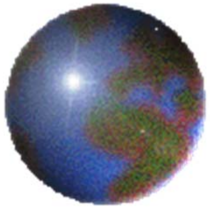


Chapter 1
Introduction

What is a Derivative?

- A derivative is an instrument whose value depends on, or is derived from, the value of another asset.
- Examples: futures, forwards, swaps, options, exotics…

Why Derivatives Are Important

- Derivatives play a key role in transferring risks in the economy
- The underlying assets include stocks, currencies, interest rates, commodities, debt instruments, electricity, insurance payouts, the weather, etc
- Many financial transactions have embedded derivatives
- The real options approach to assessing capital investment decisions has become widely accepted

How Derivatives Are Traded

- On exchanges such as the Chicago Board Options Exchange (CBOE)
- In the over-the-counter (OTC) market where traders working for banks, fund managers and corporate treasurers contact each other directly

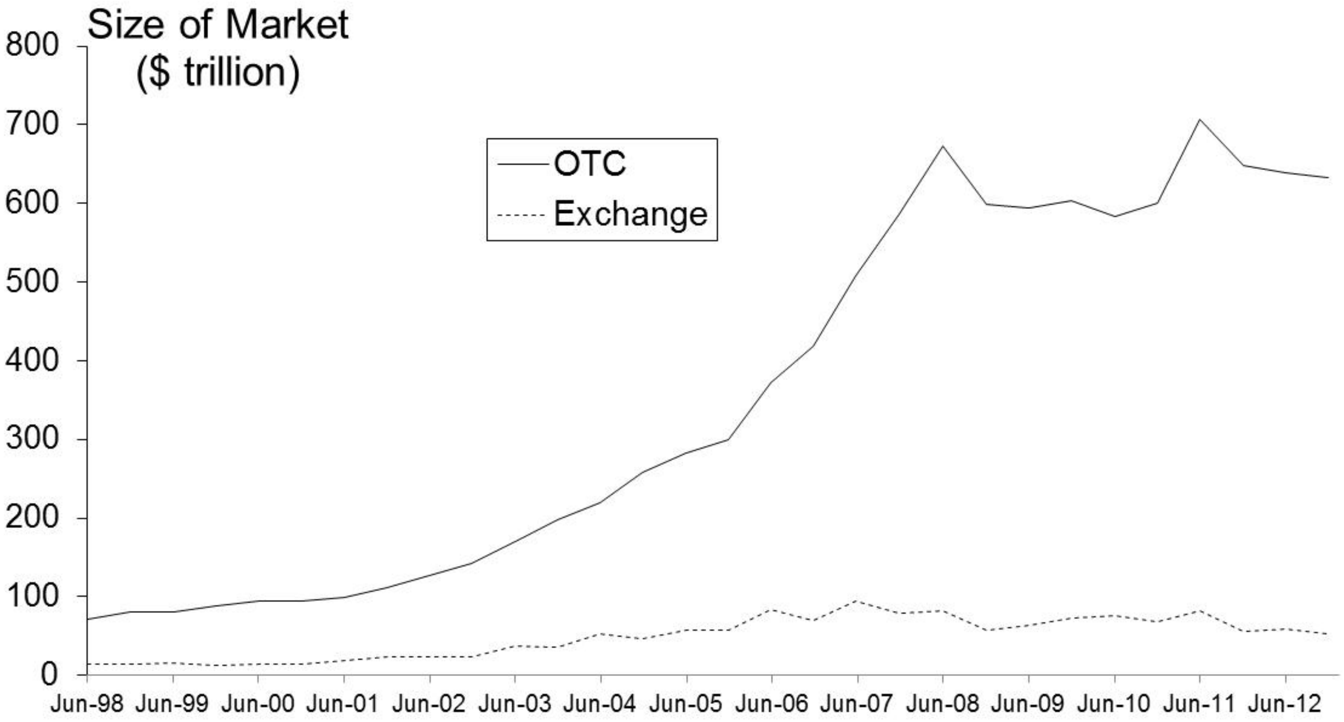
The OTC Market Prior to 2008

- Largely unregulated
- Banks acted as market makers quoting bids and offers
- Master agreements usually defined how transactions between two parties would be handled
- But some transactions were handled by central counterparties (CCPs). A CCP stands between the two sides to a transaction in the same way that an exchange does

Since 2008…

- OTC market has become regulated. Objectives:
  - Reduce systemic risk (see Business Snapshot 1.2, page 5)
  - Increase transparency
- In the U.S and some other countries, standardized OTC products must be traded on swap execution facilities (SEFs) which are similar to exchanges
- CCPs must be used for standardized transactions between dealers in most countries
- All trades must be reported to a central registry

Size of OTC and Exchange-Traded Markets (Figure 1.1, Page 5)



Source: Bank for International Settlements. Chart shows total principal amounts for OTC market and value of underlying assets for exchange market

The Lehman Bankruptcy *(Business Snapshot 1.1)*

- Lehman's filed for bankruptcy on September 15, 2008. This was the biggest bankruptcy in US history
- Lehman was an active participant in the OTC derivatives markets and got into financial difficulties because it took high risks and found it was unable to roll over its short term funding
- It had hundreds of thousands of transactions outstanding with about 8,000 counterparties
- Unwinding these transactions has been challenging for both the Lehman liquidators and their counterparties

How Derivatives are Used

- To hedge risks
- To speculate (take a view on the future direction of the market)
- To lock in an arbitrage profit
- To change the nature of a liability
- To change the nature of an investment without incurring the costs of selling one portfolio and buying another

Foreign Exchange Quotes for GBP, May 26, 2013 *(See page 6)*

	Bid	Offer
Spot	1.5541	1.5545
1-month forward	1.5538	1.5543
3-month forward	1.5533	1.5538
6-month forward	1.5526	1.5532

Forward Price

- The forward price for a contract is the delivery price that would be applicable to the contract if were negotiated today (i.e., it is the delivery price that would make the contract worth exactly zero)
- The forward price may be different for contracts of different maturities (as shown by the table)

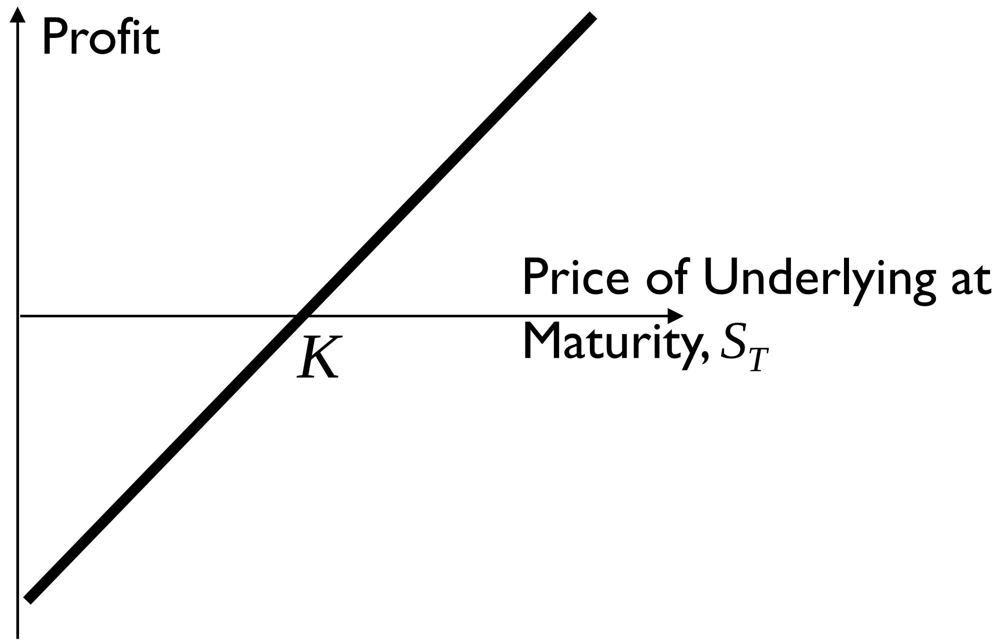
Terminology

- The party that has agreed to buy has what is termed a long position
- The party that has agreed to sell has what is termed a short position

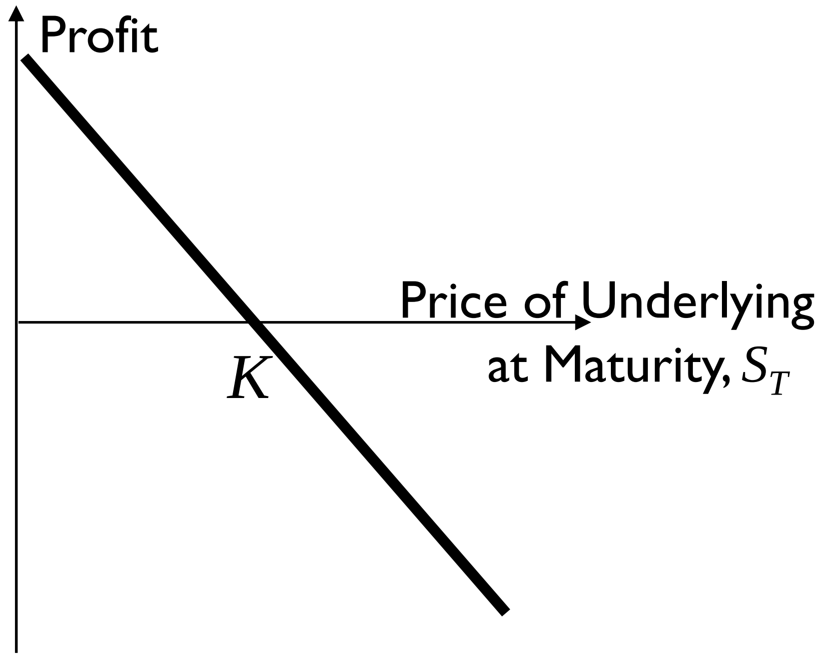
Example (page 6)

- On May 6, 2013, the treasurer of a corporation enters into a long forward contract to buy £1 million in six months at an exchange rate of 1.5532
- This obligates the corporation to pay $1,553,200 for £1 million on November 6, 2010
- What are the possible outcomes?

Profit from a Long Forward Position ***(K= delivery price=forward price at time contract is entered into)***



Profit from a Short Forward Position ***(K= delivery price=forward price at time contract is entered into)***



Futures Contracts *(page 8)*

- Agreement to buy or sell an asset for a certain price at a certain time
- Similar to forward contract
- Whereas a forward contract is traded OTC, a futures contract is traded on an exchange

Exchanges Trading Futures

- CME Group (formed when Chicago Mercantile Exchange and Chicago Board of Trade merged)
- NYSE Euronext (being acquired by bteh InterContinental Exchange)
- BM&F (Sao Paulo, Brazil)
- TIFFE (Tokyo)
- and many more (see list at end of book)

Examples of Futures Contracts

Agreement to:

- Buy 100 oz. of gold @ US$1400/oz. in December
- Sell £62,500 @ 1.5500 US$/£ in March
- Sell 1,000 bbl. of oil @ US$90/bbl. in April

1. Gold: An Arbitrage Opportunity?

Suppose that:

- The spot price of gold is US$1,400
- The 1-year forward price of gold is US$1,500
- The 1-year US$ interest rate is 5% per annum

Is there an arbitrage opportunity?

2. Gold: Another Arbitrage Opportunity?

Suppose that:

- The spot price of gold is US$1,400
- The 1-year forward price of gold is US$1,400
- The 1-year US$ interest rate is 5% per annum

Is there an arbitrage opportunity?

The Forward Price of Gold

(ignores the gold lease rate)

If the spot price of gold is S and the forward price for a contract deliverable in T years is F, then

$$F = S\,(1+r\,)^T$$

where r is the 1-year (domestic currency) risk-free rate of interest.

In our examples, $S = 1400$, $T = 1$, and $r = 0.05$ so that

$$F = 1400(1+0.05) = 1470$$

1. Oil: An Arbitrage Opportunity?

Suppose that:

- The spot price of oil is US$95
- The quoted 1-year futures price of oil is US$125
- The 1-year US$ interest rate is 5% per annum
- The storage costs of oil are 2% per annum

Is there an arbitrage opportunity?

2. Oil: Another Arbitrage Opportunity?

Suppose that:

- The spot price of oil is US$95
- The quoted 1-year futures price of oil is US$80
- The 1-year US$ interest rate is 5% per annum
- The storage costs of oil are 2% per annum

Is there an arbitrage opportunity?

Options

- A call option is an option to buy a certain asset by a certain date for a certain price (the strike price)
- A put option is an option to sell a certain asset by a certain date for a certain price (the strike price)

American vs European Options

- An American option can be exercised at any time during its life
- A European option can be exercised only at maturity

Google Call Option Prices from CBOE (May 8, 2013; Stock Price is bid 871.23, offer 871.37); See Table 1.2 page 9

Strike Price	Jun 2013 Bid	Jun 2013 Offer	Sep 2013 Bid	Sep 2013 Offer	Dec 2013 Bid	Dec 2013 Offer
820	56.00	57.50	76.00	77.80	88.00	90.30
840	39.50	40.70	62.90	63.90	75.70	78.00
860	25.70	26.50	51.20	52.30	65.10	66.40
880	15.00	15.60	41.00	41.60	55.00	56.30
900	7.90	8.40	32.10	32.80	45.90	47.20
920	n.a.	n.a.	24.80	25.60	37.90	39.40

Google Put Option Prices from CBOE (May 8, 2013; Stock Price is bid 871.23, offer 871.37); See Table 1.3 page 9

Strike Price	Jun 2013 Bid	Jun 2013 Offer	Sep 2013 Bid	Sep 2013 Offer	Dec 2013 Bid	Dec 2013 Offer
820	5.00	5.50	24.20	24.90	36.20	37.50
840	8.40	8.90	31.00	31.80	43.90	45.10
860	14.30	14.80	39.20	40.10	52.60	53.90
880	23.40	24.40	48.80	49.80	62.40	63.70
900	36.20	37.30	59.20	60.90	73.40	75.00
920	n.a.	n.a.	71.60	73.50	85.50	87.40

Options vs Futures/Forwards

- A futures/forward contract gives the holder the obligation to buy or sell at a certain price
- An option gives the holder the right to buy or sell at a certain price

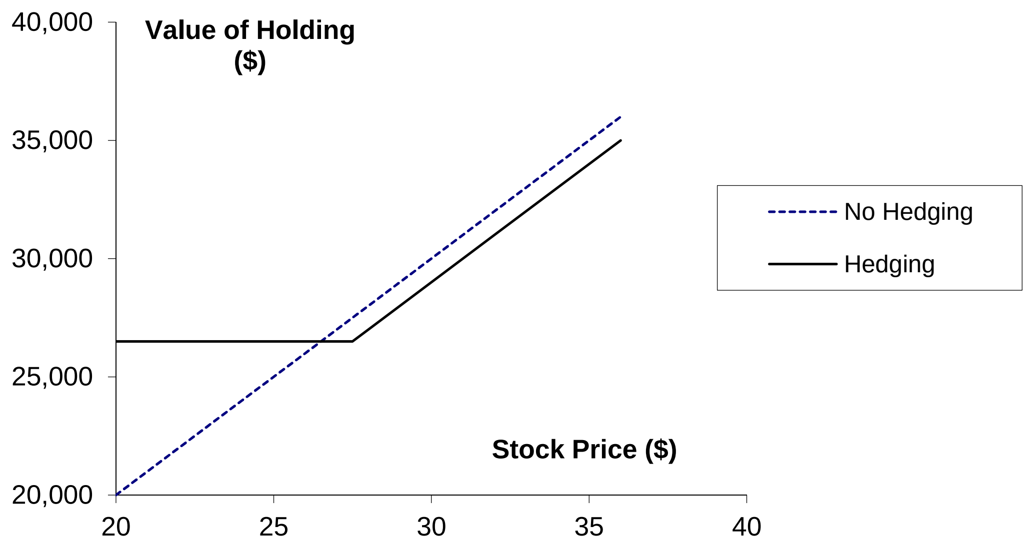
Types of Traders

- Hedgers
- Speculators
- Arbitrageurs

Hedging Examples ***(pages 11-13)***

- A US company will pay £10 million for imports from Britain in 3 months and decides to hedge using a long position in a forward contract
- An investor owns 1,000 Microsoft shares currently worth \$28 per share. A two-month put with a strike price of \$27.50 costs \$1. The investor decides to hedge by buying 10 contracts

Value of Microsoft Shares with and without Hedging *(Fig 1.4, page 13)*



Speculation Example

- An investor with $2,000 to invest feels that a stock price will increase over the next 2 months. The current stock price is $20 and the price of a 2-month call option with a strike of 22.50 is $1
- What are the alternative strategies?

Arbitrage Example

- A stock price is quoted as £100 in London and $150 in New York
- The current exchange rate is 1.5300
- What is the arbitrage opportunity?

Dangers

- Traders can switch from being hedgers to speculators or from being arbitrageurs to speculators
- It is important to set up controls to ensure that trades are using derivatives in for their intended purpose
- Soc Gen (see Business Snapshot 1.4 on page 18) is an example of what can go wrong

Hedge Funds *(see Business Snapshot 1.3, page 12)*

- Hedge funds are not subject to the same rules as mutual funds and cannot offer their securities publicly.
- Mutual funds must
  - disclose investment policies,
  - makes shares redeemable at any time,
  - limit use of leverage
- Hedge funds are not subject to these constraints.
- Hedge funds use complex trading strategies are big users of derivatives for hedging, speculation and arbitrage

Types of Hedge Funds

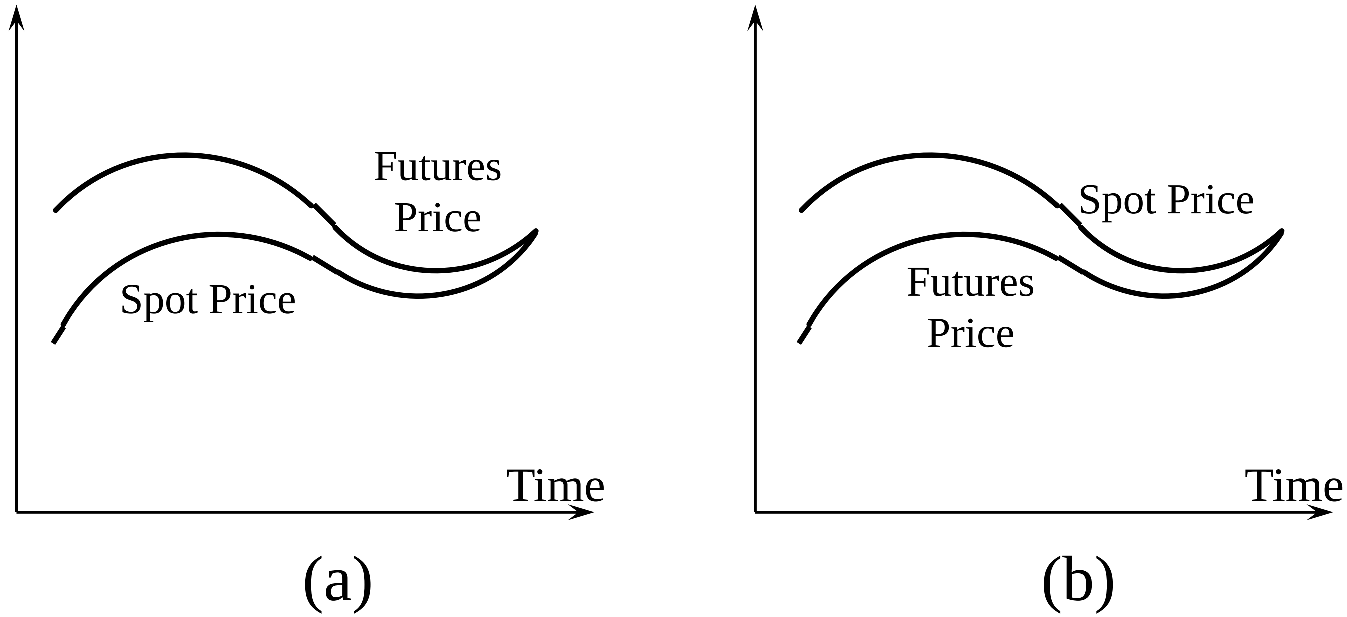
- Long/Short Equities
- Convertible Arbitrage
- Distressed Securities
- Emerging Markets
- Global Macro
- Merger Arbitrage

Chapter 2
Mechanics of Futures Markets

Futures Contracts

- Available on a wide range of assets
- Exchange traded
- Specifications need to be defined:
  - What can be delivered,
  - Where it can be delivered, &
  - When it can be delivered
- Settled daily

Convergence of Futures to Spot (Figure 2.1, page 29)



Margins

- A margin is cash or marketable securities deposited by an investor with his or her broker
- The balance in the margin account is adjusted to reflect daily settlement
- Margins minimize the possibility of a loss through a default on a contract

Margin Cash Flows

- A trader has to bring the balance in the margin account up to the initial margin when it falls below the maintenance margin level
- A member of the exchange clearing house only has an initial margin and is required to bring the balance in its account up to that level every day.
- These daily margin cash flows are referred to as variation margin
- A member is also required to contribute to a default fund

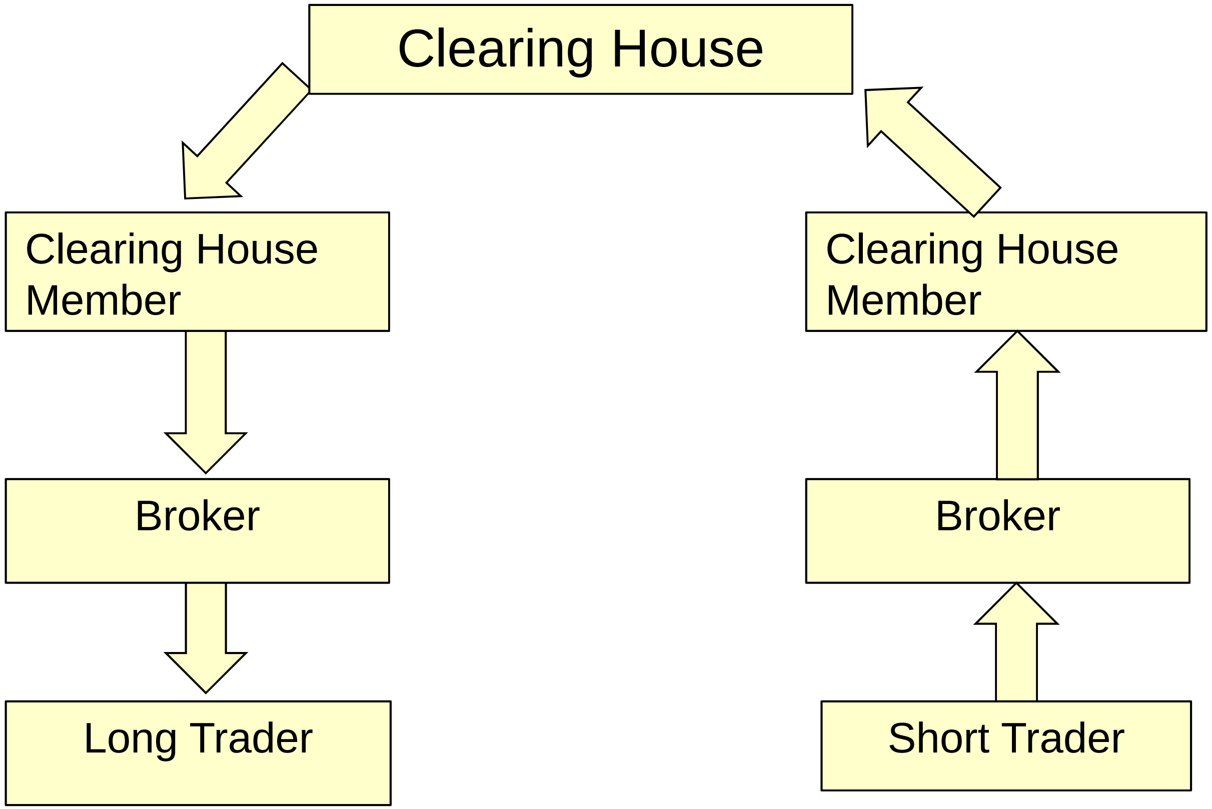
Example of a Futures Trade (page 27-29)

- An investor takes a long position in 2 December gold futures contracts on June 5
  - contract size is 100 oz.
  - futures price is US$1,450
  - initial margin requirement is US$6,000/contract (US$12,000 in total)
  - maintenance margin is US$4,500/contract (US$9,000 in total)

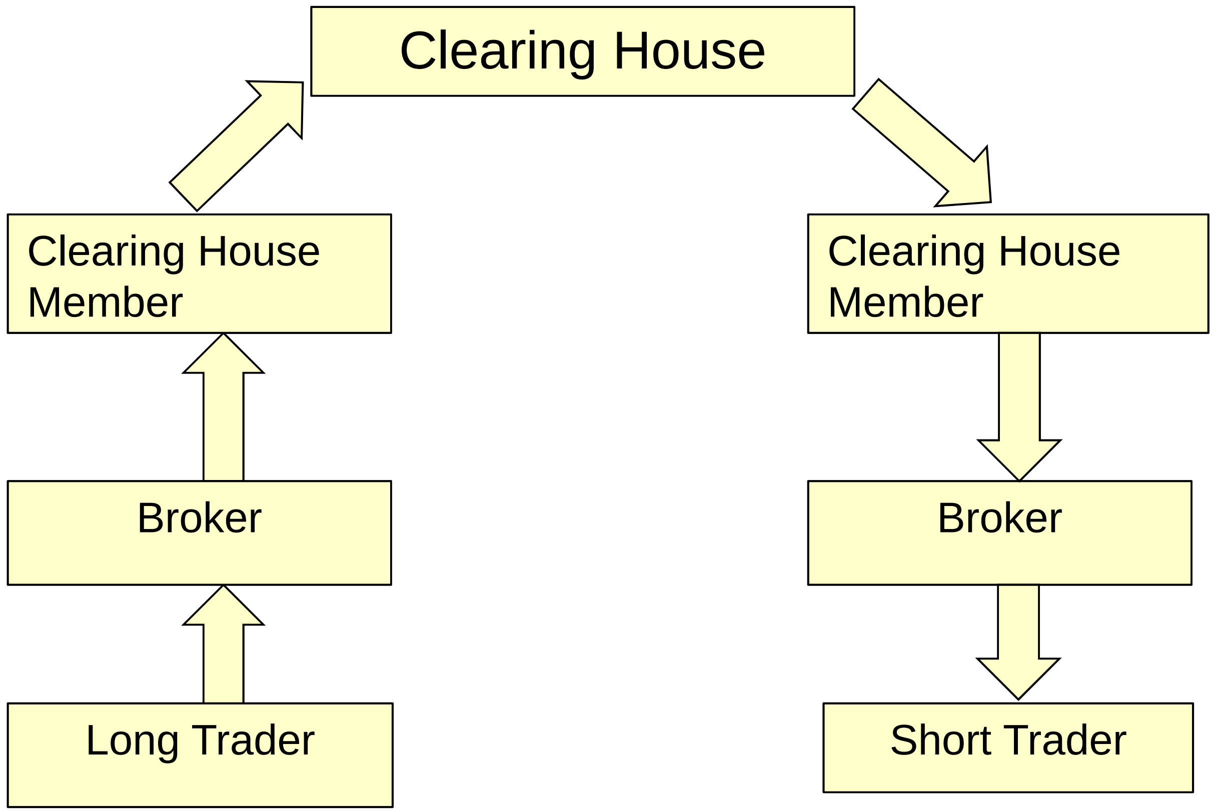
A Possible Outcome *(Table 2.1, page 30)*

Day	Trade Price ($)	Settle Price ($)	Daily Gain ($)	Cumul. Gain ($)	Margin Balance ($)	Margin Call ($)
1	1,450.00				12,000	
1		1,441.00	−1,800	− 1,800	10,200	
2		1,438.30	−540	−2,340	9,660	
…..		…..	…..	…..	……	
6		1,436.20	−780	−2,760	9,240	
7		1,429.90	−1,260	−4,020	7,980	4,020
8		1,430.80	180	−3,840	12,180	
…..		…..	…..	…..	……	
16	1,426.90		780	−4,620	15,180	

Margin Cash Flows When Futures Price Increases



Margin Cash Flows When Futures Price Decreases



Some Terminology

- Open interest: the total number of contracts outstanding
  - equal to number of long positions or number of short positions
- Settlement price: the price just before the final bell each day
  - used for the daily settlement process
- Volume of trading: the number of trades in one day

Key Points About Futures

- They are settled daily
- Closing out a futures position involves entering into an offsetting trade
- Most contracts are closed out before maturity

Crude Oil Trading on May 14, 2013 *(Table 2.2, page 36)*

	Open	High	Low	Prior Settle	Last Trade	Change	Volume
Jun 2013	94.93	95.66	94.50	95.17	94.72	−0.45	162,901
Aug 2013	95.24	95.92	94.81	95.43	95.01	−0.42	37,830
Dec 2013	93.77	94.37	93.39	93.89	93.60	−0.29	27,179
Dec 2014	89.98	90.09	89.40	89.71	89.62	−0.09	9,606
Dec 2015	86.99	87.33	86.94	86.99	86.94	−0.05	2,181

Collateralization in OTC Markets

- It is becoming increasingly common for transactions to be collateralized in OTC markets
- Consider transactions between companies A and B
- These might be governed by an ISDA Master agreement with a credit support annex (CSA)
- The CSA might require A to post collateral with B equal to the value to B of its outstanding transactions with B when this value is positive.

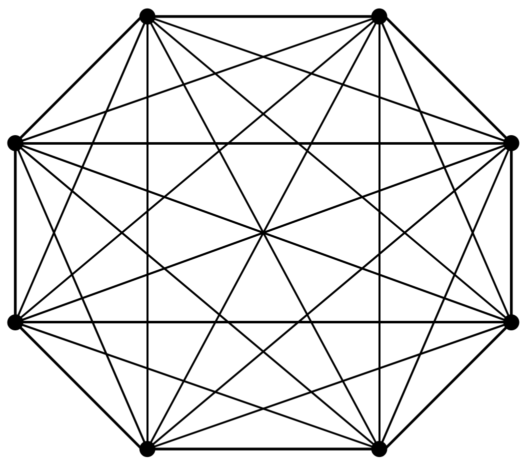
Collateralization in OTC Markets *continued*

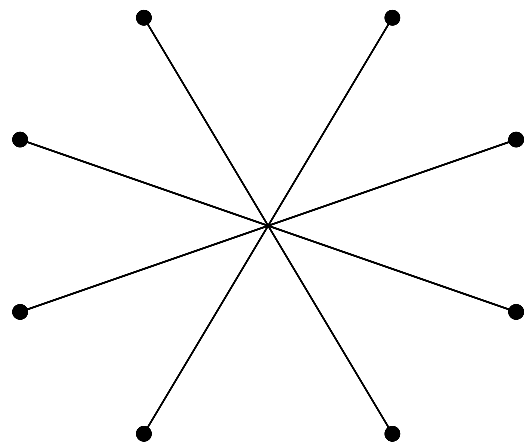
- If A defaults, B is entitled to take possession of the collateral
- The transactions are not settled daily and interest is paid on cash collateral
- See Business Snapshot 2.2 for how collateralization affected Long Term Capital Management when there was a “flight to quality” in 1998.

Clearing Houses and OTC Markets

- Traditionally most transactions have been cleared bilaterally in OTC markets
- Following the 2007-2009 crisis, the has been a requirement for most standardized OTC derivatives transactions between dealers to be cleared through central counterparties (CCPs)
- CCPs require initial margin, variation margin, and default fund contributions from members similarly to exchange clearing houses

Bilateral Clearing vs Central Clearing House





New Regulations

- New regulations for trades between dealers that are not cleared centrally require dealers to post both initial margin and daily variation margin
- The initial margin is posted with a third party

Delivery

- If a futures contract is not closed out before maturity, it is usually settled by delivering the assets underlying the contract. When there are alternatives about what is delivered, where it is delivered, and when it is delivered, the party with the short position chooses.
- A few contracts (for example, those on stock indices and Eurodollars) are settled in cash

Questions

- When a new trade is completed what are the possible effects on the open interest?
- Can the volume of trading in a day be greater than the open interest?

Types of Orders

- Limit
- Stop-loss
- Stop-limit
- Market-if touched
- Discretionary
- Time of day
- Open
- Fill or kill

Regulation of Futures

- In the US, the regulation of futures markets is primarily the responsibility of the Commodity Futures and Trading Commission (CFTC)
- Regulators try to protect the public interest and prevent questionable trading practices

Accounting & Tax

- Ideally hedging profits (losses) should be recognized at the same time as the losses (profits) on the item being hedged
- Ideally profits and losses from speculation should be recognized on a mark-to-market basis
- Roughly speaking, this is what the accounting and tax treatment of futures in the U.S. and many other countries attempt to achieve

Forward Contracts vs Futures Contracts (Table 2.3, page 43)

FORWARDS	FUTURES
Private contract between 2 parties	Exchange traded
Non-standard contract	Standard contract
Usually 1 specified delivery date	Range of delivery dates
Settled at end of contract	Settled daily
Delivery or final cash settlement usually occurs	Contract usually closed out prior to maturity
Some credit risk	Virtually no credit risk

Foreign Exchange Quotes

- Futures exchange rates are quoted as the number of USD per unit of the foreign currency
- Forward exchange rates are quoted in the same way as spot exchange rates. This means that GBP, EUR, AUD, and NZD are quoted as USD per unit of foreign currency. Other currencies (e.g., CAD and JPY) are quoted as units of the foreign currency per USD.

Chapter 3
Hedging Strategies Using Futures

Long & Short Hedges

- A long futures hedge is appropriate when you know you will purchase an asset in the future and want to lock in the price
- A short futures hedge is appropriate when you know you will sell an asset in the future and want to lock in the price

Arguments in Favor of Hedging

- Companies should focus on the main business they are in and take steps to minimize risks arising from interest rates, exchange rates, and other market variables

Arguments against Hedging

- Shareholders are usually well diversified and can make their own hedging decisions
- It may increase risk to hedge when competitors do not
- Explaining a situation where there is a loss on the hedge and a gain on the underlying can be difficult

Basis Risk

- Basis is usually defined as the spot price minus the futures price
- Basis risk arises because of the uncertainty about the basis when the hedge is closed out

Long Hedge for Purchase of an Asset

- Define

  F_1 : Futures price at time hedge is set up

  F_2 : Futures price at time asset is purchased

  S_2 : Asset price at time of purchase

  b_2 : Basis at time of purchase

Cost of asset	S_2
Gain on Futures	$F_2 - F_1$
Net amount paid	$S_2 - (F_2 - F_1) = F_1 + b_2$

Short Hedge for Sale of an Asset

- Define

  F_1 : Futures price at time hedge is set up
  F_2 : Futures price at time asset is sold
  S_2 : Asset price at time of sale
  b_2 : Basis at time of sale

Price of asset	S_2
Gain on Futures	$F_1 - F_2$
Net amount received	$S_2 + (F_1 - F_2) = F_1 + b_2$

Choice of Contract

- Choose a delivery month that is as close as possible to, but later than, the end of the life of the hedge
- When there is no futures contract on the asset being hedged, choose the contract whose futures price is most highly correlated with the asset price. This is known as cross hedging.

Optimal Hedge Ratio *(page 59)*

Proportion of the exposure that should optimally be hedged is

$$h^* = \rho \frac{\sigma_S}{\sigma_F}$$

where

σ_S is the standard deviation of ΔS, the change in the spot price during the hedging period,

σ_F is the standard deviation of ΔF, the change in the futures price during the hedging period

ρ is the coefficient of correlation between ΔS and ΔF.

Example *(Page 61)*

- Airline will purchase 2 million gallons of jet fuel in one month and hedges using heating oil futures
- From historical data σ_F =0.0313, σ_S =0.0263, and ρ= 0.928

$$h^* = 0.928 \times \frac{0.0263}{0.0313} = 0.78$$

Example *continued*

- The size of one heating oil contract is 42,000 gallons
- The spot price is 1.94 and the futures price is 1.99 (both dollars per gallon) so that

$$V_A = 1.94 \times 2{,}000{,}000 = 3{,}880{,}000$$
$$V_F = 1.99 \times 42{,}000 = 83{,}580$$

- Optimal number of contracts is

$$= 0.78 \times 2{,}000{,}000 / 42{,}000$$

which rounds to 37

Alternative Definition of Optimal Hedge Ratio

- Optimal hedge ratio is

$$\hat{h} = \hat{\rho}\,\frac{\hat{\sigma}_S}{\hat{\sigma}_F}$$

where variables are defined as follows

$\hat{\rho}$	Correlation between percentage daily changes for spot and futures
$\hat{\sigma}_S$	SD of percentage daily changes in spot
$\hat{\sigma}_F$	SD of percentage daily changes in futures

Optimal Number of Contracts

Q_A	Size of position being hedged (units)
Q_F	Size of one futures contract (units)
V_A	Value of position being hedged (=spot price time Q_A)
V_F	Value of one futures contract (=futures price times Q_F)

Optimal number of contracts if adjustment for daily settlement

$$= \frac{h^* Q_A}{Q_F}$$

Optimal number of contracts after "tailing adjustment" to allow or daily settlement of futures

$$= \frac{\hat{h} V_A}{V_F}$$

Hedging Using Index Futures

(Page 64)

To hedge the risk in a portfolio the number of contracts that should be shorted is

$$\beta \frac{V_A}{V_F}$$

where V_A is the value of the portfolio, β is its beta, and V_F is the value of one futures contract

Example

S&P 500 futures price is 1,000

Value of Portfolio is $5 million

Beta of portfolio is 1.5

What position in futures contracts on the S&P 500 is necessary to hedge the portfolio?

Changing Beta

- What position is necessary to reduce the beta of the portfolio to 0.75?
- What position is necessary to increase the beta of the portfolio to 2.0?

Why Hedge Equity Returns

- May want to be out of the market for a while. Hedging avoids the costs of selling and repurchasing the portfolio
- Suppose stocks in your portfolio have an average beta of 1.0, but you feel they have been chosen well and will outperform the market in both good and bad times. Hedging ensures that the return you earn is the risk-free return plus the excess return of your portfolio over the market.

Stack and Roll *(page 68-69)*

- We can roll futures contracts forward to hedge future exposures
- Initially we enter into futures contracts to hedge exposures up to a time horizon
- Just before maturity we close them out an replace them with new contract reflect the new exposure
- etc

Liquidity Issues *(See Business Snapshot 3.2)*

- In any hedging situation there is a danger that losses will be realized on the hedge while the gains on the underlying exposure are unrealized
- This can create liquidity problems
- One example is Metallgesellschaft which sold long term fixed-price contracts on heating oil and gasoline and hedged using stack and roll
- The price of oil fell.....

Chapter 4
Interest Rates

Types of Rates

- Treasury rate
- LIBOR
- Fed funds rate
- Repo rate

Treasury Rate

- Rate on instrument issued by a government in its own currency

LIBOR

- LIBOR is the rate of interest at which a AA bank can borrow money on an unsecured basis from another bank
- For 10 currencies and maturities ranging from 1 day to 12 months it is calculated daily by the British Bankers Association from submissions from a number of major banks
- There have been some suggestions that banks manipulated LIBOR during certain periods. Why would they do this?

The Fed Funds Rate

- Unsecured interbank overnight rate of interest
- Allows banks to adjust the cash (i.e., reserves) on deposit with the Federal Reserve at the end of each day
- The effective fed funds rate is the average rate on brokered transactions
- The central bank may intervene with its own transactions to raise or lower the rate

Repo Rate

- Repurchase agreement is an agreement where a financial institution that owns securities agrees to sell them for X and buy them bank in the future (usually the next day) for a slightly higher price, Y
- The financial institution obtains a loan.
- The rate of interest is calculated from the difference between X and Y and is known as the repo rate

Measuring Interest Rates

- The compounding frequency used for an interest rate is the unit of measurement
- The difference between quarterly and annual compounding is analogous to the difference between miles and kilometers

Impact of Compounding

When we compound m times per year at rate R an amount A grows to $A(1+R/m)^m$ in one year

Compounding frequency	*Value of $100 in one year at 10%*
Annual (m=1)	110.00
Semiannual (m=2)	110.25
Quarterly (m=4)	110.38
Monthly (m=12)	110.47
Weekly (m=52)	110.51
Daily (m=365)	110.52

Continuous Compounding

(Page 81)

- In the limit as we compound more and more frequently we obtain continuously compounded interest rates
- \$100 grows to \$100e^{RT} when invested at a continuously compounded rate R for time T
- \$100 received at time T discounts to \$100e^{-RT} at time zero when the continuously compounded discount rate is R

Conversion Formulas *(Page 81)*

Define

R_c : continuously compounded rate

R_m: same rate with compounding *m* times per year

$$R_c = m \ln\left(1 + \frac{R_m}{m}\right)$$

$$R_m = m\left(e^{R_c/m} - 1\right)$$

Examples

- 10% with semiannual compounding is equivalent to 2ln(1.05)=9.758% with continuous compounding
- 8% with continuous compounding is equivalent to $4(e^{0.08/4}-1)$=8.08% with quarterly compounding
- Rates used in option pricing are nearly always expressed with continuous compounding

Zero Rates

A zero rate (or spot rate), for maturity T is the rate of interest earned on an investment that provides a payoff only at time T

Example *(Table 4.2, page 83)*

Maturity (years)	Zero rate (cont. comp.
0.5	5.0
1.0	5.8
1.5	6.4
2.0	6.8

Bond Pricing

- To calculate the cash price of a bond we discount each cash flow at the appropriate zero rate
- In our example, the theoretical price of a two-year bond providing a 6% coupon semiannually is

$$3e^{-0.05\times0.5}+3e^{-0.058\times1.0}+3e^{-0.064\times1.5}$$
$$+103e^{-0.068\times2.0}=98.39$$

Bond Yield

- The bond yield is the discount rate that makes the present value of the cash flows on the bond equal to the market price of the bond
- Suppose that the market price of the bond in our example equals its theoretical price of 98.39
- The bond yield (continuously compounded) is given by solving

$$3e^{-y\times 0.5} + 3e^{-y\times 1.0} + 3e^{-y\times 1.5} + 103e^{-y\times 2.0} = 98.39$$

to get y=0.0676 or 6.76%.

Par Yield

- The par yield for a certain maturity is the coupon rate that causes the bond price to equal its face value.
- In our example we solve

$$\frac{c}{2}e^{-0.05\times0.5}+\frac{c}{2}e^{-0.058\times1.0}+\frac{c}{2}e^{-0.064\times1.5}$$
$$+\left(100+\frac{c}{2}\right)e^{-0.068\times2.0}=100$$

to get $c=6.87$ (with semiannual compounding)

Par Yield continued

In general if m is the number of coupon payments per year, d is the present value of \$1 received at maturity and A is the present value of an annuity of \$1 on each coupon date

$$c = \frac{(100 - 100d)m}{A}$$

(in our example, $m = 2$, $d = 0.87284$, and $A = 3.70027$)

Data to Determine Zero Curve

(Table 4.3, page 84)

Bond Principal	Time to Maturity (yrs)	Coupon per year ($)*	Bond price ($)
100	0.25	0	97.5
100	0.50	0	94.9
100	1.00	0	90.0
100	1.50	8	96.0
100	2.00	12	101.6

* Half the stated coupon is paid each year

The Bootstrap Method

- An amount 2.5 can be earned on 97.5 during 3 months.
- Because 100=97.5$e^{0.10127 \times 0.25}$ the 3-month rate is 10.127% with continuous compounding
- Similarly the 6 month and 1 year rates are 10.469% and 10.536% with continuous compounding

The Bootstrap Method continued

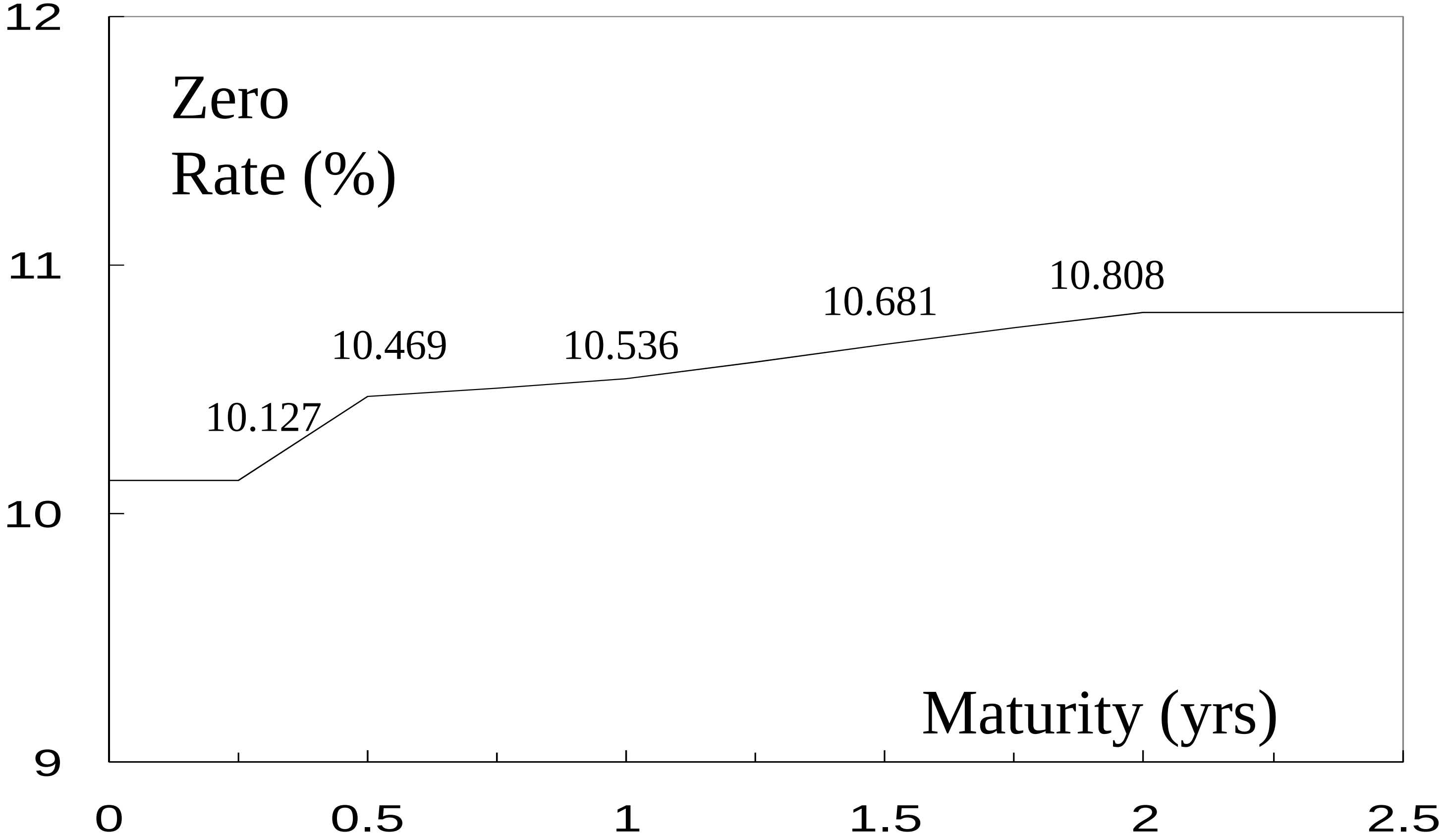
- To calculate the 1.5 year rate we solve

$$4e^{-0.10469\times 0.5} + 4e^{-0.10536\times 1.0} + 104e^{-R\times 1.5} = 96$$

to get R = 0.10681 or 10.681%

- Similarly the two-year rate is 10.808%

Zero Curve Calculated from the Data *(Figure 4.1, page 86)*



Forward Rates

- The forward rate is the future zero rate implied by today's term structure of interest rates

Formula for Forward Rates

- Suppose that the zero rates for time periods T_1 and T_2 are R_1 and R_2 with both rates continuously compounded.
- The forward rate for the period between times T_1 and T_2 is

$$\frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$$

- This formula is only approximately true when rates are not expressed with continuous compounding

Application of the Formula

Year (n)	Zero rate for n-year investment (% per annum)	Forward rate for nth year (% per annum)
1	3.0	
2	4.0	5.0
3	4.6	5.8
4	5.0	6.2
5	5.5	6.5

Instantaneous Forward Rate

- The instantaneous forward rate for a maturity T is the forward rate that applies for a very short time period starting at T. It is

$$R + T\frac{\partial R}{\partial T}$$

where R is the T-year rate

Upward vs Downward Sloping Yield Curve

- For an upward sloping yield curve:
  Fwd Rate > Zero Rate > Par Yield

- For a downward sloping yield curve
  Par Yield > Zero Rate > Fwd Rate

Forward Rate Agreement

- A forward rate agreement (FRA) is an OTC agreement that a certain rate will apply to a certain principal during a certain future time period

Forward Rate Agreement: Key Results

- An FRA is equivalent to an agreement where interest at a predetermined rate, R_K is exchanged for interest at the market rate
- An FRA can be valued by assuming that the forward LIBOR interest rate, R_F , is certain to be realized
- This means that the value of an FRA is the present value of the difference between the interest that would be paid at interest at rate R_F and the interest that would be paid at rate R_K

Valuation Formulas

- If the period to which an FRA applies lasts from T_1 to T_2, we assume that R_F and R_K are expressed with a compounding frequency corresponding to the length of the period between T_1 and T_2
- With an interest rate of R_K, the interest cash flow is $R_K(T_2 - T_1)$ at time T_2
- With an interest rate of R_F, the interest cash flow is $R_F(T_2 - T_1)$ at time T_2

Valuation Formulas *continued*

- When the rate R_K will be received on a principal of L the value of the FRA is the present value of

$$(R_K - R_F)(T_2 - T_1)$$

received at time T_2

- When the rate R_K will be received on a principal of L the value of the FRA is the present value of

$$(R_F - R_K)(T_2 - T_1)$$

received at time T_2

Example

- An FRA entered into some time ago ensures that a company will receive 4% (s.a.) on $100 million for six months starting in 1 year
- Forward LIBOR for the period is 5% (s.a.)
- The 1.5 year rate is 4.5% with continuous compounding
- The value of the FRA (in $ millions) is

$$100 \times (0.04 - 0.05) \times 0.5 \times e^{-0.045 \times 1.5} = -0.467$$

Example *continued*

- If the six-month interest rate in one year turns out to be 5.5% (s.a.) there will be a payoff (in $ millions) of

$$100 \times (0.04 - 0.055) \times 0.5 = -0.75$$

in 1.5 years

- The transaction might be settled at the one-year point for an equivalent payoff of

$$\frac{-0.75}{1.0275} = -0.730$$

Duration *(page 91-94)*

- Duration of a bond that provides cash flow c_i at time t_i is

$$D = \sum_{i=1}^{n} t_i \left[\frac{c_i e^{-yt_i}}{B} \right]$$

where B is its price and y is its yield (continuously compounded)

Key Duration Relationship

- Duration is important because it leads to the following key relationship between the change in the yield on the bond and the change in its price

$$\frac{\Delta B}{B} = -D\Delta y$$

Key Duration Relationship *continued*

- When the yield y is expressed with compounding m times per year

$$\Delta B = -\frac{BD\Delta y}{1 + y/m}$$

- The expression

$$\frac{D}{1 + y/m}$$

is referred to as the "modified duration"

Bond Portfolios

- The duration for a bond portfolio is the weighted average duration of the bonds in the portfolio with weights proportional to prices
- The key duration relationship for a bond portfolio describes the effect of small parallel shifts in the yield curve
- What exposures remain if duration of a portfolio of assets equals the duration of a portfolio of liabilities?

Convexity

The convexity, C, of a bond is defined as

$$C = \frac{1}{B}\frac{\partial^2 B}{\partial y^2} = \frac{\sum_{i=1}^{n} c_i t_i^2 e^{-yt_i}}{B}$$

This leads to a more accurate relationship

$$\frac{\Delta B}{B} = -D\Delta y + \frac{1}{2}C(\Delta y)^2$$

When used for bond portfolios it allows larger shifts in the yield curve to be considered, but the shifts still have to be parallel

Theories of the Term Structure

Page 96-98

- Expectations Theory: forward rates equal expected future zero rates
- Market Segmentation: short, medium and long rates determined independently of each other
- Liquidity Preference Theory: forward rates higher than expected future zero rates

Liquidity Preference Theory

- Suppose that the outlook for rates is flat and you have been offered the following choices

Maturity	Deposit rate	Mortgage rate
1 year	3%	6%
5 year	3%	6%

- Which would you choose as a depositor? Which for your mortgage?

Liquidity Preference Theory cont

- To match the maturities of borrowers and lenders a bank has to increase long rates above expected future short rates
- In our example the bank might offer

Maturity	Deposit rate	Mortgage rate
1 year	3%	6%
5 year	4%	7%

Chapter 5
Determination of Forward and Futures Prices

Consumption vs Investment Assets

- Investment assets are assets held by significant numbers of people purely for investment purposes (Examples: gold, silver)
- Consumption assets are assets held primarily for consumption (Examples: copper, oil)

Short Selling *(Page 105-106)*

- Short selling involves selling securities you do not own
- Your broker borrows the securities from another client and sells them in the market in the usual way

Short Selling (continued)

- At some stage you must buy the securities so they can be replaced in the account of the client
- You must pay dividends and other benefits the owner of the securities receives
- There may be a small fee for borrowing the securities

Example

- You short 100 shares when the price is $100 and close out the short position three months later when the price is $90
- During the three months a dividend of $3 per share is paid
- What is your profit?
- What would be your loss if you had bought 100 shares?

Notation for Valuing Futures and Forward Contracts

S_0: Spot price today

F_0: Futures or forward price today

T: Time until delivery date

r: Risk-free interest rate for maturity T

An Arbitrage Opportunity?

- Suppose that:
  - The spot price of a non-dividend-paying stock is \$40
  - The 3-month forward price is \$43
  - The 3-month US\$ interest rate is 5% per annum
- Is there an arbitrage opportunity?

Another Arbitrage Opportunity?

- Suppose that:
  - The spot price of nondividend-paying stock is $40
  - The 3-month forward price is US$39
  - The 1-year US$ interest rate is 5% per annum (continuously compounded)
- Is there an arbitrage opportunity?

The Forward Price

If the spot price of an investment asset is S_0 and the futures price for a contract deliverable in T years is F_0, then

$$F_0 = S_0 e^{rT}$$

where r is the T-year risk-free rate of interest.

In our examples, S_0 =40, T=0.25, and r=0.05 so that

$$F_0 = 40e^{0.05\times 0.25} = 40.50$$

If Short Sales Are Not Possible..

Formula still works for an investment asset because investors who hold the asset will sell it and buy forward contracts when the forward price is too low

When an Investment Asset Provides a Known Income *(page 110, equation 5.2)*

$$F_0 = (S_0 - I)e^{rT}$$

where I is the present value of the income during life of forward contract

When an Investment Asset Provides a Known Yield *(Page 112, equation 5.3)*

$$F_0 = S_0 \, e^{(r-q)T}$$

where q is the average yield during the life of the contract (expressed with continuous compounding)

Valuing a Forward Contract

- A forward contract is worth zero (except for bid-offer spread effects) when it is first negotiated
- Later it may have a positive or negative value
- Suppose that K is the delivery price and F_0 is the forward price for a contract that would be negotiated today

Valuing a Forward Contract (pages 112-114)

- By considering the difference between a contract with delivery price K and a contract with delivery price F_0 we can deduce that:
  - the value of a long forward contract is
    $$(F_0 - K)e^{-rT}$$
  - the value of a short forward contract is
    $$(K - F_0)e^{-rT}$$

Forward vs Futures Prices

- When the maturity and asset price are the same, forward and futures prices are usually assumed to be equal. (Eurodollar futures are an exception)
- In theory, when interest rates are uncertain, they are slightly different:
  - A strong positive correlation between interest rates and the asset price implies the futures price is slightly higher than the forward price
  - A strong negative correlation implies the reverse

Stock Index *(Page 115-117)*

- Can be viewed as an investment asset paying a dividend yield
- The futures price and spot price relationship is therefore

$$F_0 = S_0 \, e^{(r-q)T}$$

where q is the average dividend yield on the portfolio represented by the index during life of contract

Stock Index *(continued)*

- For the formula to be true it is important that the index represent an investment asset
- In other words, changes in the index must correspond to changes in the value of a tradable portfolio
- The Nikkei index viewed as a dollar number does not represent an investment asset (See Business Snapshot 5.3, page 116)

Index Arbitrage

- When $F_0 > S_0e^{(r\text{-}q)T}$ an arbitrageur buys the stocks underlying the index and sells futures
- When $F_0 < S_0e^{(r\text{-}q)T}$ an arbitrageur buys futures and shorts or sells the stocks underlying the index

Index Arbitrage
(continued)

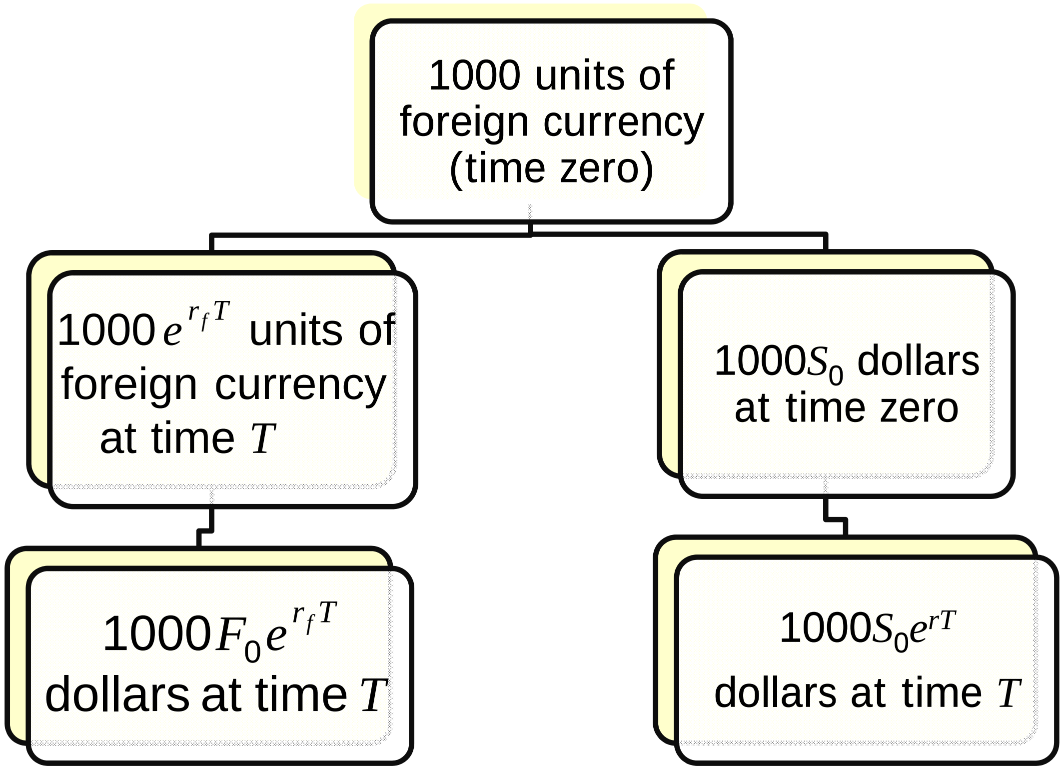
- Index arbitrage involves simultaneous trades in futures and many different stocks
- Very often a computer is used to generate the trades
- Occasionally simultaneous trades are not possible and the theoretical no-arbitrage relationship between F_0 and S_0 does not hold (see Business Snapshot 5.4 on page 117)

Futures and Forwards on Currencies *(Page 117-120)*

- A foreign currency is analogous to a security providing a yield
- The yield is the foreign risk-free interest rate
- It follows that if r_f is the foreign risk-free interest rate

$$F_0 = S_0 e^{(r - r_f)T}$$

Explanation of the Relationship Between Spot and Forward (Figure 5.1)



Consumption Assets: Storage is Negative Income

$$F_0 \leq S_0 \, e^{(r+u)T}$$

where u is the storage cost per unit time as a percent of the asset value.

Alternatively,

$$F_0 \leq (S_0 + U)e^{rT}$$

where U is the present value of the storage costs.

The Cost of Carry *(Page 123)*

- The cost of carry, c, is the storage cost plus the interest costs less the income earned
- For an investment asset $F_0 = S_0 e^{cT}$
- For a consumption asset $F_0 \leq S_0 e^{cT}$
- The convenience yield on the consumption asset, y, is defined so that
  $F_0 = S_0 \, e^{(c-y)T}$

Futures Prices & Expected Future Spot Prices *(Page 124-126)*

- Suppose k is the expected return required by investors in an asset
- We can invest $F_0 e^{-rT}$ at the risk-free rate and enter into a long futures contract to create a cash inflow of S_T at maturity
- This shows that

$$F_0 e^{-rT} e^{kT} = E(S_T)$$

or

$$F_0 = E(S_T) e^{(r-k)T}$$

Futures Prices & Future Spot Prices (continued)

No Systematic Risk	$k = r$	$F_0 = E(S_T)$
Positive Systematic Risk	$k > r$	$F_0 < E(S_T)$
Negative Systematic Risk	$k < r$	$F_0 > E(S_T)$

Positive systematic risk: stock indices
Negative systematic risk: gold (at least for some periods)

Chapter 6
Interest Rate Futures

Day Count Convention

- Defines:
  - the period of time to which the interest rate applies
  - The period of time used to calculate accrued interest (relevant when the instrument is bought of sold

Day Count Conventions in the U.S. *(Page 132)*

Treasury Bonds:	Actual/Actual (in period)
Corporate Bonds:	30/360
Money Market Instruments:	Actual/360

Examples

- Bond: 8% Actual/ Actual in period.
  - 4% is earned between coupon payment dates. Accruals on an Actual basis. When coupons are paid on March 1 and Sept 1, how much interest is earned between March 1 and April 1?
- Bond: 8% 30/360
  - Assumes 30 days per month and 360 days per year. When coupons are paid on March 1 and Sept 1, how much interest is earned between March 1 and April 1?

Examples *continued*

- T-Bill: 8% Actual/360:
  - 8% is earned in 360 days. Accrual calculated by dividing the actual number of days in the period by 360. How much interest is earned between March 1 and April 1?

The February Effect *(Business Snapshot 6.1)*

- How many days of interest are earned between February 28, 2015 and March 1, 2015 when
  - day count is Actual/Actual in period?
  - day count is 30/360?

Treasury Bill Prices in the US

$$P = \frac{360}{n}(100 - Y)$$

Y is cash price per \$100

P is quoted price

Treasury Bond Price Quotes in the U.S

Cash price = Quoted price +
Accrued Interest

Treasury Bond Futures

Pages 135-140

Cash price received by party with short position =

Most recent settlement price × Conversion factor + Accrued interest

Example

- Most recent settlement price = 90.00
- Conversion factor of bond delivered = 1.3800
- Accrued interest on bond =3.00
- Price received for bond is 1.3800×90.00+3.00 = $127.20 per $100 of principal

Conversion Factor

The conversion factor for a bond is approximately equal to the value of the bond on the assumption that the yield curve is flat at 6% with semiannual compounding

CBOT T-Bonds & T-Notes

Factors that affect the futures price:

- Delivery can be made any time during the delivery month
- Any of a range of eligible bonds can be delivered
- The wild card play

Eurodollar Futures *(Page 140-145)*

- A Eurodollar is a dollar deposited in a bank outside the United States
- Eurodollar futures are futures on the 3-month Eurodollar deposit rate (same as 3-month LIBOR rate)
- One contract is on the rate earned on $1 million
- A change of one basis point or 0.01 in a Eurodollar futures quote corresponds to a contract price change of $25

Eurodollar Futures continued

- A Eurodollar futures contract is settled in cash
- When it expires (on the third Wednesday of the delivery month) the final settlement price is 100 minus the actual three month Eurodollar deposit rate

Example

Date	Quote
Nov 1	97.12
Nov 2	97.23
Nov 3	96.98
…….	……
Dec 21	97.42

Example

- Suppose you buy (take a long position in) a contract on November 1
- The contract expires on December 21
- The prices are as shown
- How much do you gain or lose a) on the first day, b) on the second day, c) over the whole time until expiration?

