4.7 Change of Basis

Consider the bases $S = \{(1,0),(0,1)\}$ and $B = \{(1,1),(1,-1)\}$ for $\mathbb{R}^2$. We can write any vector, say $\vec{x} = (2,3)$, in terms of the basis vectors in S:

$$\vec{x} = (2,3) = 2(1,0) + 3(0,1)$$

or in terms of the basis vectors in B:

$$\vec{x} = (2,3) = \tfrac{5}{2}(1,1) - \tfrac{1}{2}(1,-1)$$

So, the scalars used in the expression for $\vec{x}$ depend on which basis we are using.

Let $B = \{\vec{v}_1, \dots, \vec{v}_n\}$ be an (ordered) basis for a vector space V. For any vector $\vec{x}$ in V, we can write:

$$\vec{x} = x_1\vec{v}_1 + \cdots + x_n\vec{v}_n$$

for some unique scalars $x_1, \dots, x_n$ (which depend on B). So, we call $x_1, \dots, x_n$ the *coordinates of $\vec{x}$ relative to B*, and define the *coordinate vector of $\vec{x}$ relative to B* as:

$$[\vec{x}]_B = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

Fact: If $B_1 = \{\vec{u}_1, \dots, \vec{u}_n\}$ and $B_2 = \{\vec{v}_1, \dots, \vec{v}_n\}$ are two bases for V, then there is a matrix P satisfying $P[\vec{x}]_{B_1} = [\vec{x}]_{B_2}$ for every vector $\vec{x}$ in V. We call P the *transition matrix from B_1 to B_2*, because it converts every coordinate vector relative to B_1 into the coordinate vector relative to B_2.

Reason: Let $\vec{x}$ be a vector in V. Write $\vec{x} = d_1\vec{u}_1 + \cdots + d_n\vec{u}_n$ in terms of the basis vectors in B_1. Write each basis vector in B_1 in terms of the basis vectors in B_2:

$$\begin{gathered}\vec{u}_1 = c_{11}\vec{v}_1 + \cdots + c_{n1}\vec{v}_n \\ \vdots \\ \vec{u}_n = c_{1n}\vec{v}_1 + \cdots + c_{nn}\vec{v}_n\end{gathered}$$

Form the matrix $P = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}$. Then we get:

$$\begin{aligned}\vec{x} = d_1\vec{u}_1 + \cdots + d_n\vec{u}_n &= d_1(c_{11}\vec{v}_1 + \cdots + c_{n1}\vec{v}_n) + \cdots + d_n(c_{1n}\vec{v}_1 + \cdots + c_{nn}\vec{v}_n) \\ &= (c_{11}d_1 + \cdots + c_{1n}d_n)\vec{v}_1 + \cdots + (c_{n1}d_1 + \cdots + c_{nn}d_n)\vec{v}_n\end{aligned}$$

and so:

$$P[\vec{x}]_{B_1} = \begin{pmatrix} c_{11} & \cdots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \cdots & c_{nn} \end{pmatrix}\begin{pmatrix} d_1 \\ \vdots \\ d_n \end{pmatrix} = \begin{pmatrix} c_{11}d_1 + \cdots + c_{1n}d_n \\ \vdots \\ c_{n1}d_1 + \cdots + c_{nn}d_n \end{pmatrix} = [\vec{x}]_{B_2}$$

Note: The transition matrix P from $B_1 = \{\vec{u}_1, \dots, \vec{u}_n\}$ to $B_2 = \{\vec{v}_1, \dots, \vec{v}_n\}$ is unique, because if another matrix Q satisfies $Q[\vec{x}]_{B_1} = [\vec{x}]_{B_2}$ for every vector $\vec{x}$ in V, then $Q[\vec{x}]_{B_1} = [\vec{x}]_{B_2} = P[\vec{x}]_{B_1}$ for every vector $\vec{x}$ in V. Applying this to each basis vector in B_1, we have:

$$\text{column } j \text{ of } Q = Q\vec{e}_j = Q\left[\vec{u}_j\right]_{B_1} = P\left[\vec{u}_j\right]_{B_1} = P\vec{e}_j = \text{column } j \text{ of } P$$

Therefore, $Q = P$.

