衍生工具模型 (金融工程风格)

2019-10-14 (修改稿)

要求

阅读本课件. 完成所有作业题, 作业题 1-6 解答在下次课前提交.

1 约定

除非特别声明, 我们约定

1. 利率 r 为正常数.
2. 假设 S 连续股息派发, 其股息派发率为非负常数 q. 称其服从几何布朗运动, 是指以下等式满足

$$\frac{dS(t)}{S(t)} = (\mu - q)dt + \sigma dB(t), \tag{1.0.1}$$

其中, μ 和 σ 为常数, $\sigma > 0$, B_t 为 (标准) 布朗运动. 注: 公式(1.0.1)隐含 $S(t) \neq 0$, $\forall t$.

3. 假设 S 连续股息派发, 其股息派发率为非负常数 q. 在本课件中我们总是假定股价 S 服从几何布朗运动.

例子 1.1. 给定两只无股息派发的股票 S_1 和 S_2. 假设其服从几何布朗运动

$$\frac{dS_i(t)}{S_i(t)} = \mu_i dt + \sigma_i dB_i(t), \quad i = 1, 2.$$

进一步假设: $\mu_1 > \mu_2$, $S_1(t_0) = S_2(t_0)$, $\exists t_0$. 由无套利假定易知: $c(S_i, t_0, 0, T) = S_1(t_0) = S_2(t_0)$, $i = 1, 2$. 然而, 从直觉上, 似乎有 $c(S_1, t_0, 0, T) > c(S_2, t_0, 0, T)$. 结论: $c(S_i, t_0, 0, T)$ 与 μ_i 的大小无关. 以 $i = 1$ 为例. 易知

$$S(T) = S(t_0) \exp\left(\left(\mu_1 - \frac{\sigma_1^2}{2}\right)(T - t_0) + \sigma_1(B_1(T) - B_1(t_0))\right).$$

于是

$$\begin{aligned}\mathbb{E}[S(T)|\mathcal{F}_{t_0}] =& \mathbb{E}\left[S(t_0) \exp\left(\left(\mu_1 - \frac{\sigma_1^2}{2}\right)(T - t_0) + \sigma_1(B_1(T) - B_1(t_0))\right) \middle| \mathcal{F}_{t_0}\right] \\ =& e^{\mu_1(T - t_0)} S(t_0).\end{aligned}$$

所以, $S(t_0) = e^{-\mu_1(T-t_0)}\mathbb{E}[S(T)|\mathcal{F}_{t_0}]$. 课堂上解释其意思.

□

作业题 1. 证明:

$$\mathbb{E}\left[S(t_0)\exp\left(\left(\mu_1-\frac{\sigma_1^2}{2}\right)(T-t_0)+\sigma_1(B_1(T)-B_1(t_0))\right)\middle|\mathcal{F}_{t_0}\right]=e^{\mu_1(T-t_0)}S(t_0).$$

□

2 Black-Scholes 方程

2.1 自融资

课上讲.

2.2 无股息派发情形下, Black-Scholes 方程

假设 S 无股息派发, 对于 $v\in\{c(S,t,E,T),p(S,t,E,T)\}$, 有

$$\frac{\partial v}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 v}{\partial S^2}+rS\frac{\partial v}{\partial S}-rv=0. \tag{2.2.1}$$

上式称为无股息派发情形下的 Black-Scholes 方程. 课上讲推导. 推导方法用动态对冲.

注释 2.1. 公式(2.2.1)中的 v 不只局限于 $c(S,t,E,T)$ 和 $p(S,t,E,T)$. 只要满足公式(2.2.1)的推导过程即可. 例如: (1) 欧式期权 $c^d(S,t,E,T)$, 其 terminal payoff 为

$$c^d(S,T,E,T)=\begin{cases}1, & S(T)\geq E,\\ 0, & S(T)<E.\end{cases}$$

(2) 欧式期权 $c^{a/n}(S,t,E,T)$, 其 terminal payoff 为

$$c^{a/n}(S,T,E,T)=\begin{cases}S(T), & S(T)\geq E,\\ 0, & S(T)<E.\end{cases}$$

易知: $v(S,t,E,T)=c^d(S,t,E,T)$ 和 $v(S,t,E,T)=c^{a/n}(S,t,E,T)$ 都满足 Black-Scholes 方程(2.2.1), 并且 $c(S,t,E,T)=c^{a/n}(S,t,E,T)-Ec^d(S,t,E,T)$ (无套利).