Example *continued*

- If on Nov. 1 you know that you will have $1 million to invest on for three months on Dec 21, the contract locks in a rate of

  100 - 97.12 = 2.88%

- In the example you earn 100 – 97.42 = 2.58% on $1 million for three months (=$6,450) and make a gain day by day on the futures contract of 30×$25 =$750

Formula for Contract Value *(equation 6.2, page 141)*

- If Q is the quoted price of a Eurodollar futures contract, the value of one contract is $10{,}000[100\text{-}0.25(100\text{-}Q)]$
- This corresponds to the \$25 per basis point rule

Forward Rates and Eurodollar Futures *(Page 143-145)*

- Eurodollar futures contracts last as long as 10 years
- For Eurodollar futures lasting beyond two years we cannot assume that the forward rate equals the futures rate

There are Two Reasons

- Futures is settled daily whereas forward is settled once
- Futures is settled at the beginning of the underlying three-month period; FRA is settled at the end of the underlying three- month period

Forward Rates and Eurodollar Futures continued

- A "convexity adjustment" often made is

  Forward Rate = Futures Rate$-0.5\sigma^2 T_1 T_2$

  - T_1 is the start of period covered by the forward/futures rate
  - T_2 is the end of period covered by the forward/futures rate (90 days later that T_1)
- σ is the standard deviation of the change in the short rate per year

Convexity Adjustment when σ=0.012 *(page 144)*

Maturity of Futures (yrs)	Convexity Adjustment (bps)
2	3.2
4	12.2
6	27.0
8	47.5
10	73.8

Extending the LIBOR Zero Curve

- LIBOR deposit rates define the LIBOR zero curve out to one year
- Eurodollar futures can be used to determine forward rates and the forward rates can then be used to bootstrap the zero curve

Example *(page 144-145)*

$$F = \frac{R_2 T_2 - R_1 T_1}{T_2 - T_1}$$

so that

$$R_2 = \frac{F(T_2 - T_1) + R_1 T_1}{T_2}$$

If the 400-day LIBOR zero rate has been calculated as 4.80% and the forward rate for the period between 400 and 491 days is 5.30 the 491 day rate is 4.893%

Duration Matching

- This involves hedging against interest rate risk by matching the durations of assets and liabilities
- It provides protection against small parallel shifts in the zero curve

Use of Eurodollar Futures

- One contract locks in an interest rate on $1 million for a future 3-month period
- How many contracts are necessary to lock in an interest rate on $1 million for a future six-month period?

Duration-Based Hedge Ratio

$$\frac{PD_P}{V_F D_F}$$

V_F	Contract price for interest rate futures
D_F	Duration of asset underlying futures at maturity
P	Value of portfolio being hedged
D_P	Duration of portfolio at hedge maturity

Example

- It is August. A fund manager has \$10 million invested in a portfolio of government bonds with a duration of 6.80 years and wants to hedge against interest rate moves between August and December
- The manager decides to use December T-bond futures. The futures price is 93-02 or 93.0625 and the duration of the cheapest to deliver bond will be 9.2 years at the futures contract maturity
- The number of contracts that should be shorted is

$$\frac{10{,}000{,}000}{93{,}062.50} \times \frac{6.80}{9.20} = 79$$

Limitations of Duration-Based Hedging

- Assumes that only parallel shift in yield curve take place
- Assumes that yield curve changes are small
- When T-Bond futures is used assumes there will be no change in the cheapest-to-deliver bond

GAP Management *(Business Snapshot 6.3)*

This is a more sophisticated approach used by banks to hedge interest rate. It involves

- Bucketing the zero curve
- Hedging exposure to situation where rates corresponding to one bucket change and all other rates stay the same

Liquidity Risk

- If a bank funds long term assets with short term liabilities such as commercial paper, it can use FRAs, futures, and swaps to hedge its interest rate exposure
- But it still has a liquidity exposure.
- It may find it impossible to roll over the commercial paper if the market loses confidence in the bank
- Northern Rock is an example of this tyep of liquidity problem

Chapter 7
Swaps

Nature of Swaps

A swap is an agreement to exchange cash flows at specified future times according to certain specified rules

An Example of a "Plain Vanilla" Interest Rate Swap

- An agreement by Microsoft to receive 6-month LIBOR & pay a fixed rate of 5% per annum every 6 months for 3 years on a notional principal of $100 million
- Next slide illustrates cash flows that could occur (Day count conventions are not considered)

One Possible Outcome for Cash Flows to Microsoft *(Table 7.1, page 155)*

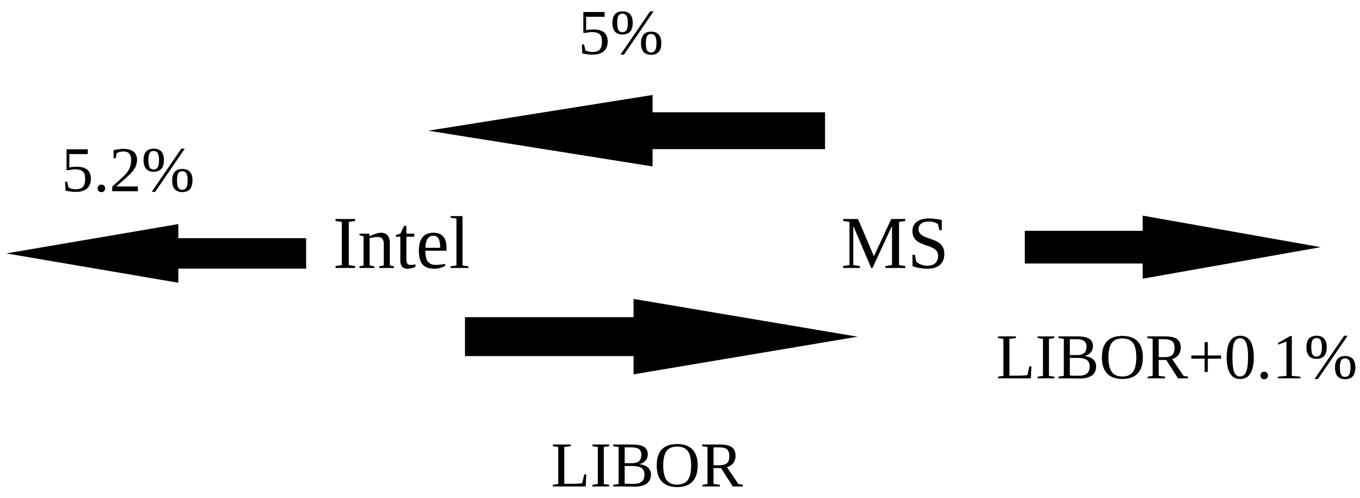
Date	LIBOR	Floating Cash Flow	Fixed Cash Flow	Net Cash Flow
Mar 5, 2014	4.20%			
Sep 5, 2014	4.80%	+2.10	−2.50	−0.40
Mar 5, 2015	5.30%	+2.40	−2.50	−0.10
Sep 5, 2015	5.50%	+2.65	−2.50	+ 0.15
Mar 5, 2016	5.60%	+2.75	−2.50	+0.25
Sep 5, 2016	5.90%	+2.80	−2.50	+0.30
Mar 5, 2017		+2.95	−2.50	+0.45

Typical Uses of an Interest Rate Swap

- Converting a liability from
  - fixed rate to floating rate
  - floating rate to fixed rate

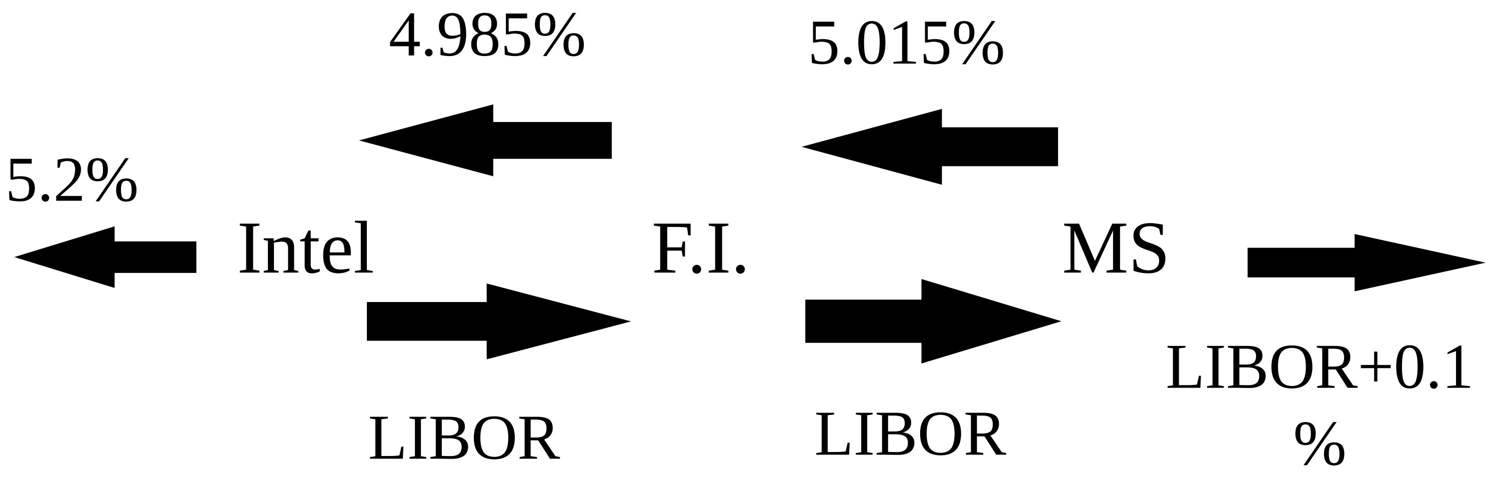
- Converting an investment from
  - fixed rate to floating rate
  - floating rate to fixed rate

Intel and Microsoft (MS) Transform a Liability *(Figure 7.2, page 155)*



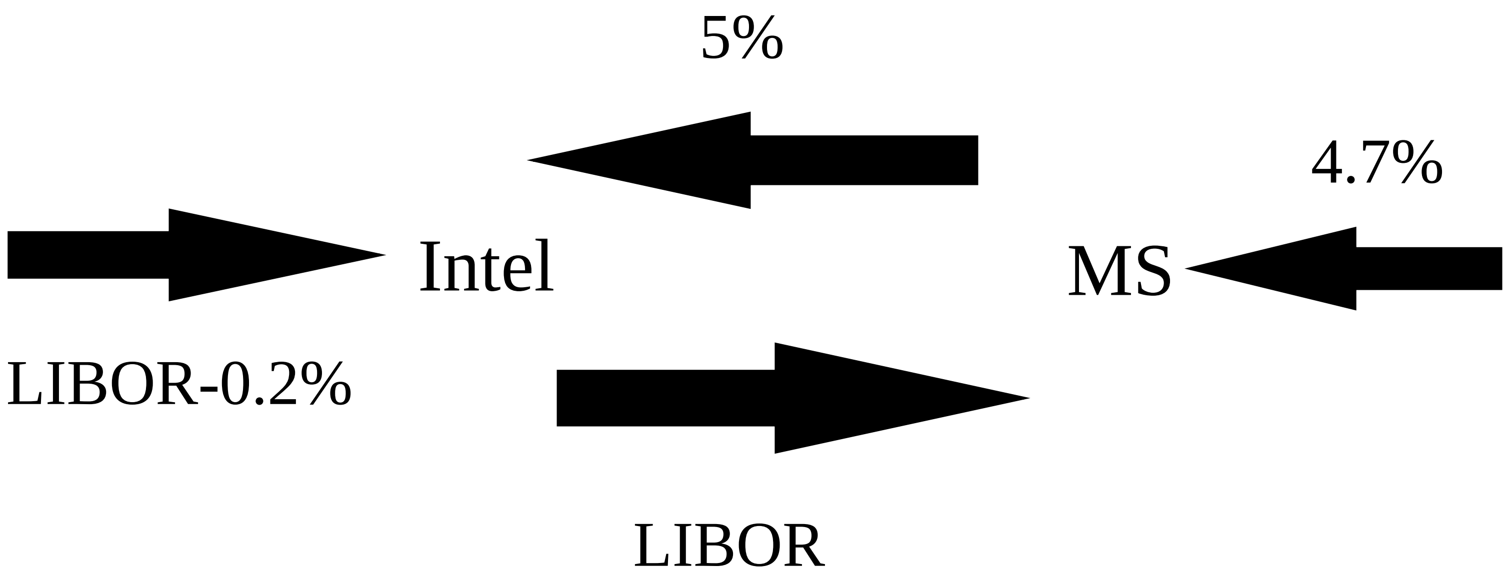
Financial Institution is Involved

(Figure 7.4, page 157)



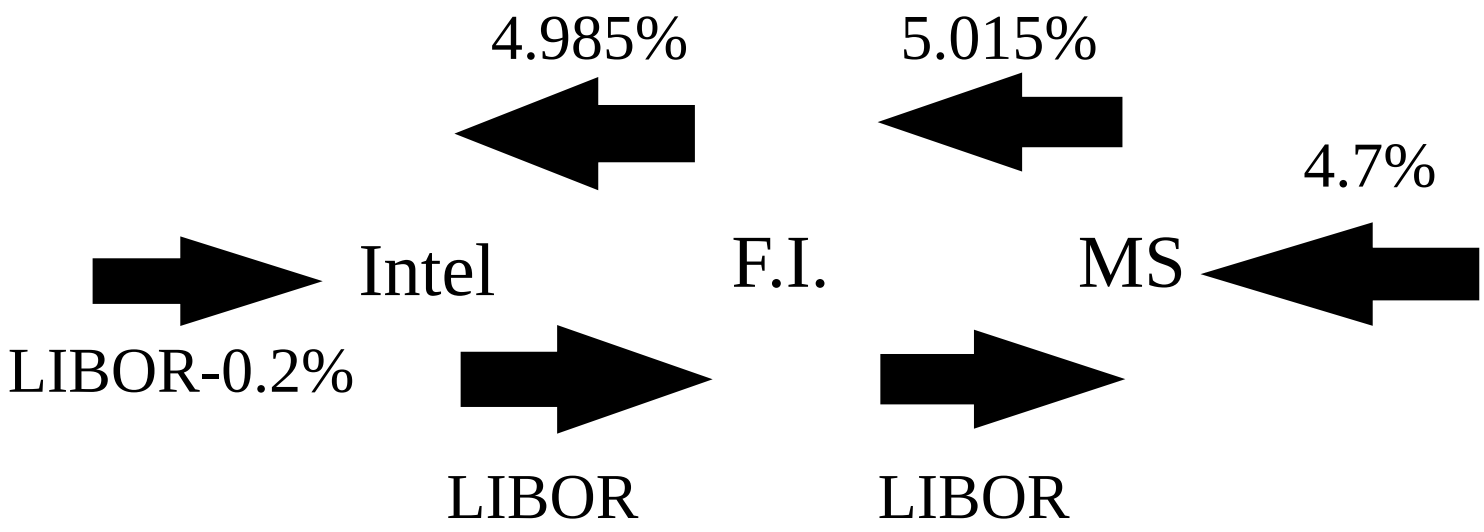
Financial Institution has two offsetting swaps

Intel and Microsoft (MS) Transform an Asset (Figure 7.3, page 156)



Financial Institution is Involved

(See Figure 7.5, page 157)



Quotes By a Swap Market Maker

(Table 7.3, page 158)

Maturity	Bid (%)	Offer (%)	Swap Rate (%)
2 years	6.03	6.06	6.045
3 years	6.21	6.24	6.225
4 years	6.35	6.39	6.370
5 years	6.47	6.51	6.490
7 years	6.65	6.68	6.665
10 years	6.83	6.87	6.850

Day Count

- A day count convention is specified for for fixed and floating payment
- For example, LIBOR is likely to be actual/360 in the US because LIBOR is a money market rate

Confirmations

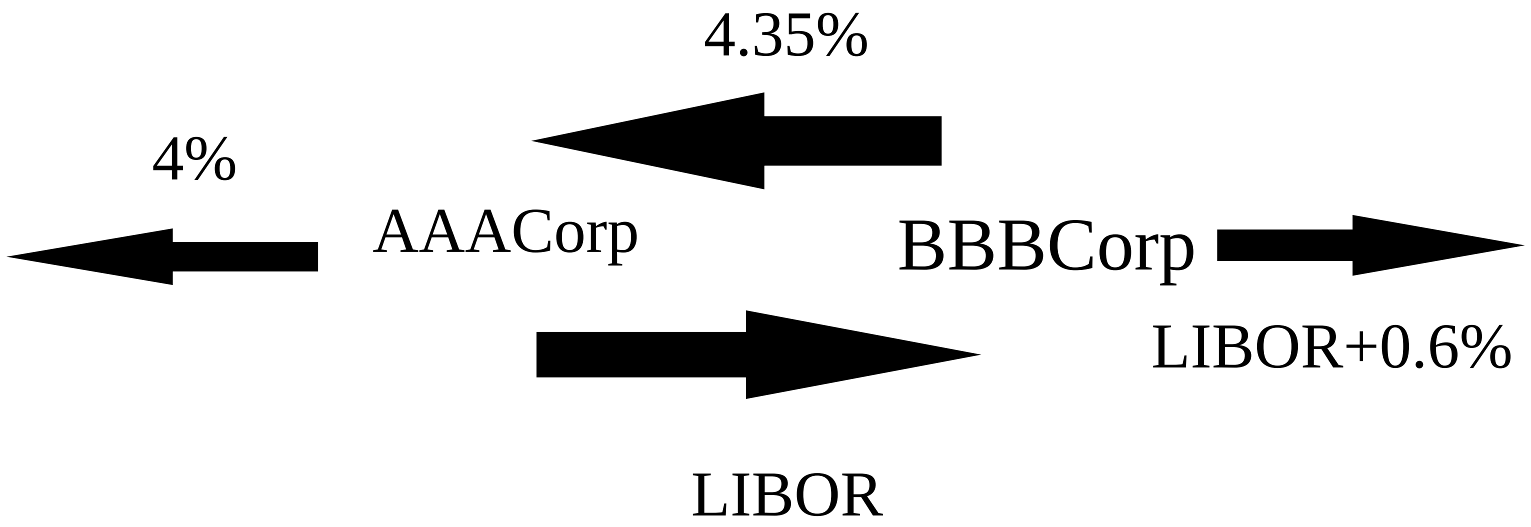
- Confirmations specify the terms of a transaction
- The International Swaps and Derivatives has developed Master Agreements that can be used to cover all agreements between two counterparties
- Many interest rate swaps are now cleared through a CCP such as LCH Clearnet or the CME Group

The Comparative Advantage Argument
(Table 7.4, page 160)

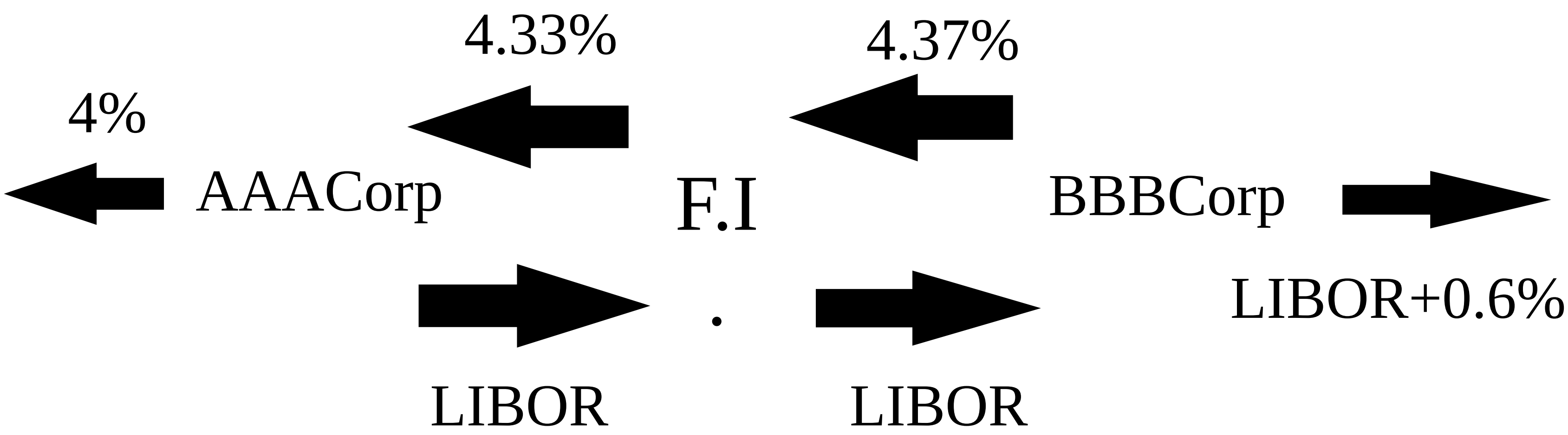
- AAACorp wants to borrow floating
- BBBCorp wants to borrow fixed

	Fixed	Floating
AAACorp	4.0%	6 month LIBOR − 0.1%
BBBCorp	5.2%	6 month LIBOR + 0.6%

The Swap *(Figure 7.6, page 161)*



The Swap when a Financial Institution is Involved *(Figure 7.7, page 162)*



Criticism of the Comparative Advantage Argument

- The 4.0% and 5.2% rates available to AAACorp and BBBCorp in fixed rate markets are 5-year rates
- The LIBOR−0.1% and LIBOR+0.6% rates available in the floating rate market are six-month rates
- BBBCorp's fixed rate depends on the spread above LIBOR it borrows at in the future

The Nature of Swap Rates

- Six-month LIBOR is a short-term AA borrowing rate
- The 5-year swap rate has a risk corresponding to the situation where 10 six-month loans are made to AA borrowers at LIBOR
- This is because the lender can enter into a swap where income from the LIBOR loans is exchanged for the 5-year swap rate

The Discount Rate

- Pre-crisis derivatives cash flows were discounted at LIBOR
- As Chapter 9 explains, this has changed
- Here we illustrate the valuation methodology by assuming that LIBOR is the discount rate

Using Swap Rates to Bootstrap the LIBOR/Swap Zero Curve

- Consider a new swap where the fixed rate is the swap rate
- When principals are added to both sides on the final payment date the swap is the exchange of a fixed rate bond for a floating rate bond
- The floating-rate rate bond is worth par assuming LIBOR discounting is used. The swap is worth zero. The fixed-rate bond must therefore also be worth par
- This shows that swap rates define par yield bonds that can be used to bootstrap the LIBOR (or LIBOR/swap) zero curve

Example of Bootstrapping the LIBOR/Swap Curve *(Example 7.1, page 164)*

- 6-month, 12-month, and 18-month LIBOR/swap rates are 4%, 4.5%, and 4.8% with continuous compounding.
- Two-year swap rate is 5% (semiannual)

$$2.5e^{-0.04\times 0.5}+2.5e^{-0.045\times 1.0}+2.5e^{-0.048\times 1.5}+102.5e^{-2R}=100$$

- The 2-year LIBOR/swap rate, R, is 4.953%

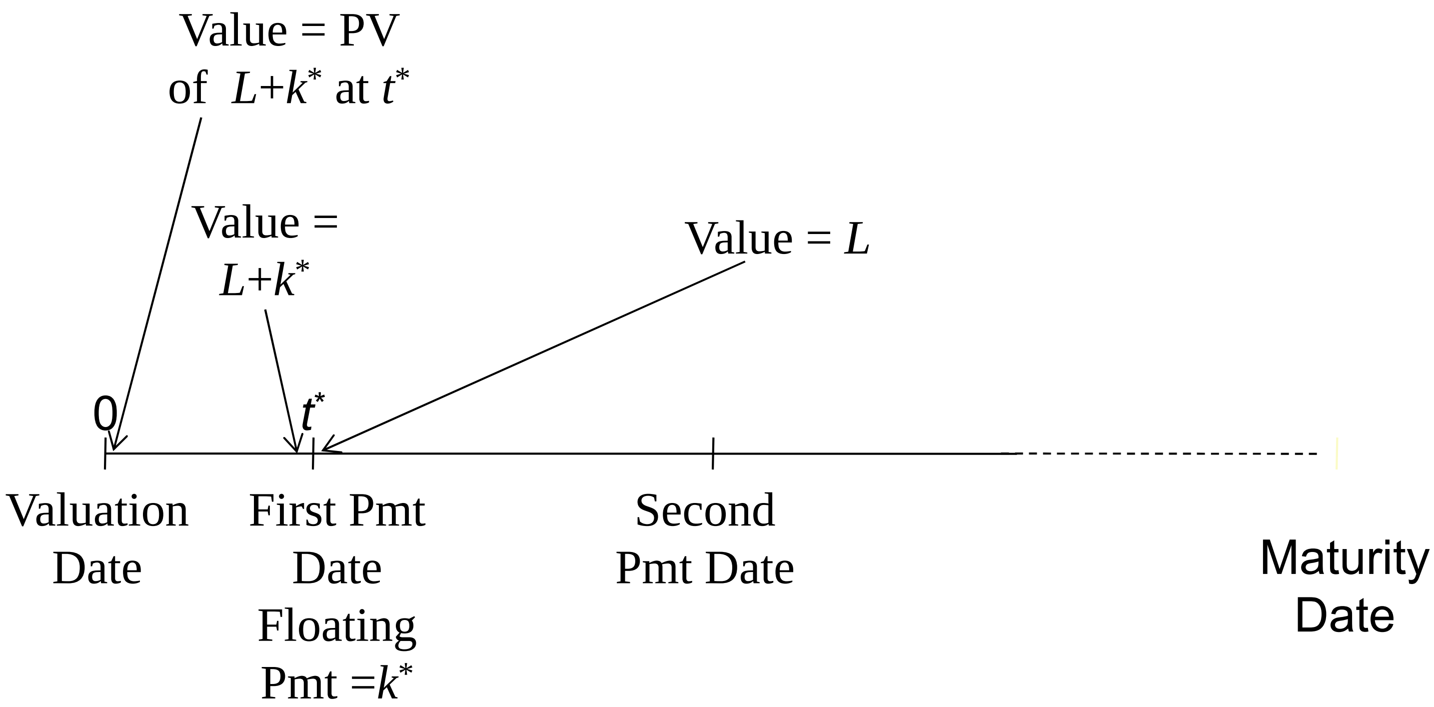
Valuation of an Interest Rate Swap

- Initially interest rate swaps are worth close to zero
- At later times they can be valued as the difference between the value of a fixed-rate bond and the value of a floating-rate bond
- Alternatively, they can be valued as a portfolio of forward rate agreements (FRAs)

Valuation in Terms of Bonds

- The fixed rate bond is valued in the usual way
- The floating rate bond is valued by noting that it is worth par immediately after the next payment date

Valution of Floating-Rate Bond



Example

- Receive six-month LIBOR, pay 3% (s.a. compounding) on a principal of $100 million
- Remaining life 1.25 years
- LIBOR zero rates for 3-months, 9-months and 15-months are 2.8%, 3%, and 3.2% (cont comp)
- 6-month LIBOR on last payment date was 12.9% (s.a. compounding)

Valuation Using Bonds *(page 166)*

Time	B_{fix} cash flow	B_{fl} cash flow	Disc factor	PV B_{fix}	PV B_{fl}
0.25	1.5	101.450	0.9930	1.4895	100.7423
0.75	1.5		0.9763	1.4644	
1.25	101.5		0.9584	97.2766	
Total				100.2306	100.7423

Swap value = 100.7423 − 100.2306 = 0.5117

Valuation in Terms of FRAs

- Each exchange of payments in an interest rate swap is an FRA
- The FRAs can be valued on the assumption that today's forward rates are realized

Valuation of Example Using FRAs
(page 167)

Time	Fixed cash flow	Floating cash flow	Net Cash Flow	Disc factor	PV B_{fl}
0.25	−1.5000	+1.4500	−0.0050	0.9930	−0.0497
0.75	−1.5000	+1.7145	+0.2145	0.9763	+0.2094
1.25	−1.5000	+1.8672	+0.3672	0.9584	+0.3519
Total					+0.5117

An Example of a Currency Swap

An agreement to pay 5% on a sterling principal of £10,000,000 & receive 6% on a US$ principal of $15,000,000 every year for 5 years

Exchange of Principal

- In an interest rate swap the principal is not exchanged
- In a currency swap the principal is usually exchanged at the beginning and the end of the swap's life

The Cash Flows *(Table 7.7, page 170)*

Date	Dollar Cash Flows (millions)	Sterling cash flow (millions)
Feb 1, 2014	−15.00	+10.0
Feb 1, 2015	+0.90	−0.50
Feb 1, 2016	+0.90	−0.50
Feb 1, 2017	+0.90	−0.50
Feb 1, 2018	+0.90	−0.50
Feb 1, 2019	+15.90	−10.50

Typical Uses of a Currency Swap

- Convert a liability in one currency to a liability in another currency
- Convert an investment in one currency to an investment in another currency

Comparative Advantage May Be Real Because of Taxes

- General Electric wants to borrow AUD
- Quantas wants to borrow USD
- Costs after adjusting for the differential impact of taxes:

	USD	AUD
General Electric	5.0%	7.6%
Quantas	7.0%	8.0%

Valuation of Currency Swaps

Like interest rate swaps, currency swaps can be valued either as the difference between 2 bonds or as a portfolio of forward contracts

Example

- All Japanese LIBOR/swap rates are 4%
- All USD LIBOR/swap rates are 9%
- 5% is received in yen; 8% is paid in dollars. Payments are made annually
- Principals are $10 million and 1,200 million yen
- Swap will last for 3 more years
- Current exchange rate is 110 yen per dollar

Valuation in Terms of Bonds *(Table 7.9, page 173)*

Time	Cash Flows ($)	PV ($)	Cash flows (yen)	PV (yen)
1	0.8	0.7311	60	57.65
2	0.8	0.6682	60	55.39
3	0.8	0.6107	60	53.22
3	10.0	7.6338	1,200	1,064.30
Total		9.6439		1,230.55

Value of Swap = 1230.55/110 − 9.6439 = 1.5430

Valuation in Terms of Forwards

(Table 7.10, page 174)

Time	$ cash flow	Yen cash flow	Forward Exch rate	Yen cash flow in $	Net Cash Flow	Present value
1	-0.8	60	0.009557	0.5734	-0.2266	-0.2071
2	-0.8	60	0.010047	0.6028	-0.1972	-0.1647
3	-0.8	60	0.010562	0.6337	-0.1663	-0.1269
3	-10.0	1200	0.010562	12.6746	+2.6746	2.0417
Total						1.5430

Swaps & Forwards

- A swap can be regarded as a convenient way of packaging forward contracts
- Although the swap contract is usually worth close to zero at the outset, each of the underlying forward contracts are not worth zero

Credit Risk: Single Uncollateralized Transaction with Counterparty

- A swap is worth zero to a company initially
- At a future time its value is liable to be either positive or negative
- The company has credit risk exposure only when ithe value is positive
- Some swaps are more likely to lead to credit risk exposure than others
- What is the situation if early forward rates have a positive value?
- What is the situation when the early forward rates have a negative value?

Other Types of Swaps

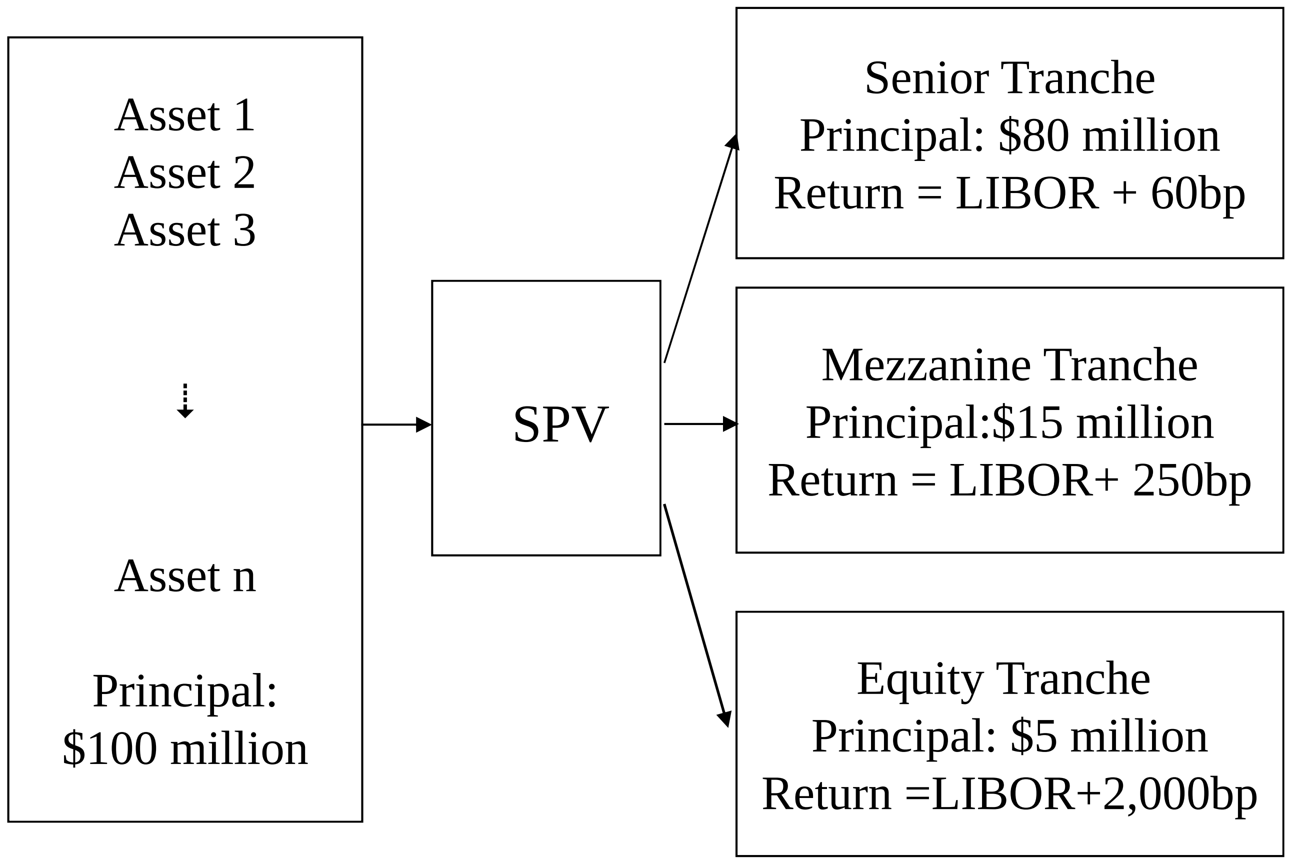
- Floating-for-floating interest rate swaps
- amortizing swaps
- step up swaps
- forward swaps
- constant maturity swaps
- compounding swaps
- LIBOR-in-arrears swaps
- accrual swaps
- diff swaps
- cross currency interest rate swaps
- equity swaps
- extendable swaps
- puttable swaps
- swaptions
- commodity swaps
- volatility swaps
- etc etc

Chapter 8
Securitization and the Credit Crisis of 2007

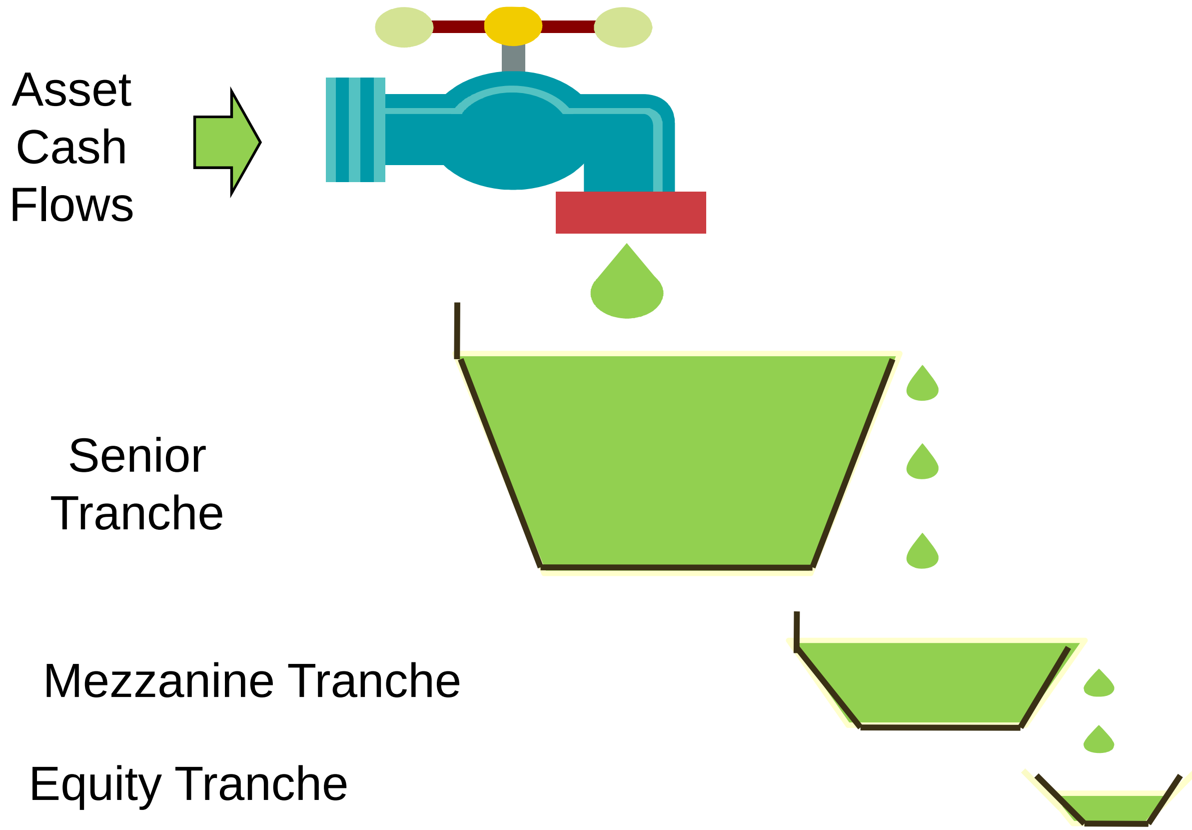
Securitization

- Traditionally banks have funded loans with deposits
- Securitization is a way that loans can increase much faster than deposits

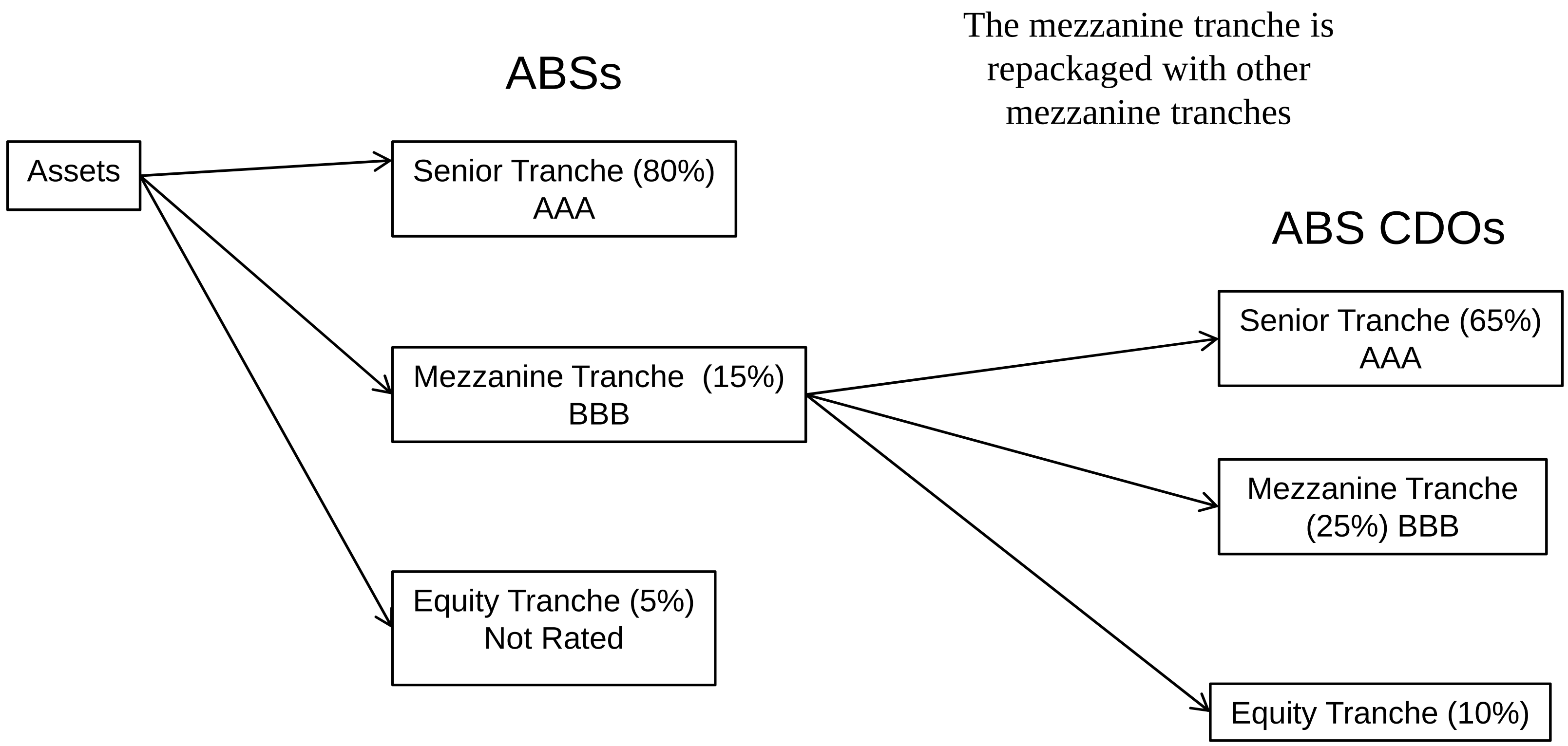
Asset Backed Security (Simplified)



The Waterfall



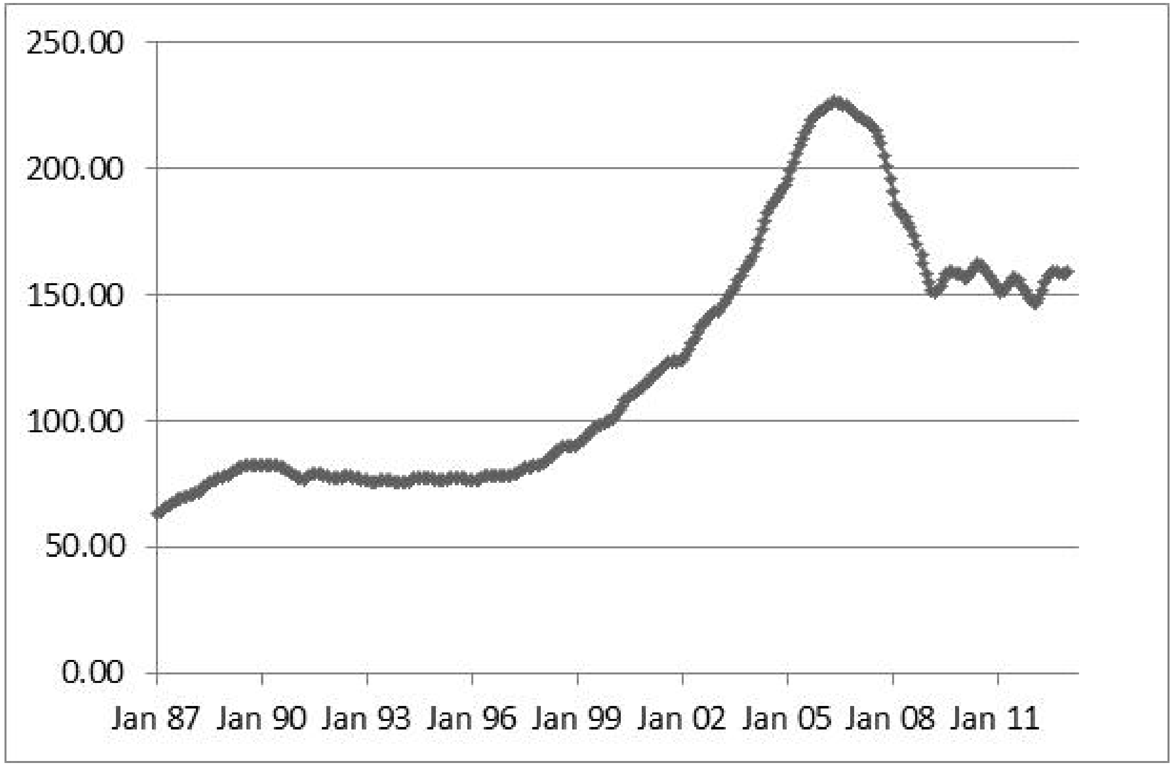
ABS CDOs or Mezz CDOs (Simplified)



Losses to AAA Senior Tranche of ABS CDO (Table 8.1, page 189)

Losses on Subprime portfolios	Losses on Mezzanine Tranche of ABS	Losses on Equity Tranche of ABS CDO	Losses on Mezzanine Tranche of ABS CDO	Losses on Senior Tranche of ABS CDO
10%	33.3%	100%	93.3%	0%
13%	53.3%	100%	100%	28.2%
17%	80.0%	100%	100%	69.2%
20%	100%	100%	100%	100%

U.S. Real Estate Prices, 1987 to 2013:
S&P/Case-Shiller Composite-10 Index



What happened…

- Starting in 2000, mortgage originators in the US relaxed their lending standards and created large numbers of subprime first mortgages.
- This, combined with very low interest rates, increased the demand for real estate and prices rose.
- To continue to attract first time buyers and keep prices increasing they relaxed lending standards further
- Features of the market: 100% mortgages, ARMs, teaser rates, NINJAs, liar loans, non-recourse borrowing
- Mortgages were packaged in financial products and sold to investors

What happened...

- Banks found it profitable to invest in the AAA rated tranches because the promised return was significantly higher than the cost of funds and capital requirements were low
- In 2007 the bubble burst. Some borrowers could not afford their payments when the teaser rates ended. Others had negative equity and recognized that it was optimal for them to exercise their put options.
- Foreclosures increased supply and caused U.S. real estate prices to fall. Products, created from the mortgages, that were previously thought to be safe began to be viewed as risky
- There was a “flight to quality” and credit spreads increased to very high levels
- Many banks incurred huge losses

What Many Market Participants Did Not Realize…

- Default correlation goes up in stressed market conditions
- Recovery rates are less in stressed market conditions
- A tranche with a certain rating cannot be equated with a bond with the same rating. For example, the BBB tranches used to create ABS CDOs were typically about 1% wide and had “all or nothing” loss distributions (quite different from BBB bond)
- This is quite different from the loss distribution for a BBB bond from a BBB bond

Regulatory Arbitrage

- The regulatory capital banks were required to keep for the tranches created from mortgages was less than that for the mortgages themselves

Incentives

- The crisis highlighted what are referred to as agency costs
  - Mortgage originators (Their prime interest was in in originating mortgages that could be securitized)
  - Valuers (They were under pressure to provide high valuations so that the loan-to-value ratios looked good)
  - Traders (They were focused on the next end-of year bonus and not worried about any longer term problems in the market)

The Aftermath…

- A huge amount of new regulation (Basel II.5, Basel III, Dodd-Frank, etc). For example:
  - Banks required to hold more equity capital with the definition of equity capital being tightened
  - Banks required to satisfy liquidity ratios
  - CCPs and SEFs for OTC derivatives
  - Bonuses limited in Europe
  - Bonuses spread over several years
  - Proprietary trading restricted

Chapter 9
OIS Discounting, Credit Issues, and Funding Costs

Treasury Rates

- Treasury rates are lower than other very low risk rates because
  - Treasury instruments must often be held by financial institutions for regulatory purposes
  - Treasury instruments require no capital
  - Treasury instruments have favorable tax treatment in the US because they are not taxed at the state level
- As a result Treasury rates are not a used by derivatives dealers as a proxy for the risk-free rate

The "Risk-Free" Discount Rate

- A risk-free discount rate is in theory necessary to value derivatives
- LIBOR and swap rates have traditionally been used as proxies for risk-free rates by derivatives dealers
- During the crisis banks were reluctant to lend to each other and LIBOR soared
- As a result, practices in the market have changed
- For collateralized transactions derivatives dealers now use the OIS rate as the discount rate (It is argued that collateralized transactions are funded by the collateral)
- For non-collateralized transactions a rate reflecting the bank's funding cost is often used

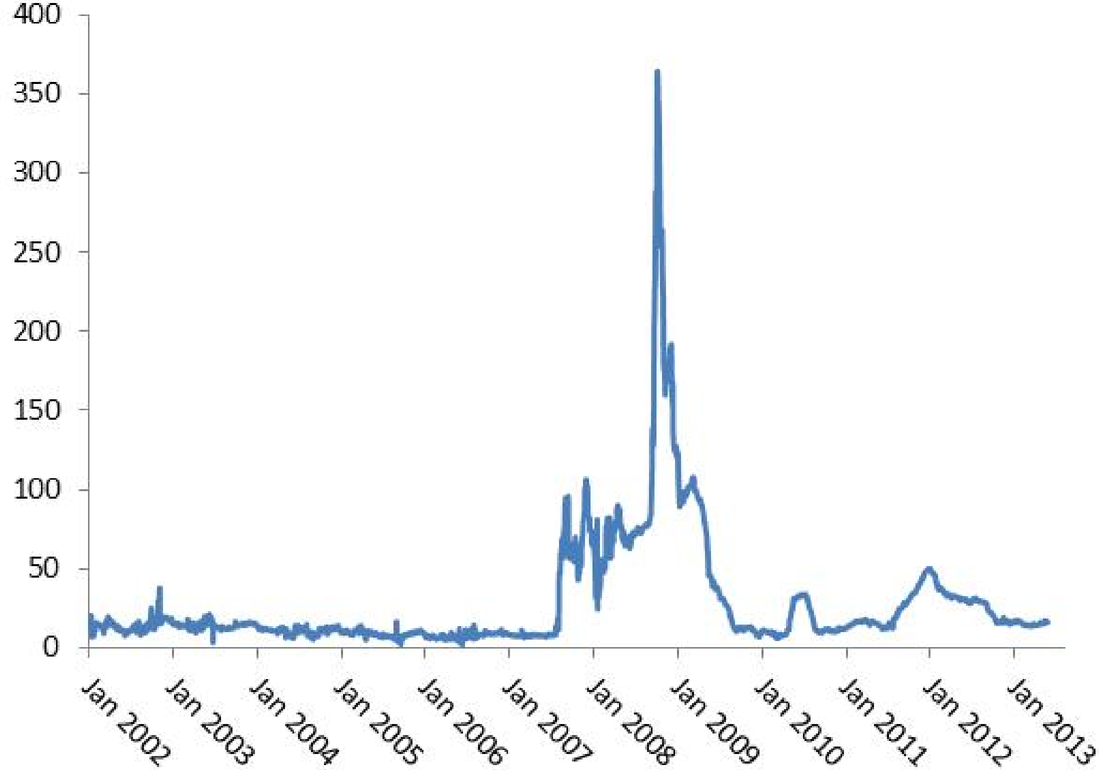
OIS Rates

- In an overnight indexed swap a fixed rate for a period is exchanged for the geometric average of the overnight rates
- Should OIS rate equal the LIBOR rate? A bank can
  - Borrow $100 million in the overnight market, rolling forward for 3 months
  - Enter into an OIS swap to convert this to the 3-month OIS rate
  - Lend the funds to another bank at LIBOR for 3 months

Overnight Indexed Swaps continued

- ...but it bears the credit risk of another bank in this arrangement
- The OIS rate is therefore less than the corresponding LIBOR rate
- The excess of LIBOR over the OIS rate is the LIBOR-OIS spread. It is usually about 10 basis points but spiked at an all time high of 364 basis points in October 2008

The 3 Month LIBOR-OIS Spread



The OIS Zero Curve

- When OIS discounting is used, it is necessary to determine the OIS zero curve
- This can be bootstrapped from OIS rates in the same way that the LIBOR/swap zero curve is bootstrapped from quotes for LIBOR-for-fixed swaps
- When long maturity OIS swaps do not exist, it is necessary to make an assumption about the spread between OIS swap rates and LIBOR-for-fixed swap rates

Swap Valuation

- LIBOR-for-fixed interest rate swaps are always valued on the assumption that forward LIBOR rates are realized
- But forward LIBOR rates depend on whether OIS or LIBOR discounting is used

Example

- One-year LIBOR rate is 5%
- Two year LIBOR-for-fixed swap rate is 6%
- Both rates are annually compounded

If LIBOR is used for discounting…

- The two year LIBOR/swap zero rate is R where

$$\frac{6}{1.05}+\frac{106}{(1+R)^2}=100$$

so that R = 6.030%

- The forward LIBOR rate for the second year is

$$\frac{1.0630^2}{1.05}-1=7.0707\%$$

If LIBOR is used for discounting *continued*

- Check:
  - When the forward rate is 7.0707 and LIBOR discounting is used the two-year swap has a value of

$$\frac{6-5}{1.05}+\frac{6-7.0707}{1.6030^2}=0$$

If OIS Discounting Is Used….

- Assume that OIS zero rates for one and two years have been bootstrapped as 4.5% and 5.5% with annual compounding
- If F is the forward LIBOR rate for the second year then

$$\frac{6-5}{1.045}+\frac{6-F}{1.055^2}=0$$

so that F is 7.0651%

Swap Valuation with OIS Discounting

- Forward LIBOR rates are calculated as for this simple example
- Typically, 1-month, 3-month, 6-month and 12-month forward rates are calculated separately from the corresponding swap quotes.
- Interpolation is used to calculate the forward rates to value a particular existing swap.

Valuing Bilaterally Cleared Derivatives Portfolios

- A dealer first calculates the value of a portfolio of derivatives with a counterparty assuming neither side will default
- It then reduces the value of the portfolio to reflect possible losses from a counterparty default. This is the credit value adjustment (CVA)
- It then increases the value to reflect possible gains from a default by itself. This is the debit value adjustment (DVA)

Valuing Bilaterally Cleared Derivatives Portfolios *continued*

- Value after credit adjustments is:

  No-default value − CVA + DVA

- CVA and DVA adjustments should reflect collateral arrangements

The CVA Calculation

Time	0	t_1	t_2	t_3	t_4	…………………	$t_n = T$
Counterparty default probability	q_1	q_2	q_3	q_4		…………………	q_n
PV of dealer's loss given default	v_1	v_2	v_3	v_4		…………………	v_n

$$\text{CVA} = \sum_{i=1}^{n} q_i v_i$$

The DVA Calculation

Time	0	t_1	t_2	t_3	t_4	………………… t_n=T
Dealer default probability		q_1^*	q_2^*	q_3^*	q_4^*	………………… q_n^*
PV of counterparty's loss given default		v_1^*	v_2^*	v_3^*	v_4^*	………………… v_n^*

$$\text{DVA} = \sum_{i=1}^{n} q_i^* v_i^*$$

Funding Costs

- Many dealers argue that they should take funding costs into account when non-collateralized transactions are valued
- This leads to what is known as a funding value adjustment (FVA)
- An FVA cannot be justified theoretically…
  - What discount rate should a company use for an investment in a Treasury bond.
  - The discount rate for a project undertaken by a company should reflect the project’s risk not the company’s funding cost

Chapter 10
Mechanics of Options Markets

Review of Option Types

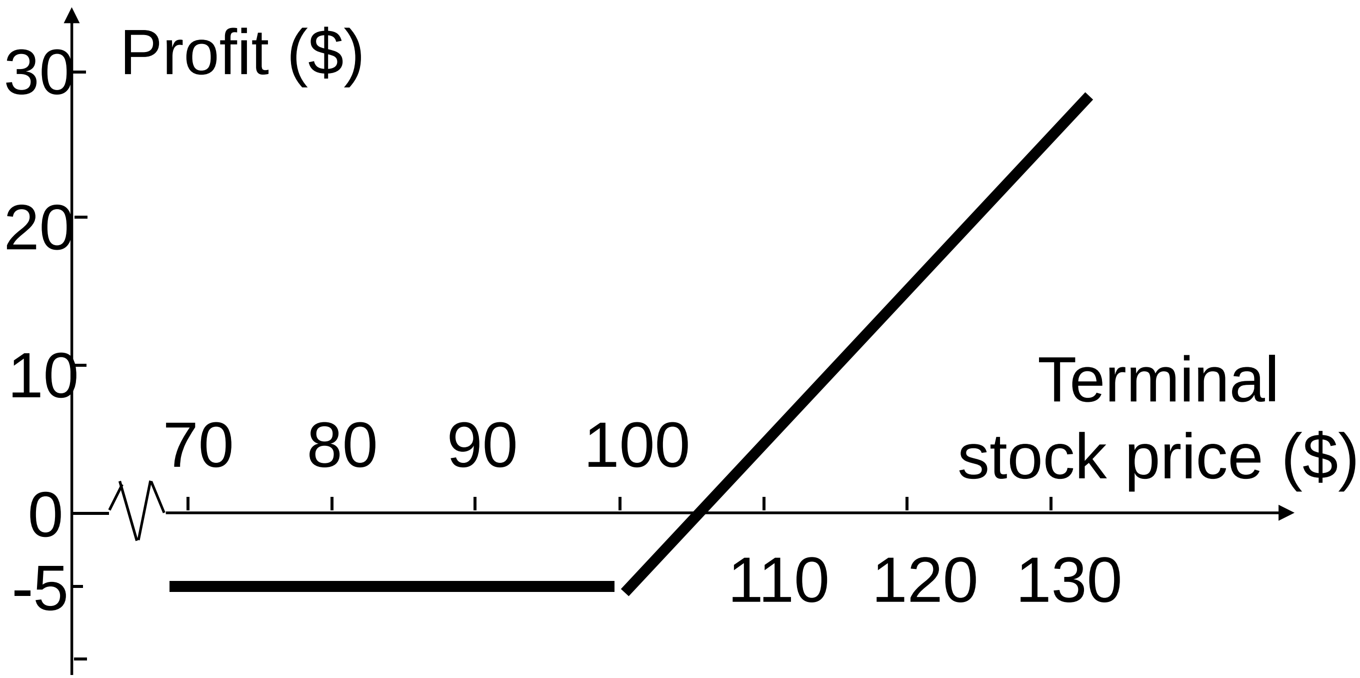
- A call is an option to buy
- A put is an option to sell
- A European option can be exercised only at the end of its life
- An American option can be exercised at any time

Option Positions

- Long call
- Long put
- Short call
- Short put

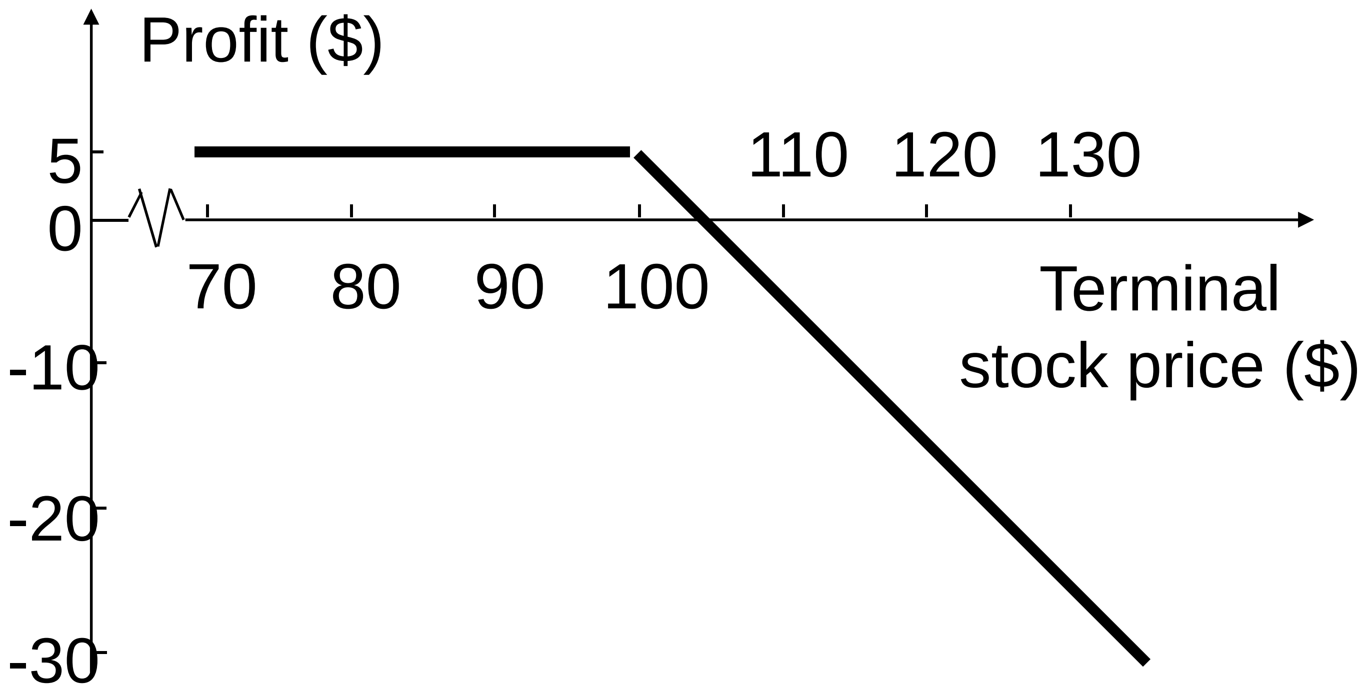
Long Call *(Figure 10.1, Page 214)*

Profit from buying one European call option: option price = $5, strike price = $100, option life = 2 months



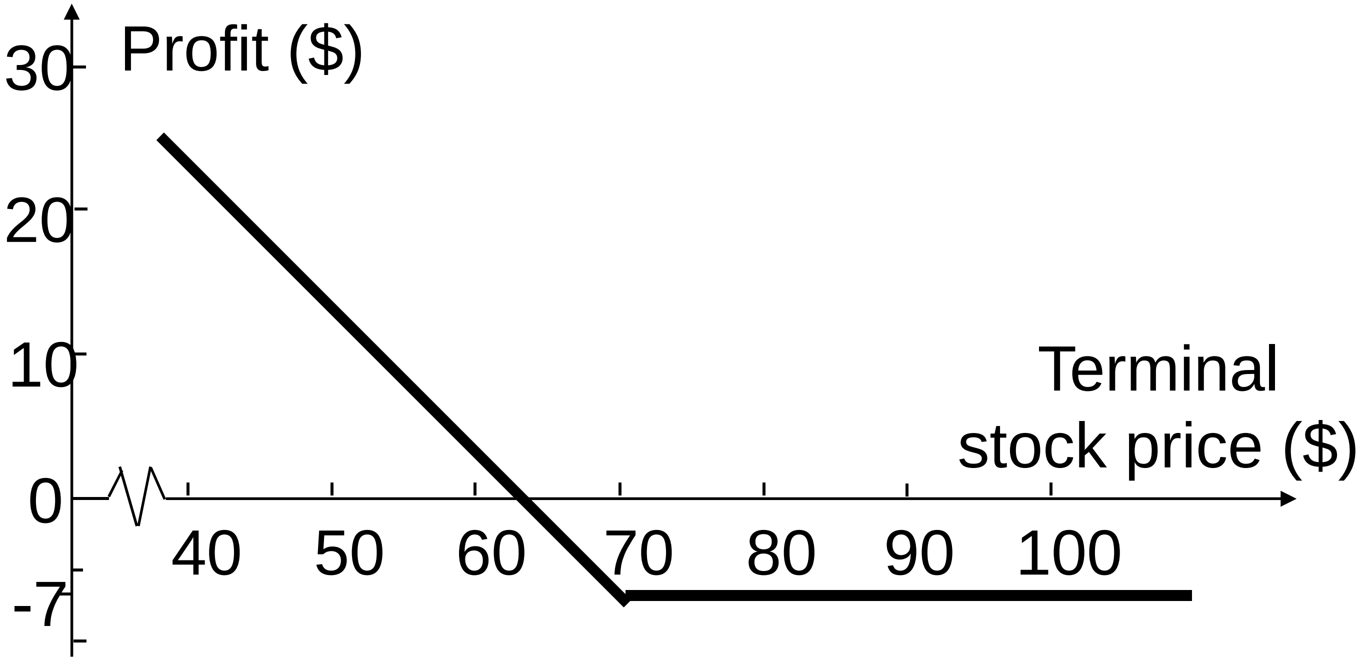
Short Call *(Figure 10.3, page 216)*

Profit from writing one European call option: option price = \$5, strike price = \$100



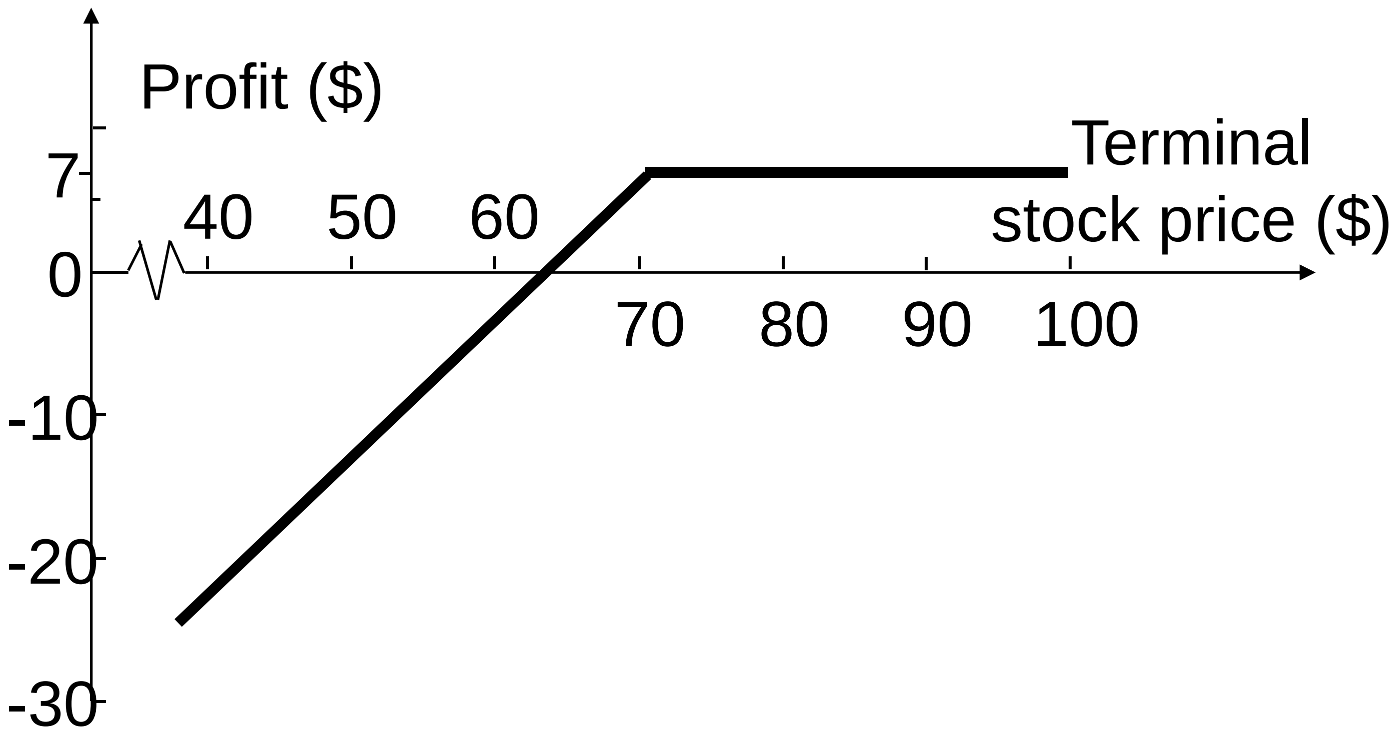
Long Put *(Figure 10.2, page 215)*

Profit from buying a European put option: option price = $7, strike price = $70



Short Put *(Figure 10.4, page 216)*

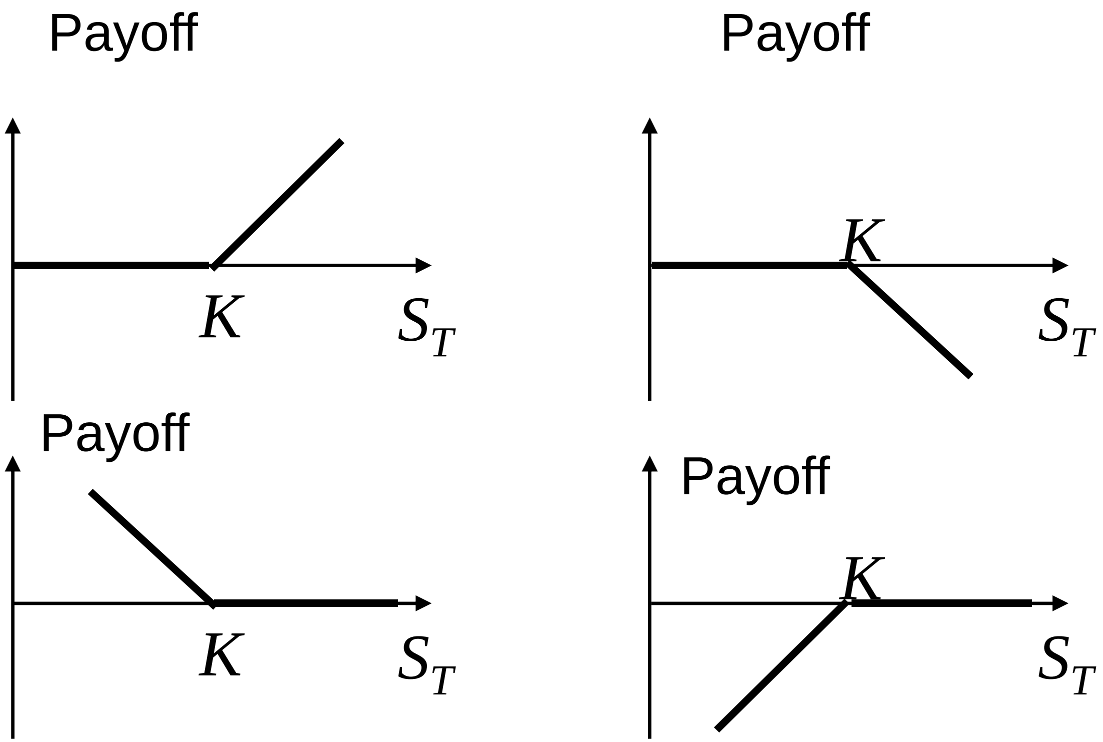
Profit from writing a European put option: option price = $7, strike price = $70



Payoffs from Options

What is the Option Position in Each Case?

