CHAPTER 14
Wiener Processes and Itô's Lemma

Practice Questions

Problem 14.1.
What would it mean to assert that the temperature at a certain place follows a Markov process? Do you think that temperatures do, in fact, follow a Markov process?

Imagine that you have to forecast the future temperature from a) the current temperature, b) the history of the temperature in the last week, and c) a knowledge of seasonal averages and seasonal trends. If temperature followed a Markov process, the history of the temperature in the last week would be irrelevant.
To answer the second part of the question you might like to consider the following scenario for the first week in May:

(i) Monday to Thursday are warm days; today, Friday, is a very cold day.
(ii) Monday to Friday are all very cold days.

What is your forecast for the weekend? If you are more pessimistic in the case of the second scenario, temperatures do not follow a Markov process.

Problem 14.2.
Can a trading rule based on the past history of a stock's price ever produce returns that are consistently above average? Discuss.

The first point to make is that any trading strategy can, just because of good luck, produce above average returns. The key question is whether a trading strategy *consistently* outperforms the market when adjustments are made for risk. It is certainly possible that a trading strategy could do this. However, when enough investors know about the strategy and trade on the basis of the strategy, the profit will disappear.
As an illustration of this, consider a phenomenon known as the small firm effect. Portfolios of stocks in small firms appear to have outperformed portfolios of stocks in large firms when appropriate adjustments are made for risk. Research was published about this in the early 1980s and mutual funds were set up to take advantage of the phenomenon. There is some evidence that this has resulted in the phenomenon disappearing.

Problem 14.3.
A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.5 per quarter and a variance rate of 4.0 per quarter. How high does the company's initial cash position have to be for the company to have a less than 5% chance of a negative cash position by the end of one year?

Suppose that the company's initial cash position is x. The probability distribution of the cash position at the end of one year is

$$\varphi(x+4\times 0.5, 4\times 4)=\varphi(x+2.0, 16)$$

where $\varphi(m,v)$ is a normal probability distribution with mean m and variance v. The probability of a negative cash position at the end of one year is

$$N\left(-\frac{x+2.0}{4}\right)$$

where $N(x)$ is the cumulative probability that a standardized normal variable (with mean zero and standard deviation 1.0) is less than x. From the properties of the normal distribution

$$N\left(-\frac{x+2.0}{4}\right)=0.05$$

when:

$$-\frac{x+2.0}{4}=-1.6449$$

i.e., when $x=4.5796$. The initial cash position must therefore be $4.58 million.

Problem 14.4.

Variables X_1 and X_2 follow generalized Wiener processes with drift rates μ_1 and μ_2 and variances σ_1^2 and σ_2^2. What process does X_1+X_2 follow if:

(a) *The changes in X_1 and X_2 in any short interval of time are uncorrelated?*

(b) *There is a correlation ρ between the changes in X_1 and X_2 in any short interval of time?*

(a) Suppose that X_1 and X_2 equal a_1 and a_2 initially. After a time period of length T, X_1 has the probability distribution

$$\varphi(a_1+\mu_1 T,\sigma_1^2 T)$$

and X_2 has a probability distribution

$$\varphi(a_2+\mu_2 T,\sigma_2^2 T)$$

From the property of sums of independent normally distributed variables, X_1+X_2 has the probability distribution

$$\varphi\left(a_1+\mu_1 T+a_2+\mu_2 T,\sigma_1^2 T+\sigma_2^2 T\right)$$

i.e.,

$$\varphi\left[a_1+a_2+(\mu_1+\mu_2)T,(\sigma_1^2+\sigma_2^2)T\right]$$

This shows that X_1+X_2 follows a generalized Wiener process with drift rate $\mu_1+\mu_2$ and variance rate $\sigma_1^2+\sigma_2^2$.

(b) In this case the change in the value of X_1+X_2 in a short interval of time Δt has the probability distribution:

$$\varphi\left[(\mu_1+\mu_2)\Delta t,(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)\Delta t\right]$$

If μ_1, μ_2, σ_1, σ_2 and ρ are all constant, arguments similar to those in Section 14.2 show that the change in a longer period of time T is

$$\varphi\left[(\mu_1+\mu_2)T,(\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2)T\right]$$

The variable, X_1+X_2, therefore follows a generalized Wiener process with drift rate $\mu_1+\mu_2$ and variance rate $\sigma_1^2+\sigma_2^2+2\rho\sigma_1\sigma_2$.

Problem 14.5.

Consider a variable, S, that follows the process

$$dS=\mu\, dt+\sigma\, dz$$

For the first three years, $\mu=2$ and $\sigma=3$; for the next three years, $\mu=3$ and $\sigma=4$. If the initial value of the variable is 5, what is the probability distribution of the value of the variable at the end of year six?

