

# Hybrid Black Hole–Disk Driven Jets: Steady Axisymmetric Ideal MHD Modeling

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## Abstract

Improved observational precision in relativistic jets has underscored the need for tractable theoretical models. In this study, we construct a semi-analytical hybrid jet model that incorporates both black hole–driven and disk-driven components within the framework of steady, axisymmetric, ideal MHD. We derive a condition that determines the launching sites of cold outflows, introducing a new constraint on the magnetic field configuration threading the accretion disk. Using the Bernoulli equation and critical point analysis, we derive flow solutions along various magnetic field lines. Our hybrid jet model shows that discontinuities in field-line angular velocity lead to clear velocity shear and density jumps at the interface between the two jet components. These features are accompanied by localized enhancements in velocity and density, potentially explaining the observed limb-brightening.

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# 1 Introduction

Relativistic jets from spinning black holes rank among the most energetic phenomena in astrophysics [1–4]. These bipolar outflows, extending over several orders of magnitude in length along the spin axis, have been observed across the electromagnetic spectrum—from radio to gamma rays—in systems such as active galactic nuclei (AGNs), X-ray binaries, and gamma-ray bursts [5, 6]. A prominent example is the jet from M87\*, which exhibits relativistic motion on kiloparsec scales [7–9]. Recent observations by the Event Horizon Telescope (EHT) have resolved the jet’s base down to the horizon scale, revealing features such as edge-brightening [10] and signs of precession [11].

Near black holes, strong gravitational fields require a general relativistic magnetohydrodynamics (GRMHD) framework to model jet dynamics [12–18]. In this context, steady, axisymmetric ideal MHD models offer essential physical insights and quantitative benchmarks [19–32], aiding both simulation calibration and observational interpretation [33]. Jet dynamics are primarily governed by two equations: the relativistic Bernoulli equation, describing energy conversion along field lines [34], and the trans-field Grad–Shafranov equation, determining the global magnetic field structure [35–39].

Horizon-scale magnetic fields are sustained by the accretion disk, with field lines anchored to its surface. As matter accretes, it advects these field lines inward, allowing them to be captured by the black hole. In the Blandford–Znajek (BZ) mechanism, black hole rotational energy is extracted via

horizon-threading field lines, producing outward Poynting flux [40]. In contrast, the Blandford–Payne (BP) mechanism launches disk-driven outflows by channeling plasma along disk-threading field lines, where centrifugal forces overcome gravity [41–43]. While the BZ process is often considered dominant, both BZ and BP mechanisms likely operate simultaneously, as magnetic fields can couple to both the black hole and the disk [12–16, 44], as is shown in Fig. 1.

Centrifugal forces and low mass loading evacuate the polar region, forming a low-density, force-free (FFE) spine where magnetic energy is efficiently converted into kinetic energy, driving ultra-relativistic outflows and high-energy emission (e.g., X-rays, gamma rays) [45–48]. Meanwhile, external pressure from the corona or disk winds confines plasma into a denser, matter-loaded sheath, producing the characteristic spine–sheath structure [12, 15, 16, 49, 50]. The weakly magnetized sheath requires full MHD treatment, accelerates less efficiently, and mainly contributes to the millimeter wavelengths observed by the EHT [51, 52]. The BZ jets typically exhibit both spine and sheath, while the BP jets, launched from larger disk radii, primarily contribute to the sheath [12, 53].

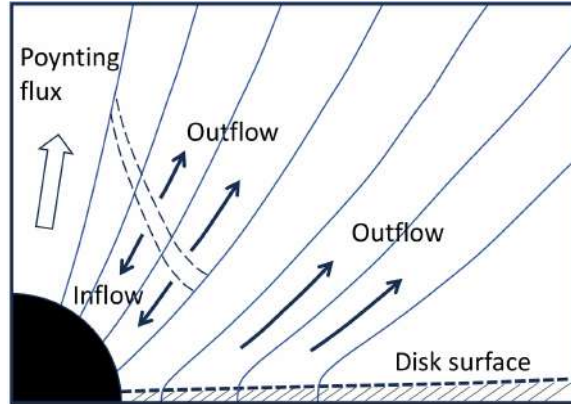


Figure 1: Schematic illustration of the jet structure. The BZ process operates along magnetic field lines that thread the black hole horizon, which carry a continuous outward Poynting flux. Along these field lines, plasma inflows occur below the loading zone, while outflows are launched above it. Concurrently, the BP process is realized along magnetic field lines anchored in the accretion disk, which drive outflows via magnetocentrifugal acceleration and magnetic pressure gradients.

Future high-resolution observations are expected to unveil finer jet structures. Though subdominant to disk near the black hole, its emission is prominent at larger radii due to collimation and the Doppler boosting [10]. In this context, studying GRMHD jets powered by different launching mechanisms is crucial for interpreting observational data, as model–data comparisons can constrain key physical parameters such as launching conditions, magnetic field configurations, and black hole spin. To develop a comprehensive yet tractable framework for jet modeling, it is essential to semi-analytically explore the steady, axisymmetric dynamics and morphology of ideal MHD jets across a range of launching mechanisms, magnetic field geometries, and magnetization levels.

In this work, we try to establish systematically a framework to investigate steady, axisymmetric, ideal MHD systems relevant to energy conversion and flow acceleration in both black hole–driven and disk–driven jets. We derive a general launching condition that identifies the permissible launching positions for cold outflows, which in turn imposes a novel constraint on the structure of disk-threading magnetic fields near the equatorial plane. Building on this foundation, we propose a simplified hybrid BZ and BP jet model under prescribed magnetic field configurations in Kerr spacetime. By solving the Bernoulli equation and performing a critical-point analysis, we obtain the flow structure along different types of field lines. Our results show that the interface between the BZ and BP components naturally gives rise to a velocity shear layer, accompanied by localized enhancements in both density and velocity—features that may offer a physical explanation for the observed limb-brightening in relativistic jets.

Our analysis is largely algebraic and relies solely on the conservation laws governing stationary, axisymmetric, ideal MHD flows. For simplicity, a geometrically thin plasma loading region is assumed to exist in the vicinity of the black hole. Additionally, the fast magnetosonic point is placed at infinity, such that the terminal Lorentz factor—or equivalently, the asymptotic magnetization—serves as the only free parameter along a given field line. All other physical quantities, such as the mass flux, density and magnetization, are uniquely determined by the Bernoulli equation in conjunction with regularity conditions at the MHD critical points.

The remaining parts of this paper are organized as follows. In Section 2 we review the basic framework, including the conserved quantities in ideal MHD, the wind equation, and the treatment of the asymptotic fast magnetosonic point. In Section 3, we focus on plasma loading and derive the launching conditions for both equatorial and off-equatorial configurations. In Section 4, we analyze jet solutions along prescribed magnetic field lines that thread either the black hole (Section 4.1) or the accretion disk (Section 4.2) in the Kerr spacetime. In Section 5, we propose the hybrid jet model and discuss its principal features in detail. We summarize our findings and provide further discussion in Section 6.

## 2 Jets in ideal MHD

### 2.1 Stationary axisymmetric flow

We begin with a non-self-gravitating magnetofluid in a general stationary and axisymmetric spacetime. The line element of the spacetime takes the form

$$ds^2 = g_{tt}dt^2 + 2g_{t\phi}dtd\phi + g_{\phi\phi}d\phi^2 + g_{\mathcal{P}\mathcal{P}}d\mathcal{P}^2. \quad (2.1)$$

Here and thereafter, we label the poloidal coordinates  $r, \theta$  by the index  $\mathcal{P}$ . The jet is composed of ionized plasma, which is supported by the baryonic matter in the accretion flow, or electron-positron pairs produced via photon-photon collisions [54]. In this study, we only consider the baryonic matter, thus the jet dynamics is governed by ions. The energy-momentum tensor takes the form of  $T^{\mu\nu} = T_{\text{m}}^{\mu\nu} + T_{\text{EM}}^{\mu\nu}$ , where

$$T_{\text{m}}^{\mu\nu} = hu^\mu u^\nu + P g^{\mu\nu}, \quad T_{\text{EM}}^{\mu\nu} = F^{\mu\rho} F_\rho^\nu - \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}, \quad (2.2)$$

represent the matter stress tensor and electromagnetic stress tensor, respectively;  $h = e + P$  is the rest-frame enthalpy density;  $e, P$  denote the rest-frame energy density and pressure,  $u^\mu$  is the flow four-velocity,  $F^{\mu\nu}$  is the electromagnetic field tensor. Throughout this work, we require infinite conductivity which is also known as ideal MHD condition, ensuring that the electric field measured by the plasma is zero,  $u_\mu F^{\mu\nu} = 0$ <sup>1</sup>.

The evolution of the magnetized fluid is governed by the GRMHD equations, namely the conservation of energy-momentum:  $\nabla_\mu T^{\mu\nu} = 0$ , and the conservation of particle number:  $\nabla_\mu(\rho u^\mu) = 0$ , where  $\rho$  denotes the rest-frame mass density. To close the system, an equation of state for the ion component is required. Under the adiabatic approximation, the total energy density is described by a polytropic relation  $e = \rho + (\Gamma - 1)^{-1}P$ , where  $\Gamma$  is the adiabatic index, taking the value  $\Gamma = 5/3$  for non-relativistic ions and  $\Gamma = 4/3$  for relativistic ions.

To proceed, we define the “pseudo-electromagnetic fields” as the contractions of  $F^{\mu\nu}$  and its dual with the Killing vector in  $t$  direction:  $E^\mu = F^{t\mu}$ ,  $B^\mu = -(\star F)^{t\mu}$ . Under stationary and axisymmetric ideal MHD, the poloidal magnetic field is aligned with the poloidal flow velocity,  $B^r u^\theta = B^\theta u^r$ , and the field components take

$$B^r = \frac{\partial_\theta \psi}{\sqrt{-g}}, \quad B^\theta = -\frac{\partial_r \psi}{\sqrt{-g}}, \quad B^\phi = (u^\phi - \Omega_F u^t) \frac{B^\mathcal{P}}{u^\mathcal{P}}, \quad (2.3)$$

where  $\sqrt{-g}$  is the metric determinant,  $\psi \equiv A_\phi(r, \theta)$  is the stream function serving as a label for magnetic field lines and plays a central role in describing the field geometry, and  $\Omega_F \equiv F_{t\theta}/F_{r\phi} = F_{t\theta}/F_{\theta\phi}$  is the field-line angular velocity, which is conserved along each magnetic field line. More details on  $B^\mu$  are included in Appendix A.2.

A nonzero field line rotation drives a generic evolution of ideal MHD flows, causing an initially arbitrary magnetic field near the black hole to gradually become toroidal at large distances. This behavior can be analyzed using a corotating observer with four-velocity  $u_{\text{co}}^\mu \propto (1, 0, 0, \Omega_F)$ . A characteristic surface, known as the light cylinder [55], is defined by the condition  $u_{\text{co}}^2 \propto -k_0 \rightarrow 0$  where

<sup>1</sup>Although finite resistivity effects—such as magnetic reconnection and current sheet formation—play a crucial role in understanding the details of magnetofluids, the ideal MHD approximation offers a well physical framework when the focus is on global fluid properties or time-averaged structures [17].

$k_0 = -(g_{\phi\phi}\Omega_F^2 + 2g_{t\phi}\Omega_F + g_{tt})$ , at which the corotating observer becomes null. Beyond the light cylinder, no timelike corotating observers exist, and the flow lags behind the rotating field lines, leading to a strong toroidal magnetic field in the fluid frame.

Moreover, it is well-established that the stationary and axisymmetric ideal MHD system further possesses three conserved quantities of the magnetofluid along each magnetic field line [19, 56]. We denote them as the mass flux  $\eta$ , specific angular momentum  $L$ , and the specific energy  $E$ :

$$\begin{aligned}\eta &\equiv \frac{\sqrt{-g}\rho u^r}{F_{\theta\phi}} = \frac{\sqrt{-g}\rho u^\theta}{F_{\phi r}} = \frac{\rho u^{\mathcal{P}}}{B^{\mathcal{P}}}, \\ L &\equiv \mu u_\phi - \frac{\kappa B^\phi}{\eta}, \quad E \equiv -\mu u_t - \frac{\kappa B^\phi \Omega_F}{\eta},\end{aligned}\tag{2.4}$$

where  $\mu \equiv h/\rho$  denotes the specific enthalpy. The existence of these conserved quantities  $\{\Omega_F, \eta, E, L\}$  helps to simplify solving the flow structure, for a given stream function  $\psi(r, \theta)$ . In Appendix A.3, we present a concise derivation of these MHD conserved quantities.

## 2.2 The wind equation

The flow four-velocity can be solved by using the MHD conserved quantities. Firstly, with the definition of  $E$  and  $L$ , the  $t$ -,  $\phi$ -components of the four-velocity can be rewritten as

$$u_t = \frac{E}{\mu} \frac{(g_{tt} + g_{t\phi}\Omega_F)(1 - \Omega_F l) + M_A^2}{k_0 - M_A^2}, \quad u_\phi = \frac{E}{\mu} \frac{(g_{t\phi} + g_{\phi\phi}\Omega_F)(1 - \Omega_F l) - M_A^2 l}{k_0 - M_A^2},\tag{2.5}$$

where  $l \equiv L/E$  denotes the angular momentum density. Here, we have also introduced a dimensionless Alfvénic Mach number in the poloidal plane [19]:

$$M_A \equiv \sqrt{\frac{\mu \rho u_p^2}{B_p^2}} = \sqrt{\frac{\mu u_p |\eta|}{B_p}},\tag{2.6}$$

where  $u_p \equiv \sqrt{u^{\mathcal{P}} u_{\mathcal{P}}} = \sqrt{u^r u_r + u^\theta u_\theta}$  and  $B_p \equiv \sqrt{B^{\mathcal{P}} B_{\mathcal{P}}} = \sqrt{B^r B_r + B^\theta B_\theta}$  are the magnitudes of the poloidal velocity and poloidal magnetic field, respectively.  $M_A$  characterizes the ratio between the flow velocity and the propagation speed of magnetic disturbances in the poloidal plane<sup>2</sup>. Combining Eqs. (2.5) with Eqs. (2.3), the toroidal component of the magnetic field is given by

$$B^\phi = \eta E \frac{(g_{tt} + g_{t\phi}\Omega_F)l + (g_{t\phi} + g_{\phi\phi}\Omega_F)}{\kappa(k_0 - M_A^2)}.\tag{2.7}$$

By substituting Eqs. (2.5) into the normalization condition  $u_\mu u^\mu = -1$ , one can derive the relativistic Bernoulli equation (also called the wind equation) that governs the evolution of the poloidal velocity

<sup>2</sup>The second equality in Eq. (2.6) follows from  $u_p/B_p = |u_{\mathcal{P}}/B_{\mathcal{P}}|$ . Since  $\eta$  is signed to indicate the flow direction, we use  $|\eta|$  in defining the Alfvénic Mach number to ensure it remains a positive-definite quantity.

along the field lines. Through ideal MHD condition,  $u^\theta = B^\theta u^r / B^r$ ,  $u^r = \pm u_p (g_{rr} + g_{\theta\theta} (B^\theta)^2 / (B^r)^2)^{-1/2}$ , where the sign “ $\pm$ ” corresponds to outflow/inflow. The wind equation can be written as

$$u_p^2 + 1 = \left(\frac{E}{\mu}\right)^2 \frac{k_0 k_2 - 2k_2 M_A^2 - k_4 M_A^4}{(k_0 - M_A^2)^2}, \quad (2.8)$$

where

$$k_0 \equiv -(g_{\phi\phi} \Omega_F^2 + 2g_{t\phi} \Omega_F + g_{tt}), \quad k_2 \equiv (1 - \Omega_F l)^2, \quad k_4 \equiv -\frac{g_{\phi\phi} + 2g_{t\phi} l + g_{tt} l^2}{\kappa}. \quad (2.9)$$

Once the magnetic field and conserved quantities  $\{\Omega_F, \eta, E, L\}$  are known,  $u_p$  (and  $M_A$ ) is determined by solving Eq. (2.8), and  $u^t, u^\phi$  are determined by Eq. (2.5). For plasma with finite temperature, the wind equation is complicated. For example, for adiabatic ions, the enthalpy density is given by  $h = \rho + \Gamma(\Gamma - 1)^{-1} P$  with  $P \propto \rho^\Gamma$ . Since  $\rho = |\eta| B_p / u_p$ , the specific enthalpy  $\mu = h / \rho$  is a function of  $u_p$  along a given field line. Substituting the equation of state into Eq. (2.8) leads to a non-integer power polynomial equation in  $u_p$ , which can only be solved numerically [57].