K = Strike price, S_T = Price of asset at maturity



Assets Underlying Exchange-Traded Options
Page 217-218

- Stocks
- Foreign Currency
- Stock Indices
- Futures

Specification of Exchange-Traded Options

- Expiration date
- Strike price
- European or American
- Call or Put (option class)

Terminology

Moneyness :

- At-the-money option
- In-the-money option
- Out-of-the-money option

Terminology
(continued)

- Option class
- Option series
- Intrinsic value
- Time value

Ways the CBOE Is Trying to Take Market Share from the OTC Market

- Flex options
- Binary options
- Credit event binary options (CEBOs)
- Doom options

Dividends & Stock Splits

(Page 221-222)

- Suppose you own N options with a strike price of K :
  - No adjustments are made to the option terms for cash dividends
  - When there is an n-for-m stock split,
    - the strike price is reduced to mK/n
    - the no. of options is increased to nN/m
  - Stock dividends are handled similarly to stock splits

Dividends & Stock Splits (continued)

- Consider a call option to buy 100 shares for $20/share
- How should terms be adjusted:
  - for a 2-for-1 stock split?
  - for a 5% stock dividend?

Market Makers

- Most exchanges use market makers to facilitate options trading
- A market maker quotes both bid and ask prices when requested
- The market maker does not know whether the individual requesting the quotes wants to buy or sell

Margins *(Page 224-226)*

- Margins are required when options are sold
- When a naked option is written the margin is the greater of:
  - A total of 100% of the proceeds of the sale plus 20% of the underlying share price less the amount (if any) by which the option is out of the money
  - A total of 100% of the proceeds of the sale plus 10% of the underlying share price (call) or exercise price (put)
- For other trading strategies there are special rules

Warrants

- Warrants are options that are issued by a corporation or a financial institution
- The number of warrants outstanding is determined by the size of the original issue and changes only when they are exercised or when they expire

Warrants
(continued)

- The issuer settles up with the holder when a warrant is exercised
- When call warrants are issued by a corporation on its own stock, exercise will usually lead to new treasury stock being issued

Employee Stock Options *(see also Chapter 16)*

- Employee stock options are a form of remuneration issued by a company to its executives
- They are usually at the money when issued
- When options are exercised the company issues more stock and sells it to the option holder for the strike price
- Expensed on the income statement

Convertible Bonds

- Convertible bonds are regular bonds that can be exchanged for equity at certain times in the future according to a predetermined exchange ratio
- Usually a convertible is callable
- The call provision is a way in which the issuer can force conversion at a time earlier than the holder might otherwise choose

Chapter 11
Properties of Stock Options

Notation

c: European call option price

p: European put option price

S_0: Stock price today

K: Strike price

T: Life of option

σ: Volatility of stock price

C: American call option price

P: American put option price

S_T: Stock price at option maturity

D: PV of dividends paid during life of option

r Risk-free rate for maturity T with cont. comp.

Effect of Variables on Option Pricing *(Table 11.1, page 235)*

Variable	c	p	C	P
S_0	+	−	+	−
K	−	+	−	+
T	?	?	+	+
σ	+	+	+	+
r	+	−	+	−
D	−	+	−	+

American vs European Options

An American option is worth at least as much as the corresponding European option

$$C \geq c$$

$$P \geq p$$

Calls: An Arbitrage Opportunity?

- Suppose that

$c = 3$	$S_0 = 20$
$T = 1$	$r = 10\%$
$K = 18$	$D = 0$

- Is there an arbitrage opportunity?

Lower Bound for European Call Option Prices; No Dividends

(Equation 11.4, page 240)

$$c \geq \max(S_0 - Ke^{-rT}, 0)$$

Puts: An Arbitrage Opportunity?

- Suppose that

$p= 1$	$S_0 = 37$
$T = 0.5$	$r = 5\%$
$K = 40$	$D = 0$

- Is there an arbitrage opportunity?

Lower Bound for European Put Prices; No Dividends

(Equation 11.5, page 241)

$$p \geq \max(Ke^{-rT} - S_0, 0)$$

Put-Call Parity: No Dividends

- Consider the following 2 portfolios:
  - Portfolio A: European call on a stock + zero-coupon bond that pays K at time T
  - Portfolio C: European put on the stock + the stock

Values of Portfolios

		$S_T > K$	$S_T < K$
Portfolio A	Call option	$S_T - K$	0
	Zero-coupon bond	K	K
	Total	S_T	K
Portfolio C	Put Option	0	$K - S_T$
	Share	S_T	S_T
	Total	S_T	K

The Put-Call Parity Result *(Equation 11.6, page 242)*

- Both are worth max(S_T , K) at the maturity of the options
- They must therefore be worth the same today. This means that

$$c + Ke^{-rT} = p + S_0$$

Arbitrage Opportunities

- Suppose that

$c = 3$ $S_0 = 31$

$T = 0.25$ $r = 10\%$

$K = 30$ $D = 0$

- What are the arbitrage possibilities when

$p = 2.25$?

$p = 1$?

Early Exercise

- Usually there is some chance that an American option will be exercised early
- An exception is an American call on a non-dividend paying stock
- This should never be exercised early

An Extreme Situation

- For an American call option:

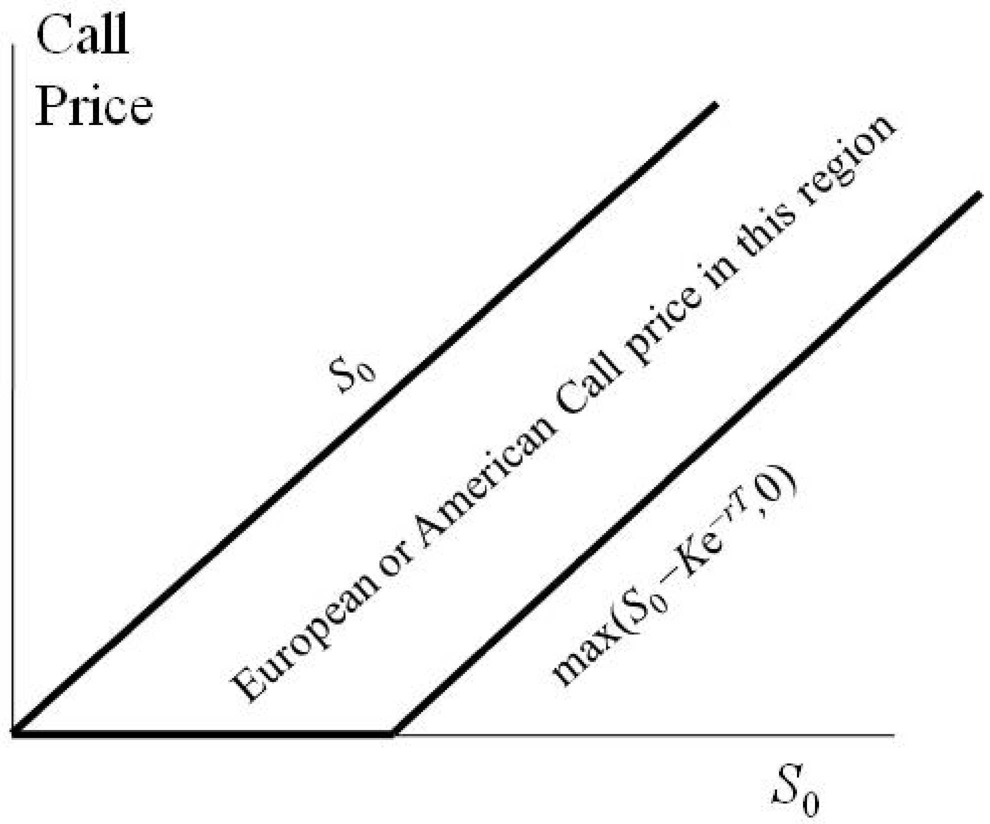
  $$S_0 = 100; T = 0.25; K = 60; D = 0$$

  Should you exercise immediately?
- What should you do if
  - You want to hold the stock for the next 3 months?
  - You do not feel that the stock is worth holding for the next 3 months?

Reasons For Not Exercising a Call Early (No Dividends)

- No income is sacrificed
- You delay paying the strike price
- Holding the call provides insurance against stock price falling below strike price

Bounds for European or American Call Options



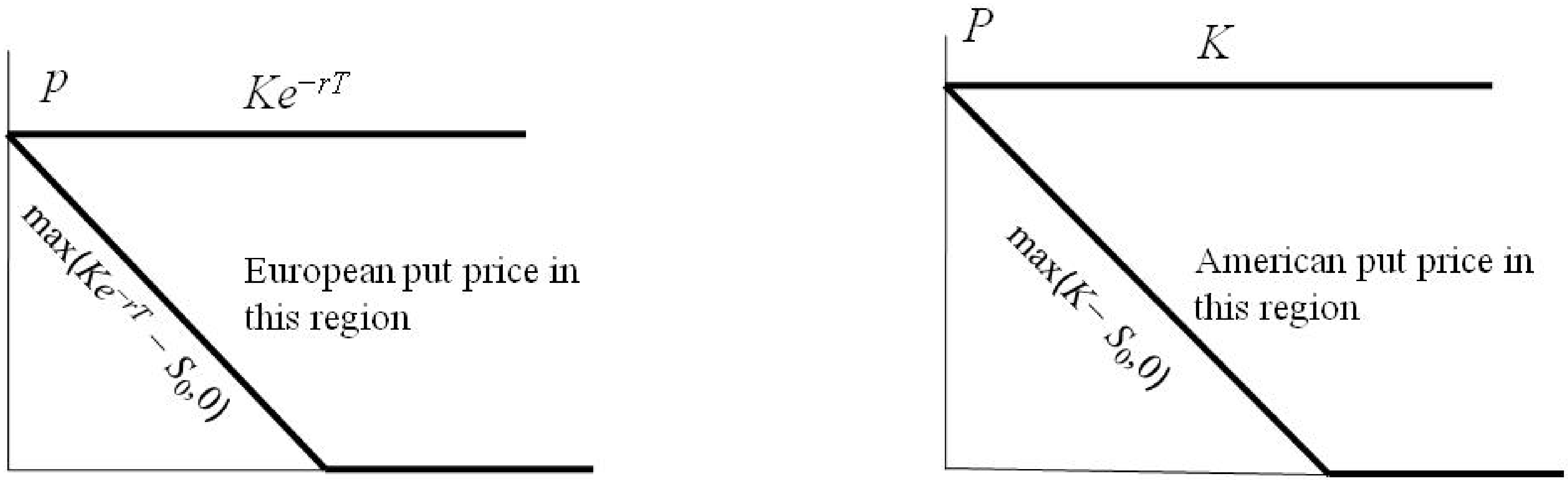
Should Puts Be Exercised Early ?

Are there any advantages to exercising an American put when

$S_0 \;\; = 60; T = 0.25; r{=}10\%$

$K = 100; D = 0$

Bounds for European and American Put Options (No Dividends)



The Impact of Dividends on Lower Bounds to Option Prices

(Equations 11.8 and 11.9, page 249)

$$c \geq S_0 - D - Ke^{-rT}$$

$$p \geq D + Ke^{-rT} - S_0$$

Extensions of Put-Call Parity

- American options; $D = 0$

  $$S_0 - K < C - P < S_0 - Ke^{-rT}$$

  Equation 11.7 p. 244

- European options; $D > 0$

  $$c + D + Ke^{-rT} = p + S_0$$

  Equation 11.10 p. 250

- American options; $D > 0$

  $$S_0 - D - K < C - P < S_0 - Ke^{-rT}$$

  Equation 11.11 p. 250

Chapter 12
Trading Strategies Involving Options

Strategies to be Considered

- Bond plus option to create principal protected note
- Stock plus option
- Two or more options of the same type (a spread)
- Two or more options of different types (a combination)

Principal Protected Note

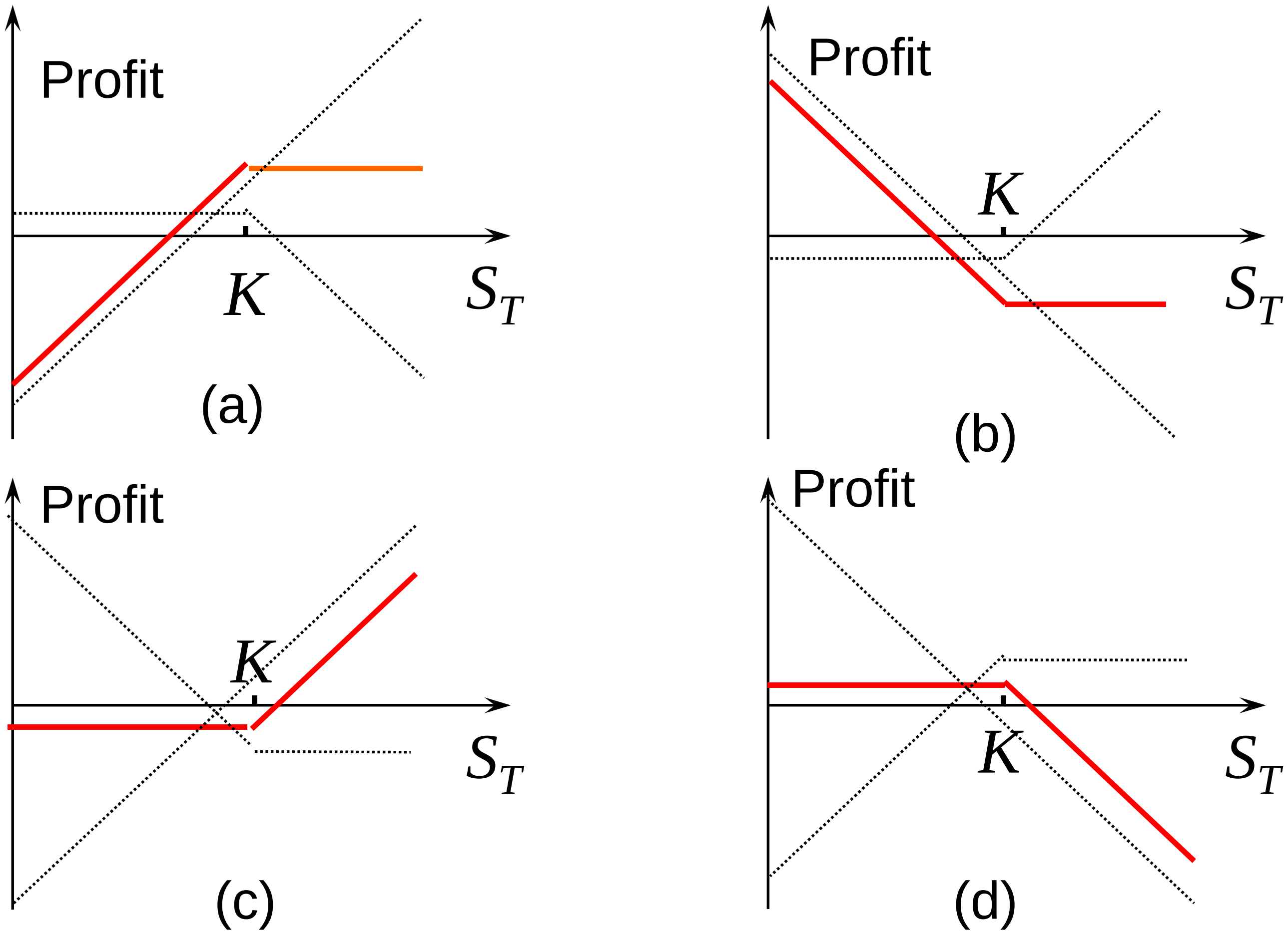
- Allows investor to take a risky position without risking any principal
- Example: $1000 instrument consisting of
  - 3-year zero-coupon bond with principal of $1000
  - 3-year at-the-money call option on a stock portfolio currently worth $1000

Principal Protected Notes continued

- Viability depends on
  - Level of dividends
  - Level of interest rates
  - Volatility of the portfolio
- Variations on standard product
  - Out of the money strike price
  - Caps on investor return
  - Knock outs, averaging features, etc

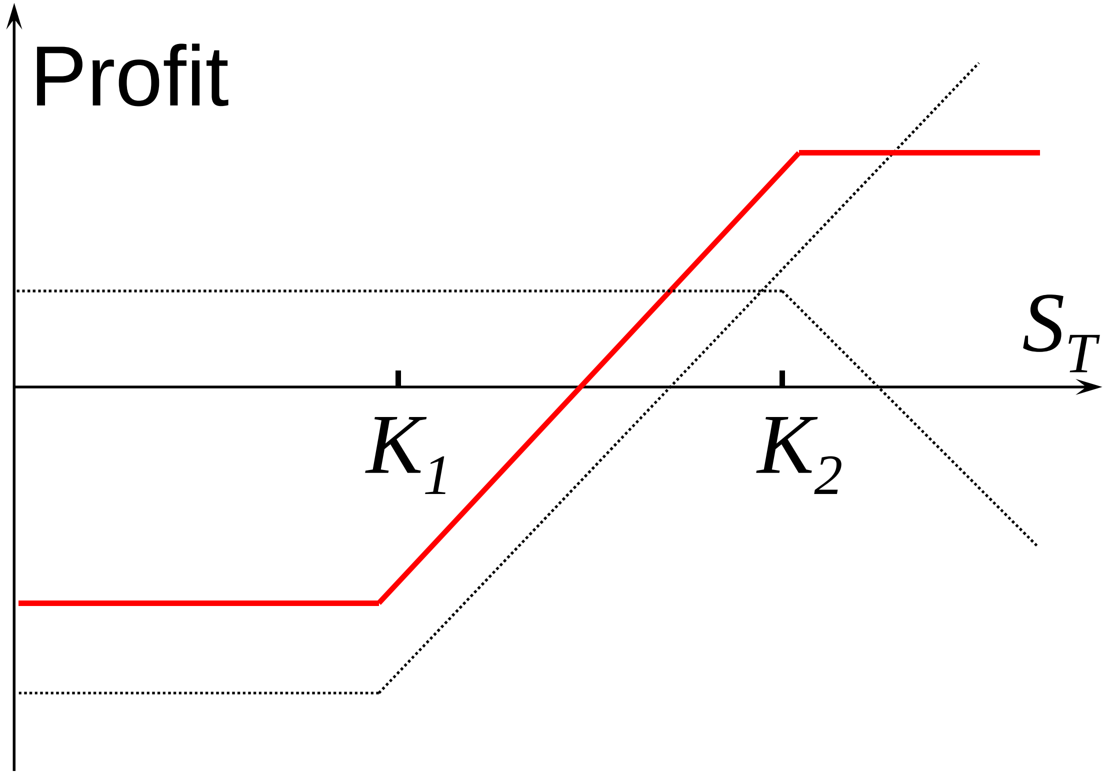
Positions in an Option & the Underlying

(Figure 12.1, page 257)



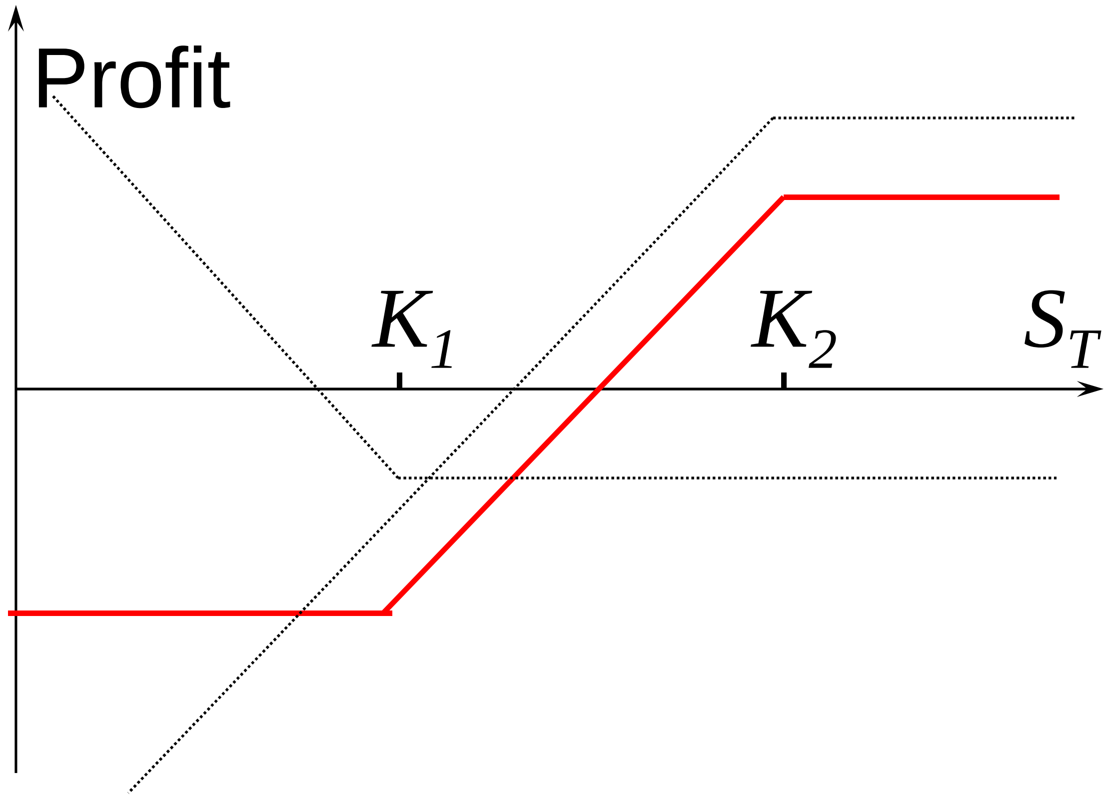
Bull Spread Using Calls

(Figure 12.2, page 258)



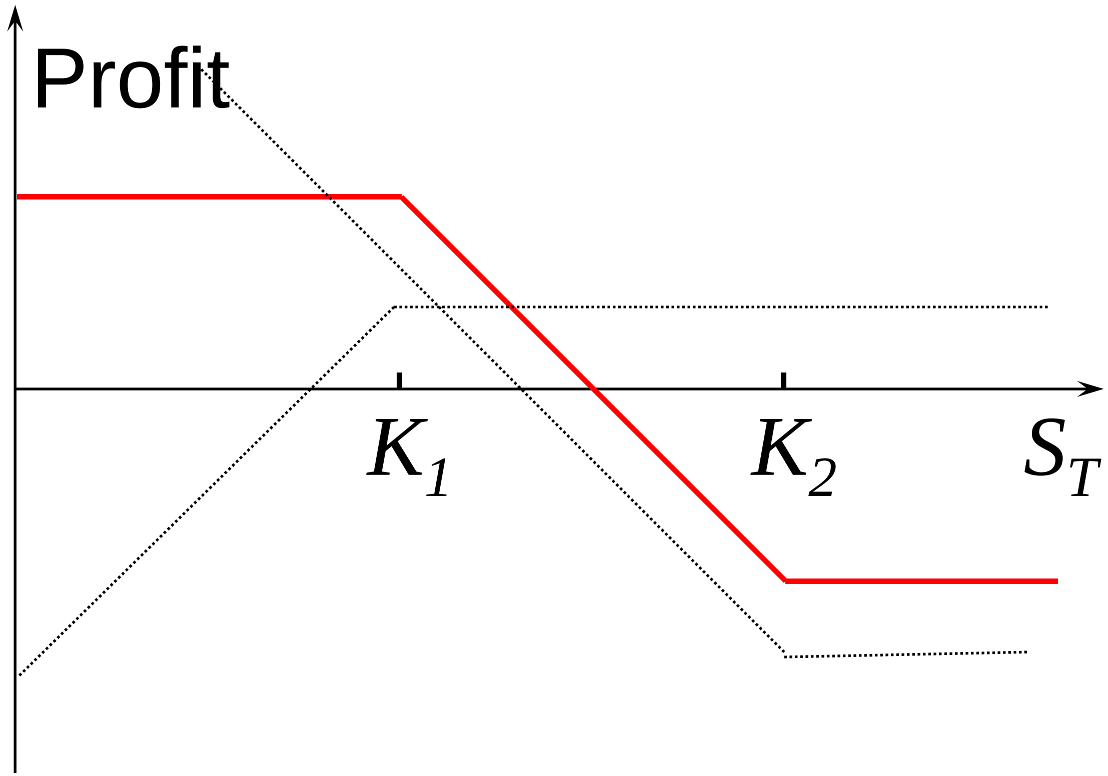
Bull Spread Using Puts

Figure 12.3, page 259



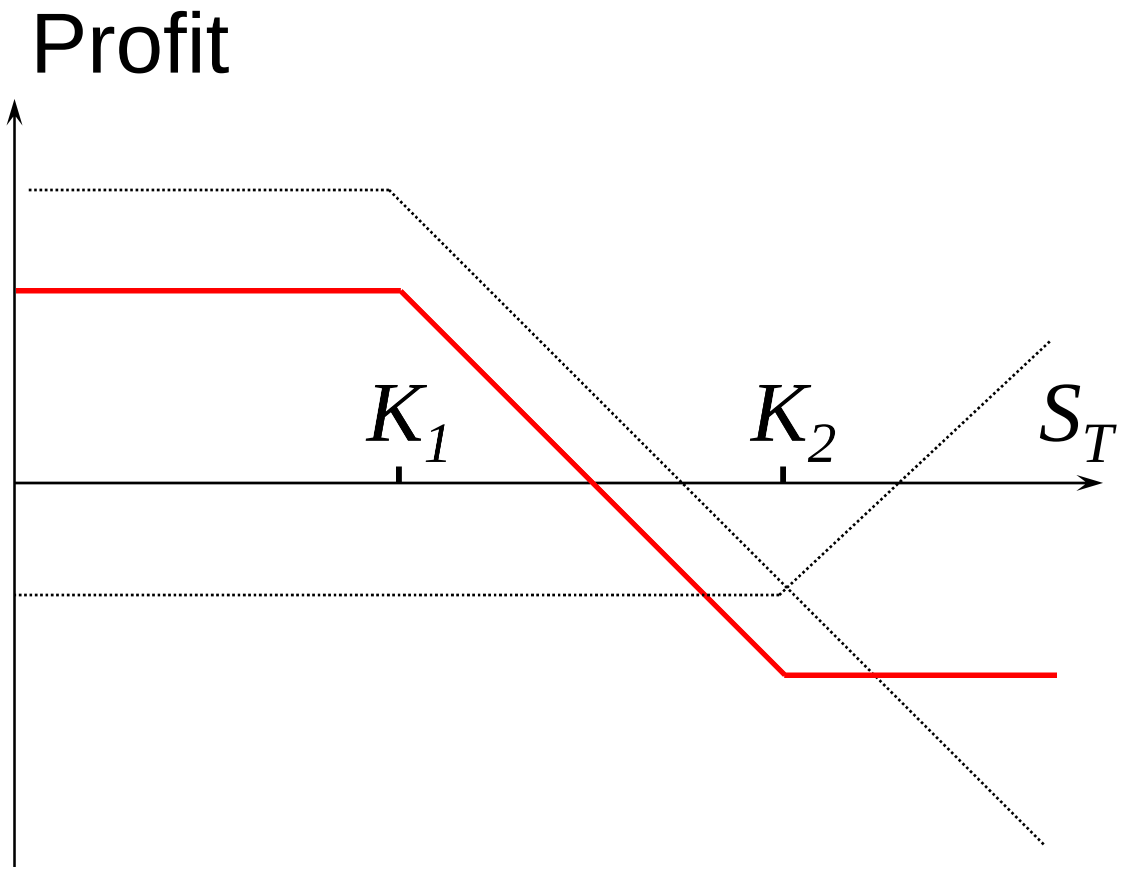
Bear Spread Using Puts

Figure 12.4, page 260



Bear Spread Using Calls

Figure 12.5, page 261

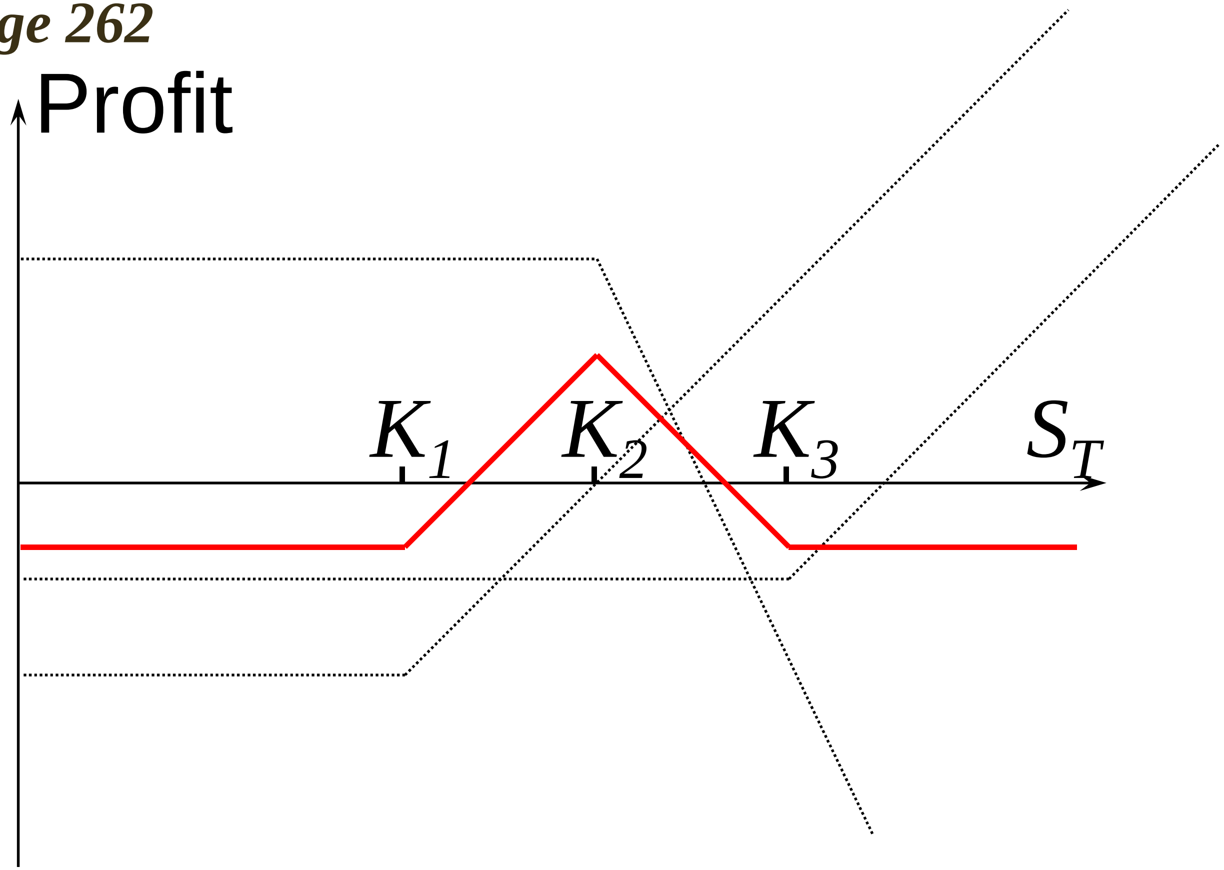


Box Spread

- A combination of a bull call spread and a bear put spread
- If all options are European a box spread is worth the present value of the difference between the strike prices
- If they are American this is not necessarily so (see Business Snapshot 11.1)

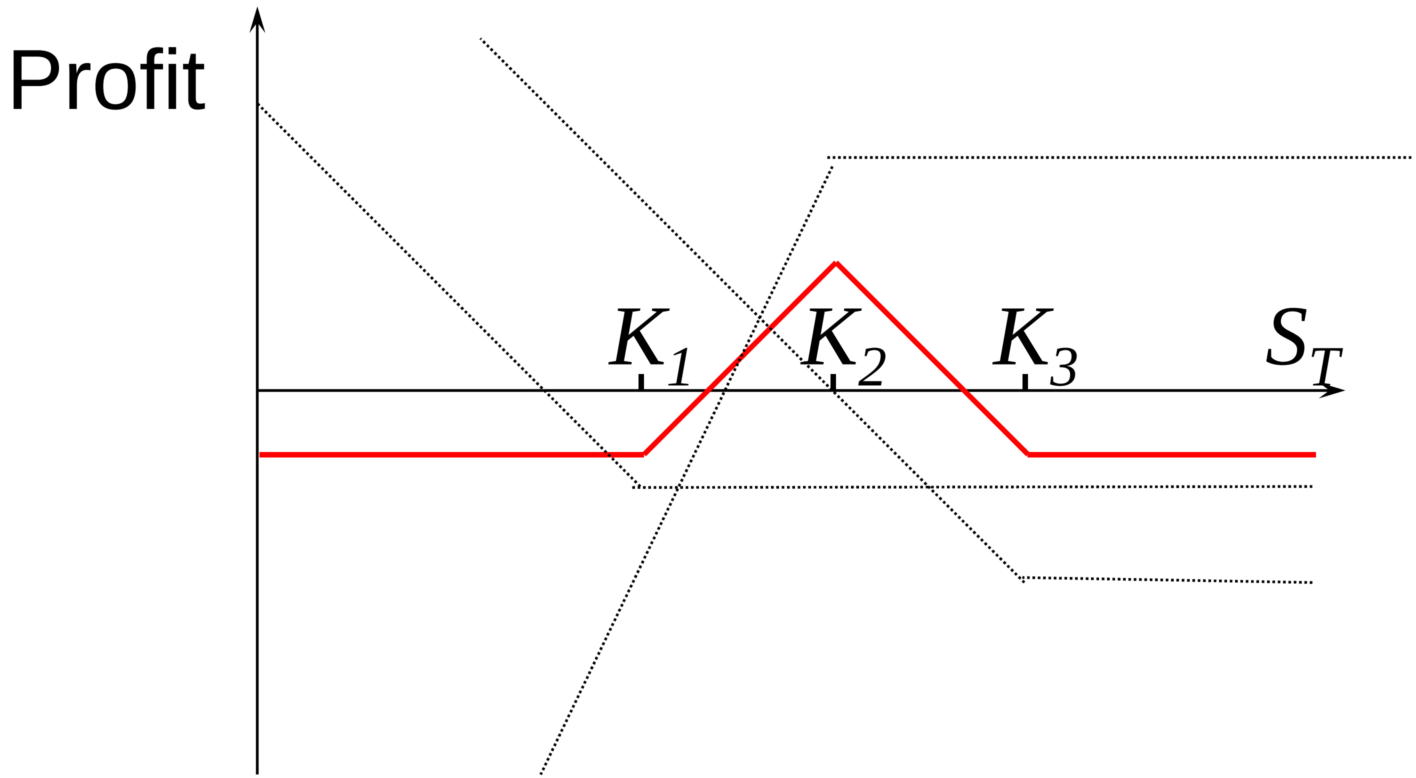
Butterfly Spread Using Calls

Figure 12.6, page 262



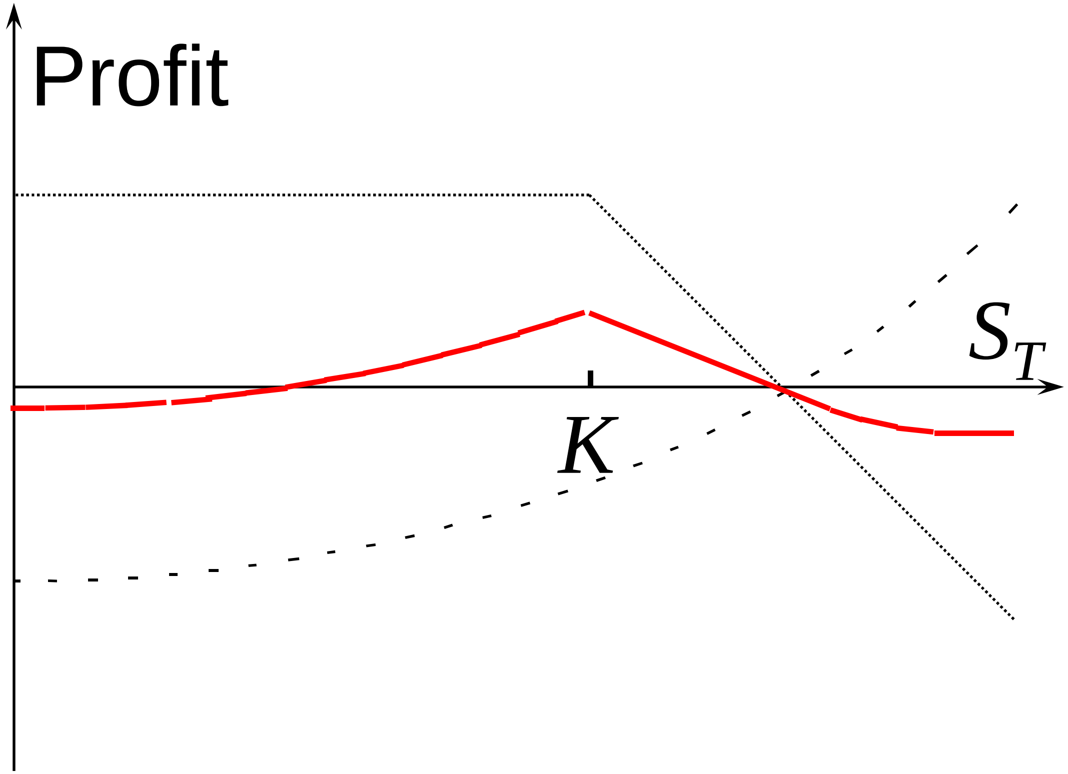
Butterfly Spread Using Puts

Figure 12.7, page 264



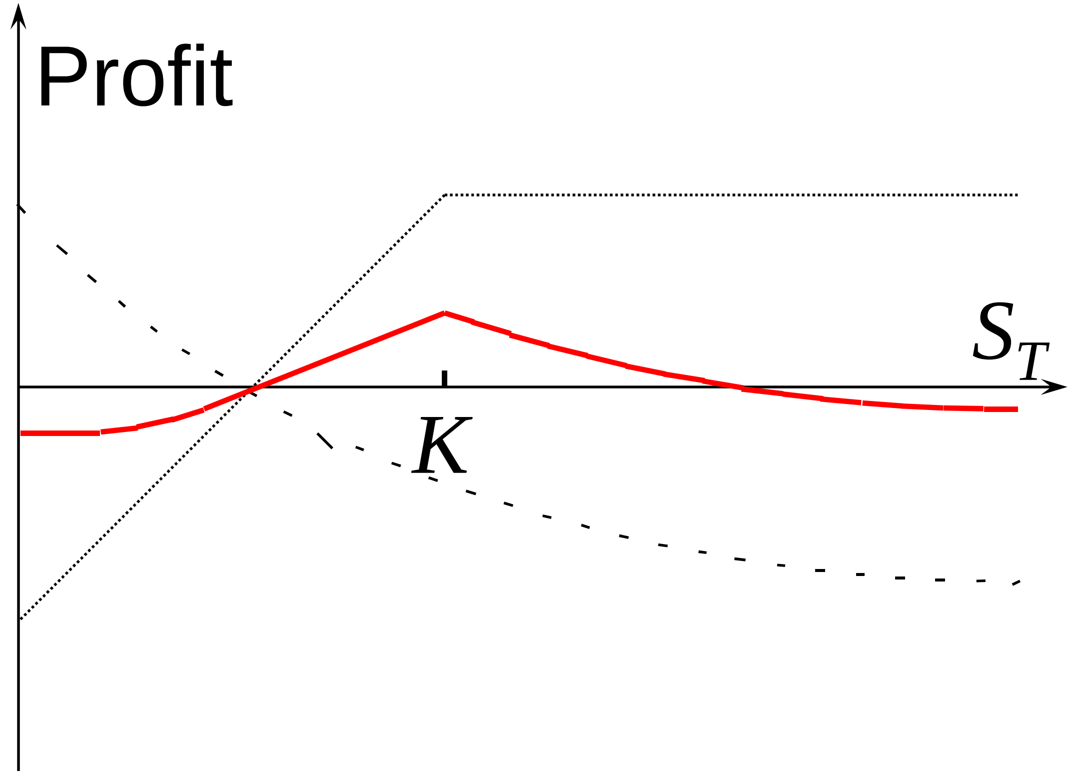
Calendar Spread Using Calls

Figure 12.8, page 265



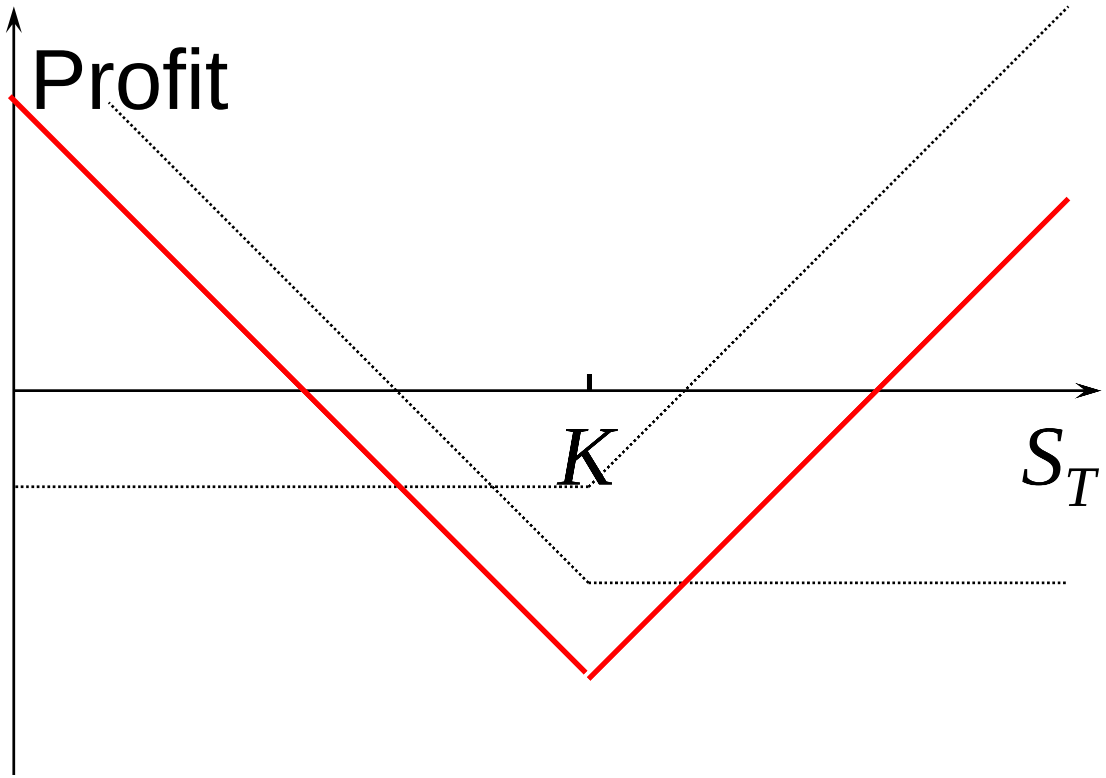
Calendar Spread Using Puts

Figure 12.9, page 266



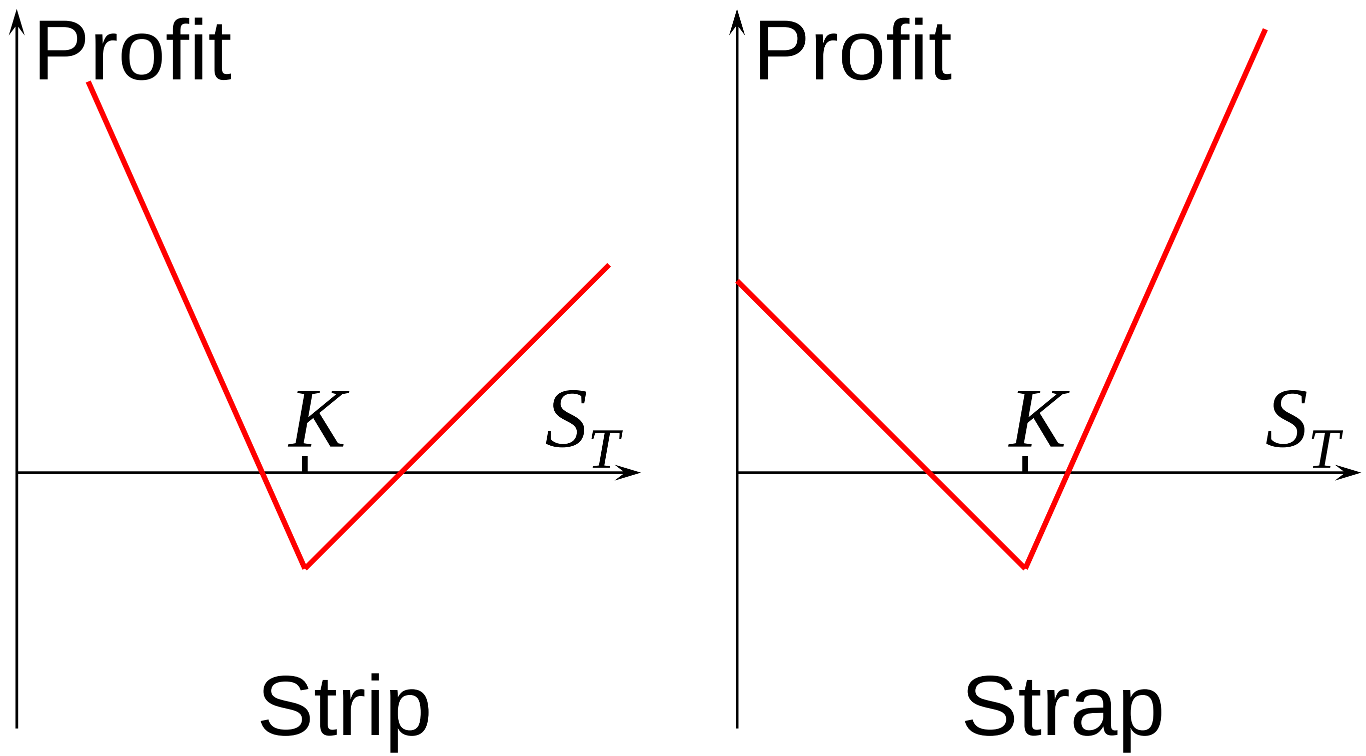
A Straddle Combination

Figure 12.10, page 267



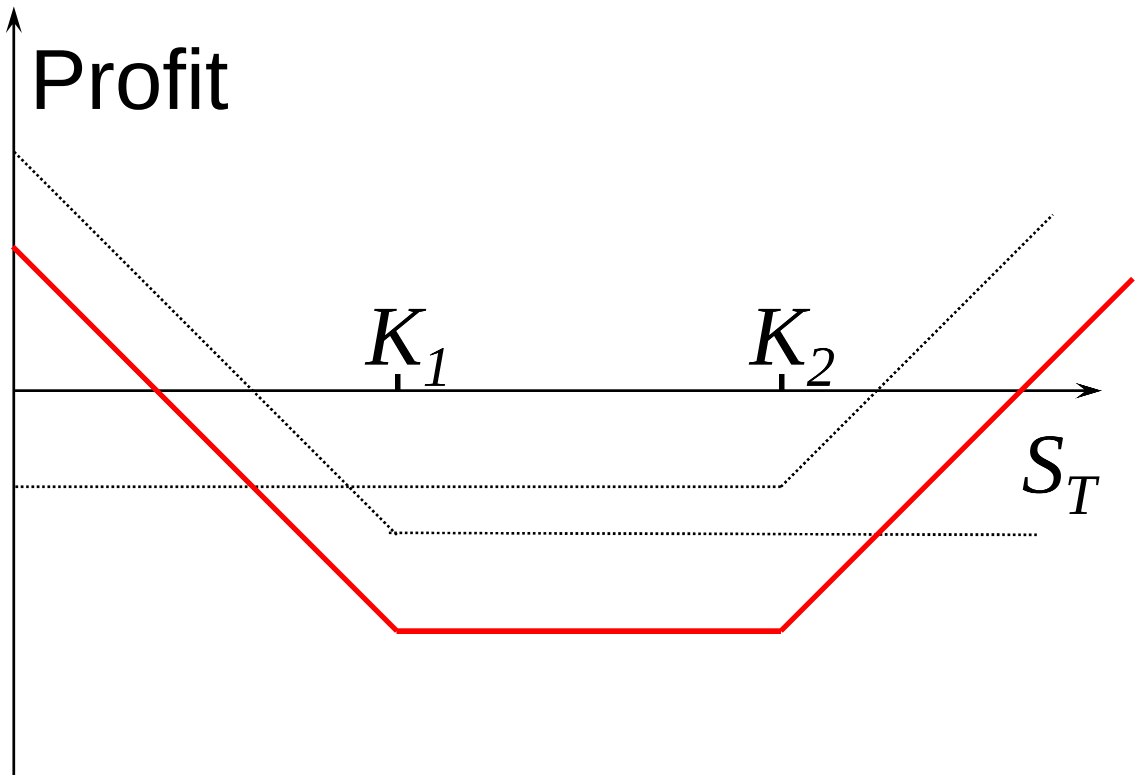
Strip & Strap

Figure 12.11, page 268



A Strangle Combination

Figure 12.12, page 269



Other Payoff Patterns

- When the strike prices are close together a butterfly spread provides a payoff consisting of a small “spike”
- If options with all strike prices were available any payoff pattern could (at least approximately) be created by combining the spikes obtained from different butterfly spreads

Chapter 13
Binomial Trees

A Simple Binomial Model

- A stock price is currently \$20
- In 3 months it will be either \$22 or \$18

Stock Price = \$22

Stock price = \$20

Stock Price = \$18

A Call Option (*Figure 13.1, page 275)*

A 3-month call option on the stock has a strike price of 21.

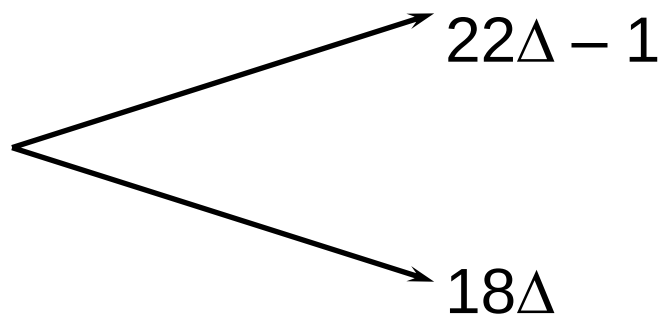
Stock Price = \$22
Option Price = \$1

Stock price = \$20
Option Price=?

Stock Price = \$18
Option Price = \$0

Setting Up a Riskless Portfolio

- For a portfolio that is long Δ shares and a short 1 call option values are



- Portfolio is riskless when $22\Delta - 1 = 18\Delta$ or $\Delta = 0.25$

Valuing the Portfolio
(Risk-Free Rate is 12%)

- The riskless portfolio is:

  long 0.25 shares

  short 1 call option

- The value of the portfolio in 3 months is

  $22 \times 0.25 - 1 = 4.50$

- The value of the portfolio today is

  $4.5e^{-0.12\times0.25} = 4.3670$

Valuing the Option

- The portfolio that is

  long 0.25 shares

  short 1 option

  is worth 4.367
- The value of the shares is
  5.000 (= 0.25 × 20)
- The value of the option is therefore
  0.633 (5.000 – 0.633 = 4.367)

Generalization *(Figure 13.2, page 276)*

A derivative lasts for time T and is dependent on a stock

$$\begin{tikzcd} & \begin{matrix} S_0u \\ f_u \end{matrix} \\ \begin{matrix} S_0 \\ f \end{matrix} \arrow[ur] \arrow[dr] & \\ & \begin{matrix} S_0d \\ f_d \end{matrix} \end{tikzcd}$$

Generalization *(continued)*

- Value of a portfolio that is long Δ shares and short 1 derivative:

$$S_0 u\Delta - f_u$$

$$S_0 d\Delta - f_d$$

- The portfolio is riskless when $S_0 u\Delta - f_u = S_0 d\Delta - f_d$ or

$$\Delta = \frac{f_u - f_d}{S_0 u - S_0 d}$$

Generalization *(continued)*

- Value of the portfolio at time T is $S_0 u\Delta - f_u$
- Value of the portfolio today is $(S_0 u\Delta - f_u)e^{-rT}$
- Another expression for the portfolio value today is $S_0\Delta - f$
- Hence

$$f = S_0\Delta - (S_0 u\Delta - f_u)e^{-rT}$$

Generalization
(continued)

Substituting for Δ we obtain

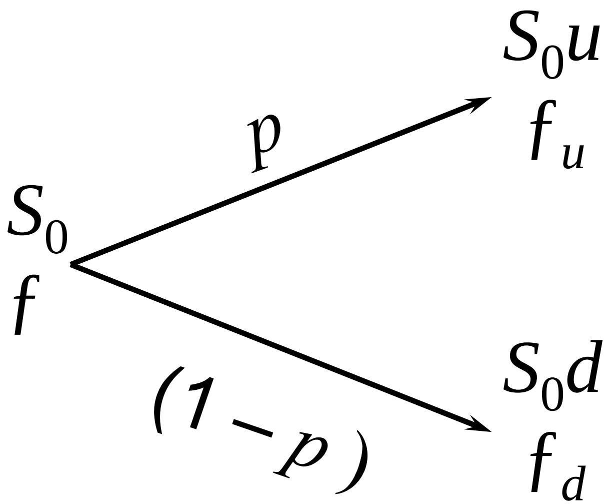
$$f = [\, pf_u + (1 - p)f_d \,]e^{-rT}$$

where

$$p = \frac{e^{rT} - d}{u - d}$$

p as a Probability

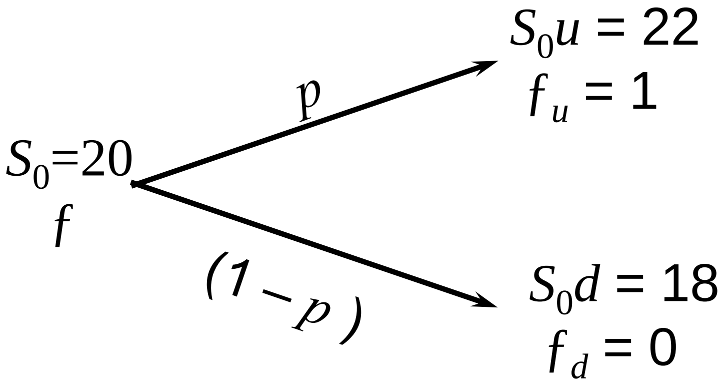
- It is natural to interpret p and 1-p as probabilities of up and down movements
- The value of a derivative is then its expected payoff in a risk-neutral world discounted at the risk-free rate



Risk-Neutral Valuation

- When the probability of an up and down movements are p and 1-p the expected stock price at time T is S_0e^{rT}
- This shows that the stock price earns the risk-free rate
- Binomial trees illustrate the general result that to value a derivative we can assume that the expected return on the underlying asset is the risk-free rate and discount at the risk-free rate
- This is known as using risk-neutral valuation

Original Example Revisited



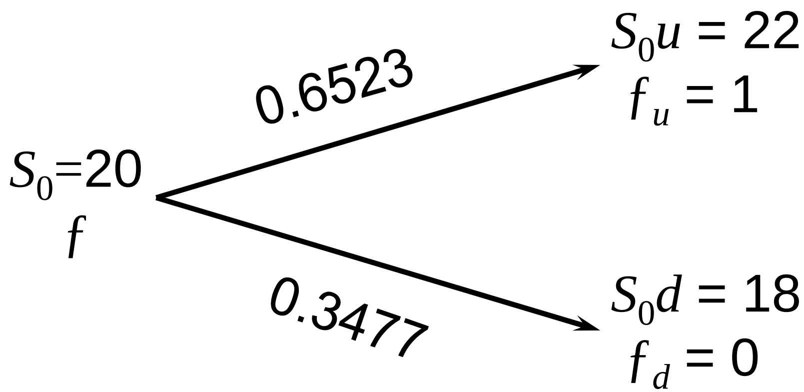
p is the probability that gives a return on the stock equal to the risk-free rate:

$20e^{0.12\times0.25} = 22p + 18(1-p)$ so that $p = 0.6523$

Alternatively:

$$p = \frac{e^{rT} - d}{u - d} = \frac{e^{0.12\times0.25} - 0.9}{1.1 - 0.9} = 0.6523$$

Valuing the Option Using Risk-Neutral Valuation



The value of the option is

$e^{-0.12\times0.25}$ (0.6523×1 + 0.3477×0)

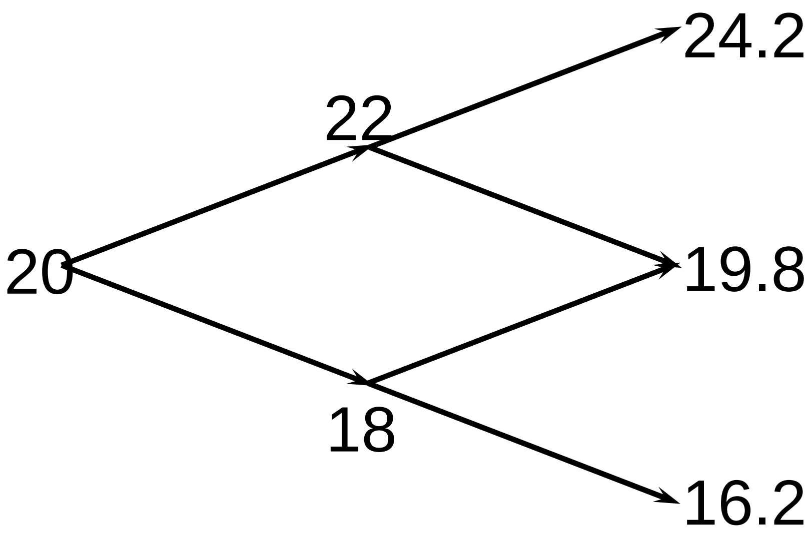
= 0.633

Irrelevance of Stock's Expected Return

- When we are valuing an option in terms of the price of the underlying asset, the probability of up and down movements in the real world are irrelevant
- This is an example of a more general result stating that the expected return on the underlying asset in the real world is irrelevant

A Two-Step Example

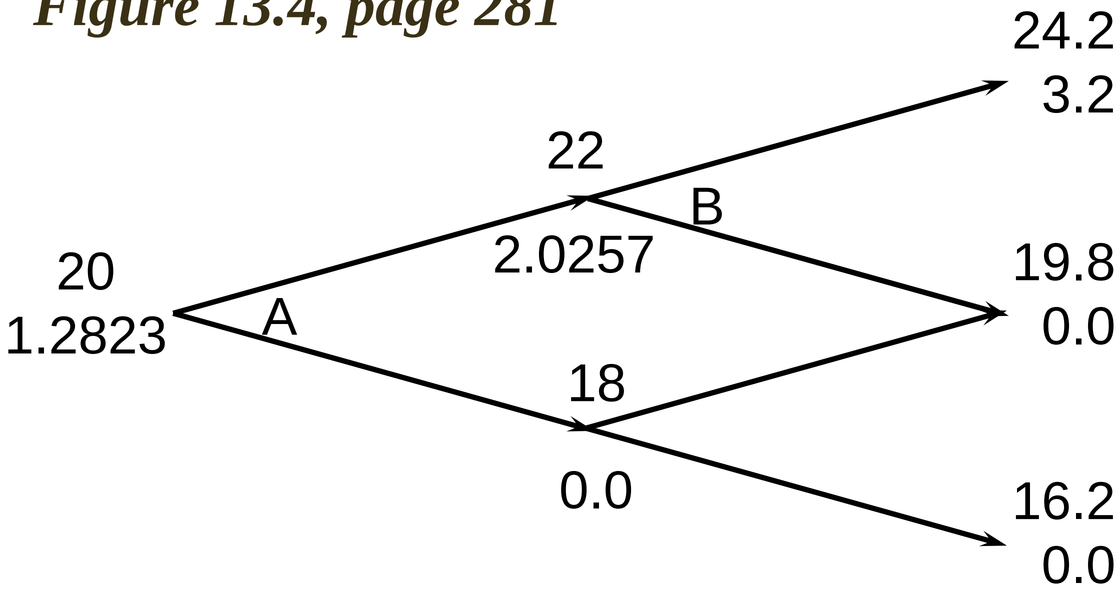
Figure 13.3, page 281



- K=21, r = 12%
- Each time step is 3 months

Valuing a Call Option

Figure 13.4, page 281



Value at node B

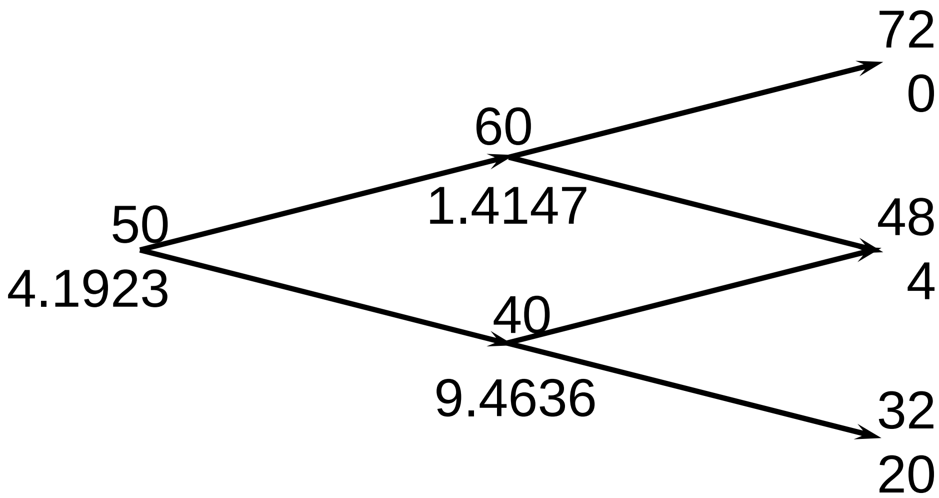
$= e^{-0.12\times0.25}(0.6523\times3.2 + 0.3477\times0) = 2.0257$

Value at node A

$= e^{-0.12\times0.25}(0.6523\times2.0257 + 0.3477\times0) = 1.2823$

A Put Option Example

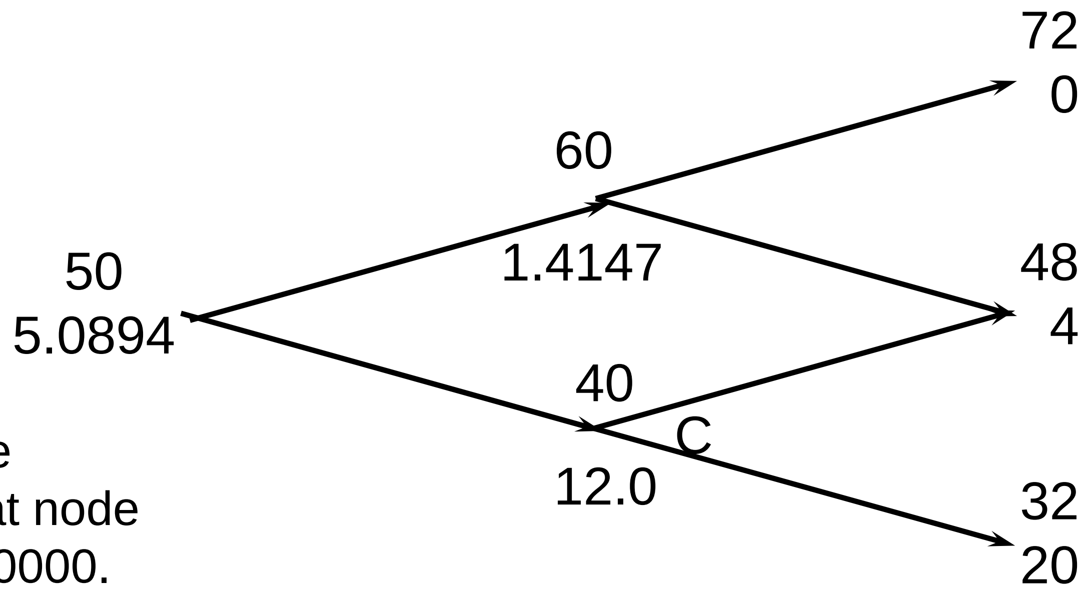
Figure 13.7, page 284



K = 52, time step =1yr

r = 5%, u =1.32, d = 0.8, p = 0.6282

What Happens When the Put Option is American *(Figure 13.8, page 285)*



The American feature increases the value at node C from 9.4636 to 12.0000.

