BOUSFIELD LOCALIZATIONS BEFORE LUNCH

RUSHIL MALLARAPU

Contents

Localization is ubiquitous as a way to solve big problems by reducing them to specific ones, and as we do homotopy theory, working one "prime" at a time will become natural and necessary to get things done. The goal is to introduce Bousfield localization of spectra, generalizing localizations and completions from classical algebra. I'll discuss some properties and useful results, go through examples of localizing/completing at primes, and touch on the chromatic picture of stable homotopy theory.

This is the second of two talks I gave at Kan Seminar in Fall 2022, with Tomer Schlank. All mistakes in these notes are my own. Please reach out to me if you spot anything!

1. Localizations of Spectra

To start, all of the following material can be set up in a *far* more general context. Bousfield's original papers – building on Quillen's homotopical algebra – specifically work with categories of spaces (vis-á-vis simplicial sets) and spectra. These days, what is normally called *Bousfield localization* refers to a model-categorical construction which adds more weak equivalences without changing cofibrant objects, and as such, perhaps the morally right way to view this theory is via (reflective) localizations of ∞-categories [Lur09, 5.2.7]. In any case, this talk will be exceedingly specific to spectra, so much of what follows only generalizes to presentable stable symmetric monoidal ∞-categories.[1] A good place to read about these notions is [Law20].

The goal of this section is to get our feet wet with definitions, universal properties, and constructions with general localizations. Let $h\mathbf{Sp}$ denote the homotopy category of spectra – pick whichever model of this suits your fancy, CW spectra if you must.

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[1]So all your favorite examples, hopefully.

Definition 1.1. Let $E \in h\mathbf{Sp}$ be a spectra, and E_* the associated homology theory.[2]

(a) A map $f\colon A \to B$ in $h\mathbf{Sp}$ is an *E_*-equivalence* if E_*f is an isomorphism.

(b) A spectrum $A \in h\mathbf{Sp}$ is *E_*-acyclic* if $E_*A \simeq 0$.

A spectrum $X \in h\mathbf{Sp}$ is *E_*-local* if one of the two equivalent conditions are satisfied:

(1) Every E_*-equivalence $f\colon A \to B$ induces a bijection $f^*\colon [B, X] \to [A, X]$, or

(2) For every E_*-acyclic A, we have $[A, X]_* = 0$.

The following presentation is roughly adapted from [Rav84, §1]. In particular, *functoriality* of localization isn't immediate in full generality, so we'll make some concessions.

Definition 1.2. E_*-localization is a functor $L_E\colon h\mathbf{Sp} \to h\mathbf{Sp}$ with image in E_*-local spectra together with a natural E_*-equivalence $\eta\colon \mathrm{id} \Longrightarrow L_E$.

This implies $\eta_X\colon X \to L_E X$ is both the terminal E_*-equivalence from X and the initial E_*-local map under X; if $f\colon X \to B$ is an E_*-equivalence or C is E_*-local, then we have

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "\eta_X"'] & B \arrow[d, "\exists!\,\tilde{f}"] \\ & L_E X \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "g"'] & L_E X \arrow[d, "\exists!\,\tilde{g}"] \\ & C \end{tikzcd}$$

As a formal consequence, any E_*-equivalence $X \to Y$ with Y E_*-local is canonically equivalent to η_X, so localization is unique. This means that if $(\tilde{L}_E, \tilde{\eta})$ was some other functor satisfying our properties, then $L_E \simeq \tilde{L}_E$, compatibly with the localization maps. This argument is nearly identical to the E_*-Whitehead theorem below. Also, L_E is idempotent, in that L_E is equivalent to $L_E L_E$. This means E_*-localizing something E_*-local does nothing.

