# One-touch Double Barrier Binary Option Values

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# *One-touch double barrier binary option values*

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The valuation and applications of one-touch double barrier binary options that include features of knock-out, knock-in, European and American style are described. Using a conventional Black–Scholes option-pricing environment, analytical solutions of the options are derived. The relationships among different types of one-touch double barrier binary options are discussed. An investor having a particular view on values of foreign exchanges, equities or commodities can use the options as directional trades or structured products in financial market.

## I. INTRODUCTION

One-touch double barrier binary options are path-dependent options in which the existence and payment of the options depend on the movement of the underlying price through their option life. We discuss two types of one-touch double barrier binary options here: (1) up-and-down out binary option, and (2) American binary knock-out option. For the first type, the option vanishes if the underlying price hits the upper barrier or the lower barrier once in the option life. Otherwise, the option buyer receives a fixed payment at maturity. This option combines the characteristics of a European binary option and knock-out barrier options together. For the second type, the option vanishes if the underlying price hits a knock-out barrier, while it gives a fixed payment if another payment barrier is touched. This option can be considered as an American binary option with a knock-out barrier.

Single-barrier European option and binary option formulae are published by Rubinstein and Reiner (1991a, 1991b). Since the one-touch double barrier options are not combinations of these options, in the following section we develop their valuation formulae in a Black–Scholes environment. In the third section we shall study the relationships among the one-touch double barrier (knock-out and knock-in) binary options and their applications in financial market.

## II. OPTION VALUATION

Our objective is to value the option in a Black–Scholes environment (Black and Scholes, 1973). The underlying is assumed to follow a lognormal random walk, and the interest rate and dividend are taken to be continuous and constant over the option life. The valuation method is a risk-neutral valuation approach. Therefore, we start the deviation from the Black–Scholes equation

$$\frac{\partial f}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}+(r-d)S\frac{\partial f}{\partial S}-rf=0 \qquad (1)$$

where $f$ is the option value, $S$ is the underlying price, $t$ is the time, $\sigma$ is the volatility, $r$ is the risk-free interest rate and $d$ is the dividend if the underlying price is a stock price. For a foreign exchange option, $r$ is the interest rate of the term currency and $d$ is the interest rate of the commodity currency.

### *Up-and-down out binary option*

If $S$ ever reaches the barriers, $H_1$ and $H_2$, then the option is worthless; thus on the lines $H_1$ and $H_2$ the option value is zero. The boundary conditions for $t < T$ (time to maturity) of an up-and-down out binary option with $H_2 > H_1$ are

$$f(H_1, t)=0 \qquad f(H_2, t)=0 \qquad (2)$$

We let the final payment of the option be $R$ if $S$ never hits the barriers. Therefore, the final condition of Equation 1 is

$$f(S, T)=R \qquad H_1 < S < H_2 \qquad (3)$$

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In order to make Equation 1 dimensionless (Wilmott *et al*., 1993), we put

$$S = H_1 e^x \qquad t = T - \frac{2\tau}{\sigma^2} \qquad f = H_1 e^{\alpha x + \beta\tau} u(x, \tau)$$

$$k_1 = \frac{2(r-d)}{\sigma^2} \tag{4}$$

where

$$\alpha = -\frac{1}{2}(k_1 - 1) \qquad \beta = -\frac{1}{4}(k_1 - 1)^2 - \frac{2r}{\sigma^2} \tag{5}$$

Then the Black–Scholes equation is transformed into a standard diffusion equation (heat equation)

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < \ln\frac{H_2}{H_1} \tag{6}$$

in which the boundary conditions are

$$u(0, \tau) = 0 \qquad u\left(\ln\frac{H_2}{H_1}, \tau\right) = 0 \tag{7}$$

for $\tau > 0$ and the initial condition is

$$u(x, 0) = \frac{Re^{-\alpha x}}{H_1} \qquad 0 < x < \ln\frac{H_2}{H_1} \tag{8}$$

This is a diffusion equation with homogeneous boundary conditions. We use the method of separation of variables to solve Equation 6. Therefore, $u(x, \tau)$ can be expressed as a series of eigenfunctions:

$$u(x, \tau) = \sum_{n=1}^{\infty} b_n(\tau) \sin\left(\frac{n\pi x}{L}\right) \tag{9}$$

where $b_n(\tau) = b_n e^{-\left(\frac{n\pi}{L}\right)^2 \tau}$ and $L = \ln(H_2/H_1)$.