This increases the value of the option from 4.1923 to 5.0894.

Delta

- Delta (Δ) is the ratio of the change in the price of a stock option to the change in the price of the underlying stock
- The value of Δ varies from node to node

Choosing u and d

One way of matching the volatility is to set

$$u = e^{\sigma\sqrt{\Delta t}}$$

$$d = 1/u = e^{-\sigma\sqrt{\Delta t}}$$

where σ is the volatility and Δt is the length of the time step. This is the approach used by Cox, Ross, and Rubinstein

Girsanov's Theorem

- Volatility is the same in the real world and the risk-neutral world
- We can therefore measure volatility in the real world and use it to build a tree for the an asset in the risk-neutral world

Assets Other than Non-Dividend Paying Stocks

- For options on stock indices, currencies and futures the basic procedure for constructing the tree is the same except for the calculation of p

The Probability of an Up Move

$$p = \frac{a - d}{u - d}$$

$a = e^{r\Delta t}$ for a nondividend paying stock

$a = e^{(r-q)\Delta t}$ for a stock index where q is the dividend yield on the index

$a = e^{(r-r_f)\Delta t}$ for a currency where r_f is the foreign risk - free rate

$a = 1$ for a futures contract

Proving Black-Scholes-Merton from Binomial Trees *(Appendix to Chapter 13)*

$$c = e^{-rT} \sum_{j=0}^{n} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j} \max(S_0 u^j d^{n-j} - K,\ 0)$$

Option is in the money when $j > \alpha$ where

$$\alpha = \frac{n}{2} - \frac{\ln(S_0/K)}{2\sigma\sqrt{T/n}}$$

so that

$$c = e^{-rT}(S_0 U_1 - K U_2)$$

where

$$U_1 = \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j} u^j d^{n-j}$$

$$U_2 = \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} p^j (1-p)^{n-j}$$

Proving Black-Scholes-Merton from Binomial Trees *continued*

- The expression for U_1 can be written

$$U_1 = [pu + (1-p)d]^n \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} \left(p^*\right)^j \left(1-p^*\right)^{n-j} = e^{rT} \sum_{j>\alpha} \frac{n!}{(n-j)!\,j!} \left(p^*\right)^j \left(1-p^*\right)^{n-j}$$

where

$$p^* = \frac{pu}{pu + (1-p)d}$$

- Both U_1 and U_2 can now be evaluated in terms of the cumulative binomial distribution
- We now let the number of time steps tend to infinity and use the result that a binomial distribution tends to a normal distribution

Chapter 14
Wiener Processes and Itô's Lemma

Stochastic Processes

- Describes the way in which a variable such as a stock price, exchange rate or interest rate changes through time
- Incorporates uncertainties

Example 1

- Each day a stock price
  - increases by $1 with probability 30%
  - stays the same with probability 50%
  - reduces by $1 with probability 20%

Example 2

- Each day a stock price change is drawn from a normal distribution with mean \$0.2 and standard deviation \$1

Markov Processes *(See pages 302-303)*

- In a Markov process future movements in a variable depend only on where we are, not the history of how we got to where we are
- Is the process followed by the temperature at a certain place Markov?
- We assume that stock prices follow Markov processes

Weak-Form Market Efficiency

- This asserts that it is impossible to produce consistently superior returns with a trading rule based on the past history of stock prices. In other words technical analysis does not work.
- A Markov process for stock prices is consistent with weak-form market efficiency

Example

- A variable is currently 40
- It follows a Markov process
- Process is stationary (i.e. the parameters of the process do not change as we move through time)
- At the end of 1 year the variable will have a normal probability distribution with mean 40 and standard deviation 10

Questions

- What is the probability distribution of the stock price at the end of 2 years?
- ½ years?
- ¼ years?
- Δt years?

Taking limits we have defined a continuous stochastic process

Variances & Standard Deviations

- In Markov processes changes in successive periods of time are independent
- This means that variances are additive
- Standard deviations are not additive

Variances & Standard Deviations
(continued)

- In our example it is correct to say that the variance is 100 per year.
- It is strictly speaking not correct to say that the standard deviation is 10 per year.

A Wiener Process *(See pages 304-305)*

- Define $\phi(\mu, v)$ as a normal distribution with mean μ and variance v

- A variable z follows a Wiener process if
  - The change in z in a small interval of time Δt is Δz
  - $\Delta z = \varepsilon\sqrt{\Delta t}$ where ε is $\phi(0,1)$
  - The values of Δz for any 2 different (non-overlapping) periods of time are independent

Properties of a Wiener Process

- Mean of $[z(T) - z(0)]$ is 0
- Variance of $[z(T) - z(0)]$ is T
- Standard deviation of $[z(T) - z(0)]$ is $\sqrt{T}$

Generalized Wiener Processes

(See page 305-308)

- A Wiener process has a drift rate (i.e. average change per unit time) of 0 and a variance rate of 1
- In a generalized Wiener process the drift rate and the variance rate can be set equal to any chosen constants

Generalized Wiener Processes
(continued)

$$\Delta x = a\,\Delta t + b\,\varepsilon\sqrt{\Delta t}$$

- Mean change in x per unit time is a
- Variance of change in x per unit time is b^2

Taking Limits . . .

- What does an expression involving dz and dt mean?
- It should be interpreted as meaning that the corresponding expression involving Δz and Δt is true in the limit as Δt tends to zero
- In this respect, stochastic calculus is analogous to ordinary calculus

The Example Revisited

- A stock price starts at 40 and has a probability distribution of $\phi(40,100)$ at the end of the year
- If we assume the stochastic process is Markov with no drift then the process is

$$dS = 10dz$$

- If the stock price were expected to grow by \$8 on average during the year, so that the year-end distribution is $\phi(48,100)$, the process would be

$$dS = 8dt + 10dz$$

Itô Process *(See pages 308)*

- In an Itô process the drift rate and the variance rate are functions of time

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

- The discrete time equivalent

$$\Delta x = a(x,t)\Delta t + b(x,t)\varepsilon\sqrt{\Delta t}$$

is true in the limit as Δt tends to zero

Why a Generalized Wiener Process Is Not Appropriate for Stocks

- For a stock price we can conjecture that its expected *percentage* change in a short period of time remains constant (not its expected actual change)
- We can also conjecture that our uncertainty as to the size of future stock price movements is proportional to the level of the stock price

An Ito Process for Stock Prices

(See pages 308-311)

$$dS = \mu S \, dt + \sigma S \, dz$$

where μ is the expected return σ is the volatility.

The discrete time equivalent is

$$\Delta S = \mu S \Delta t + \sigma S \varepsilon \sqrt{\Delta t}$$

The process is known as geometric Brownian motion

Interest Rates

- What would be a reasonable stochastic process to assume for the short-term interest rate?

Monte Carlo Simulation

- We can sample random paths for the stock price by sampling values for ε
- Suppose μ= 0.15, σ= 0.30, and Δt = 1 week (=1/52 or 0.192 years), then

$$\Delta S = 0.15 \times 0.0192 S + 0.30 \times \sqrt{0.0192}\varepsilon$$

or

$$\Delta S = 0.00288 S + 0.0416 S\varepsilon$$

Monte Carlo Simulation – Sampling one Path *(See Table 14.1, page 311)*

Week	Stock Price at Start of Period	Random Sample for ε	Change in Stock Price, ΔS
0	100.00	0.52	2.45
1	102.45	1.44	6.43
2	108.88	−0.86	−3.58
3	105.30	1.46	6.70
4	112.00	−0.69	−2.89

Correlated Processes

Suppose dz_1 and dz_2 are Wiener processes with correlation ρ

Then

$$\Delta z_1 = \varepsilon_1 \sqrt{\Delta t}$$
$$\Delta z_2 = \varepsilon_2 \sqrt{\Delta t}$$

where ε_1 and ε_2 are random samples from a bivariate standard normal distribution where correlation is ρ

Itô's Lemma ***(See pages 313-315)***

- If we know the stochastic process followed by x, Itô's lemma tells us the stochastic process followed by some function $G\,(x,\ t\ )$.
  When $dx=a(x,t)\ dt+b(x,t)\ dz$ then

$$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + ?\ \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b\ dz$$

- Since a derivative is a function of the price of the underlying asset and time, Itô's lemma plays an important part in the analysis of derivatives

Indication of Why Itô's Lemma is True

- A Taylor's series expansion of $G(x, t)$ gives

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \tfrac{1}{2}\frac{\partial^2 G}{\partial x^2}\Delta x^2$$

$$+ \frac{\partial^2 G}{\partial x \partial t}\Delta x\,\Delta t + \tfrac{1}{2}\frac{\partial^2 G}{\partial t^2}\Delta t^2 + \ldots$$

Ignoring Terms of Higher Order Than Δt

In ordinary calculus we have

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t$$

In stochastic calculus this becomes

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + ½\frac{\partial^2 G}{\partial x^2}\Delta x^2$$

because Δx has a component which is of order $\sqrt{\Delta t}$

Substituting for Δx

Suppose

$$dx = a(x,t)dt + b(x,t)dz$$

so that

$$\Delta x = a\ \Delta t + b\ \varepsilon\sqrt{\Delta t}$$

Then ignoring terms of higher order than Δt

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + ½\frac{\partial^2 G}{\partial x^2}b^2\varepsilon^2\Delta t$$

The $\varepsilon^2 \Delta t$ Term

Since $\varepsilon \approx \phi(0,1), \quad E(\varepsilon) = 0$

$$E(\varepsilon^2) - [E(\varepsilon)]^2 = 1$$

$$E(\varepsilon^2) = 1$$

It follows that $E(\varepsilon^2 \Delta t) = \Delta t$

The variance of Δt is proportional to Δt^2 and can be ignored. Hence

$$\Delta G = \frac{\partial G}{\partial x}\Delta x + \frac{\partial G}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2 G}{\partial x^2} b^2 \Delta t$$

Taking Limits

Taking limits : $$dG = \frac{\partial G}{\partial x} dx + \frac{\partial G}{\partial t} dt + ½ \frac{\partial^2 G}{\partial x^2} b^2 dt$$

Substituting : $$dx = a\ dt + b\ dz$$

We obtain : $$dG = \left(\frac{\partial G}{\partial x} a + \frac{\partial G}{\partial t} + ½ \frac{\partial^2 G}{\partial x^2} b^2 \right) dt + \frac{\partial G}{\partial x} b\ dz$$

This is Ito's Lemma

Application of Itô's Lemma to a Stock Price Process

The stock price process is

$$dS = \mu S\, dt + \sigma S\, dz$$

For a function G of S and t

$$dG = \left(\frac{\partial G}{\partial S}\mu S + \frac{\partial G}{\partial t} + ½\frac{\partial^2 G}{\partial S^2}\sigma^2 S^2 \right) dt + \frac{\partial G}{\partial S}\sigma S\, dz$$

Examples

1. The forward price of a stock for a contract maturing at time T

$$G = S\,e^{r(T-t)}$$

$$dG = (\mu - r)G\,dt + \sigma G\,dz$$

2. The log of a stock price

$$G = \ln S$$

$$dG = \left(\mu - \frac{\sigma^2}{2}\right)dt + \sigma\,dz$$

Chapter 15
The Black-Scholes-Merton Model

The Stock Price Assumption

- Consider a stock whose price is S
- In a short period of time of length Δt, the return on the stock is normally distributed:

$$\frac{\Delta S}{S} \approx \phi\left(\mu \Delta t, \sigma^2 \Delta t\right)$$

where μ is expected return and σ is volatility

The Lognormal Property

(Equations 15.2 and 15.3, page 322)

- It follows from this assumption that

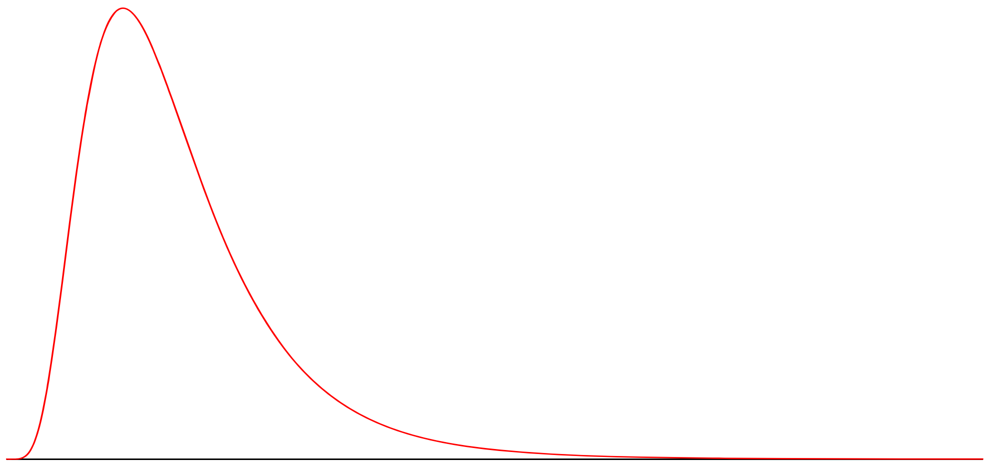
$$\ln S_T - \ln S_0 \approx \phi\left[\left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

or

$$\ln S_T \approx \phi\left[\ln S_0 + \left(\mu - \frac{\sigma^2}{2}\right)T, \sigma^2 T\right]$$

- Since the logarithm of S_T is normal, S_T is lognormally distributed

The Lognormal Distribution



$$E(S_T) = S_0\, e^{\mu T}$$

$$\text{var}(S_T) = S_0^{\,2} e^{2\mu T}(e^{\sigma^2 T} - 1)$$

Continuously Compounded Return *(Equations 15.6 and 15.7, page 324)*

If x is the realized continuously compounded return

$$S_T = S_0 \, e^{xT}$$

$$x = \frac{1}{T} \ln \frac{S_T}{S_0}$$

$$x \approx \phi\left(\mu - \frac{\sigma^2}{2}, \frac{\sigma^2}{T} \right)$$

The Expected Return

- The expected value of the stock price is $S_0 e^{\mu T}$
- The expected return on the stock is
  $\mu - \sigma^2/2$ not μ

  This is because

  $$\ln[E(S_T / S_0)] \quad \text{and} \quad E[\ln(S_T / S_0)]$$

  are not the same

μ and $\mu - \sigma^2/2$

- μ is the expected return in a very short time, Δt, expressed with a compounding frequency of Δt
- $\mu - \sigma^2/2$ is the expected return in a long period of time expressed with continuous compounding (or, to a good approximation, with a compounding frequency of Δt)

Mutual Fund Returns *(See Business Snapshot 15.1 on page 326)*

- Suppose that returns in successive years are 15%, 20%, 30%, −20% and 25% (ann. comp.)
- The arithmetic mean of the returns is 14%
- The returned that would actually be earned over the five years (the geometric mean) is 12.4% (ann. comp.)
- The arithmetic mean of 14% is analogous to μ
- The geometric mean of 12.4% is analogous to $\mu-\sigma^2/2$

The Volatility

- The volatility is the standard deviation of the continuously compounded rate of return in 1 year
- The standard deviation of the return in a short time period time Δt is approximately $\sigma\sqrt{\Delta t}$
- If a stock price is \$50 and its volatility is 25% per year what is the standard deviation of the price change in one day?

Estimating Volatility from Historical Data *(page 326-328)*

1. Take observations $S_0, S_1, \ldots, S_n$ at intervals of τ years (e.g. for weekly data τ = 1/52)
2. Calculate the continuously compounded return in each interval as:

$$u_i = \ln\left(\frac{S_i}{S_{i-1}}\right)$$

3. Calculate the standard deviation, s , of the u_i´s
4. The historical volatility estimate is: $\hat{\sigma} = \frac{s}{\sqrt{\tau}}$

Nature of Volatility *(Business Snapshot 15.2, page 329)*

- Volatility is usually much greater when the market is open (i.e. the asset is trading) than when it is closed
- For this reason time is usually measured in "trading days" not calendar days when options are valued
- It is assumed that there are 252 trading days in one year for most assets

Example

- Suppose it is April 1 and an option lasts to April 30 so that the number of days remaining is 30 calendar days or 22 trading days
- The time to maturity would be assumed to be 22/252 = 0.0873 years

The Concepts Underlying Black-Scholes-Merton

- The option price and the stock price depend on the same underlying source of uncertainty
- We can form a portfolio consisting of the stock and the option which eliminates this source of uncertainty
- The portfolio is instantaneously riskless and must instantaneously earn the risk-free rate
- This leads to the Black-Scholes-Merton differential equation

The Derivation of the Black-Scholes-Merton Differential Equation

$$\Delta S = \mu S\ \Delta t + \sigma S\ \Delta z$$

$$\Delta f = \left(\frac{\partial f}{\partial S}\mu S + \frac{\partial f}{\partial t} + ½\frac{\partial^2 f}{\partial S^2}\sigma^2 S^2 \right)\Delta t + \frac{\partial f}{\partial S}\sigma S\ \Delta z$$

We set up a portfolio consisting of

$$-1:\ \text{derivative}$$

$$+\frac{\partial f}{\partial S}:\ \text{shares}$$

This gets rid of the dependence on Δz.

The Derivation of the Black-Scholes-Merton Differential Equation *continued*

The value of the portfolio, Π, is given by

$$\Pi = -f + \frac{\partial f}{\partial S} S$$

The change in its value in time Δt is given by

$$\Delta\Pi = -\Delta f + \frac{\partial f}{\partial S} \Delta S$$

The Derivation of the Black-Scholes-Merton Differential Equation *continued*

The return on the portfolio must be the risk - free rate. Hence

$$\Delta\Pi = r\,\Pi\Delta t$$

$$-\Delta f + \frac{\partial f}{\partial S}\Delta S = r\left(-f + \frac{\partial f}{\partial S}S\right)\Delta t$$

We substitute for Δf and ΔS in this equation to get the Black - Scholes differential equation :

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \tfrac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2} = rf$$

The Differential Equation

- Any security whose price is dependent on the stock price satisfies the differential equation
- The particular security being valued is determined by the boundary conditions of the differential equation
- In a forward contract the boundary condition is $f = S - K$ when $t = T$
- The solution to the equation is

$$f = S - K e^{-r(T-t)}$$

Perpetual Derivative

- For a perpetual derivative there is no dependence on time and the differential equation becomes

$$rS\frac{df}{dS}+\frac{1}{2}\sigma^2S^2\frac{d^2f}{dS^2}=rf$$

A derivative that pays off Q when $S = H$ is worth QS/H when $S<H$ and $Q(S/H)^{-2r/\sigma^2}$ when $S>H$. (These values satisfy the differential equation and the boundary conditions)

The Black-Scholes-Merton Formulas for Options (See pages 335-336)

$$c = S_0 N(d_1) - K e^{-rT} N(d_2)$$

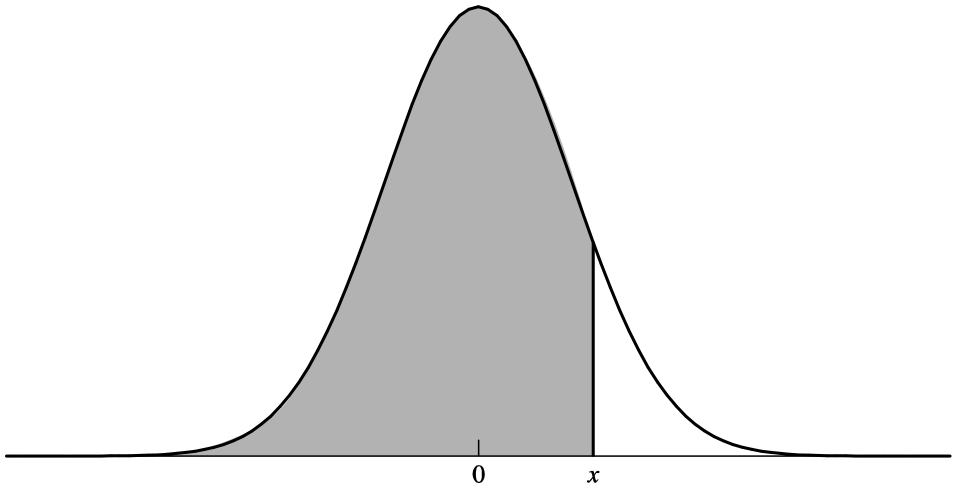
$$p = K e^{-rT} N(-d_2) - S_0 N(-d_1)$$

where $$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0/K) + (r - \sigma^2/2)T}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

The N(x) Function

- $N(x)$ is the probability that a normally distributed variable with a mean of zero and a standard deviation of 1 is less than x



- See tables at the end of the book

Properties of Black-Scholes Formula

- As S_0 becomes very large c tends to $S_0 - Ke^{-rT}$ and p tends to zero
- As S_0 becomes very small c tends to zero and p tends to $Ke^{-rT} - S_0$
- What happens as σ becomes very large?
- What happens as T becomes very large?

Understanding Black-Scholes

$$c = e^{-rT} N(d_2)\left(S_0 e^{rT} N(d_1)/N(d_2) - K\right)$$

e^{-rT} :	Discount rate
$N(d_2)$:	Probability of exercise
$e^{rT} N(d_1)/N(d_2)$:	Expected percentage increase in stock price if option is exercised
K :	Strike price paid if option is exercised

Risk-Neutral Valuation

- The variable μ does not appear in the Black-Scholes-Merton differential equation
- The equation is independent of all variables affected by risk preference
- The solution to the differential equation is therefore the same in a risk-free world as it is in the real world
- This leads to the principle of risk-neutral valuation

Applying Risk-Neutral Valuation

1. Assume that the expected return from the stock price is the risk-free rate
2. Calculate the expected payoff from the option
3. Discount at the risk-free rate

Valuing a Forward Contract with Risk-Neutral Valuation

- Payoff is $S_T - K$
- Expected payoff in a risk-neutral world is $S_0 e^{rT} - K$
- Present value of expected payoff is

$$e^{-rT}[S_0 e^{rT} - K] = S_0 - Ke^{-rT}$$

Proving Black-Scholes-Merton Using Risk-Neutral Valuation *(Appendix to Chapter 15)*

$$c = e^{-rT} \int_K^\infty \max(S_T - K,\ 0) g(S_T) dS_T$$

where $g(S_T)$ is the probability density function for the lognormal distribution of S_T in a risk-neutral world. ln S_T is $\varphi(m,\ s^2)$ where

$$m = \ln S_0 + \left(r - \sigma^2/2\right)T \qquad s = \sigma\sqrt{T}$$

We substitute

$$Q = \frac{\ln S_T - m}{s}$$

so that

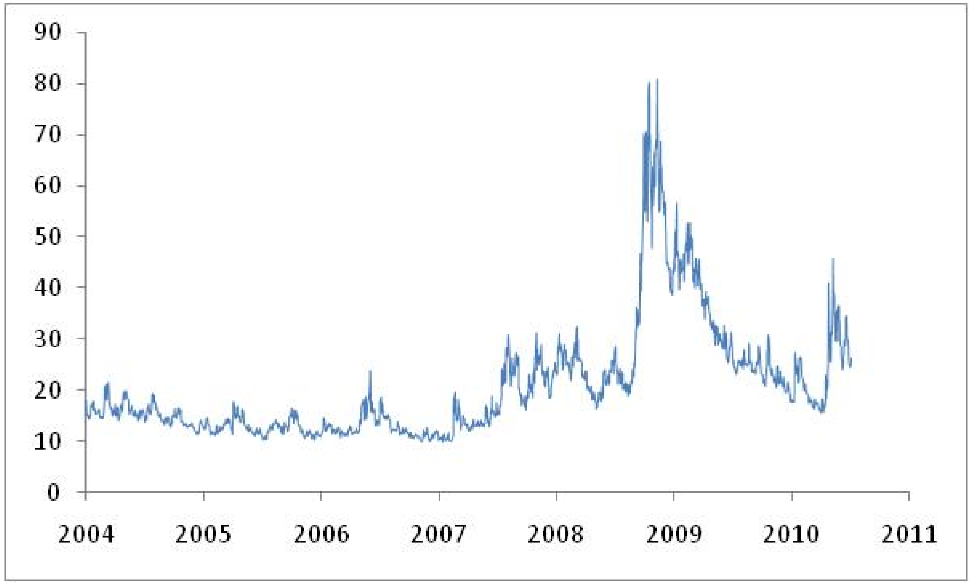
$$c = e^{-rT} \int_{(\ln K - m)/s}^\infty \max(e^{Qs+m} - K,\ 0) h(Q) dQ$$

where h is the probability density function for a standard normal. Evaluating the integral leads to the BSM result.

Implied Volatility

- The implied volatility of an option is the volatility for which the Black-Scholes-Merton price equals the market price
- There is a one-to-one correspondence between prices and implied volatilities
- Traders and brokers often quote implied volatilities rather than dollar prices

The VIX S&P500 Volatility Index



Chapter 26 explains how the index is calculated

An Issue of Warrants & Executive Stock Options

- When a regular call option is exercised the stock that is delivered must be purchased in the open market
- When a warrant or executive stock option is exercised new Treasury stock is issued by the company
- If little or no benefits are foreseen by the market the stock price will reduce at the time the issue of is announced.
- There is no further dilution (See Business Snapshot 15.3.)

The Impact of Dilution

- After the options have been issued it is not necessary to take account of dilution when they are valued
- Before they are issued we can calculate the cost of each option as $N/(N+M)$ times the price of a regular option with the same terms where N is the number of existing shares and M is the number of new shares that will be created if exercise takes place

Dividends

- European options on dividend-paying stocks are valued by substituting the stock price less the present value of dividends into Black-Scholes
- Only dividends with ex-dividend dates during life of option should be included
- The “dividend” should be the expected reduction in the stock price expected

American Calls

- An American call on a non-dividend-paying stock should never be exercised early
- An American call on a dividend-paying stock should only ever be exercised immediately prior to an ex-dividend date
- Suppose dividend dates are at times t_1, t_2, …t_n. Early exercise is sometimes optimal at time t_i if the dividend at that time is greater than

$$K[1-e^{-r(t_{i+1}-t_i)}]$$

Black's Approximation for Dealing with Dividends in American Call Options

Set the American price equal to the maximum of two European prices:

1. The 1st European price is for an option maturing at the same time as the American option
2. The 2nd European price is for an option maturing just before the final ex-dividend date

Chapter 16
Employee Stock Options

Nature of Employee Stock Options

- Employee stock options are call options issued by a company on its own stock
- They are often at-the-money at the time of issue
- They often last as long as 10 years

Typical Features of Employee Stock Options *(page 355)*

- There is a vesting period during which options cannot be exercised
- When employees leave during the vesting period options are forfeited
- When employees leave after the vesting period in-the-money options are exercised immediately and out of the money options are forfeited
- Employees are not permitted to sell options
- When options are exercised the company issues new shares

Exercise Decision

- To realize cash from an employee stock option the employee must exercise the options and sell the underlying shares
- Even when the underlying stock pays no dividend an employee stock option (unlike a regular call option) is often exercised early

Drawbacks of Employee Stock Options

- Gain to executives from good performance is much greater than the penalty for bad performance
- Executives do very well when the stock market as a whole goes up, even if their firm does relatively poorly
- Executives are encouraged to focus on short-term performance at the expense of long-term performance
- Executives are tempted to time announcements or take other decisions that maximize the value of the options

Accounting for Employee Stock Options

- Prior to 1995 the cost of an employee stock option on the income statement was its intrinsic value on the issue date
- After 1995 a “fair value” had to be reported in the notes (but expensing fair value on the income statement was optional)
- Since 2005 both FASB and IASB have required the fair value of options to be charged against income at the time of issue

Traditional At-the-Money Call Options

- The attraction of at-the-money call options used to be that they led to no expense on the income statement because they had zero intrinsic value on the exercise date
- Other plans were liable to lead an expense
- Now that the accounting rules have changed some companies are considering other types of plans

Nontraditional Plans *page 358*

- Strike price is linked to stock index so that the company’s stock price has to outperform the index for options to move in the money
- Strike price increases in a predetermined way
- Options vest only if specified profit targets are met

Valuation of Employee Stock Options

- Most common approach is to use Black-Scholes-Merton with time to maturity equal to an estimate of expected life
- There is no theoretical justification for this but it seems to give reasonable results in most circumstances

Example *(Example 16.1, page 359)*

- A company issues one million10-year ATM options
  - stock price is \$30.
  - It estimates the long term volatility using historical data to be 25% and the average time to exercise to be 4.5 years
  - The 4.5 year interest rate is 5% and dividends during the next 4.5 years are estimated to have a PV of \$4
- Using BSM with S_0 =30, K=30, r=5%, σ=25%, and T=4.5 years gives value of each option equal to \$6.31
- The income statement expense would be \$6.31 million

Other Approaches

- Estimate the probability of exercise as a function of the stock price and remaining life. Use a binomial tree with roll back rules reflecting the probabilities
  - A simple version of this is to assume that the option is exercised when the ratio of the stock price to the strike price reaches some multiple
- Use an auction to determine the market prices of securities whose payoffs mirror the payoffs from the options
  - This is an approach used by Zions Bancorp in 2007

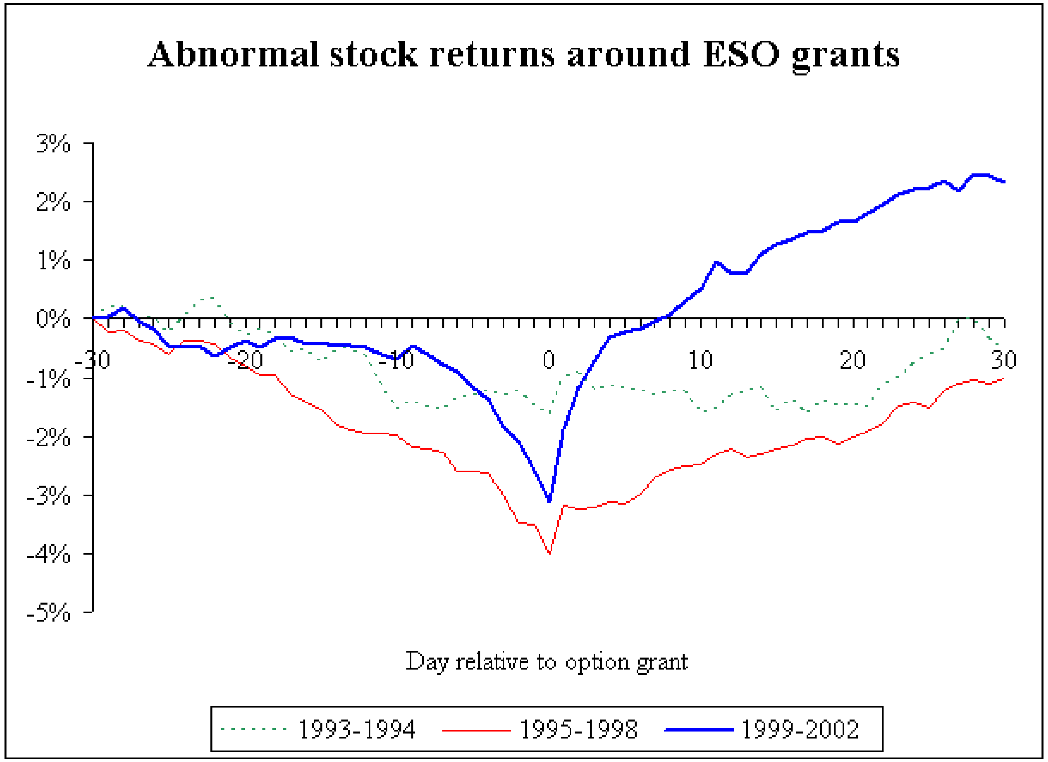
Dilution

- Employee stock options are liable to dilute the interests of shareholders because new shares are bought at below market price
- However this dilution takes place at the time the market hears that the options have been granted (Business Snapshot 15.3)
- It does not take place at the time the options are exercised

Backdating

- Backdating appears to have been a widespread practice in the United States
- A company might take the decision to issue at-the-money options on April 30 when the stock price is $50 and then backdate the grant date to April 3 when the stock price is $42
- Why would they do this?

Academic Research Exposed Backdating *(See Eric Lie's web site: www.biz.uiowa.edu/faculty/elie/backdating.htm*



Chapter 17
Options on Stock Indices and Currencies

Index Options *(page 367-369)*

- The most popular underlying indices in the U.S. are
  - The S&P 100 Index (OEX and XEO)
  - The S&P 500 Index (SPX)
  - The Dow Jones Index times 0.01 (DJX)
  - The Nasdaq 100 Index (NDX)

- Exchange-traded contracts are on 100 times index; they are settled in cash; OEX is American; the XEO and all others are European

Index Option Example

- Consider a call option on an index with a strike price of 880
- Suppose 1 contract is exercised when the index level is 900
- What is the payoff?

Using Index Options for Portfolio Insurance

- Suppose the value of the index is S_0 and the strike price is K
- If a portfolio has a β of 1.0, the portfolio insurance is obtained by buying 1 put option contract on the index for each $100S_0$ dollars held
- If the β is not 1.0, the portfolio manager buys β put options for each $100S_0$ dollars held
- In both cases, K is chosen to give the appropriate insurance level

Example 1

- Portfolio has a beta of 1.0
- It is currently worth $500,000
- The index currently stands at 1000
- What trade is necessary to provide insurance against the portfolio value falling below $450,000?

Example 2

- Portfolio has a beta of 2.0
- It is currently worth $500,000 and index stands at 1000
- The risk-free rate is 12% per annum
- The dividend yield on both the portfolio and the index is 4%
- How many put option contracts should be purchased for portfolio insurance?

Calculating Relation Between Index Level and Portfolio Value in 3 months

- If index rises to 1040, it provides a 40/1000 or 4% return in 3 months
- Total return (incl. dividends) = 5%
- Excess return over risk-free rate = 2%
- Excess return for portfolio = 4%
- Increase in Portfolio Value = 4+3−1=6%
- Portfolio value=$530,000

Determining the Strike Price (Table 17.2, page 369)

Value of Index in 3 months	Expected Portfolio Value in 3 months ($)
1,080	570,000
1,040	530,000
1,000	490,000
960	450,000
920	410,000

An option with a strike price of 960 will provide protection against a 10% decline in the portfolio value

European Options on Assets Providing a Known Yield

We get the same probability distribution for the asset price at time T in each of the following cases:

1. The asset starts at price S_0 and provides a yield = q
2. The asset starts at price S_0e^{-qT} and provides no income

European Options on Assets Providing Known Yield continued

We can value European options by reducing the asset price to S_0e^{-qT} and then behaving as though there is no income

Extension of Chapter 11 Results

(Equations 17.1 to 17.3)

Lower Bound for calls:

$$c \geq \max\left(S_0 e^{-qT} - Ke^{-rT}, 0\right)$$

Lower Bound for puts

$$p \geq \max\left(Ke^{-rT} - S_0 e^{-qT}, 0\right)$$

Put Call Parity

$$c + Ke^{-rT} = p + S_0 e^{-qT} \qquad c + Ke^{-rT} = p + F_0 e^{-rT}$$

Extension of Chapter 15 Results

(Equations 17.4 and 17.5)

$$c = S_0 e^{-qT} N(d_1) - K e^{-rT} N(d_2)$$

$$p = K e^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1)$$

where $$d_1 = \frac{\ln(S_0 / K) + (r - q + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / K) + (r - q - \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

Alternative Formulas *(page 375)*

$$c = e^{-rT}[F_0 N(d_1) - KN(d_2)]$$
$$p = e^{-rT}[KN(-d_2) - F_0 N(-d_1)]$$
$$d_1 = \frac{\ln(F_0 / K) + \sigma^2 T / 2}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

where

$$F_0 = S_0 e^{(r-q)T}$$

Valuing European Index Options

We can use these formulas for an option on an asset paying a dividend yield

Set S_0 = current index level

Set F_0= futures or forward index price for a contract maturing at the same time as the option

Set q = average dividend yield expected during the life of the option

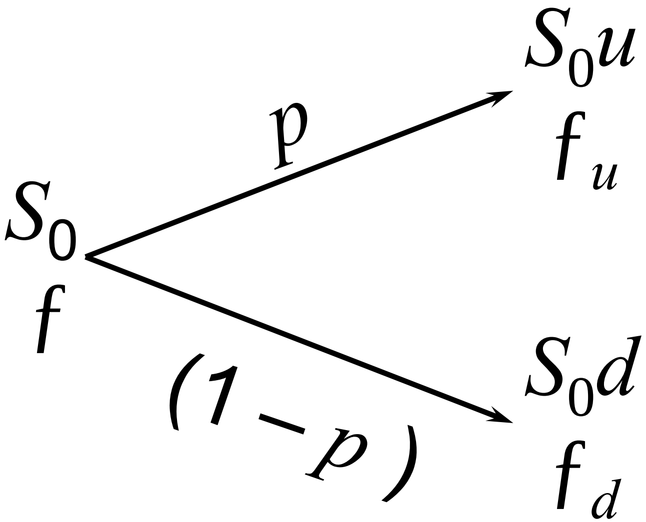
Implied Forward Prices and Dividend Yields

- From European calls and puts with the same strike price and time to maturity

$$F_0 = K + (c - p)e^{rT} \qquad q = -\frac{1}{T}\ln\frac{c - p + Ke^{-rT}}{S_0}$$

- These formulas allow term structures of forward prices and dividend yields to be estimated
- OTC European options are typically valued using the forward prices (Estimates of q are not then required)
- American options require the dividend yield term structure

The Binomial Model



$$f = e^{-rT}[pf_u + (1-p)f_d]$$

The Binomial Model *continued*

- In a risk-neutral world the asset price grows at $r-q$ rather than at r when there is a dividend yield at rate q
- The probability, p, of an up movement must therefore satisfy

$$pS_0u+(1-p)S_0d = S_0e^{(r-q)T}$$

so that

$$p=\frac{e^{(r-q)T}-d}{u-d}$$

Currency Options

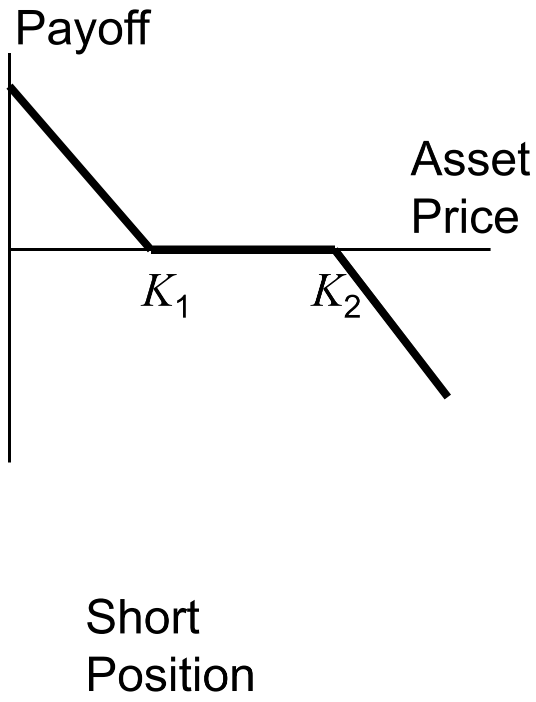
- Currency options trade on NASDAQ OMX
- There also exists a very active over-the-counter (OTC) market
- Currency options are used by corporations to buy insurance when they have an FX exposure

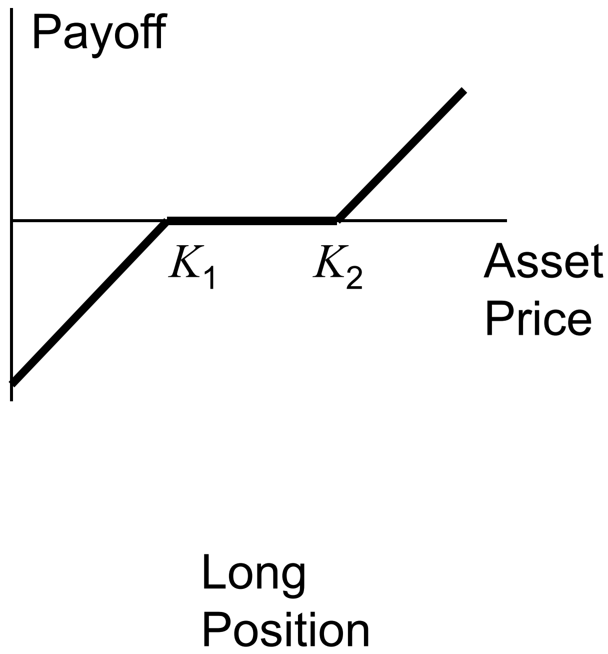
Range Forward Contracts

- Have the effect of ensuring that the exchange rate paid or received will lie within a certain range
- When currency is to be paid it involves selling a put with strike K_1 and buying a call with strike K_2 (with $K_2 > K_1$)
- When currency is to be received it involves buying a put with strike K_1 and selling a call with strike K_2
- Normally the price of the put equals the price of the call

Range Forward Contract *continued*

Figure 17.1, page 370





The Foreign Interest Rate

- We denote the foreign interest rate by r_f
- When a U.S. company buys one unit of the foreign currency it has an investment of S_0 dollars
- The return from investing at the foreign rate is $r_f S_0$ dollars
- This shows that the foreign currency provides a yield at rate r_f

Valuing European Currency Options

- A foreign currency is an asset that provides a yield equal to r_f
- We can use the formula for an option on a stock paying a dividend yield :
  - S_0 = current exchange rate
  - $q = r_f$

Formulas for European Currency Options *(Equations 17.11 and 17.12, page 377)*

$$c = S_0 e^{-r_f T} N(d_1) - K e^{-rT} N(d_2)$$

$$p = K e^{-rT} N(-d_2) - S_0 e^{-r_f T} N(-d_1)$$

$$\text{where} \quad d_1 = \frac{\ln(S_0 / K) + (r - r_f + \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

$$d_2 = \frac{\ln(S_0 / K) + (r - r_f - \sigma^2 / 2)T}{\sigma\sqrt{T}}$$

Alternative Formulas (Equations 17.13 and 17.14)

Using $F_0 = S_0 e^{(r-r_f)T}$

$$c = e^{-rT}[F_0 N(d_1) - KN(d_2)]$$

$$p = e^{-rT}[KN(-d_2) - F_0 N(-d_1)]$$

$$d_1 = \frac{\ln(F_0/K) + \sigma^2 T/2}{\sigma\sqrt{T}}$$

$$d_2 = d_1 - \sigma\sqrt{T}$$

Chapter 18
Futures Options

Options on Futures

- Referred to by the maturity month of the underlying futures
- The option is American and usually expires on or a few days before the earliest delivery date of the underlying futures contract

Mechanics of Call Futures Options

- When a call futures option is exercised the holder acquires
  - A long position in the futures
  - A cash amount equal to the excess of the futures price at the time of the most recent settlement over the strike price

Mechanics of Put Futures Option

- When a put futures option is exercised the holder acquires
  - A short position in the futures
  - A cash amount equal to the excess of the strike price over the futures price at the time of the most recent settlement

Example 18.1 *(page 383-384)*

- Sept. call option contract on copper futures has a strike of 320 cents per pound. It is exercised when futures price is 331 cents and most recent settlement is 330. One contract is on 25,000 pounds
- Trader receives
  - Long Sept. futures contract on copper
  - 25,000 times 10 cents or $2,500 in cash

Example 18.2 *(page 384)*

- Dec put option contract on corn futures has a strike price of 600 cents per bushel. It is exercised when the futures price is 580 cents per bushel and the most recent settlement price is 579 cents per bushel. One contract is on 5000 bushels
- Trader receives
  - Short Dec futures contract on corn
  - $1,050 in cash

The Payoffs

If the futures position is closed out immediately:

Payoff from call = $F - K$

Payoff from put = $K - F$

where F is futures price at time of exercise

Interest Rate Futures Options

- Options on T-Bond futures (quoted as percentage of face value to the nearest 1/64 of 1%)
- Options on Eurodollar futures. Each one basis point in the quote represents $25
- If you think interest rates will go up should you buy call or put options?

Potential Advantages of Futures Options over Spot Options

- Futures contracts may be easier to trade and more liquid than the underlying asset
- Exercise of option does not lead to delivery of underlying asset
- Futures options and futures usually trade on same exchange
- Futures options may entail lower transactions costs

European Futures Options

- European futures options and spot options are equivalent when futures contract matures at the same time as the option
- It is common to regard European spot options as European futures options when they are valued in the over-the-counter markets

Put-Call Parity for Futures Options

(Equation 18.1, page 387)

Consider the following two portfolios:

1. European call plus Ke^{-rT} of cash
2. European put plus long futures plus cash equal to F_0e^{-rT}

They must be worth the same at time T so that

$$c + Ke^{-rT} = p + F_0e^{-rT}$$

Other Relations

$$F_0 e^{-rT} - K < C - P < F_0 - Ke^{-rT}$$

$$c > (F_0 - K)e^{-rT}$$

$$p > (F_0 - K)e^{-rT}$$

Binomial Tree Example

A 1-month call option on futures has a strike price of 29.

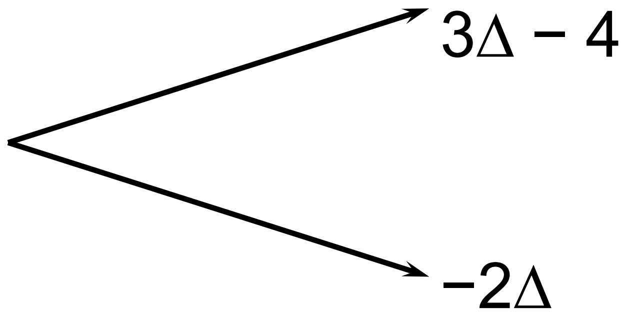
Futures Price = $33
Option Price = $4

Futures Price = $30
Option Price = ?

Futures Price = $28
Option Price = $0

Setting Up a Riskless Portfolio

- Consider the Portfolio: long Δ futures
  short 1 call option



- Portfolio is riskless when $3\Delta - 4 = -2\Delta$ or $\Delta = 0.8$

Valuing the Portfolio
(Risk-Free Rate is 6%)

- The riskless portfolio is:

  long 0.8 futures
  short 1 call option

- The value of the portfolio in 1 month is

  $$-1.6$$

- The value of the portfolio today is

  $$-1.6e^{-0.06/12} = -1.592$$

Valuing the Option

- The portfolio that is

  long 0.8 futures
  short 1 option

  is worth −1.592
- The value of the futures is zero
- The value of the option must therefore be 1.592

Generalization of Binomial Tree Example *(Figure 18.2, page 390)*

- A derivative lasts for time T and is dependent on a futures price

$$\begin{tikzcd}[row sep=small] & F_0 u \\ & f_u \\ F_0 \arrow[uur] \arrow[dr] & \\ f & F_0 d \\ & f_d \end{tikzcd}$$

Generalization
(continued)

- Consider the portfolio that is long Δ futures and short 1 derivative

$$F_0 u\,\Delta - F_0\,\Delta - f_u$$

$$F_0 d\,\Delta - F_0\Delta - f_d$$

- The portfolio is riskless when

$$\Delta = \frac{f_u - f_d}{F_0 u - F_0 d}$$

Generalization
(continued)

- Value of the portfolio at time T is

$$F_0 u\,\Delta - F_0\Delta - f_u$$

- Value of portfolio today is $-f$
- Hence

$$f = -\,[F_0 u\Delta - F_0\Delta - f_u]e^{-rT}$$

Generalization
(continued)

- Substituting for Δ we obtain

$$f = [\, p f_u + (1 - p) f_d \,] e^{-rT}$$

where

$$p = \frac{1 - d}{u - d}$$

Growth Rates For Futures Prices

- A futures contract requires no initial investment
- In a risk-neutral world the expected return should be zero
- The expected growth rate of the futures price is therefore zero
- The futures price can therefore be treated like a stock paying a dividend yield of r

Valuing European Futures Options

- We can use the formula for an option on a stock paying a dividend yield
  - S_0 = current futures price, F_0
  - q = domestic risk-free rate, r
- Setting $q = r$ ensures that the expected growth of F in a risk-neutral world is zero
- The result is referred to as Black's model because it was first suggested in a paper by Fischer Black in 1976

Black's Model *(Equations 18.9 and 18.10, page 392)*

$$c = e^{-rT}\left[F_0\ N(d_1) - K\ N(d_2)\right]$$

$$p = e^{-rT}\left[K\ N(-d_2) - F_0\ N(-d_1)\right]$$

where $d_1 = \dfrac{\ln(F_0 / K) + \sigma^2 T / 2}{\sigma\sqrt{T}}$

$$d_2 = \frac{\ln(F_0 / K) - \sigma^2 T / 2}{\sigma\sqrt{T}} = d_1 - \sigma\sqrt{T}$$

How Black's Model is Used in Practice

- Black's model is frequently used to value European options on the spot price of an asset in the over-the-counter market
- This avoids the need to estimate income on the asset

Using Black's Model Instead of Black-Scholes-Merton *(Example 18.7, page 393)*

- Consider a 6-month European call option on spot gold
- 6-month futures price is 1,240, 6-month risk-free rate is 5%, strike price is 1,200, and volatility of futures price is 20%
- Value of option is given by Black's model with F_0 = 1,240, K=1,200, r = 0.05, T=0.5, and σ = 0.2
- It is 88.37

Futures Option Price vs Spot Option Price

- If futures prices are higher than spot prices (normal market), an American call on futures is worth more than a similar American call on spot. An American put on futures is worth less than a similar American put on spot.
- When futures prices are lower than spot prices (inverted market) the reverse is true.

Futures Style Options *(page 394-95)*

- A futures-style option is a futures contract on the option payoff
- Some exchanges trade these in preference to regular futures options
- The futures price for a call futures-style option is

$$F_0 N(d_1) - KN(d_2)$$

- The futures price for a put futures-style option is

$$KN(-d_2) - F_0 N(-d_1)$$

Put-Call Parity Results

Non-Dividend-Paying Stock

$$c + Ke^{-rT} = p + S$$

Indices:

$$c + Ke^{-rT} = p + S_0 e^{-qT}$$

Foreign exchange:

$$c + Ke^{-rT} = p + S_0 e^{-r_f T}$$

Futures:

$$c + Ke^{-rT} = p + F_0 e^{-rT}$$

Summary of Key Results from Chapters 17 and 18

- We can treat stock indices, currencies, and futures like a stock paying a dividend yield of q
  - For stock indices, q is average dividend yield on the index over the option life
  - For currencies, $q = r_f$
  - For futures, $q = r$

Chapter 19
The Greek Letters

Example

- A bank has sold for \$300,000 a European call option on 100,000 shares of a non-dividend paying stock
- $S_0 = 49$, $K = 50$, $r = 5\%$, $\sigma = 20\%$,
  $T = 20$ weeks, $\mu = 13\%$
- The Black-Scholes-Merton value of the option is \$240,000
- How does the bank hedge its risk to lock in a \$60,000 profit?

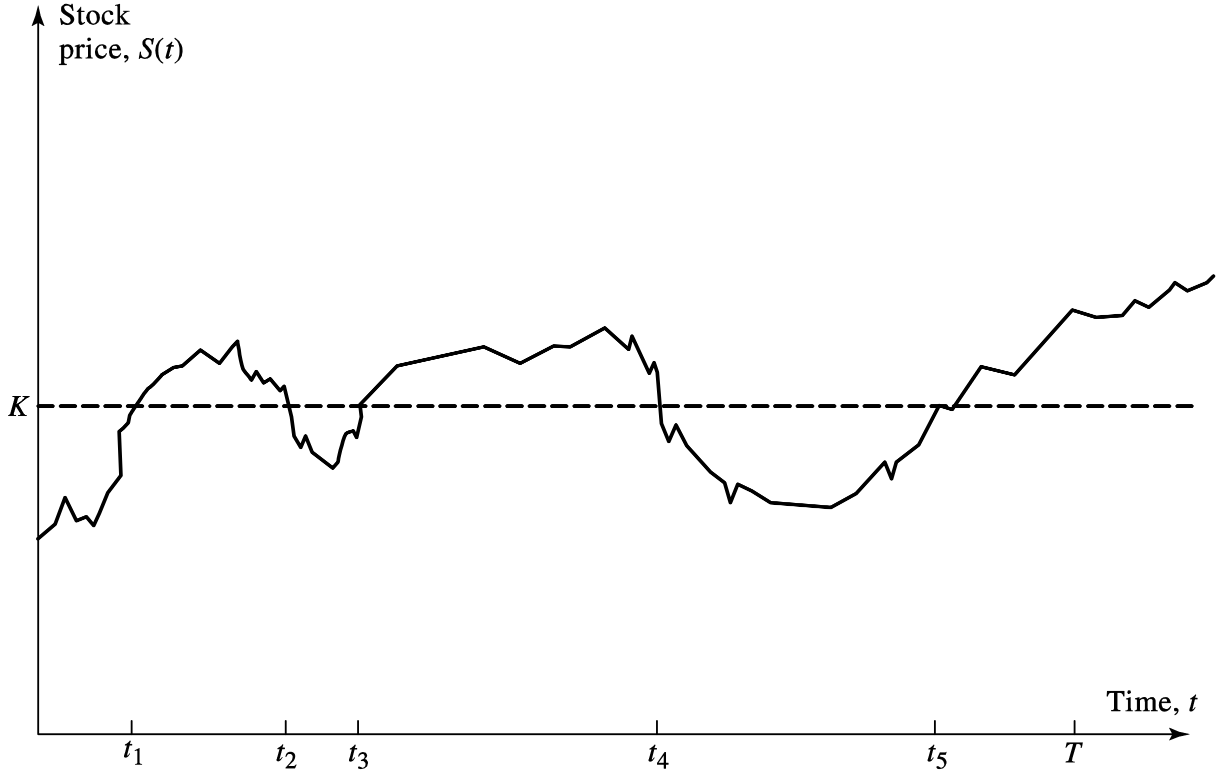
Naked & Covered Positions

- Naked position
  - Take no action
- Covered position
  - Buy 100,000 shares today
- What are the risks associated with these strategies?

Stop-Loss Strategy

- This involves:
  - Buying 100,000 shares as soon as price reaches \$50
  - Selling 100,000 shares as soon as price falls below \$50

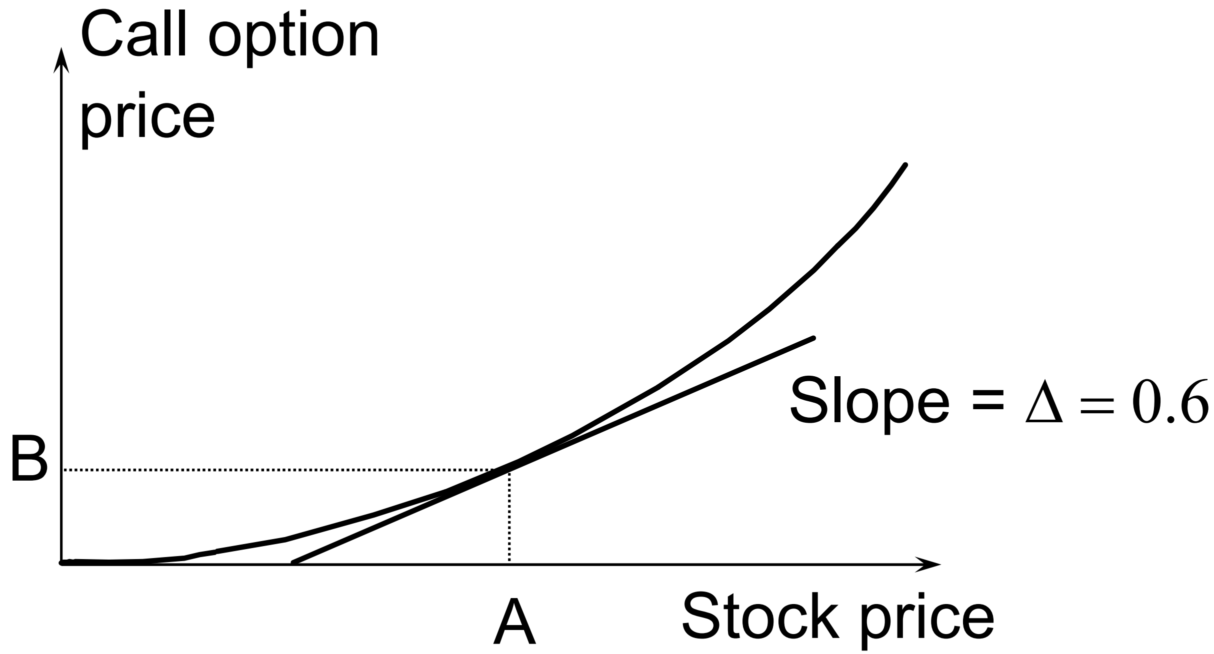
Stop-Loss Strategy *continued*



Ignoring discounting, the cost of writing and hedging the option appears to be max(S_0−K, 0). What are we overlooking?