Theorem 1.3. *E_*-localization exists for every generalized homology theory.*

The proof in Bousfield's paper is somewhat set-theoretic and goes through the *small object argument* [Bou79, 1.12-1.15]. To make a very long story short, it uses the fact that $h\mathbf{Sp}$ is generated under colimits of spheres to reduce statements about localizing with respect to a proper class (the class of E-equivalences) down to localizing with respect to an honest set. This kind of argument is pretty common once you go looking for it.

What follows are some of the nice properties of Bousfield localization. I won't prove most of them; [Bou79] or [Law20] have most of the proofs, and these are largely formal arguments you should do in private.

Theorem 1.4. *(E_*-Whitehead).*
If $f\colon X \to Y$ is an E_*-equivalence of E_*-local spectra, then $f\colon X \simeq Y$.

[2] That is, $E_*X := \pi_* E \otimes X$ for $X \in \mathbf{Sp}$, and $E_*X := E_*\Sigma^\infty_+ X$ for $X \in \mathbf{Spc}$.

Proof. This condition implies that $f^*\colon [Y,X] \to [X,X]$ is a bijection, so we get a unique map $g\colon Y \to X$ such that $gf = \mathrm{id}_X$. Now, using that $f^*\colon [Y,Y] \to [X,Y]$ is also a bijection, observe that fg and id_Y both map to $fgf = f = f^* \mathrm{id}_Y$, which tells us that $fg = \mathrm{id}_Y$. Thus, f is an equivalence as claimed. ◻

Theorem 1.5. *E_*-local spectra are closed under cofibers, products, retracts, and limits.*

(By limit I mean inverse homotopy limit, but I can't stand this "inverse" v. "direct" terminology and if someone tries to take limits in the homotopy category directly, they should be reported to, well, someone probably.)

Theorem 1.6. *E_*-localization commutes with direct sums, suspension, and cofibers; i.e., if*

$$W \to X \to Y \to \Sigma W$$

is a cofiber sequence, then so is

$$L_E W \to L_E X \to L_E Y \to \Sigma(L_E W).$$

Theorem 1.7. *If E is a ring spectrum and X is an E-module, then X is E_*-local.*

Proof. If A is E_*-acyclic and $f\colon A \to X$ is any map, we can factor it as

$$A \xrightarrow{\eta\otimes 1} E \otimes A \xrightarrow{1\otimes f} E \otimes X \xrightarrow{\mu} X,$$

and $E \otimes A \simeq 0$, so f is null. Thus, $[A,X] \simeq 0$. ◻

2. Bousfield Classes

We want to adopt the perspective that E_*-localization is "restricting our gaze to everything that E_* sees." In that case, we should be able to (1) determine when two homology theories "see" the same part of $h\mathbf{Sp}$, and (2) compare how much any two homology theories "see." Bousfield classes are a useful language for this, allowing for concise statements of many foundational results in chromatic homotopy theory.

Definition 2.1. Define an equivalence relation on $h\mathbf{Sp}$ via $E \sim G$ if for any $X \in h\mathbf{Sp}$, $E_*X = 0 \iff G_*X = 0$. Let $\langle E\rangle$ denote the equivalence class of E under this relation.

We can order these equivalence classes as follows: let $\langle E\rangle \le \langle G\rangle$ if every G_*-acyclic spectrum is E_*-acyclic. In addition, we can define

$$\langle E\rangle \wedge \langle F\rangle = \langle E \otimes F\rangle \quad \text{and} \quad \langle E\rangle \vee \langle F\rangle = \langle E \oplus F\rangle .$$

You should check that these are well-defined; it's a good exercise in the notions.

Note that this implies $\langle E\rangle \wedge \langle F\rangle \le \langle E\rangle \le \langle E\rangle \vee \langle F\rangle$, justifying this meet/join notation.

