4.5 Basis and Dimension

A set $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ of vectors in a vector space V is a *basis* for V if both of the following conditions are true:

(1) B is linearly independent

(2) B spans V

In other words, a basis for a vector space V is a linearly independent set of vectors that spans V.

Fact: $B = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis for V if and only if every vector in V can be uniquely expressed as a linear combination of the vectors in B.

This means that for each vector $\vec{v}$ in V, there is exactly one set of scalars $c_1, c_2, \dots, c_n$ for which we can express:

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_n\vec{v}_n$$

So, basis vectors are, in a sense, the “building blocks” of a vector space.

Note: Every vector space has a basis. We will only deal with bases that are finite.

Examples. Certain vector spaces have a “standard” set of basis vectors.

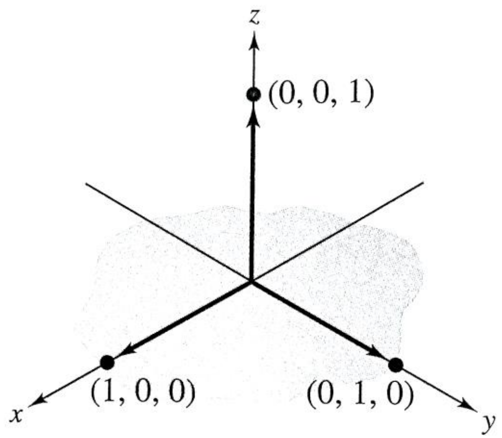
1. The standard basis for $\mathbb{R}^n$ is $S = \{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$, where $\vec{e}_i = (0, \dots, 1, \dots, 0)$ has 1 in the ith coordinate and 0’s elsewhere, for each $i = 1,2, \dots, n$, i.e.:

$$\vec{e}_1 = (1,0, \dots, 0), \vec{e}_2 = (0,1, \dots, 0), \dots, \vec{e}_n = (0,0, \dots, 1)$$

Note that each vector $\vec{x} = (x_1, x_2, \dots, x_n)$ in $\mathbb{R}^n$ can be uniquely expressed as:

$$\vec{x} = x_1(1,0, \dots, 0) + x_2(0,1, \dots, 0) + \cdots + x_n(0,0, \dots, 1)$$

- If $n = 1$, the standard basis vector for $\mathbb{R}$ is $\vec{e}_1 = 1$.
- If $n = 2$, the standard basis vectors for $\mathbb{R}^2$ are $\vec{e}_1 = (1,0), \vec{e}_2 = (0,1)$.
- If $n = 3$, the standard basis vectors for $\mathbb{R}^3$ are $\vec{e}_1 = (1,0,0), \vec{e}_2 = (0,1,0)$, $\vec{e}_3 = (0,0,1)$.



2. The standard basis for $M_{m,n}$ is $S = \{E_{11}, \dots, E_{1n}, \dots, E_{m1}, \dots, E_{mn}\}$, where:

$$E_{ij} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & 1 & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

has 1 in the (i, j)-entry and 0's elsewhere, for each $i = 1, \dots, m$ and $j = 1, \dots, n$.

- If $m = n = 2$, the standard basis vectors for $M_{2,2}$ are:

$$E_{11} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad E_{12} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad E_{21} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad E_{22} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Note that each matrix A in $M_{2,2}$ can be uniquely expressed as:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = aE_{11} + bE_{12} + cE_{21} + dE_{22}$$

3. The standard basis for P_n is $S = \{1, x, x^2, \dots, x^n\}$.

Note that each polynomial p of degree $\leq n$ can be uniquely expressed as:

$$p(x) = a_0 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$

Remark: $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$ if and only if any vector $\vec{b}$ in $\mathbb{R}^n$ is a unique linear combination of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$, which is equivalent to $A\vec{x} = \vec{b}$ having a unique solution, where A is the (square) matrix whose columns are the coordinates of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$. By the Invertible Matrix Theorem, this is equivalent to $|A| \neq 0$. Therefore, $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a basis of $\mathbb{R}^n$ if and only $|A| \neq 0$.