Delta *(See Figure 19.2, page 403)*

- Delta (Δ) is the rate of change of the option price with respect to the underlying



Hedge

- Trader would be hedged with the position:
  - short 1000 options
  - buy 600 shares
- Gain/loss on the option position is offset by loss/gain on stock position
- Delta changes as stock price changes and time passes
- Hedge position must therefore be rebalanced

Delta Hedging

- This involves maintaining a delta neutral portfolio
- The delta of a European call on a non-dividend paying stock is $N(d_1)$
- The delta of a European put on the stock is

$$N(d_1) - 1$$

The Costs in Delta Hedging continued

- Delta hedging a written option involves a "buy high, sell low" trading rule

First Scenario for the Example:

Table 19.2 page 406

Week	Stock price	Delta	Shares purchased	Cost (‘$000)	Cumulative Cost ($000)	Interest
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	48.12	0.458	(6,400)	(308.0)	2,252.3	2.2
2	47.37	0.400	(5,800)	(274.7)	1,979.8	1.9
…….	…….	…….	…….	…….	…….	…….
19	55.87	1.000	1,000	55.9	5,258.2	5.1
20	57.25	1.000	0	0	5263.3	

Second Scenario for the Example

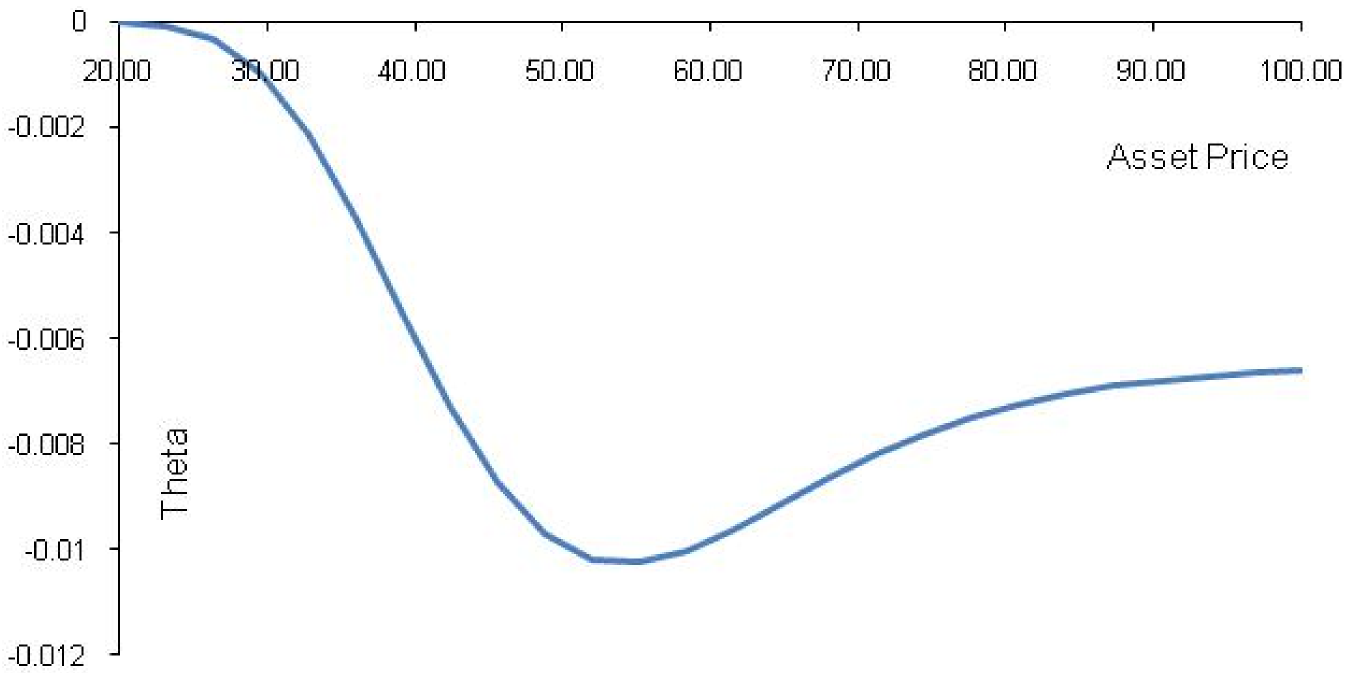
Table 19.3, page 407

Week	Stock price	Delta	Shares purchased	Cost (‘$000)	Cumulative Cost ($000)	Interest
0	49.00	0.522	52,200	2,557.8	2,557.8	2.5
1	49.75	0.568	4,600	228.9	2,789.2	2.7
2	52.00	0.705	13,700	712.4	3,504.3	3.4
....	....	....	....	....	....	....
19	46.63	0.007	(17,600)	(820.7)	290.0	0.3
20	48.12	0.000	(700)	(33.7)	256.6	

Theta

- Theta (Θ) of a derivative (or portfolio of derivatives) is the rate of change of the value with respect to the passage of time
- The theta of a call or put is usually negative. This means that, if time passes with the price of the underlying asset and its volatility remaining the same, the value of a long call or put option declines

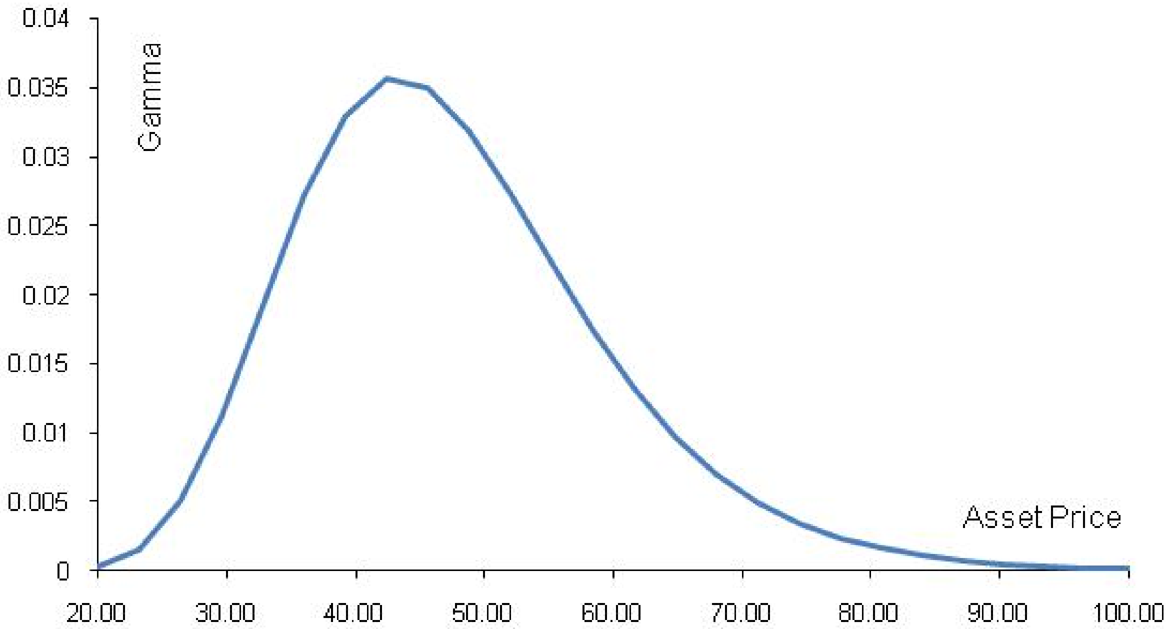
Theta for Call Option: K=50, σ = 25%, r = 5% T = 1



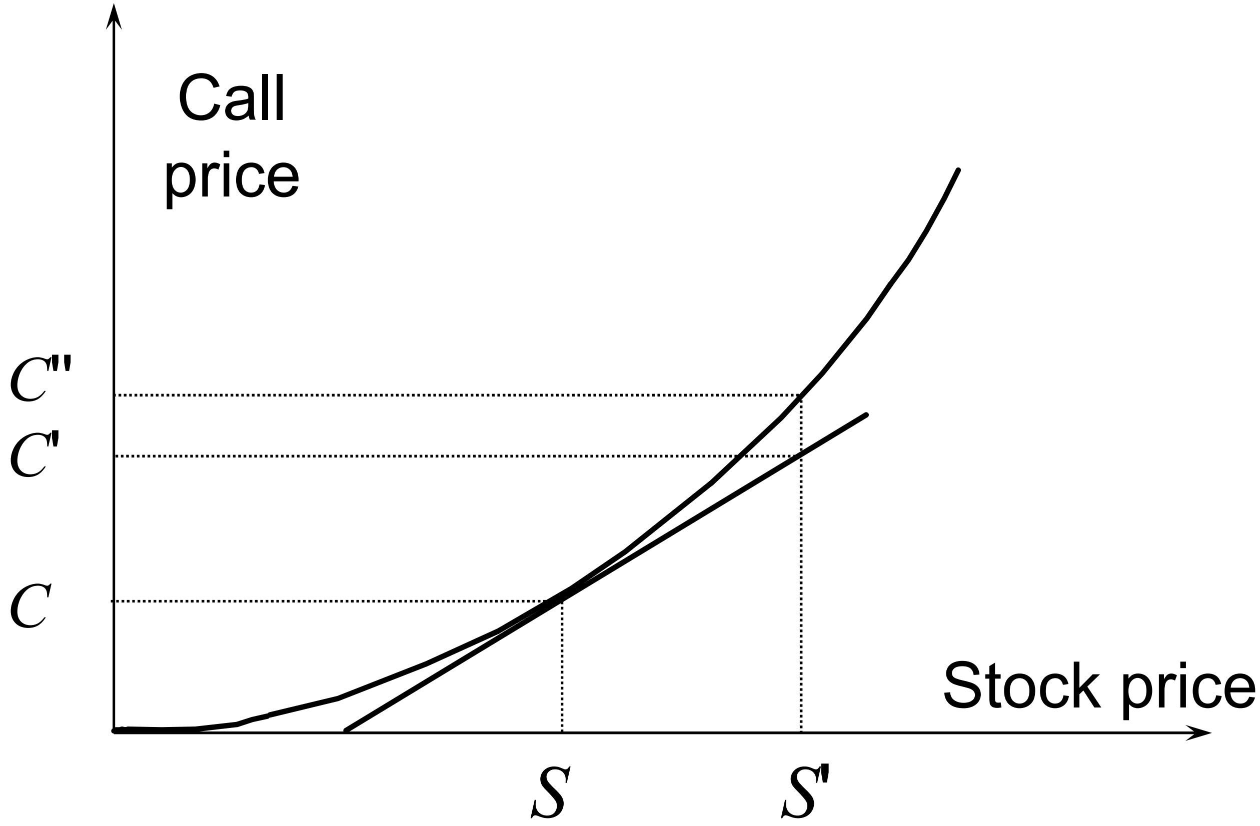
Gamma

- Gamma (Γ) is the rate of change of delta (Δ) with respect to the price of the underlying asset
- Gamma is greatest for options that are close to the money

Gamma for Call or Put Option: K=50, σ = 25%, r = 5% T = 1

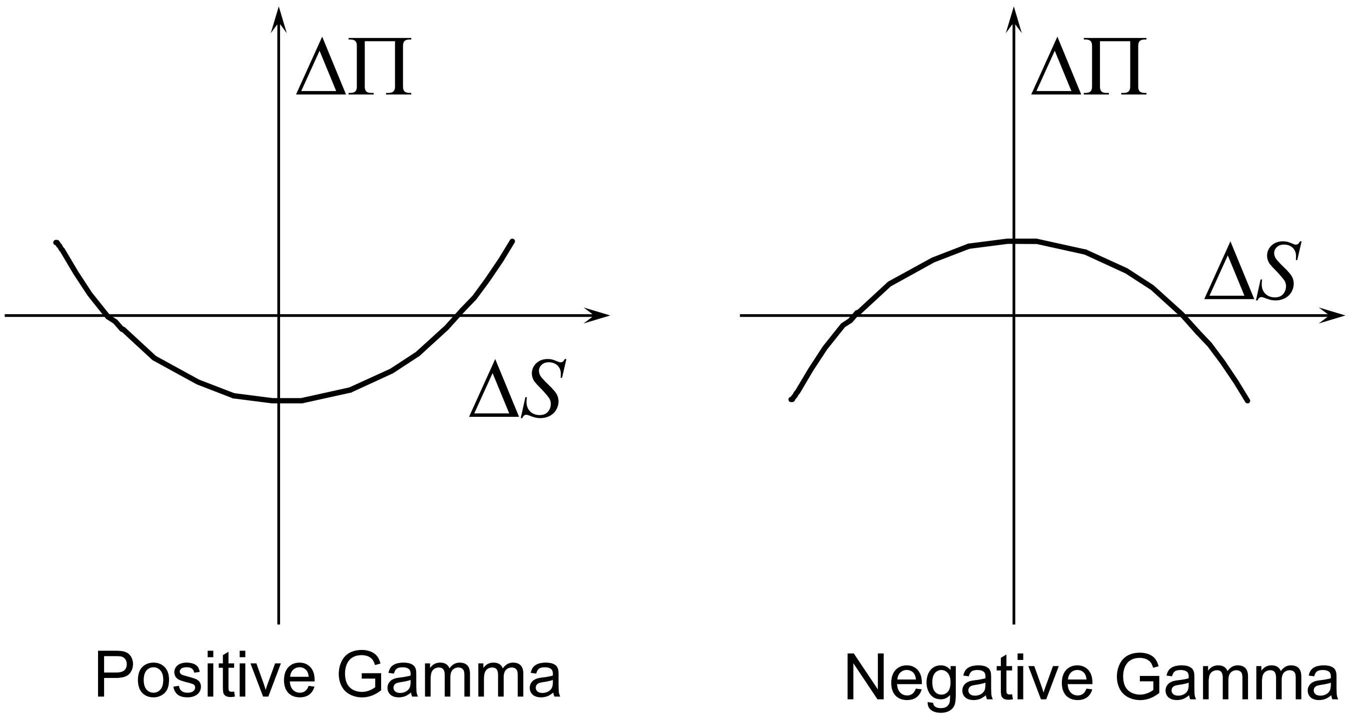


Gamma Addresses Delta Hedging Errors Caused By Curvature *(Figure 19.7, page 411)*



Interpretation of Gamma

For a delta neutral portfolio, $\Delta\Pi \approx \Theta\,\Delta t + \frac{1}{2}\Gamma\Delta S^2$



Relationship Between Delta, Gamma, and Theta *(page 415)*

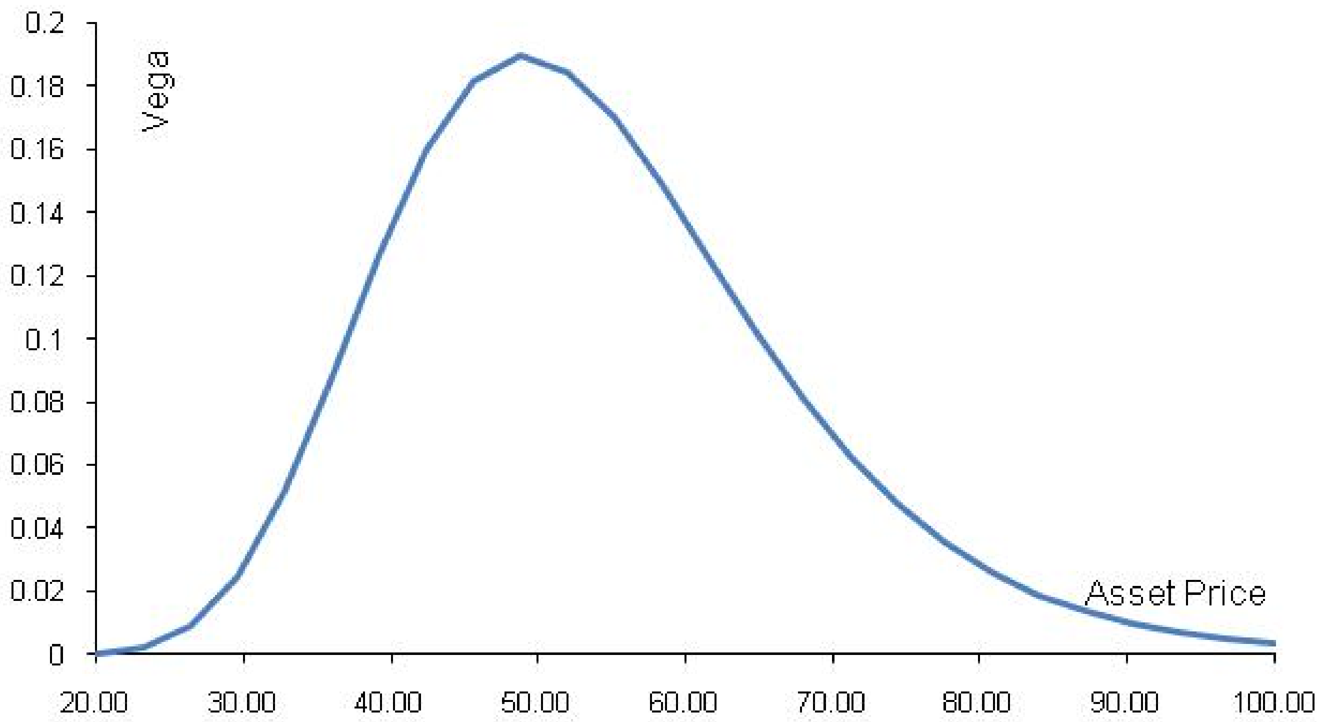
For a portfolio of derivatives on a stock paying a continuous dividend yield at rate q it follows from the Black-Scholes-Merton differential equation that

$$\Theta + rS\Delta + \frac{1}{2}\sigma^2 S^2 \Gamma = r\Pi$$

Vega

- Vega (ν) is the rate of change of the value of a derivatives portfolio with respect to volatility

Vega for Call or Put Option: K=50, σ = 25%, r = 5% T = 1



Taylor Series Expansion *(Appendix to Chapter 19)*

- The value of a portfolio of derivatives dependent on an asset is a function of of the asset price S, its volatility σ, and time t

$$\Delta\Pi = \frac{\partial\Pi}{\partial S}\Delta S + \frac{\partial\Pi}{\partial\sigma}\Delta\sigma + \frac{\partial\Pi}{\partial t}\Delta t + \frac{1}{2}\frac{\partial^2\Pi}{\partial S^2}(\Delta S)^2 + \ldots$$

$$= \text{Delta}\times\Delta S + \text{Vega}\times\Delta\sigma + \text{Theta}\times\Delta t + \frac{1}{2}\text{Gamma}\times(\Delta S)^2 + \ldots$$

Managing Delta, Gamma, & Vega

- Delta can be changed by taking a position in the underlying asset
- To adjust gamma and vega it is necessary to take a position in an option or other derivative

Example

	Delta	*Gamma*	*Vega*
Portfolio	0	−5000	−8000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

What position in option 1 and the underlying asset will make the portfolio delta and gamma neutral? Answer: Long 10,000 options, short 6000 of the asset

What position in option 1 and the underlying asset will make the portfolio delta and vega neutral? Answer: Long 4000 options, short 2400 of the asset

Example *continued*

	Delta	*Gamma*	*Vega*
Portfolio	0	−5000	−8000
Option 1	0.6	0.5	2.0
Option 2	0.5	0.8	1.2

What position in option 1, option 2, and the asset will make the portfolio delta, gamma, and vega neutral?

We solve

$$-5000+0.5w_1+0.8w_2=0$$

$$-8000+2.0w_1+1.2w_2=0$$

to get $w_1 = 400$ and $w_2 = 6000$. We require long positions of 400 and 6000 in option 1 and option 2. A short position of 3240 in the asset is then required to make the portfolio delta neutral

Rho

- Rho is the rate of change of the value of a derivative with respect to the interest rate

Hedging in Practice

- Traders usually ensure that their portfolios are delta-neutral at least once a day
- Whenever the opportunity arises, they improve gamma and vega
- There are economies of scale
  - As portfolio becomes larger hedging becomes less expensive per option in the portfolio

Scenario Analysis

A scenario analysis involves testing the effect on the value of a portfolio of different assumptions concerning asset prices and their volatilities

Greek Letters for European Options on an Asset that Provides a Yield at Rate q

Greek Letter	*Call Option*	*Put Option*
Delta	$e^{-qT}N(d_1)$	$e^{-qT}[N(d_1)-1]$
Gamma	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$	$\dfrac{N'(d_1)e^{-qT}}{S_0\sigma\sqrt{T}}$
Theta	$-S_0N'(d_1)\sigma e^{-qT}/\left(2\sqrt{T}\right)$ $+qS_0N(d_1)e^{-qT}-rKe^{-rT}N(d_2)$	$-S_0N'(d_1)\sigma e^{-qT}/\left(2\sqrt{T}\right)$ $+qS_0N(-d_1)e^{-qT}+rKe^{-rT}N(-d_2)$
Vega	$S_0\sqrt{T}N'(d_1)e^{-qT}$	$S_0\sqrt{T}N'(d_1)e^{-qT}$
Rho	$KTe^{-rT}N(d_2)$	$-KTe^{-rT}N(-d_2)$

Futures Contract Can Be Used for Hedging

- The delta of a futures contract on an asset paying a yield at rate q is $e^{(r-q)T}$ times the delta of a spot contract
- The position required in futures for delta hedging is therefore $e^{-(r-q)T}$ times the position required in the corresponding spot contract

Hedging vs Creation of an Option Synthetically

- When we are hedging we take positions that offset delta, gamma, vega, etc
- When we create an option synthetically we take positions that matchdelta, gamma, vega, etc

Portfolio Insurance

- In October of 1987 many portfolio managers attempted to create a put option on a portfolio synthetically
- This involves initially selling enough of the portfolio (or of index futures) to match the Δ of the put option

Portfolio Insurance continued

- As the value of the portfolio increases, the Δ of the put becomes less negative and some of the original portfolio is repurchased
- As the value of the portfolio decreases, the Δ of the put becomes more negative and more of the portfolio must be sold

Portfolio Insurance continued

The strategy did not work well on October 19, 1987...

Chapter 20
Volatility Smiles

What is a Volatility Smile?

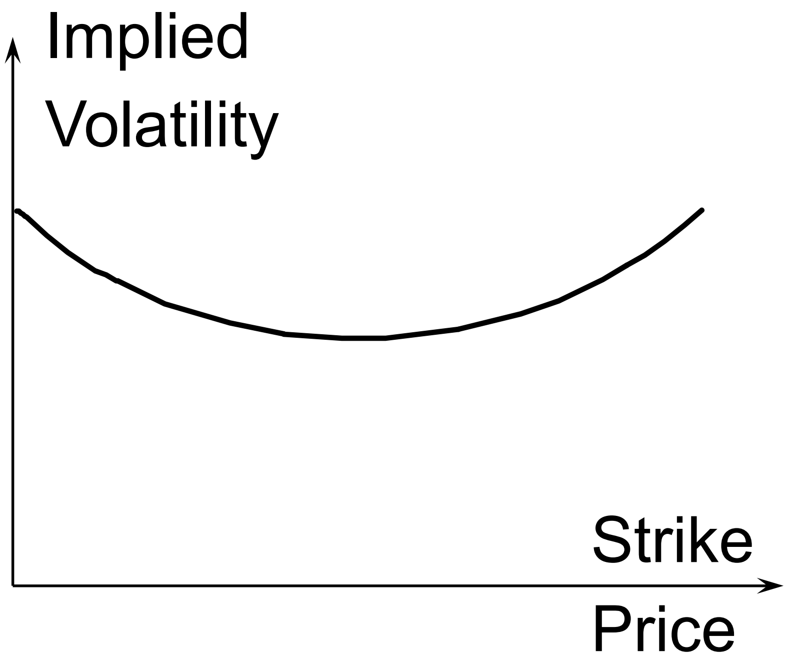
- It is the relationship between implied volatility and strike price for options with a certain maturity
- The volatility smile for European call options should be exactly the same as that for European put options
- The same is at least approximately true for American options

Why the Volatility Smile is the Same for European Calls and Put

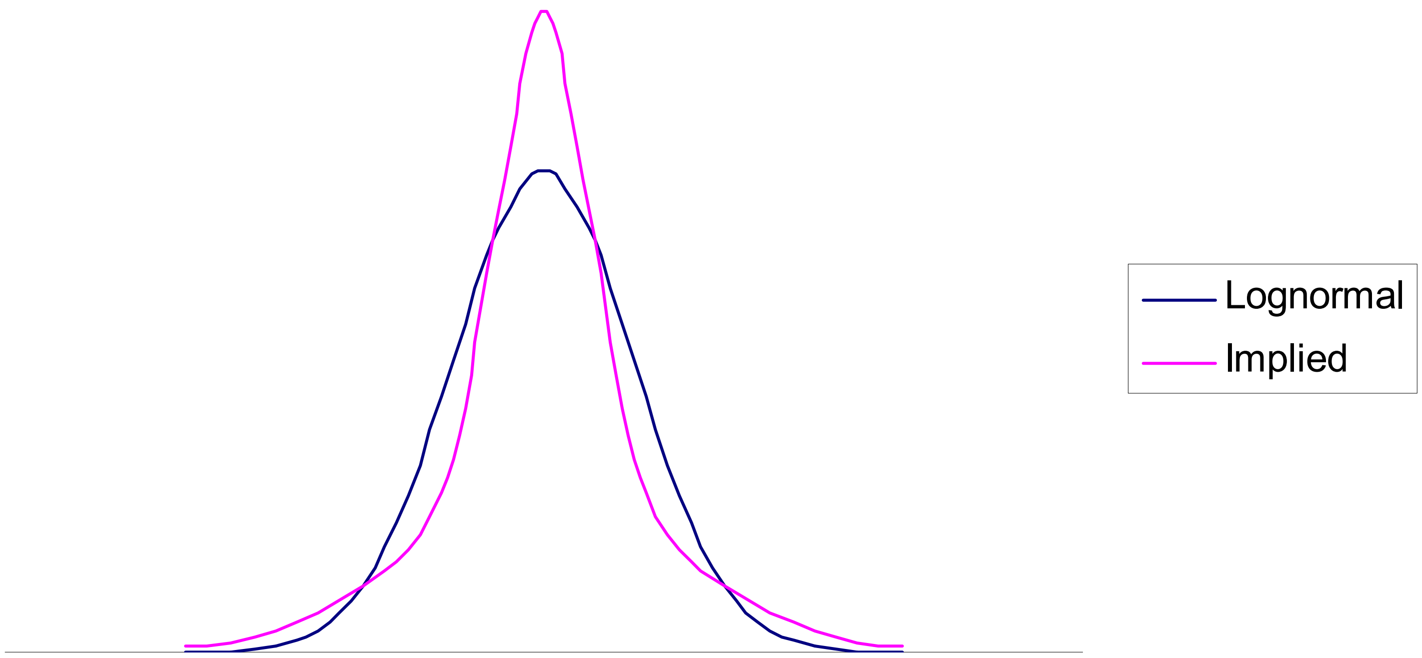
- Put-call parity $p + S_0 e^{-qT} = c + K e^{-rT}$ holds for market prices (p_{mkt} and c_{mkt}) and for Black-Scholes-Merton prices (p_{bs} and c_{bs})
- As a result, $p_{\mathrm{mkt}} - p_{\mathrm{bs}} = c_{\mathrm{mkt}} - c_{\mathrm{bs}}$
- When $p_{\mathrm{bs}} = p_{\mathrm{mkt}}$, it must be true that $c_{\mathrm{bs}} = c_{\mathrm{mkt}}$
- It follows that the implied volatility calculated from a European call option should be the same as that calculated from a European put option when both have the same strike price and maturity

The Volatility Smile for Foreign Currency Options

(Figure 20.1, page 433)



Implied Distribution for Foreign Currency Options



Properties of Implied Distribution for Foreign Currency Options

- Both tails are heavier than the lognormal distribution
- It is also "more peaked" than the lognormal distribution

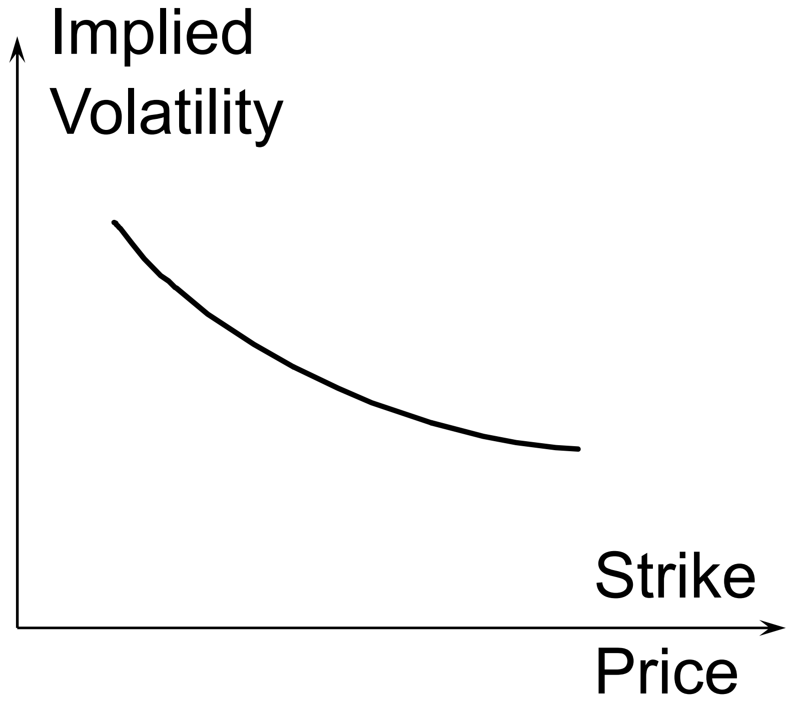
Possible Causes of Volatility Smile for Foreign Currencies

- Exchange rate exhibits jumps rather than continuous changes
- Volatility of exchange rate is stochastic

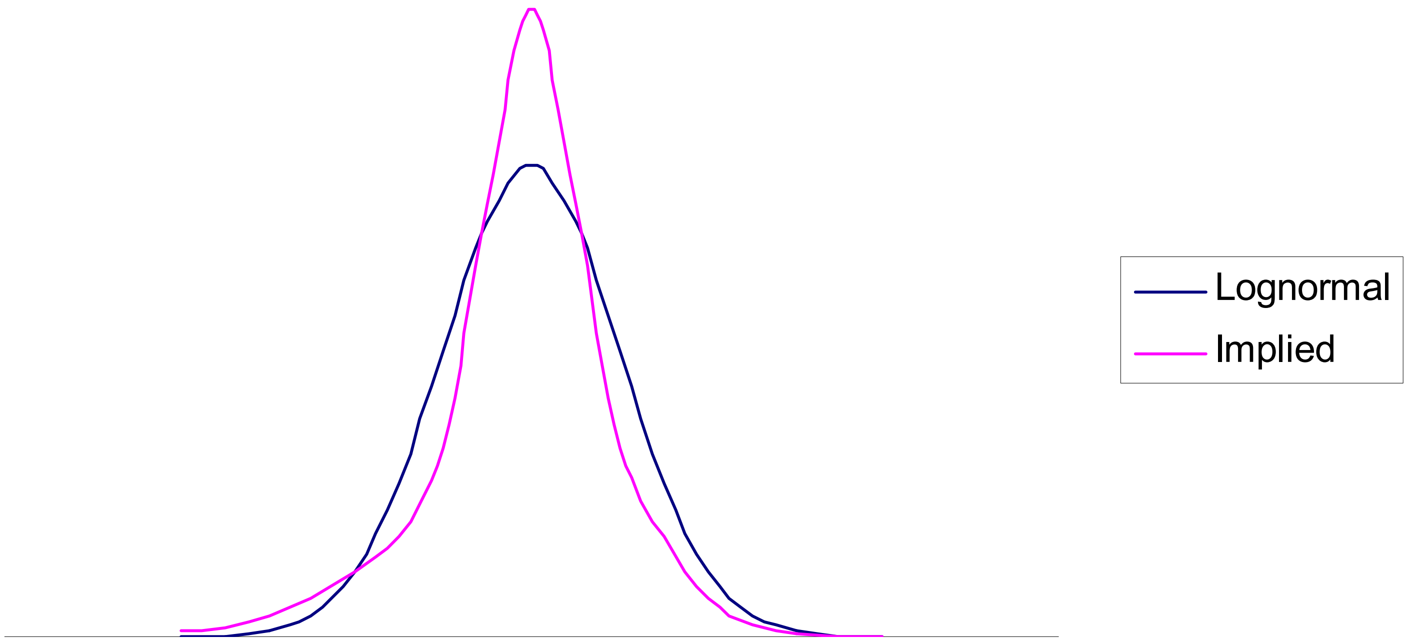
Historical Analysis of Exchange Rate Changes

	Real World (%)	Normal Model (%)
>1 SD	25.04	31.73
>2SD	5.27	4.55
>3SD	1.34	0.27
>4SD	0.29	0.01
>5SD	0.08	0.00
>6SD	0.03	0.00

The Volatility Smile for Equity Options *(Figure 20.3, page 436)*



Implied Distribution for Equity Options



Properties of Implied Distribution for Equity Options

- The left tail is heavier than the lognormal distribution
- The right tail is less heavy than the lognormal distribution

Reasons for Smile in Equity Options

- Leverage
- Crashophobia

Other Volatility Smiles?

- What is the volatility smile if
  - True distribution has a less heavy left tail and heavier right tail
  - True distribution has both a less heavy left tail and a less heavy right tail

Ways of Characterizing the Volatility Smiles

- Plot implied volatility against K/S_0
- Plot implied volatility against K/F_0
  - Note: traders frequently define an option as at-the-money when K equals the forward price, F_0, not when it equals the spot price S_0
- Plot implied volatility against delta of the option
  - Note: traders sometimes define at-the money as a call with a delta of 0.5 or a put with a delta of −0.5. These are referred to as “50-delta options”

Volatility Term Structure

- In addition to calculating a volatility smile, traders also calculate a volatility term structure
- This shows the variation of implied volatility with the time to maturity of the option
- The volatility term structure tends to be downward sloping when volatility is high and upward sloping when it is low

Volatility Surface

- The implied volatility as a function of the strike price and time to maturity is known as a volatility surface

Example of a Volatility Surface

(Table 20.2, page 439)

	K/S_0				
	0.90	0.95	1.00	1.05	1.10
1 mnth	14.2	13.0	12.0	13.1	14.5
3 mnth	14.0	13.0	12.0	13.1	14.2
6 mnth	14.1	13.3	12.5	13.4	14.3
1 year	14.7	14.0	13.5	14.0	14.8
2 year	15.0	14.4	14.0	14.5	15.1
5 year	14.8	14.6	14.4	14.7	15.0

Greek Letters

- If the Black-Scholes price, c_{BS} is expressed as a function of the stock price, S, and the implied volatility, σ_{imp}, the delta of a call is

$$\frac{\partial c_{\text{BS}}}{\partial S} + \frac{\partial c_{\text{BS}}}{\partial \sigma_{\text{imp}}} \frac{\partial \sigma_{\text{imp}}}{\partial S}$$

- Is the delta higher or lower than

$$\frac{\partial c_{\text{BS}}}{\partial S}$$

for equities?

Volatility Smiles When a Large Jump is Expected *(pages 440 to 442)*

- At the money implied volatilities are higher that in-the-money or out-of-the-money options (so that the smile is a frown!)

Determining the Implied Distribution *(Appendix to Chapter 20)*

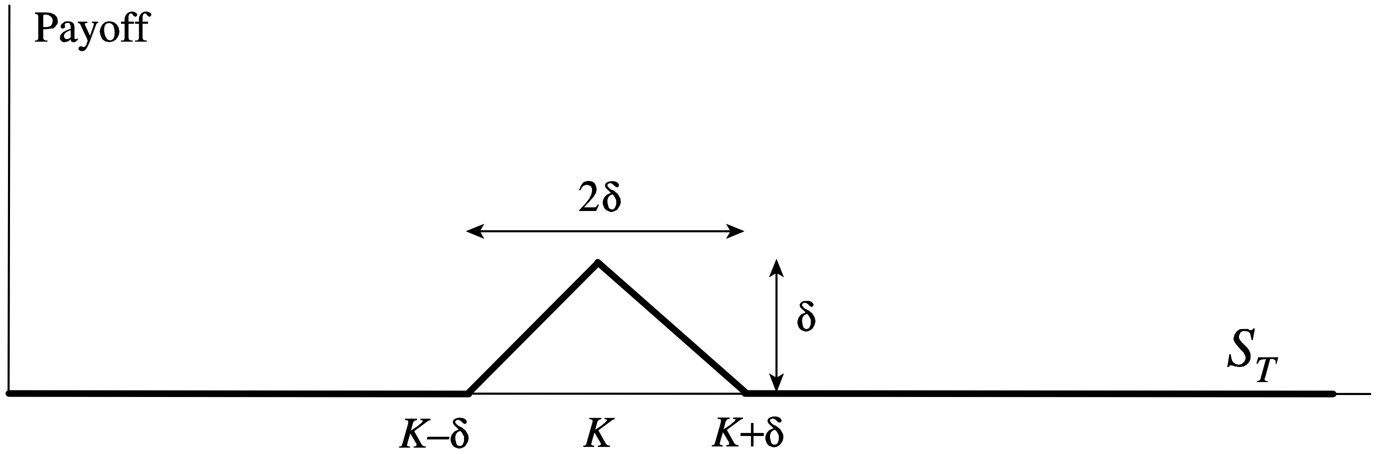
$$c = e^{-rT} \int_{S_T = K}^{\infty} (S_T - K) g(S_T) dS_T$$

$$\frac{\partial^2 c}{\partial K^2} = e^{-rT} g(K)$$

If c_1, c_2, and c_3 are call prices for strikes $K - \delta, K$, and $K + \delta$ then

$$g(K) = e^{rT} \frac{c_1 + c_3 - 2c_2}{\delta^2}$$

A Geometric Interpretation (Figure 20A.1, page 448)



Assuming that density is $g(K)$ from $K-\delta$ to $K+\delta$, $c_1 + c_3 - c_2 = e^{-rT}\,\delta^2\,g(K)$

Chapter 21
Basic Numerical Procedures

Approaches to Derivatives Valuation

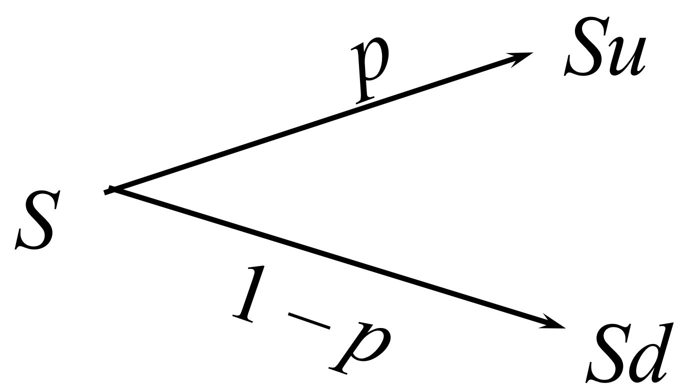
- Trees
- Monte Carlo simulation
- Finite difference methods

Binomial Trees

- Binomial trees are frequently used to approximate the movements in the price of a stock or other asset
- In each small interval of time the stock price is assumed to move up by a proportional amount u or to move down by a proportional amount d

Movements in Time Δt

(Figure 21.1, page 451)



Tree Parameters for asset paying a dividend yield of q

Parameters p, u, and d are chosen so that the tree gives correct values for the mean & variance of the stock price changes in a risk-neutral world

Mean: $e^{(r-q)\Delta t} = pu + (1-p)d$

Variance: $\sigma^2\Delta t = pu^2 + (1-p)d^2 - e^{2(r-q)\Delta t}$

A further condition often imposed is $u = 1/d$

Tree Parameters for asset paying a dividend yield of q

(continued)

When Δt is small a solution to the equations is

$$u = e^{\sigma\sqrt{\Delta t}}$$

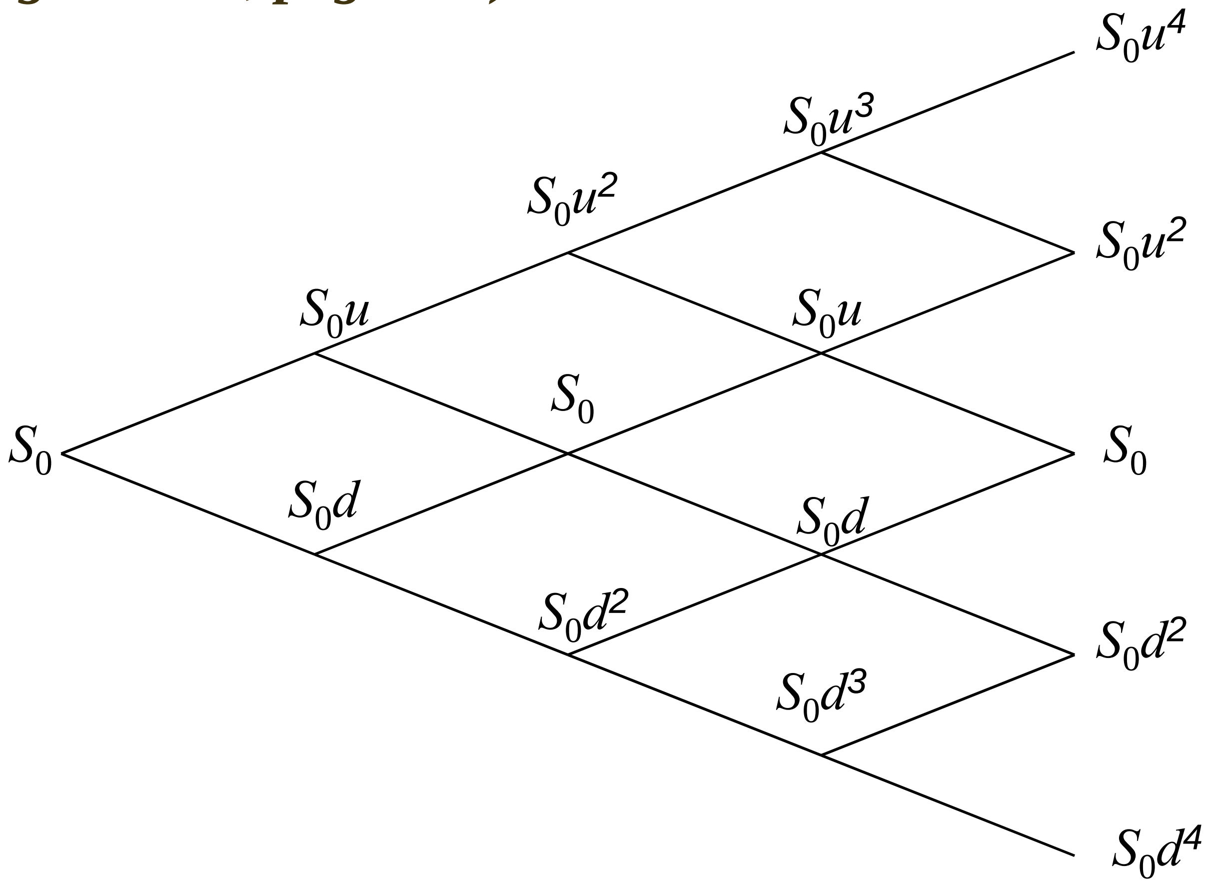
$$d = e^{-\sigma\sqrt{\Delta t}}$$

$$p = \frac{a-d}{u-d}$$

$$a = e^{(r-q)\Delta t}$$

The Complete Tree

(Figure 21.2, page 453)



Backwards Induction

- We know the value of the option at the final nodes
- We work back through the tree using risk-neutral valuation to calculate the value of the option at each node, testing for early exercise when appropriate

Example: Put Option

(Example 21.1, page 453-455)

S_0 = 50; K = 50; r =10%; σ = 40%;

T = 5 months = 0.4167; Δt = 1 month = 0.0833

In this case

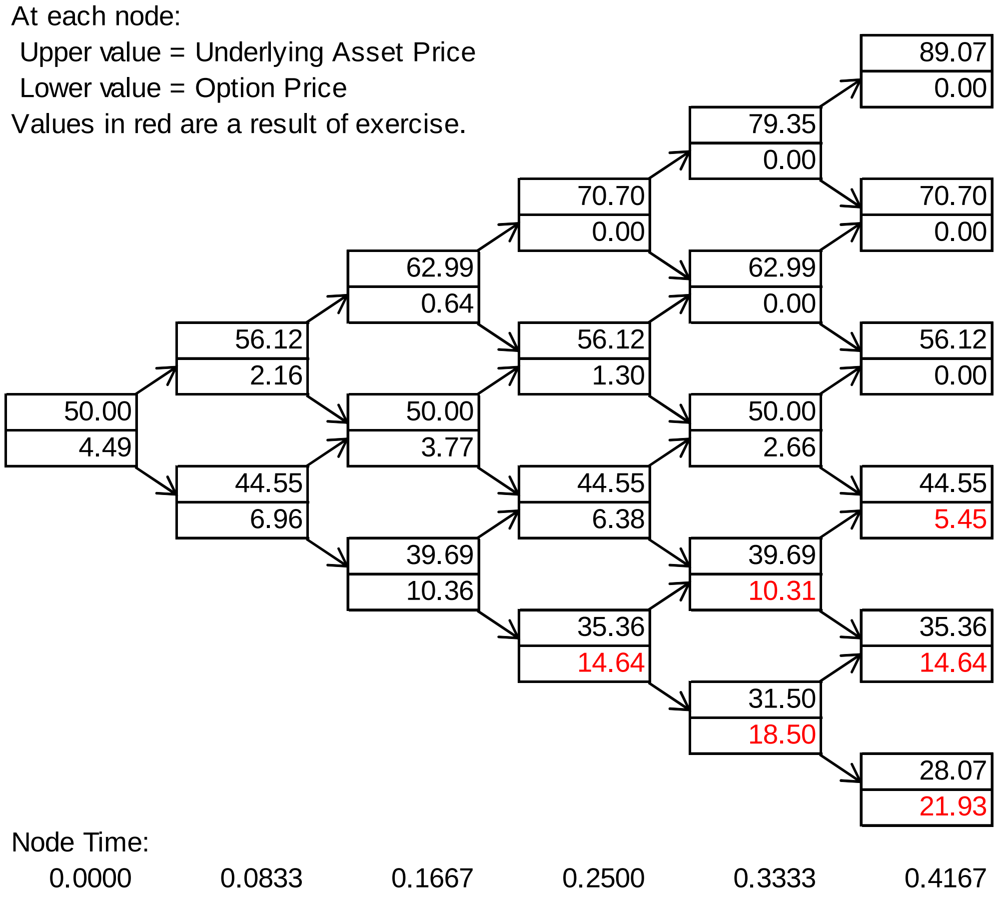
$$a = e^{0.1\times 1/12} = 1.0084$$

$$u = e^{0.4\sqrt{1/12}} = 1.1224$$

$$d = \frac{1}{u} = 0.8909$$

$$p = \frac{1.0084 - 0.8909}{1.1224 - 0.8909} = 0.5073$$

Example *(continued; Figure 21.3, page 454)*



Calculation of Delta

Delta is calculated from the nodes at time Δt

$$\text{Delta} = \frac{2.16 - 6.96}{56.12 - 44.55} = -0.41$$

Calculation of Gamma

Gamma is calculated from the nodes at time $2\Delta t$

$$\Delta_1 = \frac{0.64 - 3.77}{62.99 - 50} = -0.24; \quad \Delta_2 = \frac{3.77 - 10.36}{50 - 39.69} = -0.64$$

$$\text{Gamma} = \frac{\Delta_1 - \Delta_2}{11.65} = 0.03$$

=0.5(62.99-50)+0.5(50-39.69)

Calculation of Theta

Theta is calculated from the central nodes at times 0 and $2\Delta t$

$$\text{Theta} = \frac{3.77 - 4.49}{0.1667} = -4.3 \text{ per year}$$

or -0.012 per calendar day

Calculation of Vega

- We can proceed as follows
- Construct a new tree with a volatility of 41% instead of 40%.
- Value of option is 4.62
- Vega is $4.62 - 4.49 = 0.13$

  per 1% change in volatility

Trees for Options on Indices, Currencies and Futures Contracts

As with Black-Scholes-Merton:

- For options on stock indices, q equals the dividend yield on the index
- For options on a foreign currency, q equals the foreign risk-free rate
- For options on futures contracts $q = r$

Binomial Tree for Stock Paying Known Dividends

- Procedure:
  - Construct a tree for the stock price less the present value of the dividends
  - Create a new tree by adding the present value of the dividends at each node
- This ensures that the tree recombines and makes assumptions similar to those when the Black-Scholes-Merton model is used for European options

Control Variate Technique

- Value American option, f_A
- Value European option using same tree, f_E
- Value European option using Black-Scholes – Merton, f_{BS}
- Option price $= f_A + (f_{BS} - f_E)$

Alternative Binomial Tree

(page 465)

Instead of setting $u = 1/d$ we can set each of the 2 probabilities to 0.5 and

$$u = e^{(r-q-\sigma^2/2)\Delta t+\sigma\sqrt{\Delta t}}$$

$$d = e^{(r-q-\sigma^2/2)\Delta t-\sigma\sqrt{\Delta t}}$$

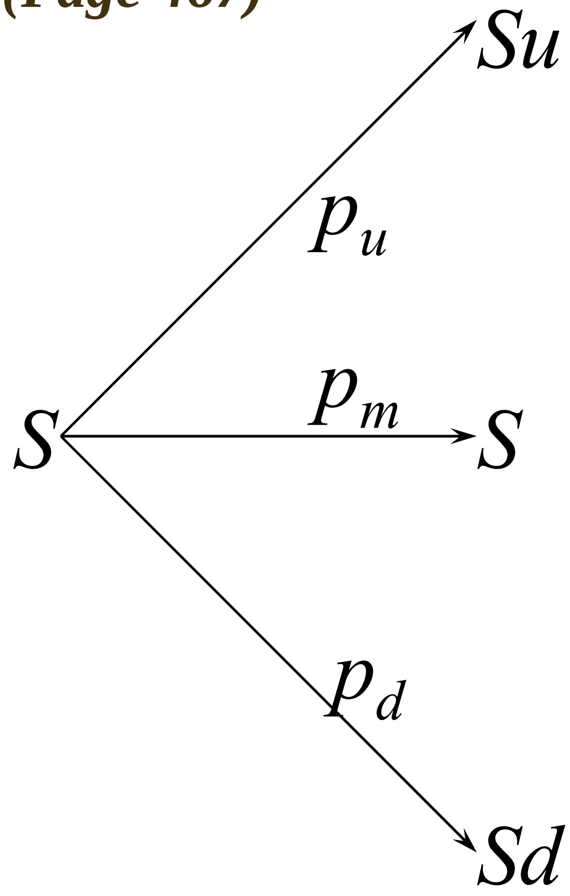
Trinomial Tree *(Page 467)*

$$u = e^{\sigma\sqrt{3\Delta t}} \qquad d = 1/u$$

$$p_u = \sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$

$$p_m = \frac{2}{3}$$

$$p_d = -\sqrt{\frac{\Delta t}{12\sigma^2}}\left(r - \frac{\sigma^2}{2}\right) + \frac{1}{6}$$

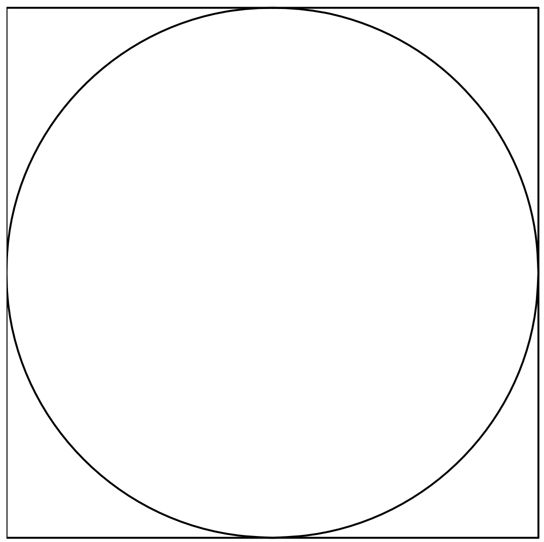


Time Dependent Parameters in a Binomial Tree *(page 468)*

- Making r or q a function of time does not affect the geometry of the tree. The probabilities on the tree become functions of time.
- We can make σ a function of time by making the lengths of the time steps inversely proportional to the variance rate.

Monte Carlo Simulation and π

- How could you calculate π by randomly sampling points in the square?



Monte Carlo Simulation and Options

When used to value European stock options, Monte Carlo simulation involves the following steps:

1. Simulate 1 path for the stock price in a risk neutral world
2. Calculate the payoff from the stock option
3. Repeat steps 1 and 2 many times to get many sample payoffs
4. Calculate mean payoff
5. Discount mean payoff at risk free rate to get an estimate of the value of the option

Sampling Stock Price Movements

- In a risk neutral world the process for a stock price is

$$dS = \hat{\mu} S\, dt + \sigma S\, dz$$

where $\hat{\mu}$ is the risk-neutral return

- We can simulate a path by choosing time steps of length Δt and using the discrete version of this

$$\Delta S = \hat{\mu} S\, \Delta t + \sigma S\, \varepsilon \sqrt{\Delta t}$$

where ε is a random sample from $\phi(0,1)$

A More Accurate Approach

(Equation 21.15, page 471)

Use

$$d \ln S = \left(\hat{\mu} - \sigma^2/2\right) dt + \sigma\, dz$$

The discrete version of this is

$$\ln S(t+\Delta t) - \ln S(t) = \left(\hat{\mu} - \sigma^2/2\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}$$

or

$$S(t+\Delta t) = S(t)\, e^{\left(\hat{\mu} - \sigma^2/2\right)\Delta t + \sigma\varepsilon\sqrt{\Delta t}}$$

Extensions

When a derivative depends on several underlying variables we can simulate paths for each of them in a risk-neutral world to calculate the values for the derivative

Sampling from Normal Distribution *(Page 473)*

- In Excel =NORMSINV(RAND()) gives a random sample from $\phi(0,1)$

To Obtain 2 Correlated Normal Samples

- Obtain independent normal samples x_1 and x_2 and set

$$\varepsilon_1 = x_1$$
$$\varepsilon_2 = \rho x_1 + x_2\sqrt{1-\rho^2}$$

- Use a procedure known as Cholesky's decomposition when samples are required from more than two normal variables (see page 473)

Standard Errors in Monte Carlo Simulation

The standard error of the estimate of the option price is the standard deviation of the discounted payoffs given by the simulation trials divided by the square root of the number of observations.

Application of Monte Carlo Simulation

- Monte Carlo simulation can deal with path dependent options, options dependent on several underlying state variables, and options with complex payoffs
- It cannot easily deal with American-style options

Determining Greek Letters

For Δ:

1. Make a small change to asset price
2. Carry out the simulation again using the same random number streams
3. Estimate Δ as the change in the option price divided by the change in the asset price

Proceed in a similar manner for other Greek letters

Variance Reduction Techniques

- Antithetic variable technique
- Control variate technique
- Importance sampling
- Stratified sampling
- Moment matching
- Using quasi-random sequences

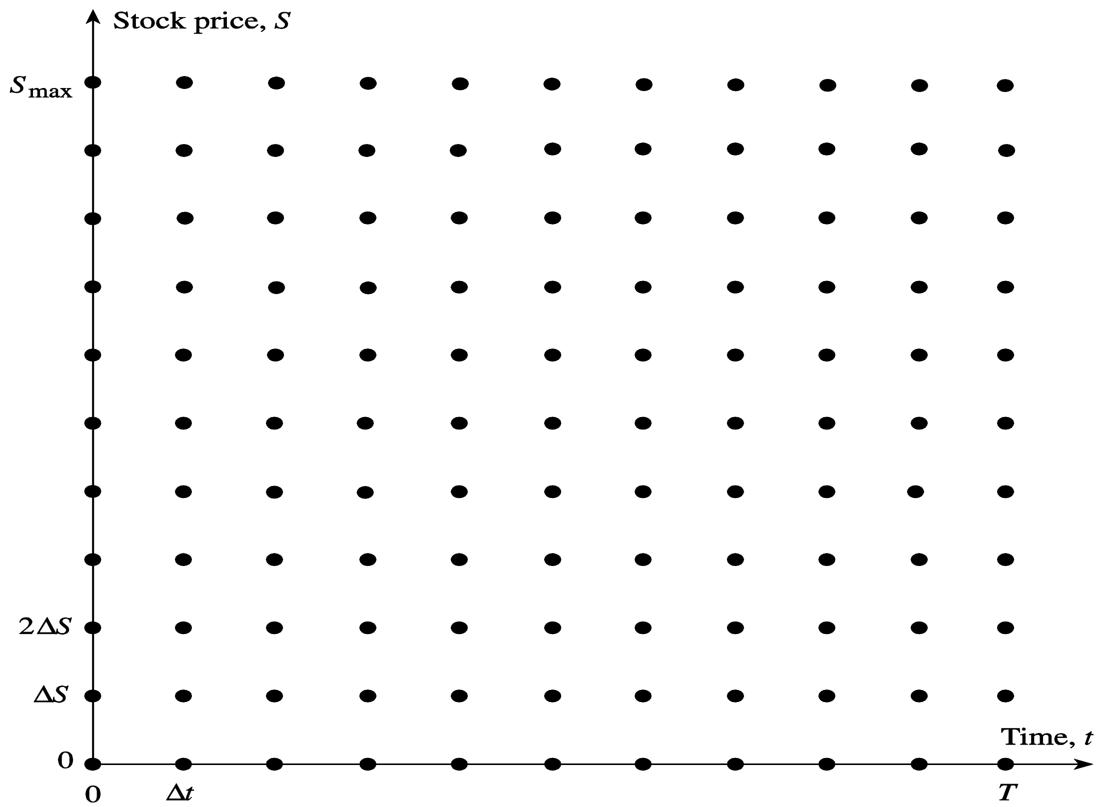
Sampling Through the Tree

- Instead of sampling from the stochastic process we can sample paths randomly through a binomial or trinomial tree to value a derivative
- At each node that is reached we sample a randon number between 0 and 1. If it is betweeb 0 and p, we take the up branch; if it is between p and 1, we take the down branch

Finite Difference Methods

- Finite difference methods aim to represent the differential equation in the form of a difference equation
- We form a grid by considering equally spaced time values and stock price values
- Define $f_{i,j}$ as the value of f at time $i\Delta t$ when the stock price is $j\Delta S$

The Grid



Finite Difference Methods
(continued)

$$\frac{\partial f}{\partial t}+(r-q)S\frac{\partial f}{\partial S}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 f}{\partial S^2}=r?$$

Set $$\frac{\partial f}{\partial S}=\frac{f_{i,j+1}-f_{i,j-1}}{2\Delta S}$$

$$\frac{\partial^2 f}{\partial S^2}=\left(\frac{f_{i,j+1}-f_{i,j}}{\Delta S}-\frac{f_{i,j}-f_{i,j-1}}{\Delta S}\right)\Big/\Delta S \qquad \text{or}$$

$$\frac{\partial^2 f}{\partial S^2}=\frac{f_{i,j+1}+f_{i,j-1}-2f_{i,j}}{\Delta S^2}$$

Implicit Finite Difference Method

Set $\qquad \dfrac{\partial f}{\partial t} = \dfrac{f_{i+1,j} - f_{i,j}}{\Delta t}$

to obtain for each node an equation of the form of the form :

$$a_j f_{i,j-1} + b_j f_{i,j} + c_j f_{i,j+1} = f_{i+1,j}$$

Explicit Finite Difference Method

If $\partial f / \partial S$ and $\partial^2 f / \partial S^2$ are assumed to be the same at the $(i+1, j)$ point as they are at the (i, j) point we obtain the explicit finite difference method

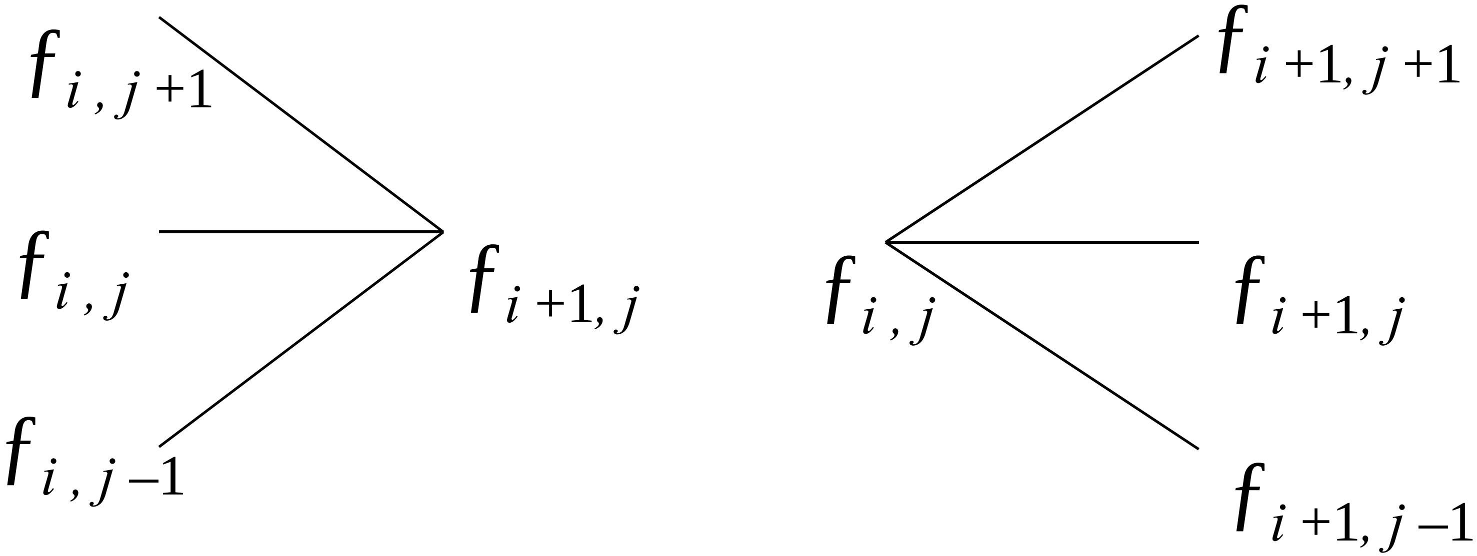
This involves solving equations of the form:

$$f_{i,j} = a_j^* f_{i+1,j-1} + b_j^* f_{i+1,j} + c_j^* f_{i+1,j+1}$$

Implicit vs Explicit Finite Difference Method

- The explicit finite difference method is equivalent to the trinomial tree approach
- The implicit finite difference method is equivalent to a multinomial tree approach

Implicit vs Explicit Finite Difference Methods *(Figure 21.16, page 484)*



Implicit Method

Explicit Method

Other Points on Finite Difference Methods

- It is better to have ln S rather than S as the underlying variable
- Improvements over the basic implicit and explicit methods:
  - Hopscotch method
  - Crank-Nicolson method

Chapter 22
Value at Risk

The Question Being Asked in VaR

"What loss level is such that we are X% confident it will not be exceeded in N business days?"

VaR vs. Expected Shortfall

- VaR is the loss level that will not be exceeded with a specified probability
- Expected Shortfall (or C-VaR) is the expected loss given that the loss is greater than the VaR level
- Although expected shortfall is theoretically more appealing, it is VaR that is used by regulators in setting bank capital requirements

Advantages of VaR

- It captures an important aspect of risk in a single number
- It is easy to understand
- It asks the simple question: “How bad can things get?”

Historical Simulation to Calculate the One-Day VaR

- Create a database of the daily movements in all market variables.
- The first simulation trial assumes that the percentage changes in all market variables are as on the first day
- The second simulation trial assumes that the percentage changes in all market variables are as on the second day
- and so on

Historical Simulation continued

- Suppose we use 501 days of historical data (Day 0 to Day 500)
- Let v_i be the value of a variable on day i
- There are 500 simulation trials
- The ith trial assumes that the value of the market variable tomorrow is

$$v_{500}\frac{v_i}{v_{i-1}}$$

Example : Calculation of 1-day, 99% VaR for a Portfolio on Sept 25, 2008 (Table 22.1, page 498)

Index	*Value ($000s)*
DJIA	4,000
FTSE 100	3,000
CAC 40	1,000
Nikkei 225	2,000

Data After Adjusting for Exchange Rates *(Table 22.2, page 498)*

Day	*Date*	*DJIA*	*FTSE 100*	*CAC 40*	*Nikkei 225*
0	Aug 7, 2006	11,219.38	11,131.84	6,373.89	131.77
1	Aug 8, 2006	11,173.59	11,096.28	6,378.16	134.38
2	Aug 9, 2006	11,076.18	11,185.35	6,474.04	135.94
3	Aug 10, 2006	11,124.37	11,016.71	6,357.49	135.44
…	……	…..	…..	……	……
499	Sep 24, 2008	10,825.17	9,438.58	6,033.93	114.26
500	Sep 25, 2008	11,022.06	9,599.90	6,200.40	112.82

Scenarios Generated (Table 22.3, page 499)

$$11{,}022.06 \times \frac{11{,}173.59}{11{,}219.38} = 10{,}977.08$$

Scenario	*DJIA*	*FTSE 100*	*CAC 40*	*Nikkei 225*	*Portfolio Value ($000s)*	*Loss ($000s)*
1	10,977.08	9,569.23	6,204.55	115.05	10,014.334	−14.334
2	10,925.97	9,676.96	6,293.60	114.13	10,027.481	−27.481
3	11,070.01	9,455.16	6,088.77	112.40	9,946.736	53.264
…	…….	…….	…….	……..	…….	……..
499	10,831.43	9,383.49	6,051.94	113.85	9,857.465	142.535
500	11,222.53	9,763.97	6,371.45	111.40	10,126.439	−126.439

Ranked Losses *(Table 22.4, page 500)*

Scenario Number	*Loss ($000s)*
494	477.841
339	345.435
349	282.204
329	277.041
487	253.385
227	217.974
131	205.256

99% one-day VaR

The N-day VaR

- The N-day VaR for market risk is usually assumed to be $\sqrt{N}$ times the one-day VaR
- In our example the 10-day VaR would be calculated as

$$\sqrt{10} \times 253{,}385 = 801{,}274$$

- This assumption is in theory only perfectly correct if daily changes are normally distributed and independent

The Model-Building Approach

- The main alternative to historical simulation is to make assumptions about the probability distributions of the return on the market variables and calculate the probability distribution of the change in the value of the portfolio analytically
- This is known as the model building approach or the variance-covariance approach

Daily Volatilities

- In option pricing we measure volatility "per year"
- In VaR calculations we measure volatility "per day"

$$\sigma_{\text{day}} = \frac{\sigma_{\text{year}}}{\sqrt{252}}$$

Daily Volatility continued

- Theoretically, σ_{day} is the standard deviation of the continuously compounded return in one day
- In practice we assume that it is the standard deviation of the percentage change in one day

Microsoft Example *(page 502)*

- We have a position worth \$10 million in Microsoft shares
- The volatility of Microsoft is 2% per day (about 32% per year)
- We use N=10 and X=99

Microsoft Example *continued*

- The standard deviation of the change in the portfolio in 1 day is $200,000
- Assume that the expected change is zero (OK for short time periods) and the probability distribution of the change is
- The 1-day 99% VaR is

$$200{,}000 \times 2.326 = \$465{,}300$$

- The 10-day 99% VaR is

$$\sqrt{10} \times 465{,}300 = 1{,}471{,}300$$

AT&T Example *(page 503)*

- Consider a position of \$5 million in AT&T
- The daily volatility of AT&T is 1% (approx 16% per year)
- The 10-day 99% VaR is

$$\sqrt{10} \times 2.326 \times 50{,}000 = 367{,}800$$

Portfolio

- Now consider a portfolio consisting of both Microsoft and AT&T
- Assume that the returns of AT&T and Microsoft are bivariate normal
- Suppose that the correlation between the returns is 0.3

S.D. of Portfolio

- A standard result in statistics states that

$$\sigma_{X+Y} = \sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho\sigma_X\sigma_Y}$$

- In this case σ_X = 200,000 and σ_Y = 50,000 and ρ = 0.3. The standard deviation of the change in the portfolio value in one day is therefore 220,200

VaR for Portfolio

- The 10-day 99% VaR for the portfolio is
  $$220{,}200 \times \sqrt{10} \times 2.326 = \$1{,}620{,}100$$
- The benefits of diversification are
  (1,471,300+367,800)–1,620,100=$219,00
- What is the incremental effect of the AT&T holding on VaR?

