Signals & Systems

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Contents

Chapter 1

Discrete Time Systems

1.1 Introduction

Definition 1.1 (Signal)**.**

A signal is a function. That is, for some signal x we take as input an element from $\mathbb{R}$ or $\mathbb{Z}$ and output either a $\mathbb{R}$ or $\mathbb{C}$ number.

- Put simply, signals follow the funcition rule that tells us what mapping the function in question should follow.
- From this, we abide by the convention that whenever the domain of our signal is $\mathbb{Z}$ we are working in discrete time. Conversely, if the domain is $\mathbb{R}$ we will be in continuous time.

Definition 1.2 (Kronecker Delta a.k.a. Discrete Time Impulse)**.**

$$\delta(n) = \begin{cases} 1, & \text{if } n = 0 \\ 1, & \text{if } n < 0 \end{cases}.$$

Definition 1.3 (Discrete Time Unit Step)**.**

$$u(n) = \begin{cases} 1, & \text{if } n \geq 0 \\ 0, & \text{if } n < 0 \end{cases}.$$

Definition 1.4 (Continuous Time Unit Step)**.**

$$u(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0 \end{cases}.$$

Theorem 1.1. Any discrete time signal can be written into a linear combination of shifted impulses. It follows that, the unit step function can be reinterpreted as

$$u(n) = \sum_{k=0}^{\infty} \delta(n-k).$$

Theorem 1.2. Immediately from above, we say that any discrete time signal can be expressed as a linear combination of shifted impulses.

$$u(n) = \sum_{k=0}^{\infty} \delta(n-k) = \sum_{m=-\infty}^{n} \delta(m).$$

- In discrete time, the unit step is the cumulative sum of Kronecker deltas. A similar concept to that of the integral.
- We can represent $\delta(n)$ in terms of shifted $u(n)$'s as

$$\delta(n) = u(n) - u(n-1).$$

Note that

$$\delta(n) = \frac{u(n) - u(n-1)}{n - (n-1)} = u(n) - u(n-1).$$

It suffices to say that this is similar to the derivative.

- Therefore, in discrete time $\delta(n)$ is known as the discrete time derivative of the impulse signal and $u(n)$ is it's integral.

1.2 LTI Systems & Impulse Response

Definition 1.5 (Systems)**.**

Similar to signals we say that systems are functions. That is, $x \longrightarrow H \longrightarrow y$ where x is our input signal and y is the corresponding output signal after x "passes" through the system H.

More formally, we define $\mathbb{X}$ to be the input signal space and $\mathbb{Y}$ to be the output signal space. Thus, $y = H(x)$ where $x \in \mathbb{X},\ y \in \mathbb{Y}$.

- **Systems Properties:** (1) Linearity (or Superposition) and (2) Time Invariance.
- **Linearity:** System H is said to be linear if it satisfies both the scaling and additivity property.
  1. Scaling: The output of a weighted input must be scaled by the weight of that input, i.e., $\alpha x \longrightarrow H \longrightarrow \alpha y,\ \forall x \in \mathbb{X},\ \forall \alpha \in \mathbb{R}/\mathbb{C}$.
  2. Additivity: The system is additive if it satisfies the below,

$$x_1 \longrightarrow H \longrightarrow y_1 \quad \text{and} \quad x_2 \longrightarrow H \longrightarrow y_1 \quad \text{then} \quad x_1 + x_2 \longrightarrow H \longrightarrow y_1 + y_2$$

  for all $x_1, x_2 \in \mathbb{X}$.

  The notion of **superposition** occurs when the scaling and additive properties are combined in one step, i.e.,

$$\alpha x_1 + \beta x_2 \longrightarrow H \longrightarrow \alpha y_1 + \beta y_2$$

  for all $\alpha, \beta \in \mathbb{R}/\mathbb{C}$ and for all $x_1, x_2 \in \mathbb{X}$.

- **Time Invariance:** If input $\hat{x}(n) = x(n-N)$ for $N \in \mathbb{Z}$, then if $\hat{y}(n) = y(n-N),\ \forall x \in \mathbb{X}$ and $\forall N \in \mathbb{Z}$, the system H is time invariant.
- **Zero Input - Zero Output (ZIZO):** If our input is zero then the output must be zero, and if this does not hold then the system is nonlinear.
- **LTI Systems:** Are systems that possess both linearity and time invariance.

