Report | HW1

CS754 - Advanced Image Processing

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Contents

Declaration: The work submitted is our own, and we have adhered to the principles of academic honesty while completing and submitting this work. We have not referred to any unauthorized sources, and we have not used generative AI tools for the work submitted here.

1 Q1

The order-s Restricted Isometry Constant (RIC), δ_s, of the matrix A is defined as the **smallest** constant such that for every subset S of indices with $|S| \leq s$, the sub-matrix A_S satisfies:

$$(1-\delta_s)\|\theta\|_2^2 \leq \|A_S\theta\|_2^2 \leq (1+\delta_s)\|\theta\|_2^2$$

for all $\theta \in \mathbb{R}^{|S|}$.
Since can equivalently state RIP property of order s as

$$(1-\delta_s)\|\theta\|^2 \leq \|A_{\mathcal{S}}\theta\|^2 \leq (1+\delta_s)\|\theta\|^2 \quad \forall\theta \in \mathbb{R}^s, \forall\mathcal{S} \subset \{1,2,\ldots,n\} \text{ s.t. } |\mathcal{S}| = s$$
$$\Rightarrow (1-\delta_s) \leq \frac{\|A_{\mathcal{S}}\theta\|^2}{\|\theta\|^2} \leq (1+\delta_s)\| \quad \forall\theta \in \mathbb{R}^s, \forall\mathcal{S} \subset \{1,2,\ldots,n\} \text{ s.t. } |\mathcal{S}| = s$$

and $A_{\mathcal{S}}$ is a column subset of A. Now, a detour to the analysis of $\frac{\|A_{\mathcal{S}}\theta\|^2}{\|\theta\|^2}$—

We can state that for any matrix B, the following holds true

$$\text{max}\theta\frac{\|B\theta\|^2}{\|\theta\|^2} = \lambda_{\text{max}} \tag{1}$$

$$\text{min}\theta\frac{\|B\theta\|^2}{\|\theta\|^2} = \lambda_{\text{min}} \tag{2}$$

Where $\lambda_{\text{max}}, \lambda_{\text{min}}$ are the maximum and minimum eigenvalues of $B^\top B$ respectively. **Why?**

$$\frac{\|B\theta\|^2}{\|\theta\|^2} = \frac{\theta^\top B^\top B\theta}{\theta^\top\theta}$$

Since $B^\top B$ is a symmetric square matrix, its eigenvector form an ortho-basis (Spectral theorem) and that all of the eigenvalues are positive (property of a positive semi-definite matrix). Thus we can express any vector of size width(B) as a sum of its eigenvectors. Thus

$$\theta = \sum_{i=1}^{\text{width}(B)} \alpha_i v_i \tag{3}$$

where $\langle v_i, \lambda_i \rangle$ are eigenvector-value pairs (v_i are all normalised).

$$\begin{aligned}
\frac{\|B\theta\|^2}{\|\theta\|^2} &= \frac{\theta^\top B^\top B\theta}{\theta^\top\theta} \\
&= \frac{(\sum_{i=1}^{\text{width}(B)} \alpha_i v_i)^\top B^\top B(\sum_{i=1}^{\text{width}(B)} \alpha_i v_i)}{(\sum_{i=1}^{\text{width}(B)} \alpha_i v_i)^\top(\sum_{i=1}^{\text{width}(B)} \alpha_i v_i)} \\
&= \frac{(\sum_{i=1}^{\text{width}(B)} \alpha_i v_i)^\top(\sum_{i=1}^{\text{width}(B)} \alpha_i B^\top B v_i)}{(\sum_{i=1}^{\text{width}(B)} \alpha_i v_i)^\top(\sum_{i=1}^{\text{width}(B)} \alpha_i v_i)} \\
&= \frac{(\sum_{i=1}^{\text{width}(B)} \alpha_i v_i)^\top(\sum_{i=1}^{\text{width}(B)} \alpha_i \lambda_i v_i)}{(\sum_{i=1}^{\text{width}(B)} \alpha_i v_i)^\top(\sum_{i=1}^{\text{width}(B)} \alpha_i v_i)} \\
&= \frac{\sum_{i=1}^{\text{width}(B)} \alpha_i^2 \lambda_i}{\sum_{i=1}^{\text{width}(B)} \alpha_i^2}
\end{aligned}$$

Thus obviously, any θ that yields the maximum value of $\frac{\|B\theta\|^2}{\|\theta\|^2}$ is aligned with the eigen-vector with the largest eigenvalue. Moreover, that "maximum" value is nothing other than that said λ_{max}.
Similarly, any θ that yields the minimum value is aligned with the v_i corresponding to λ_{min} and the "minimum" value is just λ_{min}.

