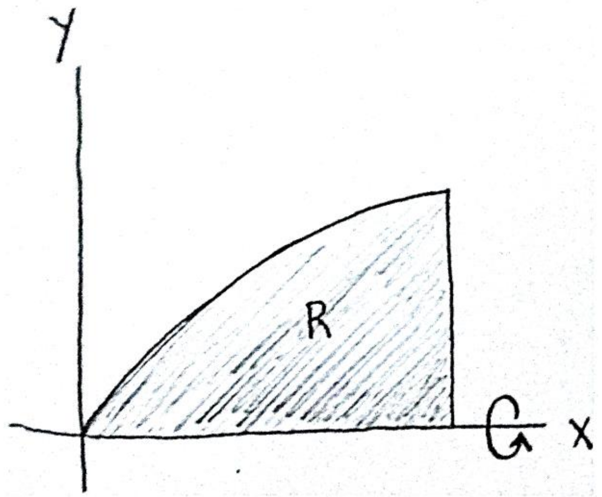
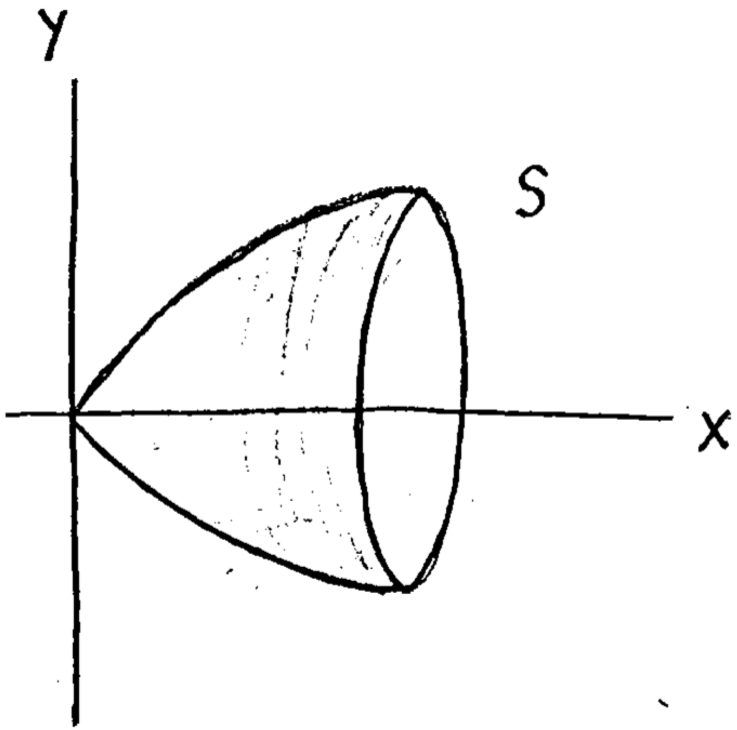
# 6.2 Volume: Disk Method

Consider a region $R$ enclosed by some curves in the $xy$-plane. By rotating $R$ around the $x$-axis, we get a 3-dimensional solid of revolution $S$:



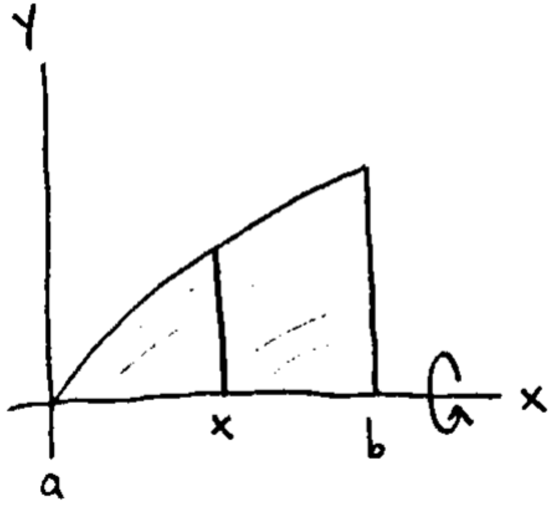

$\Rightarrow$



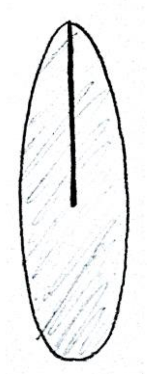

We can calculate the volume of $S$ by integrating a “cross-sectional area” function.