□

作业题 2. 假设 S 连续股息派发, 其股息派发率为非负常数 q. 对于 $v\in\{c(S,t,E,T),p(S,t,E,T)\}$, 证明:

$$\frac{\partial v}{\partial t}+\frac{\sigma^2}{2}S^2\frac{\partial^2 v}{\partial S^2}+(r-q)S\frac{\partial v}{\partial S}-rv=0. \tag{2.2.2}$$

□

2.3 Black-Scholes 方程与鞅之间的关系

2.3.1 风险中性

几何布朗运动:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dB_t, \quad (S \text{ 无股息派发}).$$

将上式改写成

$$\begin{aligned}\frac{dS_t}{S_t} =& rdt + \sigma \left(\underbrace{\frac{\mu - r}{\sigma} dt + dB_t}_{\text{记为 } dB_t^Q} \right) \\ =& rdt + \sigma dB_t^Q.\end{aligned}$$

现在要做测度变换: 将 B_t^Q 变换成标准布朗运动. 已知: B_t 的概率密度为

$$f(x,t) := \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right).$$

固定时间 $t > 0$, 假设共有 N 个样本, N 充分大. 则在区间 $[x, x + dx]$ 上的样本数为 $Nf(x,t)dx$. 现在改变计数方式 (测度变换): 将样本数 $Nf(x,t)dx$ 计为

$$N \exp\left(-\frac{\mu - r}{\sigma} x - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right) f(x,t)dx = N \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{1}{2t}\left(x + \frac{\mu - r}{\sigma} t\right)^2\right) dx.$$

容易验证: 这样改变计数方式保持归一化, 即: 对上式将 x 从 $-\infty$ 到 $+\infty$ 积分为 N. 将这个计数方式记为 Q. 现在计算在 Q 下 $B^Q = (\mu - r)t/\sigma + B_i$ 的期望:

$$\begin{aligned}\mathbb{E}^Q[B_t^Q] =& \int_{-\infty}^{+\infty} \left(\frac{\mu - r}{\sigma} t + x\right) \underbrace{\exp\left(-\frac{\mu - r}{\sigma} x - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right)}_{\text{测度变换 (改变计数方式)}} f(x,t)dx \\ =& \frac{1}{\sqrt{2\pi t}} \int_{-\infty}^{+\infty} \left(\frac{\mu - r}{\sigma} t + x\right) \exp\left(-\frac{1}{2t}\left(x + \frac{\mu - r}{\sigma} t\right)^2\right) dx \\ =& 0.\end{aligned}$$

容易验证: 在测度 Q 下, B^Q 是标准布朗运动. 如果 S 股息派发率为常数 $q \geq 0$, 那么同理有

$$\frac{dS_t}{S_t} = (r - q)dt + \sigma dB_t^Q. \tag{2.3.1}$$

上式称为股价 S 的几何布朗运动风险中性表述, 对应于 Q 测度称为风险中性测度.

由以上分析知: 上式 (2.3.1) 并不表示 μ 和 r 相等. 在概率论中, 上述因子

$$\exp\left(-\frac{\mu - r}{\sigma} x - \frac{1}{2}\left(\frac{\mu - r}{\sigma}\right)^2 t\right)$$

是 Radon-Nikodym 导数. 如果 S 股息派发率为常数 $q \geq 0$, 给定 Terminal payoff 函数 $f(S_T)$ 的欧式期权 v, 计算

$$\begin{aligned}
d(e^{-rt}v_t) =& -re^{-rt}v_t + e^{-rt}dv_t \\
=& -re^{-rt}v_t + e^{-rt}\left(\frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial S}dS + \frac{1}{2}\frac{\partial^2 v}{\partial S^2}(dS)^2\right) \\
=& -re^{-rt}v_t + e^{-r(T-t)}\left(\frac{\partial v}{\partial t}dt + \frac{\partial v}{\partial S}S((r-q)dt + \frac{\sigma^2}{2}S^2\frac{\partial^2 v}{\partial S^2}dt\right) \\
& + e^{-rt}\frac{\partial v}{\partial S}S\sigma dB^Q \\
=& e^{-rt}\left(\frac{\partial v}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 v}{\partial S^2} + (r-q)S\frac{\partial v}{\partial S} - rv_t\right)dt + e^{-rt}\frac{\partial v}{\partial S}S\sigma dB^Q.
\end{aligned}$$