Now, we can come back to our original problem—

$$(1-\delta_s)\|\theta\|^2 \leq \theta^\top A_{\mathcal{S}}^\top A_{\mathcal{S}}\theta \leq (1+\delta_s)\|\theta\|^2$$

Now, we know that any

$$\lambda_{\text{max}} = \max_{\theta \in U, \text{dim}(U) \leq S} \frac{\theta^\top A_{\mathcal{S}}^\top A_{\mathcal{S}}\theta}{\|\theta\|^2} \text{ and } \lambda_{\text{min}} = \min_{\theta \in U, \text{dim}(U) \leq S} \frac{\theta^\top A_{\mathcal{S}}^\top A_{\mathcal{S}}\theta}{\|\theta\|^2}$$

The right-hand inequality of RIP states:

$$(1-\delta_S)\|\theta\|^2 \leq \|A_{\mathcal{S}}\theta\|^2$$

Therefore, to find δ_s that satisfies the above equation, we need the maximum value of $\dfrac{\|A_{\mathcal{S}}\theta\|}{\|\theta\|}$ over all sets $\mathcal{S}$:

$$\begin{aligned}
(1+\delta_s) &\geq max_{S,\theta}\frac{\|A_{\mathcal{S}}\theta\|^2}{\|\theta\|^2} \\
\implies 1+\delta_s &\geq \lambda_{max} \\
\implies \delta_s &\geq \lambda_{max} - 1
\end{aligned} \tag{4}$$

Similarly, for the left-hand inequality of RIP, we need the minimum limit of $\dfrac{\|A_{\mathcal{S}}\theta\|}{\|\theta\|}$:

$$(1-\delta_s) \leq min_{S,\theta}\frac{\|A_{\mathcal{S}}\theta\|^2}{\|\theta\|^2}$$

$$\begin{aligned}
(1-\delta_s) &\leq \lambda_{min} \\
\delta_s &\geq 1 - \lambda_{min}
\end{aligned} \tag{5}$$

Rearranging such that δ_S always satisfies the inequalities (4) and (5),

$$\boxed{\delta_s = \text{max}(1-\lambda_{\text{min}}, \lambda_{\text{max}} - 1)}$$

2 Q2

The Correct Statement can be (i) $\delta_s < \delta_t$ or (iii) $\delta_s = \delta_t$

As proved in Question 1,

$$\delta_s = \text{max}(1 - \lambda_{\text{min}}, \lambda_{\text{max}} - 1)$$

where λ_{min} and λ_{max} are the minimum and maximum eigenvalues.

The estimated RIC, δ_t is obtained by considering larger number of subsets than that considered for δ_s

$$\delta_t = \text{max}(1 - {\lambda^t}_{\text{min}}, {\lambda^t}_{\text{max}} - 1)$$

where ${\lambda^t}_{\text{min}}$ and ${\lambda^t}_{\text{max}}$ are the new minimum and maximum eigenvalues.

Since the enumeration goes even further considering all subsets and other additional too, the new maximum and minimum eigenvalues, $(\lambda^t_{\text{min}}, \lambda^t_{\text{max}})$ might be even less and greater respectively.

$$\begin{aligned}
\lambda_{\text{min}} \geq {\lambda^t}_{\text{min}} \quad &, \quad \lambda_{\text{max}} \leq {\lambda^t}_{\text{max}} \\
\implies 1 - \lambda_{\text{min}} \leq 1 - {\lambda^t}_{\text{min}} \quad &, \quad \lambda_{\text{max}} - 1 \leq {\lambda^t}_{\text{max}} - 1 \\
\implies \text{max}(1 - \lambda_{\text{min}}, \lambda_{\text{max}} - 1) &\leq \text{max}(1 - {\lambda^t}_{\text{min}}, {\lambda^t}_{\text{max}} - 1) \\
\implies &\boxed{\delta_s \leq \delta_t}
\end{aligned}$$

3 Q3

The P_0 problem seeks the sparsest solution to $Ax = y$, i.e.,

$$P_0 : \quad \min \|x\|_0 \quad \text{subject to} \quad Ax = y.$$

The P_1 problem is its convex relaxation, replacing ℓ_0-norm minimization with ℓ_1-norm minimization:

$$P_1 : \quad \min \|x\|_1 \quad \text{subject to} \quad Ax = y.$$

For a unique solution to P_1, it is known that $\delta_{2s} < 0.41$ is a sufficient condition.

