n)}{\sqrt{n}}\sigma$$

which is determined by $b-a$, and the equal tail theorem can be applied because $t$ distribution has a symmetric unimodal pdf. By selecting $a = -t_{n-1,\alpha/2}$ and $b = t_{n-1,\alpha/2}$, we have a minimum expected length interval

$$\bar{X} - t_{n-1,\alpha/2}\frac{S}{\sqrt{n}} \le \mu \le \bar{X} + t_{n-1,\alpha/2}\frac{S}{\sqrt{n}}$$

# *Wald*-type interval of normal mean

Approximated Wald statistic:

$$Z = \frac{\bar{X} - \mu}{S \,/\, \sqrt{n}} \to N(0,1),$$

a pivotal quantity. Therefore the $1-\alpha$ confidence interval for $\mu$ is then

$$\bar{X} - z_{\alpha/2} S \,/\, \sqrt{n} \le \mu \le \bar{X} + z_{\alpha/2} S \,/\, \sqrt{n}$$

# Credible interval of normal mean

With the conjugate prior $N(\upsilon, \tau^2)$, the posterior distribution of the normal mean for $N(\mu, \sigma^2)$ is

$$\pi(\mu \mid \mathbf{x}) \sim \mathrm{N}\left(\frac{n\tau^2}{\sigma^2 + n\tau^2}\overline{x} + \frac{\sigma^2}{\sigma^2 + n\tau^2}\upsilon, \frac{\sigma^2\tau^2}{\sigma^2 + n\tau^2}\right)$$

A $1-\alpha$ credible interval for the normal mean is then given as

$$\frac{n\tau^2\overline{x} + \sigma^2\upsilon}{\sigma^2 + n\tau^2} - z_{\alpha/2}\sqrt{\frac{\sigma^2\tau^2}{\sigma^2 + n\tau^2}} \le \mu \le \frac{n\tau^2\overline{x} + \sigma^2\upsilon}{\sigma^2 + n\tau^2} + z_{\alpha/2}\sqrt{\frac{\sigma^2\tau^2}{\sigma^2 + n\tau^2}}$$

# Inverting the test of the normal variance

Let $X_1,\ldots,X_n$ be a random sample from a normal population $N(\mu,\sigma^2)$, where $\mu$ is **unknown**.

Can we obtain a closed interval $[L(\mathbf{X}),U(\mathbf{X})]$ for $\sigma^2$?