CHAPTER 19
The Greek Letters

Practice Questions

Problem 19.1.
Explain how a stop-loss trading rule can be implemented for the writer of an out-of-the-money call option. Why does it provide a relatively poor hedge?

Suppose the strike price is 10.00. The option writer aims to be fully covered whenever the option is in the money and naked whenever it is out of the money. The option writer attempts to achieve this by buying the assets underlying the option as soon as the asset price reaches 10.00 from below and selling as soon as the asset price reaches 10.00 from above. The trouble with this scheme is that it assumes that when the asset price moves from 9.99 to 10.00, the next move will be to a price above 10.00. (In practice the next move might back to 9.99.) Similarly it assumes that when the asset price moves from 10.01 to 10.00, the next move will be to a price below 10.00. (In practice the next move might be back to 10.01.) The scheme can be implemented by buying at 10.01 and selling at 9.99. However, it is not a good hedge. The cost of the trading strategy is zero if the asset price never reaches 10.00 and can be quite high if it reaches 10.00 many times. A good hedge has the property that its cost is always very close the value of the option.

Problem 19.2.
What does it mean to assert that the delta of a call option is 0.7? How can a short position in 1,000 options be made delta neutral when the delta of each option is 0.7?

A delta of 0.7 means that, when the price of the stock increases by a small amount, the price of the option increases by 70% of this amount. Similarly, when the price of the stock decreases by a small amount, the price of the option decreases by 70% of this amount. A short position in 1,000 options has a delta of -700 and can be made delta neutral with the purchase of 700 shares.

Problem 19.3.
Calculate the delta of an at-the-money six-month European call option on a non-dividend-paying stock when the risk-free interest rate is 10% per annum and the stock price volatility is 25% per annum.

In this case, $S_0 = K$, $r = 0.1$, $\sigma = 0.25$, and $T = 0.5$. Also,

$$d_1 = \frac{\ln(S_0 / K) + (0.1 + 0.25^2 / 2)0.5}{0.25\sqrt{0.5}} = 0.3712$$

The delta of the option is $N(d_1)$ or 0.64.

Problem 19.4.
What does it mean to assert that the theta of an option position is −0.1 when time is measured in years? If a trader feels that neither a stock price nor its implied volatility will change, what type of option position is appropriate?

A theta of -0.1 means that if Δt units of time pass with no change in either the stock price or its volatility, the value of the option declines by $0.1\Delta t$. A trader who feels that neither the stock price nor its implied volatility will change should write an option with as high a negative theta as possible. Relatively short-life at-the-money options have the most negative thetas.

Problem 19.5.
What is meant by the gamma of an option position? What are the risks in the situation where the gamma of a position is large and negative and the delta is zero?

The gamma of an option position is the rate of change of the delta of the position with respect to the asset price. For example, a gamma of 0.1 would indicate that when the asset price increases by a certain small amount delta increases by 0.1 of this amount. When the gamma of an option writer's position is large and negative and the delta is zero, the option writer will lose significant amounts of money if there is a large movement (either an increase or a decrease) in the asset price.

Problem 19.6.
"The procedure for creating an option position synthetically is the reverse of the procedure for hedging the option position." Explain this statement.

To hedge an option position it is necessary to create the opposite option position synthetically. For example, to hedge a long position in a put it is necessary to create a short position in a put synthetically. It follows that the procedure for creating an option position synthetically is the reverse of the procedure for hedging the option position**.**

Problem 19.7.
Why did portfolio insurance not work well on October 19, 1987?

Portfolio insurance involves creating a put option synthetically. It assumes that as soon as a portfolio's value declines by a small amount the portfolio manager's position is rebalanced by either (a) selling part of the portfolio, or (b) selling index futures. On October 19, 1987, the market declined so quickly that the sort of rebalancing anticipated in portfolio insurance schemes could not be accomplished.

Problem 19.8.
The Black-Scholes-Merton price of an out-of-the-money call option with an exercise price of \$40 is \$4. A trader who has written the option plans to use a stop-loss strategy. The trader's plan is to buy at \$40.10 and to sell at \$39.90. Estimate the expected number of times the stock will be bought or sold.

The strategy costs the trader 0.10 each time the stock is bought or sold. The total expected cost of the strategy, in present value terms, must be \$4. This means that the expected number of times the stock will be bought or sold is approximately 40. The expected number of times it will be bought is approximately 20 and the expected number of times it will be sold is also approximately 20. The buy and sell transactions can take place at any time during the life of the option. The above numbers are therefore only approximately correct because of the effects of discounting. Also the estimate is of the number of times the stock is bought or sold in the risk-neutral world, not the real world.

Problem 19.9.

Suppose that a stock price is currently \$20 and that a call option with an exercise price of \$25 is created synthetically using a continually changing position in the stock. Consider the following two scenarios:

a) *Stock price increases steadily from \$20 to \$35 during the life of the option.*

b) *Stock price oscillates wildly, ending up at \$35.*

Which scenario would make the synthetically created option more expensive? Explain your answer.

The holding of the stock at any given time must be $N(d_1)$. Hence the stock is bought just after the price has risen and sold just after the price has fallen. (This is the buy high sell low strategy referred to in the text.) In the first scenario the stock is continually bought. In second scenario the stock is bought, sold, bought again, sold again, etc. The final holding is the same in both scenarios. The buy, sell, buy, sell... situation clearly leads to higher costs than the buy, buy, buy... situation. This problem emphasizes one disadvantage of creating options synthetically. Whereas the cost of an option that is purchased is known up front and depends on the forecasted volatility, the cost of an option that is created synthetically is not known up front and depends on the volatility actually encountered.

