Report | HW 3

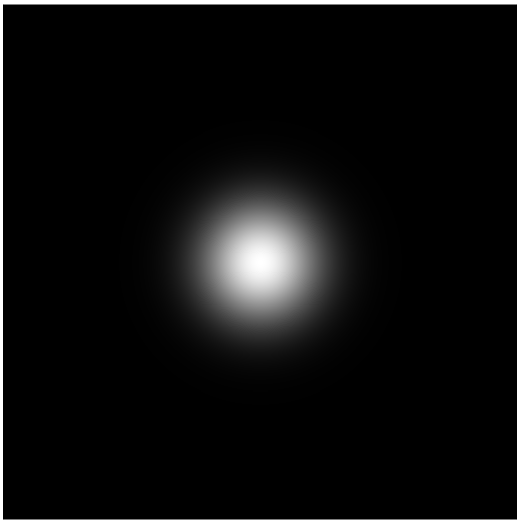
# CS663 - Digital Image Processing


Omkar Shirpure (22B0910)
Suryansh Patidar (22B1036)
Aditya Saran (22B1839)

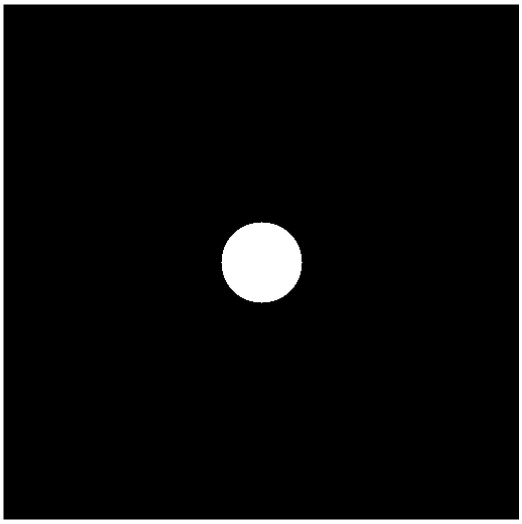
# Contents

# 1 Q1

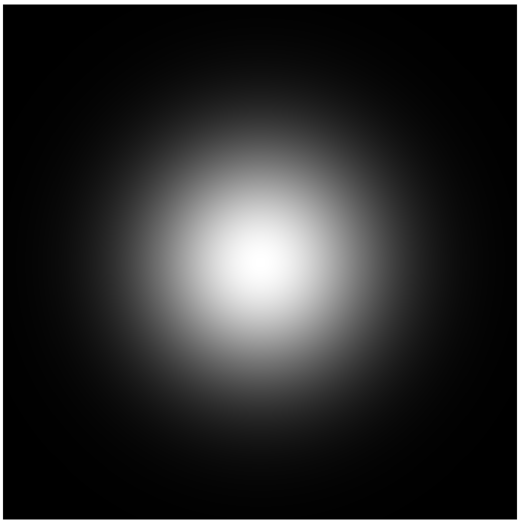
## 1.1 Ideal Filters



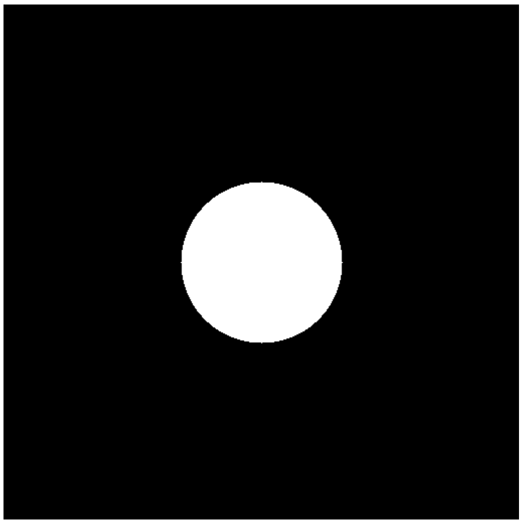
(a) Ideal Gaussian Filter ($\sigma = 40$)



(b) Ideal Low Pass Filter ($D = 40$)



