CHAPTER 21
Basic Numerical Procedures

Practice Questions

Problem 21.1.

Which of the following can be estimated for an American option by constructing a single binomial tree: delta, gamma, vega, theta, rho?

Delta, gamma, and theta can be determined from a single binomial tree. Vega is determined by making a small change to the volatility and recomputing the option price using a new tree. Rho is calculated by making a small change to the interest rate and recomputing the option price using a new tree.

Problem 21.2.

Calculate the price of a three-month American put option on a non-dividend-paying stock when the stock price is \$60, the strike price is \$60, the risk-free interest rate is 10% per annum, and the volatility is 45% per annum. Use a binomial tree with a time interval of one month.

In this case, $S_0 = 60$, $K = 60$, $r = 0.1$, $\sigma = 0.45$, $T = 0.25$, and $\Delta t = 0.0833$. Also

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.45\sqrt{0.0833}} = 1.1387$$

$$d = \frac{1}{u} = 0.8782$$

$$a = e^{r\Delta t} = e^{0.1\times 0.0833} = 1.0084$$

$$p = \frac{a-d}{u-d} = 0.4998$$

$$1-p = 0.5002$$

The output from DerivaGem for this example is shown in the Figure S21.1. The calculated price of the option is \$5.16.

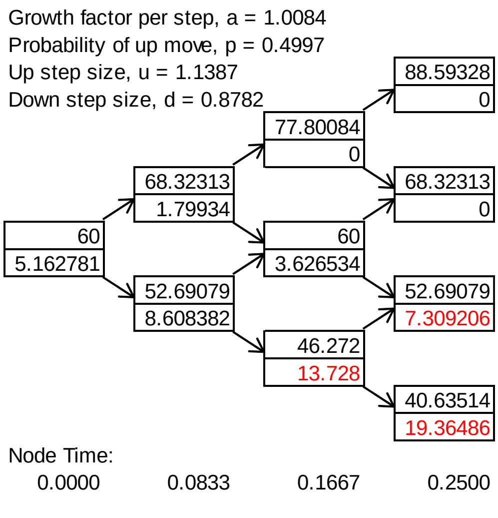


Figure S21.1: Tree for Problem 21.2

Problem 21.3.

Explain how the control variate technique is implemented when a tree is used to value American options.

The control variate technique is implemented by

1. Valuing an American option using a binomial tree in the usual way $(= f_A)$.
2. Valuing the European option with the same parameters as the American option using the same tree $(= f_E)$.
3. Valuing the European option using Black-Scholes-Merton (=f_{BSM}). The price of the American option is estimated as $f_A + f_{BSM} - f_E$.

Problem 21.4.

Calculate the price of a nine-month American call option on corn futures when the current futures price is 198 cents, the strike price is 200 cents, the risk-free interest rate is 8% per annum, and the volatility is 30% per annum. Use a binomial tree with a time interval of three months.

In this case $F_0 = 198$, $K = 200$, $r = 0.08$, $\sigma = 0.3$, $T = 0.75$, and $\Delta t = 0.25$. Also

$$u = e^{0.3\sqrt{0.25}} = 1.1618$$

$$d = \frac{1}{u} = 0.8607$$

$$a = 1$$

$$p = \frac{a-d}{u-d} = 0.4626$$

$$1 - p = 0.5373$$

The output from DerivaGem for this example is shown in the Figure S21.2. The calculated price of the option is 20.34 cents.

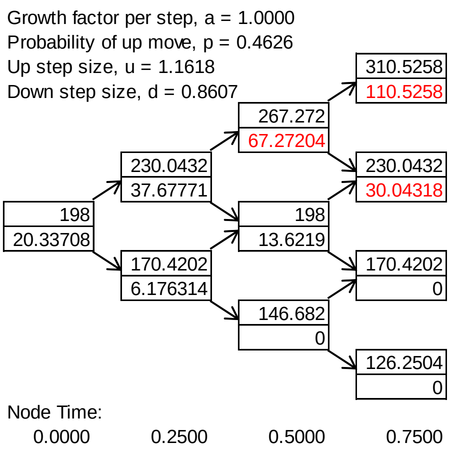


Figure S21.2: Tree for Problem 21.4

Problem 21.5.

Consider an option that pays off the amount by which the final stock price exceeds the average stock price achieved during the life of the option. Can this be valued using the binomial tree approach? Explain your answer.

A binomial tree cannot be used in the way described in this chapter. This is an example of what is known as a history-dependent option. The payoff depends on the path followed by the stock price as well as its final value. The option cannot be valued by starting at the end of the tree and working backward since the payoff at the final branches is not known unambiguously. Chapter 27 describes an extension of the binomial tree approach that can be used to handle options where the payoff depends on the average value of the stock price.

Problem 21.6.

"For a dividend-paying stock, the tree for the stock price does not recombine; but the tree for the stock price less the present value of future dividends does recombine." Explain this statement.

Suppose a dividend equal to D is paid during a certain time interval. If S is the stock price at the beginning of the time interval, it will be either $Su-D$ or $Sd-D$ at the end of the time interval. At the end of the next time interval, it will be one of $(Su-D)u$, $(Su-D)d$, $(Sd-D)u$ and $(Sd-D)d$. Since $(Su-D)d$ does not equal $(Sd-D)u$ the tree does not recombine. If S is equal to the stock price less the present value of future dividends, this problem is avoided.

Problem 21.7.

Show that the probabilities in a Cox, Ross, and Rubinstein binomial tree are negative when the condition in footnote 8 holds.

With the usual notation

$$p=\frac{a-d}{u-d}$$

$$1-p=\frac{u-a}{u-d}$$

If $a<d$ or $a>u$, one of the two probabilities is negative. This happens when

$$e^{(r-q)\Delta t}<e^{-\sigma\sqrt{\Delta t}}$$

or

$$e^{(r-q)\Delta t}>e^{\sigma\sqrt{\Delta t}}$$

This in turn happens when $(q-r)\sqrt{\Delta t}>\sigma$ or $(r-q)\sqrt{\Delta t}>\sigma$ Hence negative probabilities occur when

$$\sigma<|(r-q)\sqrt{\Delta t}\,|$$

This is the condition in footnote 8.

Problem 21.8.

Use stratified sampling with 100 trials to improve the estimate of π in Business Snapshot 21.1 and Table 21.1.

In Table 21.1 cells A1, A2, A3,..., A100 are random numbers between 0 and 1 defining how far to the right in the square the dart lands. Cells B1, B2, B3,...,B100 are random numbers between 0 and 1 defining how high up in the square the dart lands. For stratified sampling we could choose equally spaced values for the A's and the B's and consider every possible combination. To generate 100 samples we need ten equally spaced values for the A's and the B's so that there are $10\times10=100$ combinations. The equally spaced values should be 0.05, 0.15, 0.25,..., 0.95. We could therefore set the A's and B's as follows:

$$\text{A1} = \text{A2} = \text{A3} = \ldots = \text{A10} = 0.05$$

$$\text{A11} = \text{A12} = \text{A13} = \ldots = \text{A20} = 0.15$$

$$\ldots$$

$$\ldots$$

$$\text{A91} = \text{A92} = \text{A93} = \ldots = \text{A100} = 0.95$$

and

$$\text{B1} = \text{B11} = \text{B21} = \ldots. = \text{B91} = 0.05$$

$$\text{B2} = \text{B12} = \text{B22} = \ldots = \text{B92} = 0.15$$

$$\ldots$$

$$\ldots$$

$$\text{B10} = \text{B20} = \text{B30} = \ldots = \text{B100} = 0.95$$

We get a value for π equal to 3.2, which is closer to the true value than the value of 3.04 obtained with random sampling in Table 21.1. Because samples are not random we cannot easily calculate a standard error of the estimate.