The Linear Model

This assumes

- The daily change in the value of a portfolio is linearly related to the daily returns from market variables
- The returns from the market variables are normally distributed

Markowitz Result for Variance of Return on Portfolio

$$\text{Variance of Portfolio Return} = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} w_i w_j \sigma_i \sigma_j$$

w_i is weight of ith instrument in portfolio

σ_i^2 is variance of return on ith instrument in portfolio

ρ_{ij} is correlation between returns of ith and jth instruments

VaR Result for Variance of Portfolio Value ($\alpha_i = w_i P$)

$$\Delta P = \sum_{i=1}^{n} \alpha_i \Delta x_i$$

$$\sigma_P^2 = \sum_{i=1}^{n} \sum_{j=1}^{n} \rho_{ij} \alpha_i \alpha_j \sigma_i \sigma_j$$

$$\sigma_P^2 = \sum_{i=1}^{n} \alpha_i^2 \sigma_i^2 + 2 \sum_{i<j} \rho_{ij} \alpha_i \alpha_j \sigma_i \sigma_j$$

σ_i is the daily volatility of ith instrument (i.e., SD of daily return)
σ_P is the SD of the change in the portfolio value per day

Covariance Matrix ($var_i = cov_{ii}$)

$$C = \begin{pmatrix} \text{var}_1 & \text{cov}_{12} & \text{cov}_{13} & \cdots & \text{cov}_{1n} \\ \text{cov}_{21} & \text{var}_2 & \text{cov}_{23} & \cdots & \text{cov}_{2n} \\ \text{cov}_{31} & \text{cov}_{32} & \text{var}_3 & \ldots & \text{cov}_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \text{cov}_{n1} & \text{cov}_{n2} & \text{cov}_{n3} & \ldots & \text{var}_n \end{pmatrix}$$

$\text{cov}_{ij} = \rho_{ij}\,\sigma_i\,\sigma_j$ where σ_i and σ_j are the SDs of the daily returns of variables i and j, and ρ_{ij} is the correlation between them

Alternative Expressions for σ_P^2

pages 505-506

$$\sigma_P^2 = \sum_{i=1}^{n}\sum_{j=1}^{n} \mathrm{cov}_{ij}\,\alpha_i\alpha_j$$

$$\sigma_P^2 = \boldsymbol{\alpha}^{\mathrm{T}} C \boldsymbol{\alpha}$$

where $\boldsymbol{\alpha}$ is the column vector whose ith element is α_i and $\boldsymbol{\alpha}^{\mathrm{T}}$ is its transpose

Alternatives for Handling Interest Rates

- Duration approach: Linear relation between ΔP and Δy but assumes parallel shifts)
- Cash flow mapping: Cash flows are mapped to standard maturities and variables are zero-coupon bond prices with the standard maturities
- Principal components analysis: 2 or 3 independent shifts with their own volatilities

When Linear Model Can be Used

- Portfolio of stocks
- Portfolio of bonds
- Forward contract on foreign currency
- Interest-rate swap

The Linear Model and Options

Consider a portfolio of options dependent on a single stock price, S. If δ is the delta of the option, then it is approximately true that

$$\delta \approx \frac{\Delta P}{\Delta S}$$

Define

$$\Delta x = \frac{\Delta S}{S}$$

Linear Model and Options continued *(page 508)*

Then

$$\Delta P \approx \delta\,\Delta S = S\delta\,\Delta x$$

Similarly when there are many underlying market variables

$$\Delta P \approx \sum_i S_i \delta_i\,\Delta x_i$$

where δ_i is the delta of the portfolio with respect to the ith asset

Example

- Consider an investment in options on Microsoft and AT&T. Suppose the stock prices are 120 and 30 respectively and the deltas of the portfolio with respect to the two stock prices are 1,000 and 20,000 respectively
- As an approximation

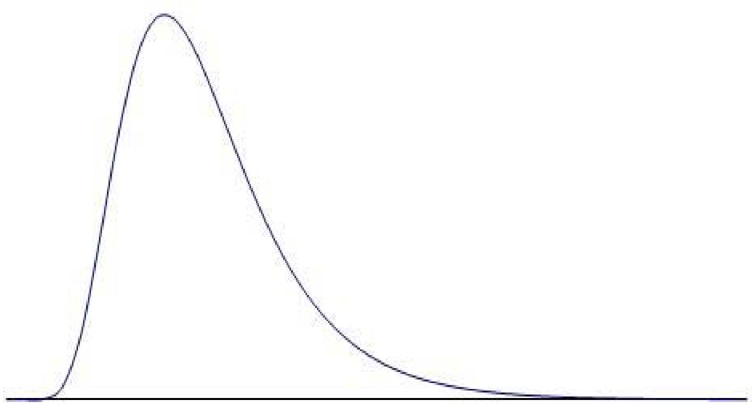
$$\Delta P = 120 \times 1{,}000 \Delta x_1 + 30 \times 20{,}000 \Delta x_2$$

where Δx_1 and Δx_2 are the percentage changes in the two stock prices

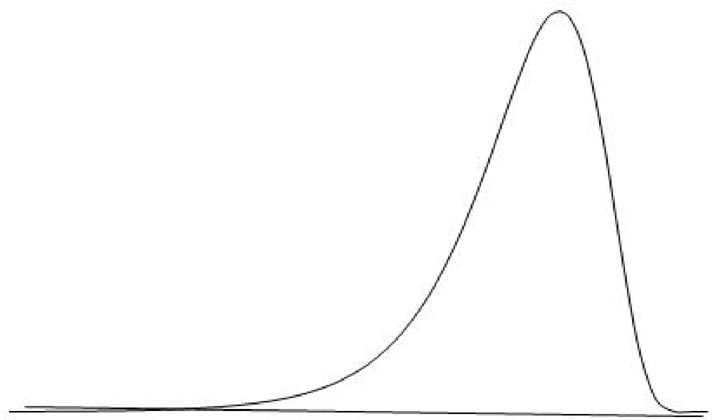
But the distribution of the daily return on an option is not normal

The linear model fails to capture skewness in the probability distribution of the portfolio value.

Impact of gamma *(Figure 22.4, page 509)*

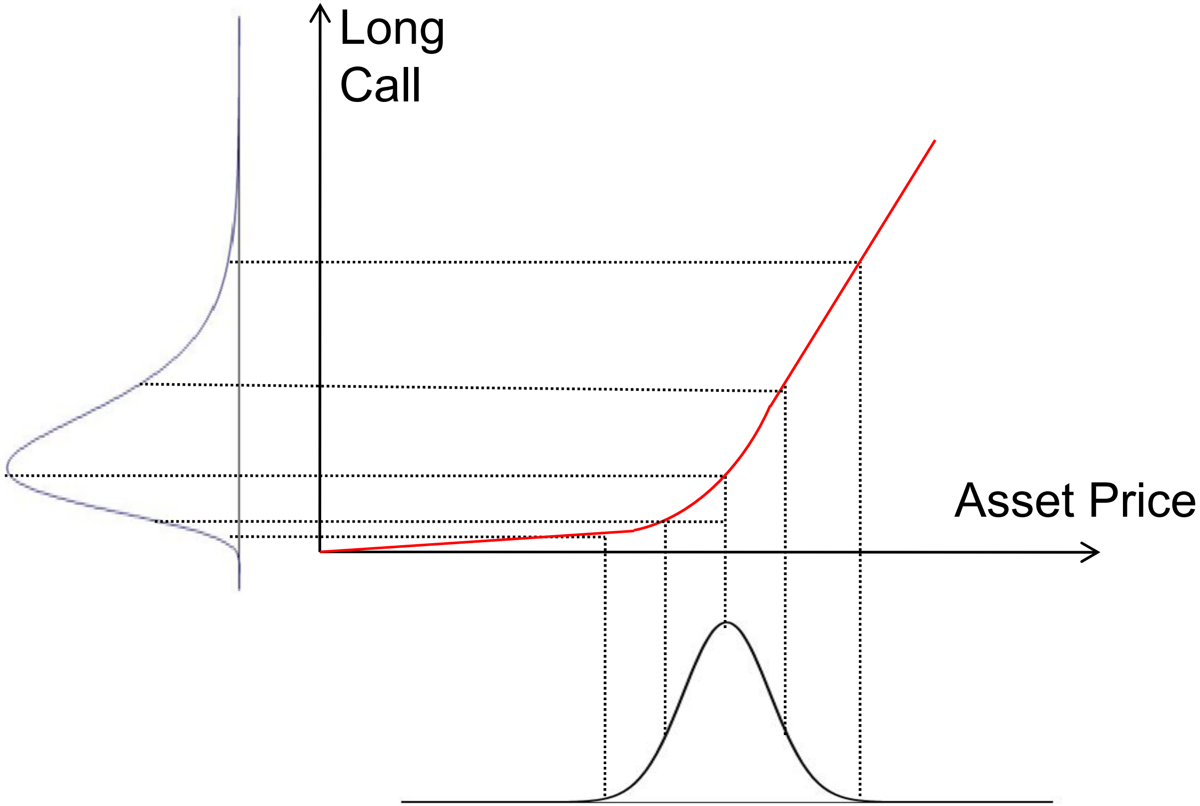


Positive Gamma

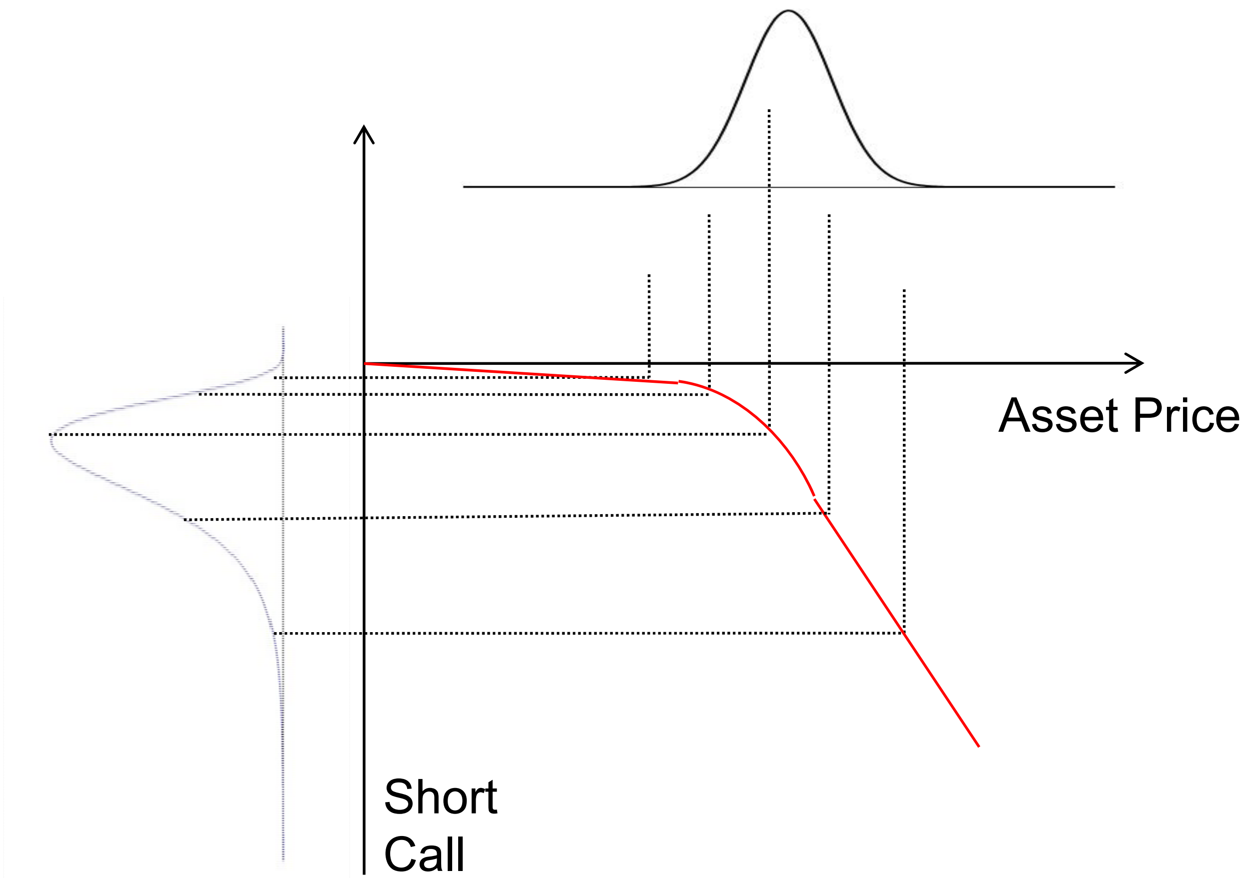


Negative Gamma

Translation of Asset Price Change to Price Change for Long Call *(Figure 22.5, page 510)*



Translation of Asset Price Change to Price Change for Short Call *(Figure 22.6, page 510)*



Quadratic Model

For a portfolio dependent on a single stock price it is approximately true that

$$\Delta P = \delta \Delta S + \frac{1}{2}\gamma(\Delta S)^2$$

this becomes

$$\Delta P = S\delta\,\Delta x + \frac{1}{2}S^2\gamma\,(\Delta x)^2$$

Quadratic Model continued

With many market variables we get an expression of the form

$$\Delta P = \sum_{i=1}^{n} S_i \delta_i \Delta x_i + \sum_{i=1}^{n} \frac{1}{2} S_i S_j \gamma_{ij} \Delta x_i \Delta x_j$$

where

$$\delta_i = \frac{\partial P}{\partial S_i} \qquad \gamma_{ij} = \frac{\partial^2 P}{\partial S_i S_j}$$

But this is much more difficult to work with than the linear model

Monte Carlo Simulation *(page 511-512)*

To calculate VaR using MC simulation we

- Value portfolio today
- Sample once from the multivariate distributions of the Δx_i
- Use the Δx_i to determine market variables at end of one day
- Revalue the portfolio at the end of day

Monte Carlo Simulation continued

- Calculate ΔP
- Repeat many times to build up a probability distribution for ΔP
- VaR is the appropriate fractile of the distribution times square root of N
- For example, with 1,000 trial the 1 percentile is the 10th worst case.

Monte Carlo Simulation

- Calculate ΔP
- Repeat many times to build up a probability distribution for ΔP
- VaR is the appropriate fractile of the distribution times square root of N
- For example, with 1,000 trial the 1 percentile is the 10th worst case.

Speeding up Calculations with the Partial Simulation Approach

- Use the approximate delta/gamma relationship between ΔP and the Δx_i to calculate the change in value of the portfolio
- This can also be used to speed up the historical simulation approach

Comparison of Approaches

- Model building approach assumes normal distributions for market variables. It tends to give poor results for low delta portfolios
- Historical simulation lets historical data determine distributions, but is computationally slower

Stress Testing

- This involves testing how well a portfolio performs under extreme but plausible market moves
- Scenarios can be generated using
  - Historical data
  - Analyses carried out by economics group
  - Senior management

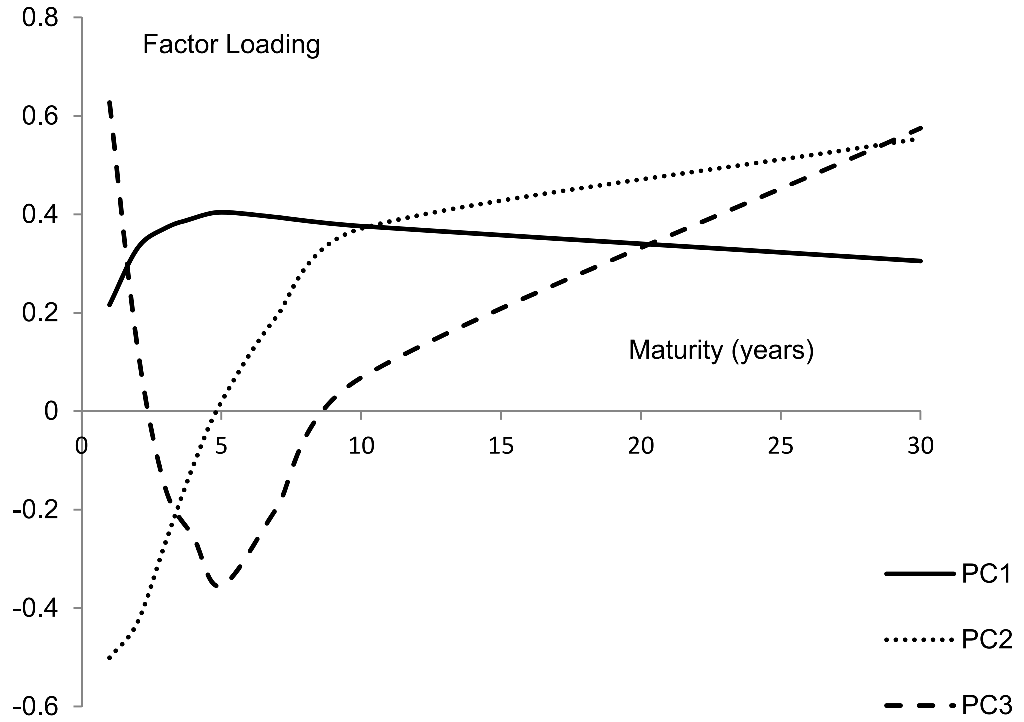
Back-Testing

- Tests how well VaR estimates would have performed in the past
- We could ask the question: How often was the actual 1-day loss greater than the 99%/1- day VaR?

Principal Components Analysis for Swap Rates

- The first factor is a roughly parallel shift (90.9% of variance in data explained)
- The second factor is a twist (another 6.8% of variance explained)
- The third factor is a bowing (another 1.3% of variation explained)

The First Three Principal Components *(Figure 22.7, page 515)*



Standard Deviation of Factor Scores (bp)

PC1	PC2	PC3	PC4	…..
17.55	4.77	2.08	1.29	….

Using PCA to Calculate VaR *(page 516)*

Example: Sensitivity of portfolio to 1 bp rate move ($m)

1 yr	2 yr	3 yr	4 yr	5 yr
+10	+4	-8	-7	+2

Sensitivity to first factor is from factor loadings:

$10\times0.372 + 4\times0.392 - 8\times0.404 - 7\times0.394 + 2\times0.376$

$= -0.05$

Similarly sensitivity to second factor = − 3.87

Using PCA to calculate VaR
continued

- As an approximation

$$\Delta P = -0.05 f_1 - 3.87 f_2$$

- The factors are independent in a PCA
- The standard deviation of ΔP is

$$\sqrt{0.05^2 \times 17.55^2 + 3.87^2 \times 4.77^2} = 18.48$$

- The 1 day 99% VaR is 18.48 × 2.326 = 42.99

Chapter 23
Estimating Volatilities and Correlations

Standard Approach to Estimating Volatility (page 521)

- Define σ_n as the volatility per day between day n-1 and day n, as estimated at end of day n-1
- Define S_i as the value of market variable at end of day i
- Define $u_i = \ln(S_i/S_{i-1})$

$$\sigma_n^2 = \frac{1}{m-1}\sum_{i=1}^{m}(u_{n-i} - \bar{u})^2$$

$$\bar{u} = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}$$

Simplifications Usually Made in Risk Management *(page 522)*

- Set $u_i = (S_i - S_{i\text{-}1})/S_{i\text{-}1}$
- Assume that the mean value of u_i is zero
- Replace $m-1$ by m

This gives $$\sigma_n^2 = \frac{1}{m}\sum_{i=1}^{m} u_{n-i}^2$$

Weighting Scheme

Instead of assigning equal weights to the observations we can set

$$\sigma_n^2 = \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

where

$$\sum_{i=1}^{m} \alpha_i = 1$$

ARCH(m) Model *(page 523)*

In an ARCH(m) model we also assign some weight to the long-run variance rate, V_L:

$$\sigma_n^2 = \gamma V_L + \sum_{i=1}^{m} \alpha_i u_{n-i}^2$$

where

$$\gamma + \sum_{i=1}^{m} \alpha_i = 1$$

EWMA Model

- In an exponentially weighted moving average model, the weights assigned to the u^2 decline exponentially as we move back through time
- This leads to

$$\sigma_n^2 = \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2$$

To Show that Weights Decline Exponentially

$$\begin{aligned}\sigma_n^2 &= \lambda\sigma_{n-1}^2 + (1-\lambda)u_{n-1}^2 \\ &= \lambda[\lambda\sigma_{n-2}^2 + (1-\lambda)u_{n-2}^2] + (1-\lambda)u_{n-1}^2 \\ &= (1-\lambda)u_{n-1}^2 + \lambda(1-\lambda)u_{n-2}^2 + \lambda^2\sigma_{n-2}^2\end{aligned}$$

Substituting for σ_{n-2}^2, then for σ_{n-3}^2 then for σ_{n-4}^2, and so on:

$$\sigma_n^2 = (1-\lambda)u_{n-1}^2 + \lambda(1-\lambda)u_{n-2}^2 + \lambda^2(1-\lambda)u_{n-3}^2 + \ldots + \lambda^{m-1}(1-\lambda)u_{n-m}^2 + \lambda^m\sigma_{n-m}^2$$

Weights start at 1 - λ and decline at rate λ

Attractions of EWMA

- Relatively little data needs to be stored
- We need only remember the current estimate of the variance rate and the most recent observation on the market variable
- Tracks volatility changes
- 0.94 is a popular choice for λ

GARCH (1,1) *page 525*

In GARCH (1,1) we assign some weight to the long-run average variance rate

$$\sigma_n^2 = \gamma V_L + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

Since weights must sum to 1

$$\gamma + \alpha + \beta = 1$$

GARCH (1,1) continued

Setting $\omega = \gamma V$ the GARCH (1,1) model is

$$\sigma_n^2 = \omega + \alpha u_{n-1}^2 + \beta \sigma_{n-1}^2$$

and

$$V_L = \frac{\omega}{1-\alpha-\beta}$$

Example *(Example 23.2, page 525)*

- Suppose

$$\sigma_n^2 = 0.000002 + 0.13u_{n-1}^2 + 0.86\sigma_{n-1}^2$$

- The long-run variance rate is 0.0002 so that the long-run volatility per day is 1.4%

Example *continued*

- Suppose that the current estimate of the volatility is 1.6% per day and the most recent percentage change in the market variable is 1%.
- The new variance rate is

$$0.000002 + 0.13 \times 0.0001 + 0.86 \times 0.000256 = 0.00023336$$

The new volatility is 1.53% per day

GARCH (p,q)

$$\sigma_n^2 = \omega + \sum_{i=1}^{p} \alpha_i u_{n-i}^2 + \sum_{j=1}^{q} \beta_j \sigma_{n-j}^2$$

Maximum Likelihood Methods

- In maximum likelihood methods we choose parameters that maximize the likelihood of the observations occurring

Example 1

- We observe that a certain event happens one time in ten trials. What is our estimate of the proportion of the time, p, that it happens?
- The probability of the event happening on one particular trial and not on the others is

$$p(1-p)^9$$

- We maximize this to obtain a maximum likelihood estimate. Result: $p = 0.1$ (as expected)

Example 2

Estimate the variance of observations from a normal distribution with mean zero

Maximize : $$\prod_{i=1}^{m}\left[\frac{1}{\sqrt{2\pi v}}\exp\left(\frac{-u_i^2}{2v}\right)\right]$$

Taking logarithms this is equivalent to maximizing :

$$\sum_{i=1}^{m}\left[-\ln(v)-\frac{u_i^2}{v}\right]$$

Result : $$v=\frac{1}{m}\sum_{i=1}^{m}u_i^2$$

Application to GARCH

We choose parameters that maximize

$$\prod_{i=1}^{m} \frac{1}{\sqrt{2\pi v_i}} \exp\left(-\frac{u_i^2}{2v_i} \right)$$

or

$$\sum_{i=1}^{m} \left[-\ln(v_i) - \frac{u_i^2}{v_i} \right]$$

S&P 500 Excel Application

- Start with trial values of ω, α, and β
- Update variances
- Calculate

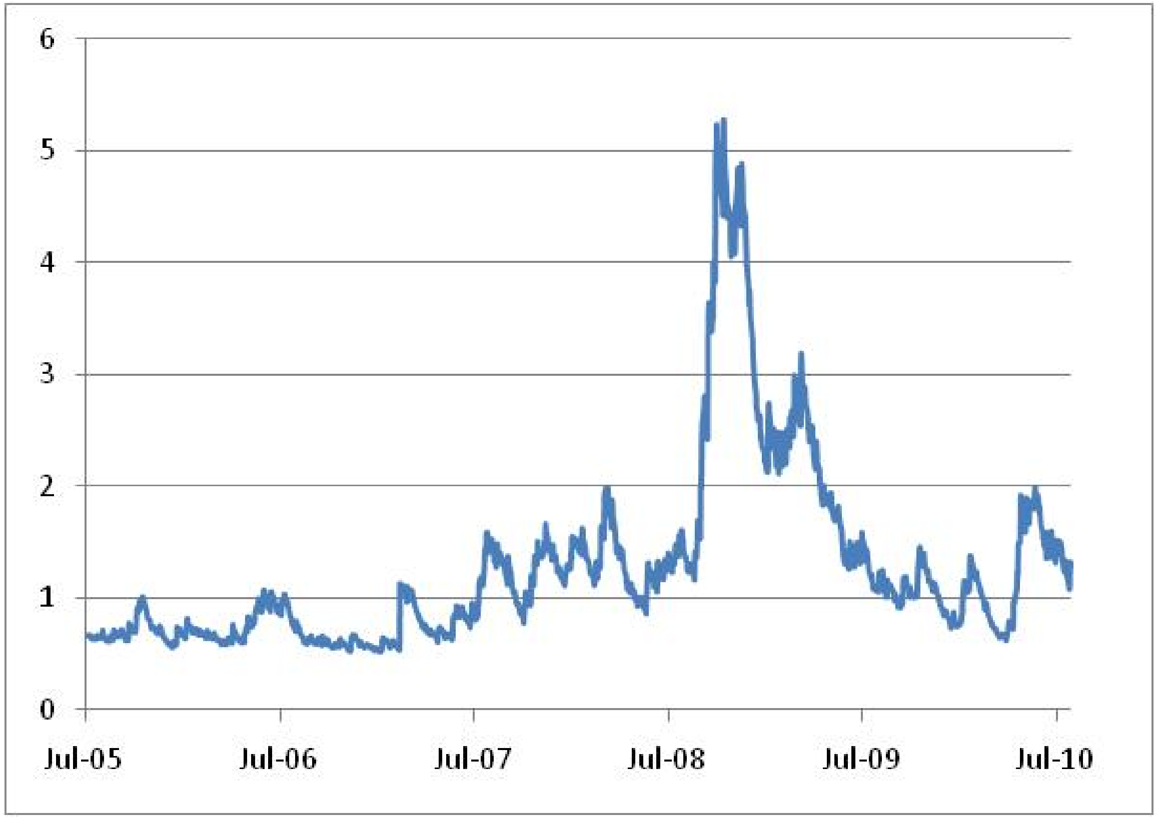
$$\sum_{i=1}^{m}\left[-\ln(v_i)-\frac{u_i^2}{v_i}\right]$$

- Use solver to search for values of ω, α, and β that maximize this objective function
- For efficient operation of Solver: set up spreadsheet so that three numbers that are the same order of magnitude are being searched for

S&P 500 Excel Application (Table 23.1)

Date	*Day*	S_i	$u_i=(S_i-S_{i-1})/S_{i-1}$	$v_i=\sigma_i^2$	$-\ln(v_i)-u_i^2/v_i$
18-Jul-2005	1	1221.13			
19-Jul-2005	2	1229.35	0.006731		
20-Jul-2005	3	1235.20	0.004759	0.00004531	9.5022
21-Jul-2005	4	1227.04	−0.006606	0.00004447	9.0393
…….	…..	…….	………..	………….	…………
13-Aug-2010	1279	1079.25	−0.004024	0.00016327	8.6209
Total					10,228.2349

The Results *(Figure 23.2, page 530)*



Variance Targeting

- One way of implementing GARCH(1,1) that increases stability is by using variance targeting
- We set the long-run average volatility equal to the sample variance
- Only two other parameters then have to be estimated

How Good is the Model?

- The Ljung-Box statistic tests for autocorrelation
- We compare the autocorrelation of the u_i^2 with the autocorrelation of the u_i^2/σ_i^2

Forecasting Future Volatility

(equation 23.13, page 532)

A few lines of algebra shows that

$$E[\sigma^2_{n+k}] = V_L + (\alpha+\beta)^k (\sigma^2_n - V_L)$$

The variance rate for an option expiring on day m is

$$\frac{1}{m}\sum_{k=0}^{m-1} E\left[\sigma^2_{n+k}\right]$$

Forecasting Future Volatility
continued (equation 23.14, page 534)

Define

$$a = \ln \frac{1}{\alpha + \beta}$$

The volatility per annum for an option lasting T days is

$$\sqrt{252\left(V_L + \frac{1 - e^{-aT}}{aT}\left[V(0) - V_L\right]\right)}$$

S&P Example

- $\omega = 0.0000013465$, $\alpha = 0.083394$, $\beta = 0.910116$

$$a = \ln \frac{1}{0.083394 + 0.910116} = 0.006511$$

Option Life (days)	10	30	50	100	500
Volatility (% per annum)	27.36	27.10	26.87	26.35	24.32

Volatility Term Structures

- GARCH (1,1) suggests that, when calculating vega, we should shift the long maturity volatilities less than the short maturity volatilities
- When instantaneous volatility changes by $\Delta\sigma(0)$, volatility for T-day option changes by

$$\frac{1-e^{-aT}}{aT}\frac{\sigma(0)}{\sigma(T)}\Delta\sigma(0)$$

Results for S&P 500 *(Table 23.4)*

- When instantaneous volatility changes by 1%

Option Life (days)	10	30	50	100	500
Volatility increase (%)	0.97	0.92	0.87	0.77	0.33

Correlations and Covariances

(page 535-537)

Define $x_i=(X_i-X_{i\text{-}1})/X_{i\text{-}1}$ and $y_i=(Y_i-Y_{i\text{-}1})/Y_{i\text{-}1}$

Also

$\sigma_{x,n}$: daily vol of X calculated on day $n-1$

$\sigma_{y,n}$: daily vol of Y calculated on day $n-1$

cov_n: covariance calculated on day $n-1$

The correlation is $\text{cov}_n/(\sigma_{x,n}\,\sigma_{y,n})$

Updating Correlations

- We can use similar models to those for volatilities
- Under EWMA

$$\text{cov}_n = \lambda\,\text{cov}_{n\text{-}1} + (1\text{-}\lambda)x_{n\text{-}1}y_{n\text{-}1}$$

Positive Finite Definite Condition

A variance-covariance matrix, Ω, is internally consistent if the positive semi-definite condition

$$\mathbf{w}^{\mathbf{T}}\Omega\mathbf{w} \geq 0$$

for all vectors **w**

Example

The variance-covariance matrix

$$\begin{pmatrix} 1 & 0 & 0.9 \\ 0 & 1 & 0.9 \\ 0.9 & 0.9 & 1 \end{pmatrix}$$

is not internally consistent

Volatilities and Correlations for Four-Index on Sept 25, 2008 with Equal Weights

	DJIA	FTSE	CAC 40	Nikkei 225
DJIA	1			
FTSE	0.489	1		
CAC 40	0.496	0.918	1	
Nikkei 225	−0.062	0.201	0.211	1

	DJIA	FTSE	CAC 40	Nikkei 225
Vol. per day (%)	1.11	1.42	1.40	1.38

Volatilities and Correlations for Four-Index on Sept 25, 2008 for EWMA and λ=0.94

	DJIA	FTSE	CAC 40	Nikkei 225
DJIA	1			
FTSE	0.611	1		
CAC 40	0.629	0.971	1	
Nikkei 225	−0.113	0.409	0.342	1

	DJIA	FTSE	CAC 40	Nikkei 225
Vol. per day (%)	2.19	3.21	3.09	1.59

One-Day 99% VaR Estimates

Historical Simulation	$253,385
Model Building Equal Weights	$217,757
Model Building EWMA	$471,025

Chapter 24
Credit Risk

Credit Ratings

- In the S&P rating system, AAA is the best rating. After that comes AA, A, BBB, BB, B, CCC, CC, and C
- The corresponding Moody's ratings are Aaa, Aa, A, Baa, Ba, B,Caa, Ca, and C
- Bonds with ratings of BBB (or Baa) and above are considered to be "investment grade"

Estimating Default Probabilities

- Alternatives:
  - use historical data
  - use credit spreads
  - use Merton’s model

Historical Data

Historical data provided by rating agencies are also used to estimate the probability of default

Cumulative Ave Default Rates (%)

(1970-2012, Moody's, Table 24.1, page 545)

	1	2	3	4	5	7	10
Aaa	0.000	0.013	0.013	0.037	0.106	0.247	0.503
Aa	0.022	0.069	0.139	0.256	0.383	0.621	0.922
A	0.063	0.203	0.414	0.625	0.870	1.441	2.480
Baa	0.177	0.495	0.894	1.369	1.877	2.927	4.740
Ba	1.112	3.083	5.424	7.934	10.189	14.117	19.708
B	4.051	9.608	15.216	20.134	24.613	32.747	41.947
Caa-C	16.448	27.867	36.908	44.128	50.366	58.302	69.483

Interpretation

- The table shows the probability of default for companies starting with a particular credit rating
- A company with an initial credit rating of Baa has a probability of 0.177% of defaulting by the end of the first year, 0.495% by the end of the second year, and so on

Do Default Probabilities Increase with Time?

- For a company that starts with a good credit rating default probabilities tend to increase with time
- For a company that starts with a poor credit rating default probabilities tend to decrease with time

Conditional vs Unconditional Default Probabilities *(page 545-546)*

- The conditional default probability is the probability of default for a certain time period conditional on no earlier default
- The unconditional default probability is the probability of default for a certain time period as seen at time zero
- What are the conditional and unconditional default probabilities for a Caa rated company in the third year?

Hazard Rate

- The hazard rate (also called default density), $\lambda(t)$, at time t is defined so that $\lambda(t)\Delta t$ is the conditional default probability for a short period between t and $t+\Delta t$
- If $V(t)$ is the probability of a company surviving to time t

$$V(t+\Delta t)-V(t)=-\lambda(t)V(t)\Delta t$$

This leads to

$$V(t)=e^{-\int_0^t \lambda(t)dt}$$

The cumulative probability of default by time t is

$$Q(t)=1-e^{-\bar{\lambda}(t)t}$$

Recovery Rate

- The recovery rate for a bond is usually defined as the price of the bond immediately after default as a percent of its face value
- Recovery rates tend to decrease as default rates increase

Recovery Rates; Moody's: 1982 to 2012

Class	Mean(%)
Senior Secured	51.6
Senior Unsecured	37.0
Senior Subordinated	30.9
Subordinated	31.5
Junior Subordinated	24.7

Using Credit Spreads *(Equation 24.2, page 547)*

- Suppose $s(T)$ is the credit spread for maturity T
- Average hazard rate between time zero and time T is approximately

$$\frac{s(T)}{1-R}$$

where R is the recovery rate

- This estimate is very accurate in most situations

Explanation

- Loss rate at time t is $\lambda(t)(1-R)$
- If the credit spread is compensation for this loss rate it should approximately equal

$$\bar{\lambda}(t)(1-R)$$

Matching Bond Prices

- For more accuracy we can work forward in time choosing hazard rates that match bond prices
- This is another application of the bootstrap method

The Risk-Free Rate

- The risk-free rate when credit spreads and default probabilities are estimated is usually assumed to be the LIBOR/swap zero rate (or sometimes 10 bps below the LIBOR/swap rate)
- Asset swaps provide a direct estimates of the spread of bond yields over swap rates

Real World vs Risk-Neutral Default Probabilities

- The default probabilities backed out of bond prices or credit default swap spreads are risk-neutral default probabilities
- The default probabilities backed out of historical data are real-world default probabilities

A Comparison

- Calculate 7-year default intensities from the Moody's data, 1970-2012, (These are real world default probabilities)
- Use Merrill Lynch data to estimate average 7-year default intensities from bond prices, 1996 to 2007 (these are risk-neutral default intensities)
- Assume a risk-free rate equal to the 7-year swap rate minus 10 basis points

Data from Moody's and Merrill Lynch

	Cumulative 7-year default probability (Moody's: 1970-2012)	Average bond yield spread in bps* (Merrill Lynch: 1996 to June 2007)
Aaa	0.247%	35.74
Aa	0.621%	43.67
A	1.441%	68.68
Baa	2.927%	127.53
Ba	14.117%	280.28
B	32.747%	481.04
Caa	58.302%	1103.70

*The benchmark risk-free rate for calculating spreads is assumed to be the swap rate minus 10 basis points. Bonds are corporate bonds with a life of approximately 7 years.

Real World vs Risk Neutral Hazard Rates *(Table 24.4, page 550)*

Rating	Historical hazard rate[1] % per annum	Hazard rate from bond prices[2] (% per annum)	Ratio	Difference
Aaa	0.04	0.60	17.0	0.56
Aa	0.09	0.73	8.2	0.64
A	0.21	1.15	5.5	0.94
Baa	0.42	2.13	5.0	1.71
Ba	2.27	4.67	2.1	2.50
B	5.67	8.02	1.4	2.35
Caa	12.50	18.39	1.5	5.89

[1] Calculated as $-[\ln(1-d)]/7$ where d is the Moody's 7 yr default rate. For example, in the case of Aaa companies, d=0.00247 and -ln(0.99753)/7=0.0004 or 4bps. For investment grade companies the historical hazard rate is approximately d/7.

[2] Calculated as $s/(1-R)$ where s is the bond yield spread and R is the recovery rate (assumed to be 40%).

Average Risk Premiums Earned By Bond Traders

Rating	Bond Yield Spread over Treasuries (bps)	Spread of risk-free rate over Treasuries (bps)[1]	Spread to compensate for historical default rate (bps)[2]	Extra Risk Premium (bps)
Aaa	78	42	2	34
Aa	86	42	5	39
A	111	42	12	57
Baa	169	42	25	102
Ba	322	42	130	150
B	523	42	340	141
Caa	1146	42	750	323

[1] Equals average spread of our benchmark risk-free rate over Treasuries.

[2] Equals historical hazard rate times (1-R) where R is the recovery rate. For example, in the case of Baa, 25bps is 0.6 times 42bps.

Possible Reasons for the Extra Risk Premium *(The third reason is the most important)*

- Corporate bonds are relatively illiquid
- The subjective default probabilities of bond traders may be much higher than the estimates from Moody's historical data
- **Bonds do not default independently of each other. This leads to systematic risk that cannot be diversified away.**
- Bond returns are highly skewed with limited upside. The non-systematic risk is difficult to diversify away and may be priced by the market

Which World Should We Use?

- We should use risk-neutral estimates for valuing credit derivatives and estimating the present value of the cost of default
- We should use real world estimates for calculating credit VaR and scenario analysis

Using Equity Prices: Merton's Model *(page 553-555)*

- Merton's model regards the equity as an option on the assets of the firm
- In a simple situation the equity value is

$$\max(V_T - D, 0)$$

where V_T is the value of the firm and D is the debt repayment required

Equity vs. Assets

The Black-Scholes-Merton option pricing model enables the value of the firm's equity today, E_0, to be related to the value of its assets today, V_0, and the volatility of its assets, σ_V

$$E_0 = V_0 N(d_1) - De^{-rT} N(d_2)$$

where

$$d_1 = \frac{\ln(V_0/D) + (r + \sigma_V^2/2)T}{\sigma_V \sqrt{T}}\;;\quad d_2 = d_1 - \sigma_V \sqrt{T}$$

Volatilities

$$\sigma_E E_0 = \frac{\partial E}{\partial V}\sigma_V V_0 = N(d_1)\sigma_V V_0$$

This equation together with the option pricing relationship enables V_0 and σ_V to be determined from E_0 and σ_E

Example

- A company's equity is $3 million and the volatility of the equity is 80%
- The risk-free rate is 5%, the debt is $10 million and time to debt maturity is 1 year
- Solving the two equations yields V_0=12.40 and σ_v=21.23%
- The probability of default is $N(-d_2)$ or 12.7%

The Implementation of Merton's Model

- Choose time horizon
- Calculate cumulative obligations to time horizon. This is termed by KMV the "default point". We denote it by D
- Use Merton's model to calculate a theoretical probability of default
- Use historical data or bond data to develop a one-to-one mapping of theoretical probability into either real-world or risk-neutral probability of default.

CVA

- Credit value adjustment (CVA) is the amount by which a dealer must reduce the total value of transactions with a counterparty because of counterparty default risk

The CVA Calculation

Time	0	t_1	t_2	t_3	t_4	…………………	t_n=T
Default probability for counterparty		q_1	q_2	q_3	q_4	…………………	q_n
PV of expected loss given default		v_1	v_2	v_3	v_4	…………………	v_n

$$\text{CVA} = \sum_{i=1}^{n} q_i v_i$$

Calculation of q_i's

- Default probabilities are calculated from credit spreads

$$q_i = \exp\left(-\frac{s(t_{i-1})t_{i-1}}{1-R}\right) - \exp\left(-\frac{s(t_i)t_i}{1-R}\right)$$

Calculation of v_i's

- The v_i are calculated by simulating the market variables underlying the portfolio in a risk-neutral world
- If no collateral is posted the loss on a particular simulation trial during the ith interval is the PV of (1-R)max(V_i, 0) where V_i is the value of the portfolio at the mid point of the interval
- v_i is the average of the losses across all simulation trials

Collateral

- It is usually assumed that the collateral is posted as agreed, and returned as agreed, until N days before a default. The N days are referred to as the “cure period” or “margin period at risk.” Usually N is 10 or 20.
- Suppose that that a portfolio is fully collateralized with no initial margin and its value moves in favor of the dealer during the cure period. Then v_i is positive because
  - If the portfolio has a positive value to the dealer at the default time, collateral posted by the counterparty is insufficient
  - If the portfolio has a negative value to the dealer at the default time, excess collateral posted by the dealer will not be returned

Incremental CVA

- Results from Monte Carlo are stored so that the incremental impact of a new trade can be calculated without simulating all the other trades.

CVA Risk

- The CVA for a counterparty can be regarded as a complex derivative
- Increasingly, dealers are managing it like any other derivative
- Two sources of risk:
  - Changes in counterparty spreads
  - Changes in market variables underlying the portfolio

Wrong Way/Right Way Risk

- Simplest assumption is that probability of default q_i is independent of net exposure v_i.
- Wrong-way risk occurs when q_i is positively dependent on v_i
- Right-way risk occurs when q_i is negatively dependent on v_i

DVA

- Debit (or debt) value adjustment (DVA) is an estimate of the cost to the counterparty of a default by the dealer
- Same formulas apply except that v is counterparty's loss given a dealer default and q is dealer's probability of default
- Value of transactions with counterparty = No default value – CVA + DVA

DVA *continued*

- What happens to the reported value of transactions as dealer's credit spread increases?

Credit Risk Mitigation

- Netting
- Collateralization
- Downgrade triggers

Simple Situation

- Suppose portfolio with a counterparty consists of a single uncollateralized transaction that always a positive value to the dealer and provides a payoff at time T
- The CVA adjustment has the effect of multiplying the value of the transaction by $e^{-s(T)T}$ where $s(T)$ is the counterparty's credit spread for maturity T

Example 25.5 *(page 560)*

- A 2-year uncollateralized option sold by a new counterparty to the dealer has a Black-Scholes-Merton value of $3
- Assume a 2 year zero coupon bond issued by the counterparty has a yield of 1.5% greater than the risk free rate
- If there is no collateral and there are no other transactions between the parties, value of option is $3e^{-0.015\times 2}=2.91$

Uncollateralized Long Forward with Counterparty *(page 560)*

For a long forward contract that matures at time T the expected exposure at time t is

$$w(t) = e^{-r(T-t)}\left[F_0 N(d_1(t)) - KN(d_2(t))\right]$$

where

$$d_1(t) = \frac{\ln(F_0 / K) + \sigma^2 t / 2}{\sigma\sqrt{t}} \qquad d_2(t) = d_1(t) - \sigma\sqrt{t}$$

so that $v_i = w(t_i)e^{-rt_i}(1-R)$

where F_0 is the forward price today, K is the delivery price, σ is the volatility of the forward price, T is the time to maturity of the forward contract, and r is the risk-free rate

Example 24.6 (page 561)

- 2 year forward. Current forward price is \$1,600 per ounce. Two one-year intervals
- K = 1,500, σ = 20%, R = 0.3, r = 5%
- t_1 =0.5, t_2=1.5
- Suppose q_1 =0.02 and q_2=0.03
- v_1 = 92.67 and v_2 = 130.65
- CVA=0.02×92.67+0.03×130.65 = 5.77
- Value after CVA =

$$(1600-1500)e^{-0.05\times 2} - 5.77 = 84.71$$

Default Correlation

- The credit default correlation between two companies is a measure of their tendency to default at about the same time
- Default correlation is important in risk management when analyzing the benefits of credit risk diversification
- It is also important in the valuation of some credit derivatives, eg a first-to-default CDS and CDO tranches.

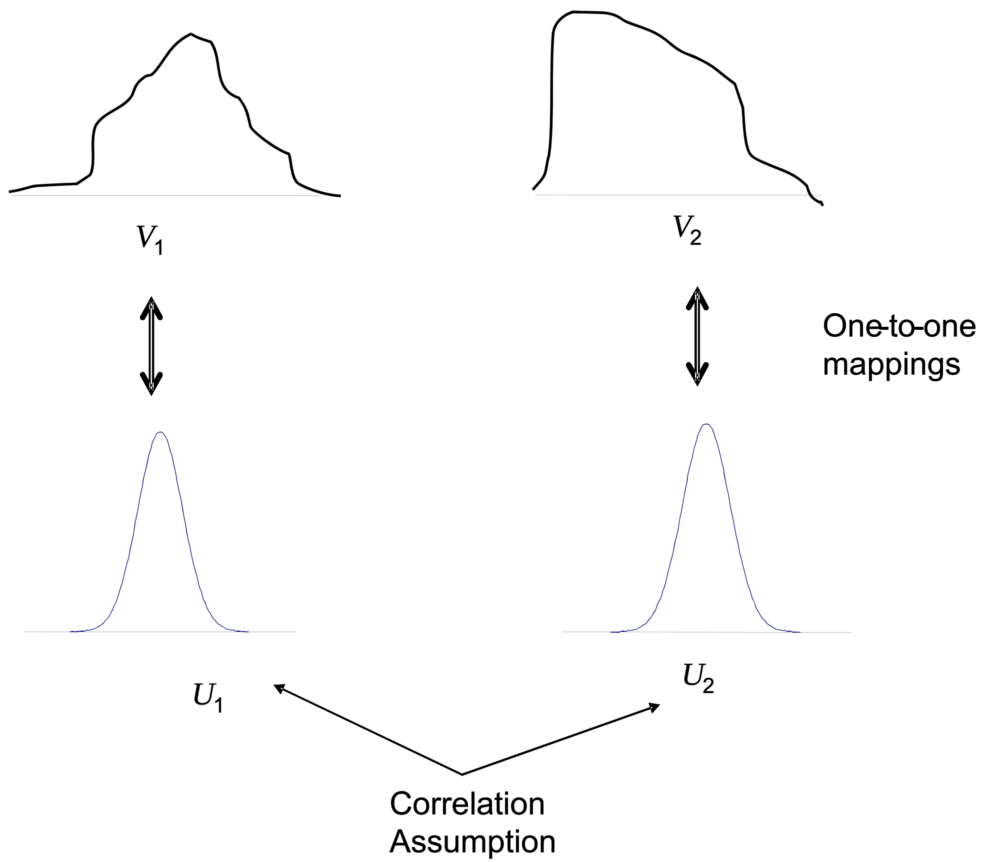
Measurement

- There is no generally accepted measure of default correlation
- Default correlation is a more complex phenomenon than the correlation between two random variables

Survival Time Correlation

- Define t_i as the time to default for company i and $Q_i(t_i)$ as the cumulative probability distribution for t_i
- The default correlation between companies i and j can be defined as the correlation between t_i and t_j
- But this does not uniquely define the joint probability distribution of default times

The Gaussian Copula Model



Gaussian Copula Model *(continued, page 562-563)*

- Define a one-to-one correspondence between the time to default, t_i, of company i and a variable x_i by

$$Q_i(t_i) = N(x_i) \quad \text{or} \quad x_i = N^{-1}[Q(t_i)]$$

  where N is the cumulative normal distribution function.
- This is a "percentile to percentile" transformation. The p percentile point of the Q_i distribution is transformed to the p percentile point of the x_i distribution. x_i has a standard normal distribution
- We assume that the x_i are multivariate normal. The default correlation measure, ρ_{ij} between companies i and j is the correlation between x_i and x_j

Example of Use of Gaussian Copula (page 563)

Suppose that we wish to simulate the defaults for n companies . For each company the cumulative probabilities of default during the next 1, 2, 3, 4, and 5 years are 1%, 3%, 6%, 10%, and 15%, respectively

Use of Gaussian Copula *continued*

- We sample from a multivariate normal distribution (with appropriate correlations) to get the x_i
- Critical values of x_i are

  $N^{-1}(0.01) = -2.33$, $N^{-1}(0.03) = -1.88$,

  $N^{-1}(0.06) = -1.55$, $N^{-1}(0.10) = -1.28$,

  $N^{-1}(0.15) = -1.04$

Use of Gaussian Copula continued

- When sample for a company is less than -2.33, the company defaults in the first year
- When sample is between -2.33 and -1.88, the company defaults in the second year
- When sample is between -1.88 and -1.55, the company defaults in the third year
- When sample is between -1,55 and -1.28, the company defaults in the fourth year
- When sample is between -1.28 and -1.04, the company defaults during the fifth year
- When sample is greater than -1.04, there is no default during the first five years

A One-Factor Model for the Correlation Structure

$$x_i = a_i F + \sqrt{1 - a_i^2}\, Z_i$$

- The correlation between x_i and x_j is $a_i a_j$
- The ith company defaults by time T when $x_i < N^{-1}[Q_i(T)]$ or

$$Z_i < \frac{N^{-1}[Q_i(T) - a_i F]}{\sqrt{1 - a_i^2}}$$

- Conditional on F the probability of this is

$$Q_i(T|F) = N\left\{\frac{N^{-1}[Q_i(T)] - a_i F}{\sqrt{1 - a_i^2}}\right\}$$

Credit VaR *(page 564-565)*

- Can be defined analogously to Market Risk VaR
- A T-year credit VaR with an X% confidence is the loss level that we are X% confident will not be exceeded over T years

Calculation from a Factor-Based Gaussian Copula Model *(equation 24.10, page 565)*

- Consider a large portfolio of loans, each of which has a probability of $Q(T)$ of defaulting by time T. Suppose that all pairwise copula correlations are ρ so that all a_i's are $\sqrt{\rho}$
- We are X% certain that F is less than

$$N^{-1}(1-X) = -N^{-1}(X)$$

- It follows that the VaR is

$$V(X,T) = N\left\{\frac{N^{-1}[Q(T)]+\sqrt{\rho}N^{-1}(X)}{\sqrt{1-\rho}}\right\}$$

Example *(page 565)*

- A bank has \$100 million of retail exposures
- 1-year probability of default averages 2% and the recovery rate averages 60%
- The copula correlation parameter is 0.1
- 99.9% worst case default rate is

$$V(0.999,1) = N\left(\frac{N^{-1}(0.02)+\sqrt{0.1}N^{-1}(0.999)\cdot}{\sqrt{1-0.1}}\right) = 0.128$$

- The one-year 99.9% credit VaR is therefore 100×0.128×(1-0.6) or \$5.13 million

CreditMetrics *(page 565-566)*

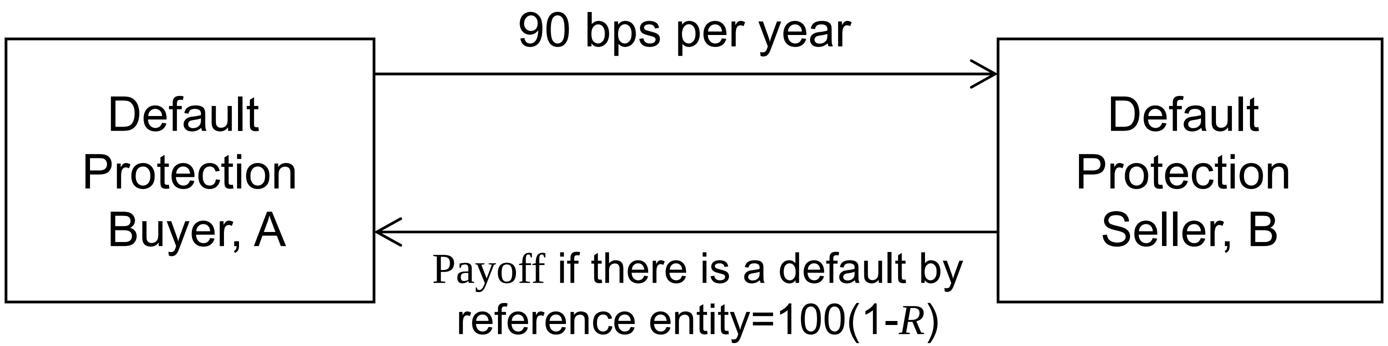
- Calculates credit VaR by considering possible rating transitions
- A Gaussian copula model is used to define the correlation between the ratings transitions of different companies

Chapter 25
Credit Derivatives

Credit Default Swaps

- Buyer of the instrument acquires protection from the seller against a default by a particular company or country (the reference entity)
- Example: Buyer pays a premium of 90 bps per year for $100 million of 5-year protection against company X
- Premium is known as the *credit default spread*. It is paid for life of contract or until default
- If there is a default, the buyer has the right to sell bonds with a face value of $100 million issued by company X for $100 million (Several bonds are typically deliverable)

CDS Structure *(Figure 25.1, page 573)*



Recovery rate, R, is the ratio of the value of the bond issued by reference entity immediately after default to the face value of the bond

Other Details

- Payments are usually made quarterly in arrears
- In the event of default there is a final accrual payment by the buyer
- Settlement can be specified as delivery of the bonds or (more usually) in cash
- An auction process usually determines the payoff
- Suppose payments are made quarterly in the example just considered. What are the cash flows if there is a default after 3 years and 1 month and recovery rate is 40%?

Attractions of the CDS Market

- Allows credit risks to be traded in the same way as market risks
- Can be used to transfer credit risks to a third party
- Can be used to diversify credit risks

Using a CDS to Hedge a Bond Position

- Portfolio consisting of a 5-year par yield corporate bond that provides a yield of 6% and a long position in a 5-year CDS costing 100 basis points per year is (approximately) a long position in a riskless instrument paying 5% per year
- This shows that bond yield spreads (measured relative to LIBOR) should be close to CDS spreads

CDS Valuation *(page 575-577)*

- Hazard rate for reference entity is 2%.
- Assume payments are made annually in arrears, that defaults always happen half way through a year, and that the expected recovery rate is 40%
- Suppose that the breakeven CDS rate is s per dollar of notional principal

Unconditional Default and Survival Probabilities *(Table 25.1)*

Time (years)	Survival Probability	Default Probability
1	0.9802	0.0198
2	0.9608	0.0194
3	0.9418	0.0190
4	0.9231	0.0186
5	0.9048	0.0183

Calculation of PV of Payments

(Table 25.2 Principal=$1)

Time (yrs)	Survival Prob	Expected Payment	Discount Factor	PV of Exp Pmt
1	0.9802	0.9802s	0.9512	0.9324s
2	0.9608	0.9608s	0.9048	0.8694s
3	0.9418	0.9418s	0.8607	0.8106s
4	0.9231	0.9231s	0.8187	0.7558s
5	0.9048	0.9048s	0.7788	0.7047s
Total				4.0728s

Present Value of Expected Payoff *(Table 25.3; Principal = $1)*

Time (yrs)	Default Probab.	Rec. Rate	Expected Payoff	Discount Factor	PV of Exp. Payoff
0.5	0.0198	0.4	0.0119	0.9753	0.0116
1.5	0.0194	0.4	0.0116	0.9277	0.0108
2.5	0.0190	0.4	0.0114	0.8825	0.0101
3.5	0.0186	0.4	0.0112	0.8395	0.0094
4.5	0.0183	0.4	0.0110	0.7985	0.0088
Total					0.0506

PV of Accrual Payment Made in Event of a Default. *(Table 25.4; Principal = $1)*

Time	Default Prob	Expected Accr Pmt	Disc Factor	PV of Pmt
0.5	0.0198	0.0099*s*	0.9753	0.0097*s*
1.5	0.0194	0.0097*s*	0.9277	0.0090*s*
2.5	0.0190	0.0095*s*	0.8825	0.0084*s*
3.5	0.0186	0.0093*s*	0.8395	0.0078*s*
4.5	0.0183	0.0091*s*	0.7985	0.0073*s*
Total				0.0422*s*

Putting it all together

- PV of expected payments is 4.0728s + 0.0422s = 4.1150s
- The breakeven CDS spread is given by

  4.1150s = 0.0506 or s = 0.0123 (123 bps)
- The value of a swap negotiated some time ago with a CDS spread of 150bps would be

  $$4.1150 \times 0.0150 - 0.0506 = 0.0111$$

  per dollar of the principal.

Implying Default Probabilities from CDS spreads

- Suppose that the mid market spread for a 5 year newly issued CDS is 100bps per year
- We can reverse engineer our calculations to conclude that the hazard is 1.63% per year.
- If probabilities are implied from CDS spreads and then used to value another CDS the result is not sensitive to the recovery rate providing the same recovery rate is used throughout

Binary CDS *(page 578)*

- The payoff in the event of default is a fixed cash amount
- In our example the PV of the expected payoff for a binary swap is 0.0844 and the breakeven binary CDS spread is 205 bps

Credit Indices

- CDX NA IG is a portfolio of 125 investment grade companies in North America
- iTraxx Europe is a portfolio of 125 European investment grade names
- The portfolios are updated on March 20 and Sept 20 each year
- The index can be thought of as the cost per name of buying protection against all 125 names

The Use of Fixed Coupons

- Increasingly CDSs and CDS indices trade like bonds
- A coupon is specified
- If spread is greater than coupon, the buyer of protection pays Notional Principal × Duration × (Spread−Coupon)
- Otherwise the seller of protection pays

  Notional Principal × Duration × (Coupon−Spread)
- Duration is the amount the spread has to be multiplied by to get the PV of spread payments

CDS Forwards and Options *(page 581)*

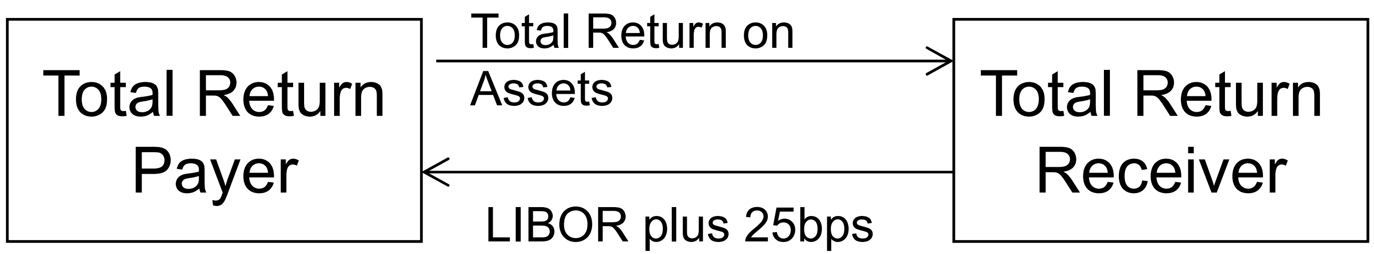
- Example: Forward contract to buy 5 year protection on Ford for 280 bps in one year. If Ford defaults during the one-year life the forward contract ceases to exist
- Example: European option to buy 5 year protection on Ford for 280 bps in one year. If Ford defaults during the one-year life of the option, the option is knocked out

Basket CDS *(page 581)*

- Similar to a regular CDS except that several reference entities are specified
- In a first to default swap there is a payoff when the first entity defaults
- Second, third, and nth to default deals are defined similarly
- Why does pricing depends on default correlation?

Total Return Swap *(page 581-582)*

- Agreement to exchange total return on a portfolio of assets for LIBOR plus a spread
- At the end there is a payment reflecting the change in value of the assets
- Usually used as financing tools by companies that want exposure to assets



Asset Backed Securities

- Securities created from a portfolio of loans, bonds, credit card receivables, mortgages, auto loans, aircraft leases, music royalties, etc
- Usually the income from the assets is tranched
- A “waterfall” defines how income is first used to pay the promised return to the senior tranche, then to the next most senior tranche, and so on.

Collateralized Debt Obligations (Page 583-585)

- A cash CDO is an ABS where the underlying assets are debt obligations
- A synthetic CDO involves forming a similar structure with short CDS contracts
- In a synthetic CDO most junior tranche bears losses first. After it has been wiped out, the second most junior tranche bears losses, and so on

Synthetic CDO Example

- Equity tranche is responsible for losses on underlying CDSs until they reach 5% of total notional principal (earns 1000 bp spread)
- Mezzanine tranche is responsible for losses between 5% and 20% (earns 200 bp spread)
- Senior tranche is responsible for losses over 20% (earns 10 bp spread)

Synthetic CDO Details

- The income is paid on the remaining tranche principal.
- Example: when losses have reached 8% of the total principal underlying the CDSs, tranche 1 has been wiped out, tranche 2 earns the promised spread (200 basis points) on 80% of its principal

Single Tranche Trading

- This involves trading tranches of portfolios of CDSs without actually forming the portfolios
- Cash flows are calculated in the same way as they would be if the portfolios had been formed

Quotes for Standard Tranches of iTraxx (Table 25.6)

Quotes are 30/360 in basis points per year except for the 0-3% tranche where the quote equals the percent of the tranche principal that must be paid upfront in addition to 500 bps per year.