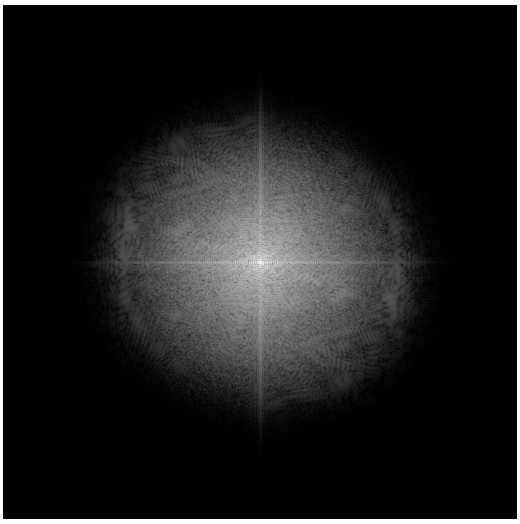
(c) Ideal Gaussian Filter ($\sigma = 80$)



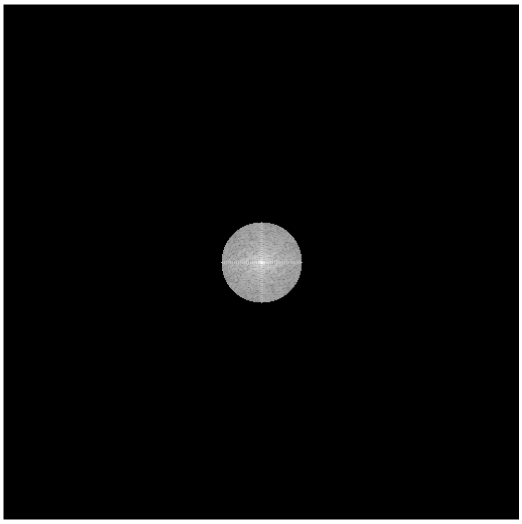
(d) Ideal Low Pass Filter ($D = 80$)

Figure 1: Ideal Filters

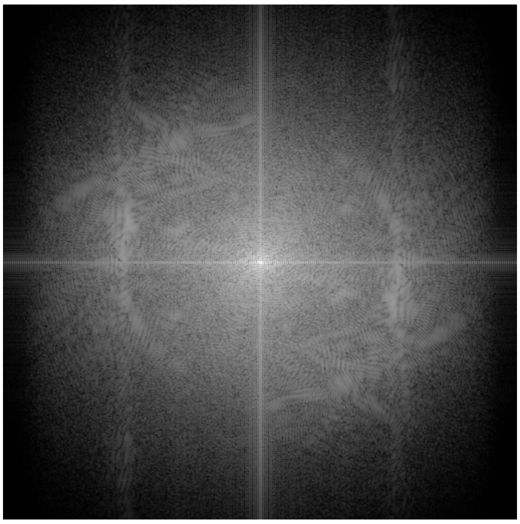
## 1.2 Log FT of filters on the image



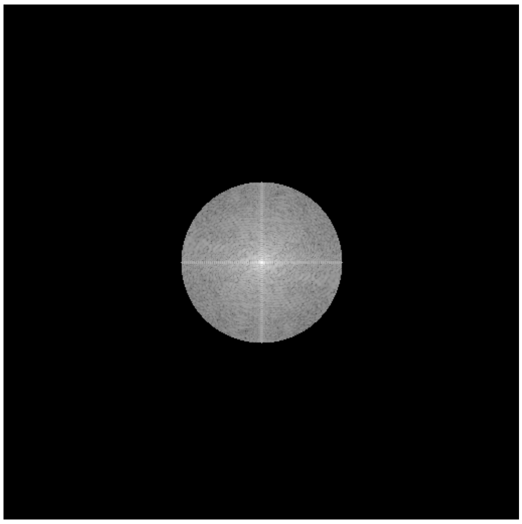
(a) Gaussian Low Pass Filter on image ($\sigma = 40$)



(b) Low Pass Filter on image ($D = 40$)



(c) Gaussian Low Pass Filter on image ($\sigma = 80$)

