11.5 Alternating Series Test

An *alternating series* is a series whose terms alternate in sign. They are of the form:

$$\sum_{n=1}^{\infty}(-1)^n b_n = -b_1 + b_2 - b_3 + \cdots$$

or:

$$\sum_{n=1}^{\infty}(-1)^{n\pm1} b_n = b_1 - b_2 + b_3 - \cdots$$

where $b_n > 0$.

Examples.

1. $\sum_{n=1}^{\infty}(-1)^n \frac{n}{n+1} = -\frac{1}{2} + \frac{2}{3} - \frac{3}{4} + \frac{4}{5} - \cdots$
2. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

Alternating Series Test

Let $\sum(-1)^n b_n$ be an alternating series, where $b_n > 0$. If both of the following conditions are satisfied:

(i) $\lim_{n\to\infty} b_n = 0$

(ii) b_n is decreasing (i.e. $b_{n+1} \leq b_n$)

then $\sum(-1)^n b_n$ converges.

Remarks:

- Recall (from section 11.1) that if $\lim_{n\to\infty} b_n \neq 0$, then $\lim_{n\to\infty} (-1)^n b_n$ does not exist (by limit fact 5), in which case $\sum(-1)^n b_n$ diverges by the Divergence Test. So, if the first condition of the AST fails, then we can immediately conclude that the series diverges by the Divergence Test.
- Recall (from Calc 1) that if $f'(x) < 0$ when $x > c$, then f is decreasing when $x > c$. This fact is often useful for showing that b_n is decreasing.

Examples. Is the alternating series convergent or divergent?

1. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$

Let $b_n = \frac{n^2}{n^3+1}$. Then:

(i) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n^2}{n^3+1} \cdot \frac{1/n^3}{1/n^3} = \lim_{n\to\infty} \frac{1/n}{1+1/n^3} = 0$

(ii) Write $f(x) = \frac{x^2}{x^3+1}$. Then:

$f'(x) = \frac{(x^3+1)2x-x^2\cdot 3x^2}{(x^3+1)^2} = \frac{2x-x^4}{(x^3+1)^2} = \frac{x(2-x^3)}{(x^3+1)^2} < 0$ when: $2 - x^3 < 0$

$$x^3 > 2$$

$$x > \sqrt[3]{2}$$

(This shows f is decreasing when $x > \sqrt[3]{2}$, and so b_n is decreasing when $n \geq 2$.)

By the AST, the series converges.

2. $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$

We have $b_n = \frac{3n}{4n-1}$. Then:

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3n}{4n-1} \cdot \frac{1/n}{1/n} = \lim_{n\to\infty} \frac{3}{4-1/n} = \frac{3}{4} \neq 0$$

So, the series diverges by the Divergence Test.

3. $\frac{2}{3} - \frac{2}{5} + \frac{2}{7} - \frac{2}{9} + \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2}{2n+1}$

Let $b_n = \frac{2}{2n+1}$. Then:

(i) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{2}{2n+1} = 0$

(ii) $b_{n+1} = \frac{2}{2(n+1)+1} = \frac{2}{2n+3} < \frac{2}{2n+1} = b_n$

So, the series converges by the AST.

4. $\frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \frac{1}{\ln 5} + \cdots$

5. $\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n^2+n+1}$

6. $\sum_{n=1}^{\infty}(-1)^n \frac{\sqrt{n}}{2n+3}$

7. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$

8. $\sum_{n=1}^{\infty}(-1)^n\left(\sqrt{n+1}-\sqrt{n}\right)$

Alternating Series Estimation Theorem

Suppose $\sum_{n=1}^{\infty}(-1)^n b_n$ is an alternating series that converges to a sum s, i.e.:

$$s = \sum_{n=1}^{\infty}(-1)^n b_n = -b_1 + b_2 - b_3 + \cdots$$

Let s_n be the sum of the first n terms of the series, i.e.:

$$s_n = -b_1 + b_2 - b_3 + \cdots \pm b_n$$

Then the nth partial sum s_n approximates the total sum s. The quantity $|s - s_n|$ is called the *error* in the approximation of s by s_n. The *Alternating Series Estimation Theorem* says:

$$\text{error} = |s - s_n| \leq b_{n+1}$$

In other words, the error in approximating the sum s of a convergent alternating series by its nth partial sum s_n is at most b_{n+1}.

Note: If $b_{n+1} \leq 0.\underbrace{00\ldots0}_{m}5$, then the approximation of s by s_n is correct to m decimal places (as long as s_n is rounded to m decimal places).

Example. Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ converges and approximate its sum correct to 3 decimal places. ($n! = n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1$ is called n *factorial*.)

Let $b_n = \frac{1}{n!}$. Then:

(i) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{1}{n!} = 0$

(ii) $b_{n+1} = \frac{1}{(n+1)!} < \frac{1}{n!} = b_n$

By the AST, $\sum_{n=1}^{\infty} \frac{(-1)^n}{n!}$ converges to a sum s. (We must approximate s by s_n for some n value satisfying $b_{n+1} \leq 0.0005$.) Observe that:

$$b_1 = 1, b_2 = \frac{1}{2} = 0.5, \dots, b_6 = \frac{1}{720} \approx 0.0014, b_7 = \frac{1}{5040} \approx 0.0002$$

By the ASET, $s_6 = -1 + \frac{1}{2} - \frac{1}{6} + \frac{1}{24} - \frac{1}{120} + \frac{1}{720} \approx -0.632$ is correct to 3 decimal places.

Example. Show that $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^6}$ converges and approximate its sum correct to 4 decimal places.

Example. Show that $\sum_{n=1}^{\infty} \frac{(-1)^n}{n3^n}$ converges. How many terms must be added so that the error in approximating the sum by the nth partial sum is < 0.0005?

Let $b_n = \frac{1}{n3^n}$. Then:

(i) $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \frac{1}{n3^n} = 0$

(ii) $b_{n+1} = \frac{1}{(n+1)3^{n+1}} < \frac{1}{n3^n} = b_n$

So, the series converges to a sum s by the AST. We need to find a n value satisfying $b_{n+1} < 0.0005$. So, we compute:

$$b_1 = \frac{1}{3}, b_2 = \frac{1}{18} \approx 0.06, \dots, b_5 = \frac{1}{5 \cdot 3^5} \approx 0.0008, b_6 = \frac{1}{6 \cdot 3^6} \approx 0.0002$$

By the ASET, we must add $n = 5$ terms of the series in order for:

$$\text{error} = |s - s_5| \leq b_6 < 0.0005$$

Example. Show that $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}$ converges. How many terms must be added so that the error in approximating the sum by the nth partial sum is < 0.00005?

Practice Exercises:

Test the series for convergence or divergence.

3. $-\frac{2}{5}+\frac{4}{6}-\frac{6}{7}+\frac{8}{8}-\frac{10}{9}+\cdots$

5. $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{3+5n}$

7. $\sum_{n=1}^{\infty}(-1)^{n}\frac{3n-1}{2n+1}$

11. $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+4}$

13. $\sum_{n=1}^{\infty}(-1)^{n-1}e^{2/n}$

23. Show that $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^6}$ converges. How many terms of the series must be added so that the error in approximating the sum by the nth partial sum is < 0.00005?

27. Show that $\sum_{n=1}^{\infty}\frac{(-1)^{n}}{(2n)!}$ converges and approximate its sum correct to 4 decimal places.