Problem 19.10.

What is the delta of a short position in 1,000 European call options on silver futures? The options mature in eight months, and the futures contract underlying the option matures in nine months. The current nine-month futures price is \$8 per ounce, the exercise price of the options is \$8, the risk-free interest rate is 12% per annum, and the volatility of silver futures prices is 18% per annum.

The delta of a European futures call option is usually defined as the rate of change of the option price with respect to the futures price (not the spot price). It is

$$e^{-rT}N(d_1)$$

In this case $F_0 = 8$, $K = 8$, $r = 0.12$, $\sigma = 0.18$, $T = 0.6667$

$$d_1 = \frac{\ln(8/8) + (0.18^2/2)\times 0.6667}{0.18\sqrt{0.6667}} = 0.0735$$

$N(d_1) = 0.5293$ and the delta of the option is

$$e^{-0.12\times 0.6667}\times 0.5293 = 0.4886$$

The delta of a short position in 1,000 futures options is therefore -488.6.

Problem 19.11.

In Problem 19.10, what initial position in nine-month silver futures is necessary for delta hedging? If silver itself is used, what is the initial position? If one-year silver futures are used, what is the initial position? Assume no storage costs for silver.

In order to answer this problem it is important to distinguish between the rate of change of the option with respect to the futures price and the rate of change of its price with respect to the spot price.

The former will be referred to as the futures delta; the latter will be referred to as the spot delta. The futures delta of a nine-month futures contract to buy one ounce of silver is by definition 1.0. Hence, from the answer to Problem 19.10, a long position in nine-month futures on 488.6 ounces is necessary to hedge the option position.

The spot delta of a nine-month futures contract is $e^{0.12\times 0.75} = 1.094$ assuming no storage costs. (This is because silver can be treated in the same way as a non-dividend-paying stock when there are no storage costs. $F_0 = S_0 e^{rT}$ so that the spot delta is the futures delta times e^{rT}) Hence the spot delta of the option position is $-488.6 \times 1.094 = -534.6$. Thus a long position in 534.6 ounces of silver is necessary to hedge the option position.
The spot delta of a one-year silver futures contract to buy one ounce of silver is $e^{0.12} = 1.1275$. Hence a long position in $e^{-0.12} \times 534.6 = 474.1$ ounces of one-year silver futures is necessary to hedge the option position.

Problem 19.12.
A company uses delta hedging to hedge a portfolio of long positions in put and call options on a currency. Which of the following would give the most favorable result?
a) *A virtually constant spot rate*
b) *Wild movements in the spot rate*
Explain your answer.

A long position in either a put or a call option has a positive gamma. From Figure 19.8, when gamma is positive the hedger gains from a large change in the stock price and loses from a small change in the stock price. Hence the hedger will fare better in case (b).

Problem 19.13.
Repeat Problem 19.12 for a financial institution with a portfolio of short positions in put and call options on a currency.

A short position in either a put or a call option has a negative gamma. From Figure 19.8, when gamma is negative the hedger gains from a small change in the stock price and loses from a large change in the stock price. Hence the hedger will fare better in case (a).

Problem 19.14.
A financial institution has just sold 1,000 seven-month European call options on the Japanese yen. Suppose that the spot exchange rate is 0.80 cent per yen, the exercise price is 0.81 cent per yen, the risk-free interest rate in the United States is 8% per annum, the risk-free interest rate in Japan is 5% per annum, and the volatility of the yen is 15% per annum. Calculate the delta, gamma, vega, theta, and rho of the financial institution's position. Interpret each number.

In this case $S_0 = 0.80$, $K = 0.81$, $r = 0.08$, $r_f = 0.05$, $\sigma = 0.15$, $T = 0.5833$

$$d_1 = \frac{\ln(0.80/0.81) + \left(0.08 - 0.05 + 0.15^2/2\right) \times 0.5833}{0.15\sqrt{0.5833}} = 0.1016$$

$$d_2 = d_1 - 0.15\sqrt{0.5833} = -0.0130$$

$$N(d_1) = 0.5405; \quad N(d_2) = 0.4998$$

The delta of one call option is $e^{-r_f T} N(d_1) = e^{-0.05\times 0.5833} \times 0.5405 = 0.5250$.

$$N'(d_1) = \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} = \frac{1}{\sqrt{2\pi}} e^{-0.00516} = 0.3969$$

so that the gamma of one call option is

$$\frac{N'(d_1)e^{-r_fT}}{S_0\sigma\sqrt{T}} = \frac{0.3969\times 0.9713}{0.80\times 0.15\times\sqrt{0.5833}} = 4.206$$

The vega of one call option is

$$S_0\sqrt{T}N'(d_1)e^{-r_fT} = 0.80\sqrt{0.5833}\times 0.3969\times 0.9713 = 0.2355$$

The theta of one call option is

$$-\frac{S_0N'(d_1)\sigma e^{-r_fT}}{2\sqrt{T}} + r_fS_0N(d_1)e^{-r_fT} - rKe^{-rT}N(d_2)$$

$$= -\frac{0.8\times 0.3969\times 0.15\times 0.9713}{2\sqrt{0.5833}}$$

$$+0.05\times 0.8\times 0.5405\times 0.9713 - 0.08\times 0.81\times 0.9544\times 0.4948$$

$$= -0.0399$$

The rho of one call option is

$$KTe^{-rT}N(d_2)$$

$$= 0.81\times 0.5833\times 0.9544\times 0.4948$$

$$= 0.2231$$

Delta can be interpreted as meaning that, when the spot price increases by a small amount (measured in cents), the value of an option to buy one yen increases by 0.525 times that amount. Gamma can be interpreted as meaning that, when the spot price increases by a small amount (measured in cents), the delta increases by 4.206 times that amount. Vega can be interpreted as meaning that, when the volatility (measured in decimal form) increases by a small amount, the option's value increases by 0.2355 times that amount. When volatility increases by 1% (= 0.01) the option price increases by 0.002355. Theta can be interpreted as meaning that, when a small amount of time (measured in years) passes, the option's value decreases by 0.0399 times that amount. In particular when one calendar day passes it decreases by $0.0399/365 = 0.000109$. Finally, rho can be interpreted as meaning that, when the interest rate (measured in decimal form) increases by a small amount the option's value increases by 0.2231 times that amount. When the interest rate increases by 1% (= 0.01), the options value increases by 0.002231.

