CHAPTER 15
The Black-Scholes-Merton Model

Practice Questions

Problem 15.1.
What does the Black–Scholes–Merton stock option pricing model assume about the probability distribution of the stock price in one year? What does it assume about the probability distribution of the continuously compounded rate of return on the stock during the year?

The Black–Scholes–Merton option pricing model assumes that the probability distribution of the stock price in 1 year (or at any other future time) is lognormal. It assumes that the continuously compounded rate of return on the stock during the year is normally distributed.

Problem 15.2.
The volatility of a stock price is 30% per annum. What is the standard deviation of the percentage price change in one trading day?

The standard deviation of the percentage price change in time Δt is $\sigma\sqrt{\Delta t}$ where σ is the volatility. In this problem $\sigma = 0.3$ and, assuming 252 trading days in one year, $\Delta t = 1/252 = 0.004$ so that $\sigma\sqrt{\Delta t} = 0.3\sqrt{0.004} = 0.019$ or 1.9%.

Problem 15.3.
Explain the principle of risk-neutral valuation.

The price of an option or other derivative when expressed in terms of the price of the underlying stock is independent of risk preferences. Options therefore have the same value in a risk-neutral world as they do in the real world. We may therefore assume that the world is risk neutral for the purposes of valuing options. This simplifies the analysis. In a risk-neutral world all securities have an expected return equal to risk-free interest rate. Also, in a risk-neutral world, the appropriate discount rate to use for expected future cash flows is the risk-free interest rate.

Problem 15.4.
Calculate the price of a three-month European put option on a non-dividend-paying stock with a strike price of $50 when the current stock price is $50, the risk-free interest rate is 10% per annum, and the volatility is 30% per annum.

In this case $S_0 = 50$, $K = 50$, $r = 0.1$, $\sigma = 0.3$, $T = 0.25$, and

$$d_1 = \frac{\ln(50/50) + (0.1 + 0.09/2)0.25}{0.3\sqrt{0.25}} = 0.2417$$

$$d_2 = d_1 - 0.3\sqrt{0.25} = 0.0917$$

The European put price is

$$50N(-0.0917)e^{-0.1\times 0.25} - 50N(-0.2417)$$

$$=50\times 0.4634e^{-0.1\times 0.25}-50\times 0.4045=2.37$$

or $2.37.

Problem 15.5.

What difference does it make to your calculations in Problem 15.4 if a dividend of $1.50 is expected in two months?

In this case we must subtract the present value of the dividend from the stock price before using Black–Scholes-Merton. Hence the appropriate value of S_0 is

$$S_0=50-1.50e^{-0.1667\times 0.1}=48.52$$

As before $K=50$, $r=0.1$, $\sigma=0.3$, and $T=0.25$. In this case

$$d_1=\frac{\ln(48.52/50)+(0.1+0.09/2)0.25}{0.3\sqrt{0.25}}=0.0414$$

$$d_2=d_1-0.3\sqrt{0.25}=-0.1086$$

The European put price is

$$50N(0.1086)e^{-0.1\times 0.25}-48.52N(-0.0414)$$

$$=50\times 0.5432e^{-0.1\times 0.25}-48.52\times 0.4835=3.03$$

or $3.03.

Problem 15.6.

What is implied volatility? How can it be calculated?

The implied volatility is the volatility that makes the Black–Scholes-Merton price of an option equal to its market price. The implied volatility is calculated using an iterative procedure. A simple approach is the following. Suppose we have two volatilities one too high (i.e., giving an option price greater than the market price) and the other too low (i.e., giving an option price lower than the market price). By testing the volatility that is half way between the two, we get a new too-high volatility or a new too-low volatility. If we search initially for two volatilities, one too high and the other too low we can use this procedure repeatedly to bisect the range and converge on the correct implied volatility. Other more sophisticated approaches (e.g., involving the Newton-Raphson procedure) are used in practice.

Problem 15.7.

A stock price is currently $40. Assume that the expected return from the stock is 15% and its volatility is 25%. What is the probability distribution for the rate of return (with continuous compounding) earned over a two-year period?

In this case $\mu=0.15$ and $\sigma=0.25$. From equation (15.7) the probability distribution for the rate of return over a two-year period with continuous compounding is:

$$\varphi\left(0.15-\frac{0.25^2}{2},\frac{0.25^2}{2}\right)$$

i.e.,

$$\varphi(0.11875, 0.03125)$$

The expected value of the return is 11.875% per annum and the standard deviation is 17.7% per annum.

Problem 15.8.