- If the input signal $x(n) = \delta(n)$, then $y(n) \triangleq h(n)$ where $h(n)$ is called the **impulse response** of the system.

Theorem 1.3. Given that we now the impulse response of an LTI system, we then know the system response to any arbitrary input x.

Proof.

$$\begin{aligned} \delta(n) &\longrightarrow H \longrightarrow h(n) \\ \delta(n-k) &\longrightarrow H \longrightarrow h(n-k) \\ x(k)\delta(n-k) &\longrightarrow H \longrightarrow x(k)h(n-k) \\ \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) &\longrightarrow H \longrightarrow \sum_{k=-\infty}^{\infty} x(k)h(n-k) \end{aligned}$$

where $x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$ and $y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$. ■

- **DT-LTI Systems:** Are a result of the above proof,

$$x(n) \longrightarrow H \longrightarrow y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

where we define y to be the convolution between x and h,

$$y(n) = (x * h)(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k). \tag{1.1}$$

- Convolution is commutative:

$$\begin{aligned} y(n) &= (x * h)(n) \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ &= \sum_{m=-\infty}^{\infty} x(n-m)h(m) \\ &= (h * x)(n) \end{aligned}$$

- If $x(n) \longrightarrow F \longrightarrow G \longrightarrow y(n)$, then we define an intermediate signal $q(n) = (x * f)(n)$. Then, $y(n) = (q * g)(n) = (x * f * g)(n) = (x * h)(n)$ where $h(n) = (f * g)(n)$ is the impulse response of a "bigger" system H.
- Two systems in series (cascaded) is an LTI system.
- $\delta(n)$ also referred as the identity element in discrete time, and $(\delta * h)(n) = (h * \delta)(n) = h(n)$.
- **Euler's Formula**: $e^{i\theta} = \cos\theta + i\sin\theta$.

- **Inverse Euler's Formula:**

$$\cos\theta = \frac{e^{i\theta} + e^{i\theta}}{2} \tag{1.2}$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} \tag{1.3}$$

- **Frequency Response of an LTI System:** From previous observations we know the following two facts:

$$x(n) = \delta(n) \longrightarrow H \longrightarrow y(n) = h(n)$$

where $x(n)$ is "activated" only when $n = 0$ and more generally,

$$x(n) \longrightarrow H \longrightarrow y(n) = (x * h)(n).$$

If $x(n)$ were to be "active" for every time value of n, then

$$x(n) = e^{i\omega n} \longrightarrow H \longrightarrow y(n)$$

where

$$\begin{aligned} y(n) &= (x * h)(n) \\ &= \sum_{k=-\infty}^{\infty} e^{i\omega k} h(n-k) \\ &= e^{i\omega n} \sum_{m=-\infty}^{\infty} h(m) e^{-i\omega m}. \end{aligned}$$

The term $\sum_{m=-\infty}^{\infty} h(m)e^{-i\omega m}$ is a constant that only depends on ω. Thus, the frequency response of the system is,

$$H(\omega) \triangleq \sum_{m=-\infty}^{\infty} h(m) e^{-i\omega m}. \tag{1.4}$$

- Similar idea to eigenvalues, $A\vec{x} = \lambda\vec{x}$, where A is the system response, $H(\omega)$ is the eigenvalue, and our complex exponential is $\vec{x}$.
- Think of $H(\omega)$ as a scaling object.
- **Eigenfunction Property:** Denote our input signal as the complex exponential $e^{i\omega n}$ which is the eigenfunction with respect to the DT-LTI system. It follows that,

$$e^{i\omega n} \longrightarrow H \longrightarrow H(\omega)e^{i\omega n}.$$

- Note that a large class of signals of interest can be written as a linear combination of complex exponentials (Fourier Analysis).

- The frequency response of a DT-LTI system is 2π periodic,

$$H(\omega+2\pi)=\sum_{n=-\infty}^{\infty} h(n)e^{-i(\omega+2\pi)n}=\sum_{n=-\infty}^{\infty} h(n)e^{-i\omega n}.$$

  When analyzing the frequency response of a system only focus on $\omega \in (0, 2\pi)$ or $\omega \in (-\pi, \pi)$.