Problem 19.15.

Under what circumstances is it possible to make a European option on a stock index both gamma neutral and vega neutral by adding a position in one other European option?

Assume that S_0, K, r, σ, T, q are the parameters for the option held and S_0, K^*, r, σ, T^*, q are the parameters for another option. Suppose that d_1 has its usual meaning and is calculated on the basis of the first set of parameters while d_1^* is the value of d_1 calculated on the basis of the second set of parameters. Suppose further that w of the second option are held for each of the first option held. The gamma of the portfolio is:

$$\alpha\left[\frac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}} + w\frac{N'(d_1^*)e^{-qT^*}}{S_0\sigma\sqrt{T^*}}\right]$$

where α is the number of the first option held.

Since we require gamma to be zero:

$$w = -\frac{N'(d_1)e^{-q(T-T^*)}}{N'(d_1^*)}\sqrt{\frac{T^*}{T}}$$

The vega of the portfolio is:

$$\alpha\left[S_0\sqrt{T}N'(d_1)e^{-q(T)} + wS_0\sqrt{T^*}N'(d_1^*)e^{-q(T^*)}\right]$$

Since we require vega to be zero:

$$w = -\sqrt{\frac{T}{T^*}}\frac{N'(d_1)e^{-q(T-T^*)}}{N'(d_1^*)}$$

Equating the two expressions for w

$$T^* = T$$

Hence the maturity of the option held must equal the maturity of the option used for hedging.

Problem 19.16.

A fund manager has a well-diversified portfolio that mirrors the performance of the S&P 500 and is worth $360 million. The value of the S&P 500 is 1,200, and the portfolio manager would like to buy insurance against a reduction of more than 5% in the value of the portfolio over the next six months. The risk-free interest rate is 6% per annum. The dividend yield on both the portfolio and the S&P 500 is 3%, and the volatility of the index is 30% per annum.

a) *If the fund manager buys traded European put options, how much would the insurance cost?*
b) *Explain carefully alternative strategies open to the fund manager involving traded European call options, and show that they lead to the same result.*
c) *If the fund manager decides to provide insurance by keeping part of the portfolio in risk-free securities, what should the initial position be?*
d) *If the fund manager decides to provide insurance by using nine-month index futures, what should the initial position be?*

The fund is worth $300,000 times the value of the index. When the value of the portfolio falls by 5% (to $342 million), the value of the S&P 500 also falls by 5% to 1140. The fund manager therefore requires European put options on 300,000 times the S&P 500 with exercise price 1140.

a) $S_0 = 1200$, $K = 1140$, $r = 0.06$, $\sigma = 0.30$, $T = 0.50$ and $q = 0.03$. Hence:

$$d_1 = \frac{\ln(1200/1140) + \left(0.06 - 0.03 + 0.3^2/2\right)\times 0.5}{0.3\sqrt{0.5}} = 0.4186$$

$$d_2 = d_1 - 0.3\sqrt{0.5} = 0.2064$$

$$N(d_1) = 0.6622;\quad N(d_2) = 0.5818$$

$$N(-d_1) = 0.3378;\quad N(-d_2) = 0.4182$$

The value of one put option is

$$\begin{aligned}&1140e^{-rT}N(-d_2) - 1200e^{-qT}N(-d_1)\\ &= 1140e^{-0.06\times 0.5}\times 0.4182 - 1200e^{-0.03\times 0.5}\times 0.3378\\ &= 63.40\end{aligned}$$

The total cost of the insurance is therefore

$$300,000 \times 63.40 = \$19,020,000$$

b) From put–call parity

$$S_0 e^{-qT} + p = c + Ke^{-rT}$$

or:

$$p = c - S_0 e^{-qT} + Ke^{-rT}$$

This shows that a put option can be created by selling (or shorting) e^{-qT} of the index, buying a call option and investing the remainder at the risk-free rate of interest. Applying this to the situation under consideration, the fund manager should:

1. Sell $360e^{-0.03\times 0.5} = \$354.64$ million of stock
2. Buy call options on 300,000 times the S&P 500 with exercise price 1140 and maturity in six months.
3. Invest the remaining cash at the risk-free interest rate of 6% per annum.

This strategy gives the same result as buying put options directly.

c) The delta of one put option is

$$\begin{aligned} &e^{-qT}[N(d_1)-1] \\ &= e^{-0.03\times 0.5}(0.6622-1) \\ &-0.3327 \end{aligned}$$

This indicates that 33.27% of the portfolio (i.e., \$119.77 million) should be initially sold and invested in risk-free securities.

d) The delta of a nine-month index futures contract is

$$e^{(r-q)T} = e^{0.03\times 0.75} = 1.023$$

The spot short position required is

$$\frac{119,770,000}{1200} = 99,808$$

times the index. Hence a short position in

$$\frac{99,808}{1.023\times 250} = 390$$

futures contracts is required.

