# 10.2 Calculus with Parametric Curves

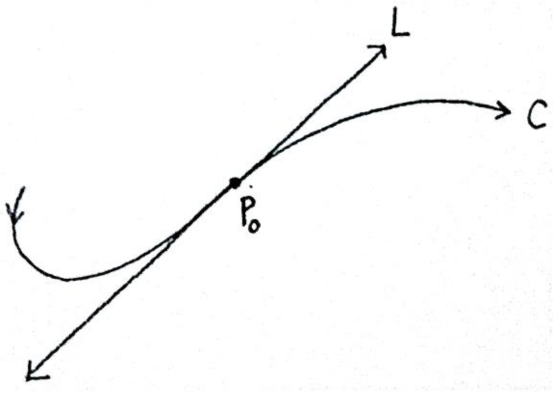
Tangent Lines

Let $C: x = f(t), y = g(t)$ be a parametric curve, and $(x_0, y_0) = \big(f(t_0), g(t_0)\big)$ be a point on $C$ corresponding to a parameter value $t = t_0$. If $f'(t_0) \neq 0$, then the slope of the tangent line to $C$ at $(x_0, y_0)$ is:

$$m = \left.\frac{y'(t)}{x'(t)}\right|_{t=t_0} = \frac{y'(t_0)}{x'(t_0)}$$

Then the equation of the tangent line $L$ to the curve $C$ at the point $P_0 = (x_0, y_0)$ is given by the point-slope form of a line:

$$y - y_0 = m(x - x_0)$$



Examples. Find the equation of the tangent line to the curve $C$ at the given point.

1. $C: x = \sqrt{t}, y = \frac{1}{4}(t^2 - 4);\ (2,3)$

The parameter value corresponding to the point $(2,3)$ on $C$ is found by setting $x = 2, y = 3$ and solving for $t$:

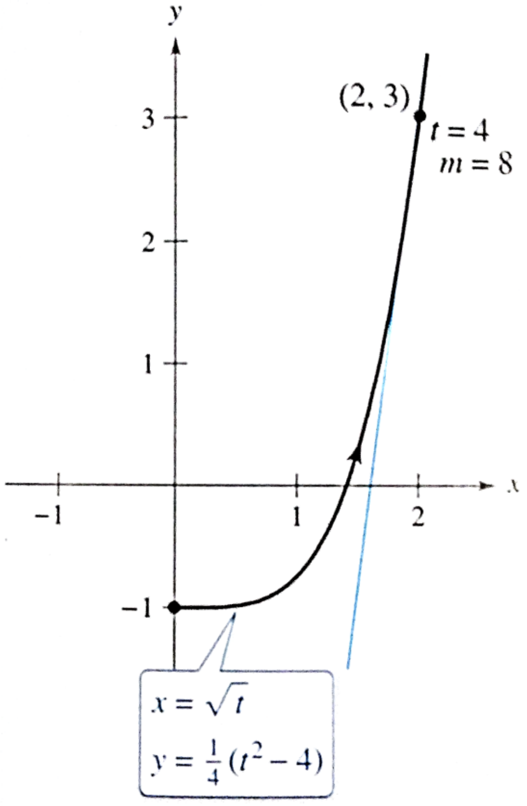
$$\sqrt{t} = 2, \qquad \frac{1}{4}(t^2 - 4) = 3 \Rightarrow t = 4$$

Then the slope of the tangent line at $(2,3)$ is:

$$m = \left.\frac{y'(t)}{x'(t)}\right|_{t=4} = \left.\frac{\frac{t}{2}}{\frac{1}{2\sqrt{t}}}\right|_{t=4} = t\sqrt{t}\Big|_{t=4} = 8$$

So, the equation of the tangent line is:

$$y - 3 = 8(x - 2) \Rightarrow y = 8x - 13$$



2. $C: x = \theta - \sin\theta\,, y = 1 - \cos\theta\,;\ \theta = \frac{\pi}{3}$

The point on $C$ corresponding to the parameter value $\theta = \frac{\pi}{3}$ is:

$$\left(\frac{\pi}{3} - \sin\frac{\pi}{3}, 1 - \cos\frac{\pi}{3}\right) = \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}, \frac{1}{2}\right)$$

The slope of the tangent line to $C$ at this point is:

$$m = \left.\frac{y'(\theta)}{x'(\theta)}\right|_{\theta=\frac{\pi}{3}} = \left.\frac{\sin\theta}{1 - \cos\theta}\right|_{\theta=\frac{\pi}{3}} = \frac{\sqrt{3}/2}{1/2} = \sqrt{3}$$