Here’s how:

1. Take a slice of the region at a point $x$, *perpendicular* to the axis of rotation.
2. Rotate the slice around the axis to produce a circular disk having area $A(x)$.
3. Integrate the cross-sectional area function $A(x)$ from the left endpoint to the right endpoint of the region.




$\Rightarrow$

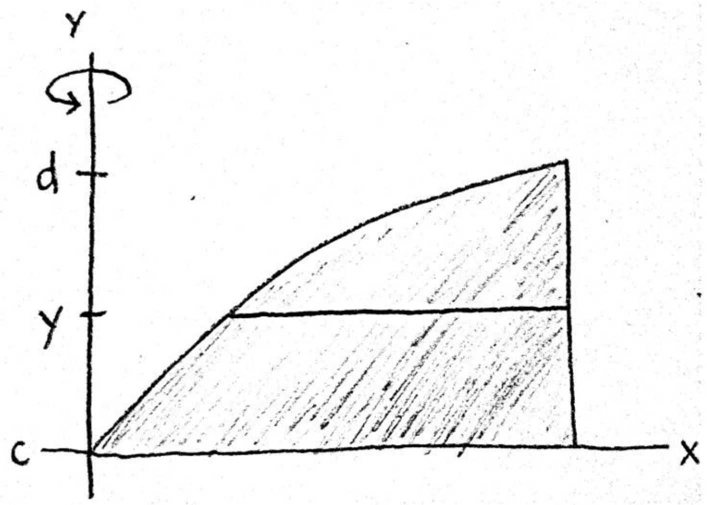


$\Rightarrow$

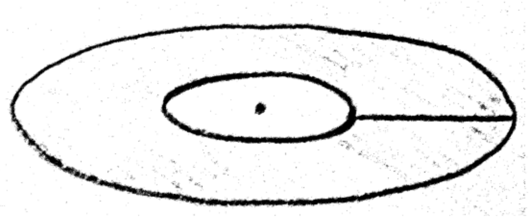
$$V = \int_a^b A(x)\,dx$$

$A(x) = \text{area of disk}$

We can also rotate $R$ around the $y$-axis. In this case, the cross-sectional area $A(y)$ is a function of $y$, which we integrate from the bottom endpoint to the top endpoint of the region.




$\Rightarrow$

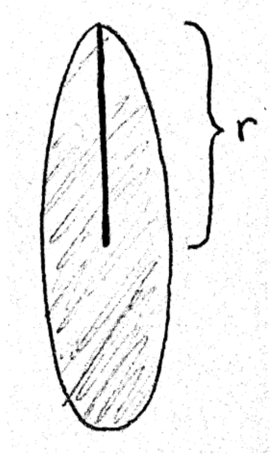


$\Rightarrow \quad V = \int_c^d A(y)\, dy$

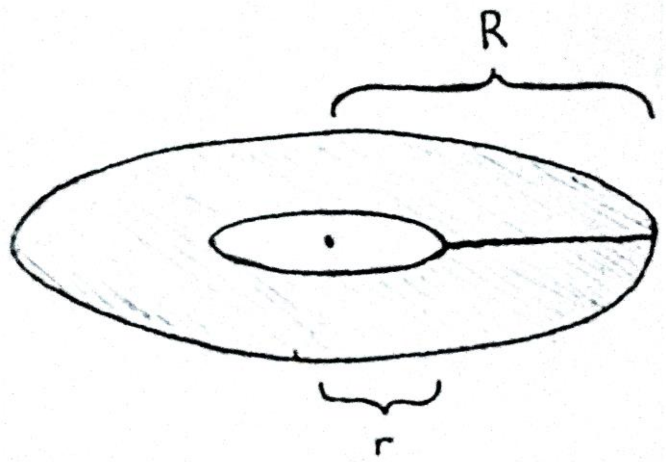
$A(y) = \text{area of disk}$

Here's how to determine the cross-sectional area function:

- If there is no gap between the region and the axis of rotation, then the area of the disk is $A = \pi r^2$, where $r$ is the radius of the disk.
- If there is a gap between the region and the axis of rotation, then the area of the disk is $A = \pi R^2 - \pi r^2 = \pi(R^2 - r^2)$, where $R$ is the outer radius and $r$ is the inner radius of the disk.