- More generally, $H(\omega)$ is $2\pi n$ periodic $\forall\, n \in \mathbb{Z}$.

- From above, $H(\omega)$ has fundamental period 2π.

1.3 BIBO Stability, Causality, & Periodicity

- **Periodicity of Complex Exponentials:** For a given continuous time signal $e^{i\omega t}$, it experiences $\frac{2\pi}{\omega}$ revolutions per second, then ω denotes the radians per second of the signal.

Theorem 1.4 (Continuous Time Periodicity)**.**

We say that, $x : \mathbb{R} \to \mathbb{R}$ is T periodic if $\exists\, T$ s.t. $x(t+T)=x(t),\ \forall t \in \mathbb{R}$.

- Smallest such positive t is called the **fundamental period** in continuous time.

- The **fundamental frequency** is given by $f_0 = \frac{1}{T}$ and $\omega = \frac{2\pi}{T}$ has units of fundamental frequency, i.e., $\omega_0 = 2\pi f_0$.

- $e^{i\omega t}$ is periodic in t with fundamental period $T = \frac{2\pi}{\omega}$, thus it is not periodic in ω.

Theorem 1.5 (Discrete Time Periodicity)**.**

Say that $x : \mathbb{Z} \to \mathbb{R}$ is N periodic if $\exists\, N$ s.t. $x(n+N)=x(n),\ \forall\, n \in \mathbb{Z}$.

- Small such positive N is the fundamental period in discrete time.

- $f_0 = \frac{1}{N}$ is the fundamental frequency in discrete time and $\omega = \frac{2\pi}{N}$ has units of fundamental frequency, i.e., $\omega_0 = w\pi f_0$.

- $e^{i\omega n}$ is periodic in ω with fundamental period 2π, i.e., it is only periodic in n if ω is a rational multiple of π:

$$\begin{aligned}
e^{i\omega(n+N))} &= e^{i\omega n}\\
e^{i\omega N} &= 1\\
\omega N &= 2\pi k\\
\omega &= \frac{2\pi k}{N}\\
&= (\frac{2\pi}{N})k
\end{aligned}$$

  therefore ω must be a rational multiple.

1.4 Discrete Time Difference Equations

- **Finite Impulse Response:** Example is the two point moving average filter,

$$y(n) = \frac{1}{2}x(n) + \frac{1}{2}x(n-1)$$

  with corresponding impulse response $h(n) = \frac{\delta(n)+\delta(n-1)}{2}$.

- **Infinite Impulse Response:** Example is the exponentially weighted average filter,

$$y(n) = x(n) + \alpha y(n-1)$$

  with impulse response $h(n) = \alpha^n u(n)$.

1.5 Discrete Time State Space Representation

Chapter 2

Continuous Time Systems

Chapter 3

Frequency Analysis

3.1 Discrete Time Fourier Series

- Start by representing signals $x(n)$ as vectors, e.g.,

$$\boldsymbol{x} = \begin{pmatrix} x(0) & x(1) & x(2) & \dots & x(p-1) \end{pmatrix}^T$$

  where p is the period of $x(n)$.

- Consider the case when $p = 2$, then $\boldsymbol{x} = \begin{pmatrix} x(0) & x(1) \end{pmatrix}^T$. But, we can further decompose $\boldsymbol{x}$ into a linear combination of weighted basis vectors. The basis vectors for when $p = 2$ are

$$\boldsymbol{\varphi}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{\varphi}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

  It follows that, $\boldsymbol{x} = \alpha_0 \, \boldsymbol{\varphi_0} + \alpha_1 \, \boldsymbol{\varphi_1}$ for some $\alpha_0, \alpha_1 \in \mathbb{Z}$.

- In Fourier analysis, instead of using the standard basis to represent the signal $x(n)$, we use another basis (i.e., change of basis) with respect to the Fourier basis vectors $\boldsymbol{\psi}_0, \ \boldsymbol{\psi}_1, \ \dots, \ \boldsymbol{\psi}_{p-1}$.