So, the equation of the tangent line is:

$$y - \frac{1}{2} = \sqrt{3}\left(x - \left(\frac{\pi}{3} - \frac{\sqrt{3}}{2}\right)\right) = \sqrt{3}x - \frac{\pi\sqrt{3}}{3} + \frac{3}{2} \Rightarrow y = \sqrt{3}x + 2 - \frac{\pi\sqrt{3}}{3}$$

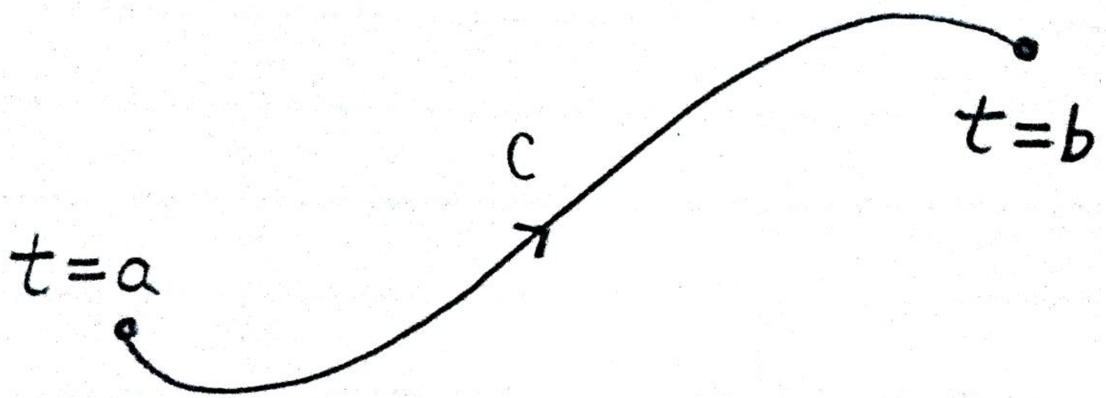
3. $C: x = 1 + \sqrt{t}, y = e^{t^2};\ (2, e)$

4. $x = \cos^3\theta, y = \sin^3\theta;\ \theta = \frac{\pi}{4}$

## Arc Length

Let $C: x = f(t), y = g(t), a \leq t \leq b$ be a parametric curve. Assume $f', g'$ are continuous and $C$ is traversed exactly once from $t = a$ to $t = b$. Then the arc length of $C$ from $t = a$ to $t = b$ is given by:

$$L = \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$$




This gives the distance traveled along $C$ from the point where $t = a$ to the point where $t = b$.

Examples. Find the arc length of the parametric curve $C$.

1. $C: x = 6t^2, y = 2t^3,\ 1 \le t \le 4$

We have:

$$\left(x'(t)\right)^2 + \left(y'(t)\right)^2 = (12t)^2 + (6t^2)^2 = 144t^2 + 36t^4 = 36t^2(4 + t^2)$$

$$\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2} = 6t\sqrt{4 + t^2}$$

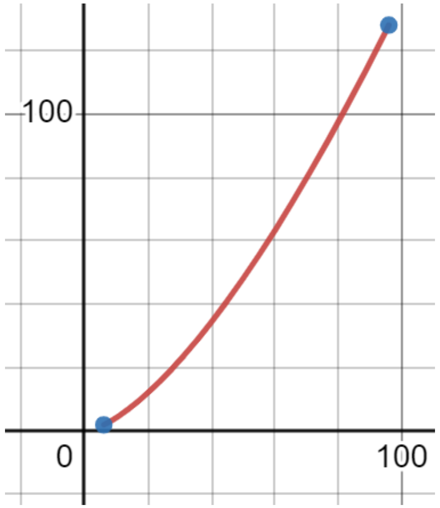
Then:

$$L = \int_1^4 6t\sqrt{4 + t^2}\, dt = \int_5^{20} 3u^{1/2}\, du = 2u^{3/2}\Big|_5^{20}$$

| |
|---|
| $u = 4 + t^2$ |
| $du = 2t\ dt$ |
| $3\ du = 6t\ dt$ |

| $x$ | $u$ |
|---|---|
| 4 | 20 |
| 1 | 5 |

$$= 2\left(20\sqrt{20} - 5\sqrt{5}\right) = 2\left(40\sqrt{5} - 5\sqrt{5}\right) = 70\sqrt{5}$$



