第4-1章 假设检验
(检验方法)

《统计推断》 第8章

感谢清华大学自动化系江瑞教授提供PPT

Introduction

A hypothesis is a statement about a population parameter. The two complementary hypotheses in a hypothesis testing problem are called the **null hypothesis** and the **alternative hypothesis**. They are denoted by H_0 and H_1, respectively.

$$H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_0^c$$
$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta \neq \theta_0$$
$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0$$
$$H_0 : \theta \geq \theta_0 \quad \text{versus} \quad H_1 : \theta < \theta_0$$

Simple Hypotheses

$$H_0 : \theta = \theta_0 \quad \text{versus} \quad H_1 : \theta = \theta_1$$

Both the null and the alternative hypotheses specify only one possible value for the parameter. Therefore, the sample has only one possible distribution for either the null or the alternative hypothesis.

Composite Hypotheses

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0$$

Both the null and the alternative hypotheses specify more than one possible value for the parameter. Therefore, the sample could have more than one possible distribution for either the null or the alternative hypothesis.

One-sided Hypotheses

$$H_0 : \theta \leq \theta_0 \quad \text{versus} \quad H_1 : \theta > \theta_0$$

The hypothesis assert that a univariate parameter is less than or greater than a specific value.

Two-sided Hypotheses

$$H_1 : \theta \neq \theta_0$$

The hypothesis assert that a univariate parameter is either greater or less than a specific value.

Research Hypothesis

The hypothesis that we expect an experiment to give support to and like the experiment data to support; that is, what we hope to prove.

Use the research hypothesis as the alternative hypothesis, use the complementary of the research hypothesis as the null hypothesis, and try to see whether the data support the research hypothesis.

Examples

In the heights of kids in China problem:

$$H_0\colon \mu \le \mu_0 \quad \text{versus} \quad H_1\colon \mu > \mu_0$$

In the clinical trial of hypertension drug problem:

$$H_0\colon \mu_{\text{post}} \ge \mu_{\text{pre}} \quad \text{versus} \quad H_1\colon \mu_{\text{post}} < \mu_{\text{pre}}$$

In the comparison of weights of kids in Beijing and Shanghai:

$$H_0\colon \mu_{\text{Beijing}} = \mu_{\text{Shanghai}} \quad \text{versus} \quad H_1\colon \mu_{\text{Beijing}} > \mu_{\text{Shanghai}}$$

In the Gaussian mixture (2 components) problem:

$$H_0\colon \lambda = 0 \quad \text{versus} \quad H_1\colon \lambda > 0$$

Hypothesis Testing Procedure

Hypothesis testing procedure

A **hypothesis testing procedure** or **hypothesis test** is a **rule** that specifies:

(1) For which smple values the decision is made to accept H_0 as true.

(2) For which smple values the decision is made to reject H_0 and accept H_1 as true.

The subset of the sample space for which H_0 will be rejected is called the **rejection region** (R) or **critical region**. The complement of the rejection region is called the **acceptance region** $(A = R^c)$.

Test Statistic

Certainly, the inference about the parameter θ is drawn by making use of the sample $\mathbf{X} = (X_1, \ldots, X_n)$, particularly, via a function of the sample, a **test statistic** $W = W(X_1, \ldots, X_n)$.

A hypothesis is a statement about the parameter, a subset of the parameter space.
A rejection region is a set of the sample observations, a subset of the sample space.

传统假设检验(I)

- Z检验：单样本均值检验（已知方差）

$$H_0 : \mu = \mu_0 \leftrightarrow H_1 : \mu \neq \mu_0$$

- 检验统计量

$$Z = \frac{\sqrt{n}(\overline{X} - \mu_0)}{\sigma}$$

- H_0下Z服从标准正态分布N(0,1)

$$Z \sim N(0, 1)$$

传统假设检验(II)

- T检验：单样本均值检验（未知方差）

$$H_0 : \mu = \mu_0 \leftrightarrow H_1 : \mu \neq \mu_0$$

- 检验统计量

$$T = \frac{\sqrt{n}(\overline{X} - \mu_0)}{S}, S^2 = \frac{1}{n-1}\sum_i (X_i - \overline{X})^2$$

- H_0下T服从自由度为n-1的t分布t(n-1)

$$T \sim t(n-1)$$

传统假设检验(III)

- 单样本方差检验（已知均值）

$$H_0 : \sigma = \sigma_0 \leftrightarrow H_1 : \sigma \neq \sigma_0$$

- 检验统计量

$$W_1 = \frac{\sum_{i=1}^{n} (X_i - \mu)^2}{\sigma_0^2}$$

- H_0下W_1服从自由度为n的卡方分布

$$W_1 \sim \chi^2(n)$$

传统假设检验(IV)

- 单样本方差检验（未知均值）

$$H_0 : \sigma = \sigma_0 \leftrightarrow H_1 : \sigma \neq \sigma_0$$

- 检验统计量

$$W_2 = \frac{\sum_{i=1}^{n}(X_i - \overline{X})^2}{\sigma_0^2}$$

- H_0下W_2服从标准自由度为 n-1的卡方分布

$$W_2 \sim \chi^2(n-1)$$

传统假设检验(V)

- 两样本均值检验

$$H_0 : \mu_X = \mu_y \leftrightarrow H_1 : \mu_X \neq \mu_y$$

- 检验统计量

$$T = \frac{\overline{X} - \overline{Y}}{S_p\sqrt{\frac{1}{m} + \frac{1}{n}}}, S_p^2 = \frac{(m-1)S_X^2 + (n-1)S_Y^2}{m+n-2}$$

- H_0下T服从标准自由度为 m+n-2的t分布

$$T \sim t(m+n-2)$$

传统假设检验(VI)

- 两样本方差检验

$$H_0 : \sigma_X^2 = \sigma_Y^2 \leftrightarrow H_1 : \sigma_X^2 \neq \sigma_Y^2$$

- 检验统计量

$$F = \frac{S_X^2/\sigma_X^2}{S_Y^2/\sigma_Y^2},$$

$$S_X^2 = \frac{1}{m-1}\sum_{i=1}^{m}(X_i - \overline{X})^2, S_Y^2 = \frac{1}{m-1}\sum_{j=1}^{n}(Y_i - \overline{Y})^2$$

- H_0下F服从标准自由度为 n-1, m-1的F分布

$$F \sim F(n-1, m-1)$$

Methods of Finding Tests

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“A random variable is a quantity whose values are random and to which a probability distribution is assigned.”