A stock price follows geometric Brownian motion with an expected return of 16% and a volatility of 35%. The current price is $38.

a) *What is the probability that a European call option on the stock with an exercise price of $40 and a maturity date in six months will be exercised?*
b) *What is the probability that a European put option on the stock with the same exercise price and maturity will be exercised?*

a) The required probability is the probability of the stock price being above $40 in six months time. Suppose that the stock price in six months is S_T

$$\ln S_T \sim \varphi\left[\ln 38 + \left(0.16 - \frac{0.35^2}{2}\right)0.5, 0.35^2 \times 0.5\right]$$

i.e.,

$$\ln S_T \sim \varphi(3.687, 0.247^2)$$

Since $\ln 40 = 3.689$, we require the probability of $\ln(S_T)>3.689$. This is

$$1 - N\left(\frac{3.689 - 3.687}{0.247}\right) = 1 - N(0.008)$$

Since N(0.008) = 0.5032, the required probability is 0.4968.

b) In this case the required probability is the probability of the stock price being less than $40 in six months time. It is

$$1 - 0.4968 = 0.5032$$

Problem 15.9.

Using the notation in the chapter, prove that a 95% confidence interval for S_T *is between*

$S_0 e^{(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}}$ and $S_0 e^{(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$

From equation (15.3):

$$\ln S_T \sim \varphi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

95% confidence intervals for $\ln S_T$ are therefore

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T - 1.96\sigma\sqrt{T}$$

and

$$\ln S_0 + (\mu - \frac{\sigma^2}{2})T + 1.96\sigma\sqrt{T}$$

95% confidence intervals for S_T are therefore

$$e^{\ln S_0+(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}} \quad \text{and} \quad e^{\ln S_0+(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$$

i.e.

$$S_0e^{(\mu-\sigma^2/2)T-1.96\sigma\sqrt{T}} \quad \text{and} \quad S_0e^{(\mu-\sigma^2/2)T+1.96\sigma\sqrt{T}}$$

Problem 15.10.

A portfolio manager announces that the average of the returns realized in each of the last 10 years is 20% per annum. In what respect is this statement misleading?

This problem relates to the material in Section 15.3. The statement is misleading in that a certain sum of money, say $1000, when invested for 10 years in the fund would have realized a return (with annual compounding) of less than 20% per annum.
The average of the returns realized in each year is always greater than the return per annum (with annual compounding) realized over 10 years. The first is an arithmetic average of the returns in each year; the second is a geometric average of these returns.

Problem 15.11.

Assume that a non-dividend-paying stock has an expected return of μ and a volatility of σ. An innovative financial institution has just announced that it will trade a derivative that pays off a dollar amount equal to ln S_T at time T where S_T denotes the values of the stock price at time T.

a) *Use risk-neutral valuation to calculate the price of the derivative at time t in term of the stock price, S, at time t*
b) *Confirm that your price satisfies the differential equation (15.16)*

a) At time *t*, the expected value of ln S_T is from equation (15.3)

$$\ln S+(\mu-\sigma^2/2)(T-t)$$

In a risk-neutral world the expected value of ln S_T is therefore

$$\ln S+(r-\sigma^2/2)(T-t)$$

Using risk-neutral valuation the value of the derivative at time *t* is

$$e^{-r(T-t)}[\ln S+(r-\sigma^2/2)(T-t)]$$

b) If

$$f=e^{-r(T-t)}[\ln S+(r-\sigma^2/2)(T-t)]$$

then

$$\frac{\partial f}{\partial t}=re^{-r(T-t)}[\ln S+(r-\sigma^2/2)(T-t)]-e^{-r(T-t)}\left(r-\sigma^2/2\right)$$
$$\frac{\partial f}{\partial S}=\frac{e^{-r(T-t)}}{S}$$
$$\frac{\partial^2 f}{\partial S^2}=-\frac{e^{-r(T-t)}}{S^2}$$

The left-hand side of the Black-Scholes-Merton differential equation is

$$e^{-r(T-t)}\left[r\ln S+r(r-\sigma^2/2)(T-t)-(r-\sigma^2/2)+r-\sigma^2/2\right]$$
$$=e^{-r(T-t)}\left[r\ln S+r(r-\sigma^2/2)(T-t)\right]$$
$$=rf$$

Hence the differential equation is satisfied.

Problem 15.12.

Consider a derivative that pays off S_T^n at time T where S_T is the stock price at that time. When the stock pays no dividends and its price follows geometric Brownian motion, it can be shown that its price at time t $(t \le T)$ has the form

$$h(t,T)S^n$$

where S is the stock price at time t and h is a function only of t and T.

(a) By substituting into the Black–Scholes–Merton partial differential equation derive an ordinary differential equation satisfied by $h(t,T)$.

(b) What is the boundary condition for the differential equation for $h(t,T)$?

(c) Show that

$$h(t,T) = e^{[0.5\sigma^2 n(n-1)+r(n-1)](T-t)}$$

where r is the risk-free interest rate and σ is the stock price volatility.