For the P_0 problem, the uniqueness of the sparse solution is guaranteed if the sensing matrix A satisfies the Restricted Isometry Property (RIP) with:

$$\delta_{2s} < 1.$$

This condition ensures that any $2s$-sparse vector has a well-conditioned restricted sub-matrix, preventing multiple solutions from satisfying $Ax = y$ with the same sparsity level.

- For P_1, a stronger condition is required to ensure that the convex relaxation still leads to the correct sparse solution, hence the stricter bound $\delta_{2s} < 0.41$.
- For P_0, the only requirement is that no two distinct s-sparse vectors can produce the same measurement y. This holds when $\delta_{2s} < 1$.

The corresponding upper bound for δ_{2s} for the uniqueness of the P_0 problem is:

$$\delta_{2s} < 1.$$

This is a much looser condition compared to the requirement for P_1, reflecting the fact that finding the exact sparsest solution is easier when directly solving P_0 (though computationally infeasible in practice).

4 Q4

Introduction

An innovative application of compressed sensing is in the development of **single-pixel cameras**, which offer a cost-effective alternative to traditional imaging systems, especially in scenarios where conventional pixelated detectors are impractical. This concept is detailed in the paper "Single-pixel imaging via compressive sampling" by Marco F. Duarte et al.

Measurement Acquisition

In a single-pixel camera, the scene is illuminated or observed through a sequence of known patterns, typically generated by a spatial light modulator (SLM) such as a digital micromirror device (DMD). For each pattern projected onto or reflected from the scene, a single-pixel detector measures the total light intensity, capturing a single scalar value per pattern. This process is repeated with multiple patterns to acquire a set of measurements.

Underlying Unknown Signal

The unknown signal in this context is the two-dimensional image of the scene, which is assumed to be sparse or compressible in a certain basis (e.g., wavelet or discrete cosine transform). This sparsity implies that the image can be represented with relatively few significant coefficients in the chosen basis, allowing for efficient reconstruction from a limited number of measurements.

Measurement Matrix

The measurement process can be mathematically modeled as:

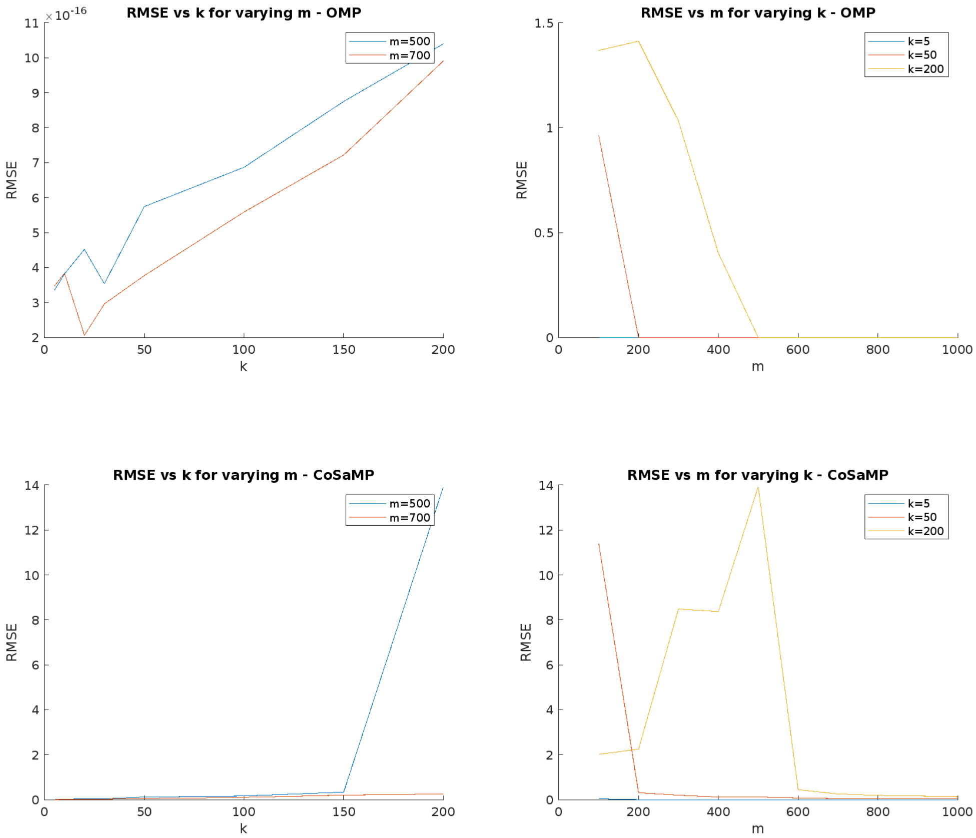
$$y = \Phi x$$

Here, y is the vector of measurements obtained from the single-pixel detector, x represents the vectorized form of the unknown image, and Φ denotes the measurement matrix. Each row of Φ corresponds to a pattern projected by the SLM, with elements indicating the weighting of each pixel in the pattern. Common choices for these patterns include random binary patterns or rows from orthogonal matrices such as the Hadamard matrix, which ensure incoherence with the sparsifying basis and facilitate accurate image reconstruction.

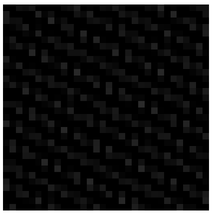
Conclusion

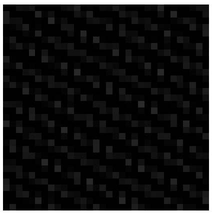
By leveraging compressed sensing principles, single-pixel cameras can reconstruct high-quality images from significantly fewer measurements than traditional cameras, making them particularly useful in applications involving wavelengths where pixelated detectors are expensive or unavailable.

5 Q5



Considering the right 2 plots, where `k` is varying, the RMSE can increase for some *k-value* initially, but decreases as `m` increases in general. On the other hand, if we consider the left 2 plots, where `m` is varying, the RMSE initially can either increase from 0 or decrease from an initial value, but for larger values of `k`, the RMSE increases in general.