However, in the cold limit  $\mu \rightarrow 1$ , substituting  $M_A = \sqrt{u_p |\eta| / B_p}$  into Eq. (2.8) reduces it to a quartic equation in  $u_p$ . By analyzing the root behaviour of this equation, one can select the physically admissible solution of  $u_p$ . For a Kerr black hole, there are typically two real roots near the event horizon—one positive and one negative. Among them, only one root remains globally non-negative and smooth across the entire domain, as illustrated in Fig. 2.

To further characterize the solution structure, we need to examine the critical behavior of the flow at the points where the flow velocity reached characteristic MHD wave speeds [21]. Firstly, the denominator on the right-hand side of Eq. (2.8) vanishes at the so-called Alfvén point, where  $M_A^2 = k_0$ . This point corresponds to the location where the poloidal velocity matches the Alfvén speed  $u_A = B_p^2 k_0 (\mu \rho)^{-1}$ , which characterizes the propagation speed of transverse magnetic disturbances along field lines. Beyond this point,  $u_p > u_A$ , Alfvén waves are unable to propagate upstream. Mathematically, the Alfvén point constitutes a general critical point of the wind equation, where multiple roots of the equation coincide, as illustrated in Fig. 2. At this critical point, both the numerator and denominator of the wind equation simultaneously vanish [58].

In addition, two other critical points commonly arise in the MHD flow: the slow magnetosonic (SM) point and the fast magnetosonic (FM) point. These points correspond to distinct dynamical transitions and impose essential regularity conditions on the solution: the requirement that the flow passes smoothly through these critical points imposes constraints on the conserved quantities and plays a key role in selecting physically viable solutions. Further details regarding the properties of these MHD wave speeds and critical points are provided in Appendix B.



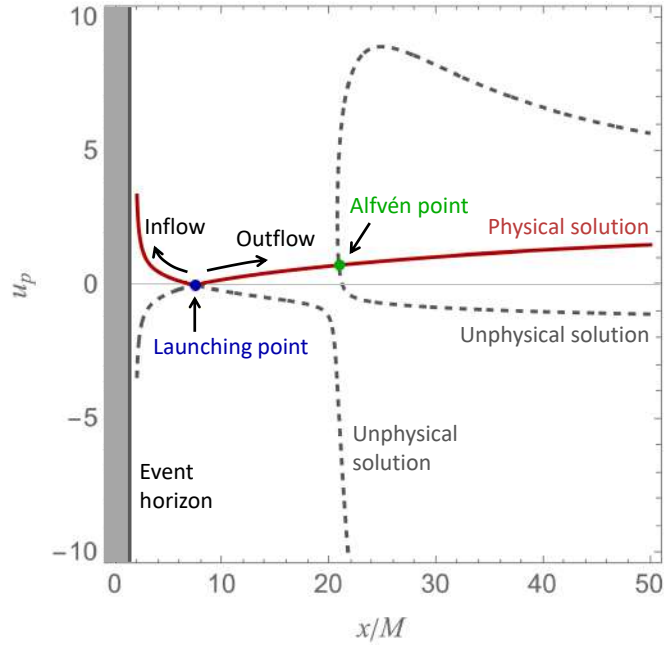


Figure 2: Real roots of the wind equation along a parabolic streamline in a cold MHD jet ( $\mu = 1$ ) in Kerr spacetime with spin  $a = 0.5$ . The streamline is defined by  $r - z = r_+$ , where  $r_+$  is the event horizon radius. Cylindrical coordinates are given by  $x = r \sin \theta$ ,  $z = r \cos \theta$ . The conserved quantities are set to  $\{\psi, \Omega_F, \eta, E, L\} = \{1.866, 0.0467, 0.00036, 27, 559.0226\}$ . The red solid curve shows the physical solution; gray dashed curves denote unphysical branches.

### 2.3 Asymptotic magnetosonic point

In this work, we adopt the assumption that the FM point of the jet outflow lies at spatial infinity. Then, FM waves can propagate freely throughout the jet interior. Such treatment is appropriate for highly relativistic, magnetically dominated jets, especially the BZ jets [59]. By imposing cold ions at the FM point at infinity, we can obtain the asymptotic expression for  $E$  and  $\eta$ :

$$E = \gamma_\infty^3, \quad \eta = \frac{\Omega_F^2 B_p x^2}{(\gamma_\infty^2 - 1)^{3/2}} \bigg|_{r \rightarrow \infty}. \quad (2.10)$$

The first equation is the Michel's minimum energy [60, 61], where the cubic dependence on  $\gamma_\infty$  arises from the relativistic conversion between magnetic and kinetic energy, and the volume compression. Eqs. (2.10) are derived under the assumption of cold ions at infinity, without requiring the entire plasma flow to be cold. It provides an effective means to characterize plasma outflows along streamlines. Especially, for  $u_p$  determined by the wind equation under a given magnetic field, the expression of  $\eta$  in Eqs. (2.10) directly yields the ion number density  $\rho = |\eta| B_p / u_p$ .

We notice that the form of  $\eta$  in Eqs. (2.10) actually implies opened magnetic field lines with the asymptotic behaviour of  $B_p \sim 1/x^2$ . The magnetic fields adopted in this work all satisfy such

condition. Note that  $d(B_p x^2)/dx < 0$  is required for efficient MHD acceleration and the existence of a finite FM point [62]. As a result, for  $B_p \sim 1/x^2$ , the FM point is automatically pushed to infinity, and the system lacks a finite-radius transition where magnetic energy is fully converted into plasma kinetic energy. This inefficiency can be reflected in the magnetization parameter, defined as the ratio of  $E_{\text{EM}} \equiv E - E_{\text{m}}$  to  $E_{\text{m}} \equiv -\mu u_t$  [63],

$$\sigma_M \equiv \frac{E_{\text{EM}}}{E_{\text{m}}} = \frac{\gamma_\infty^3 + \mu u_t}{-\mu u_t}, \quad (2.11)$$

At the asymptotic FM point where the flow is cold, Eqs. (2.5) results in  $u_t \rightarrow -\gamma_\infty$ . Substituting this into Eq. (2.11) yields  $\sigma_M|_{r \rightarrow \infty} \rightarrow \gamma_\infty^2 - 1$ , revealing that a substantial fraction of the energy remains stored in the electromagnetic field at infinity. This is a direct consequence of the field line geometry and the absence of a finite FM point. In this sense,  $\sigma_M$  acts as a diagnostic of acceleration efficiency: high magnetization indicates a magnetically dominated jet at spatial infinity.

From the asymptotic analysis of the FM point, Eq. (2.7) reduces to the following expression at large distances:

$$B^\phi \approx -\frac{\eta(\gamma_\infty^2 - 1)E}{\gamma_\infty^2 \Omega_F \kappa} \rightarrow -\frac{\Omega_F \gamma_\infty B_p}{(\gamma_\infty^2 - 1)^{1/2}}. \quad (2.12)$$

The asymptotic toroidal field magnitude is  $\sqrt{B_\phi B^\phi} \sim \sqrt{g_{\phi\phi}} B_p \rightarrow x B_p$ . Therefore, the magnetic field must take a toroidal shape at large distances. Under the limit  $\gamma_\infty \gg 1$ , we recover the result for the FFE, namely  $\kappa \Omega_F B^\phi \rightarrow -\eta E$  [57].

### 3 Jet launching

#### 3.1 The launching surface

The MHD flow is loaded through a dense plasma layer near the black hole and driven outward by magnetic forces [26, 64]. Here, we adopt a simplified model with a geometrically thin, axisymmetric launching surface (denoted by “LS”) [21], where the poloidal outflow velocity is zero [19].

Through Eqs. (2.3), the regularity of  $B^\phi$  at the launching surface with  $u_p|_{\text{LS}} \rightarrow 0$  requires the plasma flow to be corotate with the magnetic field,  $u^\phi|_{\text{LS}} = \Omega_F u^t|_{\text{LS}}$ . From the definition Eqs. (2.4), we obtain a constraint for  $E, L$ :

$$\Omega_F L = E - \mu \sqrt{k_0} \Big|_{\text{LS}}. \quad (3.1)$$

Since the flow four-velocity is always timelike, the launching surface must lie within the light cylinder.

The constraint of the launching surface location can be found by expanding the MHD conservation law (or equivalently the acceleration equation) near  $u_p = 0$ . To do this, we assume the cold jet flow for

simplicity (for the thermal jet, see Appendix C). In the cold limit, the SM point lies on the launching surface. Expanding the acceleration equation in terms of  $u_p$  yields

$$u'_p \approx k'_0 \left( -\frac{B_p^2 E^2 k_0 k_2}{2X} \frac{1}{u_p} + \frac{3B_p E^4 k_2 (k_2 + k_0 k_4) \eta^3}{2X^2} + \mathcal{O}(u_p) \right) \Big|_{\approx \text{LS}}, \quad (3.2)$$

where  $X = B_p^2 k_0^3 + E^2 (k_2 + k_0 k_4) \eta^2$ . Since Eq. (3.1) implies  $k_2 = k_0 E^{-2}$  and  $k_0 > 0$  within the light cylinder, the launching surface must satisfy  $k'_0 = 0$ . Moreover, if an outflow/inflow is to be launched from rest,  $u'_p$  must be positive/negative near the launching surface. Since  $k_2 + k_0 k_4 = \kappa^{-1} [g_{t\phi}(1 + l\Omega_F) + g_{tt}l + g_{\phi\phi}\Omega_F]^2 \geq 0$  always holds [21], we have the launching condition  $k'_0 \leq 0$  for outflow/inflow, indicating that the launching surface should be the local maximum point of  $k_0$ .

The above discussion for the launching surface is universal. When we turn to different types of jets, the difference lies in the specific location of the launching surface and the boundary condition there. In our work, the BZ and BP jets are distinguished by the topology of their magnetic fields. Once the field geometry is chosen, the field angular velocity  $\Omega_F$  is determined by the boundary conditions—either at the black hole horizon or at the disk.

For simplicity, we adopt the cold limit in the following discussion. As demonstrated in previous studies, the ions within the accretion system in AGNs are highly non-relativistic, and thus the thermal effects do not significantly influence the jet dynamics [65, 66].

### 3.2 BZ process

In the BZ jet, the magnetic field is captured by the black hole, and the plasma exhibit inward motion as it approaches the event horizon. Thus, there is a region that separates the inflow from the outflow, where the poloidal velocity goes to zero. In the ideal MHD framework, the separation is characterized by a two-dimensional surface and coincides with the launching surface [13, 25]. The differences in conserved quantities between the inflow and outflow satisfy [26]

$$\begin{aligned} (\eta_{\text{out}} E_{\text{out}} - \eta_{\text{in}} E_{\text{in}}) &= -(\eta_{\text{out}} - \eta_{\text{in}}) u_t \Big|_{\text{LS}}, \\ (\eta_{\text{out}} L_{\text{out}} - \eta_{\text{in}} L_{\text{in}}) &= (\eta_{\text{out}} - \eta_{\text{in}}) u_\phi \Big|_{\text{LS}}, \end{aligned} \quad (3.3)$$

where we use the subscript “in” “out” to label the quantities of the inflow and outflow, respectively<sup>3</sup>. Therefore,  $\eta$ ,  $E$ , and  $L$  are conserved separately within the inflow and outflow, but are different across the launching surface. In contrast, the magnetic field and flow velocity remain continuous across the launching surface.

The acceleration of the outflow is easily determined by demanding an asymptotic FM point (see Sec. 2.3). Specifically,  $E_{\text{out}}$ ,  $\eta_{\text{out}}$  are given by the asymptotic relation Eqs. (2.10), and  $L_{\text{out}}$  is fixed

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<sup>3</sup>The equations in Eqs. (3.3) are not independent; the second can be derived by combining the first with Eq. (3.1).

by the relation Eq. (3.1) at the launching surface. However, the inflow possesses finite FM point near the black hole, whose acceleration is more complicated. The existence of the inflow FM point can be understood as follows: the FM speed is the fastest propagation speed of disturbances in a magnetized fluid. Beyond the FM point, no disturbances can propagate from the downstream to the upstream. When sufficiently close to the event horizon, it is inevitable that no signal can propagate outward. Therefore, the FM point must occur outside the horizon (a detailed analysis can be found in [21]).

To determine the inflow dynamics, we need to find the location of the FM point,  $r_{\text{FM}}$ , and the corresponding conserved quantities. This is achieved by combining the matching conditions, Eqs. (3.3), between the inflow and outflow, with the relation of  $E$  and  $\eta$  at the FM point and the regularity condition that ensures the acceleration equation remains finite there. As a result, there are five equations available to determine five quantities  $\eta_{\text{in}}, E_{\text{in}}, L_{\text{in}}, u_{\text{FM}}$  and  $r_{\text{FM}}$ .

### 3.3 BP process

The BP jet originates from outflows launched at the surface of the accretion disk, where plasma is supplied along open or collimated magnetic field lines. For an ideal MHD accretion flow that is reflection-symmetric about the equatorial plane and exhibits vanishing polar velocity (i.e., zero velocity in the  $\theta$ -direction), the following boundary conditions hold:  $B^r|_{\theta=\pi/2} = E^\theta|_{\theta=\pi/2} = 0$ . If a nonzero poloidal magnetic field component is present at the equatorial plane, i.e.,  $B^\theta|_{\theta=\pi/2} \neq 0$ , then the field lines are frozen into the disk [31], implying that

$$\Omega_F = \Omega_{\text{disk}}, \quad (3.4)$$

where  $\Omega_{\text{disk}}$  is the angular velocity of the disk. In this case, the field lines are anchored in the disk and corotate with it. In the cold MHD limit, the condition for jet launching is determined by  $k'_0 = B^r \partial_r k_0 + B^\theta \partial_\theta k_0 = 0$ . Using  $\Omega'_F = 0$ , this condition becomes

$$k'_0 = -B^r (\partial_r g_{tt} + 2\Omega_F \partial_r g_{t\phi} + \Omega_F^2 \partial_r g_{\phi\phi}) - B^\theta (\partial_\theta g_{tt} + 2\Omega_F \partial_\theta g_{t\phi} + \Omega_F^2 \partial_\theta g_{\phi\phi}). \quad (3.5)$$

Interestingly, since the launching surface coincides with the disk surface, Eq. (3.5) imposes a constraint on the disk height for a given  $\psi$  and  $\Omega_F$ . Conversely, for a specified  $\psi$  and disk height, Eq. (3.5) determines the corresponding value of  $\Omega_F$ . Note that due to the reflection symmetry of the spacetime about the equatorial plane,  $\partial_\theta g_{\mu\nu}|_{\theta=\pi/2} = 0$ . Hence,  $k'_0 = 0$  is automatically satisfied at the equatorial plane, although this does not necessarily correspond to a local maximum of  $k_0$ .

For a geometrically thin disk, the jet launching surface lies close to the equatorial plane, and  $\partial_\theta g_{\mu\nu}|_{LS} \approx \partial_\theta g_{\mu\nu}|_{\theta=\pi/2} = 0$ . In this regime, the launching condition simplifies to  $0 = k'_0|_{LS} \approx -B^r (\partial_r g_{tt} + 2\Omega_F \partial_r g_{t\phi} + \Omega_F^2 \partial_r g_{\phi\phi})|_{LS}$ . Hence, if  $B^r|_{LS} = 0$ , this condition is trivially satisfied. Otherwise, for  $B^r|_{LS} \neq 0$ , solving the equation yields the angular velocity of the field-lines required for jet

launching:

$$\Omega_F = \left. \frac{-\partial_r g_{t\phi} + \sqrt{(\partial_r g_{t\phi})^2 - \partial_r g_{tt} \partial_r g_{\phi\phi}}}{\partial_r g_{\phi\phi}} \right|_{\text{LS}}, \quad (3.6)$$

which corresponds precisely to the Keplerian angular velocity. In summary, for magnetic field lines threading a thin accretion disk, jet launching is permitted under two distinct conditions: (1) the magnetic field is perpendicular to both the equatorial plane and the disk surface, or (2) the disk rotates at the Keplerian angular velocity.