- This also works for a set $S = \{p_1(x), p_2(x), \dots, p_{n+1}(x)\}$ of polynomials in P_n, where A is the (square) matrix whose columns are the coefficients of $p_1, p_2, \dots, p_{n+1}$. E.g., in P_3 this looks like:

$$\begin{aligned} p_1(x) &= a_{01} + a_{11}x + a_{21}x^2 + a_{31}x^3 \\ p_2(x) &= a_{02} + a_{12}x + a_{22}x^2 + a_{32}x^3 \\ p_3(x) &= a_{03} + a_{13}x + a_{23}x^2 + a_{33}x^3 \\ p_4(x) &= a_{04} + a_{14}x + a_{24}x^n + a_{34}x^3 \end{aligned} \Rightarrow A = \begin{pmatrix} a_{01} & a_{02} & a_{03} & a_{04} \\ a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$$

- This also works for a set $S = \{A_1, A_2, \dots, A_{mn}\}$ of matrices in M_{mn}, where A is the (square) matrix whose columns are the entries of $A_1, A_2, \dots, A_{mn}$. E.g., in M_{22} this looks like:

$$A_1 = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, A_2 = \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}, A_3 = \begin{pmatrix} a_3 & b_3 \\ c_3 & d_3 \end{pmatrix}, A_4 = \begin{pmatrix} a_4 & b_4 \\ c_4 & d_4 \end{pmatrix} \Rightarrow A = \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{pmatrix}$$

A vector space can have many different (nonstandard) bases.

Example. Show that $B = \{(1,1), (1,-1)\}$ a basis for $\mathbb{R}^2$. Then show how any vector in $\mathbb{R}^2$ can be expressed as a linear combination of the vectors in B.

Let $\vec{b} = (b_1, b_2)$ be any vector in $\mathbb{R}^2$. We must show there exist unique scalars x_1, x_2 satisfying $x_1(1,1) + x_2(1,-1) = (b_1, b_2)$. This is equivalent to showing $A\vec{x} = \vec{b}$ has a unique solution, where $A = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$, and $\vec{b} = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$.

Since $|A| = -2 \neq 0$, $A\vec{x} = \vec{b}$ has a unique solution given by:

$$\vec{x} = A^{-1}\vec{b} = -\frac{1}{2}\begin{pmatrix} -1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} b_1 + b_2 \\ b_1 - b_2 \end{pmatrix}$$

So, we can express:

$$\vec{b} = (b_1, b_2) = \frac{1}{2}(b_1 + b_2)(1,1) + \frac{1}{2}(b_1 - b_2)(1,-1)$$

Exercise. Show that $B = \{(1,2,3),(0,1,2),(-2,0,1)\}$ a basis for $\mathbb{R}^3$. Then show how any vector in $\mathbb{R}^3$ can be written as a linear combination of the vectors in B.

Fact: If $B = \{\vec{v}_1, \dots, \vec{v}_m\}$ is a basis for a vector space V, then every subset of V containing more than m vectors must be linearly dependent.

Reason: Let $S = \{\vec{u}_1, \dots, \vec{u}_n\}$ be a set of vectors in V, where $n > m$. Consider the equation:

$$x_1\vec{u}_1 + \cdots + x_n\vec{u}_n = \vec{0}$$

Since B spans V, we can express $\vec{u}_j = a_{1j}\vec{v}_1 + \cdots + a_{mj}\vec{v}_m$ for each $j = 1, \dots, n$. Then:

$$x_1(a_{11}\vec{v}_1 + \cdots + a_{m1}\vec{v}_n) + \cdots + x_n(a_{1n}\vec{v}_1 + \cdots + a_{mn}\vec{v}_m) = \vec{0}$$

$$(a_{11}x_1 + \cdots + a_{1n}x_n)\vec{v}_1 + \cdots + (a_{m1}x_1 + \cdots + a_{mn}x_n)\vec{v}_n = \vec{0}$$

Since B is linearly independent, we must have $a_{i1}x_1 + \cdots + a_{in}x_n = 0$ for each $i = 1, \dots, m$. This gives a homogeneous system of m linear equations in n variables:

$$\begin{gathered} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{gathered}$$

Since $m < n$, this has infinitely many solutions (due to the presence of a free variable).

Therefore, there must be a nontrivial solution to $x_1\vec{u}_1 + \cdots + x_n\vec{u}_n = \vec{0}$.

Fact: Any two bases for a vector space V must have the same number of vectors.
Reason: Let $B_1 = \{\vec{v}_1, \dots, \vec{v}_m\}$ and $B_2 = \{\vec{v}_1, \dots, \vec{v}_n\}$ be two bases for V.