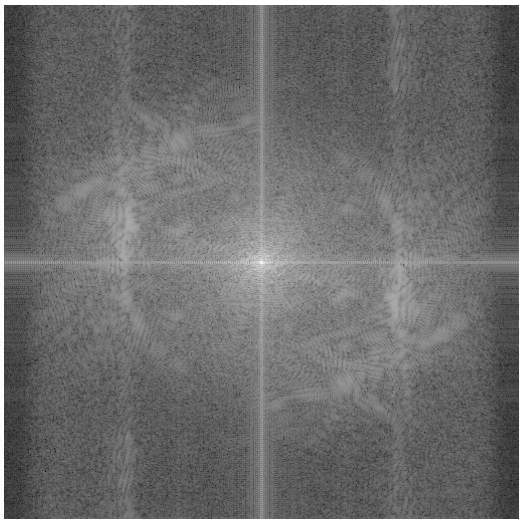


(d) Low Pass Filter on image ($D = 80$)

Figure 2: Ideal Filters

### 1.3 Filtered Images with respective parameters

(a) Original Image



(b) Log FT of Original Image

(c) Filtered Image (Gaussian Filter ($\sigma = 40$))

(d) Filtered Image (Low Pass Filter ($D = 40$))

(e) Filtered Image (Gaussian Filter ($\sigma = 80$))

(f) Filtered Image (Low Pass Filter ($D = 80$))

Figure 3: Ideal Filters

## 1.4 Comments on the Differences:

**Ideal Low Pass Filter**:

The ideal filter passes all frequencies below the cutoff $D$ and sharply cuts off frequencies above it. For $D = 40$, the image appears more blurred, removing more high-frequency components, which results in a loss of fine details. For $D = 80$, the image is less blurred compared to $D = 40$, and some fine details are still retained.

In low-pass filtered images, especially with the ideal low-pass filter, we observe noticeable striped artifacts. This occurs because the ideal filter sharply cuts off frequencies above the cutoff $D$, leading to abrupt transitions in the frequency domain.

The stripes are more prominent but thinner in the $D = 80$ filter than in $D = 40$. This is because with a higher cutoff frequency ($D = 80$), more high-frequency components are preserved, but the abrupt truncation of frequencies at the cutoff still causes visible ringing artifacts in the image.

**Gaussian Low Pass Filter**:

The Gaussian filter has a smoother frequency response compared to the ideal filter, leading to a more gradual attenuation of high frequencies. For $\sigma = 40$, the image is blurred, but the transition is smoother compared to the ideal filter. For $\sigma = 80$, the blurring effect is less pronounced, and more details are preserved in comparison to $\sigma = 40$.

# 2 Q2

## 2.1

Correlation for continuous signals in a continuous domain can be written as:

$$(h \otimes f)(t,s) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi,\eta) f(t+\xi, s+\eta)\, d\xi\, d\eta$$

Taking Fourier transform:

$$\mathcal{F}\left[(h \otimes f)(t,s)\right](u,v)$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi,\eta) f(t+\xi, s+\eta)\, d\xi\, d\eta \right) e^{-j2\pi(ut+vs)}\, dt\, ds$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi,\eta) \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(t+\xi, s+\eta) e^{-j2\pi(ut+vs)}\, dt\, ds \right) d\xi\, d\eta$$

Let $\xi + t = x$ and $s + \eta = y$.

Then, the equation becomes:

$$\mathcal{F}\left[(h \otimes f)(t,s)\right](u,v)$$

Substituting the variables:

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi,\eta) \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi[u(x-\xi)+v(y-\eta)]}\, dx\, dy \right) d\xi\, d\eta$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi,\eta) \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)}\, e^{j2\pi(u\xi+v\eta)}\, dx\, dy \right) d\xi\, d\eta$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi,\eta) \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) e^{-j2\pi(ux+vy)}\, dx\, dy \right) e^{j2\pi(u\xi+v\eta)}\, d\xi\, d\eta$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi,\eta)\, \mathcal{F}\{f(x,y)\}(u,v)\, e^{j2\pi(u\xi+v\eta)}\, d\xi\, d\eta$$

The term $\mathcal{F}\{f(x,y)\}(u,v)$ represents Fourier transform of the function $f(x,y)$ at frequencies $u$ and $v$.