- Define $\psi_0(n) = e^{i0n}$ and $\psi_1(n) = e^{i\pi n}$, then a 2-periodic signal can be written as

$$x(n) = X_0\psi_0(n) + X_1\psi_1(n) = X_0 \, e^{i0n} + X_1 \, e^{i\pi n}$$

  with respect to the Fourier coefficients X_0, X_1.

- **Note:** Only need 2 frequencies to characterize $x(n)$, both of which are integer multiples of π.

- More generally, if a signal is p-periodic, the only contributing frequencies are

$$\{0, \ \omega_0, \ 2\omega_0, \ \dots, \ (p-1)\,\omega_0\}.$$

- As a consequence, if $x(n+p) = x(n), \ \forall \, n \in \mathbb{Z}, \ \exists \, p \in \{1, 2, \dots\}$, then

$$x(n) = X_0 e^{i0\omega_0 n} + X_1 e^{i1\omega_0 n} + \dots + X_{p-1} e^{i(p-1)\omega_0 n}. \tag{3.1}$$

- **Note:** Only need $(p-1)$ Fourier coefficients since, $e^{ip\omega_0 n} = e^{ip\frac{2\pi}{p}n} = e^{i2\pi n} = 1$ for all $n \in \mathbb{Z}$.

- Going back to the case when $p = 2$, we have that

$$\boldsymbol{x} = X_0\boldsymbol{\psi}_0 + X_1\boldsymbol{\psi}_1.$$

  In order to change bases, from the standard basis of $x(n)$ to the Fourier basis, we project $\boldsymbol{x}$ onto the

Fourier basis vectors in order to find the Fourier coefficients. Projecting $\boldsymbol{x}$ onto $\boldsymbol{\psi}_0$ yields

$$\begin{aligned}\boldsymbol{x}\cdot\boldsymbol{\psi}_0 &= (X_0\boldsymbol{\psi}_0 + X_1\boldsymbol{\psi}_1)\cdot\boldsymbol{\psi}_0\\ &= X_0(\boldsymbol{\psi}_0\cdot\boldsymbol{\psi}_0) + X_1(\boldsymbol{\psi}_1\cdot\boldsymbol{\psi}_0)\\ &= X_0(\boldsymbol{\psi}_0\cdot\boldsymbol{\psi}_0).\end{aligned}$$

Therefore,

$$X_0 = \frac{\boldsymbol{x}\cdot\boldsymbol{\psi}_0}{\boldsymbol{\psi}_0\cdot\boldsymbol{\psi}_0} \qquad \text{and} \qquad X_1 = \frac{\boldsymbol{x}\cdot\boldsymbol{\psi}_1}{\boldsymbol{\psi}_1\cdot\boldsymbol{\psi}_1}.$$

- **Note:** Our signals are complex exponentials (i.e., need not be only real-valued) so the dot product does not generalize to complex vectors. Replace the dot product with the inner product to bring geometry back into the signal space.
- Define our inner product as

$$\langle \boldsymbol{f}, \boldsymbol{g}\rangle = f^T g^* = \sum_{k=0}^{p-1} f_k g_k^*.$$

- We do this because we have knowledge about vector spaces and their properties, which allows us to incorporate our vector space with an inner product to develop the notion of **norms** and **orthogonality**.
- Recall that an inner product over a vector space $\mathbb{E}$ over $\mathbb{C}$ or $\mathbb{R}$ is defined as $\langle\cdot,\cdot\rangle : \mathbb{E}\times\mathbb{E} \longrightarrow \mathbb{C}$ with the following properties:

1. $\langle \boldsymbol{x}+\boldsymbol{y}, \boldsymbol{z}\rangle = \langle \boldsymbol{x}, \boldsymbol{z}\rangle + \langle \boldsymbol{y}, \boldsymbol{z}\rangle$
2. $\langle \alpha\boldsymbol{x}, \boldsymbol{y}\rangle = \alpha\langle \boldsymbol{x}, \boldsymbol{y}\rangle$; $\langle \boldsymbol{x}, \alpha\boldsymbol{y}\rangle = \alpha^*\langle \boldsymbol{x}, \boldsymbol{y}\rangle$
3. $\langle \boldsymbol{x}, \boldsymbol{y}\rangle^* = \langle \boldsymbol{y}, \boldsymbol{x}\rangle$
4. $\langle \boldsymbol{x}, \boldsymbol{x}\rangle \geq 0$ and $\langle \boldsymbol{x}, \boldsymbol{x}\rangle = 0 \iff \boldsymbol{x} = \boldsymbol{0}$