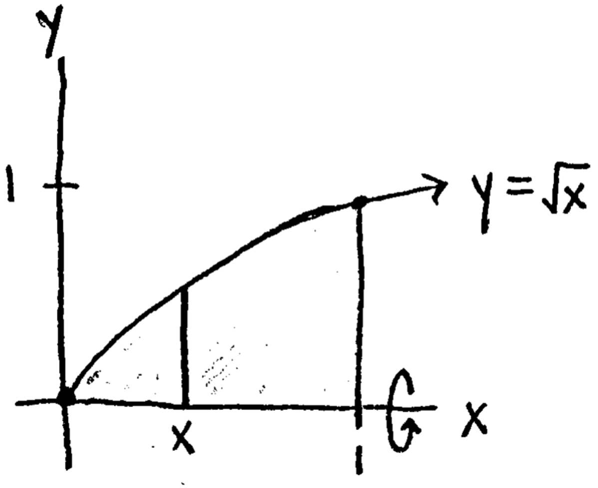

$A = \pi r^2$

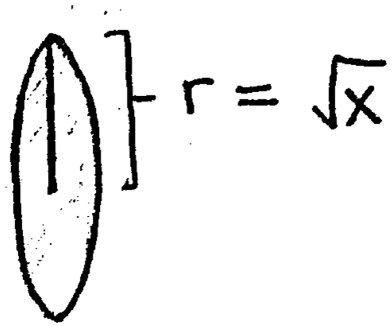



$A = \pi(R^2 - r^2)$

Examples. Find the volume of the solid obtained by rotating the region enclosed by the curves around the given axis.

1. $y = \sqrt{x}$, $y = 0$, $x = 1$; around $x$-axis

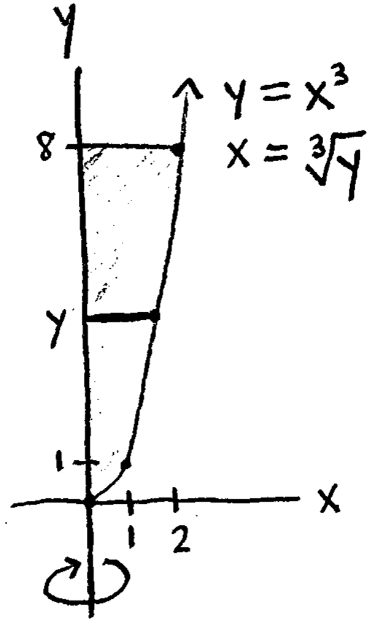


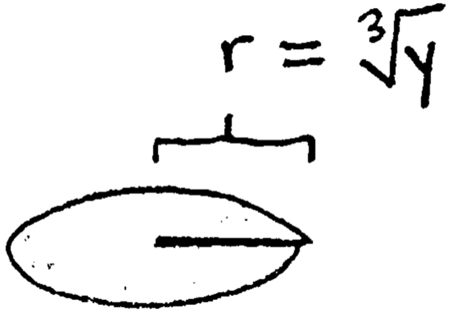




$$A(x) = \pi r^2 = \pi x$$

$$V = \int_0^1 A(x)\,dx = \int_0^1 \pi x\,dx = \frac{\pi}{2}x^2\Big|_0^1 = \frac{\pi}{2}$$

2. $y = x^3, y = 8, x = 0$; around $y$-axis




$$A(y) = \pi r^2 = \pi y^{2/3}$$

$$V = \int_0^8 A(y)\,dy = \int_0^8 \pi y^{2/3}\,dy = \frac{3\pi}{5} y^{5/3}\Big|_0^8 = \frac{96\pi}{5}$$

3. $y = \frac{1}{x}$, $y = 0$, $x = 1$, $x = 4$; around $x$-axis

4. $x = \frac{1}{2}y^2$, $x = 0$, $y = 2$; around $y$-axis

5. $y = 6 - x^2$, $y = 2$; around $x$-axis

6. $x = 2 - y^2$, $x = y^4$; around $y$-axis

Practice Exercises:

Find the volume of solid obtained by rotating the region enclosed by the curves around the given axis.

1. $y = x + 1, y = 0, x = 0, x = 2$; around $x$-axis
3. $y = \sqrt{x - 1}, y = 0, x = 5$; around $x$-axis
5. $x = 2\sqrt{y}, x = 0, y = 9$; around $y$-axis
7. $y = x^3, y = x$ (where $x \geq 0$); around $x$-axis
9. $x = y^2, x = 2y$; around $y$-axis