Let $\mathcal{F}\{f(x,y)\} = F(u,v)$, so the expression becomes:

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi,\eta)\, F(u,v)\, e^{j2\pi(u\eta+vs)}\, d\xi\, d\eta$$

$$= F(u,v) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(\xi,\eta)\; e^{j2\pi(u\xi+v\eta)}\, d\xi\, d\eta$$

Looking at the integral term, it is the Fourier transform of $h(\xi,\eta)$ at frequencies $-u$ and $-v$. Let the term $\mathcal{H}\{h(\xi,\eta)\}(-u,-v) = H(-u,-v)$ represents the fourier, so our expression becomes:

$$= F(u,v)H(-u,-v)$$

$$\boxed{\mathcal{F}\left[(h \otimes f)(t,s)\right](u,v) = H(-u,-v)F(u,v)}$$

## 2.2

Correlation for discrete signals in a discrete domain can be written as:

$$(h \otimes f)[n,m] = \sum_{\xi=-\infty}^{\infty} \sum_{\eta=-\infty}^{\infty} h[\xi,\eta] f[n+\xi, m+\eta]$$

Taking the Discrete Fourier Transform (DFT):

$$\mathcal{F}\left[(h \otimes f)[n,m]\right](u,v)$$

$$= \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left( \sum_{\xi=-\infty}^{\infty} \sum_{\eta=-\infty}^{\infty} h[\xi,\eta] f[n+\xi, m+\eta] \right) e^{-j2\pi(\frac{un}{N}+\frac{vm}{M})}$$

Now, swap the order of summation:

$$= \sum_{\xi=-\infty}^{\infty} \sum_{\eta=-\infty}^{\infty} h[\xi,\eta] \left( \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} f[n+\xi, m+\eta] e^{-j2\pi(\frac{un}{N}+\frac{vm}{M})} \right)$$

Let $\xi + n = x$ and $\eta + m = y$.

Then, the equation becomes:

$$\mathcal{F}\left[(h \otimes f)[n,m]\right](u,v)$$

Substituting the variables:

$$= \sum_{\xi=-\infty}^{\infty} \sum_{\eta=-\infty}^{\infty} h[\xi,\eta] \left( \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f[x,y] e^{-j2\pi\left(\frac{u(x-\xi)}{N}+\frac{v(y-\eta)}{M}\right)} \right)$$

$$= \sum_{\xi=-\infty}^{\infty} \sum_{\eta=-\infty}^{\infty} h[\xi,\eta] \left( \sum_{x=-\infty}^{\infty} \sum_{y=-\infty}^{\infty} f[x,y] e^{-j2\pi(\frac{ux}{N}+\frac{vy}{M})} e^{j2\pi\left(\frac{u\xi}{N}+\frac{v\eta}{M}\right)} \right)$$

Simplifying:

$$= \sum_{\xi=-\infty}^{\infty} \sum_{\eta=-\infty}^{\infty} h[\xi,\eta] \mathcal{F}\{f[x,y]\}(u,v) e^{j2\pi\left(\frac{u\xi}{N}+\frac{v\eta}{M}\right)}$$

The term $\mathcal{F}\{f[x,y]\}(u,v)$ represents the Fourier transform of $f[x,y]$ at frequencies $u, v$. Let $\mathcal{F}\{f[x,y]\} = F(u,v)$, so the expression becomes:

$$= \sum_{\xi=-\infty}^{\infty} \sum_{\eta=-\infty}^{\infty} h[\xi,\eta] F(u,v) e^{j2\pi\left(\frac{u\xi}{N}+\frac{v\eta}{M}\right)}$$

$$= F(u,v) \sum_{\xi=-\infty}^{\infty} \sum_{\eta=-\infty}^{\infty} h[\xi,\eta] e^{j2\pi\left(\frac{u\xi}{N}+\frac{v\eta}{M}\right)}$$

Looking at the sum, it is the Fourier transform of $h[\xi,\eta]$ at frequencies $-u, -v$. Let the term $\mathcal{H}\{h[\xi,\eta]\}(-u,-v) = H(-u,-v)$, so our expression becomes:

$$= F(u,v)H(-u,-v)$$

Thus, the final result is:

$$\boxed{\mathcal{F}\left[(h \otimes f)[n,m]\right](u,v) = H(-u,-v)F(u,v)}$$

# 3 Q3

## 3.1 Barbara Image: Noise ( $\sigma = 5$ )

(a) Original Image

(b) Noisy Image (Gau. noise with $\sigma = 5$)