Date	0-3%	3-6%	6-9%	9-12%	12-22%	Index
Jan 1, 2007	10.34%	41.59	11.95	5.60	2.00	23
Jan 1, 2008	30.98%	316.90	212.40	140.00	73.60	77
Jan 1, 2009	64.28%	1185.63	606.69	315.63	97.13	165

Valuation of Tranches of Synthetic CDOs and Basket CDSs (page 585-589)

- A popular approach is to use a factor-based Gaussian copula model to define correlations between times to default
- Often all pairwise correlations and all the unconditional default distributions are assumed to be the same
- Market likes to imply a pairwise correlations from market quotes.

Cumulative Default Probability Conditional on Factor

$$Q(t|F) = N\left(\frac{N^{-1}[Q(t)] - \sqrt{\rho}F}{\sqrt{1-\rho}}\right)$$

- From the binomila distribution,the probability of k defaults from n names by time t conditional on F is

$$\frac{n!}{(n-k)!k!}Q(t|F)^k[1-Q(t|F)]^{n-k}$$

Valuing CDO Tranches

- Consider times t_j (eg: t_j=0.25, 0.5, 0.75….)
- Calculate the expected tranche principal, E_j at each time
- The expected payoff between times t_i and t_{i+1} is the reduction in expected principal
- The expected payment at time t_i is proportional to the expected principal at that time

Valuation *continued. v(t) is discount factor for maturity of t*

$$A = \sum_j (t_j - t_{j-1}) E_j v(t_j)$$

$$B = \sum_j 0.5(t_j - t_{j-1})(E_j - E_{j-1}) v(0.5t_{j-1} + 0.5t_j)$$

$$C = \sum_j (E_j - E_{j-1}) v(0.5t_{j-1} + 0.5t_j)$$

$$\text{Tranche Value} = C - sA - sB \quad \text{where } s \text{ is the spread}$$

Calculation of the E's

- Discretize the distribution of F so that there are, say, 30 values with 30 weights.
- For each value of F calculate the probability that there will be 1, 2, 3…defaults on the underlying portfolio by each time t_i
- Use binomial distribution to calculate the probability of 0,1, 2, 3… defaults by each time t_i on the underlying portfolio for each value of F
- For each value of F calculate expected principal of tranche at each time t_i
- Weight value of tranche by probability of F to obtain unconditional expected principals at each time t_i

The F-values and their weights

- Calculated from Gaussian quadrature (or copied from www.rotman.utoronto.ca/~hull)

$$\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-F^2/2} g(F)\,dF \approx \sum_{k=1}^{M} w_k\, g(F_k)$$

Implied Correlations

- A compound (tranche) correlation is the correlation that is implied from the price of an individual tranche using the one-factor Gaussian copula model
- A base correlation is correlation that prices the 0 to X% tranche consistently with the market where X% is a detachment point (the end point of a standard tranche)

Procedure for Calculating Base Correlation

- Calculate compound correlation for each tranche
- Calculate PV of expected loss for each tranche
- Sum these to get PV of expected loss for base correlation tranches
- Calculate correlation parameter in one-factor gaussian copula model that is consistent with this expected loss

Implied Correlations for iTraxx on January 31, 2007

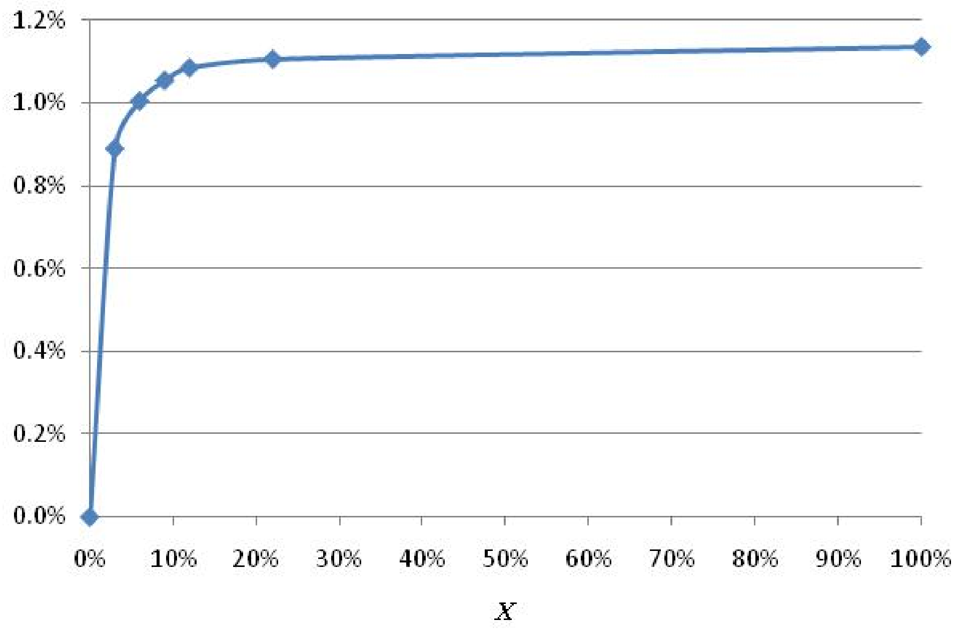
Tranche	0-3%	3-6%	6-9%	9-12%	12-22%
Compound Correlation	17.7%	7.8%	14.0%	18.2%	23.3%

Tranche	0-3%	0-6%	0-9%	0-12%	0-22%
Base Correlation	17.7%	28.4%	36.5%	43.2%	60.5%

NonStandard Tranches

- Better to interpolate expected losses rather than to interpolate base correlations
- For no arbitrage expected losses on the 0 to X% tranche must increase at a decreasing rate as a function of X

Expected Losses on 0 to X% tranche as a percent of Total Underlying Principal for iTraxx on Jan 31, 2007



More Advanced Models

- Relax assumptions that all companies have the same default probabilities
- Use a different copula
- Let copula correlation be a function of F
- Imply copula from market data
- Use a dynamic model

Chapter 26
Exotic Options

Types of Exotics

- Packages
- Perpetual American calls and puts
- Nonstandard American options
- Gap options
- Forward start options
- Cliquet options
- Compound options
- Chooser options
- Barrier options
- Binary options
- Lookback options
- Shout options
- Asian options
- Options to exchange one asset for another
- Options involving several assets
- Volatility and Variance swaps

Packages *(page 598-599)*

- Portfolios of standard options
- Examples from Chapter 11: bull spreads, bear spreads, straddles, etc
- Often structured to have zero cost
- One popular package is a range forward contract (see Chapter 17)

Perpetual American Options

- Consider first a derivative that pays off Q when $S = H$ for the first time and $S_0 < H$
- $f = Q\left(S/H\right)^{\alpha}$ ($\alpha > 0$) satisfies the boundary conditions. It satisfies the differential equation
$$\frac{\partial f}{\partial t} + (r-q)S\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf$$
when
$$(r-q)\alpha + \frac{1}{2}\alpha(\alpha-1)\sigma^2 = r$$
This has solutions $\alpha_1 > 0$ and $\alpha_2 < 0$
- The value of the derivative is therefore
$$Q\left(S/H\right)^{\alpha_1}$$

Perpetual American Options *continued*

- Consider next a perpetual American call option with strike price K
- If it is exercised when $S=H$ the value is $(H-K)\left(S/H\right)^{\alpha_1}$
- This is maximized when $H = K\alpha_1/(\alpha_1-1)$
- The value of the perpetual call is therefore

$$\frac{K}{\alpha_1-1}\left(\frac{\alpha_1-1}{\alpha_1}\frac{S}{K}\right)^{\alpha_1}$$

- The value of a perpetual put is similarly

$$\frac{K}{\alpha_2+1}\left(\frac{\alpha_2+1}{\alpha_2}\frac{S}{K}\right)^{-\alpha_2}$$

Non-Standard American Options

(page 600)

- Exercisable only on specific dates (Bermudans)
- Early exercise allowed during only part of life (initial "lock out" period)
- Strike price changes over the life (warrants, convertibles)

Gap Options

- Gap call pays $S_T - K_1$ when $S_T > K_2$
- Gap put pays off $K_1 - S_T$ when $S_T < K_2$
- Can be valued with a small modification to BSM

$$\text{Gap call} = S_0 N(d_1) - K_1 N(d_2)$$
$$\text{Gap put} = K_1 e^{-rT} N(-d_2) - S_0 e^{-qT} N(-d_1)$$

$$d_1 = \frac{\ln(S_0 / K_2) + (r - q + \sigma^2/2)T}{\sigma\sqrt{T}}$$
$$d_2 = d_1 - \sigma\sqrt{T}$$

Forward Start Options *(page 602)*

- Option starts at a future time, T_1
- Implicit in employee stock option plans
- Often structured so that strike price equals asset price at time T_1
- Value is then e^{-qT_1} times the value of similar option starting today

Cliquet Option

- A series of call or put options with rules determining how the strike price is determined
- For example, a cliquet might consist of 20 at-the-money three-month options. The total life would then be five years
- When one option expires a new similar at-the-money is comes into existence

Compound Option *(page 602-603)*

- Option to buy or sell an option
  - Call on call
  - Put on call
  - Call on put
  - Put on put
- Can be valued analytically
- Price is quite low compared with a regular option

Chooser Option "As You Like It"

(page 603-604)

- Option starts at time 0, matures at T_2
- At T_1 $(0 < T_1 < T_2)$ buyer chooses whether it is a put or call
- This is a package!

Chooser Option as a Package

At time T_1 the value is $\max(c, p)$

From put - call parity

$$p = c + e^{-r(T_2 - T_1)}K - S_1 e^{-q(T_2 - T_1)}$$

The value at time T_1 is therefore

$$c + e^{-q(T_2 - T_1)} \max(0, Ke^{-(r-q)(T_2 - T_1)} - S_1)$$

This is a call maturing at time T_2 plus a position in a put maturing at time T_1

Barrier Options *(page 604-606)*

- Option comes into existence only if stock price hits barrier before option maturity
  - 'In' options
- Option dies if stock price hits barrier before option maturity
  - 'Out' options

Barrier Options *(continued)*

- Stock price must hit barrier from below
  - ‘Up’ options
- Stock price must hit barrier from above
  - ‘Down’ options
- Option may be a put or a call
- Eight possible combinations

Parity Relations

$$c = c_{\text{ui}} + c_{\text{uo}}$$

$$c = c_{\text{di}} + c_{\text{do}}$$

$$p = p_{\text{ui}} + p_{\text{uo}}$$

$$p = p_{\text{di}} + p_{\text{do}}$$

Binary Options *(page 606-607)*

- Cash-or-nothing: pays Q if $S_T > K$, otherwise pays nothing.
  - Value = $e^{-rT} Q\, N(d_2)$
- Asset-or-nothing: pays S_T if $S_T > K$, otherwise pays nothing.
  - Value = $S_0 e^{-qT} N(d_1)$

Decomposition of a Call Option

Long: Asset-or-Nothing option

Short: Cash-or-Nothing option where payoff is K

Value = $S_0 e^{-qT} N(d_1) - e^{-rT} KN(d_2)$

Lookback Options *(page 607-609)*

- Floating lookback call pays $S_T - S_{\min}$ at time T (Allows buyer to buy stock at lowest observed price in some interval of time)
- Floating lookback put pays $S_{\max} - S_T$ at time T

  (Allows buyer to sell stock at highest observed price in some interval of time)
- Fixed lookback call pays max($S_{\max} - K$, 0)
- Fixed lookback put pays max($K - S_{\min}$, 0)
- Analytic valuation for all types

Shout Options *(page 609)*

- Buyer can 'shout' once during option life
- Final payoff is either
  - Usual option payoff, $\max(S_T - K, 0)$, or
  - Intrinsic value at time of shout, $S_\tau - K$
- Payoff: $\max(S_T - S_\tau, 0) + S_\tau - K$
- Similar to lookback option but cheaper

Asian Options *(page 609-611)*

- Payoff related to average stock price
- Average Price options pay:
  - Call: $\max(S_{\text{ave}} - K, 0)$
  - Put: $\max(K - S_{\text{ave}}, 0)$
- Average Strike options pay:
  - Call: $\max(S_T - S_{\text{ave}}, 0)$
  - Put: $\max(S_{\text{ave}} - S_T, 0)$

Asian Options

- No exact analytic valuation
- Can be approximately valued by assuming that the average stock price is lognormally distributed

Exchange Options *(page 611-612)*

- Option to exchange one asset for another
- For example, an option to exchange one unit of U for one unit of V
- Payoff is $\max(V_T - U_T, 0)$

Basket Options *(page 612)*

- A basket option is an option to buy or sell a portfolio of assets
- This can be valued by calculating the first two moments of the value of the basket at option maturity and then assuming it is lognormal

Volatility and Variance Swaps

- Volatility swap is agreement to exchange the realized volatility between time 0 and time T for a prespecified fixed volatility with both being multiplied by a prespecified principal
- Variance swap is agreement to exchange the realized variance rate between time 0 and time T for a prespecified fixed variance rate with both being multiplied by a prespecified principal
- Daily return is assumed to be zero in calculating the volatility or variance rate

Variance Swap

- The (risk-neutral) expected variance rate between times 0 and T can be calculated from the prices of European call and put options with different strikes and maturity T
- For any value of S^*

$$\hat{E}(\bar{V}) = \frac{2}{T}\ln\frac{F_0}{S^*} - \frac{2}{T}\left[\frac{F_0}{S^*} - 1\right] + \frac{2}{T}\int_{K=0}^{S^*}\frac{1}{K^2}e^{rT}p(K)dK + \frac{2}{T}\int_{K=S^*}^{\infty}\frac{1}{K^2}e^{rT}c(K)dK$$

Volatility Swap

- For a volatility swap it is necessary to use the approximate relation

$$\hat{E}(\bar{\sigma}) = \sqrt{\hat{E}\left(\bar{V}\right)}\left\{1 - \frac{1}{8}\left[\frac{\operatorname{var}(\bar{V})}{\hat{E}(\bar{V})^2}\right]\right\}$$

VIX Index *(page 615-616)*

- The expected value of the variance of the S&P 500 over 30 days is calculated from the CBOE market prices of European put and call options on the S&P 500 using the expression for $\hat{E}\left(\bar{V}\right)$
- This is then multiplied by 365/30 and the VIX index is set equal to the square root of the result

How Difficult is it to Hedge Exotic Options?

- In some cases exotic options are easier to hedge than the corresponding vanilla options (e.g., Asian options)
- In other cases they are more difficult to hedge (e.g., barrier options)

Static Options Replication

(Section 26.17, page 616-618)

- This involves approximately replicating an exotic option with a portfolio of vanilla options
- Underlying principle: if we match the value of an exotic option on some boundary , we have matched it at all interior points of the boundary
- Static options replication can be contrasted with dynamic options replication where we have to trade continuously to match the option

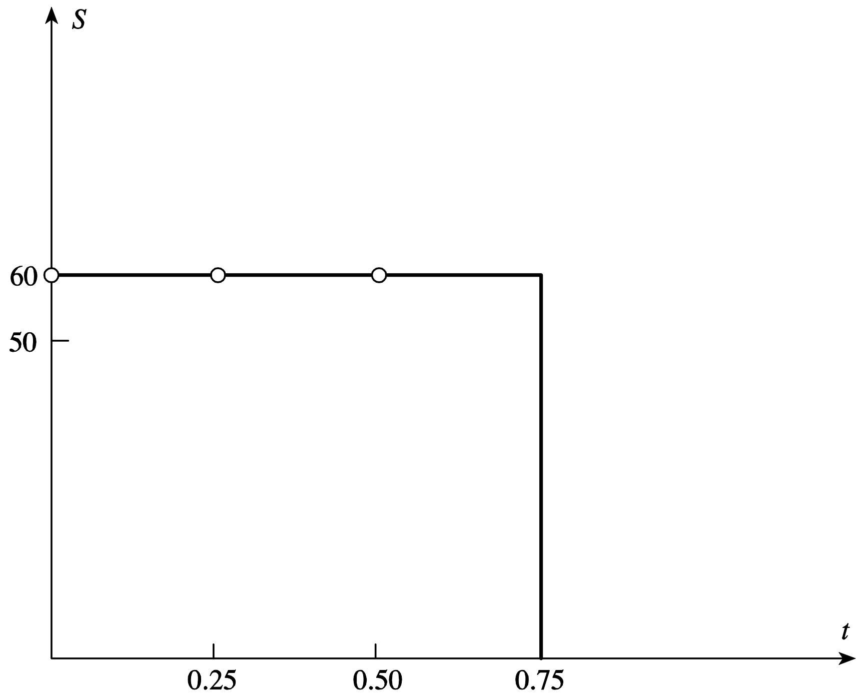
Example

- A 9-month up-and-out call option an a non-dividend paying stock where S_0 = 50, K = 50, the barrier is 60, r = 10%, and σ = 30%
- Any boundary can be chosen but the natural one is

$c\,(S, 0.75) = \max(S - 50, 0)$ when $S < 60$

$c\,(60, t) = 0$ when $0 \le t \le 0.75$

The Boundary



Example *(continued)*

We might try to match the following points on the boundary

$c(S\ , 0.75) = \text{MAX}(S - 50, 0)$ for $S < 60$

$c(60, 0.50) = 0$

$c(60, 0.25) = 0$

$c(60, 0.00) = 0$

Example *continued*

(See Table 26.1, page 619)

We can do this as follows:

+1.00 call with maturity 0.75 & strike 50

–2.66 call with maturity 0.75 & strike 60

+0.97 call with maturity 0.50 & strike 60

+0.28 call with maturity 0.25 & strike 60

Example *(continued)*

- This portfolio is worth 0.73 at time zero compared with 0.31 for the up-and out option
- As we use more options the value of the replicating portfolio converges to the value of the exotic option
- For example, with 18 points matched on the horizontal boundary the value of the replicating portfolio reduces to 0.38; with 100 points being matched it reduces to 0.32

Using Static Options Replication

- To hedge an exotic option we short the portfolio that replicates the boundary conditions
- The portfolio must be unwound when any part of the boundary is reached

Chapter 27
More on Models and Numerical Procedures

Three Alternatives to Geometric Brownian Motion

- Constant elasticity of variance (CEV)
- Mixed Jump diffusion
- Variance Gamma

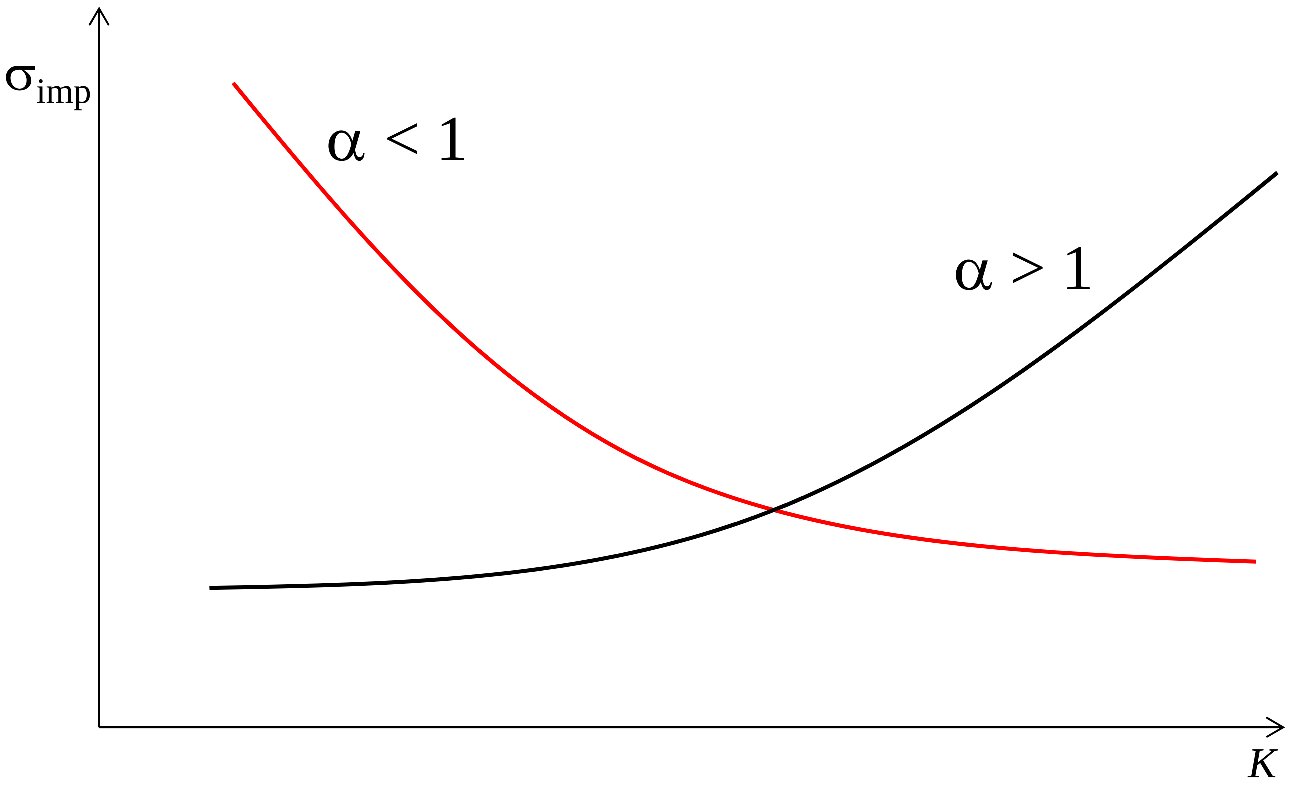
CEV Model *(pages 625-626)*

$$dS = (r - q)Sdt + \sigma S^{\alpha} dz$$

- When α = 1 the model is Black-Scholes
- When α > 1 volatility rises as stock price rises
- When α < 1 volatility falls as stock price rises

European options can be value analytically in terms of the cumulative non-central chi square distribution

CEV Models Implied Volatilities



Mixed Jump Diffusion Model

(page 626-628)

Merton produced a pricing formula when the asset price follows a diffusion process overlaid with random jumps

$$dS / S = (r - q - \lambda k)dt + \sigma dz + dp$$

- dp is the random jump
- k is the expected size of the jump
- $\lambda\, dt$ is the probability that a jump occurs in the next interval of length dt
- λk is the expected return from jumps

Simulating a Jump Process

- In each time step
  - Sample from a binomial distribution to determine the number of jumps
  - Sample to determine the size of each jump

Jumps and the Smile

- Jumps have a big effect on the implied volatility of short term options
- They have a much smaller effect on the implied volatility of long term options

The Variance-Gamma Model (page 628-630)

- Define g as change over time T in a variable that follows a gamma process. This is a process where small jumps occur frequently and there are occasional large jumps
- Conditional on g, ln S_T is normal. Its variance proportional to g
- There are 3 parameters
  - v, the variance rate of the gamma process
  - σ^2, the average variance rate of ln S per unit time
  - θ, a parameter defining skewness

Understanding the Variance-Gamma Model

- g defines the rate at which information arrives during time T (g is sometimes referred to as measuring economic time)
- If g is large the change in $\ln S$ has a relatively large mean and variance
- If g is small relatively little information arrives and the change in $\ln S$ has a relatively small mean and variance

Time Varying Volatility

- The variance rate substituted into BSM should be the average variance rate
  - Suppose the volatility is σ_1 for the first year and σ_2 for the second and third
  - Total accumulated variance at the end of three years is $\sigma_1^2 + 2\sigma_2^2$
- The 3-year average volatility is given by

$$3\bar{\sigma}^2 = \sigma_1^2 + 2\sigma_2^2; \quad \bar{\sigma} = \sqrt{\frac{\sigma_1^2 + 2\sigma_2^2}{3}}$$

Stochastic Volatility Models

(equations 27.2 and 27.3, page 631)

$$\frac{dS}{S} = (r - q)dt + \sqrt{V}\,dz_S$$

$$dV = a(V_L - V)dt + \xi V^{\alpha} dz_V$$

When V and S are uncorrelated a European option price is the Black-Scholes-Merton price integrated over the distribution of the average variance rate

Stochastic Volatility Models
continued

- When V and S are negatively correlated we obtain a downward sloping volatility skew similar to that observed in the market for equities
- When V and S are positively correlated the skew is upward sloping. (This pattern is sometimes observed for commodities)

The IVF Model *(page 632-633)*

The implied volatility function model is designed to create a process for the asset price that exactly matches observed option prices. The usual geomeric Brownian motion model

$$dS = (r - q)Sdt + \sigma Sdz$$

is replaced by

$$dS = [r(t) - q(t)]Sdt + \sigma(S, t)Sdz$$

The Volatility Function ***(equation 27.4)***

The volatility function that leads to the model matching all European option prices is

$$[\sigma(K,t)]^2 =$$

$$2\,\frac{\partial c_{mkt}/\partial t + q(t)c_{mkt} + K[r(t)-q(t)]\,\partial c_{mkt}/\partial K}{K^2(\partial^2 c_{mkt}/\partial K^2)}$$

Strengths and Weaknesses of the IVF Model

- The model matches the probability distribution of asset prices assumed by the market at each future time
- The models does not necessarily get the joint probability distribution of asset prices at two or more times correct

Convertible Bonds

- Often valued with a tree where during a time interval Δt there is
  - a probability p_u of an up movement
  - A probability p_d of a down movement
  - A probability 1-exp(-λt) that there will be a default (λ is the hazard rate)
- In the event of a default the stock price falls to zero and there is a recovery on the bond

The Probabilities

$$p_u = \frac{a - de^{-\lambda \Delta t}}{u - d}$$

$$p_d = \frac{ue^{-\lambda \Delta t} - a}{u - d}$$

$$u = e^{\sqrt{(\sigma^2 - \lambda)\Delta t}}$$

$$d = \frac{1}{u}$$

Node Calculations

Define:

$Q_{1:}$ value of bond if neither converted nor called

Q_2: value of bond if called

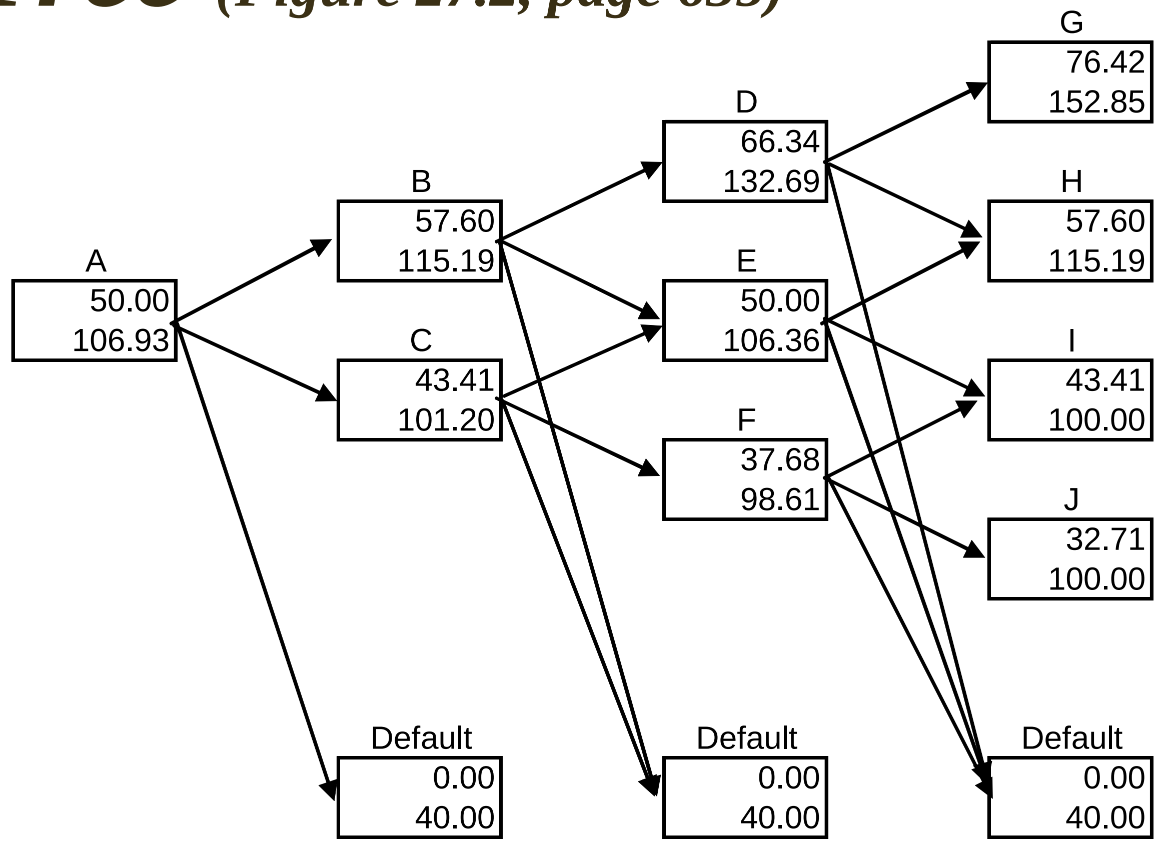
Q_3: value of bond if converted

Value at a node =max[min(Q_1,Q_2),Q_3]

Example 27.1 *(page 634)*

- 9-month zero-coupon bond with face value of \$100
- Convertible into 2 shares
- Callable for \$113 at any time
- Initial stock price = \$50,
- volatility = 30%,
- no dividends
- Risk-free rates all 5%
- Default intensity, λ, is 1%
- Recovery rate=40%

The Tree *(Figure 27.2, page 635)*



Numerical Procedures

Topics:

- Path dependent options using tree
- Barrier options
- Options where there are two stochastic variables
- American options using Monte Carlo

Path Dependence: The Traditional View

- Trees work well for American options. They cannot be used for path-dependent options
- Monte Carlo simulation works well for path-dependent options; it cannot be used for American options

Extending the Use of Trees

- Backwards induction can be used for some path-dependent options
- We will first illustrate the methodology using lookback options and then show how it can be used for Asian options

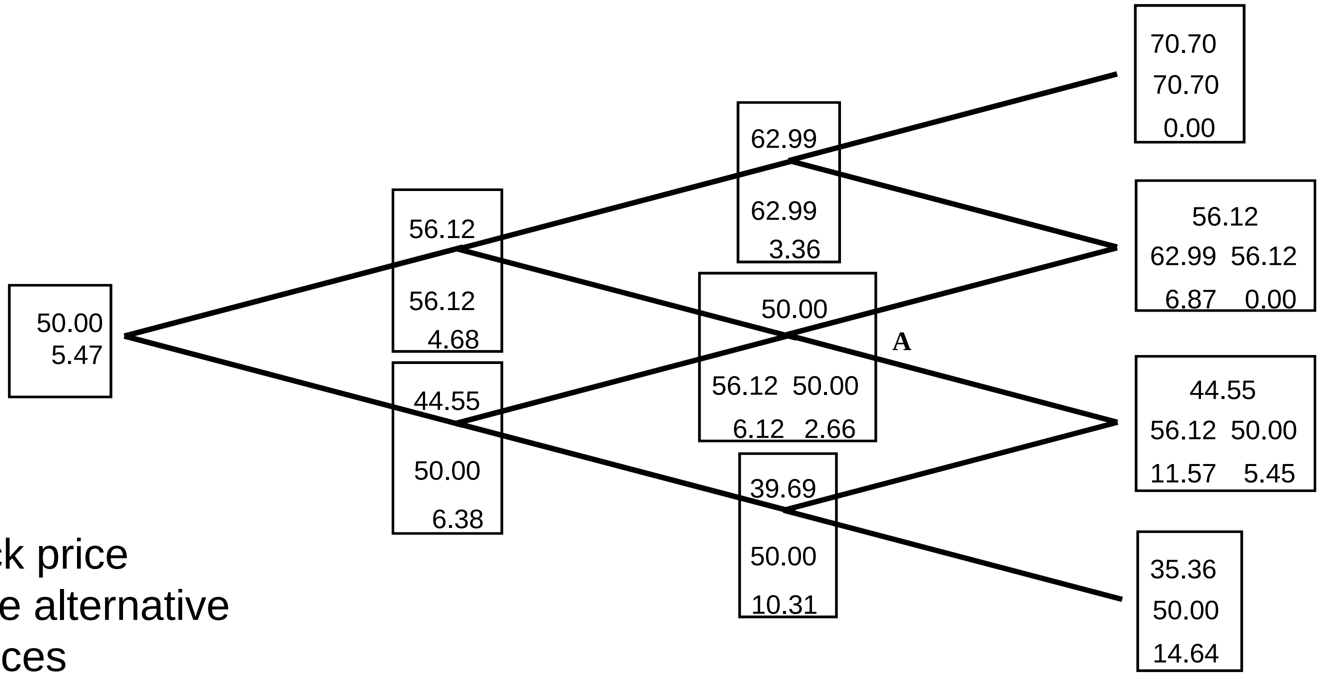
Lookback Example *(pages 636-640)*

- Consider an American lookback put on a stock where $S = 50$, $\sigma = 40\%$, $r = 10\%$, $\Delta t = 1$ month & the life of the option is 3 months
- Payoff is $S_{\text{max}}-S_T$
- We can value the deal by considering all possible values of the maximum stock price at each node

(This example is presented to illustrate the methodology. It is not the most efficient way of handling American lookbacks (See Technical Note 13)

Example: An American Lookback Put Option *(Figure 27.3, page 637)*

$S_0 = 50$, $\sigma = 40\%$, $r = 10\%$, $\Delta t = 1$ month,



Top number is stock price
Middle numbers are alternative maximum stock prices
Lower numbers are option prices

Why the Approach Works

This approach works for lookback options because

- The payoff depends on just 1 function of the path followed by the stock price. (We will refer to this as a "path function")
- The value of the path function at a node can be calculated from the stock price at the node and from the value of the function at the immediately preceding node
- The number of different values of the path function at a node does not grow too fast as we increase the number of time steps on the tree

Extensions of the Approach

- The approach can be extended so that there are no limits on the number of alternative values of the path function at a node
- The basic idea is that it is not necessary to consider every possible value of the path function
- It is sufficient to consider a relatively small number of representative values of the function at each node

Working Forward

- First work forward through the tree calculating the max and min values of the “path function” at each node
- Next choose representative values of the path function that span the range between the min and the max
  - Simplest approach: choose the min, the max, and N equally spaced values between the min and max

Backwards Induction

- We work backwards through the tree in the usual way carrying out calculations for each of the alternative values of the path function that are considered at a node
- When we require the value of the derivative at a node for a value of the path function that is not explicitly considered at that node, we use linear or quadratic interpolation

Part of Tree to Calculate Value of an Option on the Arithmetic Average

(Figure 27.4, page 639)

Node X: S = 50.00

Average S	Option Price
46.65	5.642
49.04	5.923
51.44	6.206
53.83	6.492

0.5056 (X → Y)

Node Y: S = 54.68

Average S	Option Price
47.99	7.575
51.12	8.101
54.26	8.635
57.39	9.178

0.4944 (X → Z)

Node Z: S = 45.72

Average S	Option Price
43.88	3.430
46.75	3.750
49.61	4.079
52.48	4.416

S=50, X=50, σ=40%, r =10%, T=1yr, Δt=0.05yr. We are at time $4\Delta t$

Part of Tree to Calculate Value of an Option on the Arithmetic Average (continued)

Consider Node X when the average of 5 observations is 51.44

Node Y: If this is reached, the average becomes 51.98. The option price is interpolated as 8.247

Node Z: If this is reached, the average becomes 50.49. The option price is interpolated as 4.182

Node X: value is

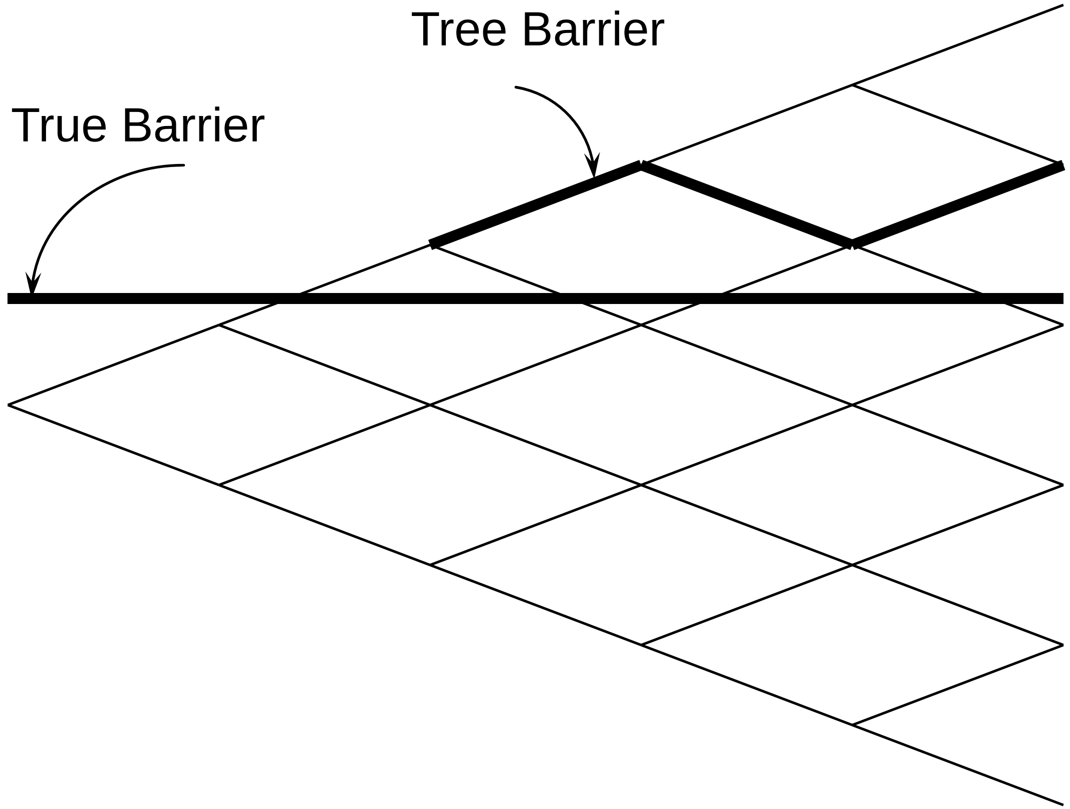
$$(0.5056\times 8.247 + 0.4944\times 4.182)e^{-0.1\times 0.05} = 6.206$$

Using Trees with Barriers

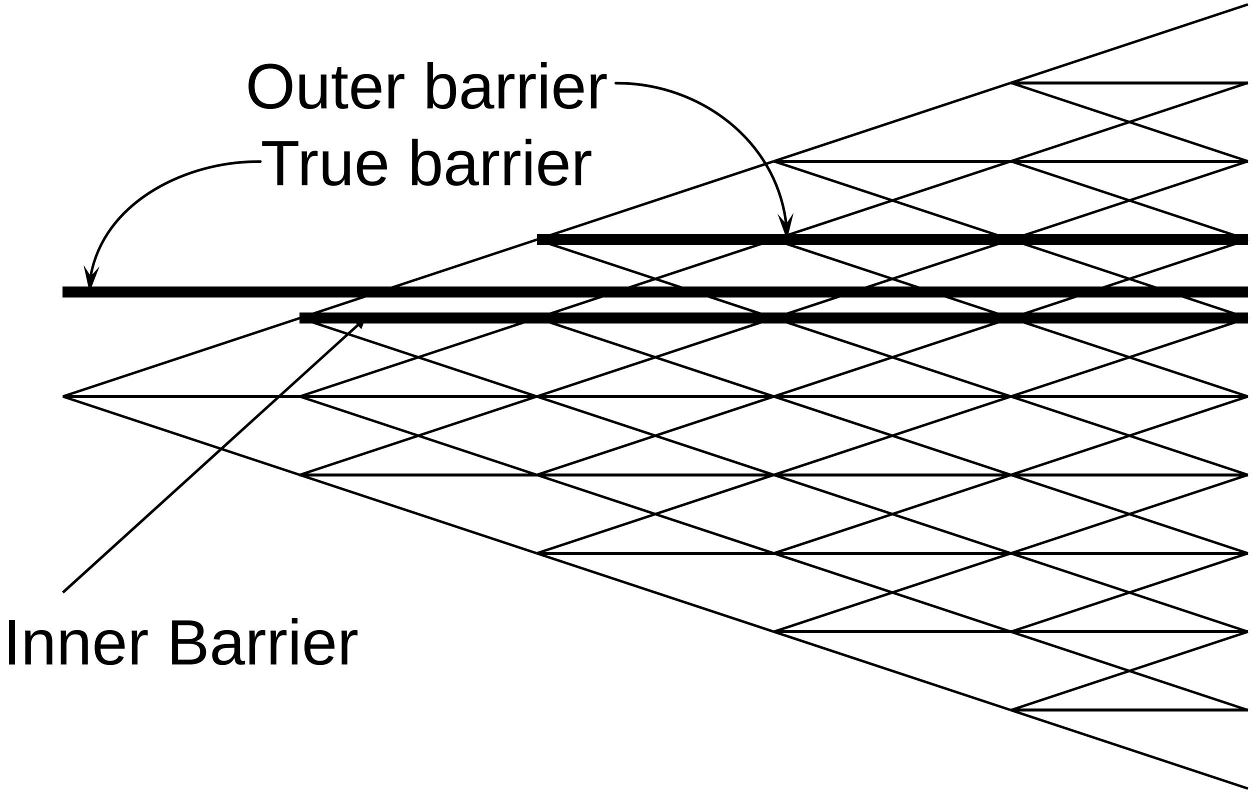
(Section 27.6, page 640-643)

- When trees are used to value options with barriers, convergence tends to be slow
- The slow convergence arises from the fact that the barrier is inaccurately specified by the tree

True Barrier vs Tree Barrier for a Knockout Option: The Binomial Tree Case



Inner and Outer Barriers for Trinomial Trees
(Figure 27.5, page 641)



Alternative Solutions to Valuing Barrier Options

- Interpolate between value when inner barrier is assumed and value when outer barrier is assumed
- Ensure that nodes always lie on the barriers
- Use adaptive mesh methodology

In all cases a trinomial tree is preferable to a binomial tree

Modeling Two Correlated Variables Using a 3-Dimensional Tree *(Section 27.7, page 643)*

- Approaches
  - Transform variables so that they are not correlated and build the tree in the transformed variables
  - Take the correlation into account by adjusting the position of the nodes
  - Take the correlation into account by adjusting the probabilities

Monte Carlo Simulation and American Options

- Two approaches:
  - The least squares approach
  - The exercise boundary parameterization approach
- Consider a 3-year put option where the initial asset price is 1.00, the strike price is 1.10, the risk-free rate is 6%, and there is no income

Sampled Paths

Path	$t = 0$	$t = 1$	$t = 2$	$t = 3$
1	1.00	1.09	1.08	1.34
2	1.00	1.16	1.26	1.54
3	1.00	1.22	1.07	1.03
4	1.00	0.93	0.97	0.92
5	1.00	1.11	1.56	1.52
6	1.00	0.76	0.77	0.90
7	1.00	0.92	0.84	1.01
8	1.00	0.88	1.22	1.34

The Least Squares Approach *(page 646-649)*

- We work back from the end using a least squares approach to calculate the continuation value at each time
- Consider year 2. The option is in the money for five paths. These give observations on S of 1.08, 1.07, 0.97, 0.77, and 0.84. The continuation values are 0.00, $0.07e^{-0.06}$, $0.18e^{-0.06}$, $0.20e^{-0.06}$, and $0.09e^{-0.06}$

The Least Squares Approach
continued

- Fitting a model of the form $V=a+bS+cS^2$ we get a best fit relation

$$V=-1.070+2.983S-1.813S^2$$

for the continuation value V

- This defines the early exercise decision at t =2. We carry out a similar analysis at $t=1$

The Least Squares Approach
continued

In practice more complex functional forms can be used for the continuation value and many more paths are sampled

The Early Exercise Boundary Parametrization Approach *(page 649-650)*

- We assume that the early exercise boundary can be parameterized in some way
- We carry out a first Monte Carlo simulation and work back from the end calculating the optimal parameter values
- We then discard the paths from the first Monte Carlo simulation and carry out a new Monte Carlo simulation using the early exercise boundary defined by the parameter values.

Application to Example

- We parameterize the early exercise boundary by specifying a critical asset price, S^*, below which the option is exercised.
- At t =1 the optimal S^* for the eight paths is 0.88. At t =2 the optimal S^* is 0.84
- In practice we would use many more paths to calculate the S^*

Chapter 28
Martingales and Measures

Derivatives Dependent on a Single Underlying Variable

Consider a variable θ (not necessarily the price of a traded security) that follows the process

$$\frac{d\theta}{\theta} = m\ dt + s\ dz$$

Imagine two derivatives dependent on θ with prices f_1 and f_2. Suppose

$$\frac{d?_1}{f_1} = \mu_1\ dt + \sigma_1\ dz$$

$$\frac{d?_2}{f_2} = \mu_2\ dt + \sigma_2\ dz$$

Forming a Riskless Portfolio

We can set up a riskless portfolio Π consisting of

$+\sigma_2 f_2$ of the 1st derivative and

$-\sigma_1 f_1$ of the 2nd derivative

$$\Pi = (\sigma_2 f_2) f_1 - (\sigma_1 f_1) f_2$$

$$\Delta\Pi = (\mu_1 \sigma_2 f_1 f_2 - \mu_2 \sigma_1 f_1 f_2)\Delta t$$

Market Price of Risk *(Page 657)*

Since the portfolio is riskless : $\Delta\Pi = r\,\Pi\Delta t$

This gives : $\mu_1\sigma_2 - \mu_2\sigma_1 = r\,\sigma_2 - r\,\sigma_1$

or $$\frac{\mu_1 - r}{\sigma_1} = \frac{\mu_2 - r}{\sigma_2}$$

- This shows that $(\mu - r)/\sigma$ is the same for all derivatives dependent on the same underlying variable, θ
- We refer to $(\mu - r)/\sigma$ as the market price of risk for θ and denote it by λ

Extension of the Analysis to Several Underlying Variables

(Equations 28.12 and 28.13, page 659)

If f depends on several underlying variables with

$$\frac{d?}{f} = \mu\ dt + \sum_{i=1}^{n} \sigma_i\ dz_i$$

then

$$\mu - r = \sum_{i=1}^{n} \lambda_i \sigma_i$$

Martingales *(Page 660-661)*

- A martingale is a stochastic process with zero drift
- A variable following a martingale has the property that its expected future value equals its value today

Alternative Worlds

In the traditional risk-neutral world

$$df = rf\,dt + \sigma f\,dz$$

In a world where the market price of risk is λ

$$df = (r + \lambda\sigma) f\,dt + \sigma f\,dz$$

The Equivalent Martingale Measure Result *(Page 660-661)*

If we set λ equal to the volatility of a security g, then Ito's lemma shows that f/g is a martingale for all derivative security prices f

Forward Risk Neutrality

We will refer to a world where the market price of risk is the volatility of g as a world that is forward risk neutral with respect to g.

If E_g denotes expectations in a world that is FRN wrt g

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

Alternative Choices for the Numeraire Security g

- Money Market Account
- Zero-coupon bond price
- Annuity factor

Money Market Account as the Numeraire

- The money market account is an account that starts at \$1 and is always invested at the short-term risk-free interest rate
- The process for the value of the account is

$$dg = rg\,dt$$

- This has zero volatility. Using the money market account as the numeraire leads to the traditional risk-neutral world where λ=0

Money Market Account continued

Since $g_0 = 1$ and $g_T = e^{\int_0^T r dt}$, the equation

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

becomes

$$f_0 = \hat{E}\left[e^{-\int_0^T r dt} f_T\right]$$

where $\hat{E}$ denotes expectations in the traditional risk - neutral world

Zero-Coupon Bond Maturing at time T as Numeraire

The equation

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

becomes

$$f_0 = P(0,T)E_T[f_T]$$

where $P(0,T)$ is the zero-coupon bond price and E_T denotes expectations in a world that is FRN wrt the bond price

Forward Prices

In a world that is FRN wrt $P(0,T)$, the expected value of a security at time T is its forward price

Interest Rates

In a world that is FRN wrt $P(0,T_2)$ the expected value of an interest rate lasting between times T_1 and T_2 is the forward interest rate

Annuity Factor as the Numeraire

The equation

$$\frac{f_0}{g_0} = E_g\left(\frac{f_T}{g_T}\right)$$

becomes

$$f_0 = A(0)E_A\left[\frac{f_T}{A(T)}\right]$$

Annuity Factors and Swap Rates

Suppose that $s(t)$ is the swap rate corresponding to the annuity factor A.

Then:

$$s(t)=E_A[s(T)]$$

Extension to Several Independent Factors

(Page 665)

In the traditional risk-neutral world

$$df(t) = r(t) f(t)dt + \sum_{i=1}^{m} \sigma_{f,i}(t) f(t)dz_i$$

$$dg(t) = r(t) g(t)dt + \sum_{i=1}^{m} \sigma_{g,i}(t) g(t)dz_i$$

For other worlds that are internally consistent

$$df(t) = \left[r(t) + \sum_{i=1}^{m} \lambda_i \sigma_{f,i}(t) \right] f(t)dt + \sum_{i=1}^{m} \sigma_{f,i}(t) f(t)dz_i$$

$$dg(t) = \left[r(t) + \sum_{i=1}^{m} \lambda_i \sigma_{g,i}(t) \right] g(t)dt + \sum_{i=1}^{m} \sigma_{g,i}(t) g(t)dz_i$$

Extension to Several Independent Factors *continued*

We define a world that is FRN wrt g as world where $\lambda_i = \sigma_{g,i}$

As in the one-factor case, f/g is a martingale and the rest of the results hold.

Applications

- Extension of Black's model to case where inbterest rates are stochastic
- Valuation of an option to exchange one asset for another

Black's Model *(page 666)*

By working in a world that is forward risk neutral with respect to a $P(0,T)$ it can be seen that Black's model is true when interest rates are stochastic providing the forward price of the underlying asset is has a constant volatility

$$c = P(0,T)[F_0 N(d_1) - KN(d_2)]$$

$$p = P(0,T)[KN(-d_2) - F_0 N(-d_1)]$$

$$d_1 = \frac{\ln(F_0/K) + \sigma_F^2 T}{\sigma_F \sqrt{T}} \quad d_2 = \frac{\ln(F_0/K) - \sigma_F^2 T}{\sigma_F \sqrt{T}}$$

Option to exchange an asset worth U for one worth V

- This can be valued by working in a world that is forward risk neutral with respect to U
- Value is

$$e^{-q_V}V_0N(d_1)-e^{-q_U}U_0N(d_2)$$

$$d_1=\frac{\ln\left(V_0/U_0\right)+(q_U-q_V+\hat{\sigma}^2/2)T}{\hat{\sigma}\sqrt{T}} \qquad d_2=d_1-\hat{\sigma}\sqrt{T}$$

$$\hat{\sigma}^2=\sigma_U^2+\sigma_V^2-2\rho\sigma_U\sigma_V$$

Change of Numeraire

(Section 28.8, page 668)

When we change the numeraire security from g to h, the drift of a variable v increases by $\rho\sigma_v\sigma_w$ where σ_v is the volatility of v, $w = h/g$, σ_w is the volatility of w, and ρ is the correlation between v and w

Chapter 29
Interest Rate Derivatives: The Standard Market Models

The Complications in Valuing Interest Rate Derivatives *(page 673)*

- We need a whole term structure to define the level of interest rates at any time
- The stochastic process for an interest rate is more complicated than that for a stock price
- Volatilities of different points on the term structure are different
- Interest rates are used for discounting the payoff as well as for defining the payoff. When OIS discounting is used for a product whose payoffs depend on LIBOR, two term structures must be considered

Approaches to Pricing Interest Rate Options

- Use a variant of Black's model
- Use a no-arbitrage (yield curve based) model

Black's Model

- Similar to the model proposed by Fischer Black for valuing options on futures in 1976
- Assumes that the value of an interest rate, a bond price, or some other variable at a particular time T in the future has a lognormal distribution

Black's Model for European Bond Options *(Equations 29.1 and 29.2, page 674)*

- Assume that the future bond price is lognormal

$$c = P(0,T)[F_B N(d_1) - KN(d_2)]$$

$$p = P(0,T)[KN(-d_2) - F_B N(-d_1)]$$

$$d_1 = \frac{\ln(F_B / K) + \sigma_B^2 T / 2}{\sigma_B \sqrt{T}}; \; d_2 = d_1 - \sigma_B \sqrt{T}$$

- Both the bond price and the strike price should be cash prices not quoted prices

Forward Bond and Forward Yield

Approximate duration relation between forward bond price, F_B, and forward bond yield, y_F

$$\frac{\Delta F_B}{F_B} \approx -D\,\Delta y_F \quad \text{or} \quad \frac{\Delta F_B}{F_B} \approx -D y_F \frac{\Delta y_F}{y_F}$$

where D is the (modified) duration of the forward bond at option maturity

Yield Vols vs Price Vols *(Equation 29.4, page 677)*

- This relationship implies the following approximation

$$\sigma_B = D y_0 \sigma_y$$

where σ_y is the forward yield volatility, σ_B is the forward price volatility, and y_0 is today's forward yield

- Often σ_y is quoted with the understanding that this relationship will be used to calculate σ_B

Theoretical Justification for Bond Option Model

Working in a world that is FRN wrt a zero-coupon bond maturing at time T, the option price is

$$P(0,T)E_T[\max(B_T - K, 0)]$$

Also

$$E_T[B_T] = F_B$$

This leads to Black's model

Caps and Floors

- A cap is a portfolio of call options on LIBOR. It has the effect of guaranteeing that the interest rate in each of a number of future periods will not rise above a certain level
- Payoff at time t_{k+1} is $L\delta_k \max(R_k - R_K, 0)$ where L is the principal, $\delta_k = t_{k+1} - t_k$, R_K is the cap rate, and R_k is the rate at time t_k for the period between t_k and t_{k+1}
- A floor is similarly a portfolio of put options on LIBOR. Payoff at time t_{k+1} is

$$L\delta_k \max(R_K - R_k, 0)$$

Caplets

- A cap is a portfolio of "caplets"
- Each caplet is a call option on a future LIBOR rate with the payoff occurring in arrears
- When using Black's model we assume that the interest rate underlying each caplet is lognormal

Black's Model for Caps *(p. 680)*

- The value of a caplet, for period (t_k, t_{k+1}) is

  $$L\delta_k P(0,t_{k+1})[F_k N(d_1) - R_K N(d_2)]$$

  where $d_1 = \dfrac{\ln(F_k / R_K) + \sigma_k^2 t_k / 2}{\sigma_k \sqrt{t_k}}$ and $d_2 = d_1 - \sigma\sqrt{t_k}$

- The value of a floorlet is

  $$L\delta_k P(0,t_{k+1})\left[R_K N(-d_2) - F_k N(-d_1)\right]$$

- F_k : forward interest rate for (t_k, t_{k+1})
- σ_k : forward rate volatility
- L: principal
- R_K : cap rate
- $\delta_k = t_{k+1} - t_k$

Black's Model continued

- When LIBOR discounting is used, the same LIBOR/swap term structure is used to calculate F_k and $P(0,t_{k+1})$
- When OIS discounting is used we calculate the OIS zero curve first and then calculate the LIBOR/swap zero curve that makes swaps currently traded have zero value
- The LIBOR/swap zero curve is used for F_k and the OIS curve is used for $P(0,t_{k+1})$

When Applying Black's Model To Caps We Must ...

- EITHER
  - Use spot volatilities
  - Volatility different for each caplet
- OR
  - Use flat volatilities
  - Volatility same for each caplet within a particular cap but varies according to life of cap

Theoretical Justification for Cap Model

Working in a world that is FRN wrt a zero-coupon bond maturing at time t_{k+1} the option price is

$$P(0,t_{k+1})E_{k+1}[\max(R_k - R_K, 0)]$$

Also

$$E_{k+1}[R_k] = F_k$$

This leads to Black's model

Swaptions

- A swaption or swap option gives the holder the right to enter into an interest rate swap in the future
- Two kinds
  - The right to pay a specified fixed rate and receive LIBOR
  - The right to receive a specified fixed rate and pay LIBOR

Black's Model for European Swaptions

- When valuing European swap options it is usual to assume that the swap rate is lognormal
- Consider a swaption which gives the right to pay s_K on an n -year swap starting at time T. The payoff on each swap payment date is

$$\frac{L}{m}\max(s_T - s_K, 0)$$

where L is principal, m is payment frequency and s_T is market swap rate at time T

Black's Model for European Swaptions

continued (equations 29.10 and 29.11)

The value of the swaption where holder has right to pay s_K is $LA[s_0N(d_1)-s_K\, N(d_2)]$

The value of a swaption where the hold has the right to receive s_K is $LA[s_KN(-d_2)-s_0\, N(-d_1)]$

where $d_1 = \dfrac{\ln(s_0/s_K)+\sigma^2T/2}{\sigma\sqrt{T}}; \qquad d_2 = d_1 - \sigma\sqrt{T}$

$$A = \frac{1}{m}\sum_{i=1}^{mn} P(0,t_i)$$

s_0 is the forward swap rate; σ is the forward swap rate volatility; t_i is the time from today until the ith swap payment.

Black's Model *continued*

- When LIBOR discounting is used, the same LIBOR/swap term structure is used to calculate s_0 and $P(0,t_{k+1})$
- When OIS discounting is used we calculate the OIS zero curve first and then calculate the LIBOR/swap zero curve that makes swaps currently traded have zero value
- The LIBOR/swap zero curve is used for s_0 and the OIS curve is used for A

Theoretical Justification for Swap Option Model

Working in a world that is FRN wrt the annuity underlying the swap, the option price is

$$LAE_A[\max(s_T - s_K, 0)]$$

Also

$$E_A[s_T] = s_0$$

This leads to Black's model

Relationship Between Swaptions and Bond Options

- An interest rate swap can be regarded as the exchange of a fixed-rate bond for a floating-rate bond
- A swaption or swap option is therefore an option to exchange a fixed-rate bond for a floating-rate bond

Relationship Between Swaptions and Bond Options *(continued)*

- At the start of the swap the floating-rate bond is worth par so that the swaption can be viewed as an option to exchange a fixed-rate bond for par
- An option on a swap where fixed is paid and floating is received is a put option on the bond with a strike price of par
- When floating is paid and fixed is received, it is a call option on the bond with a strike price of par

Deltas of Interest Rate Derivatives

Alternatives:

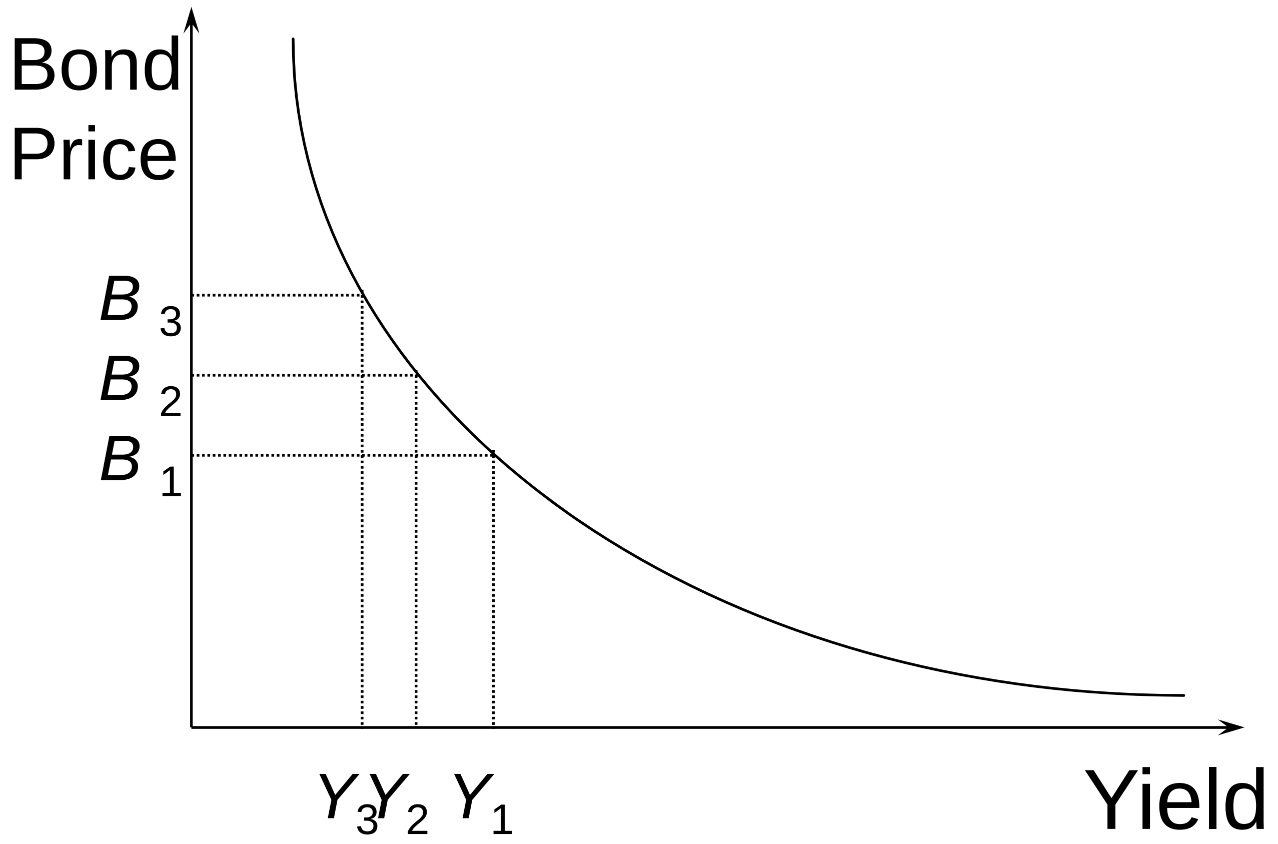
- Calculate a DV01 (the impact of a 1bps parallel shift in the zero curve)
- Calculate impact of small change in the quote for each instrument used to calculate the zero curve
- Divide zero curve (or forward curve) into buckets and calculate the impact of a shift in each bucket
- Carry out a principal components analysis for changes in the zero curve. Calculate delta with respect to each of the first two or three factors

Chapter 30
Convexity, Timing, and Timing, and Quanto Adjustments

Forward Yields and Forward Prices

- We define the forward yield on a bond as the yield calculated from the forward bond price
- There is a non-linear relation between bond yields and bond prices
- It follows that when the forward bond price equals the expected future bond price, the forward yield does not necessarily equal the expected future yield

Relationship Between Bond Yields and Prices *(Figure 30.1, page 694)*



Convexity Adjustment for Bond Yields

(Eqn 30.1, p. 695)

- Suppose a derivative provides a payoff at time T dependent on a bond yield, y_T observed at time T. Define:
- $G(y_T)$: price of the bond as a function of its yield

  y_0 : forward bond yield at time zero

  σ_y : forward yield volatility
- The expected bond price in a world that is FRN wrt $P(0,T)$ is the forward bond price
- The expected bond yield in a world that is FRN wrt $P(0,T)$ is

$$\text{Forward Bond Yield} - \frac{1}{2} y_0^2 \sigma_y^2 T \frac{G''(y_0)}{G'(y_0)}$$

Convexity Adjustment for Swap Rate

The expected value of the swap rate for the period T to $T+\tau$ in a world that is FRN wrt $P(0,T)$ is (approximately)

$$\text{Forward Swap Rate} - \frac{1}{2} y_0^2 \sigma_y^2 T \frac{G''(y_0)}{G'(y_0)}$$

where $G(y)$ defines the relationship between price and yield for a bond lasting between T and $T+\tau$ that pays a coupon equal to the forward swap rate

Example 30.1 *(page 696)*

- An instrument provides a payoff in 3 years equal to the 1-year zero-coupon rate multiplied by $1000
- Volatility is 20%
- Yield curve is flat at 10% (with annual compounding)
- The convexity adjustment is 10.9 bps so that the value of the instrument is $101.09/1.1^3 = 75.95$

Example 30.2 *(Page 696-697)*

- An instrument provides a payoff in 3 years = to the 3-year swap rate multiplied by $100
- Payments are made annually on the swap
- Volatility is 22%
- Yield curve is flat at 12% (with annual compounding)
- The convexity adjustment is 36 bps so that the value of the instrument is $12.36/1.12^3$ = 8.80

Timing Adjustments *(Equation 30.4, page 698)*

The expected value of a variable, V, in a world that is FRN wrt $P(0,T^*)$ is the expected value of the variable in a world that is FRN wrt $P(0,T)$ multiplied by

$$\exp\left[-\frac{\rho_{VR}\sigma_V\sigma_R R_0(T^*-T)}{1+R_0/m}T\right]$$

where R is the forward interest rate between T and T^* expressed with a compounding frequency of m, σ_R is the volatility of R, R_0 is the value of R today, σ_V is the volatility of F, and ρ is the correlation between R and V

Example 30.3 *(page 698)*

- A derivative provides a payoff 6 years equal to the value of a stock index in 5 years. The interest rate is 8% with annual compounding
- 1200 is the 5-year forward value of the stock index
- This is the expected value in a world that is FRN wrt $P(0,5)$
- To get the value in a world that is FRN wrt $P(0,6)$ we multiply by 1.00535
- The value of the derivative is $1200 \times 1.00535/(1.08^6)$ or 760.26

Quantos

(Section 30.3, page 699-702)

- Quantos are derivatives where the payoff is defined using variables measured in one currency and paid in another currency
- Example: contract providing a payoff of $S_T - K$ dollars ($) where S is the Nikkei stock index (a yen number)

Diff Swap

- Diff swaps are a type of quanto
- A floating rate is observed in one currency and applied to a principal in another currency

Quanto Adjustment *(page 700)*

- The expected value of a variable, V, in a world that is FRN wrt $P_X(0,T)$ is its expected value in a world that is FRN wrt $P_Y(0,T)$ multiplied by $\exp(\rho_{VW}\sigma_V\sigma_W T)$
- W is the forward exchange rate (units of Y per unit of X) and ρ_{VW} is the correlation between V and W.