This is a key consequence of these definitions:

Proposition 2.1. *The following are equivalent:*

(i) $\langle E\rangle \leq \langle G\rangle$*, i.e. there are more* E_**-acyclics than* G_**-acyclics,*

(ii) Every G_**-equivalence is an* E_**-equivalence,*

(iii) Every E_**-local spectrum is* G_**-local.*

Thus, if $\langle E\rangle = \langle G\rangle$, then $L_E \simeq L_G$, and if $\langle E\rangle \leq \langle G\rangle$, $L_E L_G \simeq L_E$, and we get a natural transformation $L_G \to L_E$. To clarify why, consider the diagonal map in the diagram below:

$$\begin{tikzcd} X \arrow[r, "\eta_X^G"] \arrow[dr] & L_G X \arrow[d, "\eta^E_{L_G X}"] \\ & L_E L_G X \end{tikzcd}$$

By property (ii), the top map, a G_*-equivalence is an E_*-equivalence, so the diagonal map is an E_*-equivalence. In addition, $L_E L_G X$ is definitely E_*-local, so this diagonal map is an E_*-localization, giving the equivalence $L_E L_G = L_E$. Of course, $L_G L_E = L_E$, and in the case that $\langle E\rangle = \langle G\rangle$, i.e. when E_*-locality is the same as G_*-locality, this tells us that $L_E = L_E L_G = L_G L_E = L_G$. Finally, our natural transformation $L_G \to L_E$ for $\langle E\rangle \leq \langle G\rangle$ is a consequence of either applying L_G to $\mathrm{id} \Longrightarrow L_E$, or realizing that if L_E is G_*-local, it admits a map from the "universal" G_*-localization, L_G.

Heuristically, the way to read a statement like $\langle E\rangle \leq \langle G\rangle$ is that G "sees" finer detail than E in $h\mathbf{Sp}$. One quick north star here is the fact that $\langle 0\rangle \leq \langle X\rangle \leq \langle S^0\rangle$ for all $X \in h\mathbf{Sp}$; i.e. the sphere sees everything, and the point sees nothing. Thus, the equivalence $L_E L_G = L_E$ tells us that if we want to reduce to what G detects and then what E detects, we can simply drop down to E to start. Similarly, the map $L_G \to L_E$ realizes the fact that if we have something G_*-local, then E_*-localizing is just forgetting *more* information.

It's worth pointing out that the *lattice* of Bousfield classes which satisfy $\langle E\rangle \wedge \langle E\rangle = \langle E\rangle$ and have complements is a strict subcollection of all Bousfield classes. These classes have the potential to behave in weird and unexpected ways, as illustrated in [Rav84]. That paper also has a ton of examples of how this notion is used, specifically in the chromatic world, although I'd claim that what most millenials[3] think when they hear Morava E-theory and K-theory feels quite distant to what appears in Ravenel's paper.

Finally, some nice properties of Bousfield classes:

Proposition 2.2. *If* $W \to X \to Y \to \Sigma W$ *is a cofiber sequence, then each of* $\langle W\rangle, \langle X\rangle, \langle Y\rangle$ *is less than the wedge of the other two.*

Proof. Note that $\langle \Sigma^n X\rangle = \langle X\rangle$ for all $n \in \mathbf{Z}$; this is because the sphere is invertible (heck, $\mathrm{Pic}(\mathbf{Sp}) = \mathbf{Z}$ for precisely this reason). Then, all you need is the five-lemma. □

[3]Read: people who think in terms of stacks, which I should do, as should you.

Proposition 2.3. *If M is a module over a ring E, then $\langle M \rangle \leq \langle E \rangle$.*

Proof. The unit and multiplication on M make it a retract of $E \otimes M$. Thus, $\langle M \rangle \leq \langle E \otimes M \rangle$, and the latter is less than $\langle E \rangle$. □

Proposition 2.4. [Rav84, 1.34], [Dev18, 5.8]
Let X be a spectrum and $f : \Sigma^d X \to X$ a self map. Then $\langle X \rangle = \langle f^{-1} X \rangle \vee \langle C(f) \rangle$.