(c) Filtered Image ($\sigma_s = 2, \sigma_r = 2$)

(d) Filtered Image ($\sigma_s = 15, \sigma_r = 3$)

(e) Filtered Image ($\sigma_s = 3, \sigma_r = 15$)

Figure 4: Barbara image with noise ( $\sigma = 5$ )

## 3.2 Barbara Image: Noise ( $\sigma = 10$ )

(a) Original Image

(b) Noisy Image (Gau. noise with $\sigma = 10$)

(c) Filtered Image ($\sigma_s = 2, \sigma_r = 2$)

(d) Filtered Image ($\sigma_s = 15, \sigma_r = 3$)

(e) Filtered Image ($\sigma_s = 3, \sigma_r = 15$)

Figure 5: Barbara image with noise ( $\sigma = 10$ )

### 3.3 Kodak Image: Noise ( $\sigma = 5$ )

(a) Original Image

(b) Noisy Image (Gau. noise with $\sigma = 5$)

(c) Filtered Image ($\sigma_s = 2, \sigma_r = 2$)

(d) Filtered Image ($\sigma_s = 15, \sigma_r = 3$)

(e) Filtered Image ($\sigma_s = 3, \sigma_r = 15$)

Figure 6: Kodak image with noise ( $\sigma = 5$ )

### 3.4 Kodak Image: Noise ( $\sigma = 10$ )

(a) Original Image

(b) Noisy Image (Gau. noise with $\sigma = 10$)

(c) Filtered Image ($\sigma_s = 2, \sigma_r = 2$)

(d) Filtered Image ($\sigma_s = 15, \sigma_r = 3$)

(e) Filtered Image ($\sigma_s = 3, \sigma_r = 15$)

Figure 7: Kodak image with noise ( $\sigma = 10$ )

## 3.5 Comparisons of the results

### 3.5.1 Barbara Image: Noise ( $\sigma = 5$ )

- ( $\sigma_s = 2$, $\sigma_r = 2$ ) : This configuration provides good spatial and intensity smoothing. Noise is reduced effectively while preserving most edge details. Fine details are slightly smoothed out.
- ( $\sigma_s = 15$, $\sigma_r = 3$ ) : The larger spatial window leads to more significant smoothing, particularly in homogeneous regions. Noise is reduced more aggressively, details are blurred significantly.
- ( $\sigma_s = 3$, $\sigma_r = 15$ ) : With a high intensity window, the filter smooths over a larger range of pixel intensities. This results in strong denoising but also leads to substantial blurring of edges and finer details, like the netting texture and facial features.

### 3.5.2 Barbara Image: Noise ( $\sigma = 10$ )

With higher noise levels, the differences between parameter configurations become more prominent.

- ( $\sigma_s = 2$, $\sigma_r = 2$ ) : Noise is reduced, and most edges are preserved. However, some noise persists in smooth regions, and edge details are slightly compromised.
- ( $\sigma_s = 15$, $\sigma_r = 3$ ) : This configuration leads to better noise reduction, but fine details and texture are more blurred than in the previous case. Some edge preservation remains, though less pronounced.
- ( $\sigma_s = 3$, $\sigma_r = 15$ ) : The higher intensity smoothing causes significant blurring of both noise and details. While the noise is mostly eliminated, this over-smoothing results in the loss of crucial details.

### 3.5.3 Kodak Image: Noise ( $\sigma = 5$ )

- ( $\sigma_s = 2$, $\sigma_r = 2$ ) : Moderate noise reduction is achieved while preserving most edge details. Fine textures, such as balcony edges and door designs, remain intact but with some slight blurring.
- ( $\sigma_s = 15$, $\sigma_r = 3$ ) : This configuration results in more aggressive smoothing, which effectively reduces noise. However, the blurring of edges and fine details becomes more noticeable, especially in regions like the balcony and architectural details.
- ( $\sigma_s = 3$, $\sigma_r = 15$ ) : The high intensity smoothing reduces noise significantly but at the cost of over-smoothing fine textures and details. The image appears softer, and important structures like edges are blurred.

### 3.5.4 Kodak Image: Noise ( $\sigma = 10$ )

The increased noise amplifies the differences between the configurations.

