CHAPTER 23
Estimating Volatilities and Correlations

Practice Questions

Problem 23.1.
Explain the exponentially weighted moving average (EWMA) model for estimating volatility from historical data.

Define u_i as $(S_i - S_{i-1})/S_{i-1}$, where S_i is value of a market variable on day i. In the EWMA model, the variance rate of the market variable (i.e., the square of its volatility) calculated for day n is a weighted average of the u_{n-i}^2 's ($i = 1, 2, 3, ?$). For some constant λ ($0 < \lambda < 1$) the weight given to u_{n-i-1}^2 is λ times the weight given to u_{n-i}^2. The volatility estimated for day n, σ_n, is related to the volatility estimated for day $n-1$, σ_{n-1}, by

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2$$

This formula shows that the EWMA model has one very attractive property. To calculate the volatility estimate for day n, it is sufficient to know the volatility estimate for day $n-1$ and u_{n-1}.

Problem 23.2.
What is the difference between the exponentially weighted moving average model and the GARCH(1,1) model for updating volatilities?

The EWMA model produces a forecast of the daily variance rate for day n which is a weighted average of (i) the forecast for day $n-1$, and (ii) the square of the proportional change on day $n-1$. The GARCH (1,1) model produces a forecast of the daily variance for day n which is a weighted average of (i) the forecast for day $n-1$, (ii) the square of the proportional change on day $n-1$. and (iii) a long run average variance rate. GARCH (1,1) adapts the EWMA model by giving some weight to a long run average variance rate. Whereas the EWMA has no mean reversion, GARCH (1,1) is consistent with a mean-reverting variance rate model.

Problem 23.3.
The most recent estimate of the daily volatility of an asset is 1.5% and the price of the asset at the close of trading yesterday was $30.00. The parameter λ in the EWMA model is 0.94. Suppose that the price of the asset at the close of trading today is $30.50. How will this cause the volatility to be updated by the EWMA model?

In this case $\sigma_{n-1} = 0.015$ and $u_n = 0.5/30 = 0.01667$, so that equation (23.7) gives

$$\sigma_n^2 = 0.94 \times 0.015^2 + 0.06 \times 0.01667^2 = 0.0002281$$

The volatility estimate on day n is therefore $\sqrt{0.0002281} = 0.015103$ or 1.5103%.

Problem 23.4.

A company uses an EWMA model for forecasting volatility. It decides to change the parameter λ *from 0.95 to 0.85. Explain the likely impact on the forecasts.*

Reducing λ from 0.95 to 0.85 means that more weight is put on recent observations of u_i^2 and less weight is given to older observations. Volatilities calculated with $\lambda = 0.85$ will react more quickly to new information and will "bounce around" much more than volatilities calculated with $\lambda = 0.95$.

Problem 23.5.
The volatility of a certain market variable is 30% per annum. Calculate a 99% confidence interval for the size of the percentage daily change in the variable.

The volatility per day is $30/\sqrt{252} = 1.89\%$. There is a 99% chance that a normally distributed variable will be within 2.57 standard deviations. We are therefore 99% confident that the daily change will be less than $2.57 \times 1.89 = 4.86\%$.

Problem 23.6.
A company uses the GARCH(1,1) model for updating volatility. The three parameters are ω, α, *and* β. *Describe the impact of making a small increase in each of the parameters while keeping the others fixed.*

The weight given to the long-run average variance rate is $1-\alpha-\beta$ and the long-run average variance rate is $\omega/(1-\alpha-\beta)$. Increasing ω increases the long-run average variance rate; increasing α increases the weight given to the most recent data item, reduces the weight given to the long-run average variance rate, and increases the level of the long-run average variance rate. Increasing β increases the weight given to the previous variance estimate, reduces the weight given to the long-run average variance rate, and increases the level of the long-run average variance rate.