If $G(S,t) = h(t,T)S^n$ then $\partial G/\partial t = h_t S^n$, $\partial G/\partial S = hnS^{n-1}$, and $\partial^2 G/\partial S^2 = hn(n-1)S^{n-2}$ where $h_t = \partial h/\partial t$. Substituting into the Black–Scholes–Merton differential equation we obtain

$$h_t + rhn + \frac{1}{2}\sigma^2 hn(n-1) = rh$$

The derivative is worth S^n when $t = T$. The boundary condition for this differential equation is therefore $h(T,T) = 1$

The equation

$$h(t,T) = e^{[0.5\sigma^2 n(n-1)+r(n-1)](T-t)}$$

satisfies the boundary condition since it collapses to $h = 1$ when $t = T$. It can also be shown that it satisfies the differential equation in (a). Alternatively we can solve the differential equation in (a) directly. The differential equation can be written

$$\frac{h_t}{h} = -r(n-1) - \frac{1}{2}\sigma^2 n(n-1)$$

The solution to this is

$$\ln h = [r(n-1) + \frac{1}{2}\sigma^2 n(n-1)](T-t)$$

or

$$h(t,T) = e^{[0.5\sigma^2 n(n-1)+r(n-1)](T-t)}$$

Problem 15.13.

What is the price of a European call option on a non-dividend-paying stock when the stock price is \$52, the strike price is \$50, the risk-free interest rate is 12% per annum, the volatility is 30% per annum, and the time to maturity is three months?

In this case $S_0 = 52$, $K = 50$, $r = 0.12$, $\sigma = 0.30$ and $T = 0.25$.

$$d_1 = \frac{\ln(52/50) + (0.12 + 0.3^2/2)0.25}{0.30\sqrt{0.25}} = 0.5365$$

$$d_2 = d_1 - 0.30\sqrt{0.25} = 0.3865$$

The price of the European call is

$$52N(0.5365) - 50e^{-0.12\times 0.25}N(0.3865)$$
$$= 52\times 0.7042 - 50e^{-0.03}\times 0.6504$$
$$= 5.06$$

or $5.06.

Problem 15.14.

What is the price of a European put option on a non-dividend-paying stock when the stock price is $69, the strike price is $70, the risk-free interest rate is 5% per annum, the volatility is 35% per annum, and the time to maturity is six months?

In this case $S_0 = 69$, $K = 70$, $r = 0.05$, $\sigma = 0.35$ and $T = 0.5$.

$$d_1 = \frac{\ln(69/70) + (0.05 + 0.35^2/2)\times 0.5}{0.35\sqrt{0.5}} = 0.1666$$

$$d_2 = d_1 - 0.35\sqrt{0.5} = -0.0809$$

The price of the European put is

$$70e^{-0.05\times 0.5}N(0.0809) - 69N(-0.1666)$$
$$= 70e^{-0.025}\times 0.5323 - 69\times 0.4338$$
$$= 6.40$$

or $6.40.

Problem 15.15.

Consider an American call option on a stock. The stock price is $70, the time to maturity is eight months, the risk-free rate of interest is 10% per annum, the exercise price is $65, and the volatility is 32%. A dividend of $1 is expected after three months and again after six months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Use DerivaGem to calculate the price of the option.

Using the notation of Section 15.12, $D_1 = D_2 = 1$, $K(1 - e^{-r(T-t_2)}) = 65(1 - e^{-0.1\times 0.1667}) = 1.07$, and $K(1 - e^{-r(t_2-t_1)}) = 65(1 - e^{-0.1\times 0.25}) = 1.60$. Since

$$D_1 < K(1 - e^{-r(T-t_2)})$$

and

$$D_2 < K(1 - e^{-r(t_2-t_1)})$$

It is never optimal to exercise the call option early. DerivaGem shows that the value of the option is 10.94.

Problem 15.16.

A call option on a non-dividend-paying stock has a market price of $2.50. The stock price is $15, the exercise price is $13, the time to maturity is three months, and the risk-free interest rate is 5% per annum. What is the implied volatility?

In the case $c = 2.5$, $S_0 = 15$, $K = 13$, $T = 0.25$, $r = 0.05$. The implied volatility must be

calculated using an iterative procedure.
A volatility of 0.2 (or 20% per annum) gives $c = 2.20$. A volatility of 0.3 gives $c = 2.32$. A volatility of 0.4 gives $c = 2.507$. A volatility of 0.39 gives $c = 2.487$. By interpolation the implied volatility is about 0.396 or 39.6% per annum.
The implied volatility can also be calculated using DerivaGem. Select equity as the Underlying Type in the first worksheet. Select Black-Scholes European as the Option Type. Input stock price as 15, the risk-free rate as 5%, time to exercise as 0.25, and exercise price as 13. Leave the dividend table blank because we are assuming no dividends. Select the button corresponding to call. Select the implied volatility button. Input the Price as 2.5 in the second half of the option data table. Hit the *Enter* key and click on calculate. DerivaGem will show the volatility of the option as 39.64%.

Problem 15.17.

With the notation used in this chapter

(a) *What is* $N'(x)$ *?*

(b) *Show that* $SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$, *where* S *is the stock price at time* t

$$d_1 = \frac{\ln(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

(c) *Calculate* $\partial d_1 / \partial S$ *and* $\partial d_2 / \partial S$.

(d) *Show that when*

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

where c *is the price of a call option on a non-dividend-paying stock.*

(e) *Show that* $\partial c / \partial S = N(d_1)$.