The change in S during the first three years has the probability distribution

$$\varphi(2\times 3,\ 9\times 3)=\varphi(6,\ 27)$$

The change during the next three years has the probability distribution

$$\varphi(3\times 3,\ 16\times 3)=\varphi(9,\ 48)$$

The change during the six years is the sum of a variable with probability distribution $\varphi(6,27)$ and a variable with probability distribution $\varphi(9,48)$. The probability distribution of the change is therefore

$$\varphi(6+9,27+48)$$

$$=\varphi(15,75)$$

Since the initial value of the variable is 5, the probability distribution of the value of the variable at the end of year six is

$$\varphi(20,75)$$

Problem 14.6.

Suppose that G is a function of a stock price, S and time. Suppose that σ_S and σ_G are the volatilities of S and G. Show that when the expected return of S increases by $\lambda\sigma_S$, the growth rate of G increases by $\lambda\sigma_G$, where λ is a constant.

From Itô's lemma

$$\sigma_G G=\frac{\partial G}{\partial S}\sigma_S S$$

Also the drift of G is

$$\frac{\partial G}{\partial S}\mu S+\frac{\partial G}{\partial t}+\frac{1}{2}\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2$$

where μ is the expected return on the stock. When μ increases by $\lambda\sigma_S$, the drift of G increases by

$$\frac{\partial G}{\partial S}\lambda\sigma_S S$$

or

$$\lambda\sigma_G G$$

The growth rate of G, therefore, increases by $\lambda\sigma_G$.

Problem 14.7.
Stock A and stock B both follow geometric Brownian motion. Changes in any short interval of time are uncorrelated with each other. Does the value of a portfolio consisting of one of stock A and one of stock B follow geometric Brownian motion? Explain your answer.

Define S_A, μ_A and σ_A as the stock price, expected return and volatility for stock A. Define S_B, μ_B and σ_B as the stock price, expected return and volatility for stock B. Define ΔS_A and ΔS_B as the change in S_A and S_B in time Δt. Since each of the two stocks follows geometric Brownian motion,

$$\Delta S_A = \mu_A S_A \Delta t + \sigma_A S_A \varepsilon_A \sqrt{\Delta t}$$

$$\Delta S_B = \mu_B S_B \Delta t + \sigma_B S_B \varepsilon_B \sqrt{\Delta t}$$

where ε_A and ε_B are independent random samples from a normal distribution.

$$\Delta S_A + \Delta S_B = (\mu_A S_A + \mu_B S_B)\Delta t + (\sigma_A S_A \varepsilon_A + \sigma_B S_B \varepsilon_B)\sqrt{\Delta t}$$

This *cannot* be written as

$$\Delta S_A + \Delta S_B = \mu(S_A + S_B)\Delta t + \sigma(S_A + S_B)\varepsilon\sqrt{\Delta t}$$

for any constants μ and σ. (Neither the drift term nor the stochastic term correspond.) Hence the value of the portfolio does not follow geometric Brownian motion.

Problem 14.8.
The process for the stock price in equation (14.8) is

$$\Delta S = \mu S\,\Delta t + \sigma S \varepsilon\sqrt{\Delta t}$$

where μ and σ are constant. Explain carefully the difference between this model and each of the following:

$$\Delta S = \mu\,\Delta t + \sigma\varepsilon\sqrt{\Delta t}$$

$$\Delta S = \mu S\,\Delta t + \sigma\varepsilon\sqrt{\Delta t}$$

$$\Delta S = \mu\,\Delta t + \sigma S\varepsilon\sqrt{\Delta t}$$

Why is the model in equation (14.8) a more appropriate model of stock price behavior than any of these three alternatives?

In:

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

the expected increase in the stock price and the variability of the stock price are constant when both are expressed as a proportion (or as a percentage) of the stock price
In:

$$\Delta S = \mu \Delta t + \sigma \varepsilon \sqrt{\Delta t}$$

the expected increase in the stock price and the variability of the stock price are constant in absolute terms. For example, if the expected growth rate is $5 per annum when the stock price is $25, it is also $5 per annum when it is $100. If the standard deviation of weekly stock price movements is $1 when the price is $25, it is also $1 when the price is $100.
In:

$$\Delta S = \mu S \Delta t + \sigma \varepsilon \sqrt{\Delta t}$$

the expected increase in the stock price is a constant proportion of the stock price while the variability is constant in absolute terms.
In:

$$\Delta S = \mu \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

the expected increase in the stock price is constant in absolute terms while the variability of the proportional stock price change is constant.
The model:

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

is the most appropriate one since it is most realistic to assume that the expected *percentage return* and the variability of the *percentage return* in a short interval are constant.