Problem 23.7.
The most recent estimate of the daily volatility of the US dollar–sterling exchange rate is 0.6% and the exchange rate at 4 p.m. yesterday was 1.5000. The parameter λ *in the EWMA model is 0.9. Suppose that the exchange rate at 4 p.m. today proves to be 1.4950. How would the estimate of the daily volatility be updated?*

The proportional daily change is $-0.005/1.5000 = -0.003333$. The current daily variance estimate is $0.006^2 = 0.000036$. The new daily variance estimate is

$$0.9 \times 0.000036 + 0.1 \times 0.003333^2 = 0.000033511$$

The new volatility is the square root of this. It is 0.00579 or 0.579%.

Problem 23.8.
Assume that S&P 500 at close of trading yesterday was 1,040 and the daily volatility of the index was estimated as 1% per day at that time. The parameters in a GARCH(1,1) model are $\omega = 0.000002$, $\alpha = 0.06$, *and* $\beta = 0.92$. *If the level of the index at close of trading today is 1,060, what is the new volatility estimate?*

With the usual notation $u_{n-1} = 20/1040 = 0.01923$ so that

$$\sigma_n^2 = 0.000002 + 0.06 \times 0.01923^2 + 0.92 \times 0.01^2 = 0.0001162$$

so that $\sigma_n = 0.01078$. The new volatility estimate is therefore 1.078% per day.

Problem 23.9.

Suppose that the daily volatilities of asset A and asset B calculated at the close of trading yesterday are 1.6% and 2.5%, respectively. The prices of the assets at close of trading yesterday were \$20 and \$40 and the estimate of the coefficient of correlation between the returns on the two assets was 0.25. The parameter λ used in the EWMA model is 0.95.
(a) Calculate the current estimate of the covariance between the assets.
(b) On the assumption that the prices of the assets at close of trading today are \$20.5 and \$40.5, update the correlation estimate.

(a) The volatilities and correlation imply that the current estimate of the covariance is $0.25 \times 0.016 \times 0.025 = 0.0001$.

(b) If the prices of the assets at close of trading are \$20.5 and \$40.5, the proportional changes are $0.5/20 = 0.025$ and $0.5/40 = 0.0125$. The new covariance estimate is

$$0.95 \times 0.0001 + 0.05 \times 0.025 \times 0.0125 = 0.0001106$$

The new variance estimate for asset A is

$$0.95 \times 0.016^2 + 0.05 \times 0.025^2 = 0.00027445$$

so that the new volatility is 0.0166. The new variance estimate for asset B is

$$0.95 \times 0.025^2 + 0.05 \times 0.0125^2 = 0.000601562$$

so that the new volatility is 0.0245. The new correlation estimate is

$$\frac{0.0001106}{0.0166 \times 0.0245} = 0.272$$

Problem 23.10.

The parameters of a GARCH(1,1) model are estimated as $\omega = 0.000004$, $\alpha = 0.05$, and $\beta = 0.92$. What is the long-run average volatility and what is the equation describing the way that the variance rate reverts to its long-run average? If the current volatility is 20% per year, what is the expected volatility in 20 days?

The long-run average variance rate is $\omega/(1-\alpha-\beta)$ or $0.000004/0.03 = 0.0001333$. The long-run average volatility is $\sqrt{0.0001333}$ or 1.155%. The equation describing the way the variance rate reverts to its long-run average is equation (23.13)

$$E[\sigma_{n+k}^2] = V_L + (\alpha+\beta)^k(\sigma_n^2 - V_L)$$

In this case

$$E[\sigma_{n+k}^2] = 0.0001333 + 0.97^k(\sigma_n^2 - 0.0001333)$$

If the current volatility is 20% per year, $\sigma_n = 0.2/\sqrt{252} = 0.0126$. The expected variance rate in 20 days is

$$0.0001333 + 0.97^{20}(0.0126^2 - 0.0001333) = 0.0001471$$

The expected volatility in 20 days is therefore $\sqrt{0.0001471} = 0.0121$ or 1.21% per day.

Problem 23.11.