Problem 21.9.

Explain why the Monte Carlo simulation approach cannot easily be used for American-style derivatives.

In Monte Carlo simulation sample values for the derivative security in a risk-neutral world are obtained by simulating paths for the underlying variables. On each simulation run, values for the underlying variables are first determined at time Δt, then at time $2\Delta t$, then at time $3\Delta t$, etc. At time $i\Delta t (i = 0, 1, 2 ?)$ it is not possible to determine whether early exercise is optimal since the range of paths which might occur after time $i\Delta t$ have not been investigated. In short, Monte Carlo simulation works by moving forward from time t to time T. Other numerical procedures which accommodate early exercise work by moving backwards from time T to time t.

Problem 21.10.

A nine-month American put option on a non-dividend-paying stock has a strike price of $49. The stock price is $50, the risk-free rate is 5% per annum, and the volatility is 30% per annum. Use a three-step binomial tree to calculate the option price.

In this case, $S_0 = 50$, $K = 49$, $r = 0.05$, $\sigma = 0.30$, $T = 0.75$, and $\Delta t = 0.25$. Also

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.30\sqrt{0.25}} = 1.1618$$
$$d = \frac{1}{u} = 0.8607$$
$$a = e^{r\Delta t} = e^{0.05\times 0.25} = 1.0126$$
$$p = \frac{a-d}{u-d} = 0.5043$$
$$1-p = 0.4957$$

The output from DerivaGem for this example is shown in the Figure S21.3. The calculated price of the option is \$4.29. Using 100 steps the price obtained is \$3.91

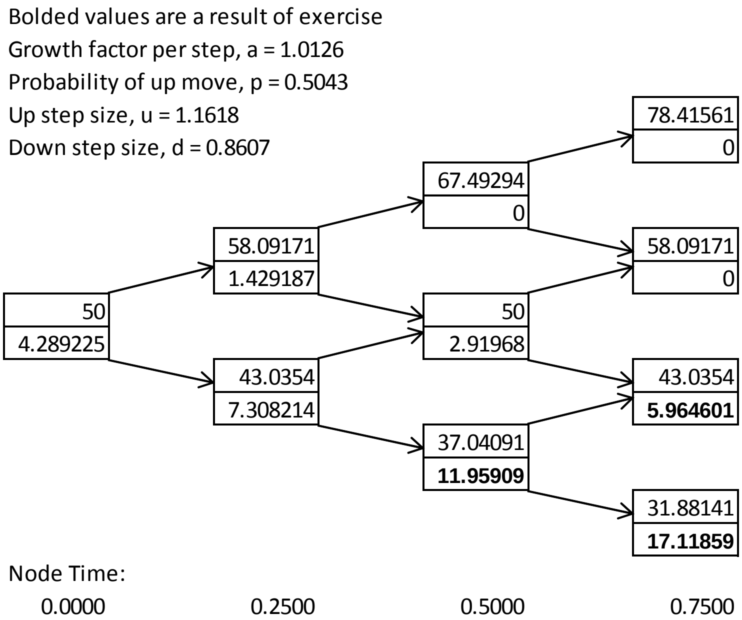


Figure S21.3: Tree for Problem 21.10

Problem 21.11.

Use a three-time-step tree to value a nine-month American call option on wheat futures. The current futures price is 400 cents, the strike price is 420 cents, the risk-free rate is 6%, and the volatility is 35% per annum. Estimate the delta of the option from your tree.

In this case $F_0 = 400$, $K = 420$, $r = 0.06$, $\sigma = 0.35$, $T = 0.75$, and $\Delta t = 0.25$. Also

$$u = e^{0.35\sqrt{0.25}} = 1.1912$$
$$d = \frac{1}{u} = 0.8395$$
$$a = 1$$
$$p = \frac{a-d}{u-d} = 0.4564$$
$$1-p = 0.5436$$

The output from DerivaGem for this example is shown in the Figure S21.4. The calculated price of the option is 42.07 cents. Using 100 time steps the price obtained is 38.64. The option's delta is calculated from the tree is

$$(79.971-11.419)/(476.498-335.783)=0.487$$

When 100 steps are used the estimate of the option's delta is 0.483.

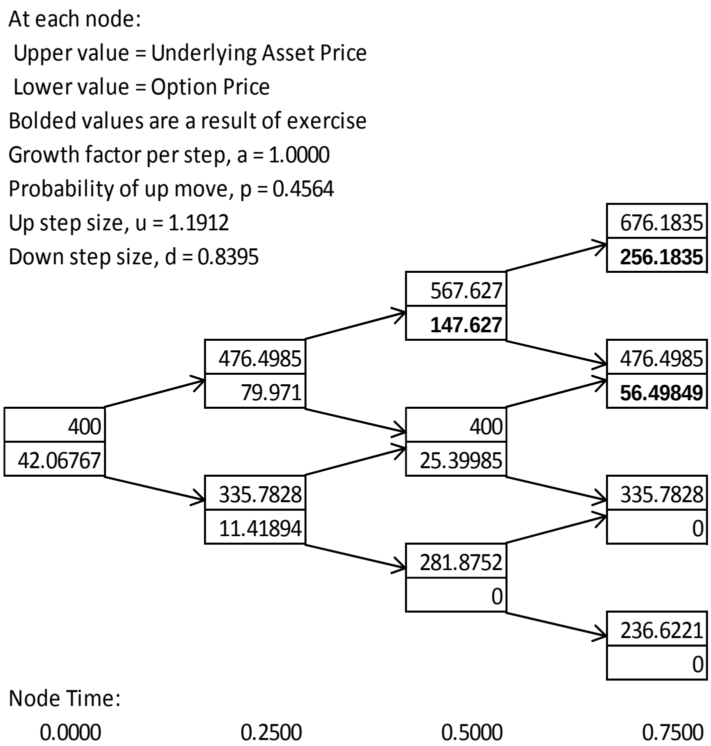


Figure S21.4: Tree for Problem 21.11

Problem 21.12.

A three-month American call option on a stock has a strike price of $20. The stock price is $20, the risk-free rate is 3% per annum, and the volatility is 25% per annum. A dividend of $2 is expected in 1.5 months. Use a three-step binomial tree to calculate the option price.

In this case the present value of the dividend is $2e^{-0.03\times 0.125}=1.9925$. We first build a tree for $S_0=20-1.9925=18.0075$, $K=20$, $r=0.03$, $\sigma=0.25$, and $T=0.25$ with $\Delta t=0.08333$. This gives Figure S21.5. For nodes between times0 and 1.5 months we then add the present value of the dividend to the stock price. The result is the tree in Figure S21.6. The price of the option calculated from the tree is 0.674. When 100 steps are used the price obtained is 0.690.

