Pulse Survey

Today we will cover:

Lecture 3: Intractability and Reductions

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- Intractable problems.

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- **Reading material:** "Computers and Intractability", sec. 1.3.: Polynomial Time Algorithms and Intractable Problems.

Intractable Problems

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We have two categories of them:

- Undecidable problems
- Decidable problems

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- Various plane tiling problem.

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  - Given a TM M and an integer k, return 1 if M terminates within k steps and 0 otherwise.

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Does M halts on input w?

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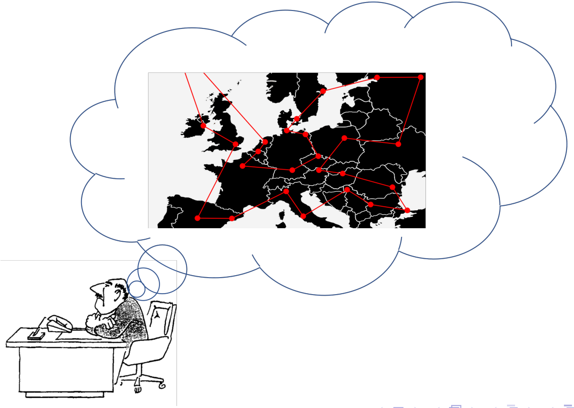

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- No-instance: otherwise.

TMs and Optimization Problems

Let M be a TM, let $w \in \Sigma^*$, and denote:

$$f_M(w) = \begin{cases} \text{Contents of } M\text{'s tape after it terminates} & \text{If } M \text{ halts on } w \\ \text{Undefined} & \text{Otherwise} \end{cases}.$$

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then M solves TSP.

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$$\text{Outcome : } f_M(w) = 1 \text{ iff there's a } (\leq d) \text{ path covering all vertices.}$$

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- Hint: How many bits are needed to encode a weighted graph?

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Similarly, if $L, L' \subseteq \Sigma^*$ are languages and there exists p-time machine M such that

$$w \in \mathcal{L} \iff f_M(w) \in \mathcal{L}'$$

then

$$\mathcal{L} \leq_p \mathcal{L}'$$

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Transitive

$$A \leq_p B \wedge B \leq_p C \Rightarrow A \leq_p C$$

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- Therefore $M \circ N$ runs in p-time and we get $A \leq_p C$.

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Yes-instance: if G has a Hamiltonian Circuit.
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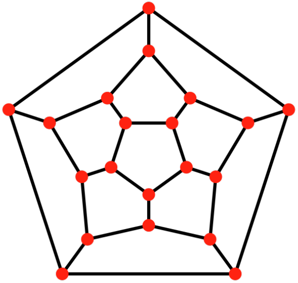

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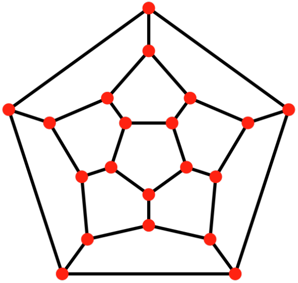

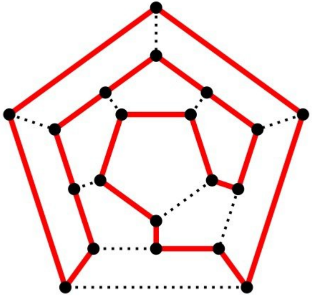

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The reduction maps $G = (V, E)$ to the instance $((V, w), |V|)$ of TSDP. Clearly, this reduction can be computed in time p-time.

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- γ is a Hamiltonian Circuit of G' and the total weight of γ is $|V|$. Therefore $(G', |V|)$ is a yes-instance of TSDP.
- Conversely, if $(G', |V|)$ is a yes-instance of TSDP then there must be a circuit ρ of G' of total weight not more than $|V|$, this can only happen if all nodes in the sequence are distinct and total weight equals $|V|$, hence ρ must be a Hamiltonian Circuit of G and so G is a yes-instance of HCP.

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- A fast algorithm that solves TSDP could be turned into a fast algorithm to solve HCP.
- If there are no fast algorithms to solve HCP then there can be no fast algorithms for TSDP.

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- How optimization and decision problems differ, and how to go from one to the other.
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Next week: Non-determinism and the $P \stackrel{?}{=} NP$ question.