Problem 19.17.

Repeat Problem 19.16 on the assumption that the portfolio has a beta of 1.5. Assume that the dividend yield on the portfolio is 4% per annum.

When the value of the portfolio goes down 5% in six months, the total return from the portfolio, including dividends, in the six months is

$$-5+2=-3\%$$

i.e., -6% per annum. This is 12% per annum less than the risk-free interest rate. Since the portfolio has a beta of 1.5 we would expect the market to provide a return of 8% per annum less than the risk-free interest rate, i.e., we would expect the market to provide a return of -2% per annum. Since dividends on the market index are 3% per annum, we would expect the market index to have dropped at the rate of 5% per annum or 2.5% per six months; i.e., we would expect the market to have dropped to 1170. A total of $450,000 = (1.5 \times 300,000)$ put options on the S&P 500 with exercise price 1170 and exercise date in six months are therefore required.

a) $S_0 = 1200$, $K = 1170$, $r = 0.06$, $\sigma = 0.3$, $T = 0.5$ and $q = 0.03$. Hence

$$d_1 = \frac{\ln(1200/1170) + (0.06 - 0.03 + 0.09/2) \times 0.5}{0.3\sqrt{0.5}} = 0.2961$$

$$d_2 = d_1 - 0.3\sqrt{0.5} = 0.0840$$

$$N(d_1) = 0.6164; \quad N(d_2) = 0.5335$$

$$N(-d_1) = 0.3836; \quad N(-d_2) = 0.4665$$

The value of one put option is

$$\begin{aligned} & Ke^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1) \\ &= 1170e^{-0.06\times0.5} \times 0.4665 - 1200e^{-0.03\times0.5} \times 0.3836 \\ &= 76.28 \end{aligned}$$

The total cost of the insurance is therefore

$$450,000 \times 76.28 = \$34,326,000$$

Note that this is significantly greater than the cost of the insurance in Problem 19.16.

b) As in Problem 19.16 the fund manager can 1) sell $354.64 million of stock, 2) buy call options on 450,000 times the S&P 500 with exercise price 1170 and exercise date in six months and 3) invest the remaining cash at the risk-free interest rate.

c) The portfolio is 50% more volatile than the S&P 500. When the insurance is considered as an option on the portfolio the parameters are as follows: $S_0 = 360$, $K = 342$, $r = 0.06$, $\sigma = 0.45$, $T = 0.5$ and $q = 0.04$

$$d_1 = \frac{\ln(360/342) + (0.06 - 0.04 + 0.45^2/2) \times 0.5}{0.45\sqrt{0.5}} = 0.3517$$

$$N(d_1) = 0.6374$$

The delta of the option is

$$\begin{aligned} & e^{-qT}[N(d_1) - 1] \\ &= e^{-0.04\times0.5}(0.6374 - 1) \\ &= -0.355 \end{aligned}$$

This indicates that 35.5% of the portfolio (i.e., $127.8 million) should be sold and invested in riskless securities.

d) We now return to the situation considered in (a) where put options on the index are required. The delta of each put option is

$$e^{-qT}(N(d_1)-1)$$
$$= e^{-0.03\times0.5}(0.6164-1)$$
$$= -0.3779$$

The delta of the total position required in put options is $-450,000\times0.3779 = -170,000$. The delta of a nine month index futures is (see Problem 19.16) 1.023. Hence a short position in

$$\frac{170,000}{1.023\times250} = 665$$

index futures contracts.

Problem 19.18.

Show by substituting for the various terms in equation (19.4) that the equation is true for:

a) *A single European call option on a non-dividend-paying stock*
b) *A single European put option on a non-dividend-paying stock*
c) *Any portfolio of European put and call options on a non-dividend-paying stock*

a) For a call option on a non-dividend-paying stock

$$\Delta = N(d_1)$$
$$\Gamma = \frac{N'(d_1)}{S_0\sigma\sqrt{T}}$$
$$\Theta = -\frac{S_0N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2)$$

Hence the left-hand side of equation (19.4) is:

$$= -\frac{S_0N'(d_1)\sigma}{2\sqrt{T}} - rKe^{-rT}N(d_2) + rS_0N(d_1) + \frac{1}{2}\sigma S_0\frac{N'(d_1)}{\sqrt{T}}$$
$$= r[S_0N(d_1) - Ke^{-rT}N(d_2)]$$
$$= r\Pi$$

b) For a put option on a non-dividend-paying stock

$$\Delta = N(d_1) - 1 = -N(-d_1)$$
$$\Gamma = \frac{N'(d_1)}{S_0\sigma\sqrt{T}}$$
$$\Theta = -\frac{S_0N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2)$$

Hence the left-hand side of equation (19.4) is:

$$-\frac{S_0N'(d_1)\sigma}{2\sqrt{T}} + rKe^{-rT}N(-d_2) - rS_0N(-d_1) + \frac{1}{2}\sigma S_0\frac{N'(d_1)}{\sqrt{T}}$$
$$= r[Ke^{-rT}N(-d_2) - S_0N(-d_1)]$$
$$= r\Pi$$

c) For a portfolio of options, Π, Δ, Θ and Γ are the sums of their values for the individual options in the portfolio. It follows that equation (19.4) is true for any portfolio of European put and call options.

Problem 19.19.
What is the equation corresponding to equation (19.4) for (a) a portfolio of derivatives on a currency and (b) a portfolio of derivatives on a futures contract?