Fact: If P is the transition matrix from $B_1 = \{\vec{u}_1, \dots, \vec{u}_n\}$ to $B_2 = \{\vec{v}_1, \dots, \vec{v}_n\}$, then P is invertible and P^{-1} is the transition matrix from B_2 to B_1.

Reason: Let Q be the transition matrix from B_2 to B_1. Then for every vector $\vec{x}$ in V, we have:

$$QP[\vec{x}]_{B_1} = Q[\vec{x}]_{B_2} = [\vec{x}]_{B_1} = I_n[\vec{x}]_{B_1}$$

By uniqueness of the transition matrix from B_1 to B_1, we get $QP = I_n \Rightarrow Q = P^{-1}$.

In this section, we will work exclusively with the vector space $\mathbb{R}^n$. We shall write $P_{A\to B}$ for the transition matrix from A to B, where A, B are any bases for $\mathbb{R}^n$. Let S be the standard basis for $\mathbb{R}^n$.

Let $B = \{\vec{u}_1, \dots, \vec{u}_n\}$ be any basis for $\mathbb{R}^n$. Write each $\vec{u}_j = (u_{1j}, \dots, u_{nj}) = u_{1j}\vec{e}_1 + \cdots + u_{nj}\vec{e}_n$ in terms of S. Then $P_{B\to S} = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix}$ is the matrix whose column vectors are the standard coordinates of $\vec{u}_1, \dots, \vec{u}_n$. So, it is simple to form the transition matrix from a nonstandard basis B into the standard basis S.

For instance, consider the (nonstandard) basis $B = \{(1,0,1), (0,-1,2), (2,3,-5)\}$ for $\mathbb{R}^3$. Then:

$$P_{B\to S} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{pmatrix}$$

For any vector $\vec{x} = x_1(1,0,1) + x_2(0,-1,2) + x_3(2,3,-5) = (x_1 + 2x_3, -x_2 + 3x_3, x_1 + 2x_2 - 5x_3)$ in $\mathbb{R}^3$, we have:

$$P_{B\to S}[\vec{x}]_B = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 3 \\ 1 & 2 & -5 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + 2x_3 \\ -x_2 + 3x_3 \\ x_1 + 2x_2 - 5x_3 \end{pmatrix} = [\vec{x}]_S$$

Let $B_1 = \{\vec{u}_1, \dots, \vec{u}_n\}$ and $B_2 = \{\vec{v}_1, \dots, \vec{v}_n\}$ be two bases for $\mathbb{R}^n$, where $\vec{u}_j = \left(u_{1j}, \dots, u_{nj}\right)$ and $\vec{v}_j = \left(v_{1j}, \dots, v_{nj}\right)$ for each $j = 1, \dots, n$.

There are two methods for finding the transition matrix from B_1 to B_2.

- Method 1: Form the transition matrices:

$$P_{B_1 \to S} = \begin{pmatrix} u_{11} & \cdots & u_{1n} \\ \vdots & & \vdots \\ u_{n1} & \cdots & u_{nn} \end{pmatrix}, \qquad P_{B_2 \to S} = \begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & & \vdots \\ v_{n1} & \cdots & v_{nn} \end{pmatrix}$$

Find $\left(P_{B_2 \to S}\right)^{-1} = P_{S \to B_2}$. Then multiply to obtain:

$$\left(P_{B_2 \to S}\right)^{-1} P_{B_1 \to S} = P_{S \to B_2} P_{B_1 \to S} = P_{B_1 \to B_2}$$

$$\begin{tikzcd} B_1 \arrow[rr, "P"] \arrow[dr, "Q"'] & & B_2 \arrow[dl, shift left, "R"] \\ & S \arrow[ur, shift left, "R^{-1}"] & \end{tikzcd} \qquad (P = R^{-1}Q)$$