Suppose that the current daily volatilities of asset X and asset Y are 1.0% and 1.2%, respectively. The prices of the assets at close of trading yesterday were \$30 and \$50 and the estimate of the coefficient of correlation between the returns on the two assets made at this *time was 0.50. Correlations and volatilities are updated using a GARCH(1,1) model. The estimates of the model's parameters are* $\alpha = 0.04$ *and* $\beta = 0.94$. *For the correlation* $\omega = 0.000001$ *and for the volatilities* $\omega = 0.000003$. *If the prices of the two assets at close of trading today are $31 and $51, how is the correlation estimate updated?*

Using the notation in the text $\sigma_{u,n-1} = 0.01$ and $\sigma_{v,n-1} = 0.012$ and the most recent estimate of the covariance between the asset returns is $\text{cov}_{n-1} = 0.01 \times 0.012 \times 0.50 = 0.00006$. The variable $u_{n-1} = 1/30 = 0.03333$ and the variable $v_{n-1} = 1/50 = 0.02$. The new estimate of the covariance, cov_n, is

$$0.000001 + 0.04 \times 0.03333 \times 0.02 + 0.94 \times 0.00006 = 0.0000841$$

The new estimate of the variance of the first asset, $\sigma^2_{u,n}$ is

$$0.000003 + 0.04 \times 0.03333^2 + 0.94 \times 0.01^2 = 0.0001414$$

so that $\sigma_{u,n} = \sqrt{0.0001414} = 0.01189$ or 1.189%. The new estimate of the variance of the second asset, $\sigma^2_{v,n}$ is

$$0.000003 + 0.04 \times 0.02^2 + 0.94 \times 0.012^2 = 0.0001544$$

so that $\sigma_{v,n} = \sqrt{0.0001544} = 0.01242$ or 1.242%. The new estimate of the correlation between the assets is therefore $0.0000841/(0.01189 \times 0.01242) = 0.569$.

Problem 23.12.

Suppose that the daily volatility of the FTSE 100 stock index (measured in pounds sterling) is 1.8% and the daily volatility of the dollar/sterling exchange rate is 0.9%. Suppose further that the correlation between the FTSE 100 and the dollar/sterling exchange rate is 0.4. What is the volatility of the FTSE 100 when it is translated to U.S. dollars? Assume that the dollar/sterling exchange rate is expressed as the number of U.S. dollars per pound sterling. (Hint: When $Z = XY$, *the percentage daily change in* Z *is approximately equal to the percentage daily change in X plus the percentage daily change in* Y*.)*

The FTSE expressed in dollars is XY where X is the FTSE expressed in sterling and Y is the exchange rate (value of one pound in dollars). Define x_i as the proportional change in X on day i and y_i as the proportional change in Y on day i. The proportional change in XY is approximately $x_i + y_i$. The standard deviation of x_i is 0.018 and the standard deviation of y_i is 0.009. The correlation between the two is 0.4. The variance of $x_i + y_i$ is therefore

$$0.018^2 + 0.009^2 + 2 \times 0.018 \times 0.009 \times 0.4 = 0.0005346$$

so that the volatility of $x_i + y_i$ is 0.0231 or 2.31%. This is the volatility of the FTSE expressed in dollars. Note that it is greater than the volatility of the FTSE expressed in sterling. This is the impact of the positive correlation. When the FTSE increases the value of sterling measured in dollars also tends to increase. This creates an even bigger increase in the value of FTSE measured in dollars. Similarly for a decrease in the FTSE.

Problem 23.13.

Suppose that in Problem 23.12 the correlation between the S&P 500 Index (measured in dollars) and the FT-SE 100 Index (measured in sterling) is 0.7, the correlation between the S&P 500 index (measured in dollars) and the dollar-sterling exchange rate is 0.3, and the daily volatility of the S&P 500 Index is 1.6%. What is the correlation between the S&P 500

Index (measured in dollars) and the FT-SE 100 Index when it is translated to dollars? (Hint: For three variables X, Y, and Z, the covariance between $X+Y$ and Z equals the covariance between X and Z plus the covariance between Y and Z.)

Continuing with the notation in Problem 23.12, define z_i as the proportional change in the value of the S&P 500 on day i. The covariance between x_i and z_i is $0.7\times0.018\times0.016=0.0002016$. The covariance between y_i and z_i is $0.3\times0.009\times0.016=0.0000432$. The covariance between x_i+y_i and z_i equals the covariance between x_i and z_i plus the covariance between y_i and z_i. It is

$$0.0002016+0.0000432=0.0002448$$

The correlation between x_i+y_i and z_i is

$$\frac{0.0002448}{0.016\times0.0231}=0.662$$

Note that the volatility of the S&P 500 drops out in this calculation.