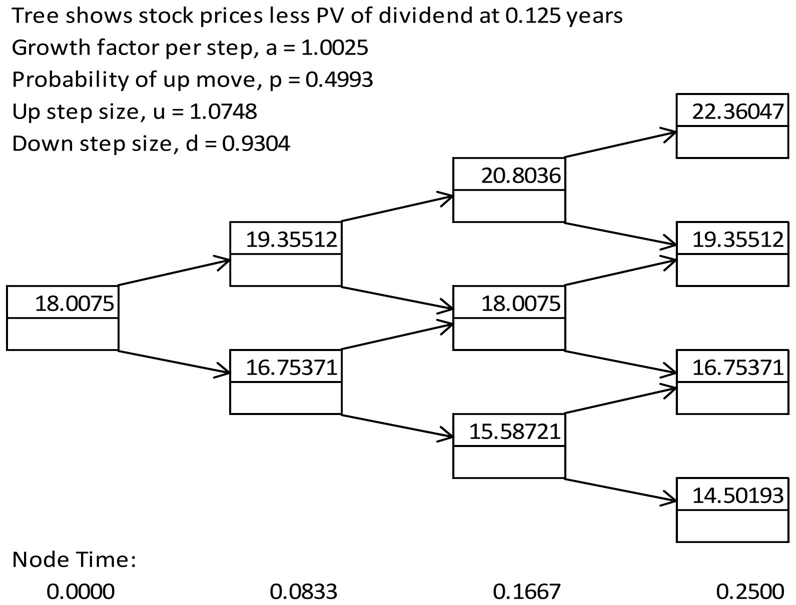


Figure S21.5: First Tree for Problem 21.12

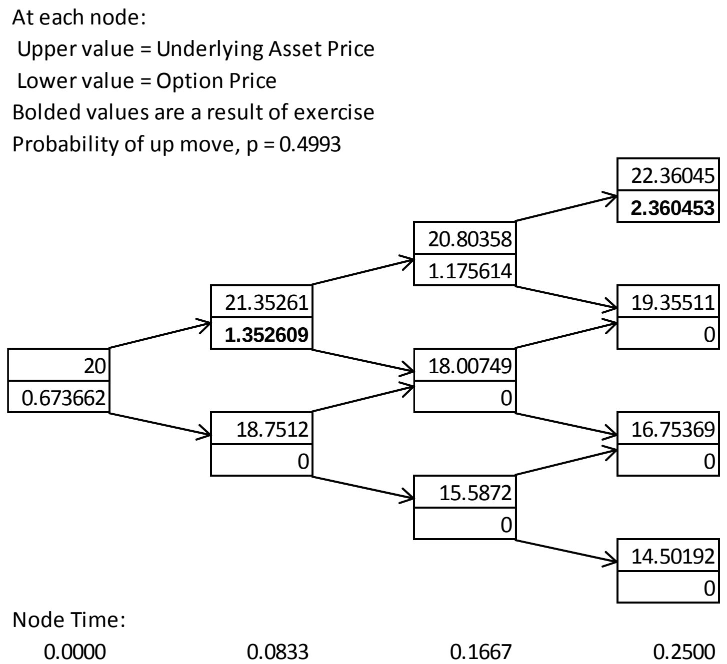


Figure S21.6: Final Tree for Problem 21.12

Problem 21.13.

A one-year American put option on a non-dividend-paying stock has an exercise price of $18. The current stock price is $20, the risk-free interest rate is 15% per annum, and the volatility of the stock is 40% per annum. Use the DerivaGem software with four three-month time steps to estimate the value of the option. Display the tree and verify that the option prices at the final and penultimate nodes are correct. Use DerivaGem to value the European version of *the option. Use the control variate technique to improve your estimate of the price of the American option.*

In this case $S_0 = 20$, $K = 18$, $r = 0.15$, $\sigma = 0.40$, $T = 1$, and $\Delta t = 0.25$. The parameters for the tree are

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.4\sqrt{0.25}} = 1.2214$$

$$d = 1/u = 0.8187$$

$$a = e^{r\Delta t} = 1.0382$$

$$p = \frac{a-d}{u-d} = \frac{1.0382 - 0.8187}{1.2214 - 0.8187} = 0.545$$

The tree produced by DerivaGem for the American option is shown in Figure S21.7. The estimated value of the American option is \$1.29.

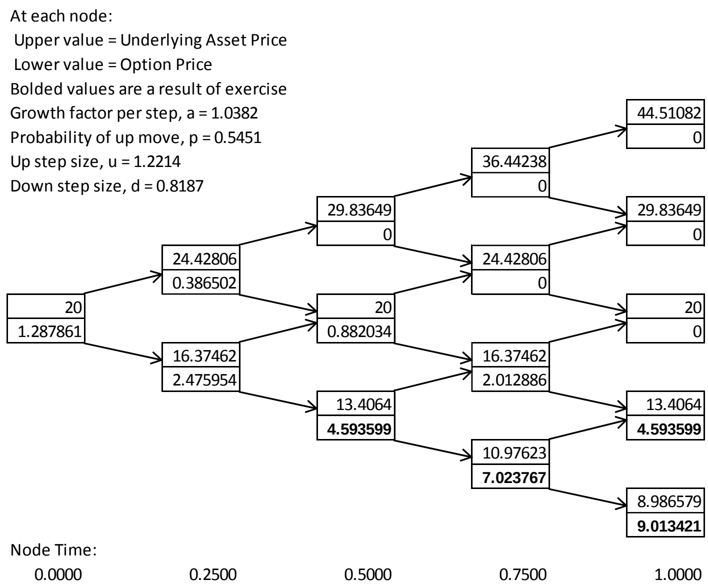


Figure S21.7: Tree to evaluate American option for Problem 21.13

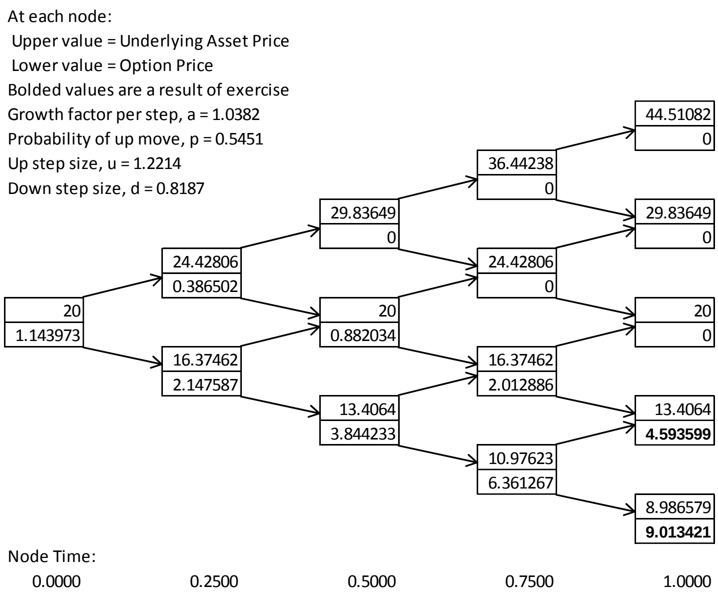


Figure S21.8: Tree to evaluate European option in Problem 21.13

As shown in Figure S21.8, the same tree can be used to value a European put option with the same parameters. The estimated value of the European option is \$1.14. The option parameters are $S_0 = 20$, $K = 18$, $r = 0.15$, $\sigma = 0.40$ and $T = 1$

$$d_1 = \frac{\ln(20/18) + 0.15 + 0.40^2/2}{0.40} = 0.8384$$

$$d_2 = d_1 - 0.40 = 0.4384$$

$$N(-d_1) = 0.2009; \quad N(-d_2) = 0.3306$$

The true European put price is therefore

$$18e^{-0.15} \times 0.3306 - 20 \times 0.2009 = 1.10$$

This can also be obtained from DerivaGem. The control variate estimate of the American put price is therefore 1.29+1.10 −1.14 = \$1.25.

Problem 21.14

A two-month American put option on a stock index has an exercise price of 480. The current level of the index is 484, the risk-free interest rate is 10% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 25% per annum. Divide the life of the option into four half-month periods and use the binomial tree approach to estimate the value of the option.