2. $C: x = e^t - t, y = 4e^{t/2},\ 0 \le t \le 2$

3. $C: x = e^t \cos t \,, y = e^t \sin t \,, 0 \le t \le \pi$

## Surface Area

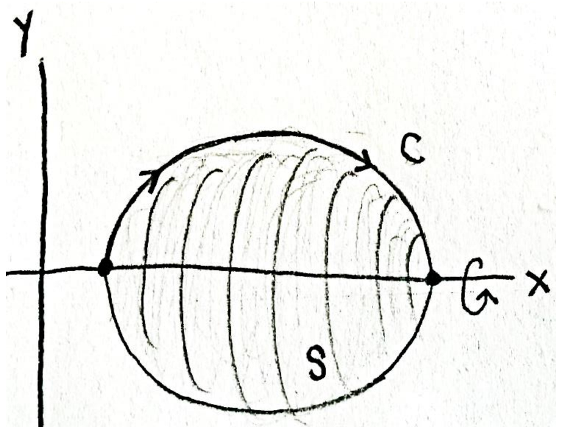
Let $C: x = f(t), y = g(t), a \leq t \leq b$ be a parametric curve. By rotating $C$ around an axis, we obtain a surface of revolution, whose surface area is given by the following formulas:

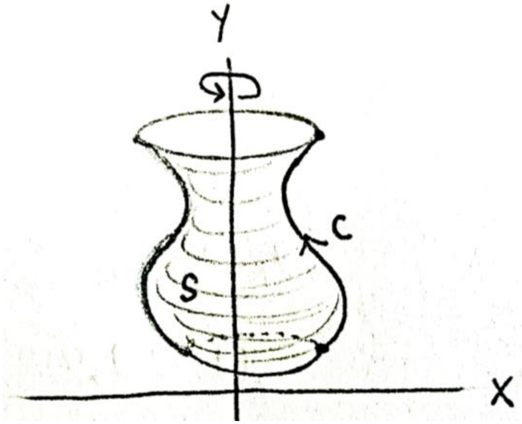
- If $C$ is rotated around the $x$-axis, then the surface area is:

$$S = \int_a^b 2\pi y(t)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$$

- If $C$ is rotated around the $y$-axis, then the surface area is:

$$S = \int_a^b 2\pi x(t)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt$$



Examples. Find the surface area of the surface obtained by rotating the parametric curve $C$ around the given axis.

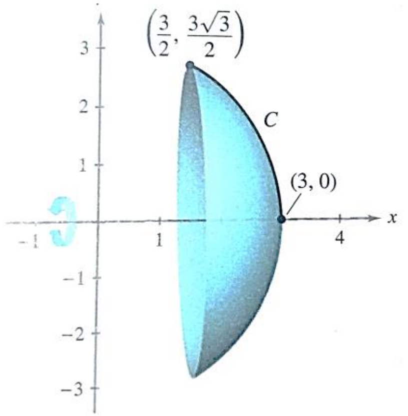
1. $C: x = 3\cos t\,, y = 3\sin t\,, 0 \leq t \leq \frac{\pi}{3}$; around $x$-axis

We have:

$$\left(x'(t)\right)^2 + \left(y'(t)\right)^2 = (-3\sin t)^2 + (3\cos t)^2 = 9(\sin^2 t + \cos^2 t) = 9$$

Then $2\pi y(t)\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2} = 2\pi \cdot 3\sin t \cdot 3 = 18\pi \sin t$, and so:

$$S = \int_0^{\pi/3} 18\pi \sin t\, dt = -18\pi \cos t\Big|_0^{\pi/3} = -18\pi\left(\tfrac{1}{2} - 1\right) = 9\pi$$



2. $C: x = 2t^2 + \frac{1}{t}, y = 8\sqrt{t}, 1 \leq t \leq 3$; around $y$-axis

Practice Exercises:

Find the equation of the tangent line to the curve $C$ at the given point.

5. $C\colon x = t\cos t\,, t = t\sin t;\, t = \pi$

7. $C\colon x = 1 + \ln t\,, y = t^2 + 2;\, (1,3)$

Find the arc length of the curve $C$ over the given interval.

41. $C\colon x = 1 + 3t^2, y = 4 + 2t^3,\ 0 \le t \le 1$

43. $C\colon x = t\sin t\,, y = t\cos t\,,\ 0 \le t \le 1$

Find the surface area of the surface obtained by rotating $C$ around the given axis.

61. $C\colon x = t^3, y = t^2, 0 \le t \le 1$; around $x$-axis

65. $C\colon x = 3t^2, y = 2t^3, 0 \le t \le 5$; around $y$-axis