2.2 Matrix Properties

Matrix addition and scalar multiplication obey the following properties.
If A, B, C are matrices of the same size, and c, d are any scalars, then:
1. $(A + B) + C = A + (B + C)$
2. $A + B = B + A$
3. $c(A + B) = cA + cB$
4. $(c + d)A = cA + dA$
5. $c(dA) = (cd)A$
6. $1A = A$

Matrix multiplication obeys the following properties.
If A, B, C are matrices (of appropriate size), and c is any scalar, then:
1. $(AB)C = A(BC)$
2. $A(B + C) = AB + AC$
3. $(A + B)C = AC + BC$
4. $c(AB) = (cA)B = A(cB)$

The *zero matrix* O_{mn} is the $m \times n$ matrix whose entries are all 0:

$$O_{mn} = \begin{pmatrix} 0 & \cdots & 0 \\ \vdots & & \vdots \\ 0 & \cdots & 0 \end{pmatrix}$$

It obeys the following properties. For any $m \times n$ matrix A and any scalar c, we have:

1. $A + O_{mn} = A$
2. $A + (-A) = O_{mn}$
3. If $cA = O_{mn}$, then either $c = 0$ or $A = O_{mn}$.

The *identity matrix* I_n is the $n \times n$ matrix whose entries are 1 along the main diagonal and 0 elsewhere:

$$I_n = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}$$

It obeys the following property. For any $m \times n$ matrix A, we have $AI_n = A = I_mA$.

Note: We sometimes abbreviate O_{mn} simply by O, or I_n by I, if the size is understood.

These properties allow us to do algebra operations with matrices (just like we do with numbers), except:

(i) Matrix multiplication is not commutative.

(ii) We can’t divide matrices.

Example. Solve the equation $3X + A = B$, where $A = \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} -3 & 4 \\ 2 & 1 \end{pmatrix}$.

We have:

$$3X + A = B$$

$$3X = B - A$$

$$X = \tfrac{1}{3}(B - A)$$

$$X = \tfrac{1}{3}\left(\begin{pmatrix} -3 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 1 & -2 \\ 0 & 3 \end{pmatrix}\right) = \tfrac{1}{3}\begin{pmatrix} -4 & 6 \\ 2 & -2 \end{pmatrix} = \begin{pmatrix} -\frac{4}{3} & 2 \\ \frac{2}{3} & -\frac{2}{3} \end{pmatrix}$$

Exercise. Let $A = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 & 2 \\ 3 & -2 & 1 \end{pmatrix}$, and $C = \begin{pmatrix} -1 & 0 \\ 3 & 1 \\ 2 & 4 \end{pmatrix}$. Find the matrix product ABC in two ways.

For any $n \times n$ square matrix A and any nonnegative integer k, we define:

$$A^k = \underbrace{AA \cdots A}_{k \text{ factors}}$$

Matrix exponentiation obeys the following properties:

1. $A^0 = I_n$
2. $A^k A^l = A^{k+l}$
3. $\left(A^k\right)^l = A^{kl}$

Example. If $A = \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}$, then:

$$\begin{aligned} A^3 = AAA &= \begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & -2 \\ 6 & -3 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 0 \end{pmatrix} \\ &= \begin{pmatrix} -4 & -1 \\ 3 & -6 \end{pmatrix} \end{aligned}$$

The *transpose* of a $m \times n$ matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

is the $n \times m$ matrix defined by:

$$A^T = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix}$$

This means that A^T is obtained by interchanging the rows and columns of A. So, the rows of A become the columns of A^T, and the columns of A become the rows of A^T.

For example, the transpose of $A = \begin{pmatrix} 0 & 1 \\ 2 & 4 \\ 1 & -1 \end{pmatrix}$ is $A^T = \begin{pmatrix} 0 & 2 & 1 \\ 1 & 4 & -1 \end{pmatrix}$.

The transpose obeys the following properties. If A, B are matrices (of appropriate size) and c is any scalar, then:

1. $(A^T)^T = A$
2. $(A+B)^T = A^T + B^T$
3. $(cA)^T = cA^T$
4. $(AB)^T = B^T A^T$

A square matrix A is called *symmetric* if $A^T = A$.

Examples.

1. $A = \begin{pmatrix} 2 & 1 & -2 \\ 1 & 0 & 3 \\ -2 & 3 & 5 \end{pmatrix}$ is symmetric.
2. The zero matrix O_{mn} is symmetric.
3. The identity matrix I_n is symmetric.

Fact: Given any system of linear equations, exactly one of the following must be true:
(1) The linear system has no solution.
(2) The linear system has a unique (or, exactly one) solution.
(3) The linear system has infinitely many solutions.

Reason: We can represent a linear system as a matrix equation $A\vec{x} = \vec{b}$. If (1) or (2) is true, then there's nothing to prove. Suppose the system has two solutions $\vec{x}_1$ and $\vec{x}_2$. Then we have:

$$A\vec{x}_1 = A\vec{x}_2 = \vec{b} \Rightarrow A(\vec{x}_1 - \vec{x}_2) = A\vec{x}_1 - A\vec{x}_2 = O$$

Let $\vec{x}_h = \vec{x}_1 - \vec{x}_2$. Then $\vec{x}_h$ is a solution to the "homogeneous" linear system $A\vec{x} = O$. Let c be any scalar. Observe that:

$$A(\vec{x}_1 + c\vec{x}_h) = A\vec{x}_1 + A(c\vec{x}_h) = \vec{b} + c(A\vec{x}_h) = \vec{b} + cO = \vec{b}$$

This shows that $\vec{x}_1 + c\vec{x}_h$ is a solution to the linear system $A\vec{x} = \vec{b}$, where c is any number. So, the linear system has infinitely many solutions.

Practice Problems:

13. Let $A = \begin{pmatrix} -4 & 0 \\ 1 & -5 \\ -3 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 2 \\ -2 & 1 \\ 4 & 4 \end{pmatrix}$. Solve the equation for X:

(a) $3X + 2A = B$

(b) $2A - 5B = 3X$

(c) $X - 3A + 2B = O_{32}$

(d) $6X - 4A - 3B = O_{32}$

25. Let $A = \begin{pmatrix} -2 & 1 \\ 0 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 0 \\ -1 & 2 \end{pmatrix}$. Show that $AB \neq BA$.

27. Let $A = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$. Show that $AC = BC$.

29. Let $A = \begin{pmatrix} 3 & 3 \\ 4 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$. Show that $AB = O_{22}$.

59. For any square matrix A and polynomial $p(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$, we define $p(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n$. For $A = \begin{pmatrix} 2 & 0 \\ 4 & 5 \end{pmatrix}$ and $p(x) = 2 - 5x + x^2$, find $p(A)$.