3. Arithmetic Localization and Completion

Time for some examples! The key ones to be aware of are p-localization, p-completion, and rationalization (here p is some prime). Let's dive right in:

Definition 3.1. For some abelian group G, the *Moore spectrum SG* is the connective spectrum with $H_0 SG = \pi_0 SG = G$ and $H_i SG = 0$ for $i \neq 0$.

There are also short exact sequences

$$0 \longrightarrow G \otimes \pi_* X \longrightarrow \pi_*(SG \otimes X) \longrightarrow \mathrm{Tor}(G, \pi_{*-1} X) \longrightarrow 0$$

$$0 \longrightarrow \mathrm{Ext}(G, \pi_{*+1} X) \longrightarrow [SG, X]_* \longrightarrow \mathrm{Hom}(G, \pi_* X) \longrightarrow 0$$

which you can think of as analogues of the universal coefficient sequences. They're proved exactly as you expect – take a free resolution of G, tensor with S^0, and take homotopy.

Two of these three cases are pretty easy:

Proposition 3.1. [Bou79, 2.4]
Let $G = \mathbf{Z}_{(J)}$ for some set J of primes. Then $X_{(J)} := L_{S\mathbf{Z}_{(J)}} X = S\mathbf{Z}_{(J)} \otimes X$ and $\pi_* X_{(J)} \cong \mathbf{Z}_{(J)} \otimes \pi_* X$ for every $X \in h\mathbf{Sp}$.

Proof. I actually think [BR20, 7.4.10] is a much clearer exposition. The key idea is that the coefficient sequence gives an iso $\mathbf{Z}_{(J)} \otimes \pi_* X \to \pi_*(S\mathbf{Z}_{(J)} \otimes X)$, which allows us to describe $S\mathbf{Z}_{(J)}$ equivalences as those which are isomorphisms on (J)-local homotopy. Then, because we have a map $X \to S\mathbf{Z}_{(J)} \otimes X$ coming from the ring structure on $S\mathbf{Z}_{(J)}$, over which $S\mathbf{Z}_{(J)} \otimes X$ is a module, we know that this map is a localization. □

In particular, when $J = \varnothing$, this is rationalization; $X_{\mathbf{Q}} = S\mathbf{Q} \otimes X$, and $\pi_* X_{\mathbf{Q}}$ are rational vector spaces. In fact, by Serre's finiteness theorem, we know $S_{\mathbf{Q}} \simeq \mathbf{Q}$ (via a map factoring the rational Hurewicz map if you like), so the rational stable homotopy category is really the derived category of $\mathbf{Q}$-vector spaces!

We usually call $X_{(p)} = X_{(\{p\})}$ the *p-localization* of X, for obvious reasons.

Bousfield showed that the only other real example of localizing with respect to some Moore spectrum comes from localizing with respect to $S^0/p := S\mathbf{Z}/p$. This is somewhat more involved.

Proposition 3.2. [BR20, 7.4.13]
The p-completion of X is

$$X_p^\wedge := L_{S^0/p}X = F(\Sigma^{-1}S\mathbf{Z}/p^\infty, X) \cong X \otimes D\Sigma^{-1}S\mathbf{Z}/p^\infty.$$

Here, $\mathbf{Z}/p^\infty = \mathbf{Z}\left[\frac{1}{p}\right]/\mathbf{Z} = \operatorname{colim}(\mathbf{Z}/p \to \mathbf{Z}/p^2 \to \cdots)$ is the *Prüfer p-group*. The function spectrum is given by

$$F(\Sigma^{-1}S\mathbf{Z}/p^\infty, X) = \operatorname{holim}\left(\cdots \to S^0/p^3 \otimes X \to S^0/p^2 \otimes X \to S^0/p \otimes X\right),$$

where the maps are induced by the projections $\mathbf{Z}/p^n \to \mathbf{Z}/p^{n-1}$. One way to see the statement with duals is by looking at the cofiber sequence

$$\Sigma^{-1}S\mathbf{Z}/p^\infty \to S^0 \to S\mathbf{Z}\left[p^{-1}\right] \to S\mathbf{Z}/p^\infty,$$

dualizing, and tensoring with X, which gives

$$X \otimes DS\mathbf{Z}[p^{-1}] \to X \to X \otimes D\Sigma^{-1}S\mathbf{Z}/p^\infty \to \Sigma X \otimes DS\mathbf{Z}[p^{-1}].$$