(f) *Show that the* c *satisfies the Black–Scholes–Merton differential equation.*

(g) *Show that* c *satisfies the boundary condition for a European call option, i.e., that* $c = \max(S-K, 0)$ *as t tends to T.*

(a) Since $N(x)$ is the cumulative probability that a variable with a standardized normal distribution will be less than x, $N'(x)$ is the probability density function for a standardized normal distribution, that is,

$$N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

(b)

$$N'(d_1) = N'(d_2 + \sigma\sqrt{T-t})$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left[-\frac{d_2^2}{2} - \sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right]$$

$$= N'(d_2)\exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right]$$

Because

$$d_2 = \frac{\ln(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

it follows that

$$\exp\left[-\sigma d_2\sqrt{T-t} - \frac{1}{2}\sigma^2(T-t)\right] = \frac{Ke^{-r(T-t)}}{S}$$

As a result

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

which is the required result.

(c)

$$d_1 = \frac{\ln\frac{S}{K} + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

$$= \frac{\ln S - \ln K + (r + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

Hence

$$\frac{\partial d_1}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Similarly

$$d_2 = \frac{\ln S - \ln K + (r - \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$$

and

$$\frac{\partial d_2}{\partial S} = \frac{1}{S\sigma\sqrt{T-t}}$$

Therefore:

$$\frac{\partial d_1}{\partial S} = \frac{\partial d_2}{\partial S}$$

(d)

$$c = SN(d_1) - Ke^{-r(T-t)}N(d_2)$$

$$\frac{\partial c}{\partial t} = SN'(d_1)\frac{\partial d_1}{\partial t} - rKe^{-r(T-t)}N(d_2) - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{\partial t}$$

From (b):

$$SN'(d_1) = Ke^{-r(T-t)}N'(d_2)$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) + SN'(d_1)\left(\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t}\right)$$

Since

$$d_1 - d_2 = \sigma\sqrt{T-t}$$

$$\frac{\partial d_1}{\partial t} - \frac{\partial d_2}{\partial t} = \frac{\partial}{\partial t}(\sigma\sqrt{T-t})$$

$$= -\frac{\sigma}{2\sqrt{T-t}}$$

Hence

$$\frac{\partial c}{\partial t} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

(e) From differentiating the Black–Scholes–Merton formula for a call price we obtain

$$\frac{\partial c}{\partial S} = N(d_1) + SN'(d_1)\frac{\partial d_1}{\partial S} - Ke^{-r(T-t)}N'(d_2)\frac{\partial d_2}{dS}$$

From the results in (b) and (c) it follows that

$$\frac{\partial c}{\partial S} = N(d_1)$$

(f) Differentiating the result in (e) and using the result in (c), we obtain

$$\frac{\partial^2 c}{\partial S^2} = N'(d_1)\frac{\partial d_1}{\partial S}$$

$$= N'(d_1)\frac{1}{S\sigma\sqrt{T-t}}$$

From the results in d) and e)

$$\frac{\partial c}{\partial t} + rS\frac{\partial c}{\partial S} + \frac{1}{2}\sigma^2 S^2\frac{\partial^2 c}{\partial S^2} = -rKe^{-r(T-t)}N(d_2) - SN'(d_1)\frac{\sigma}{2\sqrt{T-t}}$$

$$+rSN(d_1) + \frac{1}{2}\sigma^2 S^2 N'(d_1)\frac{1}{S\sigma\sqrt{T-t}}$$

$$= r[SN(d_1) - Ke^{-r(T-t)}N(d_2)]$$

$$= rc$$

This shows that the Black–Scholes–Merton formula for a call option does indeed satisfy the Black–Scholes–Merton differential equation

(g) Consider what happens in the formula for c in part (d) as t approaches T. If $S > K$, d_1 and d_2 tend to infinity and $N(d_1)$ and $N(d_2)$ tend to 1. If $S < K$, d_1 and d_2 tend to zero. It follows that the formula for c tends to $\max(S-K, 0)$.

Problem 15.18.

Show that the Black–Scholes–Merton formulas for call and put options satisfy put–call parity.

The Black–Scholes–Merton formula for a European call option is

$$c = S_0 N(d_1) - Ke^{-rT}N(d_2)$$

so that

$$c + Ke^{-rT} = S_0 N(d_1) - Ke^{-rT}N(d_2) + Ke^{-rT}$$

or

$$c + Ke^{-rT} = S_0 N(d_1) + Ke^{-rT}[1 - N(d_2)]$$

or

$$c + Ke^{-rT} = S_0 N(d_1) + Ke^{-rT}N(-d_2)$$

The Black–Scholes–Merton formula for a European put option is

$$p = Ke^{-rT}N(-d_2) - S_0N(-d_1)$$

so that

$$p + S_0 = Ke^{-rT}N(-d_2) - S_0N(-d_1) + S_0$$

or

$$p + S_0 = Ke^{-rT}N(-d_2) + S_0[1 - N(-d_1)]$$

or

$$p + S_0 = Ke^{-rT}N(-d_2) + S_0N(d_1)$$

This shows that the put–call parity result

$$c + Ke^{-rT} = p + S_0$$

holds.

Problem 15.19.

