Lectures 12: Savitch's Theorem

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- Prove Savitch's Theorem: $SPACE(f(n)) \subseteq SPACE((f(n))^2)$.
- Conclude that $PSPACE = NPSPACE$.

Savitch's Theorem

Theorem

$f(n) \geq n \implies NSPACE(f(n)) \subseteq SPACE(f^2(n))$.

Proof.

We convert a NDTM machine N into an equivalent DTM M while only squaring the space.

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	Step	**State**	**Head**	$\boldsymbol{T_0}$	$\boldsymbol{T_1}$	$\boldsymbol{T_2}$	...	$\boldsymbol{T_{n-1}}$	$\boldsymbol{T_n}$	...	$\boldsymbol{T_{f(n)}}$
c	0	q	h	w_0	w_1	w_2		w_n	w_{n+1}		$w_{f(n)}$
	0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,f(n)}$
	1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,f(n)}$
	2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,f(n)}$
	$\vdots$									$\ddots$	
	k	q_{i_k}	h_k	$w_{k,0}$	$w_{k,1}$	$w_{k,2}$		$w_{k,n-1}$	$w_{k,n}$		$w_{k,f(n)}$
c'	k	q'	h'	w'_0	w'_1	w'_2		w'_{n-1}	w'_n		$w'_{f(n)}$

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c'	k	q'	h'	w'_0	w'_1	w'_2		w'_{n-1}	w'_n		$w'_{f(n)}$

Almost looks like a ladder!

Proof (Cont.)

Function Reachable(c, c', k) # Can we reach from c to c' in k steps?

Savitch's Theorem: $f(n) \geq n \implies NSPACE(f(n)) \subseteq SPACE(f^2(n))$

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Function Reachable(c, c', k) # Can we reach from c to c' in k steps?
  If $(k = 1)$
    Return (c' can be reached from c in M) $\vee$ $(c = c')$

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  For each possible configuration c_{mid}
    If (Reachable$(c, c_{\text{mid}}, \lceil k/2 \rceil)$$\wedge$ Reachable$(c_{\text{mid}}, c', \lfloor k/2 \rfloor)$)
      Return T
  Return F

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Proof (Cont.)

- We can tweak the machine to get a single accepting c_Y configuration by reversing the head and erasing the tape before halting.

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Proof (Cont.)

- We can tweak the machine to get a single accepting c_Y configuration by reversing the head and erasing the tape before halting.
- Since N uses $f(n)$ space, there are just $C = |Q| \times f(n) \times |\Gamma|^{f(n)}$ is the number of possible configurations.

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- We start by calling Reachable(c_0, c_Y, C), where c_0 encodes the (input-specific!) starting configuration.

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- Each time we store one configuration, requiring $O(f(n))$ space.

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- The depth of the recursion is $\log_2\left(C\right) = O(f(n))$.
- Each time we store one configuration, requiring $O(f(n))$ space.
- Therefore, the total space is $O(f^2(n))$. $\square$

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- Each time we store one configuration, requiring $O(f(n))$ space.
- Therefore, the total space is $O(f^2(n))$. $\square$

Conclusion: $PSPACE = NPSPACE$.

PSPACE-Completeness

Definition

A problem B is PSPACE-Complete if:

- $B \in PSPACE$.
- For all $A \in PSPACE : A \leq_p B$.

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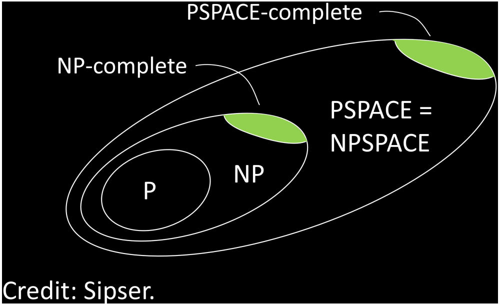
To ponder: why do we use $\leq_p$?



PSPACE-Completeness of TQBF

Knowing that $TQBF \in PSPACE - Complete$, and assuming that $TQBF \in NP$, which of the following are true:

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- $co{-}NP = PSPACE$.
- SAT is PSPACE-Complete.

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Answer on Mentimeter:

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Knowing that $TQBF \in PSPACE - Complete$, and assuming that $TQBF \in NP$, which of the following are true:

- $P = PSPACE$. No, this doesn't imply that $TQBF \in P$.

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- $P = PSPACE$. No, this doesn't imply that $TQBF \in P$.
- $NP = PSPACE$. Yes, all PSPACE problems are reducible to TQBF as its PSPACE-Complete.

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- $P = NP$. No.

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- $P = NP$. No.
- $NP = co{-}NP$. Yes! PSPACE is closed under complementation.

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- $co-NP = PSPACE$. Yes.
- SAT is PSPACE-Complete. Yes.