于是我们有

命题 2.2. *假设 S 连续股息派发, 其股息派发率为非负常数 q. Black-Scholes 方程*

$$\frac{\partial v_t}{\partial t} + \frac{\sigma^2}{2}S^2\frac{\partial^2 v_t}{\partial S^2} + (r-q)S\frac{\partial v_t}{\partial S} - rv_t = 0$$

成立的充要条件为 $e^{-rt}v_t$ 在风险中性测度 Q 下为鞅.

□

特别地 $e^{-rt}v_t = \mathbb{E}^Q[e^{-rT}f(S_T)|\mathcal{F}_t]$, 即

$$v_t = e^{-r(T-t)}\mathbb{E}^Q[f(S_T)|\mathcal{F}_t]. \tag{2.3.2}$$

易知

$$S_T = S_t \exp\left(\left(r - q - \frac{\sigma^2}{2}\right)(T-t) + \sigma(B^Q(T) - B^Q(t))\right).$$

它表示 T 时在风险中性概率 Q 下的股价. 于是

$$v_t = \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dy e^{-\frac{y^2}{2}} f\left(S_t \exp\left(\left(r - q - \frac{\sigma^2}{2}\right)(T-t) + \sigma y\sqrt{T-t}\right)\right). \tag{2.3.3}$$

上式中的 $e^{-y^2/2}/\sqrt{2\pi}$ 表示在风险中性测度 Q 下的概率密度.

例子 2.3. 假设当前时刻 $t = 0$, S 股息派发率为常数 $q \geq 0$, 求 $\mathbb{E}^Q[S_T]$.
直接计算:

$$\begin{aligned}
\mathbb{E}^Q[S(T)] =& \underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dy e^{-\frac{y^2}{2}} S_T}_{\text{没有贴现}} \\
=& \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} dy e^{-\frac{y^2}{2}} S_0 \exp\left(\left(r - q - \frac{\sigma^2}{2}\right)T + y\sigma\sqrt{T}\right) \\
=& e^{(r-q)T}S_0.
\end{aligned} \tag{2.3.4}$$

□

例子 2.4. 假设当前时刻 $t=0$, S 股息派发率为常数 $q \geq 0$, 求 $\mathrm{For}(S,0,T)$.
根据以上例子的结论, 远期合约的定义和公式(2.3.2)

$$0 = e^{-rT}\mathbb{E}^Q[S(T) - \mathrm{For}(S,0,T)|\mathcal{F}_0],$$

有

$$\mathrm{For}(S,0,T) = \mathbb{E}^Q[S(T)|\mathcal{F}_0] = e^{(r-q)T}S_0.$$

这与无套利假定下得到的结论一致.

□

例子 2.5. 假设 S 连续股息派发, 其股息派发率为非负常数 q.

$$c(S,t,E,T) = e^{-r(T-t)}\mathbb{E}^Q[\max(S_T - E, 0)\mathcal{F}_t],$$

$$p(S,t,E,T) = e^{-r(T-t)}\mathbb{E}^Q[\max(E - S_T, 0)\mathcal{F}_t],$$

□

作业题 3. 假设 S 连续股息派发, 其股息派发率为非负常数 q. 证明:

$$c(S,t,E,T) = e^{-q(T-t)}SN(d_1) - e^{-r(T-t)}EN(d_2), \tag{2.3.5}$$

再利用 Put-call parity 证明

$$p(S,t,E,T,r,q) = -Se^{-q(T-t)}N(-d_1) + e^{-r(T-t)}EN(-d_2), \tag{2.3.6}$$

其中

$$d_1 := \frac{\ln\frac{S}{E} + \left(r - q + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 := \frac{\ln\frac{S}{E} + \left(r - q - \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}},$$

$$N(x) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-\frac{y^2}{2}}dy.$$

□

作业题 4. 假设 S 连续股息派发, 其股息派发率为非负常数 q. 回答以下每个问题

1. 给定欧式期权 $c^d(S,t,E,T)$, 其 terminal payoff 为

$$c^d(S,T,E,T) = \begin{cases} 1, & S(T) \geq E, \\ 0, & S(T) < E. \end{cases}$$

求 $c^d(S,T,E,T)$ 的解析表达式.