Methods of Finding Tests

- **Neyman-Pearson tests (NPT)**
- **Likelihood ratio tests (LRT)**
- Bayesian tests
- Union-Intersection tests (UIT)
- Intersection-Union tests (IUT)

Neyman-Pearson Tests

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“A random variable is a quantity whose values are random and to which a probability distribution is assigned.”

Neyman-Pearson Tests (NPT)

Consider testing the simple hypothesis

$$H_0 : \theta = \theta_0 \text{ versus } H_1 : \theta = \theta_1,$$

where the joint pdf or pmf of the sample, corresponding to θ_i is $f(\mathbf{x} \mid \theta_i)$ $(i = 0,1)$. A **Neyman - Pearson Test (NPT)** is any test with rejection region

$$R = \{\mathbf{x} : f(\mathbf{x} \mid \theta_1) > kf(\mathbf{x} \mid \theta_0)\}$$

and acceptence region

$$R^c = \{\mathbf{x} : f(\mathbf{x} \mid \theta_1) < kf(\mathbf{x} \mid \theta_0)\},$$

where $k \geq 0$.

Binomial NPT

Let $X \sim \text{Binomial}(3, \theta)$. Consider the simple hypotheses

$$H_0 : \theta = \frac{1}{2} \text{ versus } H_1 : \theta = \frac{3}{4}.$$

The pmf of X is

$$f(x \mid \theta_i) \propto \theta_i^x (1 - \theta_i)^{3-x},$$

where $x = 0, 1, 2, 3$. Therefore

$$\frac{f(0 \mid \theta_1)}{f(0 \mid \theta_0)} = \frac{1}{8}, \frac{f(1 \mid \theta_1)}{f(1 \mid \theta_0)} = \frac{3}{8}, \frac{f(2 \mid \theta_1)}{f(2 \mid \theta_0)} = \frac{9}{8}, \frac{f(3 \mid \theta_1)}{f(3 \mid \theta_0)} = \frac{27}{8},$$

Choose

$$\begin{aligned}
{}^{27}\!/_{8} \le k, &\quad \text{we will never reject } H_0; \\
{}^{9}\!/_{8} \le k < {}^{27}\!/_{8}, &\quad \text{we will reject } H_0 \text{ if } X = 3; \\
{}^{3}\!/_{8} \le k < {}^{9}\!/_{8}, &\quad \text{we will reject } H_0 \text{ if } X = 2, 3; \\
{}^{1}\!/_{8} \le k < {}^{3}\!/_{8}, &\quad \text{we will reject } H_0 \text{ if } X = 1, 2, 3; \\
k < {}^{1}\!/_{8}, &\quad \text{we will reject } H_0 \text{ if } X = 0, 1, 2, 3.
\end{aligned}$$

NPT based on a Sufficient Statistic

Consider testing the simple hypothesis

$$H_0 : \theta = \theta_0 \text{ versus } H_1 : \theta = \theta_1.$$

Suppose $T(\mathbf{X})$ is a sufficient statistic for θ, and $g(t \mid \theta_i)$ is the pdf or pmf of T, corresponding to θ_i $(i = 0,1)$.

Then, the test with rejection region

$$S = \{t : g(t \mid \theta_1) > kg(t \mid \theta_0)\}$$

and acceptence region

$$S^c = \{t : g(t \mid \theta_1) < kg(t \mid \theta_0)\},$$

where $k \geq 0$, is a Neyman-Pearson test.

Proof

Consider the rejection region in the original sample space.

By the factorization theorem

$$f(\mathbf{x} \mid \theta_i) = g(T(\mathbf{x}) \mid \theta_i)h(\mathbf{x}) \qquad\qquad (i = 0,1)$$

Therefore,

$$S = \{t : g(t \mid \theta_1) > kg(t \mid \theta_0)\}$$

is equivalent to

$$R = \{\mathbf{x} : f(\mathbf{x} \mid \theta_1) = g(t \mid \theta_1)h(\mathbf{x}) > kg(t \mid \theta_0)h(\mathbf{x}) = kf(\mathbf{x} \mid \theta_0)\},$$

and

$$S^c = \{t : g(t \mid \theta_1) < kg(t \mid \theta_0)\}$$

is equivalent to

$$R^c = \{\mathbf{x} : f(\mathbf{x} \mid \theta_1) = g(t \mid \theta_1)h(\mathbf{x}) < kg(t \mid \theta_0)h(\mathbf{x}) = kf(\mathbf{x} \mid \theta_0)\}.$$

Bernoulli NPT

Let $X_1, \ldots, X_n \sim \text{Bernoulli}(\theta)$, iid. Consider the simple hypotheses

$$H_0 : \theta = \theta_0 \text{ versus } H_1 : \theta = \theta_1 \qquad (\theta_0 < \theta_1).$$

ΣX is a sufficient statistic of θ, and the pmf of ΣX is

$$f(\Sigma x \mid \theta_i) \propto \theta_i^{\Sigma x}(1-\theta_i)^{n-\Sigma x},$$

Therefore

$$\frac{f(\Sigma x \mid \theta_1)}{f(\Sigma x \mid \theta_0)} = \frac{\theta_1^{\Sigma x}(1-\theta_1)^{n-\Sigma x}}{\theta_0^{\Sigma x}(1-\theta_0)^{n-\Sigma x}} = \left(\frac{\theta_1}{\theta_0}\frac{1-\theta_0}{1-\theta_1}\right)^{\Sigma x}\left(\frac{1-\theta_1}{1-\theta_0}\right)^n,$$