Since the function of the initial condition and its first derivative are piecewise continuous, it is represented by its Fourier sine series, where

$$b_n = \frac{2R}{H_1 L}\int_0^L e^{-\alpha x} \sin\left(\frac{n\pi x}{L}\right) \mathrm{d}x$$

$$= \frac{2\pi n R}{H_1 L^2}\left[\frac{1 - (-1)^n L^{-\alpha}}{\alpha^2 + \left(\frac{n\pi}{L}\right)^2}\right] \tag{10}$$

Hence by combining the above results in Equations 9 and 10 and substituting back the changing variables and transformations into option value $f(S, t)$ as a function of $S$, $H_1$, $H_2$ and $t$, we obtain the option value as

$$f(S, t) = \sum_{n=1}^{\infty} \frac{2\pi n R}{L^2}\left[\frac{\left(\frac{S}{H_1}\right)^{\alpha} - (-1)^n \left(\frac{S}{H_2}\right)^{\alpha}}{\alpha^2 + \left(\frac{n\pi}{L}\right)^2}\right] \times \sin\left(\frac{n\pi}{L}\ln\frac{S}{H_1}\right) e^{-\frac{1}{2}\left[\left(\frac{n\pi}{L}\right)^2 - \beta\right]\sigma^2 (T-t)} \tag{11}$$

The option value is the probability of the underlying price staying within the barriers in the option life with the discounting effect due to $r$ and $d$.

The solution given by Equation 11 is exact. In order to judge whether a satisfactory approximation to the solution can be obtained by using only a few terms of the series, we must estimate its speed of convergence. The sine factors are all bounded as $n \to \infty$; therefore this series behaves similarly to the series

$$\sum_{n=1}^{\infty} n^{-1} e^{-n^2 (T-t)}$$

We estimate the convergence of the series as

$$\sum_{n=1}^{\infty} n^{-1} e^{-n^2(T-t)} = \sum_{n=1}^{\infty} \frac{1}{n}\left[\frac{1}{1 + n^2(T-t) + \frac{n^4 (T-t)^2}{2} + \cdots}\right]$$

$$\leqslant \sum_{n=1}^{\infty} \frac{1}{n}\left(\frac{1}{n^2 (T-t)}\right)$$

The factor $e^{-\frac{1}{2}\left(\frac{n\pi}{L}\right)^2 \sigma^2 (T-t)}$ with $(T-t) > 0$ makes the series converge similarly to the series on the right-hand side of the inequality. The only concern is a large $\alpha$ making the factors $(S/H_1)^{\alpha}$ and $(S/H_2)^{\alpha}$ so huge that it is difficult to calculate the numerical solution in some extreme conditions. Otherwise all terms after the first few terms are almost negligible when $(T-t)$ is sufficiently large.

If we put one of the barriers, for example $H_1$, sufficiently far below the spot price $S$, the presence of the barrier $H_1$ will become insignificant. Thus the solution 11 will approach the value of a knock-out binary put option with both barrier and strike at $H_2$.

### *American binary knock-out option*

In this option, if $S$ ever reaches one barrier, $H_2$, then the option is worthless; thus on the line $H_2$ the option value is zero. If $S$ ever reaches another barrier, $H_1$, the payment is $R$ at the time of touching the payment barrier. Therefore, the boundary conditions for $t < T$ of an American binary knock-out option are

$$f(H_1, t) = R \qquad f(H_2, t) = 0 \tag{12}$$

The option is worthless at maturity if the barrier $H_1$ is never touched, thus the final condition of Equation 1 is

$$f(S, T) = 0 \qquad H_1 < S < H_2 \tag{13}$$

where we assume $H_1 < S < H_2$.