Problem 14.9.
It has been suggested that the short-term interest rate, r, follows the stochastic process

$$dr = a(b-r)\,dt + rc\,dz$$

where a, b, and c are positive constants and dz is a Wiener process. Describe the nature of this process.

The drift rate is $a(b-r)$. Thus, when the interest rate is above b the drift rate is negative and, when the interest rate is below b, the drift rate is positive. The interest rate is therefore continually pulled towards the level b. The rate at which it is pulled toward this level is a. A volatility equal to c is superimposed upon the "pull" or the drift.
Suppose $a = 0.4$, $b = 0.1$ and $c = 0.15$ and the current interest rate is 20% per annum. The interest rate is pulled towards the level of 10% per annum. This can be regarded as a long run average. The current drift is -4% per annum so that the expected rate at the end of one year is about 16% per annum. (In fact it is slightly greater than this, because as the interest rate decreases, the "pull" decreases.) Superimposed upon the drift is a volatility of 15% per annum.

Problem 14.10.
Suppose that a stock price, S, follows geometric Brownian motion with expected return μ

and volatility σ *:*

$$dS = \mu S\,dt + \sigma S\,dz$$

What is the process followed by the variable S^n *? Show that* S^n *also follows geometric Brownian motion.*

If $G(S,t) = S^n$ then $\partial G/\partial t = 0$, $\partial G/\partial S = nS^{n-1}$, and $\partial^2 G/\partial S^2 = n(n-1)S^{n-2}$. Using Itô's lemma:

$$dG = [\mu nG + \frac{1}{2}n(n-1)\sigma^2 G]\,dt + \sigma nG\,dz$$

This shows that $G = S^n$ follows geometric Brownian motion where the expected return is

$$\mu n + \frac{1}{2}n(n-1)\sigma^2$$

and the volatility is $n\sigma$. The stock price S has an expected return of μ and the expected value of S_T is $S_0 e^{\mu T}$. The expected value of S_T^n is

$$S_0^n e^{[\mu n + \frac{1}{2}n(n-1)\sigma^2]T}$$

Problem 14.11.

Suppose that x *is the yield to maturity with continuous compounding on a zero-coupon bond that pays off \$1 at time* T *. Assume that* x *follows the process*

$$dx = a(x_0 - x)\,dt + sx\,dz$$

where a *,* x_0 *, and* s *are positive constants and* dz *is a Wiener process. What is the process followed by the bond price?*

The process followed by B, the bond price, is from Itô's lemma:

$$dB = \left[\frac{\partial B}{\partial x}a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}s^2x^2\right]dt + \frac{\partial B}{\partial x}sxdz$$

Since:

$$B = e^{-x(T-t)}$$

the required partial derivatives are

$$\frac{\partial B}{\partial t} = xe^{-x(T-t)} = xB$$

$$\frac{\partial B}{\partial x} = -(T-t)e^{-x(T-t)} = -(T-t)B$$

$$\frac{\partial^2 B}{\partial x^2} = (T-t)^2 e^{-x(T-t)} = (T-t)^2 B$$

Hence:

$$dB = \left[-a(x_0 - x)(T-t) + x + \frac{1}{2}s^2x^2(T-t)^2\right]Bdt - sx(T-t)Bdz$$

Problem 14.12. (Excel Spreadsheet)
A stock whose price is $30 has an expected return of 9% and a volatility of 20%. In Excel simulate the stock price path over 5 years using monthly time steps and random samples from a normal distribution. Chart the simulated stock price path. By hitting F9 observe how the path changes as the random sample change.

The process is
$\Delta S = 0.09 \times S \times \Delta t + 0.20 \times S \times \varepsilon \times \sqrt{\Delta t}$
where Δt is the length of the time step (=1/12) and ε is a random sample from a standard normal distribution.