In this case $S_0 = 484$, $K = 480$, $r = 0.10$, $\sigma = 0.25$ $q = 0.03$, $T = 0.1667$, and $\Delta t = 0.04167$

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.25\sqrt{0.04167}} = 1.0524$$

$$d = \frac{1}{u} = 0.9502$$

$$a = e^{(r-q)\Delta t} = 1.00292$$

$$p = \frac{a-d}{u-d} = \frac{1.0029 - 0.9502}{1.0524 - 0.9502} = 0.516$$

The tree produced by DerivaGem is shown in the Figure S21.9. The estimated price of the option is \$14.93.

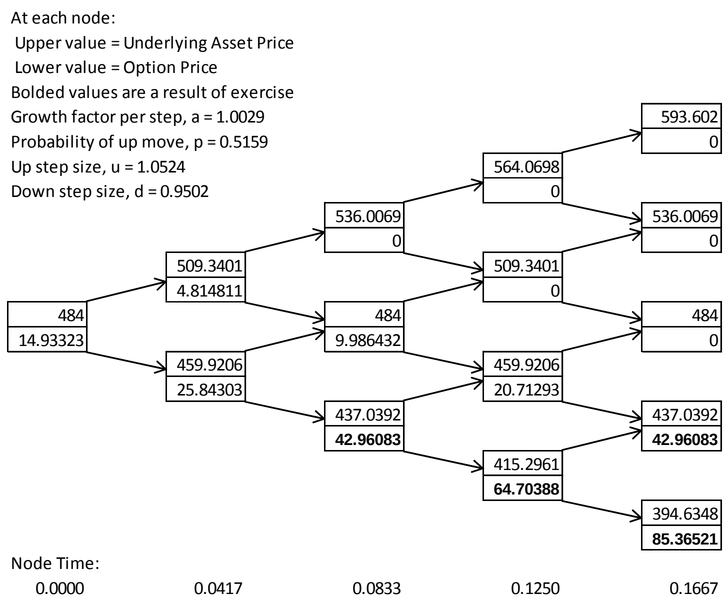


Figure S21.9: Tree to evaluate option in Problem 21.14

Problem 21.15

How can the control variate approach to improve the estimate of the delta of an American option when the binomial tree approach is used?

First the delta of the American option is estimated in the usual way from the tree. Denote this by Δ_A^*. Then the delta of a European option which has the same parameters as the American option is calculated in the same way using the same tree. Denote this by Δ_B^*. Finally the true European delta, Δ_B, is calculated using the formulas in Chapter 19. The control variate estimate of delta is then:

$$\Delta_A^* - \Delta_B^* + \Delta_B$$

Problem 21.16.

Suppose that Monte Carlo simulation is being used to evaluate a European call option on a non-dividend-paying stock when the volatility is stochastic. How could the control variate and antithetic variable technique be used to improve numerical efficiency? Explain why it is *necessary to calculate six values of the option in each simulation trial when both the control variate and the antithetic variable technique are used.*

In this case a simulation requires two sets of samples from standardized normal distributions. The first is to generate the volatility movements. The second is to generate the stock price movements once the volatility movements are known. The control variate technique involves carrying out a second simulation on the assumption that the volatility is constant. The same random number stream is used to generate stock price movements as in the first simulation. An improved estimate of the option price is

$$f_A^* - f_B^* + f_B$$

where f_A^* is the option value from the first simulation (when the volatility is stochastic), f_B^* is the option value from the second simulation (when the volatility is constant) and f_B is the true Black-Scholes-Merton value when the volatility is constant.

To use the antithetic variable technique, two sets of samples from standardized normal distributions must be used for each of volatility and stock price. Denote the volatility samples by $\{V_1\}$ and $\{V_2\}$ and the stock price samples by $\{S_1\}$ and $\{S_2\}$. $\{V_1\}$ is antithetic to $\{V_2\}$ and $\{S_1\}$ is antithetic to $\{S_2\}$. Thus if

$$\{V_1\} = +0.83, +0.41, -0.21?$$

then

$$\{V_2\} = -0.83, -0.41, +0.21?$$

Similarly for $\{S_1\}$ and $\{S_2\}$.

An efficient way of proceeding is to carry out six simulations in parallel:

Simulation 1: Use $\{S_1\}$ with volatility constant
Simulation 2: Use $\{S_2\}$ with volatility constant
Simulation 3: Use $\{S_1\}$ and $\{V_1\}$
Simulation 4: Use $\{S_1\}$ and $\{V_2\}$
Simulation 5: Use $\{S_2\}$ and $\{V_1\}$
Simulation 6: Use $\{S_2\}$ and $\{V_2\}$

If f_i is the option price from simulation i, simulations 3 and 4 provide an estimate $0.5(f_3 + f_4)$ for the option price. When the control variate technique is used we combine this estimate with the result of simulation 1 to obtain $0.5(f_3 + f_4) - f_1 + f_B$ as an estimate of the price where f_B is, as above, the Black-Scholes-Merton option price. Similarly simulations 2, 5 and 6 provide an estimate $0.5(f_5 + f_6) - f_2 + f_B$. Overall the best estimate is:

$$0.5[0.5(f_3 + f_4) - f_1 + f_B + 0.5(f_5 + f_6) - f_2 + f_B]$$

Problem 21.17.

Explain how equations (21.27) to (21.30) change when the implicit finite difference method is being used to evaluate an American call option on a currency.

For an American call option on a currency

$$\frac{\partial f}{\partial t} + (r - r_f)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$

With the notation in the text this becomes

$$\frac{f_{i+1,j}-f_{ij}}{\Delta t}+(r-r_f)\,j\Delta S\,\frac{f_{i,j+1}-f_{i,j-1}}{2\Delta S}+\frac{1}{2}\sigma^2 j^2\Delta S^2\,\frac{f_{i,j+1}-2f_{i,j}+f_{i,j-1}}{\Delta S^2}=rf_{ij}$$

for $j = 1,2? \quad M-1$ and $i = 0,1? \quad N-1$. Rearranging terms we obtain

$$a_j f_{i,j-1}+b_j f_{ij}+c_j f_{i,j+1}=f_{i+1,j}$$

where

$$a_j=\frac{1}{2}(r-r_f)\,j\Delta t-\frac{1}{2}\sigma^2 j^2\Delta t$$

$$b_j=1+\sigma^2 j^2\Delta t+r\Delta t$$

$$c_j=-\frac{1}{2}(r-r_f)\,j\Delta t-\frac{1}{2}\sigma^2 j^2\Delta t$$

Equations (21.28), (21.29) and (21.30) become

$$f_{Nj}=\max[\,j\Delta S-K,0] \qquad j=0,1\ldots$$

$$f_{i0}=0 \qquad i=0,1\ldots$$

$$f_{iM}=M\Delta S-K \qquad i=0,1\ldots$$

Problem 21.18.

An American put option on a non-dividend-paying stock has four months to maturity. The exercise price is \$21, the stock price is \$20, the risk-free rate of interest is 10% per annum, and the volatility is 30% per annum. Use the explicit version of the finite difference approach to value the option. Use stock price intervals of \$4 and time intervals of one month.

We consider stock prices of \$0, \$4, \$8, \$12, \$16, \$20, \$24, \$28, \$32, \$36 and \$40. Using equation (21.34) with $r=0.10$, $\Delta t=0.0833$, $\Delta S=4$, $\sigma=0.30$, $K=21$, $T=0.3333$ we obtain the grid shown below. The option price is \$1.56.