- Method 2: Form the partitioned matrix:

$$\left(P_{B_2\to S}\middle|P_{B_1\to S}\right) = \left(\begin{array}{ccc|ccc} v_{11} & \cdots & v_{1n} & u_{11} & \cdots & u_{1n} \\ \vdots & & \vdots & \vdots & & \vdots \\ v_{n1} & \cdots & v_{nn} & u_{n1} & \cdots & u_{nn} \end{array}\right)$$

Perform the elementary row operations that reduce $P_{B_2\to S}$ to the identity matrix I_n.
These same operations will convert $P_{B_1\to S}$ to $P_{B_1\to B_2}$.
In practice, this looks like:

$$\left(P_{B_2\to S}\middle|P_{B_1\to S}\right)\sim\cdots\sim\left(I_n\middle|P_{B_1\to B_2}\right)$$

Why it works:
Performing the elementary row operations that reduce $P_{B_2\to S}$ to I_n is equivalent to:

$$EP_{B_2\to S} = I_n$$

where E is a product of elementary matrices. Then $E = \left(P_{B_2\to S}\right)^{-1} = P_{S\to B_2}$ and so:

$$EP_{B_1\to S} = P_{S\to B_2}P_{B_1\to S} = P_{B_1\to B_2}$$

Example. Find the transition matrix from $B_1 = \{(3,2,1),(1,1,2),(1,2,0)\}$ to $B_2 = \{(1,1,-1),(0,1,2),(-1,4,0)\}$.

- Method 1: We have $P_{B_1\to S} = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$ and $P_{B_2\to S} = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 4 \\ -1 & 2 & 0 \end{pmatrix}$. Then:

$$\left|P_{B_2\to S}\right| = \begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & 4 \\ -1 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 5 \\ -1 & 2 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 5 \\ 2 & -1 \end{vmatrix} = -11$$

By the adjoint formula, we get:

$$P_{S\to B_2} = \left(P_{B_2\to S}\right)^{-1} = -\frac{1}{11}\begin{pmatrix} -8 & -4 & 3 \\ -2 & -1 & -2 \\ 1 & -5 & 1 \end{pmatrix}^T = \frac{1}{11}\begin{pmatrix} 8 & 2 & -1 \\ 4 & 1 & 5 \\ -3 & 2 & -1 \end{pmatrix}$$

So, $P_{B_1\to B_2} = P_{S\to B_2}P_{B_1\to S} = \frac{1}{11}\begin{pmatrix} 8 & 2 & -1 \\ 4 & 1 & 5 \\ -3 & 2 & -1 \end{pmatrix}\begin{pmatrix} 3 & 1 & 1 \\ 2 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix} = \frac{1}{11}\begin{pmatrix} 27 & 8 & 12 \\ 19 & 15 & 6 \\ -6 & -3 & 1 \end{pmatrix}$

- Method 2:

$$\left(P_{B_2\to S}\middle|P_{B_1\to S}\right)=\left(\begin{array}{ccc|ccc}1&0&-1&3&1&1\\1&1&4&2&1&2\\-1&2&0&1&2&0\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1&0&-1&3&1&1\\0&1&5&-1&0&1\\0&2&-1&4&3&1\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|ccc}1&0&-1&3&1&1\\0&1&5&-1&0&1\\0&0&-11&6&3&-1\end{array}\right)\sim\left(\begin{array}{ccc|ccc}1&0&-1&3&1&1\\0&1&5&-1&0&1\\0&0&1&-6/11&-3/11&1/11\end{array}\right)$$

$$\sim\left(\begin{array}{ccc|ccc}1&0&0&27/11&8/11&12/11\\0&1&0&19/11&15/11&6/11\\0&0&1&-6/11&-3/11&1/11\end{array}\right)=\left(I_3\middle|P_{B_1\to B_2}\right)$$