A currency is analogous to a stock paying a continuous dividend yield at rate r_f. The differential equation for a portfolio of derivatives dependent on a currency is (see equation 17.6)

$$\frac{\partial \Pi}{\partial t} + (r - r_f) S \frac{\partial \Pi}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \Pi}{\partial S^2} = r\Pi$$

Hence

$$\Theta + (r - r_f) S\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

Similarly, for a portfolio of derivatives dependent on a futures price (see equation 18.8)

$$\Theta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

Problem 19.20.
Suppose that $70 billion of equity assets are the subject of portfolio insurance schemes. Assume that the schemes are designed to provide insurance against the value of the assets declining by more than 5% within one year. Making whatever estimates you find necessary, use the DerivaGem software to calculate the value of the stock or futures contracts that the administrators of the portfolio insurance schemes will attempt to sell if the market falls by 23% in a single day.

We can regard the position of all portfolio insurers taken together as a single put option. The three known parameters of the option, before the 23% decline, are $S_0 = 70$, $K = 66.5$, $T = 1$. Other parameters can be estimated as $r = 0.06$, $\sigma = 0.25$ and $q = 0.03$. Then:

$$d_1 = \frac{\ln(70/66.5) + (0.06 - 0.03 + 0.25^2/2)}{0.25} = 0.4502$$

$$N(d_1) = 0.6737$$

The delta of the option is

$$\begin{aligned} & e^{-qT}[N(d_1) - 1] \\ &= e^{-0.03}(0.6737 - 1) \\ &= -0.3167 \end{aligned}$$

This shows that 31.67% or $22.17 billion of assets should have been sold before the decline. These numbers can also be produced from DerivaGem by selecting Underlying Type and Index and Option Type as Black-Scholes European.
After the decline, $S_0 = 53.9$, $K = 66.5$, $T = 1$, $r = 0.06$, $\sigma = 0.25$ and $q = 0.03$.

$$d_1 = \frac{\ln(53.9/66.5) + (0.06 - 0.03 + 0.25^2/2)}{0.25} = -0.5953$$

$$N(d_1) = 0.2758$$

The delta of the option has dropped to

$$e^{-0.03\times0.5}(0.2758-1)$$
$$=-0.7028$$

This shows that cumulatively 70.28% of the assets originally held should be sold. An additional 38.61% of the original portfolio should be sold. The sales measured at pre-crash prices are about \$27.0 billion. At post-crash prices they are about \$20.8 billion.

Problem 19.21.

Does a forward contract on a stock index have the same delta as the corresponding futures contract? Explain your answer.

With our usual notation the value of a forward contract on the asset is $S_0e^{-qT}-Ke^{-rT}$. When there is a small change, ΔS, in S_0 the value of the forward contract changes by $e^{-qT}\Delta S$. The delta of the forward contract is therefore e^{-qT}. The futures price is $S_0e^{(r-q)T}$. When there is a small change, ΔS, in S_0 the futures price changes by $\Delta Se^{(r-q)T}$. Given the daily settlement procedures in futures contracts, this is also the immediate change in the wealth of the holder of the futures contract. The delta of the futures contract is therefore $e^{(r-q)T}$. We conclude that the deltas of a futures and forward contract are not the same. The delta of the futures is greater than the delta of the corresponding forward by a factor of e^{rT}.

Problem 19.22.

A bank's position in options on the dollar–euro exchange rate has a delta of 30,000 and a gamma of $-80,000$. Explain how these numbers can be interpreted. The exchange rate (dollars per euro) is 0.90. What position would you take to make the position delta neutral? After a short period of time, the exchange rate moves to 0.93. Estimate the new delta. What additional trade is necessary to keep the position delta neutral? Assuming the bank did set up a delta-neutral position originally, has it gained or lost money from the exchange-rate movement?

The delta indicates that when the value of the euro exchange rate increases by \$0.01, the value of the bank's position increases by $0.01\times30,000=\$300$. The gamma indicates that when the euro exchange rate increases by \$0.01 the delta of the portfolio decreases by $0.01\times80,000=800$. For delta neutrality 30,000 euros should be shorted. When the exchange rate moves up to 0.93, we expect the delta of the portfolio to decrease by $(0.93-0.90)\times80,000=2,400$ so that it becomes 27,600. To maintain delta neutrality, it is therefore necessary for the bank to unwind its short position 2,400 euros so that a net 27,600 have been shorted. As shown in the text (see Figure 19.8), when a portfolio is delta neutral and has a negative gamma, a loss is experienced when there is a large movement in the underlying asset price. We can conclude that the bank is likely to have lost money.

Problem 19.23.

Use the put–call parity relationship to derive, for a non-dividend-paying stock, the relationship between:

(a) *The delta of a European call and the delta of a European put.*
(b) *The gamma of a European call and the gamma of a European put.*
(c) *The vega of a European call and the vega of a European put.*
(d) *The theta of a European call and the theta of a European put.*

(a) For a non-dividend paying stock, put-call parity gives at a general time t:

$$p + S = c + Ke^{-r(T-t)}$$

Differentiating with respect to S:

$$\frac{\partial p}{\partial S} + 1 = \frac{\partial c}{\partial S}$$

or

$$\frac{\partial p}{\partial S} = \frac{\partial c}{\partial S} - 1$$

This shows that the delta of a European put equals the delta of the corresponding European call less 1.0.

(b) Differentiating with respect to S again

$$\frac{\partial^2 p}{\partial S^2} = \frac{\partial^2 c}{\partial S^2}$$

Hence the gamma of a European put equals the gamma of a European call.

(c) Differentiating the put-call parity relationship with respect to σ

$$\frac{\partial p}{\partial \sigma} = \frac{\partial c}{\partial \sigma}$$

showing that the vega of a European put equals the vega of a European call.