## 4 Specific models in the Kerr spacetime

The ideal MHD framework applies to general stationary, axisymmetric spacetimes. In this section, we focus on Kerr spacetime and examine jet flows under typical magnetic field configurations. Although the G-S equation is complex in curved spacetime [36], gravitational effects are significant only near the black hole [27]. We therefore construct the magnetic field by adopting flat spacetime stream functions, and embedding them into the Kerr background through direct superposition.

### 4.1 Parabolic magnetic fields

#### 4.1.1 The field structure

We employ the Boyer-Lindquist (BL) coordinates  $(t, r, \theta, \phi)$  and the cylinder coordinates defined as  $x = r \sin \theta$ ,  $z = r \cos \theta$ . For simplicity, we set the black hole mass to be 1. The horizon-threading magnetic field relevant to the BZ process can be well described by the following stream function [67]:

$$\psi(r, \theta) = Cr^q(1 - \cos \theta), \quad (4.1)$$

where  $0 < q < 2$  referred to as the collimation index, controls the asymptotic collimation of field lines. Increasing  $q$  results in tighter collimation. The case  $q = 0$  corresponds to a split-monopole configuration, whereas  $q = 1$  produces a parabolic field structure. The overall factor  $C$  does not affect the jet dynamics and hereafter it will be set to 1. Eq. (4.1) is in agreement with black hole magnetospheres in numerical simulations, where the typical collimation index is  $q \simeq 0.75$  [68, 69].

According to Eq. (4.1), each field line is uniquely identified by its footpoint on the event horizon, labeled by the polar angle  $\theta_+(\psi) = \arccos \left[ 1 - \left( r_+^{-q} \psi \right) \right]$ , so  $\psi$  effectively encodes the angular position of the field-line at the event horizon. In Fig. 3, we present the poloidal magnetic field configurations with  $q = 1$  (left panel) and  $q = 0.5$  (right panel). The gray dashed lines in each panel represent field lines with footpoints  $\theta_+ = n\pi/20$ ,  $n = 1, 2, \dots, 10$ .

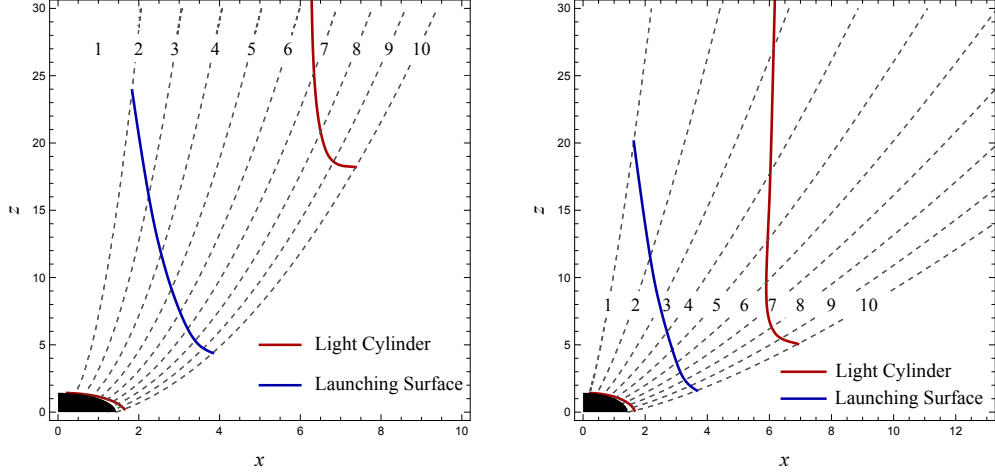


Figure 3: Poloidal magnetic field configurations corresponding to Eq. (4.1) are shown for  $q = 1$  (left) and  $q = 0.5$  (right). Gray dashed lines represent magnetic field lines anchored at various footpoint angles  $\theta_+ = n\pi/20$ ,  $n = 1, 2, \dots, 10$ ; the associated values of  $n$  are indicated in the figure. The blue curve denotes the jet launching surface, while red curves indicate the light cylinders.

For horizon-threading field lines, the requirement of finite field strength at the event horizon yields the regular condition here, also called the Znajek condition [70]:

$$\Omega_F = \left[ \frac{a}{2r} + \frac{\kappa B^\phi}{2r \sin^2 \theta B^r} \right] \Big|_{r=r_+}, \quad (4.2)$$

which serves as a constraint on the field-line angular velocity. As  $B^\phi$  depends on  $u^\mu$ , Eq. (4.2) is coupled with the wind equation. For simplicity, we adopt the FFE approximation, and neglect the plasma inertia ( $\eta \rightarrow 0$ ) when evaluating  $\Omega_F$ . Thus, the specific energy is carried entirely by the electromagnetic field,  $E \approx -\Omega_F \kappa B^\phi / \eta$ . Since  $E = \gamma_\infty^3$  and  $\eta \approx \gamma_\infty^{-3} \Omega_F^2 (B_p x^2)|_{r \rightarrow \infty}$  as given in Eqs. (2.10), we have  $\kappa B^\phi \approx -\Omega_F (B_p x^2)|_{r \rightarrow \infty}$ , where the asymptotic poloidal field takes  $B_p x^2 \rightarrow \sin^2 \theta_+$  for  $q = 0$  and  $B_p x^2 \rightarrow 2\psi$  for  $q > 0$ . Substituting this into Eq. (4.2) yields the explicit expression of  $\Omega_F$  for the stream function Eq. (4.1) :

$$\Omega_F = \begin{cases} \frac{a}{4r_+ - a^2 \sin^2 \theta_+}, & \text{for } q = 0, \\ \frac{a(1 + \cos \theta_+)}{2(3r_+ + r_+ \cos \theta_+ - a^2 \sin^2 \theta_+)}, & \text{for } 0 < q < 2. \end{cases} \quad (4.3)$$

It can be seen that for  $q = 0$ ,  $\Omega_F \geq \Omega_H/2$ ; whereas for  $0 < q < 2$ ,  $\Omega_F \leq \Omega_H/2$ , where  $\Omega_H = a(2r_+)^{-1}$  is the black hole angular velocity.

The black hole spacetime typically has an outer light cylinder far from the event horizon, and an inner light cylinder close to the event horizon [55, 71], indicated by the red curves in Fig. 3 where  $k_0 = 0$ . In the region between the inner and outer light cylinders,  $k_0 > 0$  always holds. It follows

that the point where  $k'_0 = 0$  must be a local maximum of  $k_0$ , ensuring a physical launching surface. The shapes of the launching surface and light cylinders reflect how the interplay between magnetic field rotation and spacetime rotation determine the field structures. In Appendix. D, we provide approximate expressions for light cylinders in the Kerr spacetime.

#### 4.1.2 The flow solution

Under the stream function Eq. (4.1), the conserved mass flux  $\eta$  for outflow along each field line is determined by the asymptotic expression Eqs. (2.10), resulting in

$$\eta_{\text{out}} = \begin{cases} \frac{\psi \Omega_F^2 (1 + \cos \theta_+)}{(\gamma_\infty^2 - 1)^{3/2}}, & \text{for } q = 0, \\ \frac{2\psi \Omega_F^2}{(\gamma_\infty^2 - 1)^{3/2}}, & \text{for } 0 < q < 2. \end{cases} \quad (4.4)$$

Combined with  $E_{\text{out}} = \gamma_\infty^3$  and  $\Omega_F L_{\text{out}} = E_{\text{out}} - \mu \sqrt{k_0}|_{\text{LS}}$ , the wind equation Eq. (2.8) can be solved to give  $u_p = u_p(\eta_{\text{out}}, E_{\text{out}}, L_{\text{out}}, r, \theta)$ . Then,  $u_t$ ,  $u_\phi$  and  $B^\phi$  are determined through Eqs. (2.5) and Eq. (2.7), respectively. The outflow plasma density is given by  $\rho = |\eta_{\text{out}}| B_p / u_p$ . By using the matching condition Eqs. (3.3) and the relation of  $E$  and  $\eta$  at the FM point, the inflow structure can be solved accordingly. In Fig. 4 presents a jet solution including both inflow and outflow, along a parabolic field line with  $q = 1$ ,  $\theta_+ = \pi/2$ . We adopt a black hole spin of  $a = 0.9$  and set the terminal Lorentz factor to  $\gamma_\infty = 3$ . From these inputs, the conserved quantities of outflow are  $\{E, L, \eta\}_{\text{out}} = \{27, 204.6709, 0.0021\}$ , and that of the inflow are  $\{E, L, \eta\}_{\text{in}} = \{-0.4134, -8.4023, -0.0382\}$ . The flow exhibits smooth transitions across the critical points.

As shown in the top-left panel of Fig. 4, the poloidal Alfvén speed and FM speed increase from zero at the launching surface. Since Alfvén waves are transverse and propagate along magnetic field lines, their poloidal components diminish rapidly to zero in the regions near and far from the black hole, where the magnetic field becomes predominantly toroidal. In contrast, FM waves propagate in arbitrary directions rather than being constrained to the field lines. Consequently, the poloidal FM speed remains finite on both sides of the launching surface.

The top-right panel of Fig. 4 shows the plasma density. The outflow density drops rapidly with increasing  $x$ , while the inflow density exhibits modest variation near the black hole. The bottom-left panel presents the three-velocity components,  $v^i = u^i / u^t$ . In the outflow,  $v^r$  dominates and approaches unity at large distances, while  $v^\phi$  decays to zero. In the inflow, frame dragging steadily enhances  $v^\phi$ . The polar component  $v^\theta$  remains small throughout the flow. The bottom-right panel displays the magnetic field components, which are continuous across the launching surface. At large distance, they asymptote to  $\sqrt{B_r B^r} \rightarrow 2\psi/x^2$ ,  $\sqrt{B_\theta B^\theta} \rightarrow 2\psi/x^3$  and  $\sqrt{B_\phi B^\phi} \rightarrow 2\psi/x$ , forming a highly toroidal structure with  $|B^\phi| \gg |B^r| \gg |B^\theta|$ .

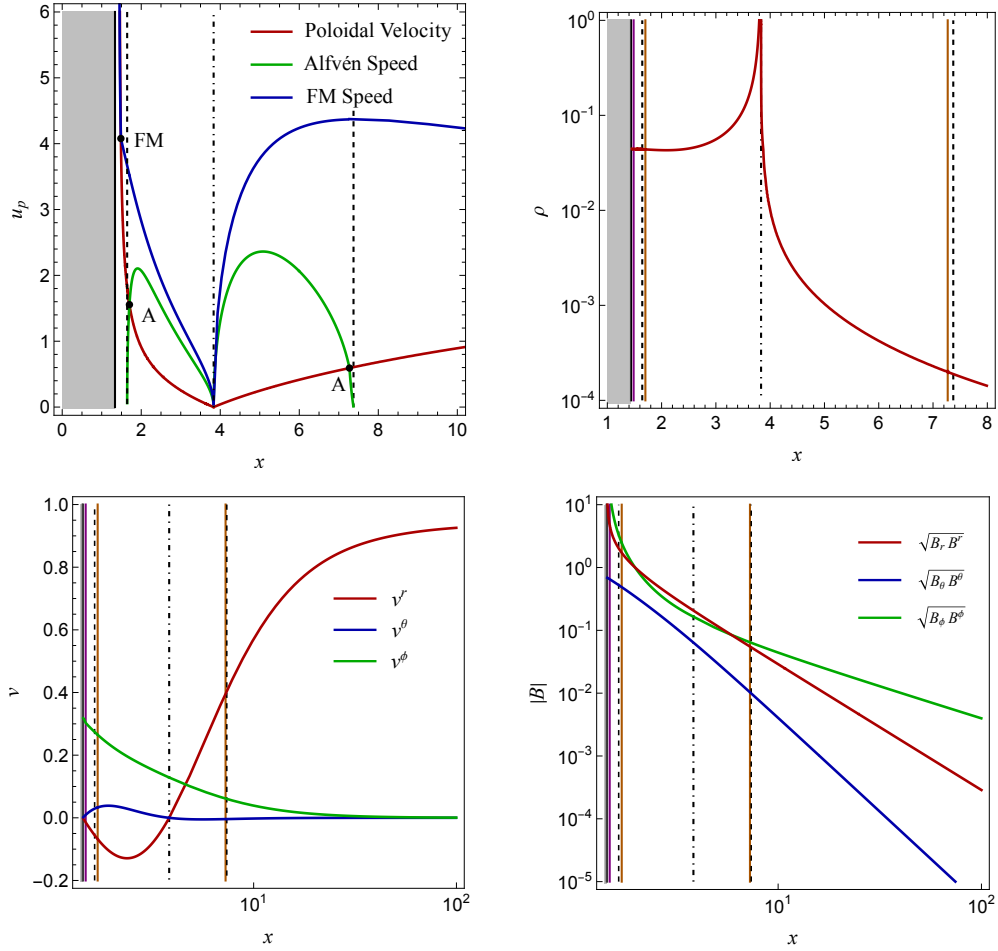


Figure 4: A representative BZ flow solution is shown, along a parabolic field line with  $q = 1$ ,  $\theta_+ = \pi/2$ . In all panels, the gray-shaded region indicates the black hole interior; the black dot-dashed line marks the launching surface; black dashed lines show the light cylinders; the purple line marks the fast magnetosonic point; and orange lines mark the Alfvén points. **Top-left:** Poloidal velocity  $u_p$  for both inflow and outflow (red), Alfvén speed (green), and fast magnetosonic speed (blue). The point labeled FM marks the fast magnetosonic point, while the points labeled A indicate the Alfvén points. **Top-right:** Plasma number density. **Bottom-left:** Components of the three-velocity:  $v^r$ ,  $v^\theta$ , and  $v^\phi$  where  $v^i = u^i/u^t$ . **Bottom-right:** Spatial components of the magnetic field.

Inflowing plasma with specific energy  $E_{\text{in}} < 0$  enables energy extraction from a rotating black hole via the MHD Penrose process [72, 73], which fundamentally relies on the presence of the ergosphere. Specifically, when  $|\Omega_H| > |\Omega_F| > 0$  and  $\text{sign}(\Omega_H) = \text{sign}(\Omega_F)$ , the existence of an Alfvén point within the ergosphere can lead to a negative inflow energy (see Appendix B.1).



## 4.2 Hyperbolic magnetic fields

For field lines anchored to the disk, their asymptotic behavior—whether collimated or open—depends on the type of the accretion system, as well as how far from the black hole they are anchored in the accretion flow [74]. In this section, we illustrate the properties of BP outflows with open field lines, by using a simplified hyperbolic-type stream function. Up to an overall normalization, the stream function in the cylinder coordinates takes

$$\psi = \frac{x^2}{b^2 + z^2}, \quad (4.5)$$

where  $b$  is a constant. Constant  $\psi, b$  parameterizes a hyperbolic field line within the poloidal plane, whose footpoint on the equatorial plane ( $z = 0$ ) is given by  $x_0(\psi) = b\sqrt{\psi}$ . At large distances, the field line satisfies  $x = \sqrt{\psi}z$ , indicating that  $(\psi)^{-1/2}$  serves as the asymptotic slope. In Appendix E.2, we examine an alternative hyperbolic field that arises as a solution to the FFE [75, 76], and we observe similar outflow behavior to that discussed in this section.

As shown in Sec. 3.3, the angular velocity of the disk-threading field line is determined by the disk rotation itself. Here, we assume a thin Keplerian disk outside the innermost stable circular orbit (ISCO), with  $\Omega_{\text{disk}} = \Omega_K = (x^{3/2} + a)^{-1}$ , which determines the field-line angular velocity as

$$\Omega_F(\psi) = \frac{1}{b^{3/2}\psi^{3/4} + a}. \quad (4.6)$$

Furthermore, the asymptotic poloidal field is  $B_p = 2\psi\sqrt{1+\psi}x^{-2}$ . Applying the asymptotic expression for  $\eta$  (Eqs. (2.10)) to the hyperbolic stream function yields

$$\eta = \frac{2\Omega_F^2\psi\sqrt{\psi+1}}{(\gamma_\infty^2 - 1)^{3/2}}. \quad (4.7)$$

By combining Eqs. (3.1) and (4.7), the wind equation (2.8) can be solved to obtain the flow velocity for given  $\gamma_\infty$ . Fig. 5 illustrates the outflow structure, along a representative field line with  $x_0 = \sqrt{20}(r_{\text{ISCO}} - 1)$  and  $b = r_{\text{ISCO}} - 1$ . The top-left panel shows the poloidal flow velocity, Alfvén speed, and FM speed, all of which increase from zero at the disk surface. Beyond the Alfvén point, the poloidal Alfvén speed rapidly declines to zero, while the poloidal FM speed remains finite—analogueous to the behavior observed in the BZ flow in Sec. 4.1.2.

