Practical Homotopy Colimits

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May 2, 2022

Intro & Outline

Hello everyone! (Thanks) My name is Rushil, & in the next 15 minutes, I'd like to tell a story about topology, category theory, & algebra, hopefully in a way that you might find useful.

Outline:

① What is a colimit?

② What problem does hocolim solve?

③ What is hocolim? Why is this familiar?

④ How to compute? Examples

In the interest of time, I'll have everything take place in Top, a convenient cat. of spaces, but this theory is all far more general, & in some cases more complicated.

What is a colimit?

Def Given a diagram $X: I \to Top$, a colimit of X is an obj. $\mathrm{colim}\, X \in Top$ & a universal cone under X:

$$\begin{tikzcd} X_i \arrow[rr] \arrow[dr] \arrow[ddr, bend right] & & X_j \arrow[dl] \arrow[ddl, bend left] \\ & \mathrm{colim}\, X \arrow[d, dashed, "\exists!"] & \\ & Y & \end{tikzcd}$$

I-Diagram

mention I is "shape"

Ex. Coproducts: $\bullet \quad \bullet$ $\quad X \amalg Y$

Initial obj.: $\emptyset$ $\quad \emptyset$

Pushout: $\bullet \leftarrow \bullet \rightarrow \bullet$

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & X \arrow[d] \\ Y \arrow[r] & X \amalg_A Y = X \amalg Y / fa \sim ga \end{tikzcd}$$

Outline:

- Def / Ex. of colimits in Top
- Problem: failure to preserve htpy equivalence.
- (Acknowledge omissions)
- Solution: homotopy colimits!
  - Def 1: Universal property
  - Def 2: Bar construction
- Examples:

  $\bullet / G = BG$

  htpy pushouts
- Computing via cofibrant replacement
- Examples:

  suspension, cofiber, cylinder,

  telescope, etc.

$$B(\mathcal{D}, \mathcal{D}, Q-) : \mathcal{M}^{\mathcal{D}} \to \mathcal{M}^{\mathcal{D}}$$

$$\bullet \otimes_I F = \int^I \bullet \otimes F_x = \mathrm{colim}\, F$$

so, colim sends WEs btwn diagrams of form $B(\mathcal{D}, \mathcal{D}, QF)$ to WEs

$$\mathrm{hocolim}_{\mathcal{D}} F = \mathrm{colim}\, B(\mathcal{D}, \mathcal{D}, QF)$$
$$= \bullet \otimes_{\mathcal{D}} B(\mathcal{D}, \mathcal{D}, QF)$$
$$= \int^{\mathcal{D}} \int^{\Delta^{op}} -$$
$$= \int^{\Delta^{op}} \int^{\mathcal{D}} -$$
$$= B(\bullet \otimes_{\mathcal{D}} \mathcal{D}_\bullet, \mathcal{D}, QF)$$
$$= B(\bullet, \mathcal{D}, QF)$$

In Top, fine w/ $B(\bullet, \mathcal{D}, F)$

Direct limit: $\bullet \to \bullet \to \bullet \to \cdots$

$$\begin{tikzcd} X_0 \arrow[r, "f_0"] \arrow[drr] & X_1 \arrow[r, "f_1"] \arrow[dr] & X_2 \arrow[r, "f_2"] \arrow[d] & \cdots \\ & & \operatorname{colim} X & \end{tikzcd}$$

$= \bigsqcup X_i / \sim$

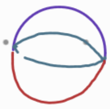
What problem does hocolim solve?

Working in Top, we'd want to do homotopy theory, but colim is badly behaved w/ weak equivalences:

Ex. $$\begin{tikzcd} D^n \arrow[d, "\simeq"] & S^{n-1} \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\simeq"] & D^n \arrow[d, "\simeq"] \\ \bullet & S^{n-1} \arrow[l] \arrow[r] & \bullet \end{tikzcd}$$ is commutative, w/ vertical arrows weak equivalences.