(d) Differentiating the put-call parity relationship with respect to t

$$\frac{\partial p}{\partial t} = rKe^{-r(T-t)} + \frac{\partial c}{\partial t}$$

This is in agreement with the thetas of European calls and puts given in Section 19.5 since $N(d_2) = 1 - N(-d_2)$.

Further Questions

Problem 19.24.
A financial institution has the following portfolio of over-the-counter options on sterling:

Type	*Position*	*Delta of Option*	*Gamma of Option*	*Vega of Option*
Call	*−1,000*	*0.5*	*2.2*	*1.8*
Call	*−500*	*0.8*	*0.6*	*0.2*
Put	*−2,000*	*-0.40*	*1.3*	*0.7*
Call	*−500*	*0.70*	*1.8*	*1.4*

A traded option is available with a delta of 0.6, a gamma of 1.5, and a vega of 0.8.

(a) *What position in the traded option and in sterling would make the portfolio both gamma neutral and delta neutral?*

(b) *What position in the traded option and in sterling would make the portfolio both vega neutral and delta neutral?*

The delta of the portfolio is

$$-1,000\times0.50-500\times0.80-2,000\times(-0.40)-500\times0.70=-450$$

The gamma of the portfolio is

$$-1,000\times2.2-500\times0.6-2,000\times1.3-500\times1.8=-6,000$$

The vega of the portfolio is

$$-1,000\times1.8-500\times0.2-2,000\times0.7-500\times1.4=-4,000$$

(a) A long position in 4,000 traded options will give a gamma-neutral portfolio since the long position has a gamma of $4,000\times1.5=+6,000$. The delta of the whole portfolio (including traded options) is then:
$4,000\times0.6-450=1,950$
Hence, in addition to the 4,000 traded options, a short position of 1,950 in sterling is necessary so that the portfolio is both gamma and delta neutral.

(b) A long position in 5,000 traded options will give a vega-neutral portfolio since the long position has a vega of $5,000\times0.8=+4,000$. The delta of the whole portfolio (including traded options) is then
$5,000\times0.6-450=2,550$
Hence, in addition to the 5,000 traded options, a short position of 2,550 in sterling is necessary so that the portfolio is both vega and delta neutral.

Problem 19.25.

Consider again the situation in Problem 19.24. Suppose that a second traded option with a delta of 0.1, a gamma of 0.5, and a vega of 0.6 is available. How could the portfolio be made delta, gamma, and vega neutral?

Let w_1 be the position in the first traded option and w_2 be the position in the second traded option. We require:

$$6,000=1.5w_1+0.5w_2$$

$$4,000=0.8w_1+0.6w_2$$

The solution to these equations can easily be seen to be $w_1=3,200$, $w_2=2,400$. The whole portfolio then has a delta of

$$-450+3,200\times0.6+2,400\times0.1=1,710$$

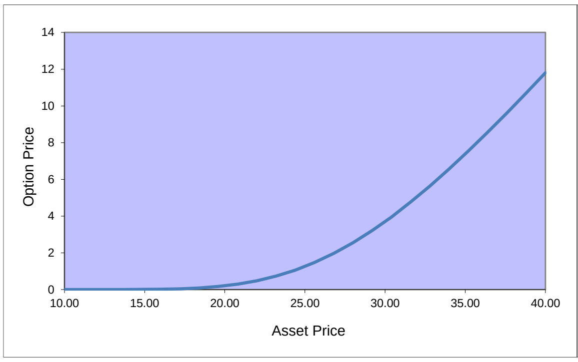
Therefore the portfolio can be made delta, gamma and vega neutral by taking a long position in 3,200 of the first traded option, a long position in 2,400 of the second traded option and a short position of 1,710 in sterling.

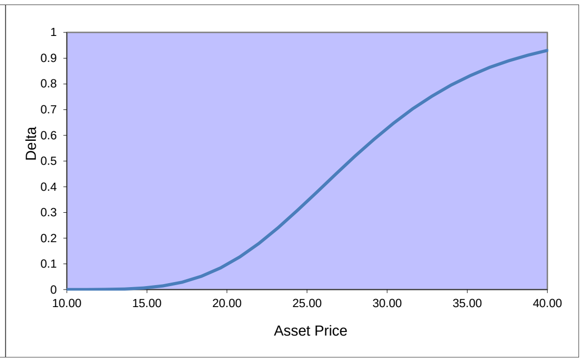
Problem 19.26.

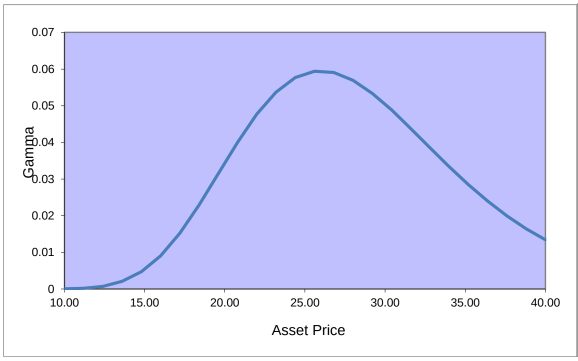
Consider a one-year European call option on a stock when the stock price is $30, the strike price is $30, the risk-free rate is 5%, and the volatility is 25% per annum. Use the DerivaGem software to calculate the price, delta, gamma, vega, theta, and rho of the option. Verify that delta is correct by changing the stock price to $30.1 and recomputing the option price. Verify that gamma is correct by recomputing the delta for the situation where the stock price is $30.1. Carry out similar calculations to verify that vega, theta, and rho are correct. Use the DerivaGem Applications Builder functions to plot the option price, delta, gamma, vega, theta, and rho against the stock price for the stock option.