Further Questions

Problem 14.13.
Suppose that a stock price has an expected return of 16% per annum and a volatility of 30% per annum. When the stock price at the end of a certain day is $50, calculate the following:
- *(a) The expected stock price at the end of the next day.*
- *(b) The standard deviation of the stock price at the end of the next day.*
- *(c) The 95% confidence limits for the stock price at the end of the next day.*

With the notation in the text

$$\frac{\Delta S}{S} \sim \phi\left(\mu \Delta t, \sigma^2 \Delta t\right)$$

In this case $S = 50$, $\mu = 0.16$, $\sigma = 0.30$ and $\Delta t = 1/365 = 0.00274$. Hence

$$\frac{\Delta S}{50} \sim \varphi(0.16 \times 0.00274, 0.09 \times 0.00274)$$
$$= \varphi(0.00044, 0.000247)$$

and

$$\Delta S \sim \varphi(50 \times 0.00044, 50^2 \times 0.000247)$$

that is,

$$\Delta S \sim \varphi(0.022, 0.6164)$$

(a) The expected stock price at the end of the next day is therefore 50.022
(b) The standard deviation of the stock price at the end of the next day is $\sqrt{0.6154} = 0.785$
(c) 95% confidence limits for the stock price at the end of the next day are

$$50.022 - 1.96 \times 0.785 \quad \text{and} \quad 50.022 + 1.96 \times 0.785$$

i.e.,

$$48.48 \quad \text{and} \quad 51.56$$

Note that some students may consider one trading day rather than one calendar day. Then $\Delta t = 1/252 = 0.00397$. The answer to (a) is then 50.032. The answer to (b) is 0.945. The answers to part (c) are 48.18 and 51.88.

Problem 14.14.

\A company's cash position, measured in millions of dollars, follows a generalized Wiener process with a drift rate of 0.1 per month and a variance rate of 0.16 per month. The initial cash position is 2.0.

(a) What are the probability distributions of the cash position after one month, six months, and one year?

(b) What are the probabilities of a negative cash position at the end of six months and one year?

(c) At what time in the future is the probability of a negative cash position greatest?

(a) The probability distributions are:

$$\varphi(2.0+0.1, 0.16) = \varphi(2.1, 0.16)$$

$$\varphi(2.0+0.6, 0.16\times 6) = \varphi(2.6, 0.96)$$
$$\varphi(2.0+1.2, 0.16\times 12) = \varphi(3.2, 1.92)$$

(b) The chance of a random sample from $\varphi(2.6, 0.96)$ being negative is

$$N\left(-\frac{2.6}{\sqrt{0.96}}\right) = N(-2.65)$$

where $N(x)$ is the cumulative probability that a standardized normal variable [i.e., a variable with probability distribution $\varphi(0,1)$] is less than x.
Because $N(-2.65) = 0.0040$, the probability of a negative cash position at the end of six months is 0.40%.
Similarly the probability of a negative cash position at the end of one year is

$$N\left(-\frac{3.2}{\sqrt{1.92}}\right) = N(-2.31) = 0.0105$$

or 1.05%.

(c) In general the probability distribution of the cash position at the end of x months is

$$\varphi(2.0+0.1x, 0.16x)$$

The probability of the cash position being negative is maximized when:

$$\frac{2.0+0.1x}{\sqrt{0.16x}}$$

is minimized. Define

$$y = \frac{2.0 + 0.1x}{0.4\sqrt{x}} = 5x^{-\frac{1}{2}} + 0.25x^{\frac{1}{2}}$$

$$\frac{dy}{dx} = -2.5x^{-\frac{3}{2}} + 0.125x^{-\frac{1}{2}}$$

$$= x^{-\frac{3}{2}}(-2.5 + 0.125x)$$

This is zero when $x = 20$ and it is easy to verify that $d^2y/dx^2 > 0$ for this value of x. It therefore gives a minimum value for y. Hence the probability of a negative cash position is greatest after 20 months.

Problem 14.15.

Suppose that x is the yield on a perpetual government bond that pays interest at the rate of \$1 per annum. Assume that x is expressed with continuous compounding, that interest is paid continuously on the bond, and that x follows the process

$$dx = a(x_0 - x)\,dt + sx\,dz$$

where a, x_0, and s are positive constants and dz is a Wiener process. What is the process followed by the bond price? What is the expected instantaneous return (including interest and capital gains) to the holder of the bond?

The process followed by B, the bond price, is from Itô's lemma:

$$dB = \left[\frac{\partial B}{\partial x}a(x_0 - x) + \frac{\partial B}{\partial t} + \frac{1}{2}\frac{\partial^2 B}{\partial x^2}s^2x^2\right]dt + \frac{\partial B}{\partial x}sxdz$$

In this case

$$B = \frac{1}{x}$$

so that:

$$\frac{\partial B}{\partial t} = 0; \quad \frac{\partial B}{\partial x} = -\frac{1}{x^2}; \quad \frac{\partial^2 B}{\partial x^2} = \frac{2}{x^3}$$

Hence

$$dB = \left[-a(x_0 - x)\frac{1}{x^2} + \frac{1}{2}s^2x^2\frac{2}{x^3}\right]dt - \frac{1}{x^2}sxdz$$

$$= \left[-a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}\right]dt - \frac{s}{x}dz$$

The expected instantaneous rate at which capital gains are earned from the bond is therefore:

$$-a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}$$

The expected interest per unit time is 1. The total expected instantaneous return is therefore:

$$1 - a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}$$

When expressed as a proportion of the bond price this is:

$$\left(1 - a(x_0 - x)\frac{1}{x^2} + \frac{s^2}{x}\right)\Bigg/\left(\frac{1}{x}\right)$$

$$= x - \frac{a}{x}(x_0 - x) + s^2$$

Problem 14.16.