The radial component of the outflow clearly dominates and asymptotically approaches unity, while the azimuthal component gradually decays to zero, as shown in the bottom-left panel in Fig. 5. The polar component remains negligible throughout the flow. The bottom-right panel shows the spatial components of the magnetic field. At large radii, the field becomes strongly toroidal, characterized by:  $\sqrt{B^r B_r} \rightarrow 2\psi\sqrt{1+\psi}/x^2$ ,  $\sqrt{B^\theta B_\theta} \rightarrow 2b^2\psi^{5/2}(1+\psi)^{-1}/x^4$  and  $\sqrt{B^\phi B_\phi} \rightarrow 2\psi\sqrt{1+\psi}/x$ .

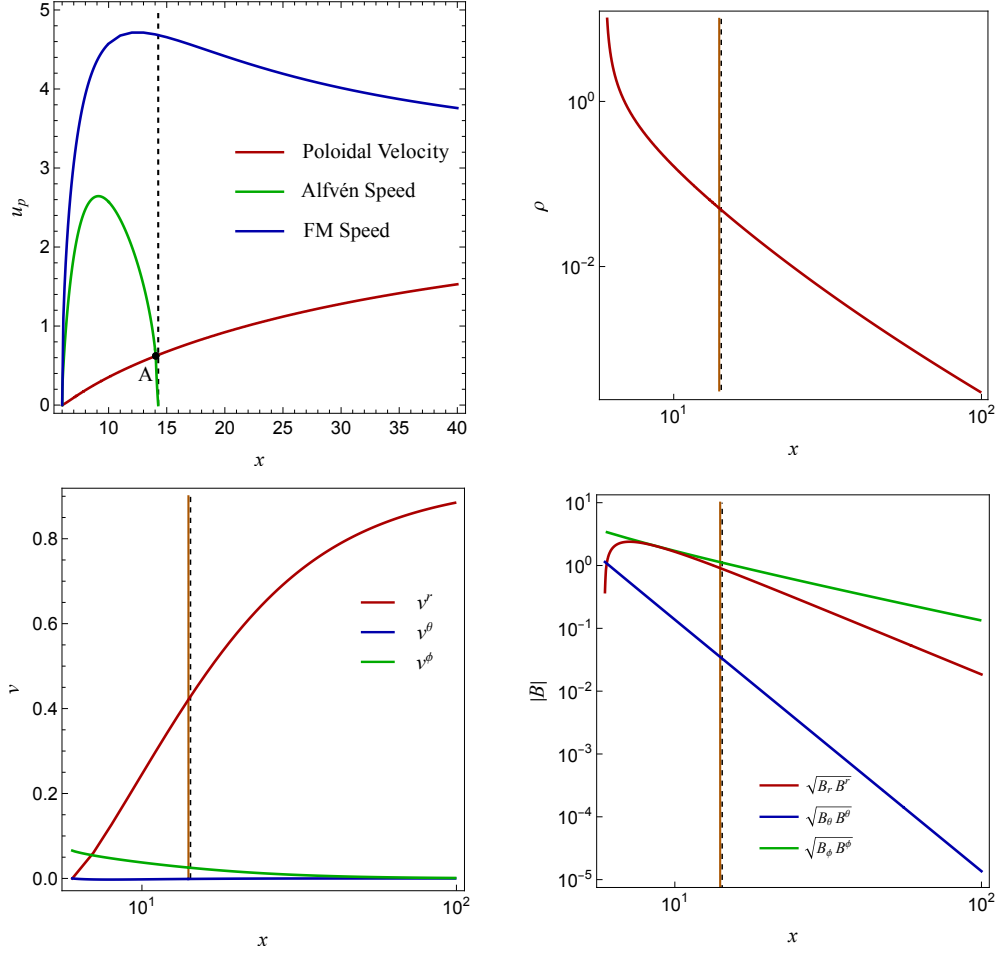


Figure 5: A representative BZ flow solution is shown, along a parabolic field line with  $x_0 = (r_{\text{ISCO}} - 1)\sqrt{20}$  and  $b = r_{\text{ISCO}} - 1$ . We set the spin  $a = 0.9$  and terminal Lorentz factor  $\gamma_\infty = 3$ . The light cylinder is shown as black dashed lines, and the Alfvén surface as an orange line. **Top-left:** Poloidal velocity  $u_p$  for outflow (red), Alfvén speed (green dashed), and FM speed (blue dashed). The point labeled A indicates the outflow Alfvén points. **Top-right:** Plasma density, which diverges near the launch point. **Bottom-left:** Components of the three-velocity  $v^r$ ,  $v^\theta$ , and  $v^\phi$ . **Bottom-right:** Magnetic field components  $\sqrt{B^r B_r}$ ,  $\sqrt{B^\theta B_\theta}$ , and  $\sqrt{B^\phi B_\phi}$ .

We briefly contrast our model with the classical BP scenario [41], where outflows are driven purely by the magneto-centrifugal mechanism: rigid field lines transmit centrifugal force while pressure gradients and magnetic tension are neglected. In that framework, outflows can be launched only if field lines emerge at angles less than  $60^\circ$  to the disk surface [41, 77]. In contrast, our ideal MHD model includes both magnetic pressure and tension. The outflow is primarily accelerated by the magnetic pressure gradient, while tension governs collimation. This allows launching from field lines at larger inclination angles, as shown in Sec. 5.

For a disk-threading magnetic field line, solving  $k'_0 = 0$  might yield two solutions: one exactly at the equatorial plane ( $z = 0$ ), and the other slightly above it. The equatorial solution usually corresponds to a local minimum of  $k_0$ , where  $B^r|_{\text{LS}} = 0$  and both the magnetic-pressure gradient and magneto-centrifugal forces are negligible. In contrast, the off-equatorial solution corresponds to a local maximum of  $k_0$  and is identified as the actual launching point, i.e., the disk surface. However, a subtle point arises: due to the geometrically thin disk assumption, only magnetic field lines with a local maximum of  $k_0$  near the equatorial plane are admissible.

In the case of a hyperbolic field, the  $k_0$  maximum always lies above the equatorial plane. For instance, under the parameters used in Fig. 5, the solutions to  $k'_0 = 0$  are  $z = 0$ ,  $z = 0.2130$ . The extent to which the launching surface deviates from the equatorial plane depends on the asymptotic slope of the hyperbola, as shown in the left panel of Fig. 6: increasing the slope shifts the  $k_0$  maximum (marked by black dots) farther from the equatorial plane. A steeper slope yields more collimated field lines, requiring a greater vertical distance for  $B^r$  to become sufficiently strong to drive an outflow. To remain consistent with the assumption of a geometrically thin disk, model parameters should be chosen such that the launching point remains close to the equatorial plane.

## 5 Hybrid jet model

### 5.1 Set up

In the preceding sections, we analyzed the flow structures along magnetic field lines that thread either the horizon or the disk independently. In realistic black hole magnetospheres, the global magnetic field is generated and sustained by the inflowing plasma. As the accretion flow advects inward toward the event horizon, it drags magnetic field lines along with it. These field lines are subsequently anchored to the spinning black hole on horizon scales. As a result, magnetic field lines can simultaneously thread both the event horizon and the accretion disk, enabling the coexistence of the BZ and BP processes [12–16, 44]. It is evident that the magnetic field line structure must remain continuous across the boundary between the BZ and BP regions. More importantly, collimation is not limited to the BZ jet; it can also occur within the BP region near the horizon, driven by the rapid rotation of magnetic field lines [16].

In this section, we present a hybrid jet model that incorporates both horizon-threading and disk-threading magnetic field lines. To maintain analytical tractability, we adopt the parabolic stream function introduced in Eq. (4.1), which generates smoothly collimated field lines extending from the event horizon to large distances. The field lines with  $\psi \leq r_+$  intersect the horizon and define the BZ region, while those with  $\psi \geq r_+$  intersect the equatorial plane outside the horizon, corresponding

to the BP region. It is important to note that, as discussed in Sec. 3.3, magnetic field lines must be orthogonal to the equatorial plane. However, since our focus is on the flow just above the disk surface—slightly offset from the equatorial plane—the magnetic field can acquire a finite radial component, thereby enabling magneto-centrifugal acceleration (as illustrated in Fig. 1). This justifies the use of the parabolic field configuration as an approximate representation of the BP region.

In the BZ region, we adopt the field-line angular velocity given in Eq. (4.3), which captures the interaction between the rotating black hole and the magnetic field. In the BP region, we assume  $\Omega_F = \Omega_K$  outside the ISCO, allowing for jet launching near the equatorial plane, as discussed in Sec. 3.3. We emphasize that this is a simplified prescription, based on the assumption of a geometrically thin, ideal MHD accretion disk with a correspondingly thin launching surface. In more realistic scenarios, sub-Keplerian flows are expected to develop near the equatorial plane. These flows are typically described in a phenomenological manner and will be investigated in future work.

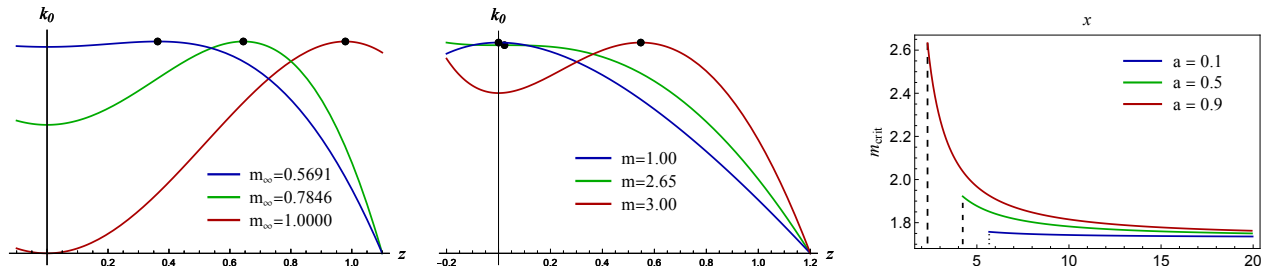


Figure 6: **Left:** Variation of  $k_0$  along disk-threading hyperbolic field lines [Eq. (4.5)], plotted for different asymptotic slopes  $m_\infty \equiv \psi^{-1/2}$ . **Middle:** Variation of  $k_0$  along disk-threading parabolic field lines with various footpoint slopes  $m$ . In both panels, the footpoints of field lines are fixed at the ISCO, where  $\Omega_F$  follows the Keplerian motion. For clarity, the values of  $k_0$  have been rescaled in each plot to highlight the trends. **Right:** Critical slope for the parabolic field, above which the maximum of  $k_0$  shifts away from the equatorial plane, shown as a function of the footpoint radius. The vertical dashed line marks the location of the ISCO.

For parabolic field configurations with  $\Omega_F = \Omega_K$ , introducing a nonzero slope at the equatorial plane allows for the existence of a near-equatorial local maximum of  $k_0$ —unlike the hyperbolic field case discussed in Sec. 4.2. To illustrate this, we examine the variation of  $k_0$  along field lines with different slopes. Specifically, we generalize the stream function in Eq. (4.1) to a parabolic form  $z = (2\psi)^{-1}m(x^2 - \psi^2)$ , where  $m$  denotes the field line slope at the equator. The middle panel of Fig. 6 shows how the slope  $m$  affects the profile of  $k_0$ . Importantly, when the slope is less than a critical value,  $m \leq m_{\text{crit}}$ , the maximum of  $k_0$  remains anchored to the equatorial plane. The right panel of Fig. 6 presents the dependence of  $m_{\text{crit}}$  on the footpoint radius of the field line. As the footpoint moves outward from the black hole,  $m_{\text{crit}}$  decreases; conversely, increasing the black hole spin leads to a higher value of  $m_{\text{crit}}$ . In this study, we focus on  $m = 1$ , which remains below the critical threshold for

all spin values considered. As a result, the launching surface consistently resides near the equatorial plane across the entire parameter space.

In the region between the ISCO and the event horizon, plasma cannot maintain stable circular motion and instead plunges inward toward the black hole. In this region, the rotation of the magnetic field lines is expected to follow the angular velocity of the infalling plasma. Here, we assume that  $\Omega_F$  equals the angular velocity of matter on a purely inspiraling trajectory originating from the ISCO, and set  $\Omega_F = (g^{\phi\phi}l_{\text{ISCO}} - g^{t\phi})(g^{t\phi}l_{\text{ISCO}} - g^{tt})^{-1}$ , where  $l_{\text{ISCO}}$  denotes the angular momentum density at the ISCO. Due to the absence of stable Keplerian orbits in this plunging region, the condition for a near-equatorial launching surface—required in Sec. 3.3—is no longer strictly satisfied. However, we find that for black hole spin  $a \gtrsim 0.75$ , the deviation of the launching surface from the equatorial plane remains moderate<sup>4</sup>. For spins  $a \geq 0.9$ , the deviation becomes negligible, allowing us to safely approximate the launching surface as near-equatorial and assign the associated conserved quantities accordingly. Therefore, in the high-spin regime favored by jet observations, our model remains applicable and provides an effective description of the hybrid jet flow.

## 5.2 Poloidal structures

Employing the strategy for calculating MHD jet flows introduced in previous sections, we now turn to visual representations of the hybrid jet flow structure. Fig. 7 shows the distribution of the poloidal velocity and the magnetization parameter (Eq. (2.11)) across magnetic field lines. Fig. 8 displays the corresponding density structure. In all figures, we set the black hole spin  $a = 0.9$ , the collimation index  $q = 1$ , and a uniform terminal Lorentz factor  $\gamma_\infty = 3$  across field lines.

We notice that a clear difference exists in the shapes of the light cylinders between the BZ and BP flows, reflecting their distinct field-line angular velocities. Moreover, since the Alfvén point lies near the light cylinder, the transition from magnetically to kinetically dominated flow also occurs nearby, making the light cylinder a qualitative boundary between slow and fast flows [59]. As shown in Fig. 7,  $u_p$  is lower and  $\sigma_M$  higher inside the light cylinder. This leads to a characteristic flow structure shaped by the light cylinder: slower speed and higher magnetization near the jet boundaries, and faster speed with lower magnetization near the BZ–BP interface.

For the density profile, the hybrid jet possesses a higher density near the center, which gradually decreases toward the outer boundaries. The density of the inflow is much higher than that of the outflow, which can be attributed to the strong gravitational capture of particles by the black hole within the loading region. If a geometrically thick loading zone, rather than the idealized thin launching

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<sup>4</sup>For example, at  $a = 0.75$ , the maximum vertical displacement of the launching surface occurs at the event horizon, where the local maximum of  $k_0$  reaches  $z = 1$ . This deviation decreases rapidly with increasing black hole spin or radial distance from the horizon—at  $a = 0.9$ , it reduces to  $z = 0.1$  at  $r = r_+ + 0.2$ .

surface assumed here, is taken into account, the density contrast between the inflow and outflow may be mitigated.