2. 给定欧式期权 $c^{a/n}(S,t,E,T)$, 其 terminal payoff 为

$$c^{a/n}(S,T,E,T) = \begin{cases} S(T), & S(T) \geq E, \\ 0, & S(T) < E. \end{cases}$$

求 $c^{a/n}(S,T,E,T)$ 的解析表达式.

□

作业题 5. 假设 S 无股息派发. 求

$$\lim_{T\to+\infty} c(S,t,E,T)$$

和

$$\lim_{T\to+\infty} p(S,t,E,T).$$

并给出以上两个极限的金融意义.

□

2.3.2 一个例子

假设 S 无股息派发. 股价 S 服从几何布朗运动

$$\frac{dS}{S} = \mu dt + \sigma dB. \tag{2.3.7}$$

假设当前时刻 $t=0$. 给定 $T>0$, 将区间 $[0,T]$ 做 N 等分 (N 充分大):

$$0 = t_0 < t_1 < \cdots t_i < \cdots < t_N = T, \quad t_i = \frac{iT}{N}, \quad (i=0,\ldots,N).$$

注释 2.6. 在上式中, N 是人为取的充分大的整数, 使得 $T = N\Delta t$, $\Delta t = t_{i+1} - t_i$. 在实际应用中, t_i 取收盘时刻. 将 T 的时间单位记为交易日, 取 $\Delta t = 1$. 所以此时在数值上, T 和 N 相等. 以下将时间单位记为交易日. 此时, $T = N$ 且 $t_i = i$, $\forall i$.

□

注释 2.7. 在以上注释中我们取 $\Delta t = 1$(天) 并且认为 Δt 非常小, 因而可以做近似计算. 然而 1 天等于 86400 秒. 那如何理解取 $\Delta t = 86400$(秒) 也是非常小呢? 事实上, 在我们的计算或推导中, Δt 总是伴随 μ, r, q 或 σ 同时出现的. 例如: $e^{r\Delta t}$. 在泰勒展开时, 我们只关心 $r\Delta t$ 是否足够小, 而与 Δt 的计算单位无关. 类似的量还有 $q\Delta t$, $\sigma\sqrt{\Delta t}$, $\mu\Delta t$ 等. 所以, 取 Δt 非常小的含义是, 取 Δt 使得上述这些量非常小.

□

考虑收益率的平方

$$\left(\frac{S_{i+1}-S_i}{S_i}\right)^2. \tag{2.3.8}$$

在几何布朗运动(2.3.7)成立的前提下, 由公式

$$\left(\frac{dS}{S}\right)^2 = \sigma^2 dt$$

知 (2.3.8)等于 $\sigma^2(i+1-i) = \sigma^2$. 但是在 $t=0$ 时, 没人可以保证未来所有区间 $[t_i, t_{i+1}]$ 上, 得到的方差 σ^2 为常数. 假设方差与时间 t 有关. 将 σ^2 看成 σ_t^2. 由(2.3.8) 知

$$\left(\frac{S_{i+1}-S_i}{S_i}\right)^2 = \sigma_i^2. \tag{2.3.9}$$

在 $t=T$ 时, 我们可以计算

$$\frac{1}{N}\sum_{i=0}^{N-1}\left(\frac{S_{i+1}-S_i}{S_i}\right)^2=\frac{1}{N}\sum_{i=0}^{N-1}\sigma_i^2.$$

如果(2.3.7)严格成立, 那么上式右边就等于 σ^2. 但是, 实证发现, 这是不对的.

现在考虑一张 (欧式) 合约:

- 合约由 A,B 双方在 $t=0$ 时签订.
- 在区间 $(0,T)$ 内双方无现金流.
- 在 $t=T$ 时, A 方从 B 方获得现金

$$\frac{1}{N}\sum_{i=0}^{N-1}\left(\frac{S_{i+1}-S_i}{S_i}\right)^2-E,$$

  其中, E 为在 $t=0$ 时给定常数.