Since $\theta_1 > \theta_0, 1-\theta_0 > 1-\theta_1$,

$$\frac{\theta_1}{\theta_0} > 1; \frac{1-\theta_0}{1-\theta_1} > 1; \frac{\theta_1}{\theta_0}\frac{1-\theta_0}{1-\theta_1} > 1$$

$f(\Sigma x \mid \theta_1) > kf(\Sigma x \mid \theta_0)$ is equivalent to

$$\Sigma x > \left[\log k + n\log\left(\frac{1-\theta_0}{1-\theta_1}\right)\right] \Big/ \log\left(\frac{\theta_1}{\theta_0}\frac{1-\theta_0}{1-\theta_1}\right) \equiv c(k).$$

Therefore, we like to reject H_0 when Σx is sufficiently large.

Normal NPT

Let $X_1, \ldots, X_n \sim \mathrm{N}(\mu, \sigma^2)$, iid, σ^2 known. Consider the simple hypothesis

$$H_0 : \mu = \mu_0 \text{ versus } H_1 : \mu = \mu_1 \quad (\mu_0 < \mu_1).$$

The pdf of $\bar{X}$ is

$$f(\bar{x} \mid \mu_i) \propto \exp\left[-\frac{n}{2\sigma^2}(\bar{x} - \mu)^2\right],$$

Therefore

$$\begin{aligned} \frac{f(\bar{x} \mid \mu_1)}{f(\bar{x} \mid \mu_0)} &= \exp\left[-\frac{n}{2\sigma^2}((\bar{x} - \mu_1)^2 - (\bar{x} - \mu_0)^2)\right] \\ &= \exp\left[-\frac{n}{2\sigma^2}(\mu_0 - \mu_1)(2\bar{x} - (\mu_0 + \mu_1))\right] \end{aligned}$$

$f(\bar{x} \mid \mu_1) > kf(\bar{x} \mid \mu_0)$ is equivalent to

$$\bar{x} > \frac{\sigma^2 \log k}{n(\mu_1 - \mu_0)} + \frac{\mu_1 + \mu_0}{2} \equiv c(k).$$

Therefore, we like to reject H_0 when $\bar{x}$ is sufficiently large.

An Example

Consider testing the simple hypotheses

$$H_0 : \mu = 0 \text{ versus } H_1 : \mu = 1.$$

Suppose $\sigma^2 = 1$ and $n = 10$. We would like to reject H_0 if

$$\bar{x} > \frac{\sigma^2 \log k}{n(\mu_1 - \mu_0)} + \frac{\mu_1 + \mu_0}{2} = \frac{1}{10}\log k + \frac{1}{2}.$$

If $k = 1$, H_0 will be rejected when $\bar{x} > 0.5$.

If $k = 10$, H_0 will be rejected when $\bar{x} > 0.73$.

If $k = 100$, H_0 will be rejected when $\bar{x} > 0.96$.

Obviously, the rejection region depends on the constant k. Varying k, we will have a family of tests. In order to ensure strong evidence of rejecting the null hypothesis, we would like to select a large k. We will see how to determine this constant later.

Likelihood Ratio Tests

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"A random variable is a quantity whose values are random and to which a probability distribution is assigned."

Likelihood Ratio Tests (LRT)

The **likelihood ratio test statistic** for a hypothesis testing

$$H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_0^c$$

is

$$\lambda(\mathbf{x}) = \frac{\sup_{\Theta_0} L(\theta \mid \mathbf{x})}{\sup_{\Theta} L(\theta \mid \mathbf{x})}$$

A **likelihood ratio test (LRT)** is any test that has a rejection region of the form

$$R = \{\mathbf{x} : \lambda(\mathbf{x}) < c\},$$

where c is any number satisfying $0 \leq c \leq 1$.

Connection to MLE

If

$$\hat{\theta}_0 = \arg\max_{\theta \in \Theta_0} L(\theta \mid \mathbf{x})$$

and

$$\hat{\theta} = \arg\max_{\theta \in \Theta} L(\theta \mid \mathbf{x})$$

exist, then

$$\lambda(\mathbf{x}) = \frac{L(\hat{\theta}_0 \mid \mathbf{x})}{L(\hat{\theta} \mid \mathbf{x})}$$

We would like to reject the null hypothesis if

$$\lambda(\mathbf{x}) < c \ (0 \le c \le 1).$$

This builds the relation between LRT and MLE.

A Six-step Procedure

Let the hypotheses be

$$H_0 : \theta \in \Theta_0 \quad \text{versus} \quad H_1 : \theta \in \Theta_0^c$$

What we need to do for a LRT include:

(1) Write the likelihood function;

(2) Determine the parameter spaces, say, Θ_0 and Θ;

(3) Calulate the restricted MLE in Θ_0;

(4) Calculate the unrestricted MLE in Θ;

(5) Obtain the likelihood ratio test statistic.

(6) Determine the rejection region.

LRT for Normal Mean, Variance Known

Let $X = (X_1, \ldots, X_n)$ be a random sample from a Normal population $N(\mu, \sigma^2)$, where σ^2 is known. Consider the hypotheses

$$H_0 : \mu = \mu_0 \text{ versus } H_1 : \mu \neq \mu_0$$

Write the likelihood Function

$$L(\mu \mid \mathbf{x}) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left[-\frac{\sum_{i=1}^{n}(x_i - \mu)^2}{2\sigma^2}\right]$$

$$l(\mu \mid \mathbf{x}) = -\frac{n}{2}\log(2\pi) - \frac{n}{2}\log(\sigma^2) - \frac{1}{2}(\sigma^2)^{-1}\sum_{i=1}^{n}(x_i - \mu)^2$$