- The **norm** of a vector is defined as $\|\boldsymbol{x}\| = \sqrt{\langle \boldsymbol{x}, \boldsymbol{x}\rangle}$, with the properties:

1. $\|\boldsymbol{x}\| \geq 0$
2. $\|\alpha\boldsymbol{x}\| = |\alpha|\,\|\boldsymbol{x}\|$
3. $\|\boldsymbol{x}+\boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$

- Two vectors are **orthogonal** to each other if $\langle \boldsymbol{x}, \boldsymbol{y}\rangle = 0$.
- With all this in mind we can formally define an inner product over a vetor space of p-periodic signals, that is

$$\forall\, n\in\mathbb{Z},\ f(n+p) = f(n) \text{ and } g(n+p) = g(n)$$

such that $\exists$ a smallest $p \in \{1, 2, 3, \ldots\}$

$$\langle \boldsymbol{f}, \boldsymbol{q}\rangle = \sum_{n=0}^{p-1} f(n) g^*(n). \tag{3.2}$$

- Given a vector $\boldsymbol{x}$ we define a basis $\{\boldsymbol{\psi}_0, \boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_{p-1}\}$ such that $\boldsymbol{\psi}_k \perp \boldsymbol{\psi}_l$ for $k \neq l$. Then

$$\boldsymbol{x} = X_0\boldsymbol{\psi}_0 + X_1\boldsymbol{\psi}_1 + \ldots + X_{p-1}\boldsymbol{\psi}_{p-1} = \sum_{k=0}^{p-1} X_k\boldsymbol{\psi}_k$$

and to find X_l we project $\boldsymbol{x}$ onto $\boldsymbol{\psi}_l$

$$\begin{aligned}\langle \boldsymbol{x}, \boldsymbol{\psi}_l\rangle &= \left\langle \sum_{k=0}^{p-1} X_k\boldsymbol{\psi}_k, \boldsymbol{\psi}_l \right\rangle \\ &= \sum_{k=0}^{p-1} X_k\langle \boldsymbol{\psi}_k, \boldsymbol{\psi}_l\rangle\end{aligned}$$

and we know that there exists two cases for this equation, either $k = l$ or $k \neq l$. When $k = l$, $\langle \boldsymbol{\psi}_k, \boldsymbol{\psi}_l\rangle \neq 0$ otherwise when $k \neq l$ then $\langle \boldsymbol{\psi}_k, \boldsymbol{\psi}_l\rangle = 0$. Therefore,

$$\begin{aligned}\langle \boldsymbol{x}, \boldsymbol{\psi}_l\rangle &= \sum_{k=0}^{p-1} X_k\langle \boldsymbol{\psi}_k, \boldsymbol{\psi}_l\rangle \\ &= X_l\langle \boldsymbol{\psi}_l, \boldsymbol{\psi}_l\rangle\end{aligned}$$

from which we can establish that

$$X_l = \frac{\langle \boldsymbol{x}, \boldsymbol{\psi}_l\rangle}{\langle \boldsymbol{\psi}_l, \boldsymbol{\psi}_l\rangle}.$$

- We have then derived two equations:

$$\text{(Synthesis)} \quad \boldsymbol{x} = \sum_{k=0}^{p-1} X_k\boldsymbol{\psi}_k \quad \text{and} \quad \text{(Analysis)} \quad X_l = \frac{\langle \boldsymbol{x}, \boldsymbol{\psi}_l\rangle}{\langle \boldsymbol{\psi}_l, \boldsymbol{\psi}_l\rangle}. \tag{3.3}$$

- Under this perspective

$$\boldsymbol{\psi}_k = \begin{bmatrix}\psi_k(0) & \psi_k(1) & \ldots & \psi_k(p-1)\end{bmatrix}^T$$

and each entry $\psi_k(n) = e^{ik\omega_0 n}$ where $\omega_0 = \frac{2\pi}{p}$. Observing the two different cases:

1. $k = l$,

$$\begin{aligned}\langle \boldsymbol{\psi}_k, \boldsymbol{\psi}_k\rangle &= \psi_k^T\psi_k^* \\ &= \sum_{k=0}^{p-1} e^{ik\omega_0 n}e^{-ik\omega_0 n} \\ &= p\end{aligned}$$

2. $k \neq l$,

$$\begin{aligned}
\langle \boldsymbol{\psi}_k, \boldsymbol{\psi}_l \rangle &= \sum_{k=0}^{p-1} e^{ik\omega_0 n} e^{il\omega_0 n} \\
&= \sum_{k=0}^{p-1} e^{i(k-l)\omega_0 n} \\
&= \frac{1 - e^{i(k-l)\omega_0 p}}{1 - e^{i(k-l)\omega}} \\
&= 0
\end{aligned}$$

since $\omega_0 p = \frac{2\pi}{p} p = 2\pi$ and $e^{i2\pi} = 1$.

- Note that the only frequencies that contribute come from the set $\{0, \omega_0, 2\omega_0, \ldots, (p-1)\omega_0\}$.
- A very important fact is that the Fourier basis is p-periodic in frequency as shown below,

$$\begin{aligned}
\psi_{k+p}(n) &= e^{i(k+p)\frac{2\pi}{p} n} \\
&= e^{ik\frac{2\pi}{p} n} \\
&= \psi_k(n).
\end{aligned}$$

- More generally, we do not confine ourselves to a fixed interval from 0 to $p-1$. Instead we use the contiguous set $\langle p \rangle$, since $\psi_k(n)$ is p-periodic in both the time index n and frequency index k.
- Putting everything together we can then formulate another pair for the Synthesis and Analysis equations of which are more computational,

$$x(n) = \sum_{k \in \langle p \rangle} X_k e^{ik\omega_0 n} \qquad \text{(Synthesis)} \tag{3.4}$$

$$X_k = \frac{1}{p} \sum_{k \in \langle p \rangle} x(n) e^{-ik\omega_0 n} \qquad \text{(Analysis)} \tag{3.5}$$

Example 1. Consider the case when $x(n) = \sum_{l=-\infty}^{\infty} \delta(n - lp)$, i.e., $x(n)$ is a running impulse train. Finding the Fourier basis coefficient X_k,

$$\begin{aligned}
X_k &= \frac{1}{p} \sum_{k \in \langle p \rangle} x(n) e^{-ik\omega_0 n} \\
&= \frac{1}{p} \sum_{k=0}^{p-1} \delta(n) e^{-ik\omega_0 n} \\
&= \frac{1}{p}
\end{aligned}$$

then $X_k = \frac{1}{p}$, $\forall k \in \mathbb{Z}$.

- This result is akin to the **Uncertainty Principle** in physics where the signals analog is: Assuming the universe is confined to a period (e.g., 0 to $p-1$) and if there is "little" support in the universe, then the spectrum will be everywhere.

Additionally, we can establish that

$$x(n) = \frac{1}{p} \sum_{k \in \langle p \rangle} e^{ik\omega_0 n}$$

for when $X_k = \frac{1}{p}$ for all k. Using this fact we have the following relation

$$\sum_{l=-\infty}^{\infty} \delta(n - lp) = \frac{1}{p} \sum_{k \in \langle p \rangle} e^{ik\omega_0 n} \tag{3.6}$$

known as **Poisson's Identity**. On a last note, we can explore two particular cases:

1. For all $m \in \mathbb{Z}$, let $n = mp$ such that $n \mod p = 0$.

$$\frac{1}{p} \sum_{k \in \langle p \rangle} e^{ik\omega_0 n} = \frac{1}{p} \sum_{k \in \langle p \rangle} e^{ik\frac{2\pi}{p}mp} = \frac{1}{p} \sum_{k \in \langle p \rangle} e^{ik2\pi m} = 1.$$

2. For all $m \in \mathbb{Z}$, let $n = mp$ such that $n \mod p \neq 0$.

$$\frac{1}{p} \sum_{k \in \langle p \rangle} e^{ik\omega_0 n} = \frac{1 - e^{ik\omega_0 p}}{1 - e^{ik\omega_0}} = 0.$$