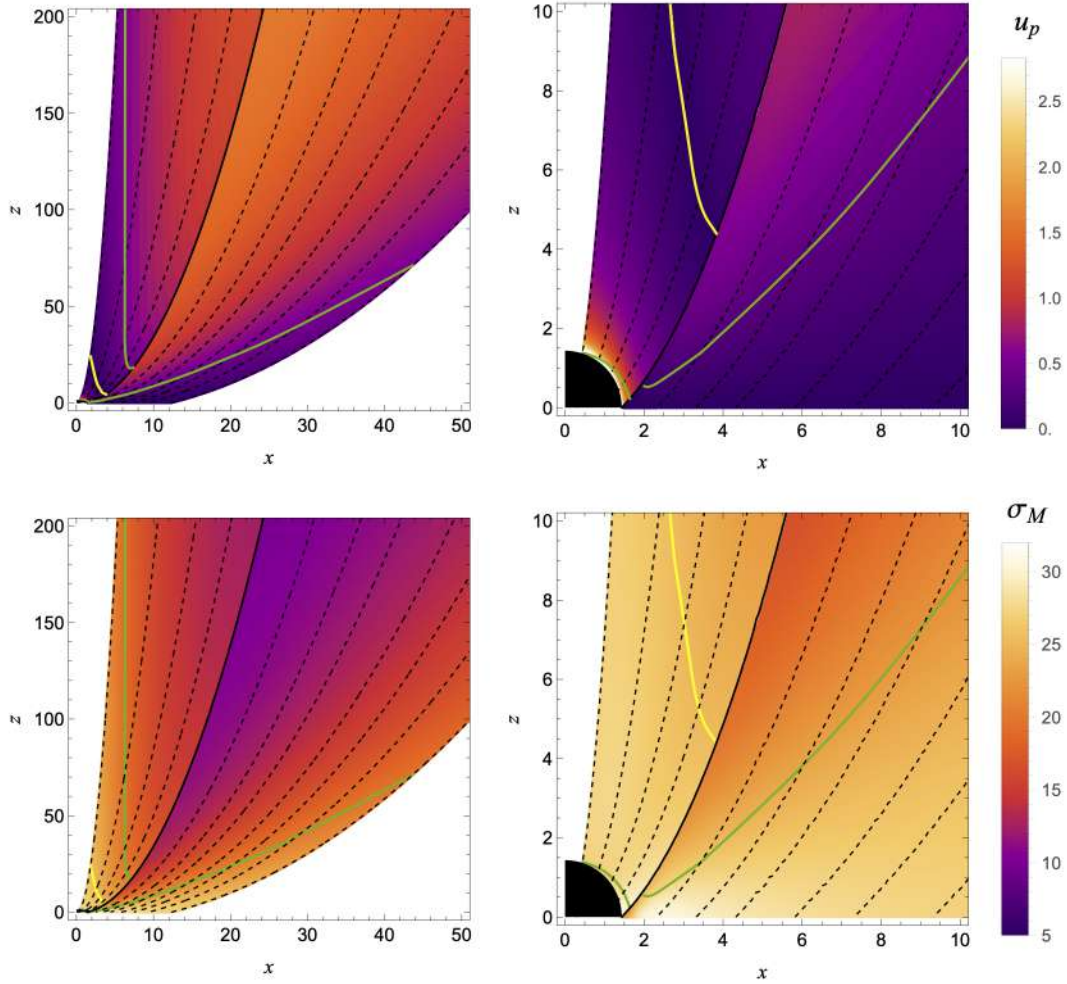


Figure 7: Distributions of the poloidal velocity (**Top**) and magnetization parameter (**Bottom**) for the hybrid model. From left to right, the panels show the global structure and successive zoom-in views toward the event horizon. In each panel, dashed lines trace magnetic field lines; the black solid line marks the BZ–BP interface; the yellow curve indicates the launching surface of horizon-threading field lines; and green curves denote the light cylinders.

A distinct feature arises at the interface between the BZ and BP jets, where the poloidal velocity, magnetization, and density exhibit discontinuous behavior. This arises from a mismatch in the rotation of magnetic field lines across the interface. Specifically, the rotation of field lines threading the black hole horizon is governed primarily by the rotation of the event horizon itself, establishing a characteristic relationship between the field-line angular velocity  $\Omega_F$  and the black hole angular velocity  $\Omega_H$ . Under the FFE approximation, efficient energy extraction requires  $\Omega_F \approx \Omega_H/2$  [40]. Although the inertia of ions can retard the rotation of magnetic field lines, it is generally insufficient to overcome



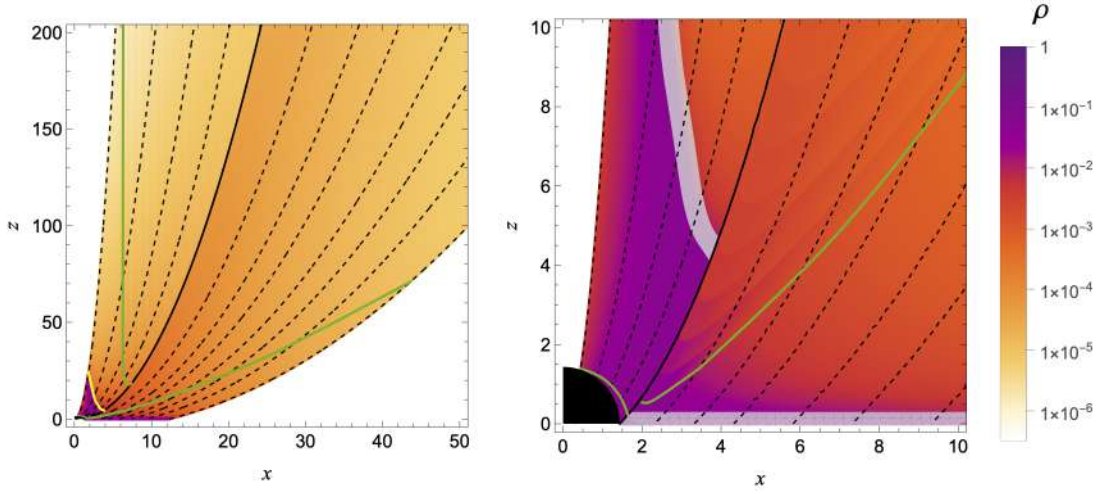


Figure 8: Plasma density distribution for the hybrid model. From left to right, the panels display the global structure and successive zoom-in views toward the event horizon. Gray-shaded bands are overlaid in the zoomed-in panel to indicate a manual cutoff near the launching surface, where the density formally diverges. All other plot elements are identical to those in Fig. 7.

the black hole’s dominant influence—a conclusion supported by dynamical simulations [15, 16]. In contrast, the field lines anchored in the accretion disk corotate with the disk plasma. As they approach the event horizon, the rotation of any equatorial disk configuration asymptotically matches that of the black hole, leading to  $\Omega_F \rightarrow \Omega_H$  [31]. Consequently, near the interface, the field-line angular velocities satisfy  $\Omega_F|_{BP} \simeq 2\Omega_F|_{BZ} \rightarrow \Omega_H$ . In our model with spin parameter  $a = 0.9$ , the field line in the BZ region has an angular velocity of  $\Omega_F = 0.1287$ , while the adjacent BP field line exhibits a higher value of  $\Omega_F = 0.3134$ .

### 5.3 Asymptotic density profile

As a next step toward understanding the role of the BZ–BP hybrid, we turn to examine the asymptotic behavior of plasma density. Starting from the asymptotic expression for the mass flux Eq. (2.10), the density far from the black hole can be approximated as  $\rho = \eta B_p / u_p \approx (\gamma_\infty^2 - 1)^{-2} \Omega_F^2 B_p^2 x^2$ . Using the asymptotic scaling relations for a parabolic stream function,  $B_p \approx 2\psi x^{-2}$ ,  $z \approx (2\psi)^{-1} x^2$ , we obtain the asymptotic expression for the density in terms of axial height as  $\rho \approx 2(\gamma_\infty^2 - 1)^{-2} \psi \Omega_F^2 z^{-1}$ . Accordingly, for a given height  $z$ , the field-line-dependent density distribution can be characterized by an effective density:

$$\rho_{\text{eff}} \equiv z\rho \Big|_{z \rightarrow \infty} = \frac{2\psi \Omega_F^2}{(\gamma_\infty^2 - 1)^2}. \quad (5.1)$$

This relation indicates that field lines with higher angular velocities yield higher  $\rho_{\text{eff}}$ , while those with larger terminal Lorentz factors correspond to lower  $\rho_{\text{eff}}$ . In realistic scenarios,  $\gamma_{\infty}$  generally varies across field lines and is dynamically constrained by the trans-field G–S equation [26, 27]. The cross-field profile of  $\gamma_{\infty}(\psi)$  thus plays a significant role in shaping both the velocity and density distributions in the poloidal plane. To qualitatively capture this effect, we adopt a synthetic profile modeled by a multipeak Gaussian function:

$$\gamma_{\infty}(\psi) = \gamma_0 + \sum_{i=1}^{\infty} \gamma_i \exp \left[ - \left( \frac{\psi - \psi_i}{\delta\psi_i} \right)^2 \right], \quad (5.2)$$

where  $\gamma_0$  sets the background Lorentz factor, and  $\gamma_i$ ,  $i \geq 1$  controls the amplitudes of the Gaussian peak centered at  $\psi_i$  with characteristic width  $\delta\psi_i$ .

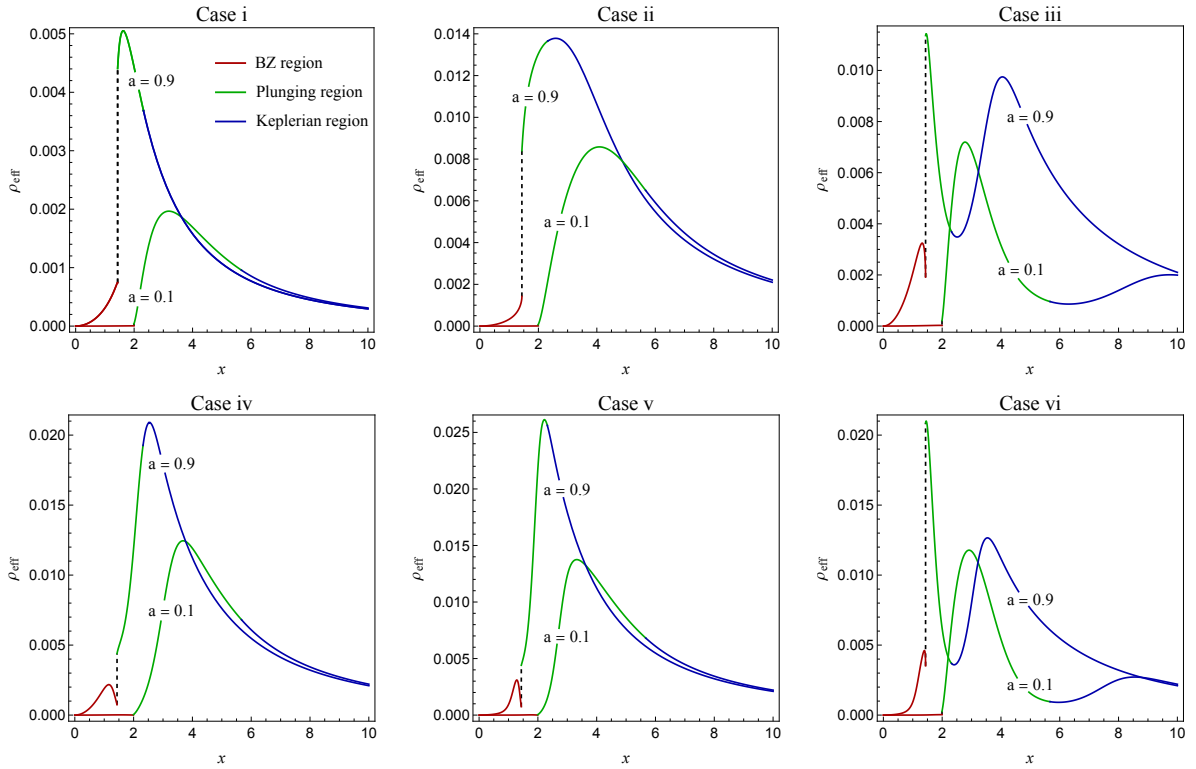


Figure 9: Effective density  $\rho_{\text{eff}}$  as a function of field-line footpoint coordinate  $x$  is shown across the BZ region, the plunging region, and the Keplerian disk region. The top row and bottom-left panels display results for four representative single-peaked Lorentz factor profiles: (i) uniform, (ii) polar-axis enhanced, (iii) ISCO-enhanced, and (iv) interface-enhanced. The bottom-middle and bottom-right panels present two double-peaked configurations: (v) polar plus interface-enhanced, and (vi) polar plus ISCO-enhanced. All cases assume a black hole spin parameter of  $a = 0.9$ .

We focus on four representative, single-peaked cases (with  $\gamma_i = 0, i \geq 2$ ): (i) a uniform profile with  $\gamma_0 = 3, \gamma_1 = 0$ ; (ii) a center-enhanced profile with  $\gamma_0 = 2, \gamma_1 = 1, \psi_0 = 0, \delta\psi = \sqrt{2}r_+$ ; (iii) an



ISCO-enhanced profile with  $\gamma_0 = 2, \gamma_1 = 1, \psi_0 = r_{\text{ISCO}}, \delta\psi = 0.3\sqrt{2}r_{\text{ISCO}}$ ; (iv) an interface-enhanced profile with  $\gamma_0 = 2, \gamma_1 = 1, \psi_0 = r_+, \delta\psi = 0.3\sqrt{2}r_+$ . Case (ii) corresponds to the scenario where the terminal Lorentz factor peaks along the polar axis, representing efficient acceleration in the jet spine. Case (iii) concentrates the peak at the ISCO-anchored field lines, while case (iv) places it near the BZ–BP interface.

In the hybrid model considered here, a stream function used to label field lines is defined as follows:  $\psi(\theta_+) = r_+(1 - \cos\theta_+)$ ,  $0 \leq \theta_+ \leq \pi/2$  in the BZ region, and  $\psi(x) = x$ ,  $x \geq r_+$  in the BP region. To allow a unified horizontal axis across the interface, we extend  $x$  into the BZ region through the substitution  $x = r_+ \sin\theta_+ \leq r_+$ , thus covering horizon-threading field lines. Fig. 9 shows the variation of the effective density  $\rho_{\text{eff}}$  across field lines, plotted against the footpoint coordinate  $x$ . For spin parameter  $a = 0.9$ , a clear discontinuity in  $\rho_{\text{eff}}$  is observed across the BZ–BP interface for all considered  $\gamma_\infty$  profiles, primarily due to the discontinuity in  $\Omega_F$  here. Near the polar axis ( $\psi \rightarrow 0$ ),  $\rho_{\text{eff}}$  approaches zero, indicating a quasi-vacuum jet spine induced by centrifugal forces. At larger  $x$  (i.e., farther from the black hole),  $\rho_{\text{eff}}$  decreases monotonically, consistent with the density profiles shown in Fig. 8. Regardless of the specific  $\gamma_\infty$  profile, the density generally peaks in the inner disk region (i.e.,  $x \lesssim r_{\text{ISCO}}$ ). However, the detailed structure of  $\rho_{\text{eff}}$  remains sensitive to the particular shape of  $\gamma_\infty(\psi)$ .

Finally, we examine the influence of double-Gaussian-peak configurations by considering the following two cases: (v) a polar-enhanced plus interface-enhanced profile, characterized by  $\gamma_0 = 2, \gamma_1 = 1, \gamma_2 = 1, \psi_1 = 0, \delta\psi_1 = 0.2\sqrt{2}r_+, \psi_2 = r_+$ , and  $\delta\psi_2 = 0.2\sqrt{2}r_+$ ; (vi) a polar-enhanced plus ISCO-enhanced profile, with  $\gamma_0 = 2, \gamma_1 = 1, \gamma_2 = 1, \psi_1 = 0, \delta\psi_1 = 0.2\sqrt{2}r_+, \psi_2 = r_{\text{ISCO}}$ , and  $\delta\psi_2 = 0.2\sqrt{2}r_{\text{ISCO}}$ . The resulting effective density distributions are shown in the bottom-middle and bottom-right panels of Fig. 9. In both cases, the polar-axis peak of  $\gamma_\infty$  contributes little to the overall density profile, as it corresponds to a highly magnetized yet quasi-vacuum jet spine.

## 6 Summary and discussion

In this work, we investigated steady, axisymmetric, ideal MHD jets by developing a unified framework that encompasses both black hole-driven and disk-driven outflows. The analysis begins with prescribed magnetic field geometries—either threading the event horizon or anchored in the accretion disk—and proceeds by solving the relativistic Bernoulli equation along each field line, accompanied by a critical-point analysis. This approach enables a detailed examination of jet acceleration and energy conversion processes under ideal MHD conditions. Based on the conservation laws of ideal MHD, we derived a general criterion for the successful launching of cold outflows, applicable to both BZ and BP type jets. This framework enables a tractable, semi-analytic investigation of jet launching,

acceleration, and energy conversion processes.