Determine the Parameter Spaces

Under the null, the parameter space is

$$\Theta_0 = \{\mu_0\}$$

The entire parameter space is

$$\Theta = (-\infty, \infty).$$

Calculate the Restricted MLE

Θ_0 contains a single point, say, μ_0. So it must be the maximum likelihood estimate of μ. Therefore,

$$\hat{\mu}_0 = \mu_0.$$

We have the restricted maximum likelihood as

$$L(\hat{\mu}_0 \mid \mathbf{x}) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left[-\frac{\sum_{i=1}^{n}(x_i - \mu_0)^2}{2\sigma^2}\right].$$

Calculate the Unrestricted MLE

The entire parameter space is

$$\Theta = (-\infty, \infty).$$

Maximizing the likelihood function yields

$$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} x_i = \overline{x}.$$

The unrestricted maximum likelihood is therefore

$$L(\hat{\mu} \mid \mathbf{x}) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left[-\frac{\sum_{i=1}^{n}(x_i - \overline{x})^2}{2\sigma^2}\right].$$

Obtain the Likelihood Ratio Test Statistic

The likelihood ratio test statistic is

$$\lambda(\mathbf{x}) = \frac{L(\hat{\mu}_0 \mid \mathbf{x})}{L(\hat{\mu} \mid \mathbf{x})} = \frac{(2\pi\sigma^2)^{-n/2} \exp\left[-\sum_{i=1}^{n} (x_i - \mu_0)^2 / (2\sigma^2)\right]}{(2\pi\sigma^2)^{-n/2} \exp\left[-\sum_{i=1}^{n} (x_i - \overline{x})^2 / (2\sigma^2)\right]}$$

Notice that

$$\begin{aligned} \sum_{i=1}^{n} (x_i - \mu_0)^2 &= \sum_{i=1}^{n} (x_i - \overline{x} + \overline{x} - \mu_0)^2 \\ &= \sum_{i=1}^{n} (x_i - \overline{x})^2 + n(\overline{x} - \mu_0)^2 \end{aligned}$$

We have

$$\lambda(\mathbf{x}) = \frac{L(\hat{\mu}_0 \mid \mathbf{x})}{L(\hat{\mu} \mid \mathbf{x})} = \exp\left[-\frac{(\overline{x} - \mu_0)^2}{2\sigma^2 / n}\right]$$

Determine the Rejection Region

The rejection region for LRT is $\lambda(\mathbf{x}) < c, 0 \le c \le 1.$

In our problem, it is

$$\lambda(\mathbf{x}) = \exp\left[-\frac{(\bar{x}-\mu_0)^2}{2\sigma^2/n}\right] < c, 0 \le c \le 1.$$

Or equivalently,

$$\frac{|\,\bar{x}-\mu_0\,|}{\sigma/\sqrt{n}} > \sqrt{-2\log c} = z, z \ge 0.$$

Therefore, the rejection region is

$$R = \left\{\mathbf{x} : \lambda(\mathbf{x}) < c\right\} = \left\{\mathbf{x} : \frac{|\,\bar{x}-\mu_0\,|}{\sigma/\sqrt{n}} > z\right\}.$$

You may want to write it as

$$R = \left\{\mathbf{x} : \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} > z\right\} \cup \left\{\mathbf{x} : \frac{\bar{x}-\mu_0}{\sigma/\sqrt{n}} < -z\right\}.$$

One-sample *Z* Test, Two-sided

H_0: $\mu = \mu_0$ **versus** H_1: $\mu \neq \mu_0$

The parameter spaces are $\Theta_0 = \{\mu_0\}$ and $\Theta = (-\infty, \infty)$.

The restricted MLE yields

$$\hat{\mu}_0 = \mu_0,\ \ L(\mu_0 \mid \mathbf{x}) = (2\pi\sigma^2)^{-n/2} \exp\left[-\sum_{i=1}^{n} (x_i - \mu_0)^2 / (2\sigma^2)\right].$$

The unrestricted MLE yields

$$\hat{\mu} = \bar{x},\ \ L(\mu_0 \mid \mathbf{x}) = (2\pi\sigma^2)^{-n/2} \exp\left[-\sum_{i=1}^{n} (x_i - \bar{x})^2 / (2\sigma^2)\right].$$

We have

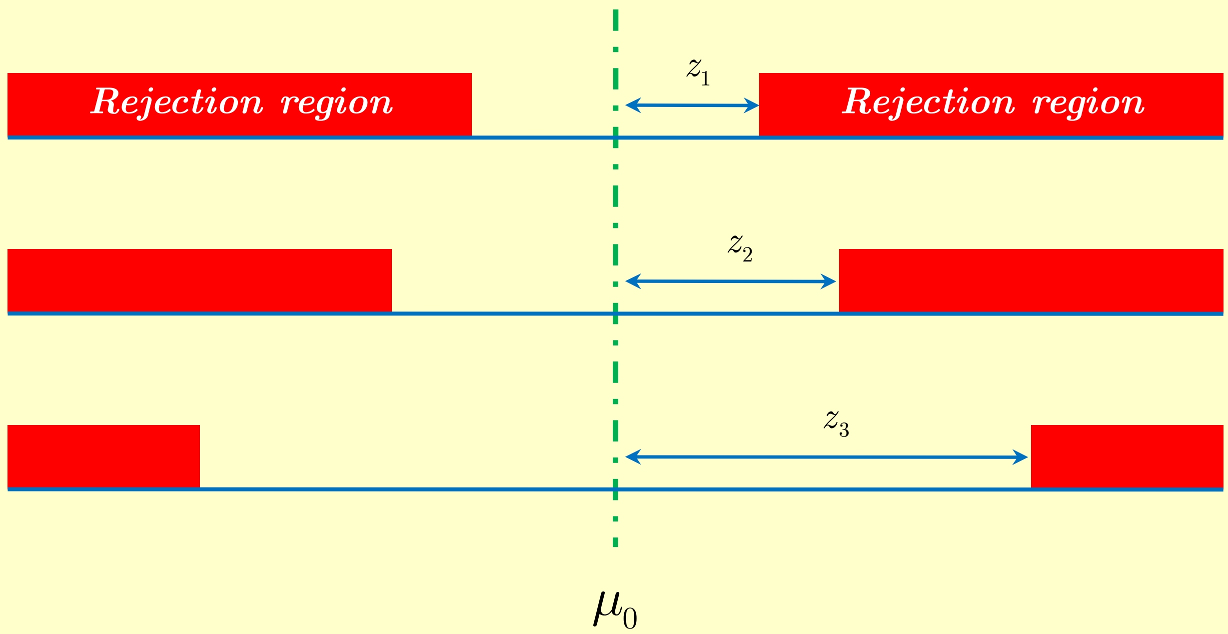
$$\lambda(\mathbf{x}) = \frac{L(\hat{\mu}_0 \mid \mathbf{x})}{L(\hat{\mu} \mid \mathbf{x})} = \exp\left[-\frac{(\bar{x} - \mu_0)^2}{2\sigma^2 / n}\right].$$