问: 在 $t=0$ 时, A 要付给 B 多少钱? 此类合约属于方差互换 (Variance swap). 如果 E 使得在 $t=0$ 时, A,B 双方无现金流, 那么此时的 E 记为 E_{var}, 对应远期合约的价格. 问题: 标的

$$\frac{1}{N}\sum_{i=0}^{N-1}\left(\frac{S_{i+1}-S_i}{S_i}\right)^2$$

如何买卖? 现在做些推导.

$$\begin{aligned}\ln S_{i+1}-\ln S_i&=\ln\frac{S_{i+1}}{S_i}\\&=\ln\left(1+\frac{S_{i+1}-S_i}{S_i}\right)\\&=\frac{S_{i+1}-S_i}{S_i}-\frac{1}{2}\left(\frac{S_{i+1}-S_i}{S_i}\right)^2.\end{aligned}$$

再将上式对于 i 边边相加:

$$\ln S_N-\ln S_0=\sum_{i=0}^{N-1}\frac{S_{i+1}-S_i}{S_i}-\frac{1}{2}\sum_{i=0}^{N-1}\left(\frac{S_{i+1}-S_i}{S_i}\right)^2.$$

即

$$\begin{aligned}\sum_{i=0}^{N-1}\left(\frac{S_{i+1}-S_i}{S_i}\right)^2&=2\ln\frac{S_0}{S_N}+2\sum_{i=0}^{N-1}\frac{S_{i+1}-S_i}{S_i}\\&=2\ln\frac{S_0}{S_N}+2\int_{t=0}^{t=T}\frac{dS}{S}\end{aligned}$$

假设存在唯一风险中性测度 Q，使得

$$\frac{dS}{S}=rdt+\sigma_t dB_t^Q.$$

上式内容不必深究. 于是

$$
\begin{aligned}
\sum_{i=0}^{N-1}\left(\frac{S_{i+1}-S_i}{S_i}\right)^2 &= 2\ln\frac{S_0}{S_N}+2\int_{t=0}^{t=T}\left(rdt+\sigma_t dB_t^Q\right)\\
&= 2\ln\frac{S_0}{S_N}+2rT+2\int_{t=0}^{t=T}\sigma_t dB_t^Q\\
&= 2\ln\frac{S_0}{S_*}+2\ln\frac{S_*}{S_T}+2rT+2\int_{t=0}^{t=T}\sigma_t dB_t^Q, \quad (S_N=S_T),
\end{aligned}
\tag{2.3.10}
$$

其中 S_* 为任意正参数, 易证

$$
\begin{aligned}
-\ln\frac{S_T}{S_*} = &-\frac{S_T-S_*}{S_*}\\
&+\int_0^{S_*}\frac{1}{E^2}\max(E-S_T,0)dE\\
&+\int_{S_*}^{+\infty}\frac{1}{E^2}\max(S_T-E,0)dE
\end{aligned}
\tag{2.3.11}
$$

在 $t=0$ 时, A 要付给 B 现金为

$$
\begin{aligned}
w_0 :=& e^{-rT}\mathbb{E}^Q\left[\frac{1}{N}\sum_{i=0}^{N-1}\left(\frac{S_{i+1}-S_i}{S_i}\right)^2-E\middle|\mathcal{F}_0\right]\\
=& e^{-rT}\mathbb{E}^Q\left[\frac{1}{N}\left(2\ln\frac{S_0}{S_*}+2\ln\frac{S_*}{S_T}+2rT+2\int_{t=0}^{t=T}\sigma_t dB_t^Q\right)-E\middle|\mathcal{F}_0\right]\\
&(\text{利用}(2.3.10)).
\end{aligned}
$$

在上式中,

$$
\mathbb{E}^Q\left[2\int_{t=0}^{t=T}\sigma_t dB_t^Q\middle|\mathcal{F}_0\right]=0.
$$