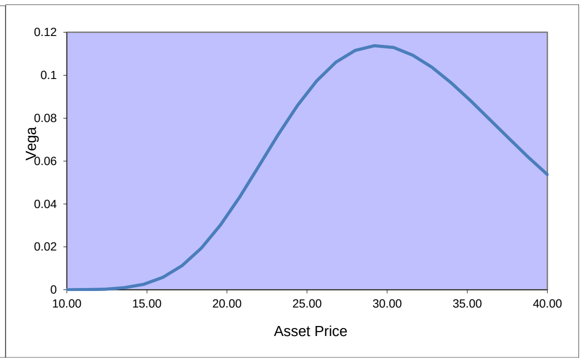
The price, delta, gamma, vega, theta, and rho of the option are 3.7008, 0.6274, 0.050, 0.1135,

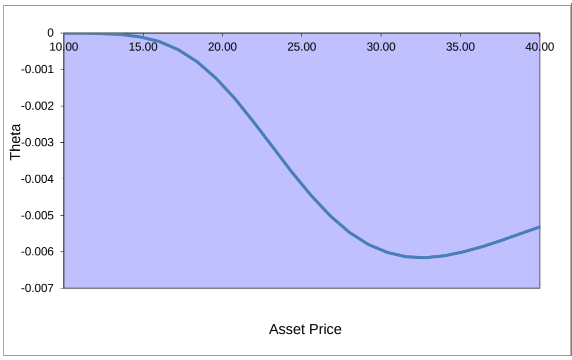
-0.00596, and 0.1512. When the stock price increases to 30.1, the option price increases to 3.7638. The change in the option price is $3.7638-3.7008=0.0630$.Delta predicts a change in the option price of $0.6274\times0.1=0.0627$ which is very close. When the stock price increases to 30.1, delta increases to 0.6324. The size of the increase in delta is $0.6324-0.6274=0.005$. Gamma predicts an increase of $0.050\times0.1=0.005$ which is the same. When the volatility increases from 25% to 26%, the option price increases by 0.1136 from 3.7008 to 3.8144. This is consistent with the vega value of 0.1135. When the time to maturity is changed from 1 to 1-1/365 the option price reduces by 0.006 from 3.7008 to 3.6948. This is consistent with a theta of -0.00596. Finally when the interest rate increases from 5% to 6% the value of the option increases by 0.1527 from 3.7008 to 3.8535. This is consistent with a rho of 0.1512.
Charts from DerivaGem showing the variation of the option price, delta, gamma, vega, theta and rho with the stock price are shown below.

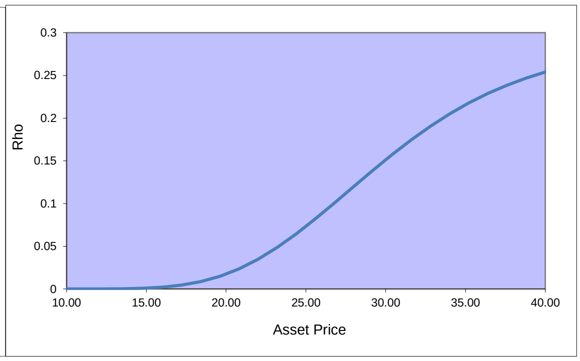












Problem 19.27.

A deposit instrument offered by a bank guarantees that investors will receive a return during a six-month period that is the greater of (a) zero and (b) 40% of the return provided by a market index. An investor is planning to put $100,000 in the instrument. Describe the payoff *as an option on the index. Assuming that the risk-free rate of interest is 8% per annum, the dividend yield on the index is 3% per annum, and the volatility of the index is 25% per annum, is the product a good deal for the investor?*

The product provides a six-month return equal to

$$\max(0, 0.4R)$$

where R is the return on the index. Suppose that S_0 is the current value of the index and S_T is the value in six months.

When an amount A is invested, the return received at the end of six months is:

$$A\max(0, 0.4\frac{S_T - S_0}{S_0})$$

$$= \frac{0.4A}{S_0}\max(0, S_T - S_0)$$

This is $0.4A/S_0$ of at-the-money European call options on the index. With the usual notation, they have value:

$$\frac{0.4A}{S_0}[S_0e^{-qT}N(d_1) - S_0e^{-rT}N(d_2)]$$

$$= 0.4A[e^{-qT}N(d_1) - e^{-rT}N(d_2)]$$

In this case $r = 0.08$, $\sigma = 0.25$, $T = 0.50$ and $q = 0.03$

$$d_1 = \frac{\left(0.08 - 0.03 + 0.25^2/2\right)0.50}{0.25\sqrt{0.50}} = 0.2298$$

$$d_2 = d_1 - 0.25\sqrt{0.50} = 0.0530$$

$$N(d_1) = 0.5909; \quad N(d_2) = 0.5212$$

The value of the European call options being offered is

$$0.4A(e^{-0.03\times0.5}\times0.5909 - e^{-0.08\times0.5}\times0.5212)$$

$$= 0.0325A$$

This is the present value of the payoff from the product. If an investor buys the product he or she avoids having to pay $0.0325A$ at time zero for the underlying option. The cash flows to the investor are therefore

Time 0: $-A+0.0325A=-0.9675A$

After six months: $+A$

The return with continuous compounding is $2\ln(1/0.9675) = 0.066$ or 6.6% per annum. The product is therefore slightly less attractive than a risk-free investment.

Problem 19.28.