Stock Price	Time to Maturity (Months)				
(\$)	4	3	2	1	0
40	0.00	0.00	0.00	0.00	0.00
36	0.00	0.00	0.00	0.00	0.00
32	0.01	0.00	0.00	0.00	0.00
28	0.07	0.04	0.02	0.00	0.00
24	0.38	0.30	0.21	0.11	0.00
20	1.56	1.44	1.31	1.17	1.00
16	5.00	5.00	5.00	5.00	5.00
12	9.00	9.00	9.00	9.00	9.00
8	13.00	13.00	13.00	13.00	13.00
4	17.00	17.00	17.00	17.00	17.00
0	21.00	21.00	21.00	21.00	21.00

Problem 21.19.

The spot price of copper is \$0.60 per pound. Suppose that the futures prices (dollars per pound) are as follows:

3 months 0.59
6 months 0.57
9 months 0.54
12 months 0.50

The volatility of the price of copper is 40% per annum and the risk-free rate is 6% per annum.

Use a binomial tree to value an American call option on copper with an exercise price of \$0.60 and a time to maturity of one year. Divide the life of the option into four 3-month periods for the purposes of constructing the tree. (Hint: As explained in Section 18.7, the futures price of a variable is its expected future price in a risk-neutral world.)

In this case $\Delta t = 0.25$ and $\sigma = 0.4$ so that

$$u = e^{0.4\sqrt{0.25}} = 1.2214$$

$$d = \frac{1}{u} = 0.8187$$

The futures prices provide estimates of the growth rate in copper in a risk-neutral world. During the first three months this growth rate (with continuous compounding) is

$$4\ln\frac{0.59}{0.60} = -6.72\% \text{ per annum}$$

The parameter p for the first three months is therefore

$$\frac{e^{-0.0672\times0.25} - 0.8187}{1.2214 - 0.8187} = 0.4088$$

The growth rate in copper is equal to -13.79%, -21.63% and -30.78% in the following three quarters. Therefore, the parameter p for the second three months is

$$\frac{e^{-0.1379\times0.25} - 0.8187}{1.2214 - 0.8187} = 0.3660$$

For the third quarter it is

$$\frac{e^{-0.2163\times0.25} - 0.8187}{1.2214 - 0.8187} = 0.3195$$

For the final quarter, it is

$$\frac{e^{-0.3078\times0.25} - 0.8187}{1.2214 - 0.8187} = 0.2663$$

The tree for the movements in copper prices in a risk-neutral world is shown in Figure S21.10. The value of the option is \$0.037.

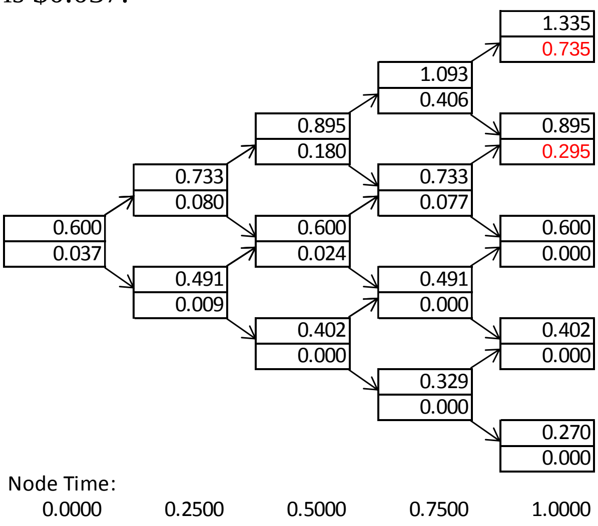


Figure S21.10: Tree to value option in Problem 21.19: At each node, upper number is price of copper and lower number is option price

Problem 21.20.

Use the binomial tree in Problem 21.19 to value a security that pays off x^2 in one year where x is the price of copper.

In this problem we use exactly the same tree for copper prices as in Problem 21.19. However, the values of the derivative are different. On the final nodes the values of the derivative equal the square of the price of copper. On other nodes they are calculated in the usual way. The current value of the security is $0.275 (see Figure S21.11).

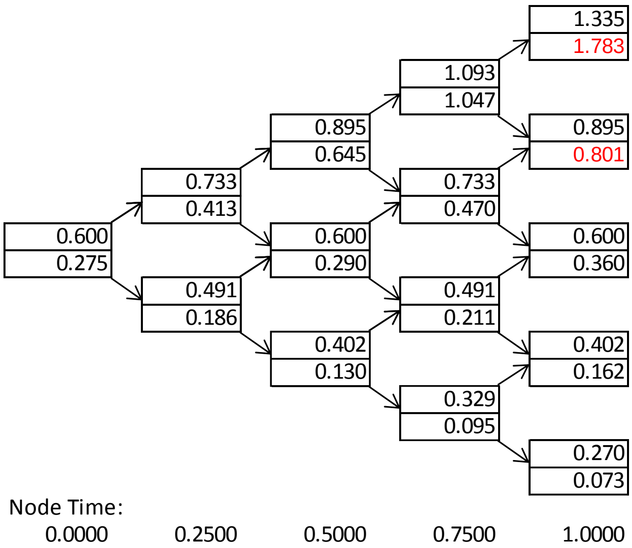


Figure S21.11: Tree to value derivative in Problem 21.20. At each node, upper number is price of copper and lower number is derivative's price.

Problem 21.21.

When do the boundary conditions for $S = 0$ and $S \rightarrow \infty$ affect the estimates of derivative prices in the explicit finite difference method?

Define S_t as the current asset price, $S_{\max}$ as the highest asset price considered and $S_{\min}$ as the lowest asset price considered. (In the example in the text $S_{\min} = 0$). Let

$$Q_1 = \frac{S_{\max} - S_t}{\Delta S} \quad \text{and} \quad Q_2 = \frac{S_t - S_{\min}}{\Delta S}$$

and let N be the number of time intervals considered. From the triangular structure of the calculations in the explicit version of the finite difference method, we can see that the values assumed for the derivative security at $S = S_{\min}$ and $S = S_{\max}$ affect the derivative's value if

$$N \geq \max(Q_1, Q_2)$$

Problem 21.22.

How would you use the antithetic variable method to improve the estimate of the European option in Business Snapshot 21.2 and Table 21.2?

The following changes could be made. Set LI as

= NORMSINV(RAND())

A1 as

=C*EXP(($E$2-$F$2*$F$2/2)*$G$2+$F$2*L2*SQRT($G$2))

H1 as

=C*EXP(($E$2-$F$2*$F$2/2)*$G$2-$F$2*L2*SQRT($G$2))

I1 as

= EXP(-$E$2*$G$2)*MAX(H1-$D$2,0)

and J1 as

=0.5*(B1+J1)

Other entries in columns L, A, H, and I are defined similarly. The estimate of the value of the option is the average of the values in the J column.

Problem 21.23.

A company has issued a three-year convertible bond that has a face value of $25 and can be exchanged for two of the company's shares at any time. The company can call the issue, forcing conversion, when the share price is greater than or equal to $18. Assuming that the company will force conversion at the earliest opportunity, what are the boundary conditions for the price of the convertible? Describe how you would use finite difference methods to value the convertible assuming constant interest rates. Assume there is no risk of the company defaulting.

The basic approach is similar to that described in Section 21.8. The only difference is the boundary conditions. For a sufficiently small value of the stock price, $S_{\min}$, it can be assumed that conversion will never take place and the convertible can be valued as a straight bond. The highest stock price which needs to be considered, $S_{\max}$, is $18. When this is reached the value of the convertible bond is $36. At maturity the convertible is worth the greater of $2S_T$ and $25 where S_T is the stock price.