original (k=5)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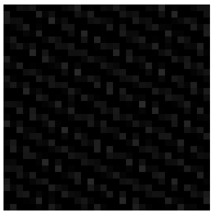
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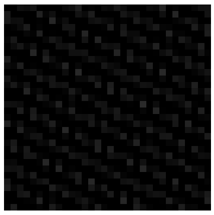
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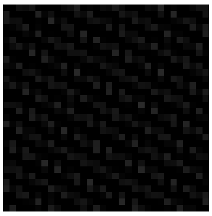
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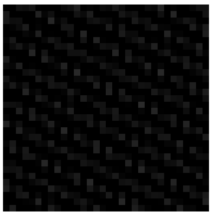
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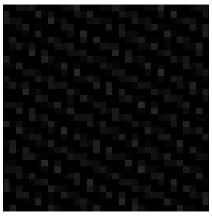
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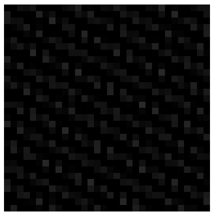
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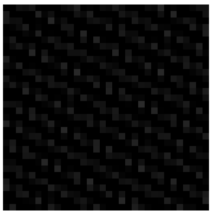
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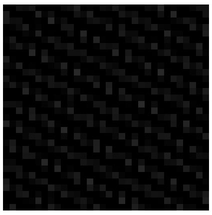
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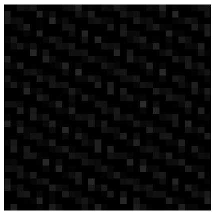
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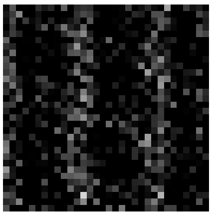
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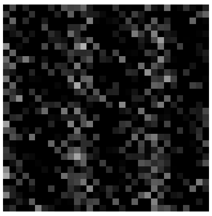
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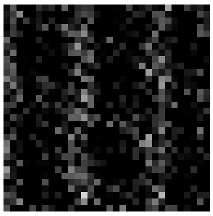
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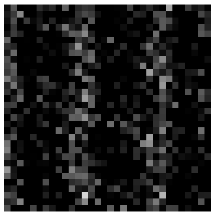
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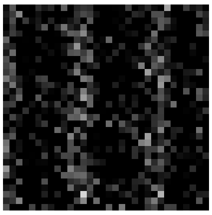
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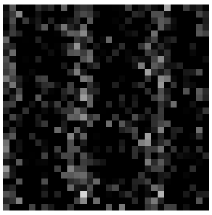
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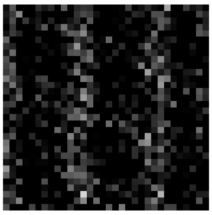
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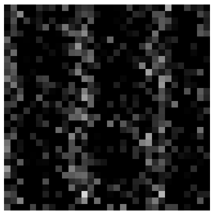
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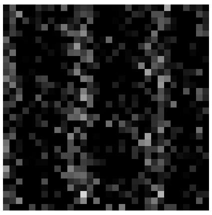
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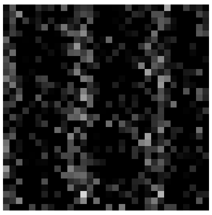
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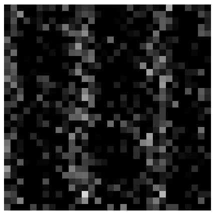
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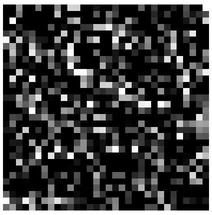
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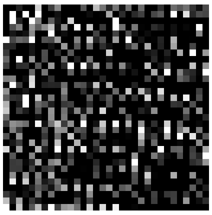
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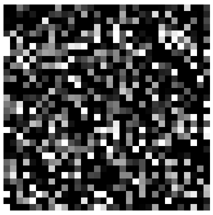
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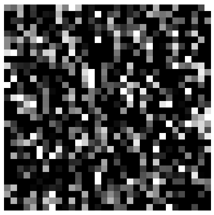
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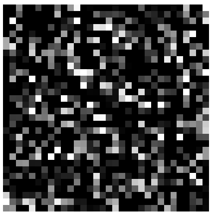
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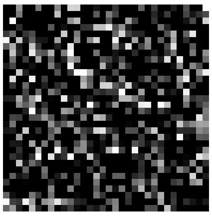
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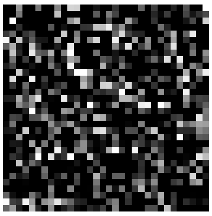
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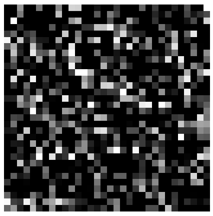
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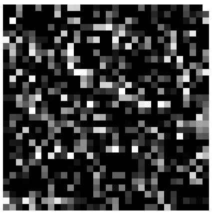
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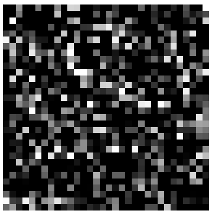
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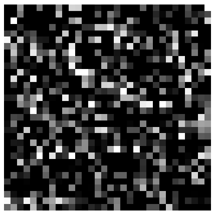
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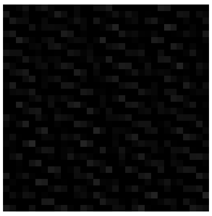
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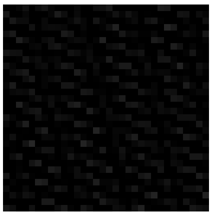
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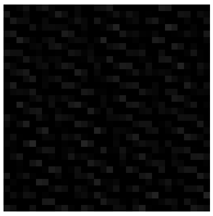
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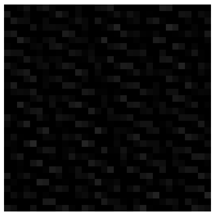
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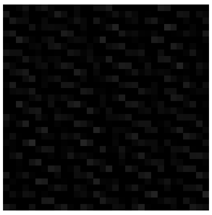
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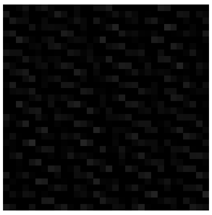
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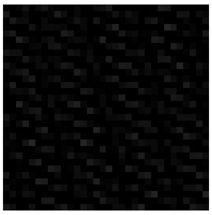
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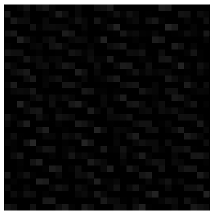
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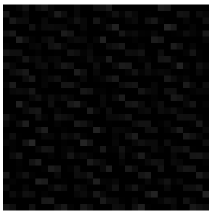
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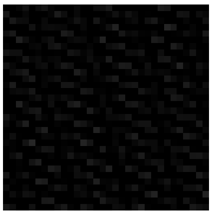
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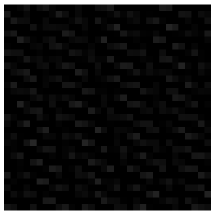
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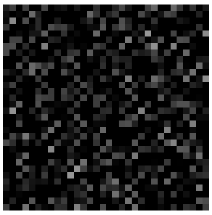
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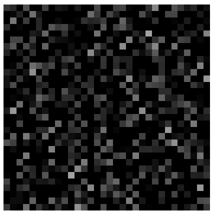
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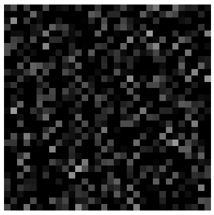
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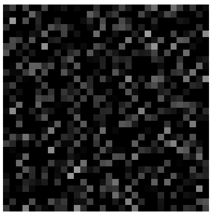
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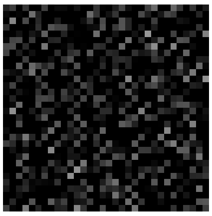
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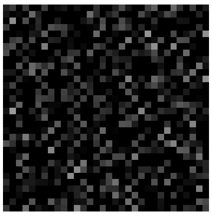
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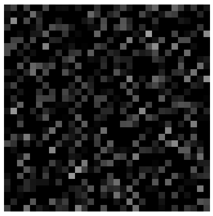
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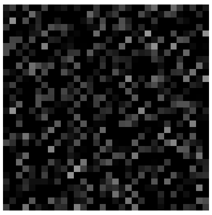
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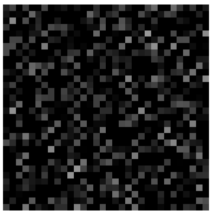
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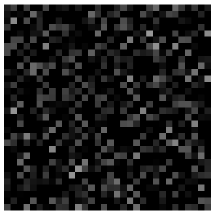
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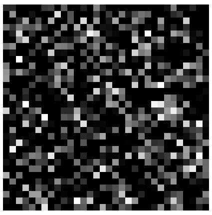
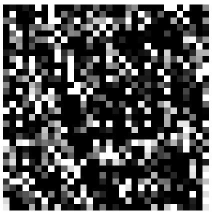

