CHAPTER 28
Martingales and Measures

Practice Questions

Problem 28.1.
How is the market price of risk defined for a variable that is not the price of an investment asset?

The market price of risk for a variable that is not the price of an investment asset is the market price of risk of an investment asset whose price is instantaneously perfectly positively correlated with the variable.

Problem 28.2.
Suppose that the market price of risk for gold is zero. If the storage costs are 1% per annum and the risk-free rate of interest is 6% per annum, what is the expected growth rate in the price of gold? Assume that gold provides no income.

If its market price of risk is zero, gold must, after storage costs have been paid, provide an expected return equal to the risk-free rate of interest. In this case, the expected return after storage costs must be 6% per annum. It follows that the expected growth rate in the price of gold must be 7% per annum.

Problem 28.3.
Consider two securities both of which are dependent on the same market variable. The expected returns from the securities are 8% and 12%. The volatility of the first security is 15%. The instantaneous risk-free rate is 4%. What is the volatility of the second security?

The market price of risk is

$$\frac{\mu - r}{\sigma}$$

This is the same for both securities. From the first security we know it must be

$$\frac{0.08 - 0.04}{0.15} = 0.26667$$

The volatility, σ for the second security is given by

$$\frac{0.12 - 0.04}{\sigma} = 0.26667$$

The volatility is 30%.

Problem 28.4.
An oil company is set up solely for the purpose of exploring for oil in a certain small area of Texas. Its value depends primarily on two stochastic variables: the price of oil and the quantity of proven oil reserves. Discuss whether the market price of risk for the second of these two variables is likely to be positive, negative, or zero.

It can be argued that the market price of risk for the second variable is zero. This is because the risk is unsystematic, i.e., it is totally unrelated to other risks in the economy. To put this another way, there is no reason why investors should demand a higher return for bearing the risk since the risk can be totally diversified away.

Problem 28.5.

Deduce the differential equation for a derivative dependent on the prices of two non-dividend-paying traded securities by forming a riskless portfolio consisting of the derivative and the two traded securities.

Suppose that the price, f, of the derivative depends on the prices, S_1 and S_2, of two traded securities. Suppose further that:

$$dS_1 = \mu_1 S_1 dt + \sigma_1 S_1 dz_1$$

$$dS_2 = \mu_2 S_2 dt + \sigma_2 S_2 dz_2$$

where dz_1 and dz_2 are Wiener processes with correlation ρ. From Ito's lemma

$$df = \left(\mu_1 S_1 \frac{\partial f}{\partial S_1} + \mu_2 S_2 \frac{\partial f}{\partial S_2} + \frac{\partial f}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 f}{\partial S_1 \partial S_2} \right) dt + \sigma_1 S_1 \frac{\partial f}{\partial S_1} dz_1 + \sigma_2 S_2 \frac{\partial f}{\partial S_2} dz_2$$

To eliminate the dz_1 and dz_2 we choose a portfolio, Π, consisting of

$$\begin{array}{ll} -1: & \text{derivative} \\ +\dfrac{\partial f}{\partial S_1}: & \text{first traded security} \\ +\dfrac{\partial f}{\partial S_2}: & \text{second traded security} \end{array}$$

$$\Pi = -f + \frac{\partial f}{\partial S_1} S_1 + \frac{\partial f}{\partial S_2} S_2$$

$$d\Pi = -df + \frac{\partial f}{\partial S_1} dS_1 + \frac{\partial f}{\partial S_2} dS_2$$

$$= -\left(\frac{\partial f}{\partial t} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 f}{\partial S_1^2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 f}{\partial S_2^2} + \rho\sigma_1\sigma_2 S_1 S_2 \frac{\partial^2 f}{\partial S_1 \partial S_2} \right) dt$$

Since the portfolio is instantaneously risk-free it must instantaneously earn the risk-free rate of interest. Hence

$$d\Pi = r\Pi dt$$

Combining the above equations

$$-\left[\frac{\partial f}{\partial t}+\frac{1}{2}\sigma_1^2 S_1^2\frac{\partial^2 f}{\partial S_1^2}+\frac{1}{2}\sigma_2^2 S_2^2\frac{\partial^2 f}{\partial S_2^2}+\rho\sigma_1\sigma_2 S_1 S_2\frac{\partial^2 f}{\partial S_1\partial S_2}\right]dt$$

$$=r\left[-f+\frac{\partial f}{\partial S_1}S_1+\frac{\partial f}{\partial S_2}S_2\right]dt$$

so that:

$$\frac{\partial f}{\partial t}+rS_1\frac{\partial f}{\partial S_1}+rS_2\frac{\partial f}{\partial S_2}+\frac{1}{2}\sigma_1^2 S_1^2\frac{\partial^2 f}{\partial S_1^2}+\frac{1}{2}\sigma_2^2 S_2^2\frac{\partial^2 f}{\partial S_2^2}+\rho\sigma_1\sigma_2 S_1 S_2\frac{\partial^2 f}{\partial S_1\partial S_2}=rf$$

Problem 28.6.