Then $\pi_*(X \otimes DS\mathbf{Z}[p^{-1}]) \cong \pi_*X \otimes \mathbf{Z}[p^{-1}]$ is S^0/p-acyclic, as p is a unit, so the middle map is a S^0/p-localization. I'm leaving out a lot of details here because this is rather technical, and people usually don't think of computing p-completions in exactly this manner. However, a useful result is the following:

Proposition 3.3. [Bou79, 2.5]
There is a split short exact sequence for any $X \in h\mathbf{Sp}$

$$0 \to \operatorname{Ext}(\mathbf{Z}/p^\infty, \pi_*X) \to \pi_*X_p^\wedge \to \operatorname{Hom}(\mathbf{Z}/p^\infty, \pi_{*-1}X) \to 0.$$

Thus, if π_*X is finitely generated in each degree, then $\pi_*X_p^\wedge \cong \mathbf{Z}_p^\wedge \otimes \pi_*X$.

This follows by recognizing that $\pi_*X_p^\wedge = [S^0, F(\Sigma^{-1}S\mathbf{Z}/p^\infty, X)]_* \cong [\Sigma^{-1}S\mathbf{Z}/p^\infty, X]_*$, and applying the cohomological coefficient sequence. When π_*X is f.g. the Hom term vanishes, and some homological algebra recognizes the Ext group as $\pi_* \otimes \mathbf{Z}_p^\wedge$.

An important remark: if X is *connective*, then $L_{HG}X = L_{SG}X$. This is why we can identify $L_{\mathbf{F}_p}S^0$ with $(S^0)_p^\vee$, which is of some importance for computations with the Adams spectral sequence![4] However, for X nonconnective, this fails dramatically. In general, $\langle HG\rangle \leq \langle SG\rangle$, so we get a map $L_{SG} \to L_{HG}$. However, if we let $X = K(1)$, then $K(1)_p^\vee = K(1)$, but $K(1)$ is $\mathbf{F}_p$-acyclic!

[4]There's a lot more to say here – in particular, there's a notion of E-nilpotent completion which one gets by taking inverse limits of Adams towers, and a key contribution of Bousfield was working out the conditions under which the E-Adams spectral sequence converged to an E-localization. This is also related to why you can use the cobar complex to compute localizations, c.f. [MNN17, 2.23].

4. Fracture Squares

We have all these localizations, but how do we use them to simplify our study of $h\mathbf{Sp}$? This is where *fracture theorems* come in. To start, something general.

Theorem 4.1. *Suppose that E and K are spectra such that $L_K L_E X$ is always zero – i.e. E_*-local objects are K_*-acyclic. Then, the following is a (homotopy) pullback:*

$$\begin{tikzcd} L_{E\oplus K}X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & L_K X \arrow[d] \\ L_E X \arrow[r] & L_E L_K X \end{tikzcd}$$

Proof. This is adapted from [Law20, 9.26]. The cospan

$$L_E X \to L_E L_K X \leftarrow L_K X$$

consists of objects which are either E_*-local or K_*-local. So, the pullback is $E \oplus K$-local. It thus suffices to show that the fiber of X mapping to the pullback is $E \oplus K$-acyclic, which is equivalent to showing that

$$\begin{tikzcd} X \arrow[r] \arrow[d] & L_K X \arrow[d] \\ L_E X \arrow[r] & L_E L_K X \end{tikzcd}$$

is a pullback after tensoring everything with $E \oplus K$. After tensoring with E, the vertical maps are equivalences, so the diagram is a pullback. After tensoring with K, the top map is an equivalence and the bottom map is zero (as everything is E_*-local), so the diagram is *also* a pullback, as desired. □