The convertible can be valued by working backwards through the grid using either the explicit or the implicit finite difference method in conjunction with the boundary conditions. In formulas (21.25) and (21.32) the present value of the income on the convertible between time $t+i\,\Delta t$ and $t+(i+1)\,\Delta t$ discounted to time $t+i\,\Delta t$ must be added to the right-hand side. Chapter 27 considers the pricing of convertibles in more detail.

Problem 21.24.

Provide formulas that can be used for obtaining three random samples from standard normal distributions when the correlation between sample i and sample j is $\rho_{i,j}$.

Suppose x_1, x_2, and x_3 are random samples from three independent normal distributions. Random samples with the required correlation structure are ε_1, ε_2, ε_3 where

$$\varepsilon_1 = x_1$$

$$\varepsilon_2 = \rho_{12}x_1 + x_2\sqrt{1-\rho_{12}^2}$$

and

$$\varepsilon_3 = \alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$$

where

$$\alpha_1 = \rho_{13}$$

$$\alpha_1\rho_{12} + \alpha_2\sqrt{1-\rho_{12}^2} = \rho_{23}$$

and

$$\alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1$$

This means that

$$\alpha_1 = \rho_{13}$$

$$\alpha_2 = \frac{\rho_{23} - \rho_{13}\rho_{12}}{\sqrt{1-\rho_{12}^2}}$$

$$\alpha_3 = \sqrt{1-\alpha_1^2 - \alpha_2^2}$$

Further Questions

Problem 21.25.

An American put option to sell a Swiss franc for dollars has a strike price of \$0.80 and a time to maturity of one year. The Swiss franc' volatility is 10%, the dollar interest rate is 6%, the Swiss franc interest rate is 3%, and the current exchange rate is 0.81. Use a three-time-step tree to value the option. Estimate the delta of the option from your tree.

The tree is shown in Figure S21.12. The value of the option is estimated as 0.0207 and its delta is estimated as

$$\frac{0.006221 - 0.041153}{0.858142 - 0.764559} = -0.373$$

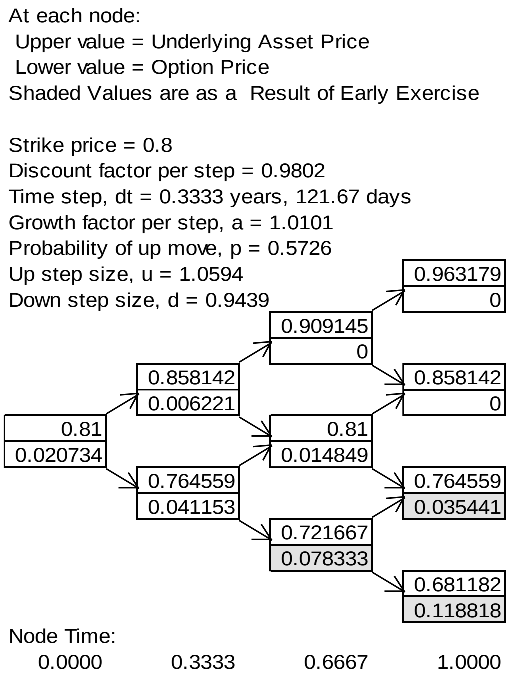


Figure S21.12: Tree for Problem 21.25

Problem 21.26.

A one-year American call option on silver futures has an exercise price of \$9.00. The current futures price is \$8.50, the risk-free rate of interest is 12% per annum, and the volatility of the futures price is 25% per annum. Use the DerivaGem software with four three-month time *steps to estimate the value of the option. Display the tree and verify that the option prices at the final and penultimate nodes are correct. Use DerivaGem to value the European version of the option. Use the control variate technique to improve your estimate of the price of the American option.*

In this case $F_0 = 8.5$, $K = 9$, $r = 0.12$, $T = 1$, $\sigma = 0.25$, and $\Delta t = 0.25$. The parameters for the tree are

$$u = e^{\sigma\sqrt{\Delta t}} = e^{0.25\sqrt{0.25}} = 1.1331$$

$$d = \frac{1}{u} = 0.8825$$

$$a = 1$$

$$p = \frac{a-d}{u-d} = \frac{1-0.8825}{1.1331-0.8825} = 0.469$$

The tree output by DerivaGem for the American option is shown in Figure S21.13. The estimated value of the option is \$0.596. The tree produced by DerivaGem for the European version of the option is shown in Figure S21.14. The estimated value of the option is \$0.586. The Black–Scholes-Merton price of the option is \$0.570. The control variate estimate of the price of the option is therefore

$$0.596 + 0.570 - 0.586 = 0.580$$

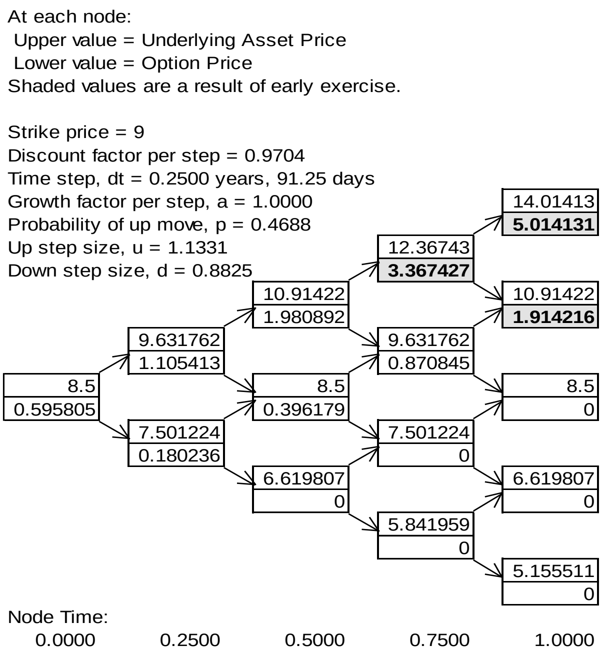


Figure S21.13: Tree for American option in Problem 21.26

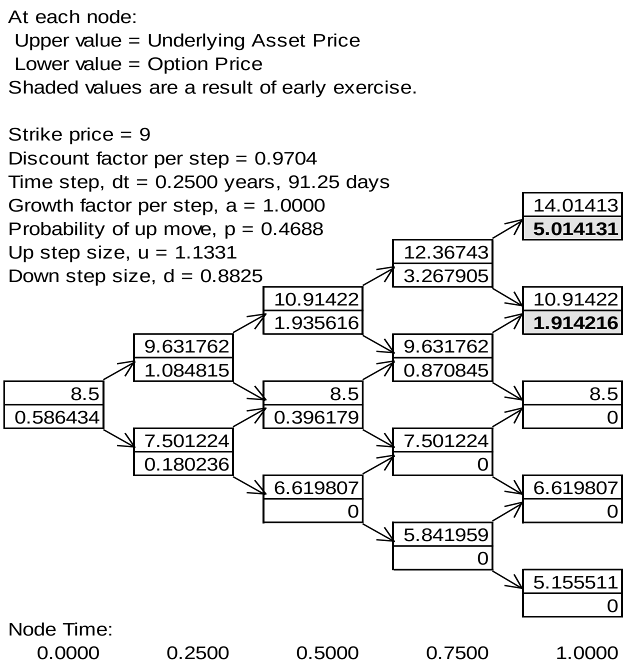


Figure S21.14: Tree for European option in Problem 21.26

Problem 21.27.