于是

$$
w_0=e^{-rT}\mathbb{E}^Q\left[\frac{1}{N}\left(2\ln\frac{S_0}{S_*}+2\ln\frac{S_*}{S_T}+2rT\right)-E\middle|\mathcal{F}_0\right].
$$

将(2.3.11)代入上式, 整理得

$$
\begin{aligned}
w_0 =& e^{-rT}\mathbb{E}^Q\left[\frac{1}{N}\sum_{i=0}^{N-1}\left(\frac{S_{i+1}-S_i}{S_i}\right)^2-E\middle|\mathcal{F}_0\right]\\
&= e^{-rT}\mathbb{E}^Q\left[\frac{1}{T}\left(2\ln\frac{S_0}{S_*}-2\frac{S_T-S_*}{S_*}+2rT\right)-E\middle|\mathcal{F}_0\right]\\
&+\int_0^{S_*}\frac{2e^{-rT}}{TE^2}\mathbb{E}^Q[\max(E-S_T,0)|\mathcal{F}_0]dE\\
&+\int_{S_*}^{+\infty}\frac{2e^{-rT}}{TE^2}\mathbb{E}^Q[\max(S_T-E,0)|\mathcal{F}_0]dE.
\end{aligned}
$$

对一 T 时到期的欧式衍生证券 v_t, 假设 $e^{-rt}v_t$ 是一个鞅. 进一步化简上式

$$w_0 = e^{-rT}\left(\frac{2}{T}\ln\frac{S_0}{S_*} - \frac{2}{T}\frac{e^{rT}S_0 - S_*}{S_*} + 2r - E\right) + \int_0^{S_*}\frac{2p(S,0,E,T)}{TE^2}dE + \int_{S_*}^{+\infty}\frac{2c(S,0,E,T)}{TE^2}dE.$$

由于 S_* 为任意正参数, 取 S_* 为远期合约的价格 $F_0 = S_0e^{rT}$, 我们可以进一步化简上式:

$$w_0 = -e^{-rT}E + \int_0^{F_0}\frac{2p(S,0,E,T)}{TE^2}dE + \int_{F_0}^{+\infty}\frac{2c(S,0,E,T)}{TE^2}dE.$$

令 $w_0 = 0$, 我们得到 $t = 0$ 时远期合约的价格

$$E_{\text{var}} = \frac{2e^{rT}}{T}\left(\int_0^{F_0}\frac{p(S,0,E,T)}{E^2}dE + \int_{F_0}^{+\infty}\frac{c(S,0,E,T)}{E^2}dE\right). \tag{2.3.12}$$

问题 1. *假设 S 连续股息派发, 其股息派发率为非负常数 q. 那么公式(2.3.12)是否成立?*

□

作业题 6. 本作业题目的是利用 A 股市场中 50ETF 期权数据, 得到式 (2.3.12)中的 E_{var}. 建议采取以下步骤:

1. 阅读课件和参考文献 (见课件最后), 用自己的方式, 叙述方差互换问题.
2. 了解上证 50ETF 期权合约基本条款, 建议看一下
   `http://www.sse.com.cn/assortment/options/contract/c/c_20151016_3999892.shtml`
   如对有关名词不太懂, 如: 认购期权, 认沽期权等, 也请去百度查找.
3. 去东方财富网 (`http://www.eastmoney.com/`) 找寻 50ETF 期权和期货数据.
4. 自己确定你认为合理的无风险利率 r 和股息派发率 q.
5. 要求用 C++ 计算. 计算 3 到 5 个 E_{var} 即可. 主要为了让大家了解计算流程.
6. 讨论你得到的计算结果.
7. 阅读本课件最后的参考文献中的 vsv and Volatility trading (这部分不必提交).

此题的数据处理方法不是唯一的, 也没有标准答案. 在遇到困难时, 希望大家独立克服.

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VARIANCE SWAP VOLATILITY AND OPTION STRATEGIES

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Born in the over-the-counter derivatives market, variance swap volatility (VSV) is slowly but surely gaining recognition as a useful tool for managing option positions. Let’s begin with this concept by looking at a variance swap contract.