By using the same transformation in the previous section, we get a standard diffusion Equation 6 with boundary conditions as

$$u(0, \tau) = \frac{Re^{-\beta\tau}}{H_1} \qquad u\left(\ln\frac{H_2}{H_1}, \tau\right) = 0 \tag{14}$$

and the initial condition is

$$u(x, 0) = 0 \qquad 0 < x < \ln\frac{H_2}{H_1} \tag{15}$$

It is a diffusion equation with non-homogeneous time-dependent boundary conditions. Since the function in the boundary conditions is differentiable, the problem can be reduced to one with homogeneous boundary conditions. We subtract from $u$ a function $y$ that is chosen to satisfy the boundary conditions. Then the difference $w = u - y$ satisfies a problem with homogeneous boundary conditions but with a modified forcing term and initial condition. We thus find that if

$$y(x, \tau) = \frac{Re^{-\beta\tau}}{H_1}\left(1 - \frac{x}{L}\right) \tag{16}$$

then the boundary conditions of $w$ become

$$w(0, \tau) = 0 \qquad w(L, \tau) = 0 \tag{17}$$

which are now homogeneous and the initial condition is

$$w(x, 0) = \frac{R}{H_1}\left(\frac{x}{L} - 1\right) \tag{18}$$

Therefore, the equation is modified as

$$\frac{\partial w}{\partial \tau} = \frac{\partial^2 w}{\partial x^2} + \frac{R\beta}{H_1}\left(1 - \frac{x}{L}\right) e^{-\beta\tau} \tag{19}$$

with a non-homogeneous forcing term in the diffusion equation.

According to Sturm–Liouville theory (Churchill and Brown, 1985), $w$ can be expressed by Equation 9 where the sine functions are the normalized eigenfunctions of the boundary value problem. Since the functions of the initial condition and the forcing term, and their first derivatives are piecewise continuous, they are represented by their Fourier sine series. The coefficients $b_n$ are determined from the differential equation

$$b_n' + \frac{n\pi}{L} b_n = \gamma_n(\tau) \tag{20}$$

and the $\gamma_n(\tau)$ are the expansion coefficients of the non-homogeneous forcing term in term of the eigenfunctions. Thus we have

$$\gamma_n(\tau) = \frac{2}{L}\int_0^L \frac{R\beta}{H_1}\left(1 - \frac{x}{L}\right) e^{-\beta\tau} \sin\left(\frac{n\pi x}{L}\right) dx = \frac{2R\beta}{n\pi H_1} e^{-\beta\tau} \tag{21}$$

The initial condition is expanded as

$$b_n(0) = \frac{2}{L}\int_0^L \frac{R}{H_1}\left(\frac{x}{L} - 1\right) \sin\left(\frac{n\pi x}{L}\right) dx = -\frac{2R}{n\pi H_1} \tag{22}$$

By combining the results in Equations 21 and 22, the solution of Equation 20 is

$$b_n(\tau) = -\frac{2R}{n\pi H_1} e^{-\left(\frac{n\pi}{L}\right)^2 \tau} + \frac{2R\beta}{n\pi H_1}\int_0^\tau e^{-\left(\frac{n\pi}{L}\right)^2(\tau - s)} e^{-\beta s} ds = \frac{2R}{n\pi H_1}\left[\frac{\beta e^{-\beta\tau} - \left(\frac{n\pi}{L}\right)^2 e^{-\left(\frac{n\pi}{L}\right)^2 \tau}}{\left(\frac{n\pi}{L}\right)^2 - \beta}\right] \tag{23}$$