- ( $\sigma_s = 2$, $\sigma_r = 2$ ) : This configuration reduces noise while still preserving some edge details. However, some noise persists in the smoother regions of the image.
- ( $\sigma_s = 15$, $\sigma_r = 3$ ) : The larger spatial smoothing window leads to more aggressive noise reduction, but at the cost of blurring fine details. Architectural elements become less distinct.
- ( $\sigma_s = 3$, $\sigma_r = 15$ ) : While this configuration eliminates most noise, the result is a highly smoothed image where significant fine details and edges are lost.

# 4 Q4

## 4.1 Analytical Derivation

### Image Structure

The image is of size $201 \times 201$, and all pixels are black (value = 0) except for the central column (column 101) where all pixel values are 255.

The image can be mathematically described as:

$$f(x, y) = \begin{cases} 255 & \text{if } x = 101,\ 1 \leq y \leq 201 \\ 0 & \text{otherwise} \end{cases}$$

### Fourier Transform

The 2D Fourier transform of the image $f(x, y)$ can be computed using the following general formula:

$$F(u, v) = \sum_{x=1}^{201} \sum_{y=1}^{201} f(x, y) e^{-j2\pi\left(\frac{(x-1)u}{201} + \frac{(y-1)v}{201}\right)}$$

Since the image is only non-zero at $x = 101$, the Fourier transform simplifies to:

$$F(u, v) = 255 \sum_{y=1}^{201} e^{-j2\pi \frac{(y-1)v}{201}}$$

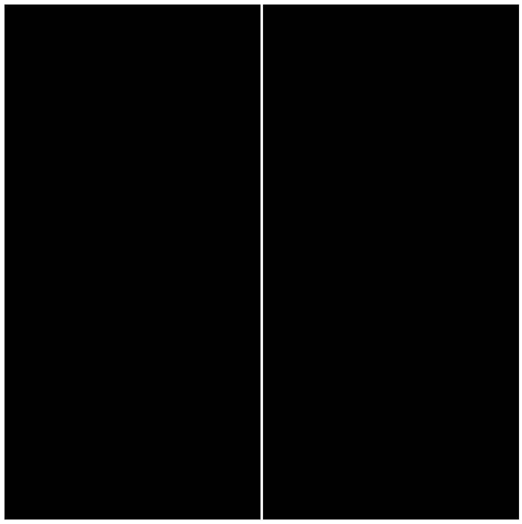
This is a summation of a complex exponential, which is a geometric series. The result of the summation for any $v$ is:

$$F(u, v) = 255 \cdot 201 \cdot \delta(u) \cdot \text{sinc}\left(\frac{v}{201}\right)$$

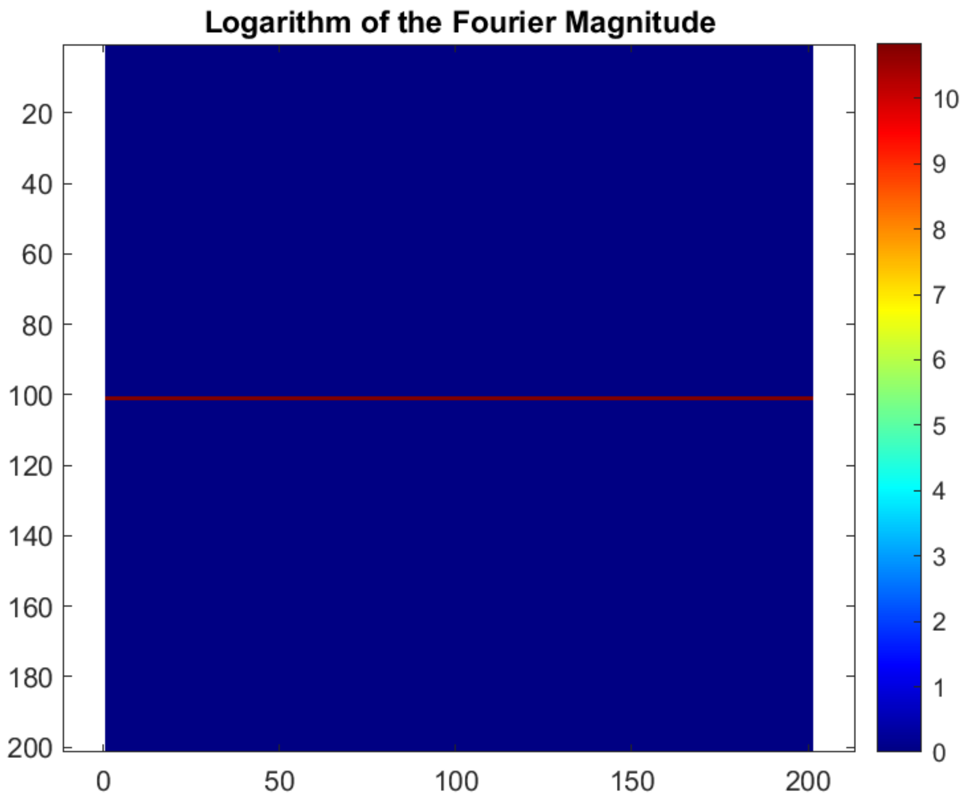
where $\delta(u)$ indicates that the Fourier transform is non-zero only when $u = 0$, and sinc represents the sinc function.