However, they have diff colimits:

$$\begin{tikzcd} S^{n-1} \arrow[r, hook] \arrow[d, hook] & D^n \arrow[d] \\ D^n \arrow[r] & S^n \end{tikzcd}$$



$S^n \not\simeq pt$

But $$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$ gives a point!

This is sad; thus, we want a homotopy-invariant version of colim, i.e. a homotopy colimit!

Instead of "gluing", we want to "glue up to homotopy" (this will be made precise)

What is a homotopy colimit?

For the rest, I'll define a homotopy colimit, give a powerful general device for computing them, explain the merits & analogues of this definition, & finally give formulas for computing them.

Def Given a diagram I, colim is a functor $\mathrm{Top}^I \to \mathrm{Top}$. hocolim is the "left derived functor" of colim, in the following sense:

Right Kan extension!

$$\begin{tikzcd} \mathrm{Top}^I \arrow[r, "\mathrm{colim}"] \arrow[d, "\gamma"] \arrow[dr, "\mathrm{hocolim}"] & \mathrm{Top} \arrow[d, "\gamma"] \\ \mathrm{Ho}(\mathrm{Top}^I) \arrow[r] & \mathrm{Ho}(\mathrm{Top}) \end{tikzcd}$$

So hocolim is "most general approx to colim"

Constr.

Given $X : I \to \mathrm{Top}$, we can form the

Simplicial bar construction:

$$B_n(\bullet, I, X) = \coprod_{i_0 \to \cdots \to i_n} X(i_0)$$

The geometric realization of this gives a model for the homotopy colimit:

$$\mathrm{hocolim}_I X \simeq B(\bullet, I, X)$$

Note, this also tells us there is a natural map $\mathrm{hocolim}_I X \to \mathrm{colim}_I X$

Dig. Derived functors & homotopical algebra:

This should be familiar if you know about derived functors from homological algebra.

$$\mathrm{Tor}_i^R(A,B) = H_i\left(A \overset{L}{\underset{R}{\otimes}} B\right) = L_i(A \otimes B)$$

$F_\bullet \to A$, $\quad \cdots \to F_1 \to F_0 \to A \to 0$

$- \otimes_R B$ $\quad \cdots \to F_1 \otimes B \to F_0 \otimes B \to A \otimes B \to 0$

$H_\bullet(-)$ $\quad \cdots \mathrm{Tor}_1(A,B) \quad \mathrm{Tor}_0(A,B) = A \otimes B$

① Derived, ie. it's the most general approx to $- \otimes_R -$ that is "exact," ie. preserves quasi-isomorphisms

② Def. requires picking a "flat res", but which one?

In genl, any k-alg R admits the "bar resolution"

$$\cdots \to R \otimes_k R \otimes_k R \to R \otimes_k R \to R \xrightarrow{\varepsilon} k \to 0$$

which resolves k by free R-modules;

This lets us resolve any R-module M by tensoring $- \otimes_k M$;

$$\beta(R,M)_n = R^{\otimes(n+1)} \otimes_k M, \quad \beta(R,M) \to M$$

So, we could define $\mathrm{Tor}_i^R(A,B) = H_i(\beta(R,A) \otimes_R B)$.

③ Practicality: this is hard to work with, & in nice situations, we have much better

colim $B(G,D,F)$

$$= \mathrm{coeq}\left(\coprod_{d_0 \to d_1} G d_1 \otimes F d_0 \rightrightarrows \coprod \ldots\right) \text{ c.f. } 8.3.8$$

$$= G \otimes_D F$$

so $\Delta^\circ \Rightarrow \bullet$

applied to $- \otimes_{\Delta^{op}} B_\bullet(G,D,F)$

gives $B(G,D,F) \to \mathrm{colim}\, B_\bullet(G,D,F)$

Exact at level of derived category;

send triangles to triangles!

$$0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0$$

$x \mapsto 2x$

$x \mapsto x + 2\mathbb{Z}$

apply $- \otimes \mathbb{Z}/2$

$$\mathbb{Z}/2 \xrightarrow{0} \mathbb{Z}/2 \to \mathbb{Z}/2 \to 0$$

$x + 2\mathbb{Z} \mapsto 2x + 2\mathbb{Z}$

$x + 2\mathbb{Z} \to x + 2\mathbb{Z}$

$$d(r_0 \otimes r_1 \otimes \cdots \otimes m)$$

$$= r_0 r_1 | r_2 | \cdots | m$$

$$+ \sum_{i=1}^{n-1} (-1)^i r_0 | \cdots | r_i r_{i+1} | \cdots | m$$

$$+ (-1)^n r_0 | \cdots | r_n m$$

resolutions ... we, ... give the same

answer!