Thus the solution of the modified Equation 19 is

$$w(x, t) = \sum_{n=1}^{\infty} \frac{2R}{n\pi H_1}\left[\frac{\beta e^{-\beta\tau} - \left(\frac{n\pi}{L}\right)^2 e^{-\left(\frac{n\pi}{L}\right)^2 \tau}}{\left(\frac{n\pi}{L}\right)^2 - \beta}\right] \sin\left(\frac{n\pi x}{L}\right) \tag{24}$$

We obtain the solution of the non-homogeneous boundary value problem Equation 6 from

$$u(x, \tau) = w(x, \tau) + y(x, \tau) \tag{25}$$

After putting back the changing variables and transformations, the value of the American binary knock-out option is

$$f(S, t) = R\left(\frac{S}{H_1}\right)^{\alpha} \left\{\sum_{n=1}^{\infty} \frac{2}{n\pi}\left[\frac{\beta - \left(\frac{n\pi}{L}\right)^2 e^{-\frac{1}{2}\left[\left(\frac{n\pi}{L}\right)^2 - \beta\right]\sigma^2(T - t)}}{\left(\frac{n\pi}{L}\right)^2 - \beta}\right] \times \sin\left(\frac{n\pi}{L}\ln\frac{S}{H_1}\right) + \left(1 - \frac{\ln\frac{S}{H_1}}{L}\right)\right\} \tag{26}$$

From the derivation, we find that the assumption $H_1 < S < H_2$ does not affect the result. Thus the same solution can be applied to the condition $H_1 > S > H_2$.

By splitting the sine series in Equation 26 into two parts, the first and second series behave similarly to the series

$$\sum_{n=1}^{\infty} n^{-3} \text{ and } \sum_{n=1}^{\infty} n^{-1} e^{-n^2(T-t)} \text{ respectively}$$

The factor $e^{-\frac{1}{2}\left(\frac{n\pi}{L}\right)^2 \sigma^2 (T-t)}$ makes the second series converge. The same concern as in Equation 11 is a large $\alpha$ making the factor $(S/H_1)^\alpha$ too large to be handled by a computer, otherwise a satisfactory approximation to the solution can be obtained by using a few terms in the series without too much computation.

If we put the knock-out barrier $H_2$ sufficiently far away from the spot price $S$, the option will then behave as without the knock-out barrier. Thus the solution 26 will approach the value of an American binary option.

## III. RELATIONSHIPS TO OTHER OPTIONS AND DISCUSSION

A reverse of an up-and-down out binary is an up-and-down exercise binary option. This option lets a buyer receive a fixed payment at the time of the underlying price touching one of the barriers (American style). It can be structured by buying two American binary knock-out options in which the payment and knock-out barriers of the two options are opposite to each other.

In single barrier options, it is easy to show that barrier options with a knock-in feature can be priced by buying an option without any knock-out feature and selling a knock-out option. The same approach can be used in one-touch double barrier binary options. For example, an American binary option with a knock-in barrier $H$, the option premium is equal to buying an American binary option and selling an American binary knock-out option with a barrier at $H$. All the options have the same payment barrier.

Since the payment of the one-touch double barrier binary option is binary, they are not ideal hedging instruments. However, they are suitable for investment. Recently structured accrual range notes are popular in financial market. The notes are linked to either foreign exchanges, equities or commodities. Up-and-down out binary options are used in these kind of notes. For example, in a daily accrual USD-DEM exchange rate range note, it pays a fixed daily accrual interest if the exchange rate remains within the range $H_1$ and $H_2$. However, if the rate moves outside the range at any time, the coupon will stop accruing for the rest of the period. The note is structured by using the up-front deposit interest to buy a series of up-and-down out binary options with barriers $H_1$ and $H_2$ and maturity on each interest accrual day. This note is suitable for an investor who has a strong view of the USD-DEM exchange rate movement within a certain range. On the other hand, if an investor has a view that the underlying price will move outside a range, an up-and-down exercise binary option is suitable for him. Other creative investment products can be structured by using one-touch double barrier binary options.

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