Lecture 11: The Ladder Game

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Answer on Mentimeter:

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- $NP \cap co-NP = \emptyset$. No. For example, $P \subseteq NP \cap co-NP$.
- $NP \cap co-NP = P$. This is an open question.
- If $L \in co-NP$ and $L \leq_p SAT$ then $NP = co-NP$. No. For example, any $L \in P \subseteq co-NP$ is reducible to SAT.

Lecture 11: The Ladder Game

We will see:

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We will see:

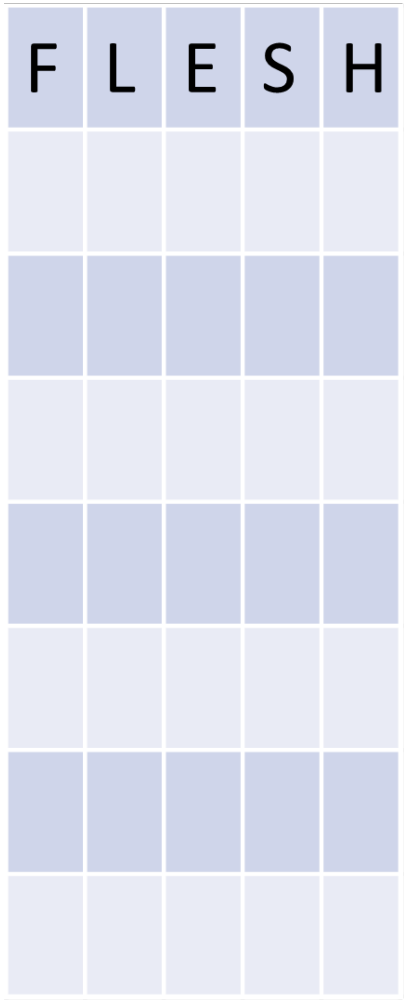
- A game.

Lecture 11: The Ladder Game

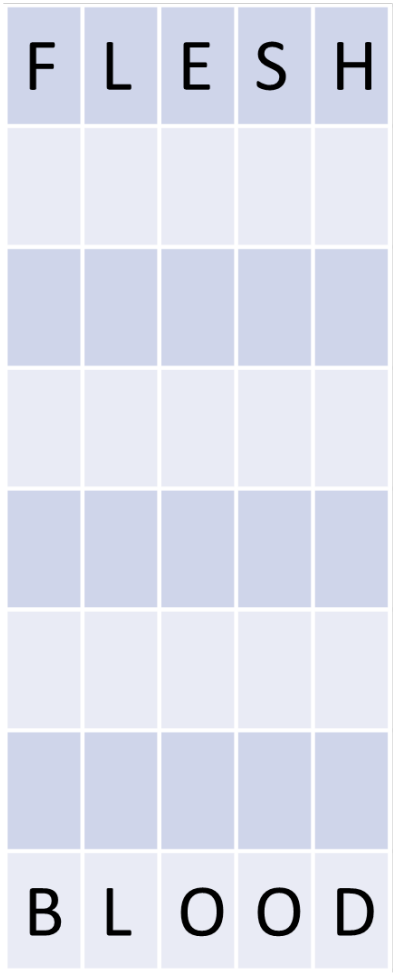
We will see:

- A game.
- And analyze its space complexity.

Word ladder



Word ladder



Word ladder

F	L	E	S	H
F	L	A	S	H
B	L	O	O	D

Word ladder

F	L	E	S	H
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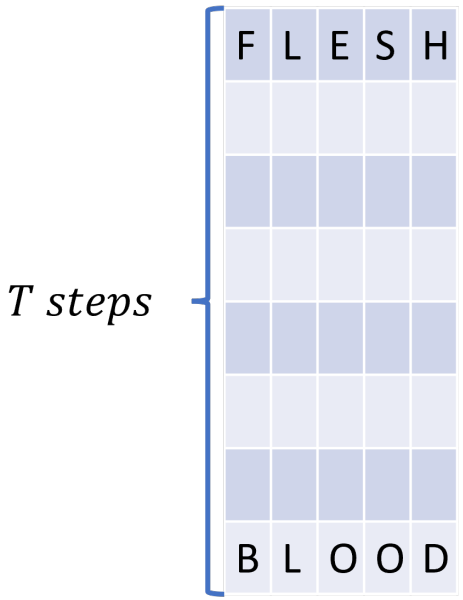
Observation: it is enough to consider ladders of length $T \leq |\Sigma|^{|s|}$.



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Proof.

An algorithm:

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Set $\ell = 0, w = s$
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    If ($w' \in L(M)$ and differs from w by one symbol)
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This shows $LADDER \in NSPACE(n)$.

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This shows $LADDER \in NSPACE(n)$.
(We store 2 words and $\log(|\Sigma|^{|s|}) = |s| \log |\Sigma| = O(n)$-bits counter). □

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Hint: what's the size of the ladder?

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    Return F

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To check if $\langle M, s, t\rangle \in$ LADDER we invoke Reachable$(s, t, |\Sigma|^{|s|})$.

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Therefore, we showed that $LADDER \in SPACE(n^2)$, and in particular, $LADDER \in PSPACE$.

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Therefore, we showed that $LADDER \in SPACE(n^2)$, and in particular, $LADDER \in PSPACE$. We will see that this phenomenon is much more general.

Lecture 11: Recap

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Next: Savitch's Theorem and PSPACE=NPSPACE.