Furthermore, we proposed a hybrid jet model based on a global parabolic magnetic field configuration that smoothly connects the BZ and BP regions. The model naturally gives rise to a velocity shear and density discontinuity at the BZ–BP interface, resulting from the discontinuity in field-line angular velocity across the transition. We illustrated these features through poloidal velocity and density profiles, and we further demonstrated that the light cylinder plays a key role in shaping the poloidal flow structure.

We quantitatively analyzed the influence of field-line angular velocity and terminal Lorentz factor on jet structure by examining the effective density, defined as  $\rho_{\text{eff}} = 2\psi\Omega_F^2/(\gamma_\infty^2 - 1)^2$ . To explore the impact of different distributions of  $\gamma_\infty(\psi)$ , we employed synthetic profiles modeled by multi-peaked Gaussians. In all cases,  $\rho_{\text{eff}}$  exhibits a pronounced discontinuity across the BZ–BP interface, driven by the jump in  $\Omega_F$ , and features a vacuum-like spine near the polar axis. A distinct peak in  $\rho_{\text{eff}}$  typically emerges in the inner disk region. These findings may provide insight into observed edge-brightened jet morphologies and transverse stratification. Moreover, the sharp transition between the BZ and BP regions identified in the model may act as a favorable origin for MHD instabilities—such as magnetic reconnection—thereby potentially enabling non-thermal particle acceleration.

Looking ahead, the hybrid model can be further developed by solving the G–S equation under appropriate boundary conditions, which would yield a more realistic terminal Lorentz factor profile,  $\gamma_\infty(\psi)$ . In addition, since the light cylinder serves as a critical surface for jet dynamics, a detailed investigation of the flow structure in its vicinity is essential. To more accurately capture the physics of jet launching, it is necessary to incorporate a finite-thickness mass-loading region in future work.

Recent advances in GRMHD simulations and polarized radiative transfer modeling have substantially improved our understanding of black hole imaging and jet formation mechanisms [44, 78–81]. The detection of ordered magnetic fields and synchrotron polarization in M87\* [82, 83], together with growing efforts in polarimetric jet modeling [59, 76, 84], has motivated the development of jet models that connect MHD theory to observable structures on scales of hundreds of gravitational radii [85, 86]. The hybrid BZ–BP model proposed in this work describes the transition from the magnetically dominated jet base to the asymptotic outflow at large distances. The resulting jet morphology can be computed and used to generate synthetic images, enabling direct comparisons with GRMHD simulations and high-resolution observations in future.

## Acknowledgments

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## A MHD conservation laws

In this section, we present a detailed analysis of the stationary, axisymmetric magnetic field, and derive the explicit forms of the conserved quantities.

### A.1 Field transformation

In the main text, we follow the convention commonly adopted in the literature and introduce the “pseudo-electromagnetic fields”, which are defined as

$$E^\mu = (dt)_\nu F^{\mu\nu} = F^{t\mu}, \quad B^\mu = -(dt)_\nu (\star F)^{\mu\nu} = -(\star F)^{t\mu}. \quad (\text{A.1})$$

The pseudo-fields differ from the fields measured by the normal observer (given by the tangent vector  $n_\mu = (-\alpha, 0, 0, 0)$ ) by a lapse function  $\alpha = (-g^{tt})^{-1/2}$  [87]. While the pseudo-fields do not represent the physical fields directly, they are widely used in GRMHD analyses due to their computational convenience. For ideal MHD, we have  $F^{\mu\nu}u_\nu = 0$ , and the transformation between the pseudo-fields and the comoving-frame magnetic field,  $b^\mu = -u_\nu (\star F)^{\mu\nu}$ , is given by

$$E^\mu = F^{t\mu} = -\epsilon^{t\mu\alpha\beta} b_\alpha u_\beta, \quad B^\mu = b^{[\mu} u^{t]}, \quad (\text{A.2})$$

$$b^t = B^\mu u_\mu, \quad u^t b^i = B^i + B^t u^i, \quad (\text{A.3})$$

where  $i$  represents the spatial indices.

### A.2 Stationary, axisymmetric magnetic field

In a stationary and axisymmetric system, physical quantities vary only in the  $(r, \theta)$  plane—denoted as the poloidal plane. In this section, we would like to briefly review the derivation of the field structure by imposing symmetry requirements and the ideal MHD condition. Since the electromagnetic

tensor only depends on  $r, \theta$ , the azimuthal electric field vanishes,  $E^\phi = \kappa^{-1}(\partial_\phi A_t - \partial_t A_\phi) = 0$ , where  $\kappa \equiv g_{t\phi}^2 - g_{tt}g_{\phi\phi}$ . Employing the ideal MHD condition in the  $t$  and  $\phi$  directions, we obtain

$$E^\mathcal{P} u_\mathcal{P} = 0, \quad B^r u^\theta - B^\theta u^r = 0. \quad (\text{A.4})$$

The first equation implies that the electric field is orthogonal to the plasma velocity in the poloidal plane. The second equation shows that the poloidal magnetic field is exactly aligned with the poloidal velocity, i.e., the magnetic field vector lies along the direction of fluid motion in the  $(r, \theta)$  plane. This alignment is a direct consequence of the infinite conductivity assumption in ideal MHD. The exceptional case is  $u^r = u^\theta = 0$ , where the electromagnetic field cannot be determined by ideal MHD alone and additional conditions are required. Using the second equation in Eq. (A.4), we have

$$B^r = \frac{\partial_\theta A_\phi}{\sqrt{-g}}, \quad B^\theta = -\frac{\partial_r A_\phi}{\sqrt{-g}}, \quad u^\mathcal{P} \partial_\mathcal{P} A_\phi = 0, \quad (\text{A.5})$$

where  $g$  is the determinant of the spacetime metric. We see that  $A_\phi$  is conserved along the fluid streamlines. In fact,  $A_\phi(r, \theta)$  marks the magnetic flux enclosed by a circular loop of radius  $r$  and polar angle  $\theta$ , and commonly referred to as the stream function  $\psi \equiv A_\phi(r, \theta)$ . The stream function serves as a label for magnetic field lines and plays a central role in describing the field geometry.

The ideal MHD condition in the poloidal directions leads to

$$\begin{aligned} g_{rr} u^t E^r &= g^{tt} u^\theta F_{r\theta} + (g^{tt} u^\phi - g^{t\phi} u^t) F_{r\phi}, \\ g_{\theta\theta} u^t E^\theta &= g^{tt} u^r F_{\theta r} + (g^{tt} u^\phi - g^{t\phi} u^t) F_{\theta\phi}. \end{aligned} \quad (\text{A.6})$$

Multiply the first equation of Eq. (A.6) by  $B^r$ , and the second equation by  $B^\theta$ , then add the two resulting equations together to obtain

$$E^\mathcal{P} B_\mathcal{P} = 0, \quad (\text{A.7})$$

meaning that the pseudo-electric field and pseudo-magnetic field are orthogonal within the poloidal plane. Combining Eq. (A.7) with the index lowering relation  $g^{tt} F_{t\mathcal{P}} = (g_{\mathcal{P}\mathcal{P}} E^\mathcal{P} + g^{t\phi} F_{\mathcal{P}\phi})$ , we can deduce  $F_{t\theta} F_{r\phi} = F_{tr} F_{\theta\phi}$ . Hence, we can introduce the following field-line angular velocity:

$$\Omega_F \equiv \frac{F_{t\mathcal{P}}}{F_{\mathcal{P}\phi}}, \quad \mathcal{P} = r, \theta, \quad (\text{A.8})$$

representing the global rotation of the magnetic field line. By substituting Eq. (A.8) in the Bianchi identity in the  $\phi$  direction, we arrive at  $F_{r\phi} \partial_\theta \Omega_F - F_{\theta\phi} \partial_r \Omega_F + (\partial_\theta F_{r\phi} - \partial_r F_{\theta\phi}) \Omega_F = 0$ . Since  $F_{\mathcal{P}\phi} = \partial_\mathcal{P} \psi$ , the last term on the left-hand side of the equation vanishes automatically. Therefore, we have  $B^\mathcal{P} \partial_\mathcal{P} \Omega_F = 0$ , indicating that  $\Omega_F$  is conserved along a field line within the poloidal plane.

The field-line angular velocity is induced by the dragging from the rotating black hole and the accretion disk. For horizon-threading field lines,  $\Omega_F$  is constrained by the regularity condition at the

event horizon, known as the Znajek condition [40]. In contrast, for field lines anchored in the disk  $\Omega_F$  is primarily determined by the local angular velocity of the accretion flow [41]. Additionally, the pseudo-electric field can be rewritten as

$$E^{\mathcal{P}} = - \left( \frac{g^{\mathcal{P}\mathcal{P}} \sqrt{-g}}{g^{tt}} \right) \left( \Omega_F - \frac{g^{t\phi}}{g^{tt}} \right) \tilde{\epsilon}_{\mathcal{P}\mathcal{P}'} B^{\mathcal{P}'} . \quad (\text{A.9})$$

It becomes clear that the pseudo-electric field is induced by both the field line rotation and the spacetime frame dragging, represented by  $g^{t\phi}$ .

### A.3 The MHD conserved quantities

In stationary, axisymmetric ideal MHD, in addition to the stream function  $\psi$  and field line angular velocity  $\Omega_F$ , the conservation of energy-momentum and particle number also yield three conserved quantities along field lines [56]. Firstly, the ideal MHD condition in the  $\phi$  direction yields:  $u^r F_{r\phi} + u^\theta F_{\theta\phi} = 0$ . This allows us to define the mass flux  $\eta$  as

$$\eta \equiv \frac{\sqrt{-g} \rho u^r}{F_{\theta\phi}} = \frac{\sqrt{-g} \rho u^\theta}{F_{\phi r}} = \frac{\rho u^{\mathcal{P}}}{B^{\mathcal{P}}} , \quad (\text{A.10})$$

which describes the particle number flux per unit magnetic flux. Substituting Eq.(A.8) into Eq.(A.10), we obtain  $F_{tr} = -\sqrt{-g} \rho u^\theta \Omega_F \eta^{-1}$  and  $F_{t\theta} = \sqrt{-g} \rho u^r \Omega_F \eta^{-1}$ . Then, applying the Bianchi identity in the  $\phi$  direction,  $\partial_r F_{\theta t} + \partial_\theta F_{tr} = 0$ , we have

$$\partial_{\mathcal{P}} (\sqrt{-g} \rho u^{\mathcal{P}} \Omega_F \eta^{-1}) = \sqrt{-g} \rho u^{\mathcal{P}} \partial_{\mathcal{P}} (\Omega_F \eta^{-1}) = 0 , \quad (\text{A.11})$$

where we have used the particle number conservation,  $\nabla_\mu (\rho u^\mu) = \partial_{\mathcal{P}} (\sqrt{-g} \rho u^{\mathcal{P}}) / \sqrt{-g} = 0$ . Since  $\Omega_F$  is conserved along field lines, we have  $B^{\mathcal{P}} \partial_{\mathcal{P}} \eta = u^{\mathcal{P}} \partial_{\mathcal{P}} \eta = 0$ , which means that  $\eta$  is also a conserved mass flux along field lines. Furthermore, with the help of field-line angular velocity and mass flux, the energy conservation can be written as

$$\begin{aligned} \nabla_\mu T^\mu_t &= \frac{1}{\sqrt{-g}} \partial_{\mathcal{P}} [\sqrt{-g} (h u^{\mathcal{P}} u_t + F^{\mathcal{P}\rho} F_{t\rho})] \\ &= \frac{1}{\sqrt{-g}} \left[ \partial_\theta \psi \partial_r (\eta \mu u_t + \kappa B^\phi \Omega_F) - \partial_r \psi \partial_\theta (\eta \mu u_t + \kappa B^\phi \Omega_F) \right] \\ &= B^{\mathcal{P}} \partial_{\mathcal{P}} \left[ \eta \left( \mu u_t + \frac{\kappa B^\phi \Omega_F}{\eta} \right) \right] . \end{aligned} \quad (\text{A.12})$$

Through the same procedure, one can get a similar equation for the angular momentum conservation  $\nabla_\mu T^\mu_\phi = 0$ . As a result, the specific energy and specific angular momentum

$$E \equiv -\mu u_t - \frac{\kappa B^\phi \Omega_F}{\eta} , \quad L \equiv \mu u_\phi - \frac{\kappa B^\phi}{\eta} \quad (\text{A.13})$$

are two conserved quantities along a field line, i.e.,  $B^{\mathcal{P}} \partial_{\mathcal{P}} E = B^{\mathcal{P}} \partial_{\mathcal{P}} L = 0$ .

Therefore, for a stationary and axisymmetric ideal MHD system, we have five conserved quantities:  $\{\psi, \Omega_F, \eta, E, L\}$ , each of which is associated with a underlying symmetry of the MHD system. The stream function  $\psi$  specifies the shape of magnetic field lines (as well as the streamlines) in the poloidal plane;  $\Omega_F, \eta, E, L$  are invariant along each field line.

## B Critical points

In Sec. 2.2, we briefly outlined the existence of three critical points in the wind equation, and in Sec. 2.3, we discussed how the asymptotic FM point constrains the conserved quantities. In this section, we provide a detailed analysis of the properties and roles of these critical points.

### B.1 The Alfvén point

The Alfvén point is defined as the location where the denominator on the right-hand side of Eq. (2.8) vanishes, marking the transition at which the poloidal plasma velocity equals the Alfvén speed:

$$u_p^2 = u_A^2 \equiv \frac{B_p^2 k_0}{\mu \rho} \Big|_A, \quad (\text{B.1})$$

where the subscript “A” denotes the Alfvén point. Since the wind equation must remain regular at the Alfvén point, the numerator in Eq. (2.8) must also vanish at this location:

$$(k_0 k_2 - 2k_2 M_A^2 - k_4 M_A^4) \Big|_A = -k_0 (k_4 k_0 + k_2) \Big|_A = 0. \quad (\text{B.2})$$

This condition determines the location of the Alfvén point along each streamline, forming a two-dimensional surface (a curve in the poloidal plane). For a monopole-like magnetic field in flat space-time, the Alfvén point is given by  $x_A^2 = \Omega_F L / E$ , which is often treated as a prescribed input parameter [19, 22, 23]. In contrast, we do not fix the Alfvén point a priori. Instead, we specify the set of conserved quantities and require that the flow pass smoothly through all critical points.

The requirement that  $u_t$  and  $u_\phi$  (given by Eqs. (2.5)) remain regular at the Alfvén point, imposes the following constraint on the specific angular momentum:

$$l = - \frac{g_{t\phi} + g_{\phi\phi} \Omega_F}{g_{tt} + g_{t\phi} \Omega_F} \Big|_A, \quad (\text{B.3})$$

indicating that the value of  $l$  is fixed by the location of the Alfvén point. Furthermore, the Alfvén point constrains the sign of the specific energy  $E$  of the plasma. To clarify this, we express the specific

energy in the form

$$E = -k_0^{-1}(g_{tt} + \Omega_F g_{t\phi})(E - \Omega_F L)|_{r_A}. \quad (\text{B.4})$$

Given that  $E - \Omega_F L = \sqrt{k_0}|_{\text{LS}}$ , the sign of  $E$  determined by the sign of the metric combination  $(g_{tt} + \Omega_F g_{t\phi})|_{r_A}$ .

The possibility of negative specific energy for inflowing plasma is intrinsically tied to the structure of the black hole ergosphere. In a rotating black hole spacetime, the ergosphere admits regions where  $(g_{tt} + \Omega_F g_{t\phi})|_{r_A} > 0$ , thereby allowing for  $E < 0$ . To make this explicit, we define a critical surface  $r = r_C(\theta)$  at which  $g_{tt} + \Omega_F g_{t\phi} = 0$ . It can be shown that  $r_C(\theta)$  always lies within the ergosphere (where  $g_{tt} = 0$ ), and this critical surface exists only when  $|\Omega_H| > |\Omega_F| > 0$  and  $\text{sign}(\Omega_H) = \text{sign}(\Omega_F)$  [21]. Therefore, if the Alfvén point lies inside the critical surface, then  $(g_{tt} + \Omega_F g_{t\phi})|_{r_A} > 0$ , leading to a negative  $E$  for the inflow. Such flows correspond to the extraction of energy from the black hole via the MHD Penrose process [72].