- **LTI Filtering of periodic DT signals:** given a signal x that passes through some filter with frequency response $H(\omega)$ we have that

$$\begin{aligned} x(n) &\longrightarrow y(n) \\ e^{ik\omega_0 n} &\longrightarrow H(k\omega_0) e^{ik\omega_0 n} \\ X_k e^{ik\omega_0 n} &\longrightarrow H(k\omega_0) X_k e^{ik\omega_0 n} \\ \sum_{k \in \langle p \rangle} X_k e^{ik\omega_0 n} &\longrightarrow \sum_{k \in \langle p \rangle} H(k\omega_0) X_k e^{ik\omega_0 n} \end{aligned}$$

  where $Y_k = H(k\omega_0) X_k$ such that $y(n) = \sum_{k \in \langle p \rangle}^{n} Y_k e^{k\omega_0 n}$

- Recall that if $x(n) = (n + lp),\ l \in \mathbb{Z} \implies x(n)$ is p-periodic. and if, $X_k = X_{k+lp} \implies X_k$ is also p-periodic. Therefore, it is redundant to keep an infinite number of copies of $x(n)$ and X_k.
- Keeping one copy of $\{x(n)\}_{n=0}^{p-1}$ and $\{X_k\}_{k=0}^{p-1}$ gives rise to the **Discrete Fourier Transform (DFT)**.
- **Matrix-Vector Formation of DTFS (DFT):** We know that $\boldsymbol{x} = \sum_{k \in \langle p \rangle} X_k \boldsymbol{\psi}_k$, which can be represented in terms of matrices as

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(p-1) \end{bmatrix} = \begin{bmatrix} \psi_0(0) & \psi_1(0) & \cdots & \psi_{p-1}(0) \\ \psi_0(1) & \psi_1(1) & \cdots & \psi_{p-1}(1) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_0(p-1) & \psi_1(p-1) & \cdots & \psi_{p-1}(p-1) \end{bmatrix} \begin{bmatrix} X_0 \\ X_1 \\ \vdots \\ X_{p-1} \end{bmatrix}.$$

From here we denote the matrix containing all Fourier basis functions as

$$\boldsymbol{\Psi} = \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \boldsymbol{\psi}_0 & \boldsymbol{\psi}_1 & \cdots & \boldsymbol{\psi}_{p-1} \\ \downarrow & \downarrow & & \downarrow \end{bmatrix}.$$

From all of this, we can now write the Synthesis and Analysis equations in matrix-vector form as:

$$\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{X} \qquad \text{(Synthesis)}$$

$$\boldsymbol{X} = \boldsymbol{\Psi}^{-1}\boldsymbol{x} \qquad \text{(Analysis)}$$

- Note that we know that the columns of $\boldsymbol{\Psi}$ are orthogonal, then

$$\boldsymbol{\Psi}^T\boldsymbol{\Psi}^* = \begin{bmatrix} \leftarrow & \boldsymbol{\psi}_0 & \rightarrow \\ \leftarrow & \boldsymbol{\psi}_1 & \rightarrow \\ & \vdots & \\ \leftarrow & \boldsymbol{\psi}_{p-1} & \rightarrow \end{bmatrix} \begin{bmatrix} \uparrow & \uparrow & & \uparrow \\ \boldsymbol{\psi}_0^* & \boldsymbol{\psi}_1^* & \cdots & \boldsymbol{\psi}_{p-1}^* \\ \downarrow & \downarrow & & \downarrow \end{bmatrix} = \begin{bmatrix} p & 0 & \cdots & 0 \\ 0 & p & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & p \end{bmatrix} = p\boldsymbol{I}$$

since $\langle \boldsymbol{\psi}_k, \boldsymbol{\psi}_l \rangle = 0$ when $k \neq l$.

- Through further analysis, we now have that

$$\boldsymbol{\Psi}^T\boldsymbol{\Psi}^* = p\boldsymbol{I} \implies \left(\boldsymbol{\Psi}^T\right)^*\boldsymbol{\Psi} = p\boldsymbol{I} \implies \frac{1}{p}\boldsymbol{\Psi}^H\boldsymbol{\Psi} = \boldsymbol{I} \implies \boldsymbol{\Psi}^{-1} = \frac{1}{p}\boldsymbol{\Psi}^H$$

where $\boldsymbol{\Psi}^H = \left(\boldsymbol{\Psi}^T\right)^*$.