A six-month American call option on a stock is expected to pay dividends of \$1 per share at the end of the second month and the fifth month. The current stock price is \$30, the exercise price is \$34, the risk-free interest rate is 10% per annum, and the volatility of the part of the stock price that will not be used to pay the dividends is 30% per annum. Use the DerivaGem software with the life of the option divided into 100 time steps to estimate the value of the option. Compare your answer with that given by Black's approximation (see Section 15.12.)

DerivaGem gives the value of the option as 1.0349. Black's approximation sets the price of the American call option equal to the maximum of two European options. The first lasts the full six months. The second expires just before the final ex-dividend date. In this case the software shows that the first European option is worth 0.957 and the second is worth 0.997. Black's model therefore estimates the value of the American option as 0.997.

Problem 21.28.

The current value of the British pound is \$1.60 and the volatility of the pound–dollar exchange rate is 15% per annum. An American call option has an exercise price of \$1.62 and a time to maturity of one year. The risk-free rates of interest in the United States and the United Kingdom are 6% per annum and 9% per annum, respectively. Use the explicit finite difference method to value the option. Consider exchange rates at intervals of 0.20 between 0.80 and 2.40 and time intervals of 3 months.

In this case equation (21.34) becomes

$$f_{ij} = a_j^* f_{i+1,j-1} + b_j^* f_{i+1,j} + c_j^* f_{i+1,j+1}$$

where

$$a_j^* = \frac{1}{1+r\Delta t}\left[-\frac{1}{2}(r-r_f)j\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t\right]$$
$$b_j^* = \frac{1}{1+r\Delta t}(1-\sigma^2 j^2 \Delta t)$$
$$c_j^* = \frac{1}{1+r\Delta t}\left[\frac{1}{2}(r-r_f)j\Delta t + \frac{1}{2}\sigma^2 j^2 \Delta t\right]$$

The parameters are $r = 0.06$, $r_f = 0.09$, $\sigma = 0.15$, $S = 1.60$, $K = 1.62$, $T = 1$, $\Delta t = 0.25$, $\Delta S = 0.2$ and we obtain the grid shown below. The option price is \$0.062.

Exchange rate	Time to Maturity (Months)				
(\$)	12	9	6	3	0
2.40	0.780	0.780	0.780	0.780	0.780
2.20	0.580	0.580	0.580	0.580	0.580
2.00	0.380	0.380	0.380	0.380	0.380
1.80	0.183	0.180	0.180	0.180	0.180
1.60	0.062	0.055	0.043	0.027	0.000
1.40	0.011	0.007	0.003	0.000	0.000
1.20	0.001	0.000	0.000	0.000	0.000
1.00	0.000	0.000	0.000	0.000	0.000
0.80	0.000	0.000	0.000	0.000	0.000

Problem 21.29.

Answer the following questions concerned with the alternative procedures for constructing trees in Section 21.4.

(a) *Show that the binomial model in Section 21.4 is exactly consistent with the mean and variance of the change in the logarithm of the stock price in time Δt.*

(b) *Show that the trinomial model in Section 21.4 is consistent with the mean and variance of the change in the logarithm of the stock price in time Δt when terms of order $(\Delta t)^2$ and higher are ignored.*

(c) *Construct an alternative to the trinomial model in Section 21.4 so that the probabilities are 1/6, 2/3, and 1/6 on the upper, middle, and lower branches emanating from each node. Assume that the branching is from S to Su, Sm, or Sd with $m^2 = ud$. Match the mean and variance of the change in the logarithm of the stock price exactly.*

(a) For the binomial model in Section 21.4 there are two equally likely changes in the logarithm of the stock price in a time step of length Δt. These are $(r-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}$ and $(r-\sigma^2/2)\Delta t - \sigma\sqrt{\Delta t}$. The expected change in the logarithm of the stock price is

$$0.5[(r-\sigma^2/2)\Delta t + \sigma\sqrt{\Delta t}] + 0.5[(r-\sigma^2/2)\Delta t - \sigma\sqrt{\Delta t}] = (r-\sigma^2/2)\Delta t$$

This is correct. The variance of the change in the logarithm of the stock price is

$$0.5\sigma^2\Delta t + 0.5\sigma^2\Delta t = \sigma^2\Delta t$$

This is correct.

(b) For the trinomial tree model in Section 21.4, the change in the logarithm of the stock price in a time step of length Δt is $+\sigma\sqrt{3\Delta t}$, 0, and $-\sigma\sqrt{3\Delta t}$ with probabilities

$$\sqrt{\frac{\Delta t}{12\sigma^2}}\left(r-\frac{\sigma^2}{2}\right)+\frac{1}{6}, \qquad \frac{2}{3}, \qquad -\sqrt{\frac{\Delta t}{12\sigma^2}}\left(r-\frac{\sigma^2}{2}\right)+\frac{1}{6}$$

The expected change is

$$\left(r-\frac{\sigma^2}{2}\right)\Delta t$$

It variance is $\sigma^2\Delta t$ plus a term of order $(\Delta t)^2$. These are correct.

(c) To get the expected change in the logarithm of the stock price in time Δt correct we require

$$\frac{1}{6}(\ln u)+\frac{2}{3}(\ln m)+\frac{1}{6}(\ln d)=\left(r-\frac{\sigma^2}{2}\right)\Delta t$$

The relationship $m^2 = ud$ implies $\ln m = 0.5(\ln u + \ln d)$ so that the requirement becomes

$$\ln m = \left(r-\frac{\sigma^2}{2}\right)\Delta t$$

or

$$m = e^{(r-\sigma^2)\Delta t}$$

The expected change in $\ln S$ is $\ln m$. To get the variance of the change in the logarithm of the stock price in time Δt correct we require

$$\frac{1}{6}(\ln u - \ln m)^2 + \frac{1}{6}(\ln d - \ln m)^2 = \sigma^2\Delta t$$

Because $\ln u - \ln m = -(\ln d - \ln m)$ it follows that

$$\ln u = \ln m + \sigma\sqrt{3\Delta t}$$
$$\ln d = \ln m - \sigma\sqrt{3\Delta t}$$

These results imply that

$$m = e^{(r-\sigma^2)\Delta t}$$
$$u = e^{(r-\sigma^2)\Delta t+\sigma\sqrt{3\Delta t}}$$
$$d = e^{(r-\sigma^2)\Delta t-\sigma\sqrt{3\Delta t}}$$

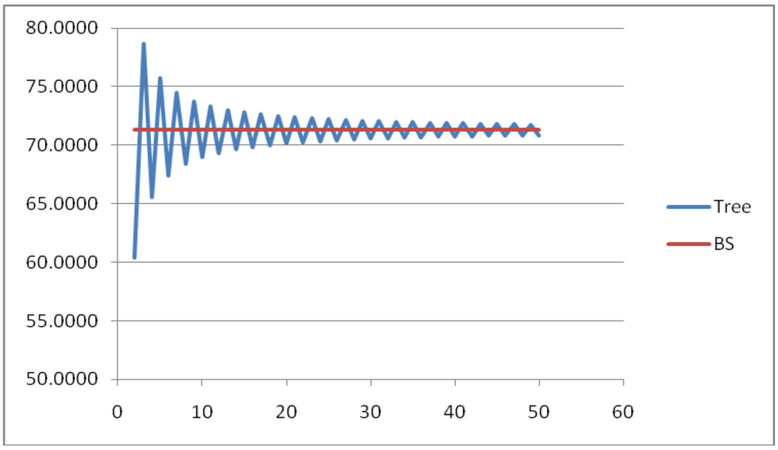
Problem 21.30. (Excel file)