original (k=200)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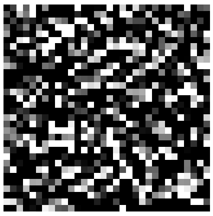
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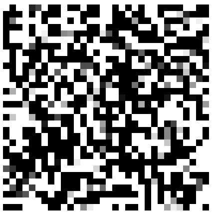
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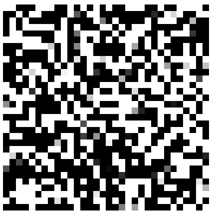
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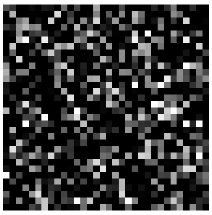
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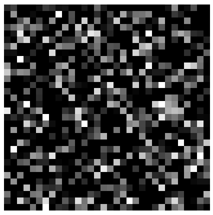
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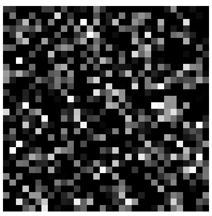
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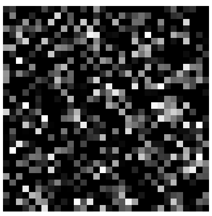
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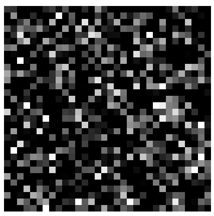
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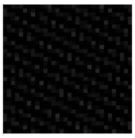
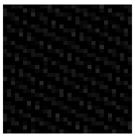