Suppose that an interest rate, x, follows the process

$$dx=a(x_0-x)\,dt+c\sqrt{x}\,dz$$

where a, x_0, and c are positive constants. Suppose further that the market price of risk for x is λ. What is the process for x in the traditional risk-neutral world

The process for x can be written

$$\frac{dx}{x}=\frac{a(x_0-x)}{x}dt+\frac{c}{\sqrt{x}}dz$$

Hence the expected growth rate in x is:

$$\frac{a(x_0-x)}{x}$$

and the volatility of x is

$$\frac{c}{\sqrt{x}}$$

In a risk neutral world the expected growth rate should be changed to

$$\frac{a(x_0-x)}{x}-\lambda\frac{c}{\sqrt{x}}$$

so that the process is

$$\frac{dx}{x}=\left[\frac{a(x_0-x)}{x}-\lambda\frac{c}{\sqrt{x}}\right]dt+\frac{c}{\sqrt{x}}dz$$

i.e.

$$dx=\left[a(x_0-x)-\lambda c\sqrt{x}\right]dt+c\sqrt{x}dz$$

Hence the drift rate should be reduced by $\lambda c\sqrt{x}$.

Problem 28.7.

Prove that when the security f provides income at rate q equation (28.9) becomes $\mu+q-r=\lambda\sigma$. (Hint: Form a new security, f^* that provides no income by assuming that all the income from f is reinvested in f.)

As suggested in the hint we form a new security f^* which is the same as f except that all income produced by f is reinvested in f. Assuming we start doing this at time zero, the relationship between f and f^* is

$$f^* = fe^{qt}$$

If μ^* and σ^* are the expected return and volatility of f^*, Ito's lemma shows that

$$\mu^* = \mu + q$$

$$\sigma^* = \sigma$$

From equation (28.9)

$$\mu^* - r = \lambda\sigma^*$$

It follows that

$$\mu + q - r = \lambda\sigma$$

Problem 28.8.

Show that when f and g provide income at rates q_f and q_g respectively, equation (28.15) becomes

$$f_0 = g_0 e^{(q_f - q_g)T} E_g\left(\frac{f_T}{g_T}\right)$$

(Hint: Form new securities f^* and g^* that provide no income by assuming that all the income from f is reinvested in f and all the income in g is reinvested in g.)

As suggested in the hint, we form two new securities f^* and g^* which are the same as f and g at time zero, but are such that income from f is reinvested in f and income from g is reinvested in g. By construction f^* and g^* are non-income producing and their values at time t are related to f and g by

$$f^* = fe^{q_f t} \qquad\qquad g^* = ge^{q_g t}$$

From Ito's lemma, the securities g and g^* have the same volatility. We can apply the analysis given in Section 28.3 to f^* and g^* so that from equation (28.15)

$$f_0^* = g_0^* E_g\left(\frac{f_T^*}{g_T^*}\right)$$

or

$$f_0 = g_0 E_g\left(\frac{f_T e^{q_f T}}{g_T e^{q_g T}}\right)$$

or

$$f_0 = g_0 e^{(q_f - q_g)T} E_g\left(\frac{f_T}{g_T}\right)$$

Problem 28.9.

"The expected future value of an interest rate in a risk-neutral world is greater than it is in the real world." What does this statement imply about the market price of risk for (a) an interest rate and (b) a bond price. Do you think the statement is likely to be true? Give reasons.

This statement implies that the interest rate has a negative market price of risk. Since bond prices and interest rates are negatively correlated, the statement implies that the market price of risk for a bond price is positive. The statement is reasonable. When interest rates increase, there is a tendency for the stock market to decrease. This implies that interest rates have negative systematic risk, or equivalently that bond prices have positive systematic risk.

Problem 28.10.