## B.2 The acceleration equation

In addition to the Alfvén point, additional critical points can be identified by analyzing the flow acceleration equation. Differentiating Eq. (2.8) along the magnetic field lines yields [21]

$$(\ln u_p)' = \frac{N}{D} = \frac{N}{D_A D_{\text{SF}}}, \quad (\text{B.5})$$

where

$$\begin{aligned} N = & \left[ -(1 + u_p^2)(k_0 - M_A^2)^3 \frac{a_s^2}{1 - a_s^2} + \left( \frac{E}{\mu} \right)^2 M_A^4 (k_0 k_4 + k_2) \right] (\ln B_p)' \\ & - \frac{1}{2(1 - a_s^2)} \left( \frac{E}{\mu} \right)^2 [M_A^4 (k_0 - M_A^2) k_4' + (k_0 k_2 - 3k_2 M_A^2 - 2k_4 M_A^4) k_0'], \\ D_A = & (k_0 - M_A^2)^2 = \left( \frac{\mu \rho}{B_p^2} \right)^2 (u_p^2 - u_A^2)^2, \\ D_{\text{SF}} = & \left[ (u_p^2 - c_s^2)(k_0 - M_A^2) + \left( \frac{E}{\mu} \right)^2 M_A^4 \frac{(k_0 k_4 + k_2)}{(k_0 - M_A^2)^2} \right] = -\frac{\mu \rho}{B_p^2} (u_p^2 - u_{FM}^2)(u_p^2 - u_{SM}^2). \end{aligned} \quad (\text{B.6})$$

Here, we have introduced the sound velocity  $c_s$ :

$$c_s^2 = \frac{a_s^2}{1 - a_s^2}, \quad a_s^2 = \frac{\partial \ln h}{\partial \ln n}. \quad (\text{B.7})$$

In Eq. (B.5),  $D_A D_{SF} = 0$  determines three critical points, where  $u_p$  takes one of the following characteristic speeds:

$$\begin{aligned} u_A^2 &\equiv \frac{B_p^2}{\mu\rho} k_0, \\ u_F^2 &\equiv \frac{1}{2} \left[ (u_0^2 + c_s^2) + \sqrt{(u_0^2 + c_s^2)^2 - 4c_s^2 u_A^2} \right], \\ u_S^2 &\equiv \frac{1}{2} \left[ (u_0^2 + c_s^2) - \sqrt{(u_0^2 + c_s^2)^2 - 4c_s^2 u_A^2} \right], \end{aligned} \quad (\text{B.8})$$

representing the Alfvén speed, fast magnetosonic (FM) speed, and slow magnetosonic (SM) speed, respectively, where  $u_0^2 = u_A^2 + \frac{\kappa(B^\phi)^2}{\mu\rho}$ . The FM speed represents the maximum propagation speed of compressive MHD waves, incorporating both thermal and magnetic contributions. In contrast, the SM speed corresponds to the lower-speed branch of compressive modes.

Each critical point along the flow corresponds to a distinct dynamical transition. The SM point typically arises near the jet base (the loading zone) and marks the onset of acceleration, where the action of plasma pressure and magnetic forces overcomes gravitational confinement. The FM point, by contrast, defines the outermost critical surface; crossing this point signifies that the flow has become causally disconnected from the central engine and has reached its terminal acceleration regime.

The requirement of smooth transitions across critical points imposes regularity conditions, which translate into algebraic constraints on the conserved quantities of the flow. In the cold plasma limit ( $c_s^2 = 0$ ), the FM speed reduces to  $u_F = u_0$  and the SM speed becomes zero. Thus, the smoothness condition at the SM point is set by the boundary conditions at the launching surface. In contrast, at the FM point, smooth crossing requires both the numerator and denominator of Eq. (B.5) to vanish simultaneously. This selects the physically admissible solutions.

### B.3 The asymptotic FM point

At the FM point, by combining the wind equation Eq. (2.8) with the condition  $u_p = u_F$ , we obtain:

$$E^2 = \frac{\mu^2(1 + c_s^2)(k_0 - M_A^2)^3}{k_0^2 k_2 - 3k_0 k_2 M_A^2 + 3k_2 M_A^4 + k_4 M_A^6} \Big|_F, \quad (\text{B.9})$$

$$\eta^2 = -\frac{B_p^2}{\mu^2} \left( -\frac{c_s^2}{M_A^4} + \frac{(1 + c_s^2)(k_0 k_4 + k_2)}{k_0^2 k_2 - 3k_0 k_2 M_A^2 + 3k_2 M_A^4 + k_4 M_A^6} \right)^{-1} \Big|_F, \quad (\text{B.10})$$

where the Mach number reduces to  $M_A^2 \rightarrow \mu|\eta|u_F B_p^{-1}|_F$ . Together with the regularity condition  $N|_F = 0$ , these equations constrain both the location of the FM point and the values of  $E$  and  $\eta$ .

To characterize the outflow at large distances, we adopt cylindrical coordinates defined by  $x \equiv r \sin \theta$ ,  $z \equiv r \cos \theta$ . We assume a cold plasma at infinity, such that  $P = 0$ ,  $\mu = 1$ , and  $c_s^2 = 0$ . Under



this condition, the Mach number at infinity ( $x \rightarrow \infty$ ) becomes

$$M_A^2 \Big|_F = k_0 + \kappa \frac{(B^\phi)^2}{B_p^2} \rightarrow -x^2 \Omega_F^2 + \frac{x^2 \Omega_F^2}{v_\infty^2} = \frac{x^2 \Omega_F^2}{\gamma_\infty^2 - 1}, \quad (\text{B.11})$$

where  $v_\infty$  is the asymptotic, spatial poloidal velocity, and  $\gamma_\infty = (1 - v_\infty^2)^{-1/2}$  is the terminal Lorentz factor of the outflow. By expanding Eq. (B.9) and Eq. (B.10) in the limit of large  $x$  and substituting Eq. (B.11), we derive the asymptotic forms of  $E$  and  $\eta$  as given in Eq. (2.10). The asymptotic expression for  $B^\phi$  can be derived by substituting Eqs. (B.11) (2.10) into the large distance limit of Eq. (2.7).

## C Launching Condition

### C.1 The plasma loading

The MHD flow is launched from a mass-loading zone [26, 64]. Within the loading zone, there is a source term for plasma injection,  $\nabla_\mu T^{\mu\nu} = S^\nu$ . We consider that during the loading process, there is only particle number loading [26], resulting in  $S^\mu = \eta' \mu u^\mu$ . Thus, according to Eq. (A.12), the energy and angular momentum transport equations are written as

$$\nabla_\mu T^\mu_t = -(\eta E)' = \eta' \mu u_t, \quad \nabla_\mu T^\mu_\phi = (\eta L)' = \eta' \mu u_\phi. \quad (\text{C.1})$$

In the main text, we adopt a geometrically thin, axisymmetric launching surface with a zero poloidal outflow velocity [19]. At the launching surface, the mass density  $\rho = |\eta| B_p / u_p$  diverges due to vanishing  $u_p$ , while  $\eta$ ,  $B_p$  remain finite. This singularity can be regularized by introducing a finite-width loading zone, where plasma is supplied either from the external accretion flow or via pair production within the jet [54, 88]. However, modeling such processes lies beyond the scope of this study.

According to Eq. (C.1), there is a jump in the conserved quantities across the launching surface:

$$\delta(\eta E) = \delta\eta(-\mu u_t), \quad \delta(\eta L) = \delta\eta(\mu u_\phi), \quad (\text{C.2})$$

where  $\delta(X) = X_{\text{flow}} - X_{\text{loading}}$ ,  $X \in \{\eta E, \eta L, \eta\}$ . Note that the limit  $S_t \rightarrow 0$  gives a special case of an electromagnetically dominated jet with the matching condition  $(\eta E)|_{\text{in}} = (\eta E)|_{\text{out}}$ , as employed in [89]. In this work, we adopt the general formulation of Eqs. (C.2), which incorporates plasma injection in the loading zone. The details within the loading zone are beyond the scope of this study. Instead, Eq. (C.2) serves solely to provide a matching condition: for field lines threading the horizon, it leads to the relation between the conserved quantities of the inflow and outflow, as given in Eq. (3.3).

## C.2 The launching condition

We derive the launching condition by analyzing the plasma acceleration near the launching surface. From energy-momentum conservation, we have  $\nabla_\mu T_m^{\mu\nu} = -\nabla_\mu T_{\text{EM}}^{\mu\nu} = f^\nu$ , where  $f^\mu$  denotes the Lorentz force density. In ideal MHD, where the electric field vanishes in the comoving frame,  $f^\mu$  is orthogonal to the comoving magnetic field, leading to

$$b^\mu f_\mu = b_\nu \nabla_\mu (h u^\mu u^\nu + P g^{\mu\nu}) = 0. \quad (\text{C.3})$$

Using  $b^\nu u_\nu = 0$ , this can be rewritten as  $h b_\nu u^\mu \nabla_\mu u^\nu + b_\mu \partial^\mu P = 0$ . Near the launching surface, the plasma motion is predominantly toroidal, with  $u^\phi \approx \Omega_F u^t$  and  $u_p \approx 0$ . The derivative term  $b_\nu u^\mu \nabla_\mu u^\nu$  expanded in powers of  $u_p$  as (also see Appendix A.4 in [59])

$$b_\nu u^\mu \nabla_\mu u^\nu \approx B^{\mathcal{P}} \partial_{\mathcal{P}} \left( \frac{1}{2} u_p^2 + \frac{1}{2} \log k_0 \right) \approx \frac{1}{2} B^{\mathcal{P}} \partial_{\mathcal{P}} \log k_0, \quad (\text{C.4})$$

where we have used Eqs. (A.2) to relate  $b^{\mathcal{P}} \propto B^{\mathcal{P}}$  near the launching surface. Substituting Eq.(C.4) into Eq.(C.3), we obtain the launching condition:

$$P' \Big|_{\text{LS}} + \frac{h}{2} (\log k_0)' \Big|_{\text{LS}} = 0. \quad (\text{C.5})$$

This equation constrains the location of the launching surface, determined by the interplay of gravity, magnetic field structure, and thermal pressure gradients. In the cold limit, the launching condition reduces to  $k'_0|_{\text{LS}} = 0$ . This follows directly from the acceleration equation (Eq.(B.5)): to ensure regularity at the launching surface, the numerator  $N \propto k_0 k_2 k'_0$  must vanish as  $u_p \rightarrow 0$ . Using  $k_2 = k_0 E^{-2}$  from Eq.(3.1) and noting that  $k_0 > 0$  within the light cylinder, we conclude that the launching surface must lie at a local extremum of  $k_0$ .

## D Kerr light cylinders

Outside a black hole, by numerically solving the condition  $k_0 = 0$  for proper values of  $\Omega_F$ , one obtains two characteristic surfaces [55, 71]: an outer light cylinder,  $r = R_O(\theta)$ , located far from the event horizon; an inner light cylinder,  $r = R_I(\theta)$ , situated near the horizon, arising from the interplay between the gravity and magnetic field rotation.

Here, we derive approximate analytical expressions for the locations of the light cylinders in the Kerr spacetime. By expanding  $k_0 = 0$  in terms of  $r^{-1}$ , we find a simple asymptotic expression:

$$R_O(\theta) \approx \frac{\sqrt{1 - a^2 \Omega_F^2 \sin^2 \theta}}{\Omega_F \sin \theta} + \mathcal{O}\left(\frac{1}{r}\right), \quad (\text{D.1})$$

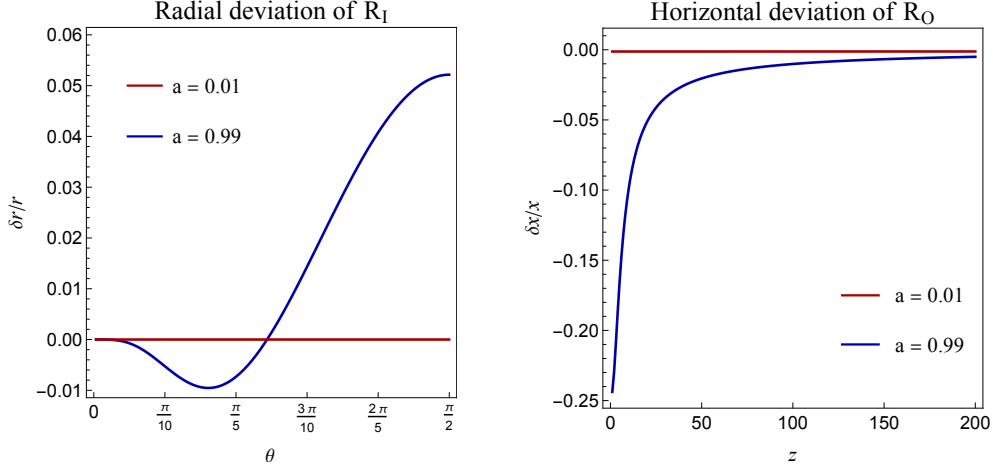


Figure 10: Comparison between the precise and approximate expressions of the light cylinders in Kerr spacetime, assuming  $\Omega_F = \Omega_H/2$ , where  $\Omega_H = a/(2r_+)$  is the black hole angular velocity. The red curves correspond to a low-spin case ( $a = 0.01$ ), and the blue curves correspond to a high-spin case ( $a = 0.99$ ). **Left:** radial deviation  $\delta r/r \equiv 1 - R_I^{\text{approx}}/R_I^{\text{precise}}$  of the inner light cylinder in polar coordinates. **Right:** horizontal deviation  $\delta x/x \equiv 1 - x_O^{\text{approx}}/x_O^{\text{precise}}$  ( $x_O = R_O \sin \theta$ ) of the outer light cylinder in cylindrical coordinates.

indicating how the outer light cylinder is affected by the black hole spin  $a$  and  $\Omega_F$ . For the inner light cylinder, by taking the near-horizon expansion for  $k_0 = 0$ , we have

$$R_I(\theta) \approx r_+ - \frac{2r_+ - \Sigma_+ + 4r_+\Omega_F(r_+\Omega_F - a)\sin^2\theta}{2r_+\Sigma_+\Omega_F^2\sin^2\theta + 2(1 - \frac{2r_+^2}{\Sigma_+})(1 - a\Omega_F\sin^2\theta)^2} + \mathcal{O}(r - r_+), \quad (\text{D.2})$$

where  $r_+ = 1 + \sqrt{1 - a^2}$  is the event horizon radius, and  $\Sigma_+ = r_+^2 + a^2 \cos^2 \theta$ . In Fig. 10, we present the relative error between the exact light cylinders and the approximate expression results. Due to the nearly circular shape of  $R_I(\theta)$ , we plot the variation of the relative error in radius with respect to  $\theta$ . From the left panel of Fig. 10, it can be seen that the approximate expression for the inner light cylinder is highly accurate in low-spin cases, whereas for high spin, it introduces an error of about 2% near the equatorial plane.

For the outer light cylinder, owing to its cylindrical geometry, we plot the relative error in  $x$  as a function of  $z$ . The approximate expression remains accurate in low-spin cases, but for high-spin black holes, it shows substantial deviations near the equatorial plane, with errors reaching up to  $\sim 25\%$ . Despite this, the approximation remains valid in the outflow region, as physically relevant open or collimated magnetic field lines tend to avoid the equator and extend toward higher latitudes.

## E Jet modeling

This section provides supplementary material for jet modeling discussed in Sec. 4.

### E.1 The parabolic field

In Boyer–Lindquist coordinates, the poloidal components of the magnetic field corresponding to the stream function Eq. (4.1) are given by

$$B^r = \frac{r^q}{\Sigma}, \quad B^\theta = -\frac{qr^{q-1}(1 - \cos \theta)}{\Sigma \sin \theta}, \quad (\text{E.1})$$

where  $\Sigma = r^2 + a^2 \cos^2 \theta$ . The large-distance behavior of the poloidal field depends on the collimation index. For  $q = 0$ , the field lines become radial and the poloidal magnetic field scales as  $B_p = \sin^2 \theta / x^2$ . For  $0 < q < 2$ , the stream function asymptotically satisfies  $z = (2\psi)^{1/(q-2)} x^{2/(2-q)}$ , and the poloidal field behaves as  $B_p = 2\psi/x^2$ .