Here’s how it works: Counterparty A agrees to pay Counterparty B a fixed notional amount due at the settlement date. In exchange, Counterparty B will pay Counterparty A an amount proportional to the sum of the daily squared returns

$$\frac{1}{n}\sum_{i=0}^{n-1}\left(\frac{S_{i+1}-S_i}{S_i}\right)^2$$

where n= number of business days until settlement, and S_i, i=0,2,...,n represent daily closing prices of the underlying stock or index. The sum of the daily returns squared can be interpreted as an estimator of the realized variance of stock returns from now to the settlement of the contract.

For instance, a contract can be stipulated as follows: using the **Standard & Poor’s** 500 as the underlying index, a volatility level of * =23% is fixed for one year. This corresponds to a nominal variance =5.29%. Counterparty B agrees to pay A USD100,000 for each percentage point of realized variance *above* ^{*2} =5.29% (where the variance is computed using the above formula) and A agrees to pay B USD100,000 per variance point *below* this value. In this case, the notional value of the contract, or fixed leg payment, is USD529,000.

What is the "fair value'' of the parameter * in such a swap? If no cash flows are exchanged initially, * should be equal to the *discounted expectation of the realized variance over the time-horizon of the contract*. This is the definition of the variance swap volatility. We will next derive a formula for computing it.

VARIANCE SWAPS AND LOG-CONTRACTS

A key observation, separately noted by Neuberger and Dupire, is that a variance swap is financially equivalent to--and can be replicated by--a European-style derivatives security with a logarithmic payoff. To see this, we make a Taylor-series expansion of the logarithm of the price up to second-order derivatives, which gives

$$\log S_{i+1} - \log S_i \approx \frac{S_{i+1}-S_i}{S_i} - \frac{1}{2}\left(\frac{S_{i+1}-S_i}{S_i}\right)^2$$

Summing both sides of this equation over the total number of days in the contract, and rearranging terms, we obtain

$$\sum_{i=0}^{n-1}\left(\frac{S_{i+1}-S_i}{S_i}\right)^2 \approx 2\log\frac{S_0}{S_n}-2\sum_{i=0}^{n-1}\frac{S_{i+1}-S_i}{S_i}$$

It follows that the "floating leg'' of the swap (the left-hand side) can be replicated by holding a derivative contract with payoff 2 log (S_0/S_n) --a *log-contract*--and a position in foward contracts. We conclude that the value of the variance swap is equal to that of a log-contract. Secondly, we see that a short position in the variance swap can be hedged by holding a log-contract and maintaining a neutral delta at the close of each trading day, assuming that funding costs are zero.

A FORMULA FOR THE VSV

To price the log-contract, we approximate the payoff with a function that is piecewise-linear, that is to say, a function whose graph consists of line segments. We use the fact that a contingent claim with such a payoff can be replicated with a portfolio of plain-vanilla options with the same maturity and different strikes. This leads to a closed-form expression for the VSV

$$\sigma = \sqrt{\frac{2}{T}\left(\int_0^F \frac{P(K,T)dK}{K^2}+\int_F^\infty \frac{C(K,T)dK}{K^2}\right)}$$

where T =duration of the contract measured in years, F=forward price of the underlying security, and $P(K,T)$, $C(K,T)$ represent the prices of European puts and calls with strike K expiring in T years. This formula can be interepreted as an arbitrage relationship between the implied volatilities of traded options and the variance swap volatility. Simply put: *the option volatility curve contains the market's expectation on the realized variance of the stock from today to the expiration date of the options.*

VSV AND VOLATILITY TRADING

The VSV is not only useful for the over-the-counter market, but serves as a "benchmark'' for analyzing options in general and the volatility skew in particular. Often, the implied volatility of at-the-money options is interpreted as the market's expectation of future volatility. We have shown here that this is wrong: the correct value is the VSV. The ATM volatility represents instead the expected volatility conditionally on the underlying price remaining at the same level. Therefore, the VSV is the natural "benchmark'' for analyzing option volatility.

Writer and money-manager **Nassim Taleb**, from the Greenwich, Conn.-based fund **Empirica Capital**, has been a strong proponent of this interpretation of the variance swap volatility, recognizing the implications that it has for option trading. In his Ph. D. thesis, he describes a hypothetical conversation between two traders that runs as follows:

"What do you believe the volatility will be for the next month?''

"20%.''

"And what market are you making on one month at-the-money calls?''