Example 30.4 *(page 700)*

- Current value of Nikkei index is 15,000
- This gives one-year forward as 15,150.75
- Suppose the volatility of the Nikkei is 20%, the volatility of the dollar-yen exchange rate is 12% and the correlation between the two is 0.3
- The one-year forward value of the Nikkei for a contract settled in dollars is $15{,}150.75e^{0.3\times0.2\times0.12\times1}$ or 15,260.23

Quantos *continued*

When we move from the traditional risk neutral world in currency Y to the tradional risk neutral world in currency X, the growth rate of a variable V increases by

$$\rho\sigma_V\sigma_S$$

where σ_V is the volatility of V, σ_S is the volatility of the exchange rate (units of Y per unit of X) and ρ is the correlation between the two

Siegel's Paradox

An exchange rate S (units of currency Y per unit of currency X) follows the risk-neutral process

$$dS = [r_Y - r_X]Sdt + \sigma_S S dz$$

This implies from Ito's lemma that

$$d(1/S) = [r_X - r_Y + \sigma_S^2](1/S)dt - \sigma_S (1/S)dz$$

Given that the process for S has a drift rate of $r_Y - r_X$, we expect the process for $1/S$ to have a drift of $r_X - r_Y$.

Can you explain this?

When is a Convexity, Timing, or Quanto Adjustment Necessary

- A convexity or timing adjustment is necessary when interest rates are used in a nonstandard way for the purposes of defining a payoff
- No adjustment is necessary for a vanilla swap, a cap, or a swap option

Chapter 31
Interest Rate Derivatives: Model of the Short Rate

Term Structure Models

- Black's model is concerned with describing the probability distribution of a single variable at a single point in time
- A term structure model describes the evolution of the whole yield curve

The Zero Curve

- The process for the instantaneous short rate, r, in the traditional risk-neutral world defines the process for the whole zero curve in this world
- If $P(t, T)$ is the price at time t of a zero-coupon bond maturing at time T

$$P(t,T) = \hat{E}\left[e^{-\bar{r}(T-t)}\right]$$

where $\bar{r}$ is the average r between times t and T

Equilibrium Models (Risk Neutral World)

Rendleman & Bartter:

$$dr = \mu r\, dt + \sigma r\, dz$$

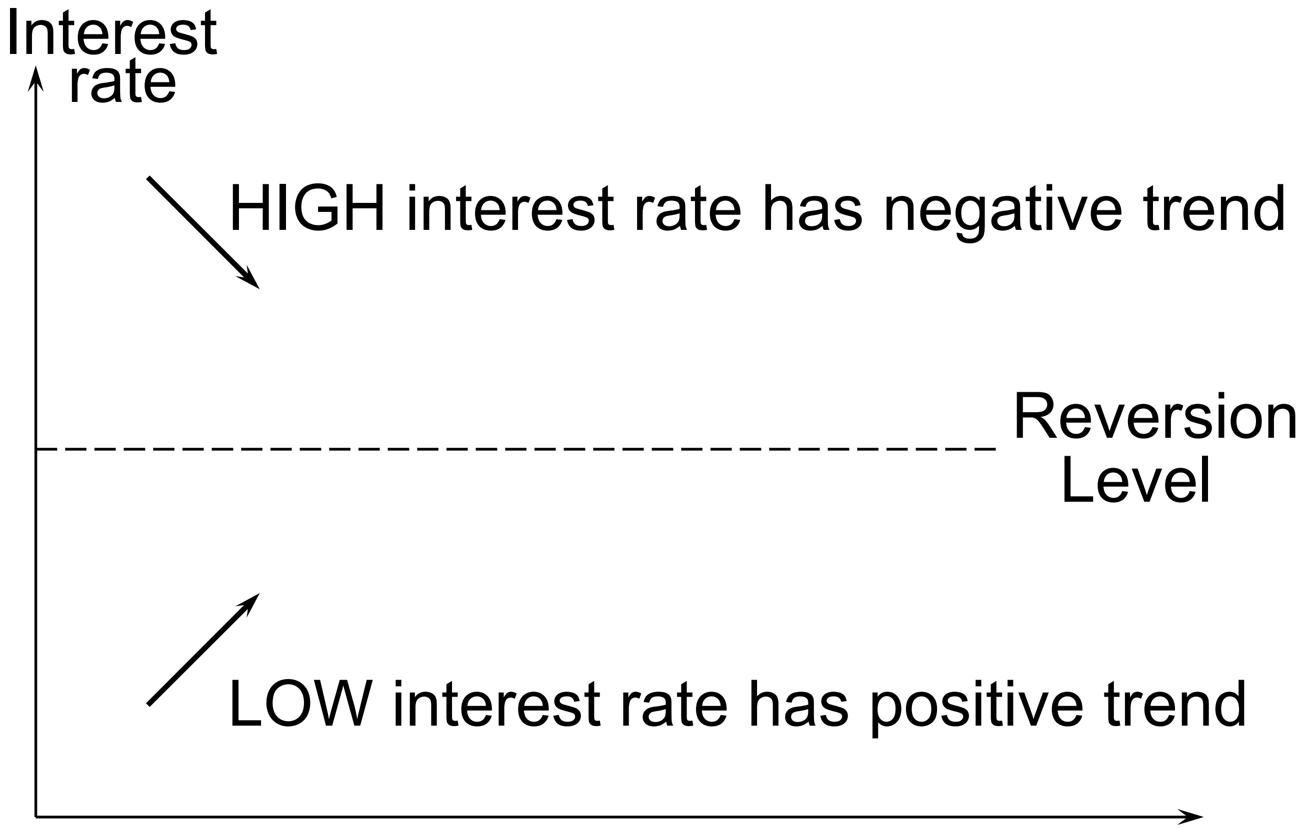
Vasicek:

$$dr = a(b - r)\, dt + \sigma dz$$

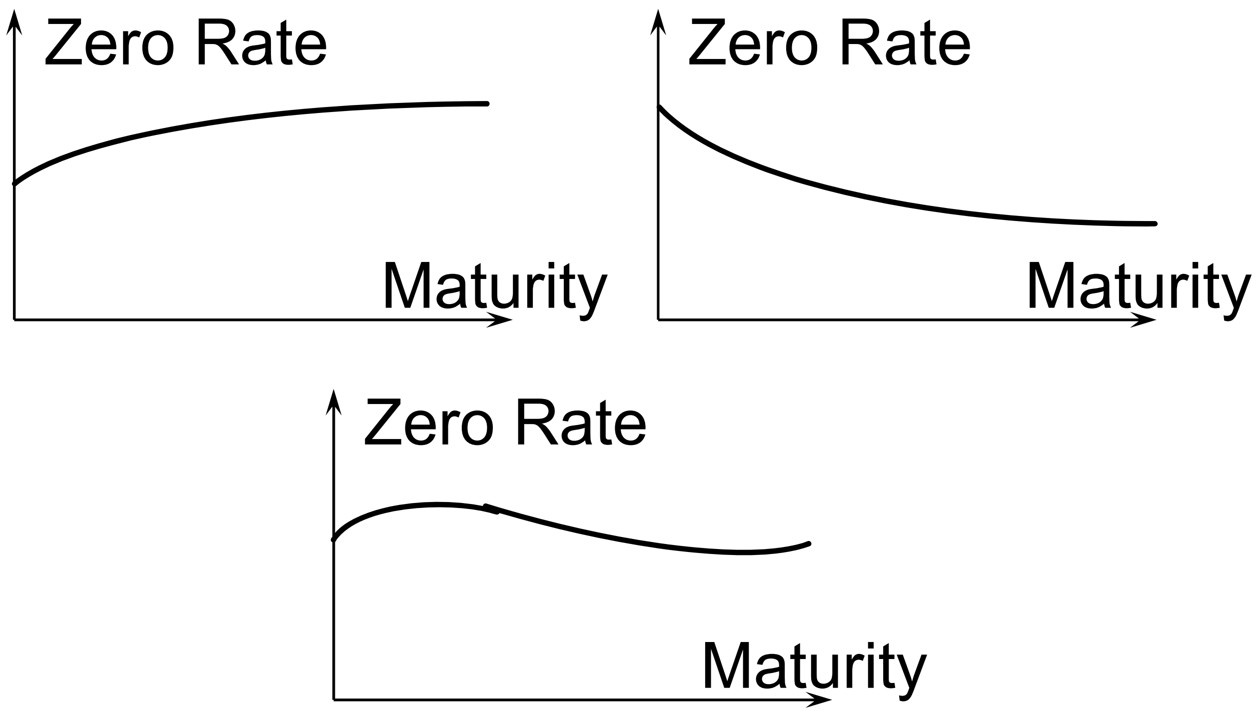
Cox, Ingersoll, & Ross (CIR):

$$dr = a(b - r)\, dt + \sigma \sqrt{r}\, dz$$

Mean Reversion *(Figure 31.1, page 709)*



Alternative Term Structures in Vasicek & CIR *(Figure 31.2, page 711)*



Properties of Vasicek and CIR

- $P(t,T) = A(t,T)e^{-B(t,T)r}$
- The A and B functions are different for the two models

$$\frac{\partial P(t,T)}{\partial r} = -B(t,T)P(t,T)$$

$$\frac{\partial^2 P(t,T)}{\partial r^2} = B(t,T)^2 P(t,T)$$

- These can be used to provide alternative duration and convexity measures

Bond Price Processes in a Risk Neutral World

- From Ito's lemma, risk neutral processes are

  Vasicek : $dP(t,T) = rP(t,T) - \sigma B(t,T)P(t,T)dz$

  CIR : $dP(t,T) = rP(t,T) - \sigma\sqrt{r(t)}B(t,T)P(t,T)dz$

- An estimate of the market price of interest rate risk, λ can be used to convert a risk-neutral process to a real-world process (or vice versa)
- What are the above processes in the real world?

Equilibrium vs No-Arbitrage Models

- In an equilibrium model today's term structure is an output
- In a no-arbitrage model today's term structure is an input

Developing No-Arbitrage Model for r

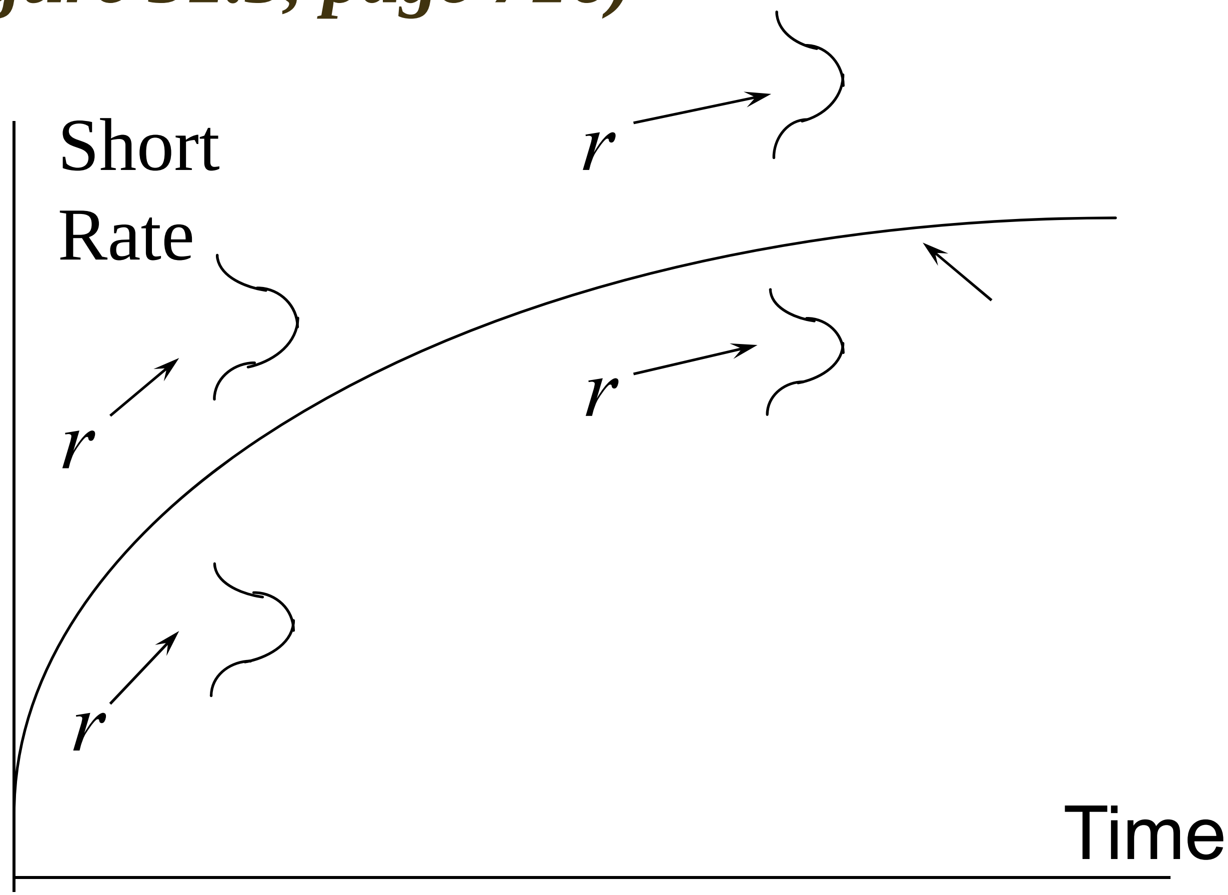
A model for r can be made to fit the initial term structure by including a function of time in the drift

Ho-Lee Model

$$dr = \theta(t)dt + \sigma dz$$

- Many analytic results for bond prices and option prices
- Interest rates normally distributed
- One volatility parameter, σ
- All forward rates have the same standard deviation

Diagrammatic Representation of Ho-Lee (Figure 31.3, page 716)

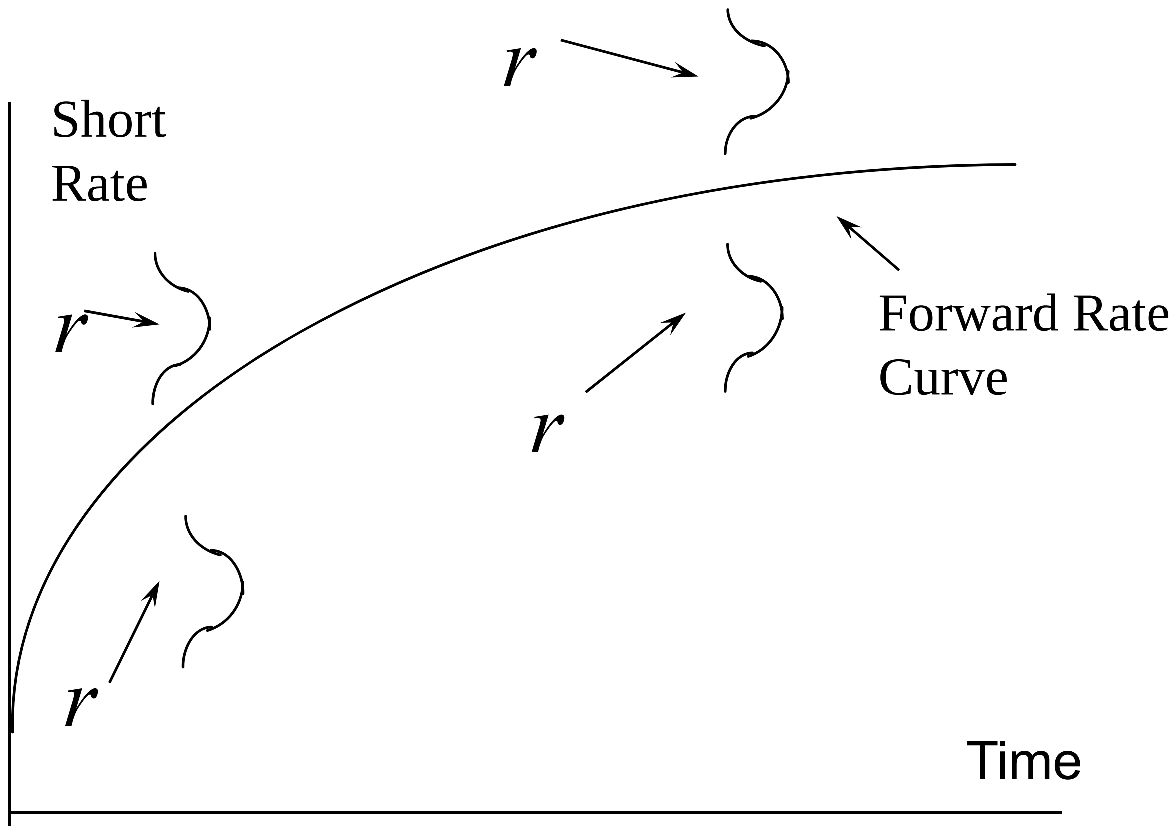


Hull-White Model

$$dr = [\theta(t) - ar]dt + \sigma dz$$

- Many analytic results for bond prices and option prices
- Two volatility parameters, a and σ
- Interest rates normally distributed
- Standard deviation of a forward rate is a declining function of its maturity

Diagrammatic Representation of Hull and White *(Figure 31.4, page 717)*



Black-Karasinski Model *(equation 31.18)*

$$d\ln(r) = \left[\theta(t) - a(t)\ln(r)\right]dt + \sigma(t)\,dz$$

- Future value of r is lognormal
- Very little analytic tractability

Options on Zero-Coupon Bonds
(equation 31.20, page 719)

- In Vasicek and Hull-White model, price of call maturing at T on a zero-coupon bond lasting to s is

$$LP(0,s)N(h) - KP(0,T)N(h - \sigma_P)$$

- Price of put is

$$KP(0,T)N(-h + \sigma_P) - LP(0,s)N(h)$$

where $$h = \frac{1}{\sigma_P}\ln\frac{LP(0,s)}{P(0,T)K} + \frac{\sigma_P}{2} \qquad \sigma_P = \frac{\sigma}{a}\left[1 - e^{-a(s-T)}\right]\sqrt{\frac{1 - e^{-2aT}}{2a}}$$

L is the principal and K is the strike price.

For Ho-Lee $\sigma_P = \sigma(s - T)\sqrt{T}$

Options on Coupon-Bearing Bonds

- In a one-factor model a European option on a coupon-bearing bond can be expressed as a portfolio of options on zero-coupon bonds.
- We first calculate the critical interest rate at the option maturity for which the coupon-bearing bond price equals the strike price at maturity
- The strike price for each zero-coupon bond is set equal to its value when the interest rate equals this critical value

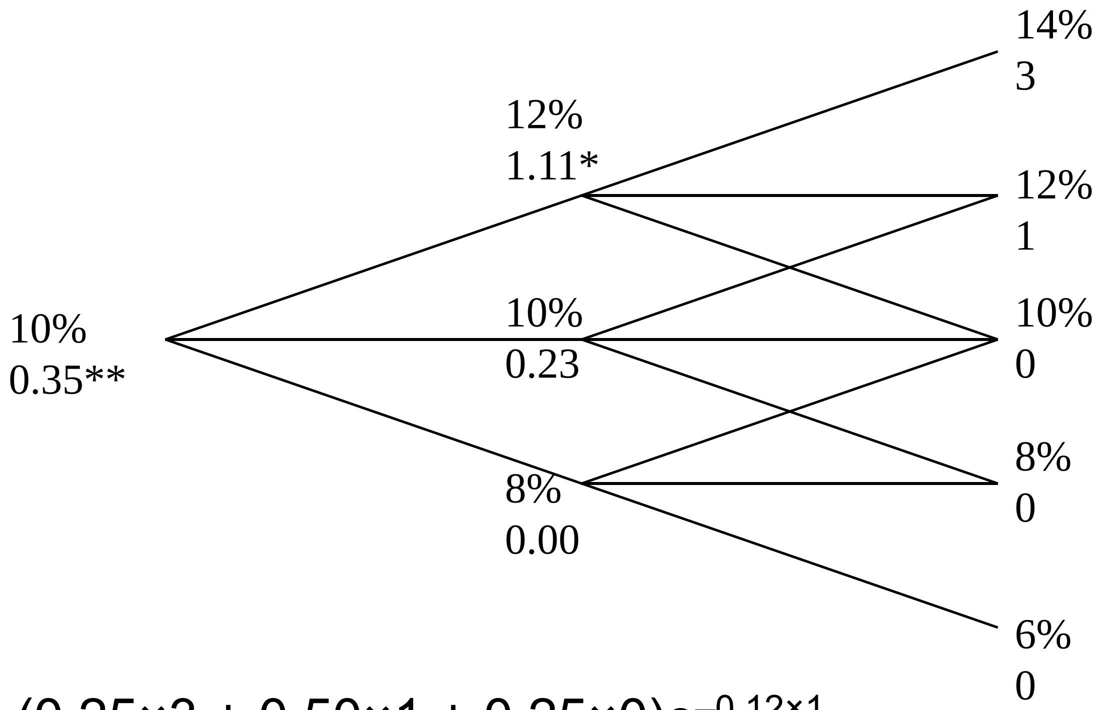
Interest Rate Trees vs Stock Price Trees

- The variable at each node in an interest rate tree is the Δt-period rate
- Interest rate trees work similarly to stock price trees except that the discount rate used varies from node to node

Two-Step Tree Example *(Figure 31.6, page 722)*

Payoff after 2 years is MAX[100(r – 0.11), 0]

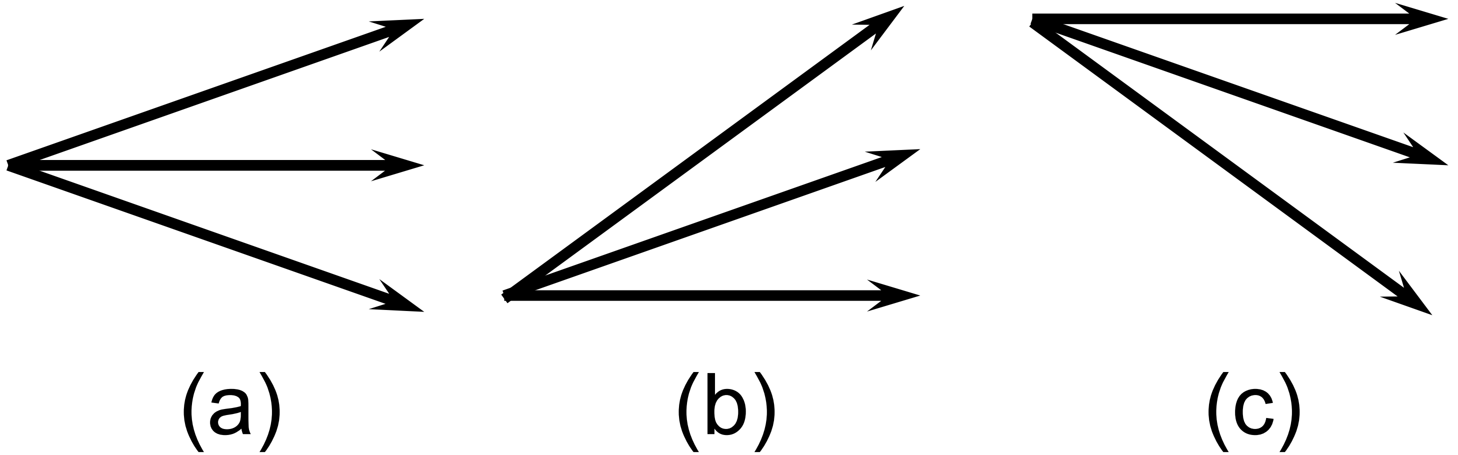
p_u=0.25; p_m=0.5; p_d=0.25; Time step=1yr



*: $(0.25\times3 + 0.50\times1 + 0.25\times0)e^{-0.12\times1}$

**: $(0.25\times1.11 + 0.50\times0.23 + 0.25\times0)e^{-0.10\times1}$

Alternative Branching Processes in a Trinomial Tree (Figure 31.7, page 723)



Procedure for Building Tree

$$dr = [\theta(t) - ar]dt + \sigma dz$$

1. Assume $\theta(t) = 0$ and $r(0) = 0$
2. Draw a trinomial tree for r to match the mean and standard deviation of the process for r
3. Determine $\theta(t)$ one step at a time so that the tree matches the initial term structure

Example *(page 725 to 730)*

$\sigma = 0.01$

$a = 0.1$

$\Delta t = 1$ year

Maturity	Zero Rate
0.5	3.430
1	3.824
1.5	4.183
2	4.512
2.5	4.812
3	5.086

Building the First Tree for the Δt rate R

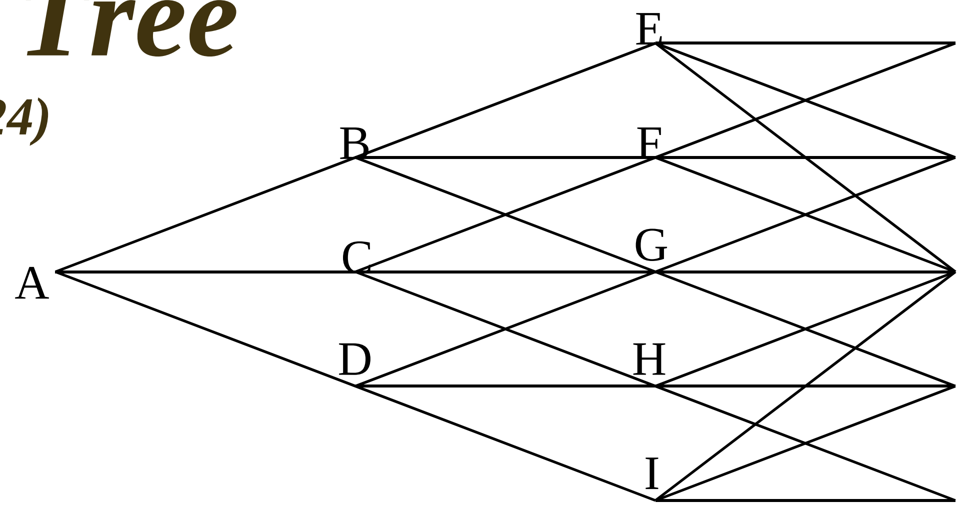
- Set vertical spacing:

$$\Delta R = \sigma\sqrt{3\Delta t}$$

- Change branching when $j_{\max}$ nodes from middle where $j_{\max}$ is smallest integer greater than 0.184/($a\Delta t$)
- Choose probabilities on branches so that mean change in R is -$aR\Delta t$ and S.D. of change is $\sigma\sqrt{\Delta t}$

The First Tree

(Figure 31.8, page 724)



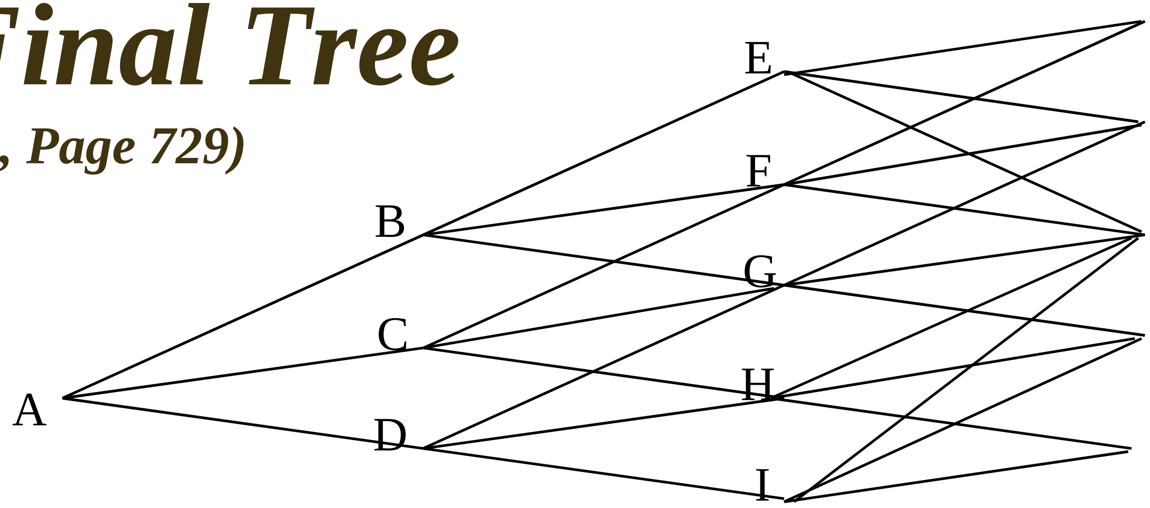
Node	A	B	C	D	E	F	G	H	I
R	0.000%	1.732%	0.000%	-1.732%	3.464%	1.732%	0.000%	-1.732%	-3.464%
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Shifting Nodes

- Work forward through tree
- Remember Q_{ij} the value of a derivative providing a $1 payoff at node j at time $i\Delta t$
- Shift nodes at time $i\Delta t$ by α_i so that the $(i+1)\Delta t$ bond is correctly priced

The Final Tree

(Figure 31.9, Page 729)



Node	A	B	C	D	E	F	G	H	I
R	3.824%	6.937%	5.205%	3.473%	9.716%	7.984%	6.252%	4.520%	2.788%
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Extensions

The tree building procedure can be extended to cover more general models of the form:

$$df(r) = [\theta(t) - a\, f(r)]dt + \sigma dz$$

We set $x=f(r)$ and proceed similarly to before

$x=\ln(r)$ gives the Black-Karasinski modelk

Calibration to Determine a and σ

- The volatility parameters a and σ (perhaps functions of time) are chosen so that the model fits the prices of actively traded instruments such as caps and European swap options as closely as possible
- We minimize a function of the form

$$\sum_{i=1}^{n}(U_i - V_i)^2 + P$$

where U_i is the market price of the ith calibrating instrument, V_i is the model price of the ith calibrating instrument and P is a function that penalizes big changes or curvature in a and σ

Chapter 32 HJM, LMM, and Multiple Zero Curves

HJM Model: Notation

$P(t,T)$: price at time t of a discount bond with principal of $1 maturing at T

Ω_t : vector of past and present values of interest rates and bond prices at time t that are relevant for determining bond price volatilities at that time

$v(t,T,\Omega_t)$: volatility of $P(t,T)$

Notation continued

$f(t,T_1,T_2)$: forward rate as seen at t for the period between T_1 and T_2

$F(t,T)$: instantaneous forward rate as seen at t for a contract maturing at T

$r(t)$: short-term risk-free interest rate at t

$dz(t)$: Wiener process driving term structure movements

Modeling Bond Prices *(Equation 32.1, page 741)*

$$dP(t,T) = r(t)P(t,T)dt + v(t,T,\Omega_t)P(t,T)dz(t)$$

We can choose any v function providing

$$v(t,t,\Omega_t) = 0 \qquad \text{for all } t$$

Because

$$f(t,T_1,T_2) = \frac{\ln[P(t,T_1)] - \ln[P(t,T_2)]}{T_2 - T_1}$$

we can use Ito's lemma to determine the process for $f(t,T_1,T_2)$. Letting T_2 approach T_1 we get a process for $F(t,T)$

The process for F(t,T)

Equation 32.4 and 32.5, page 742)

$$dF(t,T) = v(t,T,\Omega_t)v_T(t,T,\Omega_t)dt - v_T(t,T,\Omega_t)dz(t)$$

This result means that if we write

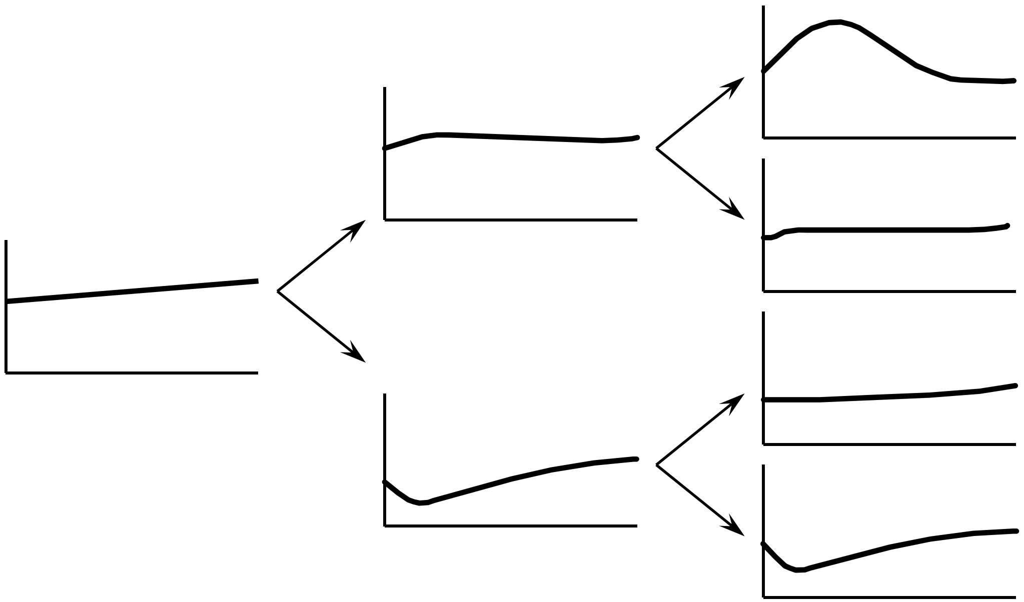
$$dF(t,T) = m(t,T,\Omega_t)dt + s(t,T,\Omega_t)dz$$

we must have

$$m(t,T,\Omega_t) = s(t,T,\Omega_t)\int_t^T s(t,\tau,\Omega_t)\,d\tau$$

Similar results hold when there is more than one factor

Tree Evolution of Term Structure is Non-Recombining



Tree for the short rate r is non-Markov

The LIBOR Market Model

The LIBOR market model is a model constructed in terms of the forward rates underlying caplet prices

Notation

t_k : kth reset date

$F_k(t)$: forward rate between times t_k and t_{k+1}

$m(t)$: index for next reset date at time t

δ_k : $t_{k+1} - t_k$

Volatility Structure

We assume a stationary volatility structure where the volatility of $F_k(t)$ depends only on the number of accrual periods between the next reset date and t_k [i.e., it is a function only of $k-m(t)$]

In Theory the Λ's can be Determined from Cap Prices

Define Λ_i as the volatility of $F_k(t)$ when $k - m(t) = i$

If σ_k is the volatility for the (t_k, t_{k+1}) caplet. If the model provides a perfect fit to cap prices we must have

$$\sigma_k^2 t_k = \sum_{i=1}^{k} \Lambda_{k-i}^2 \delta_{i-1}$$

This allows the Λ's to be determined inductively

Example 32.1 *(Page 745)*

- If Black volatilities for the first three caplets are 24%, 22%, and 20%, then

$$\Lambda_0=24.00\%$$
$$\Lambda_1=19.80\%$$
$$\Lambda_2=15.23\%$$

Example 32.2 *(Page 745-746)*

n	1	2	3	4	5
σ_n(%)	15.50	18.25	17.91	17.74	17.27
$\Lambda_{n\text{-}1}$(%)	15.50	20.64	17.21	17.22	15.25

n	6	7	8	9	10
σ_n(%)	16.79	16.30	16.01	15.76	15.54
$\Lambda_{n\text{-}1}$(%)	14.15	12.98	13.81	13.60	13.40

The Process for F_k in a One-Factor LIBOR Market Model

$$dF_k = \ldots + \Lambda_{k-m(t)} F_k \, dz$$

The drift depends on the world chosen

In a world that is forward risk-neutral with respect to $P(t, t_{i+1})$, the drift is zero

Rolling Forward Risk-Neutrality

(Equation 32.12, page 746)

It is often convenient to choose a world that is always FRN wrt a bond maturing at the next reset date. In this case, we can discount from t_{i+1} to t_i at the δ_i rate observed at time t_i. The process for F_k is

$$\frac{dF_k}{F_k} = \sum_{j=m(t)}^{i} \frac{\delta_i F_i \Lambda_{i-m(t)} \Lambda_{k-m(t)}}{1+\delta_i F_i} dt + \Lambda_{k-m(t)}\, dz$$

The LIBOR Market Model and HJM

In the limit as the time between resets tends to zero, the LIBOR market model with rolling forward risk neutrality becomes the HJM model in the traditional risk-neutral world

Monte Carlo Implementation of LMM Model

(Equation 32.14, page 746)

We assume no change to the drift between reset dates so that

$$F_k(t_{j+1}) = F_k(t_j)\exp\left[\left(\sum_{j=k}^{i}\frac{\delta_i F_i(t_j)\Lambda_{i-j}\Lambda_{k-j}}{1+\delta_j L_j} - \frac{\Lambda_{k-j}^2}{2}\right)\delta_k + \Lambda_{k-j}\varepsilon\sqrt{\delta_j}\right]$$

Multifactor Versions of LMM

- LMM can be extended so that there are several components to the volatility
- A factor analysis can be used to determine how the volatility of F_k is split into components

Ratchet Caps, Sticky Caps, and Flexi Caps

- A plain vanilla cap depends only on one forward rate. Its price is not dependent on the number of factors.
- Ratchet caps, sticky caps, and flexi caps depend on the joint distribution of two or more forward rates. Their prices tend to increase with the number of factors

Valuing European Options in the LIBOR Market Model

There is an analytic approximation that can be used to value European swap options in the LIBOR market model.

Calibrating the LIBOR Market Model

- In theory the LMM can be exactly calibrated to cap prices as described earlier
- In practice we proceed as for short rate models to minimize a function of the form

$$\sum_{i=1}^{n}(U_i - V_i)^2 + P$$

where U_i is the market price of the ith calibrating instrument, V_i is the model price of the ith calibrating instrument and P is a function that penalizes big changes or curvature in a and σ

Modeling LIBOR and OIS Term Structures Simultaneously *(pages 753-755)*

- Necessary when American swap options and other complex derivatives are valued using OIS discounting
- Need to ensure that LIBOR-OIS spread is positive
- First OIS zero curve is modeled (e.g., using a short-rate model or a LMM type of model)
- Then spreads are modeled as non-negative variable. An LMM type of model can be used for the evolution of forward spreads

Types of Agency Mortgage-Backed Securities (MBSs)

- Pass-Through
- Collateralized Mortgage Obligation (CMO)
- Interest Only (IO)
- Principal Only (PO)

Option-Adjusted Spread (OAS)

- To calculate the OAS for an interest rate derivative we value it assuming that the initial yield curve is the Treasury curve + a spread
- We use an iterative procedure to calculate the spread that makes the derivative’s model price = market price.
- This spread is the OAS.

Chapter 33
Swaps Revisited

Valuation of Swaps

- The standard approach is to assume that forward rates will be realized
- This works for plain vanilla interest rate and plain vanilla currency swaps, but does not necessarily work for non-standard swaps

Variations on Vanilla Interest Rate Swaps

- Principal different on two sides
- Payment frequency different on two sides
- Can be floating-for-floating instead of floating-for-fixed
- It is still correct to assume that forward rates are realized
- How should a swap exchanging the 3-month LIBOR for 3-month CP rate be valued?

Compounding Swaps *(Business Snapshot 33.2, page 762-763)*

- Interest is compounded instead of being paid
- Example: the fixed side is 6% compounded forward at 6.3% while the floating side is LIBOR plus 20 bps compounded forward at LIBOR plus 10 bps.
- This type of compounding swap can be valued (approximately) using the "assume forward rates are realized" rule.
- Approximation is exact if spread over LIBOR for compounding is zero

Currency Swaps

- In theory, a swap where LIBOR in one currency is exchanged for LIBOR in another currency is worth zero
- In practice it is sometimes the case that LIBOR in currency A is exchanged for LIBOR plus a spread in currency B
- This necessitates a small adjustment to the "assume forward LIBOR rate are realized" rule

More Complex Swaps

- LIBOR-in-arrears swaps
- CMS and CMT swaps
- Differential swaps

These cannot be accurately valued by assuming that forward rates will be realized

LIBOR-in Arrears Swap *(Equation 33.1, page 764-765)*

- Rate is observed at time t_i and paid at time t_i rather than time t_{i+1}
- It is necessary to make a convexity adjustment to each forward rate underlying the swap
- Suppose that F_i is the forward rate between time t_i and t_{i+1} and σ_i is its volatility
- We should increase F_i by

$$\frac{F_i^2 \sigma_i^2 (t_{i+1} - t_i) t_i}{1 + F_i \tau_i}$$

when valuing a LIBOR-in-arrears swap

CMS swaps

- Swap rate observed at time t_i is paid at time t_{i+1}
- We must
  - make a convexity adjustment because payments are swap rates (= yield on a par yield bond)
  - Make a timing adjustment because payments are made at time t_{i+1} not t_i

Differential Swaps

- Rate is observed in currency Y and applied to a principal in currency X
- We must make a quanto adjustment to the rate

Equity Swaps *(page 767-768)*

- Total return on an equity index is exchanged periodically for a fixed or floating return
- When the return on an equity index is exchanged for LIBOR the value of the swap is always zero immediately after a payment. This can be used to value the swap at other times.

Swaps with Embedded Options
(page 769-771)

- Accrual swaps
- Cancelable swaps
- Cancelable compounding swaps

Other Swaps *(page 771-772)*

- Indexed principal swap
- Commodity swap
- Volatility swap
- Bizarre deals (for example, the P&G 5/30 swap in Business Snapshot 33.4 on page 772)

Chapter 34
Energy and Commodity Derivatives

Agricultural Commodities

- Corn, wheat, soybeans, cocoa, coffee, sugar, cotton, frozen orange juice, cattle, hogs, pork bellies, etc
- Supply-demand measured by stocks-to-use ratio
- Seasonality and mean reversion in prices (farmers have a choice about what they produce)
- Weather important

Metals

- Gold, silver, platinum, palladium, copper, tin, lead, zinc, nickel, aluminium, etc
- No seasonality; weather unimportant
- Investment vs consumption metals
- Some mean reversion (It can become uneconomic to extract a metal)
- Recycling

Energy Commodities

- Main energy sources
  - Oil
  - Gas
  - Electricity
- All have mean reverting prices
- Gas and electricity exhibit jumps

Crude Oil

- Largest commodity market in the world
- Many grades. For example
  - Brent crude oil (sourced from North Sea)
  - West Texas Intermediate (WTI) crude
- Refined products, for example:
  - Gasoline
  - Heating oil
  - Kerosene

Oil Derivatives (page 777)

- Virtually all derivatives available on stocks and stock indices are also available in the OTC market with oil as the underlying asset
- Futures and futures options traded on the New York Mercantile Exchange (NYMEX) and the International Petroleum Exchange (IPE) are also popular

Natural Gas and Electricity

- Deregulated
- Elimination of government monopolies
- Producer and supplier not necessarily the same

Natural Gas Derivatives *(page 777-778)*

- A typical OTC contract is for the delivery of a specified amount of natural gas at a roughly uniform rate to specified location during a month.
- NYMEX and IPE trade contracts that require delivery of 10,000 million British thermal units of natural gas to a specified location

Electricity Derivatives (page 778-779)

- Electricity is an unusual commodity in that it cannot be stored
- The U.S is divided into about 140 control areas and a market for electricity is created by trading between control areas.

Electricity Derivatives *continued*

- A typical contract allows one side to receive a specified number of megawatt hours for a specified price at a specified location during a particular month
- Types of contracts:

  5x8, 5x16, 7x24, daily or monthly exercise,

  swing options

Commodity Prices

- Futures prices can be used to define the process followed by a commodity price in a risk-neutral world.
- We can build in mean reversion and use a process for constructing trinomial trees that is analogous to that used for interest rates in Chapter 30

The Process for the Commodity Price

A simple mean reverting process is

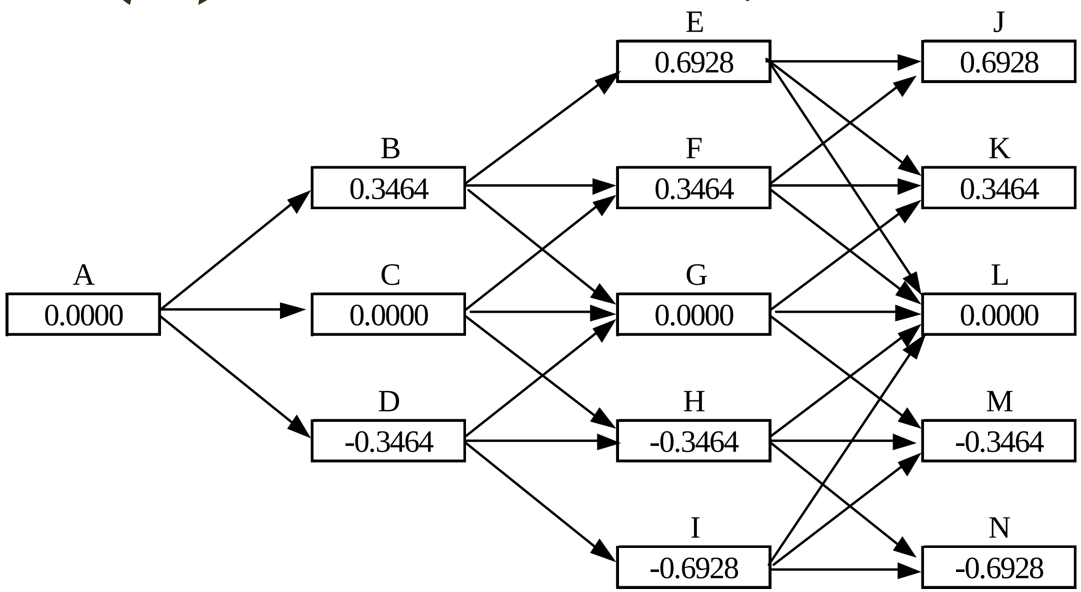
$d \ln(S) = [\theta(t) - a\ln(S)]\, dt + \sigma\, dz$

Can also be written

$$\frac{dS}{S} = \left[\theta^*(t) - a \ln S\right] dt + \sigma dz$$

Assume $a = 0.1$, $\sigma = 0.2$, and $\Delta t = 1$ year

Tree for ln(S) Assuming $\theta(t)=0$; Fig 34.1

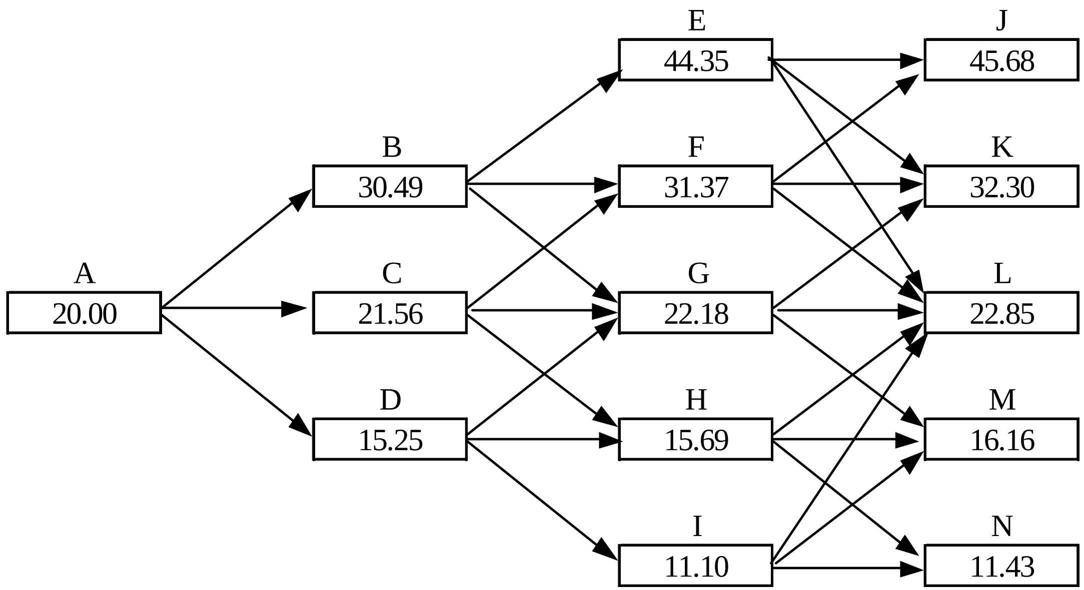


Node	A	B	C	D	E	F	G	H	I
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Determining $\theta(t)$

- The nodes on the tree are moved so that the expected commodity price equals the futures price
- Assume that the one-year, two-year and three-years futures price for the commodity are \$22, \$23, and \$24, respectively

Final Tree; Fig 34.2



Node	A	B	C	D	E	F	G	H	I
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Interpolation and Seasonality

- A simple approach
  - Use a 12 month moving average of spot prices to determine a percentage seasonality factor for each month
  - De-seasonalize the futures prices that are known
  - Interpolate to determine other de-seasonalized futures prices
  - Re-seasonalize all futures prices and construct tree

Jumps

- Some commodity prices such as gas and electricity exhibit jumps
- A process that can be assumed is then

$$d \ln S = [\theta(t) - a \ln S]dt + \sigma dz + dp$$

  where dp is a Poisson process generating jumps
- If Poisson process is known we can use tree to model process without jumps and thereby determine θ(t)
- Can be implemented with Monte Carlo simulation

Other Models

- Convenience yield follows a mean reverting process (Gibson and Schwartz)
- Volatility stochastic (Eydeland and Geman)
- Reversion level stochastic (Geman)

Weather Derivatives: Definitions

(page 785)

- Heating degree days (HDD): For each day this is max(0, 65 – A) where A is the average of the highest and lowest temperature in ºF.
- Cooling Degree Days (CDD): For each day this is max(0, A – 65)
- Contracts specify the weather station to be used

Weather Derivatives: Products

- A typical product is a forward contract or an option on the cumulative CDD or HDD during a month
- Weather derivatives are often used by energy companies to hedge the volume of energy required for heating or cooling during a particular month
- How would you value an option on August CDD at a particular weather station?

How an Energy Producer Hedges Risks

- Estimate a relationship of the form

$$Y = a + bP + cT + \varepsilon$$

where Y is the monthly profit, P is the average energy prices, T is temperature, and ε is an error term

- Take a position of $-b$ in energy forwards and $-c$ in weather forwards.

Insurance Derivatives *(page 786)*

- CAT bonds are an alternative to traditional reinsurance
- This is a bond issued by a subsidiary of an insurance company that pays a higher-than-normal interest rate.
- If claims of a certain type are in a certain range, the interest and possibly the principal on the bond are used to meet claims

Pricing Issues

- To a good approximation many underlying variables in insurance, weather, and energy derivatives contracts can be assumed to have zero systematic risk.
- This means that we can calculate expected payoff in the real world and discount at the risk-free rate

Chapter 35
Real Options

An Alternative to the NPV Rule for Capital Investments

- Define stochastic processes for the key underlying variables and use risk-neutral valuation
- This approach (known as the real options approach) is likely to do a better job at valuing growth options, abandonment options, etc than NPV

The Problem with using NPV to Value Options

- Consider the example from Chapter 13: risk-free rate =12%; strike price = \$21

Stock price = \$20 → Stock Price = \$22

Stock price = \$20 → Stock Price=\$18

- Suppose that the expected return required by investors in the real world on the stock is 16%. What discount rate should we use to value an option with strike price \$21?

Correct Discount Rates are Counter-Intuitive

- Correct discount rate for a call option is 42.6%
- Correct discount rate for a put option is –52.5%

General Approach to Valuation

- We can value any asset dependent on a variable θ by
  - Reducing the expected growth rate of θ by λs where λ is the market price of θ-risk and s is the volatility of θ
  - Assuming that all investors are risk-neutral

Extension to Many Underlying Variables

- When there are several underlying variables θ_i we reduce the growth rate of each one by its market price of risk times its volatility and then behave as though the world is risk-neutral
- Note that the variables do not have to be prices of traded securities

Estimating the Market Price of Risk Using CAPM *(equation 35.2, page 795)*

The market price of risk of a variable is given by

$$\lambda = \frac{\rho}{\sigma_m}(\mu_m - r)$$

where ρ is the instantaneous correlation between percentage changes in the variable and returns on the market; σ_m is the volatility of the market's return; μ_m is the expected return on the market; and r is the short-term risk-free rate

Types of Options

- Abandonment
- Expansion
- Contraction
- Option to defer
- Option to extend life

Example of Application of Real Options Approach to Valuing Amazon.com at end of 1999

(Business Snapshot 35.1; Schwartz and Moon)

- Estimate stochastic processes for the company's sales revenue and its average growth rate.
- Estimated the market price of risk and other key parameters (cost of goods sold as a percent of sales, variable expenses as a percent of sales, fixed expenses, etc.)
- Use Monte Carlo simulation to generate different scenarios in a risk-neutral world.
- The stock price is the average of the present values of the net cash flows discounted at the risk-free rate.

Example *(page 798)*

- A company has to decide whether to invest \$15 million to obtain 6 million units of a commodity at the rate of 2 million units per year for three years.
- The fixed operating costs are \$6 million per year and the variable costs are \$17 per unit.
- The spot price of the commodity is \$20 per unit and 1, 2, and 3-year futures prices are \$22, \$23, and \$24, respectively.
- The risk-free rate is 10% per annum for all maturities.

The Process for the Commodity Price

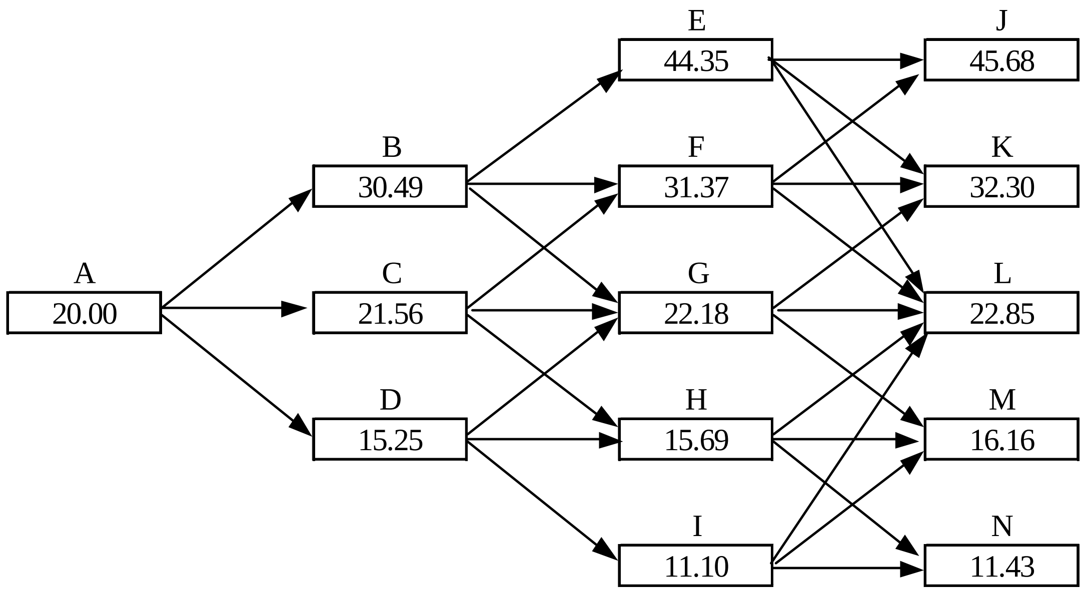
- We assume that this is

$$d\ln(S) = [\theta(t) - a\ln(S)]\,dt + \sigma\,dz$$

where $a = 0.1$ and $\sigma = 0.2$

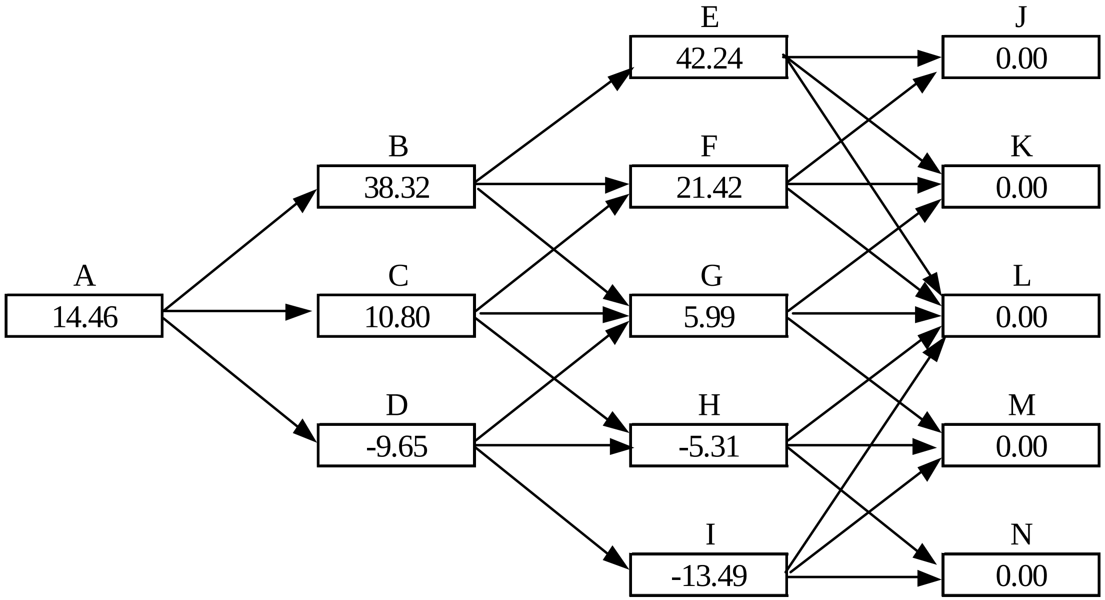
- We build a tree as in Chapter 33

The Tree of Commodity Prices (Figure 35.1)



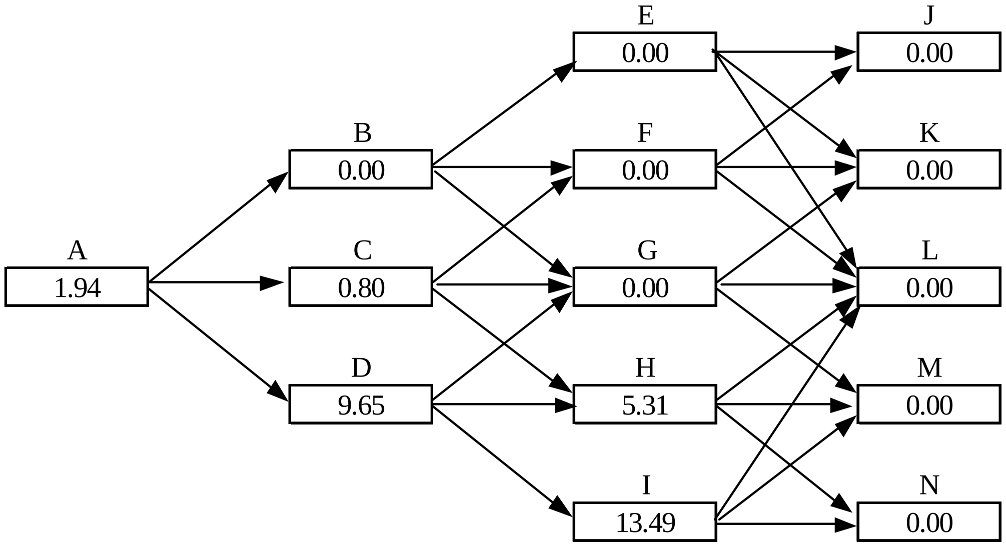
Node	A	B	C	D	E	F	G	H	I
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Valuation of Base Project; Fig 35.2



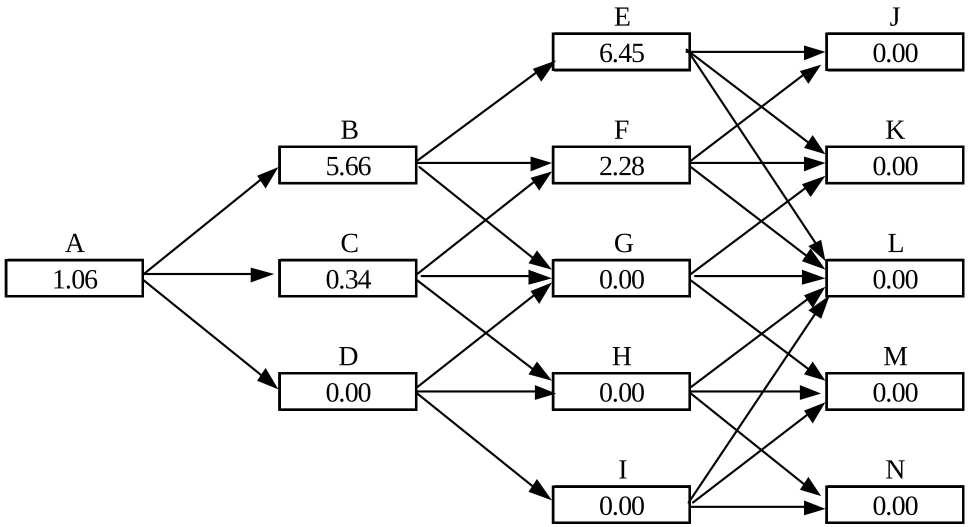
Node	A	B	C	D	E	F	G	H	I
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Valuation of Option to Abandon; Fig 35.3 (No Salvage Value; No Further Payments)



Node	A	B	C	D	E	F	G	H	I
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Value of Expansion Option; Fig 35.4 (Company Can Increase Scale of Project by 20% for $2 million)



Node	A	B	C	D	E	F	G	H	I
p_u	0.1667	0.1217	0.1667	0.2217	0.8867	0.1217	0.1667	0.2217	0.0867
p_m	0.6666	0.6566	0.6666	0.6566	0.0266	0.6566	0.6666	0.6566	0.0266
p_d	0.1667	0.2217	0.1667	0.1217	0.0867	0.2217	0.1667	0.1217	0.8867

Chapter 36
Derivatives Mishaps and What We Can Learn From Them

Big Losses by Financial Institutions

- Allied Irish Bank ($700 million)
- Amaranth ($6 billion)
- Barings ($1 billion)
- Enron Counterparties (Several over $1 billion)
- Kidder Peabody ($350 million)
- LTCM ($4 billion)
- Midland Bank ($500 million)
- Société Générale ($7 billion)
- Subprime Mortgages (many billions)
- UBS ($2.3 billion)

Big Losses by Non-Financial Corporations

- Allied Lyons ($150 million)
- Gibsons Greetings ($20 million)
- Hammersmith and Fulham ($600 million)
- Metallgesellschaft ($1.8 billion)
- Orange County ($2 billion)
- Procter and Gamble ($90 million)
- Shell ($1 billion)
- Sumitomo ($2 billion)

Lessons for All Users of Derivatives

- Risk must be quantified and risk limits defined
- Exceeding risk limits not acceptable even when profits result
- Do not assume that a trader with a good track record will always be right
- Be diversified
- Scenario analysis and stress testing is important

Lessons for Financial Institutions

- Do not give too much independence to star traders
- Separate the front middle and back office
- Models can be wrong
- Be conservative in recognizing inception profits
- Do not sell clients inappropriate products
- Beware of easy profits
- Liquidity risk is important
- There are dangers when many are following the same strategy

Lessons for Financial Institutions *continued*

- Beware of potential liquidity problems when long-term funding requirements are financed with short-term liabilities
- Market transparency is important
- Manage incentives
- Never ignore risk management, even when times are good

Lessons for Non-Financial Corporations

- It is important to fully understand the products you trade
- Beware of hedgers becoming speculators
- It can be dangerous to make the Treasurer's department a profit center