The variable S is an investment asset providing income at rate q measured in currency A. It follows the process

$$dS = \mu_S S\,dt + \sigma_S S\,dz$$

in the real world. Defining new variables as necessary, give the process followed by S, and the corresponding market price of risk, in

(a) A world that is the traditional risk-neutral world for currency A.
(b) A world that is the traditional risk-neutral world for currency B.
(c) A world that is forward risk neutral with respect to a zero-coupon currency A bond maturing at time T.
(d) A world that is forward risk neutral with respect to a zero coupon currency B bond maturing at time T.

(a) In the traditional risk-neutral world the process followed by S is

$$dS = (r - q)S\,dt + \sigma_S S\,dz$$

where r is the instantaneous risk-free rate. The market price of dz-risk is zero.

(b) In the traditional risk-neutral world for currency B the process is

$$dS = (r - q + \rho_{QS}\sigma_S\sigma_Q)S\,dt + \sigma_S S\,dz$$

where Q is the exchange rate (units of A per unit of B), σ_Q is the volatility of Q and ρ_{QS} is the coefficient of correlation between Q and S. The market price of dz-risk is $\rho_{QS}\sigma_Q$.

(c) In a world that is forward risk neutral with respect to a zero-coupon bond in currency A maturing at time T

$$dS = (r - q + \rho_{SP}\sigma_S\sigma_P)Sdt + \sigma_S Sdz$$

where σ_P is the bond price volatility and ρ_{SP} is the correlation between the stock and bond. The market price of dz-risk is $\rho_{SP}\sigma_P$.

(d) In a world that is forward risk neutral with respect to a zero-coupon bond in currency B maturing at time T

$$dS = (r - q + \rho_{SP}\sigma_S\sigma_P + \rho\sigma_S\sigma_F)Sdt + \sigma_S Sdz$$

where F is the forward exchange rate, σ_F is the volatility of F (units of A per unit of B, and ρ_{FS} is the correlation between F and S. The market price of dz-risk is $\rho_{SP}\sigma_P + \rho_{FS}\sigma_F$.

Problem 28.11.
Explain the difference between the way a forward interest rate is defined and the way the forward values of other variables such as stock prices, commodity prices, and exchange rates are defined.

The forward value of a stock price, commodity price, or exchange rate is the delivery price in a forward contract that causes the value of the forward contract to be zero. A forward bond price is calculated in this way. However, a forward interest rate is the interest rate implied by the forward bond price.

Problem 28.12.
Prove the result in Section 28.5 that when

$$df = \left[r + \sum_{i=1}^{n}\lambda_i\sigma_{f,i}\right] f\,dt + \sum_{i=1}^{n}\sigma_{f,i} f\,dz_i$$

and

$$dg = \left[r + \sum_{i=1}^{n}\lambda_i\sigma_{g,i}\right] g\,dt + \sum_{i=1}^{n}\sigma_{g,i} g\,dz_i$$

with the dz_i uncorrelated, f/g is a martingale for $\lambda_i = \sigma_{g,i}$. (Hint: Start by using equation (14A.11) to get the processes for ln f and ln g.)

$$d\ln f = \left[r + \sum_{i=1}^{n} (\lambda_i \sigma_{f,i} - \sigma_{f,i}^2 / 2) \right] dt + \sum_{i=1}^{n} \sigma_{f,i} dz_i$$
$$d\ln g = \left[r + \sum_{i=1}^{n} (\lambda_i \sigma_{g,i} - \sigma_{g,i}^2 / 2) \right] dt + \sum_{i=1}^{n} \sigma_{g,i} dz_i$$

so that

$$d\ln \frac{f}{g} = d(\ln f - \ln g) = \left[\sum_{i=1}^{n} (\lambda_i \sigma_{f,i} - \lambda_i \sigma_{g,i} - \sigma_{f,i}^2 / 2 + \sigma_{g,i}^2 / 2) \right] dt + \sum_{i=1}^{n} (\sigma_{f,i} - \sigma_{g,i}) dz_i$$

Applying Ito's lemma again

$$d\frac{f}{g} = \frac{f}{g} \left[\sum_{i=1}^{n} (\lambda_i \sigma_{f,i} - \lambda_i \sigma_{g,i} - \sigma_{f,i}^2 / 2 + \sigma_{g,i}^2 / 2) + (\sigma_{f,i} - \sigma_{g,i})^2 / 2 \right] dt + \frac{f}{g} \sum_{i=1}^{n} (\sigma_{f,i} - \sigma_{g,i}) dz_i$$

When $\lambda_i = \sigma_{g,i}$ the coefficient of dt is zero and f/g is a martingale.