"18% bid, 18.50% offered.''

This exchange seems paradoxical from the point of view of option pricing theory but it is not. In fact, "Taleb's paradox'' is resolved by observing that the expectations of volatility correspond to the VSV--not the implied volatility of any given (fixed-strike) option.

Traders often view option-implied volatility as a a "break-even level'' for daily movements of the underlying stock price (*e.g*. if the implied volatility is 16%, the daily break-even is an absolute move of 16/ 255=1%). However, they cannot be certain of the levels of implied volatility that they will face when they revise their portfolios as time passes. For example, if a trader wants to run an options book with a constant time -decay exposure for the next three months, he or she could compare his or her forecast of future implied volatility with the current three-month VSV to determine the profitability of such trading strategy.

Another application is to dynamic hedging. Option pricing theory tells us that profits/losses proportional to the stock returns *squared* can be achieved by holding a delta-neutral option position. Because a variance swap delivers a similar payoff, the VSV can be used for relative-value analysis of implied volatilities and for designing option strategies. For example, liquid contracts trading at levels deeply below VSV should be attractive for buyers of volatility, while contracts trading above the VSV should be viewed as expensive from this perspective.

As a general rule, shifts in the overall levels of volatility (i.e. the entire volatility curve) play a crucial role in option trading. These shifts cannot be predicted in advance. Notwithstanding this uncertainty, the VSV provides an interesting tool to analyze the volatility skew and to detect buying/selling opportunities using available market information. Indices like the S&P 500 and the Nasdaq 100 have large skews and hence are interesting from the point of view of VSV. Options on many individual stocks trade with a large skew and can also be analyzed in this way.

References: Dupire, B, 1993, Model Art, Risk, Sept.1993, 118-120 Neuberger, A., 1990,Volatility Trading, London Business School working paper Taleb, N., 1998, Ph.D. Dissertation, University of Paris

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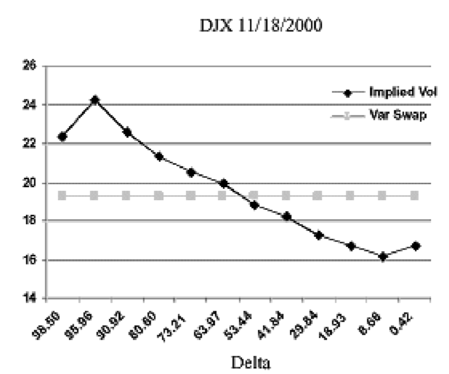
This week's Learning Curve was written by Marco Avellaneda, a professor of mathematics and director of the division of financial mathematics at New York University in New York.



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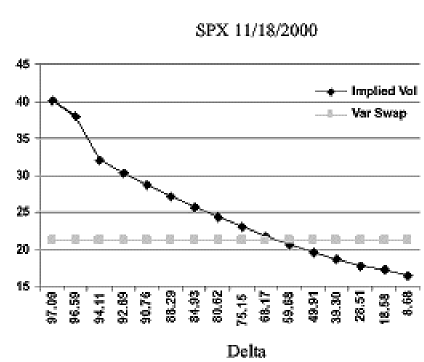

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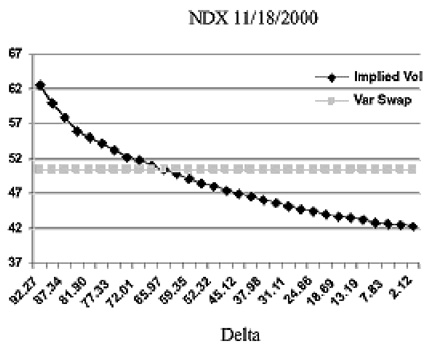

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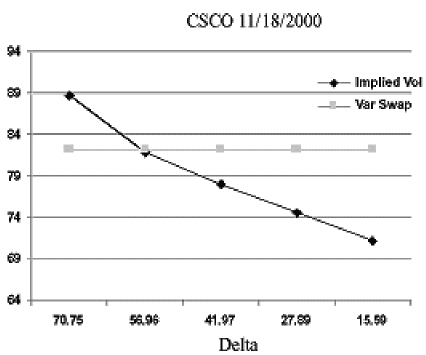

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