Problem 23.14.

Show that the GARCH (1,1) model

$$\sigma_n^2=\omega+\alpha u_{n-1}^2+\beta\sigma_{n-1}^2$$

in equation (23.9) is equivalent to the stochastic volatility model

$$dV=a(V_L-V)\,dt+\xi V\,dz$$

where time is measured in days and V is the square of the volatility of the asset price and

$$a=1-\alpha-\beta$$

$$V_L=\frac{\omega}{1-\alpha-\beta}$$

$$\xi=\alpha\sqrt{2}$$

What is the stochastic volatility model when time is measure in years?
(Hint: The variable u_{n-1} is the return on the asset price in time Δt. It can be assumed to be normally distributed with mean zero and standard deviation σ_{n-1}. It follows from the moments of the normal distribution that the mean and variance of u_{n-1}^2 are σ_{n-1}^2 and $2\sigma_{n-1}^4$, respectively.)

$$\sigma_n^2=\omega V_L+\alpha u_{n-1}^2+\beta\sigma_{n-1}^2$$

so that

$$\sigma_n^2=(1-\alpha-\beta)V_L+\alpha u_{n-1}^2+\beta\sigma_{n-1}^2$$

$$\sigma_n^2-\sigma_{n-1}^2=(1-\alpha-\beta)(V_L-\sigma_{n-1}^2)+\alpha(u_{n-1}^2-\sigma_{n-1}^2)$$

The variable u_{n-1}^2 has a mean of σ_{n-1}^2 and a variance of

$$E(u_{n-1})^4-[E(u_{n-1}^2)]^2=2\sigma_{n-1}^4$$

The standard deviation of u_{n-1}^2 is $\sqrt{2}\sigma_{n-1}^2$.

We can write $\Delta V=\sigma_n^2-\sigma_{n-1}^2$ and $V=\sigma_{n-1}^2$. Substituting for u_{n-1}^2 into the equation for $\sigma_n^2-\sigma_{n-1}^2$ we get

$$\Delta V = a(V_L - V) + Z$$

where Z is a variable with mean zero and standard deviation $\alpha\sqrt{2}V$. This equation defines the change in the variance over one day. It is consistent with the stochastic process

$$dV = a(V_L - V)dt + \alpha\sqrt{2}Vdz$$

or

$$dV = a(V_L - V)dt + \xi Vdz$$

when time is measured in days.

Discretizing the process we obtain

$$\Delta V = a(V_L - V)\Delta t + \xi V\varepsilon\sqrt{\Delta t}$$

where ε is a random sample from a standard normal distribution.

Note that we are not assuming Z is normally distributed. It is the sum of many small changes $\xi V\varepsilon\sqrt{\Delta t}$.

When time is measured in years,

$$\Delta V = a(V_L - V)252\Delta t + \xi V\varepsilon\sqrt{252}\sqrt{\Delta t}$$

and the process for V is

$$dV = 252a(V_L - V)\,dt + \xi V\sqrt{252}\,dz$$

Problem 23.15.

At the end of Section 23.8, the VaR for the four-index example was calculated using the model-building approach. How does the VaR calculated change if the investment is \$2.5 million in each index? Carry out calculations when a) volatilities and correlations are estimated using the equally weighted model and b) when they are estimated using the EWMA model with λ=0.94. Use the spreadsheets on the author's web site.

The alphas (row 21 for equal weights and row 7 for EWMA) should be changed to 2,500. This changes the one-day 99% VaR to \$226,836 when volatilities and correlations are estimated using the equally weighted model and to \$487,737 when EWMA with $\lambda = 0.94$ is used.

Problem 23.16.

What is the effect of changing λ from 0.94 to 0.97 in the EWMA calculations in the four-index example at the end of Section 23.8? Use the spreadsheets on the author's website.

The parameter λ is in cell N3 of the EWMA worksheet. Changing it to 0.97 reduces the one-day 99% VaR from \$471,025 to \$389,290. This is because less weight is given to recent observations.