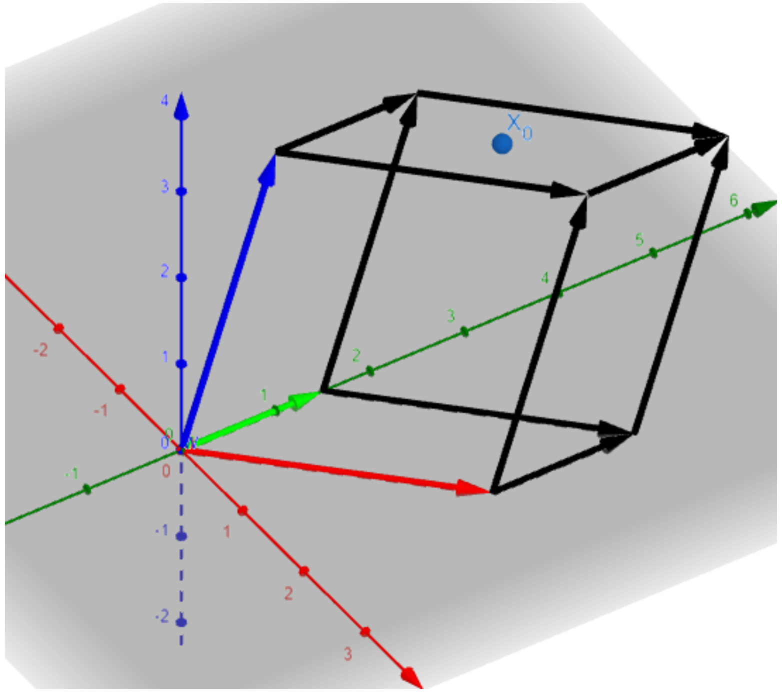
So, $P_{B_1\to B_2}=\frac{1}{11}\begin{pmatrix}27&8&12\\19&15&6\\-6&-3&1\end{pmatrix}$.

Exercises. Find the transition matrix from B_1 to B_2.

1. $B_1 = \{(-2,1),(3,2)\}$, $B_2 = \{(1,2),(-1,0)\}$

2. $B_1 = \{(1,0,1), (0,-1,2), (2,3,-5)\}$, $B_2 = \{(2,-1,4), (0,2,1), (-3,2,1)\}$

Exercise. The vectors $\vec{u} = (2,2,0)$ (red), $\vec{v} = (0, \frac{3}{2}, 0)$ (green), $\vec{w} = (0,1,3)$ (blue) determine a parallelepiped (3d parallelogram) and they form a basis B for $\mathbb{R}^3$.



(a) Find the transition matrix from S to $B = \{\vec{u}, \vec{v}, \vec{w}\}$.

(b) Find the coordinate vector of any vector $\vec{x} = (x, y, z)$ relative to B.

(c) Find the coordinate vector of the vector $\vec{x}_0 = (1, \frac{11}{4}, 3)$ relative to B.

(a)

(b)

(c)

Practice Problems:

Find the transition matrix from B to B'.

25. $B = \{(2,5),(1,2)\}$, $B' = \{(2,1),(-1,2)\}$
27. $B = \{(-3,4),(3,-5)\}$, $B' = \{(-5,-6),(7,-8)\}$
29. $B = \{(1,0,0),(0,1,0),(0,0,1)\}$, $B' = \{(1,3,3),(1,5,6),(1,4,5)\}$
31. $B = \{(1,2,4),(-1,2,0),(2,4,0)\}$, $B' = \{(0,2,1),(-2,1,0),(1,1,1)\}$

Find the transition matrices from B to B' and from B' to B. Verify that they're inverses of each other. Given $[\vec{x}]_{B'}$, find $[\vec{x}]_B$.

37. $B = \{(1,3),(-2,-2)\}$, $B' = \{(-12,0),(-4,4)\}$, $[\vec{x}]_{B'} = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$
39. $B = \{(1,0,2),(0,1,3),(1,1,1)\}$, $B' = \{(2,1,1),(1,0,0),(0,2,1)\}$, $[\vec{x}]_{B'} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$