If S follows the geometric Brownian motion process in equation (14.6), what is the process followed by (a) y = 2S, (b) $y=S^2$, (c) $y=e^S$, and (d) $y=e^{r(T-t)}/S$. In each case express the coefficients of dt and dz in terms of y rather than S.

(a) In this case $\partial y/\partial S = 2$, $\partial^2 y/\partial S^2 = 0$, and $\partial y/\partial t = 0$ so that Itô's lemma gives

$$dy = 2\mu S\,dt + 2\sigma S\,dz$$

or

$$dy = \mu y\,dt + \sigma y\,dz$$

(b) In this case $\partial y/\partial S = 2S$, $\partial^2 y/\partial S^2 = 2$, and $\partial y/\partial t = 0$ so that Itô's lemma gives

$$dy = (2\mu S^2 + \sigma^2 S^2)\,dt + 2\sigma S^2\,dz$$

or

$$dy = (2\mu + \sigma^2)y\,dt + 2\sigma y\,dz$$

(c) In this case $\partial y/\partial S = e^S$, $\partial^2 y/\partial S^2 = e^S$, and $\partial y/\partial t = 0$ so that Itô's lemma gives

$$dy = (\mu S e^S + \sigma^2 S^2 e^S/2)\,dt + \sigma S e^S\,dz$$

or

$$dy = [\mu y \ln y + \sigma^2 y(\ln y)^2/2]\,dt + \sigma y \ln y\,dz$$

(d) In this case $\partial y/\partial S = -e^{r(T-t)}/S^2 = -y/S$, $\partial^2 y/\partial S^2 = 2e^{r(T-t)}/S^3 = 2y/S^2$, and $\partial y/\partial t = -re^{r(T-t)}/S = -ry$ so that Itô's lemma gives

$$dy = (-ry - \mu y + \sigma^2 y)\,dt - \sigma y\,dz$$

or

$$dy = -(r + \mu - \sigma^2)y\,dt - \sigma y\,dz$$

Problem 14.17.

A stock price is currently 50. Its expected return and volatility are 12% and 30%, respectively. What is the probability that the stock price will be greater than 80 in two years? (Hint $S_T > 80$ when $\ln S_T > \ln 80$.)

The variable $\ln S_T$ is normally distributed with mean $\ln S_0 + (\mu - \sigma^2/2)T$ and standard deviation $\sigma\sqrt{T}$. In this case $S_0 = 50$, $\mu = 0.12$, $T = 2$, and $\sigma = 0.30$ so that the mean and standard deviation of $\ln S_T$ are $\ln 50 + (0.12 - 0.3^2/2)2 = 4.062$ and $0.3\sqrt{2} = 0.424$, respectively. Also, $\ln 80 = 4.382$. The probability that $S_T > 80$ is the same as the probability that $\ln S_T > 4.382$. This is

$$1 - N\left(\frac{4.382 - 4.062}{0.424}\right) = 1 - N(0.754)$$

where $N(x)$ is the probability that a normally distributed variable with mean zero and standard deviation 1 is less than x. Because $N(0.754) = 0.775$, the required probability is 0.225.

Problem 14.18. (See Excel Worksheet)

Stock A, whose price is \$30, has an expected return of 11% and a volatility of 25%. Stock B, whose price is \$40, has an expected return of 15% and a volatility of 30%. The processes driving the returns are correlated with correlation parameter ρ. In Excel, simulate the two stock price paths over three months using daily time steps and random samples from normal distributions. Chart the results and by hitting F9 observe how the paths change as the random samples change. Consider values of ρ equal to 0.50, 0.75, and 0.95.

The processes are

$$\Delta S_A = 0.11 \times S_A \times \Delta t + 0.25 \times S_A \times \varepsilon_A \times \sqrt{\Delta t}$$

$$\Delta S_B = 0.15 \times S_B \times \Delta t + 0.30 \times S_B \times \varepsilon_B \times \sqrt{\Delta t}$$

Where Δt is the length of the time step (=1/252) and the ε's are correlated samples from standard normal distributions.