- If $m < n$, then B_2 is linearly dependent by the previous fact, which contradicts that B_2 is a basis.
- If $m > n$, then B_1 is linearly dependent by the previous fact, which contradicts that B_1 is a basis.

So, we must have $m = n$.

This allows us to make the following definition.
The *dimension* of a vector space V, denoted by $\dim(V)$, is the number of vectors in a basis for V. If $\dim(V) = n$, we say that V is n*-dimensional*.

Examples.
1. The vector space $\mathbb{R}^n$ is n-dimensional because $\dim(\mathbb{R}^n) = n$.
2. The vector space P_n is $(n+1)$-dimensional because $\dim(P_n) = n + 1$.
3. The vector space $M_{m,n}$ is mn-dimensional because $\dim(M_{m,n}) = mn$.

Exercise. Determine whether S is a basis for P_2.

(a) $S = \{2, x, 3 + x, 4x^2 - x\}$

(b) $S = \{-3 + 6x, 3x^2, 1 - 2x - x^2\}$

Exercise. Determine whether S is a basis for $M_{2,2}$.

(a) $S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \right\}$

(b) $S = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & -1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} \right\}$

Facts: Let $S = \{\vec{v}_1, \dots, \vec{v}_m\}$ be a set of vectors in a n-dimensional vector space V.
(1) If S is linearly independent, then $m \leq n$ and S can be extended to a basis for V.
(2) If S spans V, then $m \geq n$ and S can be reduced to a basis for V.

Reasons:
(1) Assume S is linearly independent. If S spans V, then S is already a basis for V.
Otherwise, there exists a vector $\vec{v}_{m+1}$ in V not in $\mathrm{span}(S)$. Then $S_1 = \{\vec{v}_1, \dots, \vec{v}_m, \vec{v}_{m+1}\}$ is linearly independent. If S_1 spans V, then S_1 is a basis for V.
Otherwise, there exists a vector $\vec{v}_{m+2}$ in V not in $\mathrm{span}(S_1)$. Then $S_2 = \{\vec{v}_1, \dots, \vec{v}_m, \vec{v}_{m+1}, \vec{v}_{m+2}\}$ is linearly independent. If S_2 spans V, then S_2 is a basis for V. Otherwise, continue this process.
Since $\dim(V) = n$, this process must end in $\leq n - m$ steps.

(2) Assume S spans V. If S is linearly independent, then S is already a basis for V.
Otherwise, some vector in S, say $\vec{v}_m$, is a linear combination of the other vectors in S. Then $S_1 = \{\vec{v}_1, \dots, \vec{v}_{m-1}\}$ spans V. If S_1 is linearly independent, then S_1 is a basis for V.
Otherwise, some vector in S_1, say $\vec{v}_{m-1}$, is a linear combination of the other vectors in S_1. Then $S_2 = \{\vec{v}_1, \dots, \vec{v}_{m-2}\}$ spans V. If S_2 is linearly independent, then S_2 is a basis for V. Otherwise, continue this process, which must end in $\leq m$ steps.

Exercise. Extend $S = \{(1,0,2), (0,1,1)\}$ to a basis for $\mathbb{R}^3$.

Exercise. Reduce $S = \{(1,5,3), (0,1,2), (0,0,6), (2,1,0)\}$ to a basis for $\mathbb{R}^3$.

Practice Problems:

Determine whether S is a basis for V. Justify your answers.

29. $S = \{1 - x, 1 - x^2, -1 - 2x - x^2\}, V = P_2$

31. $S = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\right\}, V = M_{2,2}$

33. $S = \left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 8 & -4 \\ -4 & 3 \end{pmatrix}\right\}, V = M_{2,2}$

39. $S = \{(4, -3), (5,2)\}, V = \mathbb{R}^2$

41. $S = \{(1,5,3), (0,1,2), (0,0,6)\}, V = \mathbb{R}^3$

45. $S = \{(-1,2,0,0), (2,0,-1,0), (3,0,0,4), (0,0,5,0)\}, V = \mathbb{R}^4$

47. $S = \{1 - 2t^2 + t^3, -4 + t^2, 2t + t^3, 5t\}, V = P_3$

65. Find a basis for the vector space of all 3×3 diagonal matrices. What is its dimension?

67. Find all subsets of $S = \{(1,0), (0,1), (1,1)\}$ that form a basis for $\mathbb{R}^2$.

69. Find a basis for $\mathbb{R}^2$ that includes the vector $(2,2)$.