A stock price is currently $50 and the risk-free interest rate is 5%. Use the DerivaGem software to translate the following table of European call options on the stock into a table of implied volatilities, assuming no dividends. Are the option prices consistent with the assumptions underlying Black–Scholes–Merton?

Stock Price	*Maturity = 3 months*	*Maturity = 6 months*	*Maturity = 12 months*
45	*7.00*	*8.30*	*10.50*
50	*3.50*	*5.20*	*7.50*
55	*1.60*	*2.90*	*5.10*

Using DerivaGem we obtain the following table of implied volatilities

Stock Price	*Maturity = 3 months*	*Maturity = 6 months*	*Maturity = 12 months*
45	37.78	34.99	34.02
50	34.15	32.78	32.03
55	31.98	30.77	30.45

To calculate first number, select equity as the Underlying Type in the first worksheet. Select Black-Scholes European as the Option Type. Input stock price as 50, the risk-free rate as 5%, time to exercise as 0.25, and exercise price as 45. Leave the dividend table blank because we are assuming no dividends. Select the button corresponding to call. Select the implied volatility button. Input the Price as 7.0 in the second half of the option data table. Hit the *Enter* key and click on calculate. DerivaGem will show the volatility of the option as 37.78%. Change the strike price and time to exercise and recompute to calculate the rest of the numbers in the table.

The option prices are not exactly consistent with Black–Scholes–Merton. If they were, the implied volatilities would be all the same. We usually find in practice that low strike price options on a stock have significantly higher implied volatilities than high strike price options on the same stock. This phenomenon is discussed in Chapter 20.

Problem 15.20.

Explain carefully why Black's approach to evaluating an American call option on a dividend-paying stock may give an approximate answer even when only one dividend is anticipated. Does the answer given by Black's approach understate or overstate the true option value? Explain your answer.

Black's approach in effect assumes that the holder of option must decide at time zero whether it is a European option maturing at time t_n (the final ex-dividend date) or a European option maturing at time T. In fact the holder of the option has more flexibility than this. The holder can choose to exercise at time t_n if the stock price at that time is above some level but not otherwise. Furthermore, if the option is not exercised at time t_n, it can still be exercised at time T.

It appears that Black's approach should understate the true option value. This is because the holder of the option has more alternative strategies for deciding when to exercise the option than the two strategies implicitly assumed by the approach. These alternative strategies add value to the option.

However, this is not the whole story! The standard approach to valuing either an American or a European option on a stock paying a single dividend applies the volatility to the stock price less the present value of the dividend. (The procedure for valuing an American option is explained in Chapter 21.) Black's approach when considering exercise just prior to the dividend date applies the volatility to the stock price itself. Black's approach therefore assumes more stock price variability than the standard approach in some of its calculations. In some circumstances it can give a higher price than the standard approach.

Problem 15.21.

Consider an American call option on a stock. The stock price is \$50, the time to maturity is 15 months, the risk-free rate of interest is 8% per annum, the exercise price is \$55, and the volatility is 25%. Dividends of \$1.50 are expected in 4 months and 10 months. Show that it can never be optimal to exercise the option on either of the two dividend dates. Calculate the price of the option.

With the notation in the text

$$D_1 = D_2 = 1.50, \quad t_1 = 0.3333, \quad t_2 = 0.8333, \quad T = 1.25, \quad r = 0.08 \quad \text{and} \quad K = 55$$

$$K\left[1 - e^{-r(T-t_2)}\right] = 55(1 - e^{-0.08\times0.4167}) = 1.80$$

Hence

$$D_2 < K\left[1 - e^{-r(T-t_2)}\right]$$

Also:

$$K\left[1 - e^{-r(t_2-t_1)}\right] = 55(1 - e^{-0.08\times0.5}) = 2.16$$

Hence:

$$D_1 < K\left[1 - e^{-r(t_2-t_1)}\right]$$

It follows from the conditions established in Section 15.12 that the option should never be exercised early.

The present value of the dividends is

$$1.5e^{-0.3333\times0.08} + 1.5e^{-0.8333\times0.08} = 2.864$$

The option can be valued using the European pricing formula with:

$$S_0 = 50 - 2.864 = 47.136, \quad K = 55, \quad \sigma = 0.25, \quad r = 0.08, \quad T = 1.25$$

$$d_1 = \frac{\ln(47.136/55) + (0.08 + 0.25^2/2)1.25}{0.25\sqrt{1.25}} = -0.0545$$

$$d_2 = d_1 - 0.25\sqrt{1.25} = -0.3340$$

$$N(d_1) = 0.4783, \quad N(d_2) = 0.3692$$

and the call price is

$$47.136 \times 0.4783 - 55e^{-0.08\times1.25} \times 0.3692 = 4.17$$

or \$4.17.

Problem 15.22.

Show that the probability that a European call option will be exercised in a risk-neutral world is, with the notation introduced in this chapter, $N(d_2)$. What is an expression for the value of a derivative that pays off \$100 if the price of a stock at time T is greater than K ?