Problem 28.13.

Show that when $w = h/g$ and h and g are each dependent on n Wiener processes, the i th component of the volatility of w is the i th component of the volatility of h minus the i th component of the volatility of g. Use this to prove the result that if σ_U is the volatility of U and σ_V is the volatility of V then the volatility of U/V is $\sqrt{\sigma_U^2 + \sigma_V^2 - 2\rho\sigma_U\sigma_V}$. (Hint: Start by using equation (14A.11) to get the processes for ln g and ln h.)

$$d\ln h = \ldots + \sum_{i=1}^{n} \sigma_{h,i} \; z_i$$
$$d\ln g = \ldots + \sum_{i=1}^{n} \sigma_{g,i} \; z_i$$

so that

$$d\ln \frac{h}{g} = \ldots + \sum_{i=1}^{n} (\sigma_{h,i} - \sigma_{g,i}) \; z_i$$

Applying Ito's lemma again

$$d\frac{h}{g} = \ldots + \frac{h}{g} \sum_{i=1}^{n} (\sigma_{h,i} - \sigma_{g,i}) \; z_i$$

This proves the result.

Problem 28.14.

"If X is the expected value of a variable, X follows a martingale.`` Explain this statement

If the expected value of a variable at time *t* was expected to be greater (less) than its expected value at time zero, the expected value at time zero would be wrong. It would be too low (too high). A general result is that the expected value of a variable today is the expectation of its expected value at a future time.

Further Questions

Problem 28.15.

A security's price is positively dependent on two variables: the price of copper and the yen–dollar exchange rate. Suppose that the market price of risk for these variables is 0.5 and 0.1, respectively. If the price of copper were held fixed, the volatility of the security would be 8% per annum; if the yen–dollar exchange rate were held fixed, the volatility of the security would be 12% per annum. The risk-free interest rate is 7% per annum. What is the expected rate of return from the security? If the two variables are uncorrelated with each other, what is the volatility of the security?

Suppose that S is the security price and μ is the expected return from the security. Then:

$$\frac{dS}{S} = \mu dt + \sigma_1 dz_1 + \sigma_2 dz_2$$

where dz_1 and dz_2 are Wiener processes, $\sigma_1 dz_1$ is the component of the risk in the return attributable to the price of copper and $\sigma_2 dz_2$ is the component of the risk in the return attributable to the yen-dollar exchange rate.

If the price of copper is held fixed, $dz_1 = 0$ and:

$$\frac{dS}{S} = \mu dt + \sigma_2 dz_2$$

Hence σ_2 is 8% per annum or 0.08. If the yen-dollar exchange rate is held fixed, $dz_2 = 0$ and:

$$\frac{dS}{S} = \mu dt + \sigma_1 dz_1$$

Hence σ_1 is 12% per annum or 0.12.

From equation (28.13)

$$\mu - r = \lambda_1 \sigma_1 + \lambda_2 \sigma_2$$

where λ_1 and λ_2 are the market prices of risk for copper and the yen-$ exchange rate. In this case, $r = 0.07$, $\lambda_1 = 0.5$ and $\lambda_2 = 0.1$. Therefore

$$\mu - 0.07 = 0.5 \times 0.12 + 0.1 \times 0.08$$

so that

$$\mu = 0.138$$

i.e., the expected return is 13.8% per annum.

If the two variables affecting S are uncorrelated, we can use the result that the sum of normally distributed variables is normal with variance of the sum equal to the sum of the variances. This leads to:

$$\sigma_1 dz_1 + \sigma_2 dz_2 = \sqrt{\sigma_1^2 + \sigma_2^2}\, dz_3$$

where dz_3 is a Wiener process. Hence the process for S becomes:

$$\frac{dS}{S} = \mu dt + \sqrt{\sigma_1^2 + \sigma_2^2}\, dz_3$$

If follows that the volatility of S is $\sqrt{\sigma_1^2 + \sigma_2^2}$ or 14.4% per annum.

Problem 28.16.

Suppose that the price of a zero-coupon bond maturing at time T follows the process

$$dP(t,T) = \mu_P P(t,T)\, dt + \sigma_P P(t,T)\, dz$$

and the price of a derivative dependent on the bond follows the process

$$df = \mu_f\, f\, dt + \sigma_f\, f\, dz$$

Assume only one source of uncertainty and that f provides no income.