Therefore,

$$R = \{\mathbf{x} : \lambda(\mathbf{x}) < c\} = \left\{\frac{|\, \bar{x} - \mu_0 \,|}{\sigma / \sqrt{n}} > z\right\}, z \geq 0.$$

A Family of Tests

$$R = \left\{ \frac{|\,\overline{x} - \mu_0\,|}{\sigma / \sqrt{n}} > z \right\}, z \geq 0$$



One-sample Z Test, Greater

H_0: $\mu \leq \mu_0$ versus H_1: $\mu > \mu_0$

The likelihood function is

...

The parameter spaces are

...

The restricted region MLE yields

...

The unrestricted MLE yields

...

Therefore,

$$\lambda(\mathbf{x}) = \ldots$$

In conclusion,

$$R = \left\{ \mathbf{x} : \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} > z \right\}, z \geq 0.$$

Restricted and Unrestricted MLE

$\Theta_0 = (-\infty, \mu_0]$.

When $\bar{x} \le \mu_0$, $\hat{\mu}_0 = \bar{x}$.

When $\bar{x} > \mu_0$, $\hat{\mu}_0 = \mu_0$.

The restricted maximum likelihood is therefore

$$L(\hat{\mu}_0 \mid \mathbf{x}) = \begin{cases} (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left[-\dfrac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{2\sigma^2}\right] & \bar{x} \le \mu_0; \\ (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left[-\dfrac{\sum_{i=1}^{n}(x_i - \mu_0)^2}{2\sigma^2}\right] & \bar{x} > \mu_0. \end{cases}$$

The unrestricted maximum likelihood is still

$$L(\hat{\mu} \mid \mathbf{x}) = (2\pi\sigma^2)^{-\frac{n}{2}} \exp\left[-\frac{\sum_{i=1}^{n}(x_i - \bar{x})^2}{2\sigma^2}\right].$$

Rejection Region

The likelihood ratio test statistic is

$$\lambda(\mathbf{x}) = \frac{L(\hat{\mu}_0 \mid \mathbf{x})}{L(\hat{\mu} \mid \mathbf{x})} = \begin{cases} 1 & \bar{x} \le \mu_0 \\ \exp\left[-\dfrac{(\bar{x}-\mu_0)^2}{2\sigma^2 / n}\right] & \bar{x} > \mu_0 \end{cases}$$

Clearly, when $\bar{x} \le \mu_0$, the rejection region contains no element.

when $\bar{x} > \mu_0$, the rejection region is

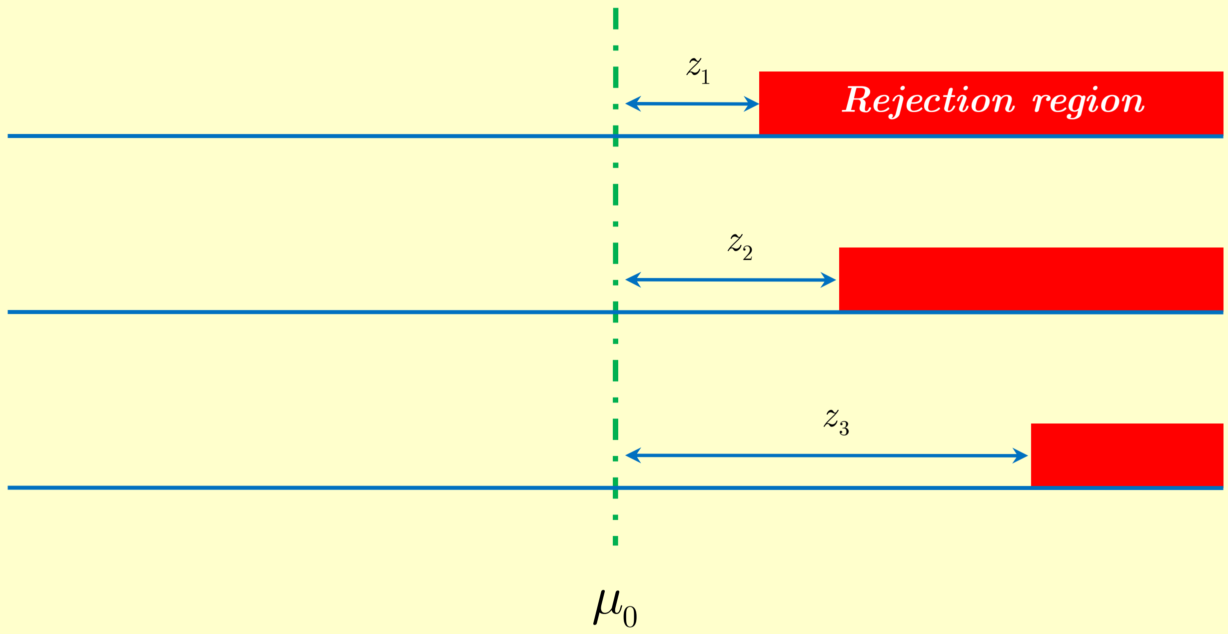
$$\lambda(\mathbf{x}) = \exp\left[-\frac{(\bar{x}-\mu_0)^2}{2\sigma^2 / n}\right] < c \ (0 \le c \le 1).$$

Or equivalently,

$$\frac{\bar{x}-\mu_0}{\sigma / \sqrt{n}} > \sqrt{-2\log c} = z, z \ge 0.$$

A Family of Tests

$$R = \left\{ \frac{\overline{x} - \mu_0}{\sigma / \sqrt{n}} > z \right\}, z \geq 0$$



One-sample Z Test, Less

H_0: $\mu \geq \mu_0$ **versus** H_1: $\mu < \mu_0$

The likelihood function is

...