The probability that the call option will be exercised is the probability that $S_T > K$ where S_T is the stock price at time T. In a risk neutral world

$$\ln S_T \sim \varphi\left[\ln S_0 + (r - \sigma^2/2)T, \sigma^2 T\right]$$

The probability that $S_T > K$ is the same as the probability that $\ln S_T > \ln K$. This is

$$1 - N\left[\frac{\ln K - \ln S_0 - (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right]$$

$$= N\left[\frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}}\right]$$

$$= N(d_2)$$

The expected value at time T in a risk neutral world of a derivative security which pays off \$100 when $S_T > K$ is therefore

$$100N(d_2)$$

From risk neutral valuation the value of the security at time zero is

$$100e^{-rT}N(d_2)$$

Problem 15.23.

Use the result in equation (15.17) to determine the value of a perpetual American put option on a non-dividend-paying stock with strike price K if it is exercised when the stock price equals H where H<K. Assume that the current stock price S is greater than H. What is the value of H that maximizes the option value? Deduce the value of a perpetual American put option with strike price K.

If the perpetual American put is exercised when $S=H$, it provides a payoff of $(K-H)$. We obtain its value, by setting $Q=K-H$ in equation (15.17), as

$$V = (K - H)\left(\frac{S}{H}\right)^{-2r/\sigma^2} = (K - H)\left(\frac{H}{S}\right)^{2r/\sigma^2}$$

Now

$$\frac{dV}{dH} = -\left(\frac{H}{S}\right)^{2r/\sigma^2} + \frac{K-H}{S}\left(\frac{2r}{\sigma^2}\right)\left(\frac{H}{S}\right)^{2r/\sigma^2-1}$$
$$= \left(\frac{H}{S}\right)^{2r/\sigma^2}\left(-1+\frac{2r(K-H)}{H\sigma^2}\right)$$
$$\frac{d^2V}{dH^2} = -\frac{2rK}{H^2\sigma^2}\left(\frac{H}{S}\right)^{2r/\sigma^2} + \left(-1+\frac{2r(K-H)}{H\sigma^2}\right)\frac{2r}{\sigma^2 S}\left(\frac{H}{S}\right)^{2r/\sigma^2-1}$$

dV/dH is zero when

$$H = \frac{2rK}{2r+\sigma^2}$$

and, for this value of *H*, d^2V/dH^2 is negative indicating that it gives the maximum value of *V*.

The value of the perpetual American put is maximized if it is exercised when *S* equals this value of *H*. Hence the value of the perpetual American put is

$$(K-H)\left(\frac{S}{H}\right)^{-2r/\sigma^2}$$

when $H=2rK/(2r+\sigma^2)$. The value is

$$\frac{\sigma^2 K}{2r+\sigma^2}\left(\frac{S(2r+\sigma^2)}{2rK}\right)^{-2r/\sigma^2}$$

This is consistent with the more general result produced in Chapter 26 for the case where the stock provides a dividend yield.

Problem 15.24.
A company has an issue of executive stock options outstanding. Should dilution be taken into account when the options are valued? Explain you answer.

The answer is no. If markets are efficient they have already taken potential dilution into account in determining the stock price. This argument is explained in Business Snapshot 15.3.

Problem 15.25.
A company's stock price is $50 and 10 million shares are outstanding. The company is considering giving its employees three million at-the-money five-year call options. Option exercises will be handled by issuing more shares. The stock price volatility is 25%, the five-year risk-free rate is 5% and the company does not pay dividends. Estimate the cost to the company of the employee stock option issue.

The Black-Scholes-Merton price of the option is given by setting $S_0 = 50$, $K = 50$, $r = 0.05$, $\sigma = 0.25$, and $T = 5$. It is 16.252. From an analysis similar to that in Section 15.10 the cost to the company of the options is

$$\frac{10}{10+3}\times 16.252 = 12.5$$

or about $12.5 per option. The total cost is therefore 3 million times this or $37.5 million. If the market perceives no benefits from the options the stock price will fall by $3.75.

Further Questions

Problem 15.26.
If the volatility of a stock is 18% per annum, estimate the standard deviation of the percentage price change in (a) one day, (b) one week, and (c) one month.

(a) $18/\sqrt{252} = 1.13\%$
(b) $18/\sqrt{52} = 2.50\%$
(c) $18/\sqrt{12} = 5.20\%$

Problem 15.27.
A stock price is currently $50. Assume that the expected return from the stock is 18% per annum and its volatility is 30% per annum. What is the probability distribution for the stock price in two years? Calculate the mean and standard deviation of the distribution. Determine the 95% confidence interval.