Thus, the Fourier transform is concentrated along the horizontal axis, with the sinc function defining its behavior in the $v$ direction.

## 4.2 Code Results



(a) Specified Image (201x201) with a column of white pixels




(b) Fourier Magnitude Plot

Figure 8: Plots

# 5 Q5

**Case 1: Function f(x,y) is real**

$$F(u,v) = \sum_{x=0}^{W_1-1}\sum_{y=0}^{W_2-1} f(x,y)e^{-j2\pi(\frac{ux}{W_1}+\frac{vy}{W_2})}$$

$$F^*(u,v) = \sum_{x=0}^{W_1-1}\sum_{y=0}^{W_2-1} f^*(x,y)e^{j2\pi(\frac{ux}{W_1}+\frac{vy}{W_2})} = \sum_{x=0}^{W_1-1}\sum_{y=0}^{W_2-1} f(x,y)e^{j2\pi(\frac{ux}{W_1}+\frac{vy}{W_2})}$$
$$= \sum_{x=0}^{W_1-1}\sum_{y=0}^{W_2-1} f(x,y)e^{-j2\pi(\frac{-ux}{W_1}+\frac{-vy}{W_2})}$$

Therefore,

$$\boxed{F^*(u,v) = F(-u,-v)}$$

**Case 2: Function f(x,y) is real and even**

We know that if f(x,y) is real, then

$$F(-u,-v) = \sum_{x=0}^{W_1-1}\sum_{y=0}^{W_2-1} f(x,y)e^{j2\pi(\frac{ux}{W_1}+\frac{vy}{W_2})} \tag{1}$$

Now replacing $x \to -t$ and $y \to -s$ in the above equation,

$$F(-u,-v) = \sum_{t=1-W_1}^{0}\sum_{s=1-W_2}^{0} f(-t,-s)e^{j2\pi(\frac{-ut}{W_1}+\frac{-vs}{W_2})}$$

Since $f(x,y)$ is even, and Periodicity property of DFT,

$$= \sum_{t=0}^{W_1-1}\sum_{s=0}^{W_2-1} f(t,s)e^{j2\pi(\frac{-ut}{W_1}+\frac{-vs}{W_2})}$$

$$= F(u,v)$$

So we finally got,

$$\boxed{F(-u,-v) = F(u,v)}$$

And for the real part, we already proved above that is $f(x,y)$ is real, then

$$F^*(u,v) = F(-u,-v)$$

Hence I can write,

$$\boxed{F^*(u,v) = F(-u,-v) = F(u,v)}$$

Hence We Proved that $F(u,v)$ should both be real and even if $f(x,y)$ is real and even.