The parameter spaces are

...

The restricted region MLE yields

...

The unrestricted MLE yields

...

Therefore,

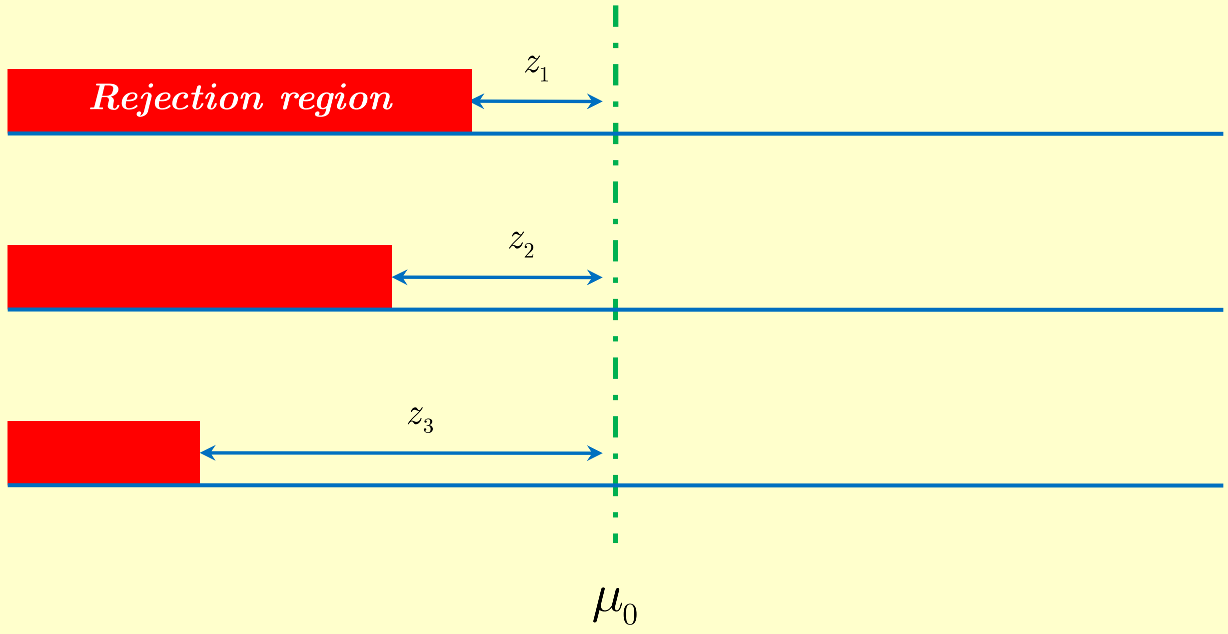
$\lambda(\mathbf{x})=...$

In conclusion,

$$R = \left\{ \mathbf{x} : \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} < -z \right\}, z \geq 0.$$

A family of Tests Instead of One Test

$$R = \left\{ \frac{\overline{x} - \mu_0}{\sigma / \sqrt{n}} < -z \right\}, z \geq 0$$



One Sample T Test

Let $X = (X_1, \ldots, X_n)$ be a random sample from a Normal population $N(\mu, \sigma^2)$, where σ^2 is ***unknown***. Consider the hypotheses

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu > \mu_0$$

The rejection region is $\quad R = \left\{ \mathbf{x} : \dfrac{\overline{x} - \mu_0}{s / \sqrt{n}} > t \right\}, \ t \geq 0.$

One-sample *t* test, an exercise

Paired-sample *T* Test

Let $(X_1, Y_1), \ldots, (X_n, Y_n)$ be a random sample from a bivariate normal population $N(\mu_X, \sigma_X^2, \mu_Y, \sigma_Y^2, \rho)$, where σ_X^2, σ_Y^2 and ρ are $\boldsymbol{unknown}$. Consider the hypotheses

$$H_0 : \mu_X = \mu_Y \quad \text{versus} \quad H_1 : \mu_X < \mu_Y$$

The rejection region is $R = \left\{ \mathbf{x} : \dfrac{\overline{x} - \overline{y}}{s_{x-y} / \sqrt{n}} < -t \right\}, \; t \geq 0,$

where $s_{x-y}^2 = \dfrac{1}{n-1} \displaystyle\sum_{i=1}^{n} [(x_i - \overline{x}) - (y_i - \overline{y})]^2$

Paired-sample *t* test, an exercise

Two-sample *T* Test

Let $\mathbf{X} = (X_1,\ldots,X_m)$ and $\mathbf{Y} = (Y_1,\ldots,Y_n)$ be a random sample from two independent Normal population $N(\mu_X,\sigma_X^2)$ and $N(\mu_Y,\sigma_Y^2)$, respectively, where $\sigma_X^2 = \sigma_Y^2 = \sigma^2$ is **unknown**. Consider the hypotheses

$$H_0 : \mu_X = \mu_Y \quad \text{versus} \quad H_1 : \mu_X \neq \mu_Y$$

Rejection region is $\quad R = \left\{ \mathbf{x} : \dfrac{|\,\overline{x} - \overline{y}\,|}{s_p\sqrt{\frac{1}{m} + \frac{1}{n}}} > t \right\}, t \geq 0,$

where $s_p^2 = \dfrac{(m-1)s_x^2 + (n-1)s_y^2}{m+n-2}$ is the pooled variance estimate.