original (k=5)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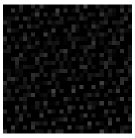
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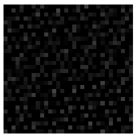
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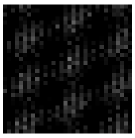
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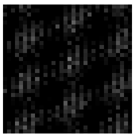
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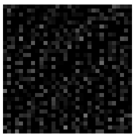
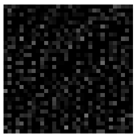

original (k=30)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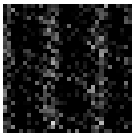
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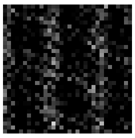
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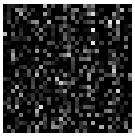
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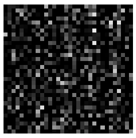
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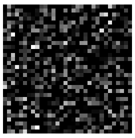
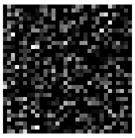

original (k=150)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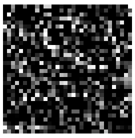
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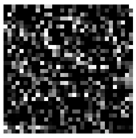
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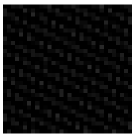
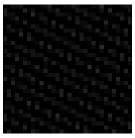

original (k=5)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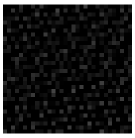
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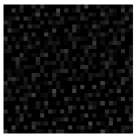
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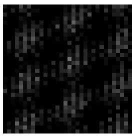
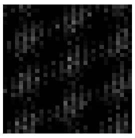

original (k=20)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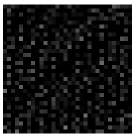
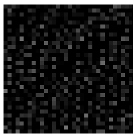

original (k=30)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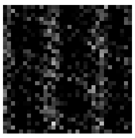
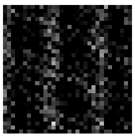

original (k=50)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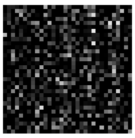
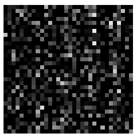

original (k=100)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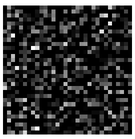
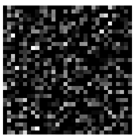

original (k=150)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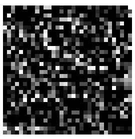
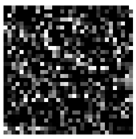

original (k=200)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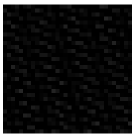
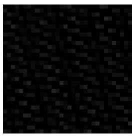

original (k=5)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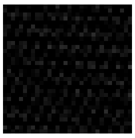
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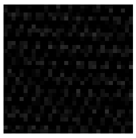
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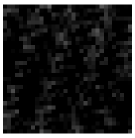
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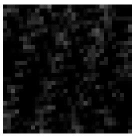
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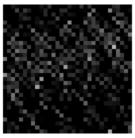
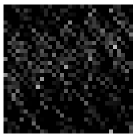

original (k=30)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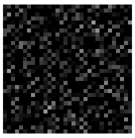
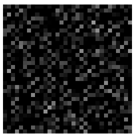

original (k=50)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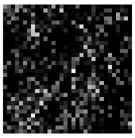
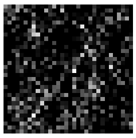

original (k=100)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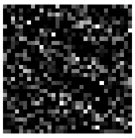
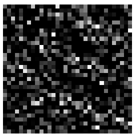

original (k=150)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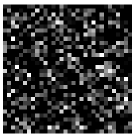
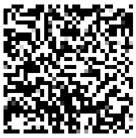
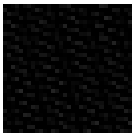
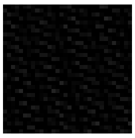
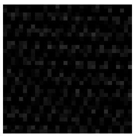
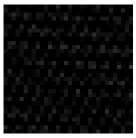

original (k=200)