In this case $S_0 = 50$, $\mu = 0.18$ and $\sigma = 0.30$. The probability distribution of the stock price in two years, S_T, is lognormal and is, from equation (15.3), given by:

$$\ln S_T \sim \varphi\left[\ln 50 + \left(0.18 - \frac{0.09}{2}\right)2,\ 0.3^2 \times 2\right]$$

i.e.,

$$\ln S_T \sim \varphi(4.18,\ 0.42^2)$$

The mean stock price is from equation (15.4)

$$50e^{0.18\times 2} = 50e^{0.36} = 71.67$$

and the standard deviation is from equation (15.5)

$$50e^{0.18\times 2}\sqrt{e^{0.09\times 2} - 1} = 31.83$$

95% confidence intervals for $\ln S_T$ are

$$4.18 - 1.96\times 0.42 \quad \text{and} \quad 4.18 + 1.96\times 0.42$$

i.e.,

$$3.35 \quad \text{and} \quad 5.01$$

These correspond to 95% confidence limits for S_T of

$$e^{3.35} \quad \text{and} \quad e^{5.01}$$

i.e.,

$$28.52 \quad \text{and} \quad 150.44$$

Problem 15.28. (Excel file)
Suppose that observations on a stock price (in dollars) at the end of each of 15 consecutive weeks are as follows:

30.2, 32.0, 31.1, 30.1, 30.2, 30.3, 30.6, 33.0,
32.9, 33.0, 33.5, 33.5, 33.7, 33.5, 33.2

Estimate the stock price volatility. What is the standard error of your estimate?

The calculations are shown in the table below

$$\sum u_i = 0.09471 \quad \sum u_i^2 = 0.01145$$

and an estimate of standard deviation of weekly returns is:

$$\sqrt{\frac{0.01145}{13} - \frac{0.09471^2}{14 \times 13}} = 0.02884$$

The volatility per annum is therefore $0.02884\sqrt{52} = 0.2079$ or 20.79%. The standard error of this estimate is

$$\frac{0.2079}{\sqrt{2 \times 14}} = 0.0393$$

or 3.9% per annum.

Week	*Closing Stock Price ($)*	*Price Relative* $= S_i / S_{i-1}$	*Weekly Return* $u_i = \ln(S_i / S_{i-1})$
1	30.2		
2	32.0	1.05960	0.05789
3	31.1	0.97188	–0.02853
4	30.1	0.96785	–0.03268
5	30.2	1.00332	0.00332
6	30.3	1.00331	0.00331
7	30.6	1.00990	0.00985
8	33.0	1.07843	0.07551
9	32.9	0.99697	–0.00303
10	33.0	1.00304	0.00303
11	33.5	1.01515	0.01504
12	33.5	1.00000	0.00000
13	33.7	1.00597	0.00595
14	33.5	0.99407	–0.00595
15	33.2	0.99104	–0.00900

Problem 15.29.

A financial institution plans to offer a security that pays off a dollar amount equal to S_T^2 *at time* T.

(a) *Use risk-neutral valuation to calculate the price of the security at time* t *in terms of the stock price,* S, *at time* t. *(Hint: The expected value of* S_T^2 *can be calculated from the mean and variance of* S_T *given in section 15.1.)*

(b) *Confirm that your price satisfies the differential equation (15.16).*

(a) The expected value of the security is $E[(S_T)^2]$ From equations (15.4) and (15.5), at time t :

$$E(S_T) = Se^{\mu(T-t)}$$

$$\text{var}(S_T) = S^2 e^{2\mu(T-t)}[e^{\sigma^2(T-t)} - 1]$$

Since $\text{var}(S_T) = E[(S_T)^2] - [E(S_T)]^2$, it follows that $E[(S_T)^2] = \text{var}(S_T) + [E(S_T)]^2$ so that

$$E[(S_T)^2] = S^2 e^{2\mu(T-t)}[e^{\sigma^2(T-t)} - 1] + S^2 e^{2\mu(T-t)}$$

$$= S^2 e^{(2\mu+\sigma^2)(T-t)}$$

In a risk-neutral world $\mu = r$ so that

$$\hat{E}[(S_T)^2] = S^2 e^{(2r+\sigma^2)(T-t)}$$

Using risk-neutral valuation, the value of the derivative security at time t is

$$e^{-r(T-t)}\hat{E}[(S_T)^2]$$

$$= S^2 e^{(2r+\sigma^2)(T-t)} e^{-r(T-t)}$$

$$= S^2 e^{(r+\sigma^2)(T-t)}$$

(b) If:

$$f = S^2 e^{(r+\sigma^2)(T-t)}$$

$$\frac{\partial f}{\partial t} = -S^2(r+\sigma^2)e^{(r+\sigma^2)(T-t)}$$

$$\frac{\partial f}{\partial S} = 2Se^{(r+\sigma^2)(T-t)}$$

$$\frac{\partial^2 f}{\partial S^2} = 2e^{(r+\sigma^2)(T-t)}$$

The left-hand side of the Black-Scholes–Merton differential equation is:

$$-S^2(r+\sigma^2)e^{(r+\sigma^2)(T-t)} + 2rS^2 e^{(r+\sigma^2)(T-t)} + \sigma^2 S^2 e^{(r+\sigma^2)(T-t)}$$

$$= rS^2 e^{(r+\sigma^2)(T-t)}$$

$$= rf$$

Hence the Black-Scholes equation is satisfied.

Problem 15.30.