One application of this, which Bousfield proved with basically the same argument[5], is the *arithmetic fracture square*:

Theorem 4.2. *Let X be any spectrum. There is a pullback*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \prod_{p\ prime} X_p^\wedge \arrow[d] \\ X_{\mathbb{Q}} \arrow[r] & \left(\prod_{p\ prime} X_p^\wedge\right)_{\mathbb{Q}} \end{tikzcd}$$

[5]See [hr] for a written-out explanation of why being an equivalence rationally and mod p for all p implies being an equivalence integrally.

What the arithmetic fracture square shows is that it's possible to analyze spectra by dividing and conquering – once we know what something looks like rationally and p-completed at each prime, we win. Thus, the natural question to ask is *how can we understand (p)-local spectra*? This is one of the many questions answered by *chromatic homotopy theory.*

5. Glimpses of the Chromatic Realm

This section is going to be dramatically oversimplified. Depending on who you ask, the correct construction of the following objects is either complicated or *really* complicated, and is one of those stories I suspect is easiest to tell in the language of stacks. That's a different talk, so take what follows on a healthy dose of faith.

A key tenet of chromatic homotopy theory is that we can decompose p-local spectra via "higher primes." The following spectra you should think of as "detecting these primes," in much the same way that $\mathbf{Z}/p$ detects behavior at p. Thus, fix some prime $p > 0$.

Definition 5.1. There exist spectra $K(n)$ for all integers $n \geq 0$, with $K(0) = \mathbf{Q}$ and $K(1)$ a retract of mod p complex K-theory. In addition, write L_n for $L_{K(0)\oplus\cdots\oplus K(n)}$.

Clearly, $\langle K(0) \oplus \cdots \oplus K(n)\rangle \geq \langle K(0) \oplus \cdots \oplus K(n-1)\rangle$, which implies there are maps $L_n \to L_{n-1}$. The magic powder here is the following, which says that these $K(n)$ can, working together, see all of the p-local stable homotopy category.

Theorem 5.2. *(Chromatic Convergence –* [Dev18, 1.4]*).*
Let X be a finite p-local spectrum. Then,

$$X \simeq \operatorname{holim}(L_0X \leftarrow L_1X \leftarrow L_2X \leftarrow \cdots).$$

In particular, we can write the p-local sphere as

$$S^0_{(p)} \simeq \operatorname{holim}(\cdots \to L_2S^0 \to L_1S^0 \to L_0S^0).$$

Recall that $L_0S^0 = S^0_{\mathbf{Q}} = \mathbf{Q}$. Thus, the lower layers of this *chromatic tower* are relatively simple objects, and as we climb higher and higher, we can see more complicated and rich information. Level 1 roughly corresponds to things that we can see with topological K-theory, level 2 can be accessed with *elliptic cohomology*, and it gets more complicated.

However, there is some hope: as it turns out, something which is $E(n-1)$-local is $K(n)$-acyclic (the quick and dirty proof I know uses that $E(n-1)$-localization is smashing, and $K(n) \otimes E(n-1) \simeq 0$ for formal group reasons). Thus, we get the chromatic fracture square:

$$\begin{tikzcd} L_nX \arrow[r] \arrow[d] & L_{K(n)}X \arrow[d] \\ L_{n-1}X \arrow[r] & L_{n-1}L_{K(n)}X \end{tikzcd}$$

This tells us that we have a chance of putting together the chromatic layers if we know what the $K(n)$-local stable homotopy category looks like, and this turns out to be somewhat more doable. (Insert obligatory remarks about why everything here should be subsumed by discussion of the geometry of $\mathcal{M}_{fg}$.)

Acknowledgments

Thanks to (people) for reviewing this talk beforehand and making sure it tried to be coherent.

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Email address: rushil_mallarapu@college.harvard.edu