# 6 Q6

$$\mathcal{F}(f(t))(u) = \int_{-\infty}^{\infty} e^{-j2\pi ut} f(t) dt$$

$$\mathcal{F}(\mathcal{F}(f(t)) = \int_{-\infty}^{\infty} e^{-j2\pi ut} \left[ \int_{-\infty}^{\infty} e^{-j2\pi u\tau} f(\tau) d\tau \right] du$$

$$= \int_{-\infty}^{\infty} f(\tau) \left[ \int_{-\infty}^{\infty} e^{-j2\pi(\tau+t)u} du \right] d\tau$$
$$= \int_{-\infty}^{\infty} \delta(\tau + t) f(\tau) d\tau = f(-t)$$

Now we can write,

$$\mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}(f(t))))) = \mathcal{F}(\mathcal{F}(f(-t)))$$

$$= \int_{-\infty}^{\infty} e^{-j2\pi ut} \left[ \int_{-\infty}^{\infty} e^{-j2\pi u(-\tau)} f(-\tau) d\tau \right] du$$

$$= \int_{-\infty}^{\infty} f(\tau) \left[ \int_{-\infty}^{\infty} e^{-j2\pi(\tau-t)u} du \right] d\tau$$
$$= \int_{-\infty}^{\infty} \delta(\tau - t) f(\tau) d\tau = f(t)$$

So we proved,

$$\boxed{\mathcal{F}(\mathcal{F}(\mathcal{F}(\mathcal{F}(f(t))))) = f(t)}$$

The relationship $F(F(f(t))) = f(-t)$ is useful in simplifying the process of computing Fourier transforms, especially when dealing with time-reversed signals. This property can be employed in the following practical scenarios:

- **Inverse Fourier Transform**: The operation

  $$F(F(f(t))) = f(-t)$$

  is similar to taking an inverse Fourier transform, as it effectively reverses the time domain. This can be useful when computing the inverse Fourier transform by reversing the time and frequency variables.

- **Efficient Calculation of Fourier Transforms**: If you need to compute the Fourier transform of a time-reversed signal,

  $$F(F(f(t))) = f(-t)$$

  simplifies the computation. Instead of manually reversing the signal and performing the Fourier transform, you can directly apply the relationship to obtain the result more efficiently.

- **Signal Processing and Filtering**: In signal processing, when performing operations such as convolution in the frequency domain, this relationship can simplify operations involving time-reversed filters or signals, reducing the computational effort.

# 7 Q7

$$\frac{\partial I(x,y;t)}{\partial t} = c\left(\frac{\partial^2 I(x,y;t)}{\partial x^2} + \frac{\partial^2 I(x,y;t)}{\partial y^2}\right) \tag{2}$$

Taking 2D Fourier transform on both sides, and using the differentiation theorem: Let $\tilde{I}(u,v;t)$ be the Fourier Transform of $I(c,y;t)$

$$\Rightarrow \frac{\partial \tilde{I}(u,v;t)}{\partial t} = -4\pi^2 c(u^2+v^2)\tilde{I}(u,v;t) \tag{3}$$

This is a Linear Differential equation which after solving result in,

$$\Rightarrow \tilde{I}(u,v;t) = A(u,v;t=0)e^{-4\pi^2 c(u^2+v^2)t} \tag{4}$$

A(u,v;t=0) will come from the boundary condition of $\tilde{I}$. Now taking the Inverse Fourier Transform on both the sides, (Note: $\tilde{I}$ will become $I$) and:

$$I(x,y;t) = F^{-1}(A(u,v;t=0)e^{-4\pi^2 c(u^2+v^2)t}) \tag{5}$$

Using the convolution theorem:

$$\Rightarrow I(x,y;t) = \tilde{A}(x,y;t=0) * F^{-1}(e^{-4\pi^2 c(u^2+v^2)t}) \tag{6}$$

And we know that $\tilde{A}(x,y;t=0) = I(x,y;t=0)$

$$F^{-1}(e^{-4\pi^2 c(u^2+v^2)t}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{j2\pi(ux+vy)}e^{-4\pi^2 c(u^2+v^2)t} \tag{7}$$

$$= \left(\sqrt{\frac{\pi}{4\pi^2 ct}}\right)e^{-\frac{1}{4ct}(x^2+y^2)} = \frac{1}{\sqrt{4\pi ct}}e^{-\frac{x^2+y^2}{4ct}} \tag{8}$$

$$\Rightarrow I(x,y;t) = I(x,y;0) * \left(\frac{1}{\sqrt{4\pi ct}}e^{-\frac{x^2+y^2}{4ct}}\right) \tag{9}$$

Considering $I(x,y;0)$ as the original image, we can see that the image at time t is like convolving the original image with a Gaussian filter of mean 0 and standard deviation of $\sqrt{2ct}$.