Consider an option on a non-dividend-paying stock when the stock price is $30, the exercise price is $29, the risk-free interest rate is 5% per annum, the volatility is 25% per annum, and the time to maturity is four months.

a. *What is the price of the option if it is a European call?*
b. *What is the price of the option if it is an American call?*
c. *What is the price of the option if it is a European put?*
d. *Verify that put–call parity holds.*

In this case $S_0 = 30$, $K = 29$, $r = 0.05$, $\sigma = 0.25$ and $T = 4/12$

$$d_1 = \frac{\ln(30/29) + (0.05 + 0.25^2/2) \times 4/12}{0.25\sqrt{0.3333}} = 0.4225$$

$$d_2 = \frac{\ln(30/29) + (0.05 - 0.25^2/2) \times 4/12}{0.25\sqrt{0.3333}} = 0.2782$$

$$N(0.4225) = 0.6637, \quad N(0.2782) = 0.6096$$

$$N(-0.4225) = 0.3363, \quad N(-0.2782) = 0.3904$$

a. The European call price is

$$30 \times 0.6637 - 29e^{-0.05\times4/12} \times 0.6096 = 2.52$$

or $2.52.

b. The American call price is the same as the European call price. It is $2.52.

c. The European put price is

$$29e^{-0.05\times4/12} \times 0.3904 - 30 \times 0.3363 = 1.05$$

or $1.05.

d. Put-call parity states that:

$$p + S = c + Ke^{-rT}$$

In this case $c = 2.52$, $S_0 = 30$, $K = 29$, $p = 1.05$ and $e^{-rT} = 0.9835$ and it is easy to verify that the relationship is satisfied,

Problem 15.31.

Assume that the stock in Problem 15.30 is due to go ex-dividend in 1.5 months. The expected dividend is 50 cents.

a. *What is the price of the option if it is a European call?*
b. *What is the price of the option if it is a European put?*
c. *If the option is an American call, are there any circumstances when it will be exercised early?*

a. The present value of the dividend must be subtracted from the stock price. This gives a new stock price of:

$$30 - 0.5e^{-0.125\times0.05} = 29.5031$$

and

$$d_1 = \frac{\ln(29.5031/29) + (0.05 + 0.25^2/2) \times 0.3333}{0.25\sqrt{0.3333}} = 0.3068$$

$$d_2 = \frac{\ln(29.5031/29) + (0.05 - 0.25^2/2) \times 0.3333}{0.25\sqrt{0.3333}} = 0.1625$$

$$N(d_1) = 0.6205; \quad N(d_2) = 0.5645$$

The price of the option is therefore

$$29.5031 \times 0.6205 - 29e^{-0.05\times4/12} \times 0.5645 = 2.21$$

or $2.21.

b. Because

$$N(-d_1) = 0.3795, \quad N(-d_2) = 0.4355$$

the value of the option when it is a European put is

$$29e^{-0.05\times4/12} \times 0.4355 - 29.5031 \times 0.3795 = 1.22$$

or $1.22.

c. If t_1 denotes the time when the dividend is paid:

$$K(1-e^{-r(T-t_1)}) = 29(1-e^{-0.05\times 0.2083}) = 0.3005$$

This is less than the dividend. Hence the option should be exercised immediately before the ex-dividend date for a sufficiently high value of the stock price.

Problem 15.32.

Consider an American call option when the stock price is $18, the exercise price is $20, the time to maturity is six months, the volatility is 30% per annum, and the risk-free interest rate is 10% per annum. Two equal dividends are expected during the life of the option, with ex-dividend dates at the end of two months and five months. Assume the dividends are 40 cents. Use Black's approximation and the DerivaGem software to value the option. Suppose now that the dividend is D on each ex-dividend date. Use the results in the Appendix to determine how high D can be without the American option being exercised early.

We first value the option assuming that it is not exercised early, we set the time to maturity equal to 0.5. There is a dividend of 0.4 in 2 months and 5 months. Other parameters are $S_0 = 18$, $K = 20$, $r = 10\%$, $\sigma = 30\%$. DerivaGem gives the price as 0.7947. We next value the option assuming that it is exercised at the five-month point just before the final dividend. DerivaGem gives the price as 0.7668. The price given by Black's approximation is therefore 0.7947. (DerivaGem also shows that the American option price calculated using the binomial model with 100 time steps is 0.8243.)

It is never optimal to exercise the option immediately before the first ex-dividend date when

$$D_1 \le K[1-e^{-r(t_2-t_1)}]$$

where D_1 is the size of the first dividend, and t_1 and t_2 are the times of the first and second dividend respectively. Hence we must have:

$$D_1 \le 20[1-e^{-(0.1\times 0.25)}]$$

that is,

$$D_1 \le 0.494$$

It is never optimal to exercise the option immediately before the second ex-dividend date when:

$$D_2 \le K(1-e^{-r(T-t_2)})$$

where D_2 is the size of the second dividend. Hence we must have:

$$D_2 \le 20(1-e^{-0.1\times 0.0833})$$

that is,

$$D_2 \le 0.166$$

It follows that the dividend can be as high as 16.6 cents per share without the American option being worth more than the corresponding European option.