Under the field line angular velocity specified by Eq. (4.3), the locations of the light cylinders and the launching surface can be determined by solving  $k_0 = 0$  and  $k'_0 = 0$ , respectively. Fig. 11 illustrates the dependence of these critical surfaces on the black hole spin, for a representative magnetic field line with paraboloidal geometry (i.e.,  $q = 1$ ), anchored at the equator with  $\theta_+ = \pi/2$ . As shown, the frame-dragging effect induced by black hole rotation substantially alters magnetic field rotation. As a result, both the launching surface and the outer light cylinder exhibit pronounced sensitivity to the spin parameter. In contrast, the location of the inner light cylinder—primarily determined by the presence of the black hole itself—remains largely insensitive to variations in spin.

### E.2 The alternative hyperbolic field

As an alternative study to Sec. 4.2, here we consider a hyperbolic field configuration presented in [75], which is a force-free solution. The stream function takes:

$$\psi = \psi_0 \int \frac{u du}{\sqrt{(1-u^2)(1-b^2\Omega_F^2(u)u^4)}}, \quad u = \frac{\sqrt{(x+b)^2+z^2} - \sqrt{(x-b)^2+z^2}}{2b}, \quad (\text{E.2})$$

Typically,  $\psi$  is a single-valued function with respect to  $u$ . For  $u < 1$ , the launching point of a given field line is located at  $x_0(u) = bu$ . In Boyer-Lindquist coordinates, the poloidal magnetic field components take the form of

$$B^r = \frac{r \cot \theta}{2\Sigma} \left( \frac{1}{\sqrt{b^2 + r^2 - 2br \sin \theta}} + \frac{1}{\sqrt{b^2 + r^2 + 2br \sin \theta}} \right) \frac{d\psi}{du},$$

$$B^\theta = -\frac{1}{2b\Sigma \sin \theta} \left( \frac{-r + b \sin \theta}{\sqrt{b^2 + r^2 - 2br \sin \theta}} + \frac{r + b \sin \theta}{\sqrt{b^2 + r^2 + 2br \sin \theta}} \right) \frac{d\psi}{du}. \quad (\text{E.3})$$

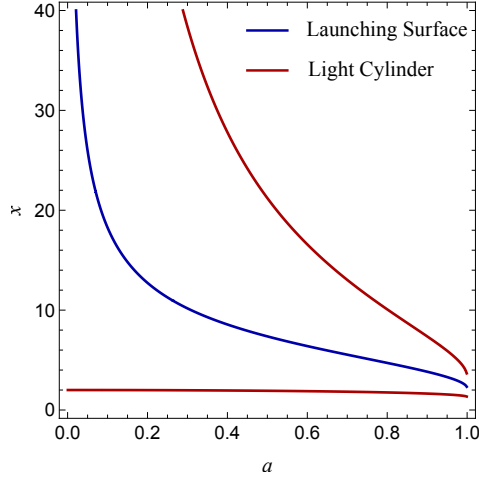


Figure 11: Dependence of the launching surfaces and light cylinders on black hole spin for the BZ magnetic field configuration. The blue curve indicates the launching surface, defined by  $k'_0 = 0$ ; the upper and lower red curves mark the outer and inner light cylinders, defined by  $k_0 = 0$ . The magnetic field line with a paraboloidal geometry (i.e.,  $q = 1$ ) and anchored at the equator of the event horizon ( $\theta_+ = \pi/2$ ).

where  $d\psi/du = u(1-u^2)^{-1/2}(1-b^2\Omega_F^2(u)u^4)^{-1/2}$ . At large distances, we have the asymptotic relation  $u^2z^2 \approx (1-u^2)x^2$ , and the field components take

$$B^z \rightarrow \frac{u(1-u^2)}{x^2} \frac{d\psi}{du}, \quad B^x \rightarrow \frac{u^2\sqrt{1-u^2}}{x^2} \frac{d\psi}{du}. \quad (\text{E.4})$$

The asymptotic poloidal field magnitude takes

$$B_p = \frac{u^2}{\sqrt{(1-b^2\Omega_F^2u^4)}} \frac{1}{x^2}. \quad (\text{E.5})$$

To ensure that the field line is physically well-defined, the parameter  $u$  must satisfy  $0 < u < 1$ . This constraint follows from Eq. (E.2), where  $u > 0$  by construction. If  $u \geq 1$ , the corresponding expression for  $z^2$  becomes non-positive. Consequently, the launching radius must satisfy  $x_0 < b$  to guarantee that  $u < 1$ . Finally, using the expression for the mass flux Eq. (2.10), we obtain

$$\eta = \frac{\Omega_F^2}{(\gamma_\infty^2 - 1)^{3/2}} \frac{u^2}{\sqrt{(1-b^2\Omega_F^2u^4)}}. \quad (\text{E.6})$$

We illustrate typical features of jet solutions in Fig. 12. The top-left panel presents the poloidal velocity, the Alfvén speed, and the FM speed of the outflow. The top-right panel shows the plasma density. We also plot the three-velocity components and the magnetic field components in the bottom-

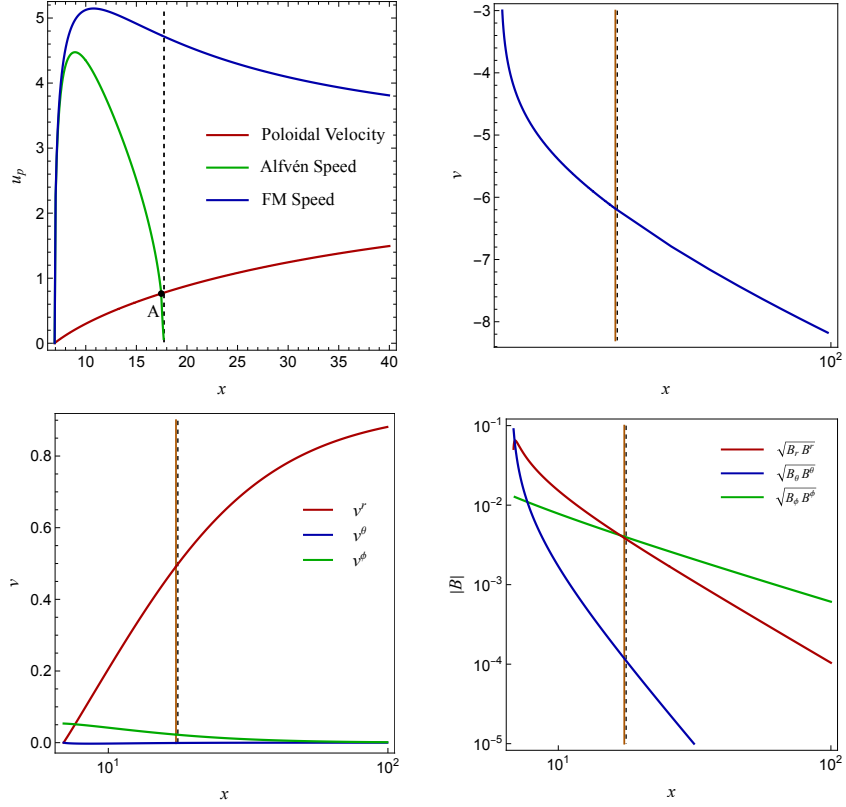


Figure 12: A representative BP flow solution is shown, along a hyperbolic field line given by Eq. (E.2) with  $x_0 = 2.94r_{\text{ISCO}}$  and  $b = 3r_{\text{ISCO}}$ . We set the spin  $a = 0.9$  and terminal Lorentz factor  $\gamma_\infty = 3$ . The light cylinder is shown as black dashed lines, and the Alfvén surface as an orange line. **Top-left:** Poloidal velocity  $u_p$  for outflow (red), Alfvén speed (green dashed), and FM speed (blue dashed). The point labeled A indicates the outflow Alfvén points. **Top-right:** Plasma density. **Bottom-left:** Components of the three-velocity  $v^r$ ,  $v^\theta$ , and  $v^\phi$ . **Bottom-right:** Magnetic field components  $\sqrt{B^r B_r}$ ,  $\sqrt{B^\theta B_\theta}$ , and  $\sqrt{B^\phi B_\phi}$ .

left and bottom-right panels, respectively. At large distances, the field has the asymptotic scalings

$$\begin{aligned}
 \sqrt{B^r B_r} &\rightarrow u^2(1 - b^2\Omega_F^2 u^4)^{-1/2}/x^2, \\
 \sqrt{B^\theta B_\theta} &\rightarrow b^2 u^5(1 - u^2)^{1/2}(1 - b^2\Omega_F^2 u^4)^{-1/2}/x^4, \\
 \sqrt{B^\phi B_\phi} &\rightarrow u^2(1 - b^2\Omega_F^2 u^4)^{-1/2}/x,
 \end{aligned}
 \tag{E.7}$$

which also exhibits strongly a toroidal structure.

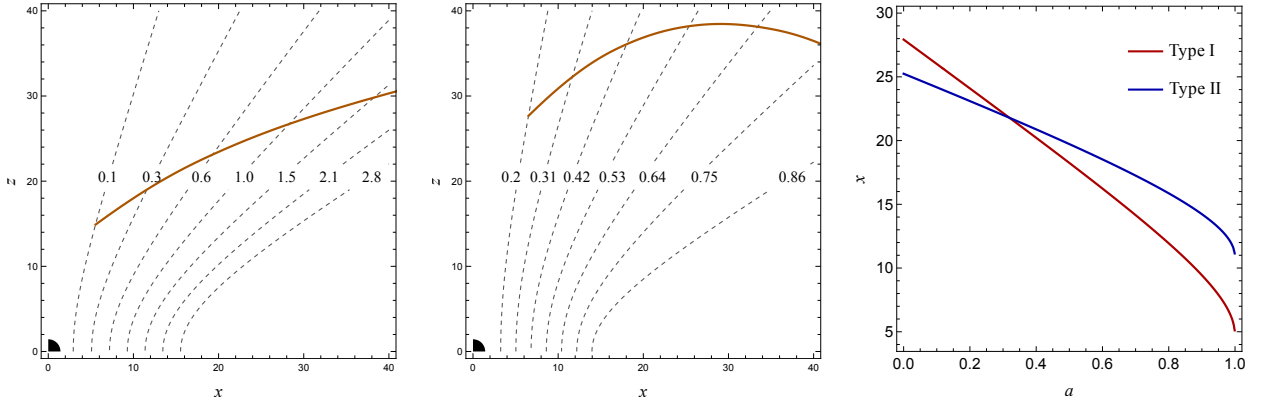


Figure 13: Comparison of magnetic field structures and light cylinder locations for two types of hyperbolic field configurations. **Left:** Type I field with  $b = 4r_{\text{ISCO}}$ . **Middle:** Type II field with  $b = 7r_{\text{ISCO}}$ . Gray dashed lines represent magnetic field lines corresponding to constant values of  $\psi$ . The orange curves denote the light cylinders. **Right:** Light cylinder radius as a function of black hole spin for both Type I (red) and Type II (blue) configurations. Parameters are set to  $b = r_{\text{ISCO}} + 1$ ,  $\psi = 1.8$  for Type I, and  $b = r_{\text{ISCO}} + 5$ ,  $u = 0.8$  for Type II.

Fig. 13 compares the poloidal field structures and light cylinder locations generated by two hyperbolic stream functions, given by Eqs. (4.5) (Type I) and (E.2) (Type II). The parameters in each configuration are chosen to produce broadly similar field geometries, facilitating a direct comparison. The light cylinder in the Type I field exhibits a stronger dependence on black hole spin compared to the Type II case. Nevertheless, for both hyperbolic fields, the spin sensitivity remains weaker than that observed in the parabolic field scenario (see Appendix (E.1)). This reduced dependence may be attributed to the anchoring of hyperbolic field lines to the accretion disk, which leads to a weaker coupling between the magnetic field and the black hole.

### E.3 Supplementary figures

As a complement to the main text, we present the poloidal velocity and density profiles for different magnetic field configurations in Fig. 14, in order to examine how the jet solutions depend on the black hole spin and the terminal Lorentz factor.

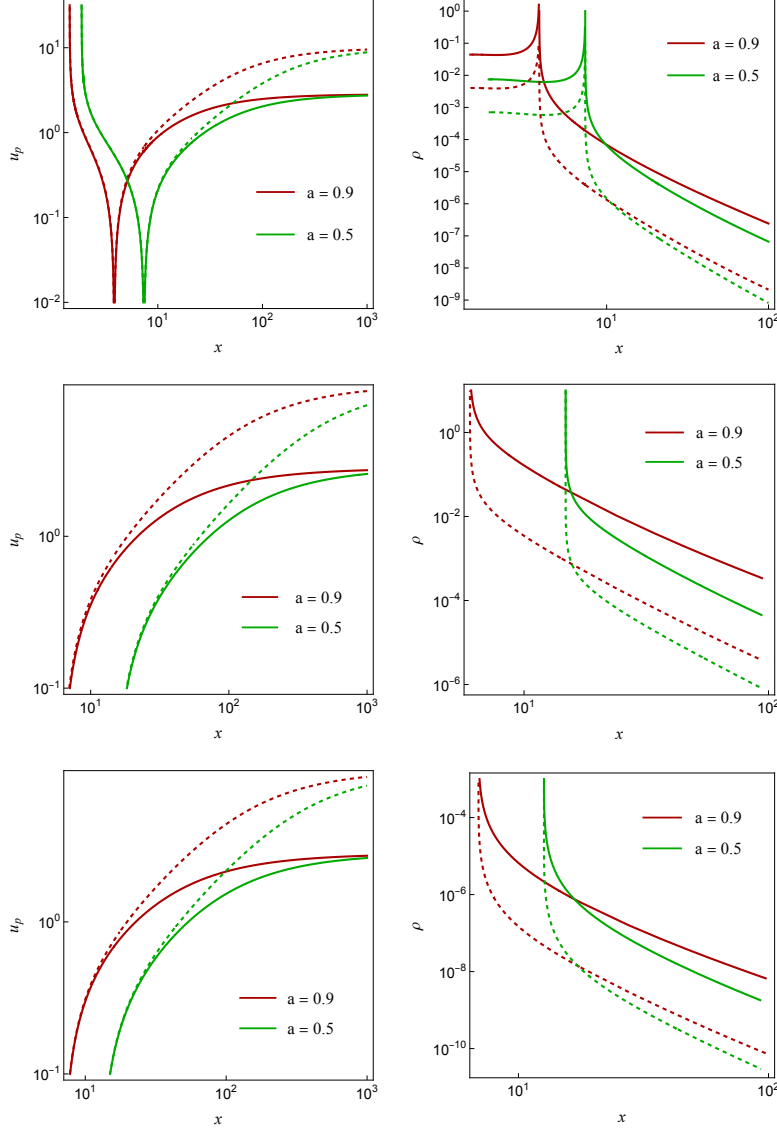


Figure 14: Poloidal velocity (**Left**) and plasma density (**Right**) profiles along magnetic field lines of three different geometries, showing the dependence on black hole spin  $a$  and terminal Lorentz factor  $\gamma_\infty$ . **Top row:** Parabolic field with  $q = 1$ ,  $\theta_+ = \pi/2$ . **Middle row:** Type I hyperbolic field anchored at  $x_0 = (r_{\text{ISCO}} - 1)\sqrt{20}$ , with  $b = r_{\text{ISCO}} - 1$ . **Bottom row:** Type II hyperbolic field anchored at  $x_0 = 2.94r_{\text{ISCO}}$ , with  $b = 3r_{\text{ISCO}}$ . In all panels, solid lines correspond to  $\gamma_\infty = 3$ , dashed lines to  $\gamma_\infty = 10$ . Red curves represent spin  $a = 0.9$ ; green curves represent  $a = 0.5$ .

From top to bottom, the rows correspond to the parabolic, Type I hyperbolic, and Type II hyperbolic magnetic field configurations, respectively. In the parabolic case, the primary distinctions among the cases stem from the differing locations of the launching surfaces, which are governed by variations in black hole spin. Additionally, differences in the terminal Lorentz factor  $\gamma_\infty$  lead to dis-



tinct asymptotic behaviors in the outflow poloidal velocities. In contrast, the inflow solutions exhibit minimal sensitivity to  $\gamma_\infty$ , rendering the differences between the two Lorentz factors barely discernible. The right panels show the corresponding density profiles. The dominant variations arise from shifts in the launching surfaces and changes in the overall density normalization. For the two hyperbolic configurations, both the velocity and density profiles share nearly identical shapes, with the primary difference in density being a uniform scaling factor.

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