The formula for the price of a European call futures option in terms of the futures price, F_0, is given in Chapter 18 as

$$c = e^{-rT}[F_0N(d_1) - KN(d_2)]$$

where

$$d_1 = \frac{\ln(F_0/K) + \sigma^2T/2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

and K, r, T, and σ are the strike price, interest rate, time to maturity, and volatility, respectively.

(a) Prove that $F_0 N'(d_1) = KN'(d_2)$

(b) Prove that the delta of the call price with respect to the futures price is $e^{-rT}N(d_1)$.

(c) Prove that the vega of the call price is $F_0\sqrt{T}N'(d_1)e^{-rT}$

(d) Prove the formula for the rho of a call futures option given in Section 19.12.

The delta, gamma, theta, and vega of a call futures option are the same as those for a call option on a stock paying dividends at rate q with q replaced by r and S_0 replaced by F_0. Explain why the same is not true of the rho of a call futures option.

(a)

$$FN'(d_1) = \frac{F}{\sqrt{2\pi}}e^{-d_1^2/2}$$

$$KN'(d_2) = KN'(d_1 - \sigma\sqrt{T}) = \frac{K}{\sqrt{2\pi}}e^{-(d_1^2/2)+d_1\sigma\sqrt{T}-\sigma^2 T/2}$$

Because $d_1\sigma\sqrt{T} = \ln(F/K) + \sigma^2 T/2$ the second equation reduces to

$$KN'(d_2) = \frac{K}{\sqrt{2\pi}}e^{-(d_1^2/2)+ln(F/K)} = \frac{F}{\sqrt{2\pi}}e^{-d_1^2/2}$$

The result follows.

(b)

$$\frac{\partial c}{\partial F} = e^{-rT}N(d_1) + e^{-rT}FN'(d_1)\frac{\partial d_1}{\partial F} - e^{-rT}KN'(d_2)\frac{\partial d_2}{\partial F}$$

Because

$$\frac{\partial d_1}{\partial F} = \frac{\partial d_2}{\partial F}$$

it follows from the result in (a) that

$$\frac{\partial c}{\partial F} = e^{-rT}N(d_1)$$

(c)

$$\frac{\partial c}{\partial \sigma} = e^{-rT}FN'(d_1)\frac{\partial d_1}{\partial \sigma} - e^{-rT}KN'(d_2)\frac{\partial d_2}{\partial \sigma}$$

Because $d_1 = d_2 + \sigma\sqrt{T}$

$$\frac{\partial d_1}{\partial \sigma} = \frac{\partial d_2}{\partial \sigma} + \sqrt{T}$$

From the result in (a) it follows that

$$\frac{\partial c}{\partial \sigma} = e^{-rT}FN'(d_1)\sqrt{T}$$

(d)

Rho is given by

$$\frac{\partial c}{\partial r} = -Te^{-rT}[FN(d_1) - KN(d_2)]]$$

or $-cT$.

Because $q = r$ in the case of a futures option there are two components to rho. One arises from differentiation with respect to r, the other from differentiation with respect to q.

Problem 19.29.
Use DerivaGem to check that equation (19.4) is satisfied for the option considered in Section 19.1. (Note: DerivaGem produces a value of theta "per calendar day." The theta in equation (19.4) is "per year.")

For the option considered in Section 19.1, $S_0 = 49$, $K = 50$, $r = 0.05$, $\sigma = 0.20$, and $T = 20/52$. DerivaGem shows that $\Theta = -0.011795 \times 365 = -4.305$, $\Delta = 0.5216$, $\Gamma = 0.065544$, $\Pi = 2.4005$. The left hand side of equation (19.4)

$$-4.305 + 0.05 \times 49 \times 0.5216 + \frac{1}{2} \times 0.2^2 \times 49^2 \times 0.065544 = 0.120$$

The right hand side is

$$0.05 \times 2.4005 = 0.120$$

This shows that the result in equation (19.4) is satisfied.

Problem 19.30. (Excel file)
Use the DerivaGem Application Builder functions to reproduce Table 19.2. (Note that in Table 19.2 the stock position is rounded to the nearest 100 shares.) Calculate the gamma and theta of the position each week. Calculate the change in the value of the portfolio each week and check whether equation (19.3) is approximately satisfied. (Note: DerivaGem produces a value of theta "per calendar day." The theta in equation (19.3) is "per year.")

Consider the first week. The portfolio consists of a short position in 100,000 options and a long position in 52,200 shares. The value of the option changes from \$240,053 at the beginning of the week to \$188,760 at the end of the week for a gain of \$51,293. The value of the shares changes from $52,200 \times 49 = \$2,557,800$ to $52,200 \times 48.12 = \$2,511,864$ for a loss of \$45,936. The net gain is $51,293 - 45,936 = \$5,357$. The gamma and theta (per year) of the portfolio are -6554.4 and 430,533 so that equation (19.3) predicts the gain as

$$430533 \times \frac{1}{52} - \frac{1}{2} \times 6554.4 \times (48.12 - 49)^2 = 5742$$

The results for all 20 weeks are shown in the following table.

Week	*Actual Gain*	*Predicted Gain*
1	5,357	5,742
2	5,689	6,093
3	−19,742	−21,084
4	1,941	1,572
5	3,706	3,652
6	9,320	9,191
7	6,249	5,936
8	9,491	9,259
9	961	870
10	−23,380	−18,992
11	1,643	2,497
12	2,645	1,356
13	11,365	10,923
14	−2,876	−3,342
15	12,936	12,302
16	7,566	8,815
17	−3,880	−2,763
18	6,764	6,899
19	4,295	5,205
20	4,806	4,805