- Consequently, the Synthesis and Analysis equations derived earlier can be rewritten as

$$\boldsymbol{x} = \boldsymbol{\Psi}\boldsymbol{X} \qquad \text{(Synthesis)} \tag{3.7}$$

$$\boldsymbol{X} = \frac{1}{p}\boldsymbol{\Psi}^H\boldsymbol{x} \qquad \text{(Analysis)} \tag{3.8}$$

- As an aside,

$$\left(\frac{1}{\sqrt{p}}\boldsymbol{\Psi}^H\right)\left(\frac{1}{\sqrt{p}}\boldsymbol{\Psi}\right) = \boldsymbol{I}$$

and we define $\frac{1}{\sqrt{p}}\boldsymbol{\Psi}$ to be the **unitary matrix** (i.e., a generalization of orthonormality to deal with complex valued elements).

Theorem 3.1 (Parsevals' Theorem)**.**

$$\sum_{n \in \langle p \rangle} |x(n)|^2 = p \sum_{k \in \langle p \rangle} |X_k|^2 .$$

Proof.

$$\begin{aligned}
\|x(n)\|^2 &= \langle \boldsymbol{x}, \boldsymbol{x} \rangle \\
&= \boldsymbol{x}^T \boldsymbol{x}^* \\
&= \left(\boldsymbol{X}^T \boldsymbol{\Psi}^T\right) \left(\boldsymbol{\Psi}^* \boldsymbol{X}\right) \\
&= \boldsymbol{X}^T p \boldsymbol{I} \boldsymbol{X} \\
&= p \boldsymbol{X}^T \boldsymbol{X} \\
&= p \langle \boldsymbol{X}, \boldsymbol{X} \rangle \\
&= p \left\|\boldsymbol{X}\right\|^2
\end{aligned}$$

■

3.2 Continuous Time Fourer Series

- Note that from here on out we will use much of the geometric intuition developed in the last section regarding a signals representation as vectors in both the time and frequency domain.
- We will now discuss signals $x : \mathbb{R} \longrightarrow \mathbb{R}$ or $\mathbb{C}$ that are p-periodic, i.e., $x(t+p) = x(t),\ \forall\, t \in \mathbb{R}$ s.t. $\exists$ a smallest positive real number p.
- Similarly to a discrete time signal the Synthesis equation of a continuous time signal is

  $$x(t) = \sum_{k=-\infty}^{\infty} X_k \psi_k(t)$$

  where $\psi_k(t) = e^{ik\omega_0 t}$.
- Notice that we are now only periodic in the time index and not in the frequency index.
- It should be obvious that the only contributing frequencies are $\{\dots, -2\omega_0, -\omega_0, 0, \omega_0, 2\omega_0, \dots\}$.
- Thus, unlike in discrete time, in continuous time we have an **uncountably infinite** set of points $x(t)$ that can be decomposed into a **countably** infinite set of frequency components.

Proof. (Periodic in the time index).

$$\begin{aligned}
\psi_k(t+p) &= e^{ik\omega_0(t+p)} \\
&= e^{ik\omega_0 t} e^{ik\omega_0 p} \\
&= e^{ikw\omega_0 t} \\
&= \psi_k(t)
\end{aligned}$$

■

Proof. (Not periodic in the frequency index).

$$\omega_{k+p}(t) = e^{i(k+p)\omega_0 t} = e^{ik\omega_0 t} e^{ip\omega_0 t}$$

where $e^{ip\omega_0 t} = e^{i2\pi t} \neq 1\ \forall t \in \mathbb{R}$, since $e^{i2\pi t} = \cos(2\pi t) + i\sin(2\pi t) \neq 1\ \forall t$.

$\therefore\ \psi_{k+p}(t) \neq \psi_k(t)$. ■

3.3 Discrete Time Fourier Transform

3.4 Continuous Time Fourier Transform

Chapter 4

Sampling and Modulation