(a) *What is the forward price, F, of f for a contract maturing at time T ?*

(b) *What is the process followed by F in a world that is forward risk neutral with respect to $P(t,T)$?*

(c) *What is the process followed by F in the traditional risk-neutral world?*

(d) *What is the process followed by f in a world that is forward risk neutral with respect to a bond maturing at time T^* where $T^* \neq T$? Assume that σ_P^* is the volatility of this bond*

(a) The no-arbitrage arguments in Chapter 5 show that

$$F(t) = \frac{f(t)}{P(t,T)}$$

(b) From Ito's lemma:

$$d \ln P = (\mu_P - \sigma_P^2 / 2)\, dt + \sigma_P\, dz$$

$$d \ln f = (\mu_f - \sigma_f^2 / 2)\, dt + \sigma_f\, dz$$

Therefore

$$d \ln \frac{f}{P} = d(\ln f - \ln P) = (\mu_f - \sigma_f^2 / 2 - \mu_P + \sigma_P^2 / 2)\, dt + (\sigma_f - \sigma_P)\, dz$$

so that

$$d\, \frac{f}{P} = (\mu_f - \mu_P + \sigma_P^2 - \sigma_f \sigma_P)\, \frac{f}{P}\, dt + (\sigma_f - \sigma_P)\, \frac{f}{P}\, dz$$

or

$$dF = (\mu_f - \mu_P + \sigma_P^2 - \sigma_f \sigma_P) F\, dt + (\sigma_f - \sigma_P) F\, dz$$

In a world that is forward risk neutral with respect to $P(t,T)$, F has zero drift. The process for F is

$$dF = (\sigma_f - \sigma_P) F\, dz$$

(c) In the traditional risk-neutral world, $\mu_f = \mu_P = r$ where r is the short-term risk-free rate and

$$dF = (\sigma_P^2 - \sigma_f \sigma_P)F\,dt + (\sigma_f - \sigma_P)F\,dz$$

Note that the answers to parts (b) and (c) are consistent with the market price of risk being zero in (c) and σ_P in (b). When the market price of risk is σ_P, $\mu_f = r + \sigma_f \sigma_P$ and $\mu_P = r + \sigma_P^2$.

(d) In a world that is forward risk-neutral with respect to a bond maturing at time T^*, $\mu_P = r + \sigma_P^* \sigma_P$ and $\mu_f = r + \sigma_P^* \sigma_f$ so that

$$dF = [\sigma_P^2 - \sigma_f \sigma_P + \sigma_P^*(\sigma_f - \sigma_P)]F\,dt + (\sigma_f - \sigma_P)F\,dz$$

or

$$dF = (\sigma_f - \sigma_P)(\sigma_P^* - \sigma_P)F\,dt + (\sigma_f - \sigma_P)F\,dz$$

Problem 28.17.

Consider a variable that is not an interest rate

(a) *In what world is the futures price of the variable a martingale*

(b) *In what world is the forward price of the variable a martingale*

(c) *Defining variables as necessary derive an expression for the difference between the drift of the futures price and the drift of the forward price in the traditional risk-neutral world*

(d) *Show that your result is consistent with the points made in Section 5.8 about the circumstances when the futures price is above the forward price.*

(a) The futures price is a martingale in the traditional risk-neutral world.

(b) The forward price for a contract maturing at time T is a martingale is a world that is forward risk neutral with respect to $P(t,T)$.

(c) Define σ_P as the volatility of $P(t,T)$ and σ_F as the volatility of the forward price. The forward rate has zero drift in a world that is forward risk neutral with respect to $P(t,T)$. When we move from the traditional world to a world that is forward risk neutral with respect to $P(t,T)$ the volatility of the numeraire ratio is σ_P and the drift increases by $\rho_{PF}\sigma_P\sigma_F$ where ρ_{PF} is the correlation between $P(t,T)$ and the forward price. It follows that the drift of the forward price in the traditional risk neutral world is $-\rho_{PF}\sigma_P\sigma_F$. The drift of the futures price is zero in the traditional risk neutral world. It follows that the excess of the drift of the futures price over the forward price is $\rho_{PF}\sigma_P\sigma_F$.

(d) P is inversely correlated with interest rates. It follows that when the correlation between interest rates and F is positive the futures price has a lower drift than the forward price. The futures and forward prices are the same at maturity. It follows that the futures price is above the forward price prior to maturity. This is consistent with Section 5.8. Similarly when the correlation between interest rates and F is negative the future price is below the forward price prior to maturity.