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- $P = NP$. No.
- $NP = co{-}NP$. Yes! PSPACE is closed under complementation.
- $co{-}NP = PSPACE$. Yes.
- SAT is PSPACE-Complete. Yes.
- BEQ is NP-Complete. Yes! $\overline{BEQ}$ is NP-Complete (why?). For any $L \in NP$, we have thus $\overline{L} \leq_p TQBF \leq_p \overline{BEQ}$ which gives $L \leq_p BEQ$.

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Theorem

TQBF is PSPACE-Complete.

Proof Sketch.

Recall the proof of the Cook-Levin theorem:

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Step	**State**	**Head**	$\boldsymbol{T}_0$	$\boldsymbol{T}_1$	$\boldsymbol{T}_2$	…	$\boldsymbol{T}_{n-1}$	$\boldsymbol{T}_n$	…	$\boldsymbol{T}_N$
0	q_0	0	w_0	w_1	w_2		w_{n-1}	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
$f(N)$	$q_{i_{f(N)}}$	$h_{f(N)}$	$w_{f(N),0}$	$w_{f(N),1}$	$w_{f(N),2}$		$w_{f(N),f(N)-1}$	$w_{f(N),n}$		$w_{f(N),N}$

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Theorem

TQBF is PSPACE-Complete.

Proof Sketch.

Recall the proof of the Cook-Levin theorem:

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_{n-1}	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
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The challenge is that now we have a PSPACE machine, so while space is bounded by $N = n^{o(1)}$, the time could be $f(N) = 2^{O(N)} = 2^{n^{O(1)}}$, so the same reduction would not work (the formula will have exponential size).

PSPACE-Completeness of TQBF

Theorem

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Proof Sketch. (Cont.)

Step	**State**	**Head**	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_{n-1}	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
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$f(N)$	$q_{i_{f(N)}}$	$h_{f(N)}$	$w_{f(N),0}$	$w_{f(N),1}$	$w_{f(N),2}$		$w_{f(N),f(N)-1}$	$w_{f(N),n}$		$w_{f(N),N}$

Instead, we can think about this similarly to a word ladder.

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TQBF is PSPACE-Complete.

Proof Sketch. (Cont.)

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Instead, we can think about this similarly to a word ladder.
Namely, given configurations c, c' and a bound on the number of steps b, we construct the following QBF $\phi_{c,c',b}$, which is true if and only if c' is reachable from c within at most b steps.

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$$\phi_{c,c',b} = \exists c_{\text{mid}}$$

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PSPACE-Completeness of TQBF

Theorem

TQBF is PSPACE-Complete.

Proof Sketch. (Cont.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_{n-1}	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
$f(N)$	$q_{i_{f(N)}}$	$h_{f(N)}$	$w_{f(N),0}$	$w_{f(N),1}$	$w_{f(N),2}$		$w_{f(N),f(N)-1}$	$w_{f(N),n}$		$w_{f(N),N}$

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1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
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Each configuration is encoded with the same $x_{t,i,\gamma}, y_{t,s}, z_{t,h}$ variables as in the CL theorem.

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For $b = 1$ we explicitly write the formula of the t'th step yielding the $t+1$'th

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The reduction outputs $\phi_{c_0,c_Y,f(N)}$.

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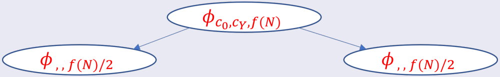

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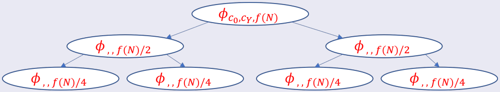

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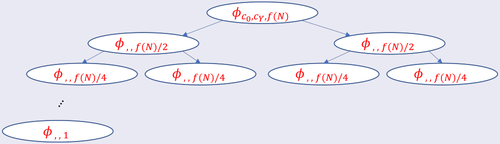

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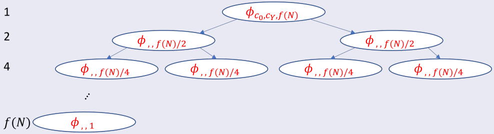

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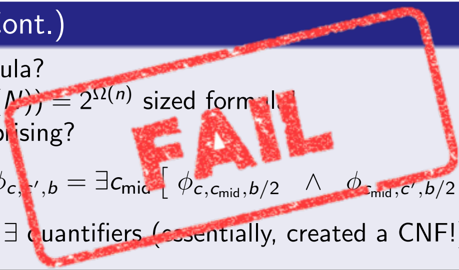

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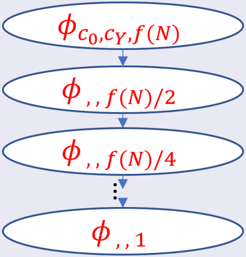

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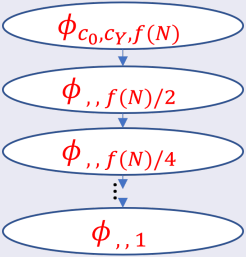

We have $\log f(N) = n^{O(1)}$ levels, each has a polynomial size QBF.

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Next lecture: the Log-space class.