CoSAMP (k=200,m=500)


original (k=5)


CoSAMP (k=5,m=700)


original (k=10)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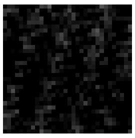
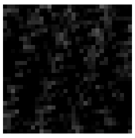

original (k=20)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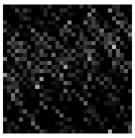
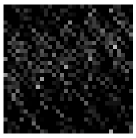

original (k=30)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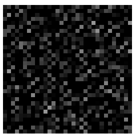
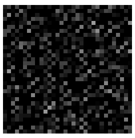
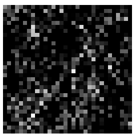
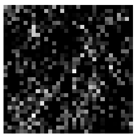
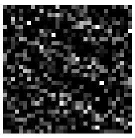
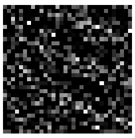

original (k=50)

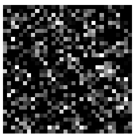

CoSAMP (k=50,m=700)

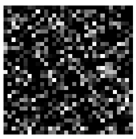

original (k=100)


CoSAMP (k=100,m=700)


original (k=150)


CoSAMP (k=150,m=700)


original (k=200)


CoSAMP (k=200,m=700)

6 Q6

(a) Probability that the intersection of Ω with F is a null set

We are given:

- n: Total number of elements in the signal g.
- F: Support set of g in the Fourier domain, with $|F|$ elements.
- Ω: Set of frequencies at which the Fourier transform of g is measured, with $|\Omega|$ elements.

We want to compute the probability that Ω and F do not intersect, i.e., $\Omega \cap F = \emptyset$.

The total number of ways to choose $|\Omega|$ frequencies out of n is:

$$\binom{n}{|\Omega|}.$$

If Ω must not intersect F, then all $|\Omega|$ frequencies must be chosen from the $n - |F|$ frequencies that are **not** in F. The number of ways to do this is:

$$\binom{n-|F|}{|\Omega|}.$$

So,

$$P(\Omega \cap F = \emptyset) = \frac{\binom{n-|F|}{|\Omega|}}{\binom{n}{|\Omega|}}.$$

(b)

I am not getting a lower bound in this question, but rather an upper bound. So i am writing the solution for that

We can write

$$\frac{\binom{n-|F|}{|\Omega|}}{\binom{n}{|\Omega|}} = \frac{(n-|\Omega|)(n-|\Omega|-1)\cdots(n-|\Omega|-|F|+1)}{n(n-1)\cdots(n-|F|+1)}$$

$$= \prod_{i=0}^{|F|-1} \frac{n-|\Omega|-i}{n-i}$$

$$= \prod_{i=0}^{|F|-1} 1 - \frac{|\Omega|}{n-i}$$

And we know that $n - i \leq n$ for $i \geq 0$. So,

$$1 - \frac{|\Omega|}{n-i} \leq 1 - \frac{|\Omega|}{n}$$

So we finally got

$$\prod_{i=0}^{|F|-1} \frac{n-|\Omega|-i}{n-i} \leq \left(1 - \frac{|\Omega|}{n}\right)^{|F|}$$

(c) Deriving a lower bound on $|\Omega|$

We are given:

- The probability of failure (i.e., $\Omega \cap F = \emptyset$) must be at most n^{-M}, where $M > 0$.
- $|\Omega| \ll n$, so we can use the approximation $\log(1-x) \approx -x$ for small x.

$$\left(1 - \frac{|\Omega|}{n}\right)^{|F|} \leq n^{-M}.$$

Taking the natural logarithm of both sides:

$$|F| \cdot \log\left(1 - \frac{|\Omega|}{n}\right) \leq -M \log n.$$

Use the approximation $\log(1-x) \approx -x$

Since $|\Omega| \ll n$, we can approximate:

$$\log\left(1 - \frac{|\Omega|}{n}\right) \approx -\frac{|\Omega|}{n}.$$

Substituting this into the inequality:

$$|F| \cdot \left(-\frac{|\Omega|}{n}\right) \leq -M \log n.$$

Multiply both sides by -1 (and reverse the inequality):

$$|F| \cdot \frac{|\Omega|}{n} \geq M \log n.$$

$$|\Omega| \geq \frac{Mn \log n}{|F|}.$$