Two-sample *t* test, an exercise

LRT based on Sufficient Statistics

If $T(\mathbf{X})$ is a sufficient statistic for θ and $\lambda^{\star}(t)$ and $\lambda(\mathbf{x})$ are the likelihood ratio test statistic based on $T(\mathbf{X})$ and $\mathbf{X}$, respectively, then $\lambda^{\star}(T(\mathbf{x})) = \lambda(\mathbf{x})$ for every $\mathbf{x}$ in the sample space.

Proof

According to the Factorization Theorem, $f(\mathbf{x} \mid \theta) = g(T(\mathbf{x}) \mid \theta)h(x)$ when $T(\mathbf{X})$ is a sufficient statistic of θ, where $g(T(\mathbf{x}) \mid \theta)$ is the pdf or pmf of T and $h(\mathbf{x})$ does not depend on θ. Therefore

$$
\begin{aligned}
\lambda(\mathbf{x}) &= \frac{\sup_{\Theta_0} L(\theta \mid \mathbf{x})}{\sup_{\Theta} L(\theta \mid \mathbf{x})} \\
&= \frac{\sup_{\Theta_0} f(\mathbf{x} \mid \theta)}{\sup_{\Theta} f(\mathbf{x} \mid \theta)} \\
&= \frac{\sup_{\Theta_0} g(T(\mathbf{x}) \mid \theta)h(\mathbf{x})}{\sup_{\Theta} g(T(\mathbf{x}) \mid \theta)h(\mathbf{x})} \\
&= \frac{\sup_{\Theta_0} g(T(\mathbf{x}) \mid \theta)}{\sup_{\Theta} g(T(\mathbf{x}) \mid \theta)} \\
\lambda^{\star}(T(\mathbf{x})) &= \frac{\sup_{\Theta_0} L^{\star}(\theta \mid T(\mathbf{x}))}{\sup_{\Theta} L^{\star}(\theta \mid T(\mathbf{x}))}
\end{aligned}
$$

An Example

Because $\bar{X}$ is a sufficient statistic of μ, and $\bar{X} \sim N(\mu, \sigma^2 / n)$.

For a normal population, the likelihood function of μ in terms of $\bar{X}$ is

$$L(\mu \mid \bar{x}) = p(\bar{x} \mid \mu) = \frac{1}{\sqrt{2\pi}\sigma / \sqrt{n}} \exp\left[-\frac{(\bar{x}-\mu)^2}{2\sigma^2 / n}\right].$$

For the test of

$$H_0 : \mu = \mu_0 \text{ versus } \mu \neq \mu_0$$

The restricted MLE yields

$$L(\hat{\mu}_0 \mid \bar{x}) = (2\pi\sigma^2 / n)^{-1/2} \exp\left[-(\bar{x}-\mu_0)^2 / (2\sigma^2 / n)\right].$$

The unrestricted MLE yields

$$L(\hat{\mu} \mid \bar{x}) = (2\pi\sigma^2 / n)^{-1/2}.$$

Therefore,

$$\lambda(\bar{x}) = \exp\left[-\frac{(\bar{x}-\mu_0)^2}{2\sigma^2 / n}\right].$$

and $$R = \left\{\mathbf{x} : \lambda(\bar{x}) < c\right\} = \left\{\mathbf{x} : \frac{|\,\bar{x}-\mu_0\,|}{\sigma / \sqrt{n}} > \sqrt{-2\log c} = z\right\}, z \geq 0.$$

Relation Between NPT and LRT

Consider testing the simple hypotheses

$$H_0 : \theta = \theta_0 \text{ versus } H_1 : \theta = \theta_1$$

LRT suggests to reject the null hypothesis when the ratio

$$\lambda(\mathbf{x}) = \frac{L(\theta_0 \mid \mathbf{x})}{\max\{L(\theta_0 \mid \mathbf{x}), L(\theta_1 \mid \mathbf{x})\}} = \begin{cases} 1 & L(\theta_0 \mid \mathbf{x}) \geq L(\theta_1 \mid \mathbf{x}) \\ \dfrac{L(\theta_0 \mid \mathbf{x})}{L(\theta_1 \mid \mathbf{x})} & L(\theta_0 \mid \mathbf{x}) < L(\theta_1 \mid \mathbf{x}) \end{cases}$$

is sufficiently small ($< c$).

NPT suggests to reject the null hypothesis when the ratio

$$\frac{f(\mathbf{x} \mid \theta_1)}{f(\mathbf{x} \mid \theta_0)} = \frac{L(\theta_1 \mid \mathbf{x})}{L(\theta_0 \mid \mathbf{x})}$$

is sufficiently large ($> k$).

If $k = 1 / c > 1$, LRT is equivalent to NPT.

If $c \geq 1$ or $k \leq 1$, LRT and NPT are different.

Because c is usually chosen to be small (k large) to ensure strong evidence, in practice the two tests are often the same.

Bayes Test

- 回顾Bayes估计, 设 $X_1, \cdots, X_n \sim f(x|\theta)$
- 先验分布：$\theta \sim \pi(\theta)$
- 后验分布：$\theta \sim \pi(\theta|x)$

$$\pi(\theta|x) = \frac{L(\theta|x)\pi(\theta)}{\int_\Theta L(\theta|x)\pi(\theta)d\theta}$$

- 对检验问题

$$H_0 : \theta \in \Theta_0 \leftrightarrow H_1 : \theta \in \Theta_0^c$$

Bayesian Tests

Suppose the posterior distribution of the parameter is

$$p(\theta \mid \mathbf{x})$$

The rejection rule is

$$P(\theta \in \Theta_0 \mid \mathbf{x}) \leq P(\theta \in \Theta_0^c \mid \mathbf{x})$$

The rejection region is

$$R = \left\{ \mathbf{x} : P(\theta \in \Theta_0 \mid \mathbf{x}) < 1/2 \right\}$$