The DerivaGem Application Builder functions enable you to investigate how the prices of options calculated from a binomial tree converge to the correct value as the number of time steps increases. (See Figure 21.4 and Sample Application A in DerivaGem.) Consider a put option on a stock index where the index level is 900, the strike price is 900, the risk-free rate is 5%, the dividend yield is 2%, and the time to maturity is 2 years

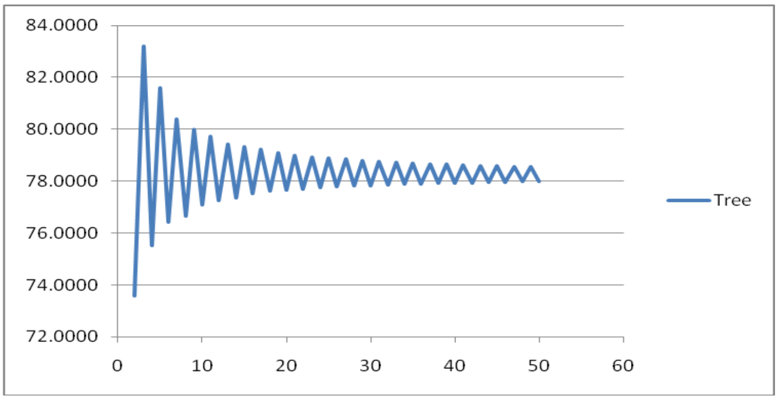
a. *Produce results similar to Sample Application A on convergence for the situation where the option is European and the volatility of the index is 20%.*
b. *Produce results similar to Sample Application A on convergence for the situation where the option is American and the volatility of the index is 20%.*
c. *Produce a chart showing the pricing of the American option when the volatility is 20% as a function of the number of time steps when the control variate technique is used.*
d. *Suppose that the price of the American option in the market is 85.0. Produce a chart showing the implied volatility estimate as a function of the number of time steps.*

The results, produced by making small modifications to Sample Application A, are shown in Figure S21.15

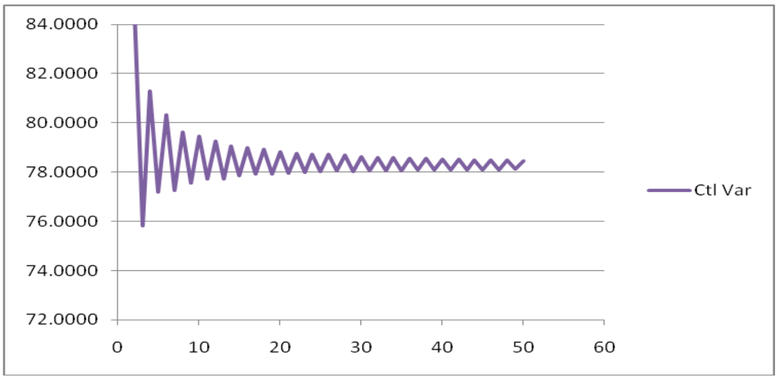
(a)



(b)



(c)



(d)

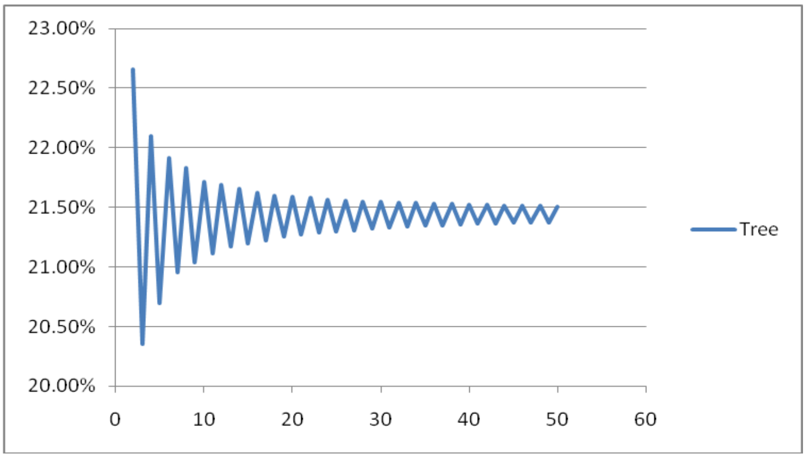


Figure S21.15: Convergence Charts for Problem 21.30

Problem 21.31.
Estimate delta, gamma, and theta from the tree in Example 21.3. Explain how each can be interpreted.

From Figure 21.5, delta is (33.64−6.13)/(327.14−275.11) = 0.5288. This is the rate of change of the option price with respect to the futures price. Gamma is

$$\frac{(56.73-12.90)/(356.73-300)-(12.90-0)/(300-252.29)}{0.5\times(356.73-252.29)}=0.009$$

This is the rate of change of delta with respect to the futures price. Theta is (12.9−19.16)/0.16667=−37.59 per year or −0.1029 per calendar day.

Problem 21.32.
How much is gained from exercising early at the lowest node at the nine-month point in Example 21.4?

Without early exercise the option is worth 0.2535 at the lowest node at the 9 month point. With early exercise it is worth 0.2552. The gain from early exercise is therefore 0.0017.

Problem 21.33.
A four-step Cox–Ross–Rubinstein binomial tree is used to price a one-year American put option on an index when the index level is 500, the strike price is 500, the dividend yield is 2%, the risk-free rate is 5%, and the volatility is 25% per annum. What is the option price, delta, gamma, and theta? Explain how you would calculate vega and rho.

The tree is shown in Figure S21.16. the option price is 41.27. Delta, gamma and theta are −0.436, 0.0042, and −0.067. To calculate vega, the volatility could be increased to 26% and the tree reconstructed to get a new option value (43.09). The increase in the option value (1.82) is the vega per 1% change in volatility. To calculate rho, the interest rate could be increased from 5% to 6% and the tree reconstructed to get a new option value (39.67). The change in the option value (−1.60) is the rho per 1% change in the interest rate.

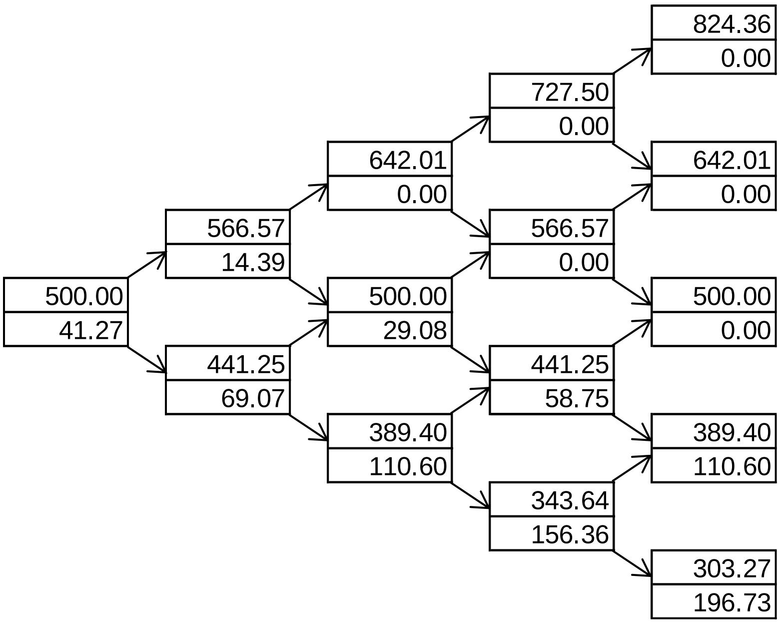


Figure S21.16: Tree for Problem 21.33