Further Questions

Problem 23.17.

Suppose that the price of gold at close of trading yesterday was $600 and its volatility was estimated as 1.3% per day. The price at the close of trading today is $596. Update the volatility estimate using

(a) *The EWMA model with* $\lambda = 0.94$

(b) *The GARCH(1,1) model with* $\omega = 0.000002$*,* $\alpha = 0.04$*, and* $\beta = 0.94$.

The proportional change in the price of gold is $-4/600 = -0.00667$. Using the EWMA model the variance is updated to

$$0.94 \times 0.013^2 + 0.06 \times 0.00667^2 = 0.00016153$$

so that the new daily volatility is $\sqrt{0.00016153} = 0.01271$ or 1.271% per day. Using GARCH (1,1) the variance is updated to

$$0.000002 + 0.94 \times 0.013^2 + 0.04 \times 0.00667^2 = 0.00016264$$

so that the new daily volatility is $\sqrt{0.00016264} = 0.1275$ or 1.275% per day.

Problem 23.18.

Suppose that in Problem 23.17 the price of silver at the close of trading yesterday was $16, its volatility was estimated as 1.5% per day, and its correlation with gold was estimated as 0.8. The price of silver at the close of trading today is unchanged at $16. Update the volatility of silver and the correlation between silver and gold using the two models in Problem 23.17. In practice, is the ω *parameter likely to be the same for gold and silver?*

The proportional change in the price of silver is zero. Using the EWMA model the variance is updated to

$$0.94 \times 0.015^2 + 0.06 \times 0 = 0.0002115$$

so that the new daily volatility is $\sqrt{0.0002115} = 0.01454$ or 1.454% per day. Using GARCH (1,1) the variance is updated to

$$0.000002 + 0.94 \times 0.015^2 + 0.04 \times 0 = 0.0002135$$

so that the new daily volatility is $\sqrt{0.0002135} = 0.01461$ or 1.461% per day. The initial covariance is $0.8 \times 0.013 \times 0.015 = 0.000156$ Using EWMA the covariance is updated to

$$0.94 \times 0.000156 + 0.06 \times 0 = 0.00014664$$

so that the new correlation is $0.00014664/(0.01454 \times 0.01271) = 0.7934$ Using GARCH (1,1) the covariance is updated to

$$0.000002 + 0.94 \times 0.000156 + 0.04 \times 0 = 0.00014864$$

so that the new correlation is $0.00014864/(0.01461 \times 0.01275) = 0.7977$.

For a given α and β, the ω parameter defines the long run average value of a variance or a covariance. There is no reason why we should expect the long run average daily variance for gold and silver should be the same. There is also no reason why we should expect the long run average covariance between gold and silver to be the same as the long run average variance of gold or the long run average variance of silver. In practice, therefore, we are likely to want to allow ω in a GARCH(1,1) model to vary from market variable to market variable. (Some instructors may want to use this problem as a lead in to multivariate GARCH models.)

Problem 23.19. (Excel file)

An Excel spreadsheet containing over 900 days of daily data on a number of different exchange rates and stock indices can be downloaded from the author's website: http://www.rotman.utoronto.ca/~hull/data.

Choose one exchange rate and one stock index. Estimate the value of λ in the EWMA model that minimizes the value of

$$\sum_i (v_i - \beta_i)^2$$

where v_i is the variance forecast made at the end of day $i-1$ and β_i is the variance calculated from data between day i and day $i+25$. Use the Solver tool in Excel. Set the variance forecast at the end of the first day equal to the square of the return on that day to start the EWMA calculations.

In the spreadsheet the first 25 observations on $(v_i - \beta_i)^2$ are ignored so that the results are not unduly influenced by the choice of starting values. The best values of λ for EUR, CAD, GBP and JPY were found to be 0.947, 0.898, 0.950, and 0.984, respectively. The best values of λ for S&P500, NASDAQ, FTSE100, and Nikkei225 were found to be 0.874, 0.901, 0.904, and 0.953, respectively.

Problem 23.20.

Suppose that the parameters in a GARCH (1,1) model are $\alpha = 0.03$, $\beta = 0.95$ and $\omega = 0.000002$.