Or equivalently

$$R = \left\{ \mathbf{x} : P(\theta \in \Theta_0^c \mid \mathbf{x}) \geq 1/2 \right\}$$

Note that

$$P(\theta \in \Theta_0 \mid \mathbf{x}) = \int_{\theta \in \Theta_0} p(\theta \mid \mathbf{x}) d\theta$$

$$P(\theta \in \Theta_0^c \mid \mathbf{x}) = \int_{\theta \in \Theta_0^c} p(\theta \mid \mathbf{x}) d\theta$$

Bayesian Test of Normal Mean

For a normal distribution $N(\mu, \sigma^2)$, the conjugate prior for μ is $N(\nu, \tau^2)$, provided that the variance σ^2 is known. With this prior, denoted by $\pi(\mu)$, the posterior distribution of μ, $\pi(\mu \mid \mathbf{x})$, is given by

$$N(\tilde{\mu}, \tilde{\sigma}^2) = N\left(\frac{n\tau^2\overline{x} + \sigma^2\nu}{n\tau^2 + \sigma^2}, \frac{\sigma^2\tau^2}{n\tau^2 + \sigma^2}\right)$$

Now we want to test

$$H_0 : \mu \le \mu_0 \quad \text{versus} \quad H_1 : \mu > \mu_0$$

According to the Bayesian test rule, the rejection region is the set of all $\mathbf{x}$ that can produce the following inequality

$$P(\mu \le \mu_0 \mid \mathbf{x}) = \int_{-\infty}^{\mu_0} \pi(\mu \mid \mathbf{x}) d\mu < \frac{1}{2} = \int_{-\infty}^{\tilde{\mu}} \pi(\mu \mid \mathbf{x}) d\mu$$

which is equivalent to $\mu_0 < \tilde{\mu}$. Therefore, the rejection region is

$$R = \left\{\mathbf{x} : \overline{x} > \mu_0 + \frac{\sigma^2(\mu_0 - \nu)}{n\tau^2}\right\}$$

复杂检验

- 某些情况下，对复杂的零假设的检验能够通过对较简单的零假设的检验得到。
- 并-交检验（Union-intersection Test, UIT）
- 交-并检验（Intersection-union Test, IUT）

Union-intersection Tests (UIT)

If the null hypothesis is

$$H_0 : \theta \in \bigcap_{\gamma \in \Gamma} \Theta_\gamma$$

Then the rejection region is

$$R = \bigcup_{\gamma \in \Gamma} \{\mathbf{x} : T(\mathbf{x}) \in R_\gamma\}$$

R_γ is the rejection region for the hypothesis testing problem

$$H_{0\gamma} : \theta \in \Theta_\gamma \quad \text{versus} \quad H_{1\gamma} : \theta \in \Theta_\gamma^c$$

An Example

Obviously, In the hypotheses

$$H_0 : \mu = \mu_0 \quad \text{versus} \quad H_1 : \mu \neq \mu_0$$

We have H_0 is equal to

$$H_0 : \{\mu : \mu \leq \mu_0\} \cap \{\mu : \mu \geq \mu_0\}$$

This is an intersection.

We have also derived that

$$R_{\mu \leq \mu_0} = \{\mathbf{x} : (\bar{x} - \mu_0) / (\sigma / \sqrt{n}) > z_U\}$$
$$R_{\mu \geq \mu_0} = \{\mathbf{x} : (\bar{x} - \mu_0) / (\sigma / \sqrt{n}) < -z_L\}$$

Therefore, the rejection region R is

$$R = R_{\mu \leq \mu_0} \cup R_{\mu \geq \mu_0} = \left\{\mathbf{x} : \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} > z_U \text{ or } \frac{\bar{x} - \mu_0}{\sigma / \sqrt{n}} < -z_L\right\}.$$

In the case that $z_U = -z_L = z > 0$, we return back to the two-sided normal mean LRT, when the variance is known.

Intersection-union Tests (IUT)

If the null hypothesis is

$$H_0 : \theta \in \bigcup_{\gamma \in \Gamma} \Theta_\gamma$$

Then the rejection region is

$$R = \bigcap_{\gamma \in \Gamma} \{\mathbf{x} : T(\mathbf{x}) \in R_\gamma\}$$

R_γ is the rejection region for the hypothesis testing problem

$$H_{0\gamma} : \theta \in \Theta_\gamma \quad \text{versus} \quad H_{1\gamma} : \theta \in \Theta_\gamma^c$$

An Example

Suppose that

$$H_0 : \{\mu_{\mathrm{a}} \le \mu' \text{ or } \mu_{\mathrm{t}} \ge \mu''\} \text{ versus } H_1 : \{\mu_{\mathrm{a}} > \mu' \text{ and } \mu_{\mathrm{t}} < \mu''\}$$

Because

$$R_{\mathrm{a}} = \left\{ \mathbf{x} : \frac{\bar{x}_a - \mu'}{\sigma_a / \sqrt{n}} > z_{\mathrm{a}} \right\}, z_{\mathrm{a}} \ge 0$$

and

$$R_{\mathrm{t}} = \left\{ \mathbf{x} : \frac{\bar{x}_{\mathrm{t}} - \mu''}{\sigma_{\mathrm{t}} / \sqrt{n}} < -z_{\mathrm{t}} \right\}, z_{\mathrm{t}} \ge 0,$$

the rejection region R is

$$R = R_{\mathrm{a}} \cap R_{\mathrm{t}} = \left\{ \mathbf{x} : \frac{\bar{x}_a - \mu'}{\sigma_a / \sqrt{n}} > z_{\mathrm{a}} \text{ and } \frac{\bar{x}_{\mathrm{t}} - \mu''}{\sigma_{\mathrm{t}} / \sqrt{n}} < -z_{\mathrm{t}} \right\}.$$

That is, H_1 will be accepted if and only if each criterion meets it standard.