This analogy goes far deeper than I've outlined, but the punchline is that you don't need to fear the above unwieldy constructions; there are shortcuts & formulas!

How to Compute:

Thrm If I is "sufficiently nice," i.e. admitting proj. cofib. replacement, & $X : I \to Top$ is a proj. cofib. diagram, then $\operatorname{hocolim} X \xrightarrow{\sim} \operatorname{colim} X$.

In some cases, if I is a Reedy category, we have explicit cofibrant replacements:

Def A cofibration is a map which is a closed inclusion / neighborhood deformation retract

e.g. retract of rel. CW cplxs

A cofibrant obj. is one s.t. $\emptyset \to X$ is cofib; e.g. CW cplxs.

Ex. Nice diagrams for pushouts:

①
$$\begin{tikzcd} A_0 \arrow[r, "f"] \arrow[d, "g"'] & A_1 \\ A_2 & \end{tikzcd}$$

A_0 cofib, f, g cofib

OR A_0, A_1 cofib, & g cofibration

Back to $S^{n-1} \to \bullet$, $\bullet$; S^{n-1} cw / replace map w/ CW inclusion, e.g. w/ cone

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & CS^{n-1} \arrow[d] \\ CS^{n-1} \arrow[r] & \Sigma S^{n-1} = S^n \end{tikzcd}$$

Yay!

Ex. Colim of $X \xrightarrow{f} Y$ is Y, but hocolim is mapping cylinder, $X \times I \sqcup Y /_{(x,1) \sim fx} = M_f$

(literal cofib replacement)

Ex. $A \xrightarrow{f} X$ Double mapping cylinder.

Derived categories

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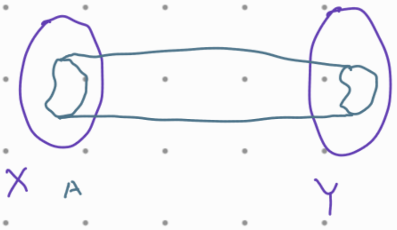
$$\begin{tikzcd} \arrow[d, "g"'] & \arrow[d] \\ Y \arrow[r] & M_{f,g} \end{tikzcd}$$

$$M_{f,g} = X \sqcup A \times I \sqcup Y \Big/ \begin{smallmatrix} fa \sim (a,0) \\ (a,1) \sim ga \end{smallmatrix}$$



$\therefore$
$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d] & X \\ \bullet & \end{tikzcd} \rightsquigarrow \begin{tikzcd} A \arrow[r, "f"] \arrow[d] & X \arrow[d] \\ CA \arrow[r] & X \cup CA = C_f \end{tikzcd}$$ "homotopy cofiber"

Ex. $\bullet/G \simeq BG$, classifying space of principal G-bundles

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & EG \arrow[d] \\ Z \arrow[r] & BG \end{tikzcd}$$

Ex. $X_1 \to X_2 \to \cdots$

$\therefore$ mapping telescope

Thats all I have time for, but I hope you have two takeaways: ① hocolim are a not-unfamiliar solution to a familiar problem, & ② lots of things are hocolims in disguise. Thanks & questions!

Remark, in Top, don't need cofib replacement! (cat level pointwise)

Natalie comments:

- Too long... can't talk faster
- Cut derived tensor products
- Rephrase technical def.
- Start w/ outline is good – can cut
- Too little redundancy
- Say existence & uniqueness for colimits
- Get rid of bar const.