(a) *What is the long-run average volatility?*
(b) *If the current volatility is 1.5% per day, what is your estimate of the volatility in 20, 40, and 60 days?*
(c) *What volatility should be used to price 20-, 40-, and 60-day options?*
(d) *Suppose that there is an event that increases the current volatility by 0.5% to 2% per day. Estimate the effect on the volatility in 20, 40, and 60 days.*
(e) *Estimate by how much does the event increase the volatilities used to price 20-, 40-, and 60-day options?*

(a) The long-run average variance, V_L, is

$$\frac{\omega}{1-\alpha-\beta} = \frac{0.000002}{0.02} = 0.0001$$

The long run average volatility is $\sqrt{0.0001} = 0.01$ or 1% per day

(b) From equation (23.13) the expected variance in 20 days is

$$0.0001 + 0.98^{20}(0.015^2 - 0.0001) = 0.000183$$

The expected volatility per day is therefore $\sqrt{0.000183} = 0.0135$ or 1.35%. Similarly the expected volatilities in 40 and 60 days are 1.25% and 1.17%, respectively.

(c) In equation (23.14) $a = \ln(1/0.98) = 0.0202$ the variance used to price options in 20 days is

$$252\left(0.0001 + \frac{1-e^{-0.0202\times 2}}{0.0202\times 20}\left[0.015^2 - 0.0001\right]\right) = 0.051$$

so that the volatility per annum is 22.61%. Similarly, the volatilities that should be used for 40- and 60-day options are 22.63% and 20.85% per annum, respectively.

(d) From equation (23.13) the expected variance in 20 days is

$$0.0001 + 0.98^{20}(0.02^2 - 0.0001) = 0.0003$$

The expected volatility per day is therefore $\sqrt{0.0003} = 0.0173$ or 1.73%. Similarly the expected volatilities in 40 and 60 days are 1.53% and 1.38% per day, respectively.

(e) When today's volatility increases from 1.5% per day (23.81% per year) to 2% per day (31.75% per year) the equation (23.15) gives the 20-day volatility increase as

$$\frac{1-e^{-0.0202\times 20}}{0.0202\times 20}\times\frac{23.81}{22.61}\times(31.75-23.81) = 6.88$$

or 6.88% bringing the volatility up to 29.49%. Similarly the 40- and 60-day volatilities increase to 27.37% and 25.70%. A more exact calculation using equation (23.14) gives 29.56%, 27.76%, and 26.27% as the three volatilities.

Problem 23.21. (Excel file)

The calculations for the four-index example at the end of Section 23.8 assume that the investments in the DJIA, FTSE 100, CAC40, and Nikkei 225 are $4 million, $3 million, \$1 million, and $2 million, respectively. How does the VaR calculated change if the investment are $3 million, $3 million, $1 million, and $3 million, respectively? Carry out calculations when a) volatilities and correlations are estimated using the equally weighted model and b) when they are estimated using the EWMA model. What is the effect of changing λ from 0.94 to 0.90 in the EWMA calculations? Use the spreadsheets on the author's website.

(a) The portfolio investment amounts have to be changed in row 21 of the Equal Weights worksheet for the model building approach. The worksheet shows that the new one-day 99% VaR is $215,007.

(b) The portfolio investment amounts have to be changed in row 7 of the EWMA worksheet. The worksheet shows that one-day 99% VaR is $447,404. Changing λ to 0.90 (see cell N3) changes this VaR to $500,403. This is higher because recent returns are given more weight.

Problem 23.22. (Excel file)

Estimate parameters for EWMA and GARCH (1,1) from data on the euro-USD exchange rate between July 27, 2005 and July 27, 2010. This data can be found on the author's web site www.rotman.utoronto.ca/~hull/data

As the spreadsheets show the optimal value of λ in the EWMA model is 0.958 and the log likelihood objective function is 11,806.4767. In the GARCH (1,1) model, the optimal values of ω, α, and β are 0.0000001330, 0.04447, and0.95343, respectively. The long-run average daily volatility is 0.7954% and the log likelihood objective function is 11,811.1955.