Lecture 6: The Cook-Levin Theorem

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- That SAT is NP-hard.

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Theorem

SAT is NP-complete.

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- That SAT is NP-hard.
- Proving NP-hardness by reduction from SAT.

Theorem

SAT is NP-complete.

Last lecture, we saw The Cook-Levin theorem part I: SAT is in NP.

Proof (Cont.)

Let M be a NDTM for L, let $w_0 w_1 \ldots w_{n-1}$ be the input, and suppose that M is guaranteed to terminate within $N = n^{O(1)}$ steps.

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (Cont.)

Let M be a NDTM for L, let $w_0 w_1 \ldots w_{n-1}$ be the input, and suppose that M is guaranteed to terminate within $N = n^{O(1)}$ steps. We first start with visualizing one possible computation:

Step

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (Cont.)

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Step	**State**

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (Cont.)

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Step	**State**	**Head**

Proof (Cont.)

Let M be a NDTM for L, let $w_0 w_1 \ldots w_{n-1}$ be the input, and suppose that M is guaranteed to terminate within $N = n^{O(1)}$ steps. We first start with visualizing one possible computation:

Step	**State**	**Head**	$\boldsymbol{T_0}$	$\boldsymbol{T_1}$	$\boldsymbol{T_2}$	...	$\boldsymbol{T_{n-1}}$

Proof (Cont.)

Let M be a NDTM for L, let $w_0 w_1 \ldots w_{n-1}$ be the input, and suppose that M is guaranteed to terminate within $N = n^{O(1)}$ steps. We first start with visualizing one possible computation:

Step	**State**	**Head**	$\boldsymbol{T}_0$	$\boldsymbol{T}_1$	$\boldsymbol{T}_2$	...	$\boldsymbol{T}_{n-1}$	$\boldsymbol{T}_n$	...	$\boldsymbol{T}_N$

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (Cont.)

Let M be a NDTM for L, let $w_0 w_1 \ldots w_{n-1}$ be the input, and suppose that M is guaranteed to terminate within $N = n^{O(1)}$ steps. We first start with visualizing one possible computation:

Step	**State**	**Head**	$\boldsymbol{T_0}$	$\boldsymbol{T_1}$	$\boldsymbol{T_2}$	...	$\boldsymbol{T_{n-1}}$	$\boldsymbol{T_n}$	...	$\boldsymbol{T_N}$
0										

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (Cont.)

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Step	**State**	**Head**	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0									

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Proof (Cont.)

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0	q_0	0								

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Step	**State**	**Head**	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n			

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0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$

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Proof (Cont.)

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0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$

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0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$

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0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

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Proof (Cont.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

Proof (Cont.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

We define binary variable sets:

Proof (Cont.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

We define binary variable sets:

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma :$ $\quad x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (Cont.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

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Proof (Cont.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

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The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (Cont.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

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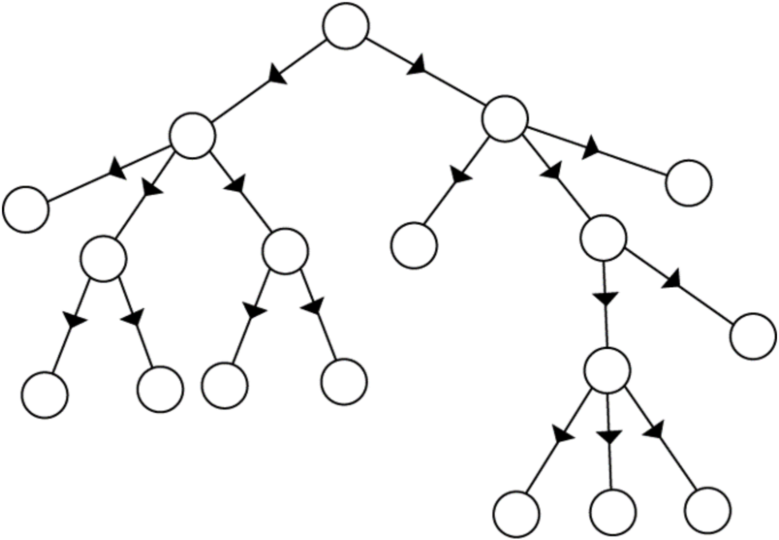
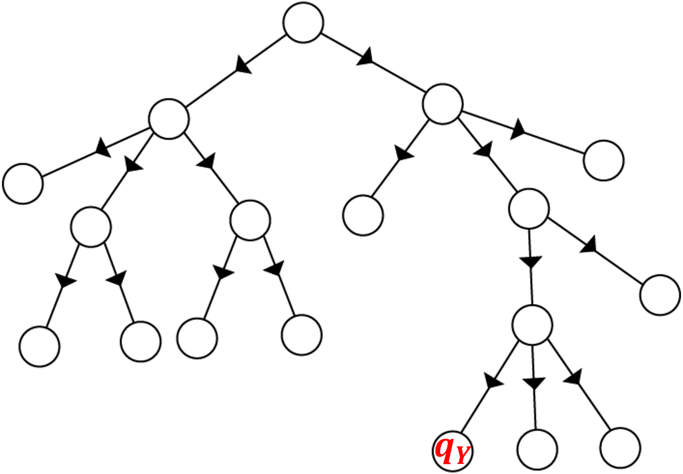
Goal: constructing a CNF formula ϕ that is satisfiable $\iff$ M accepts w.

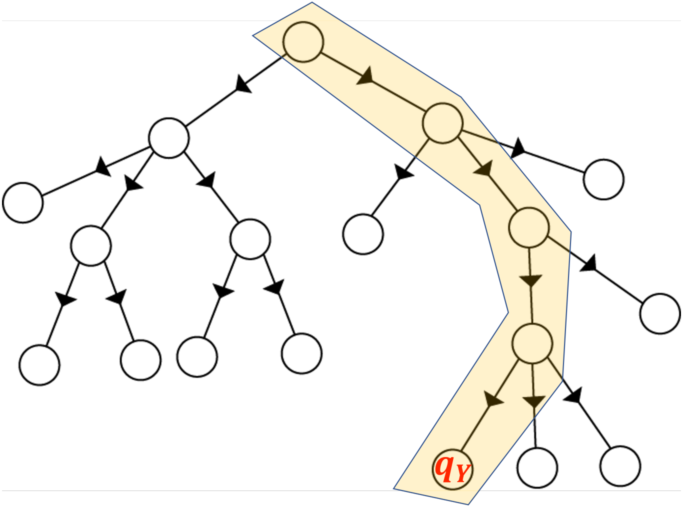
An NDTM configuration tree



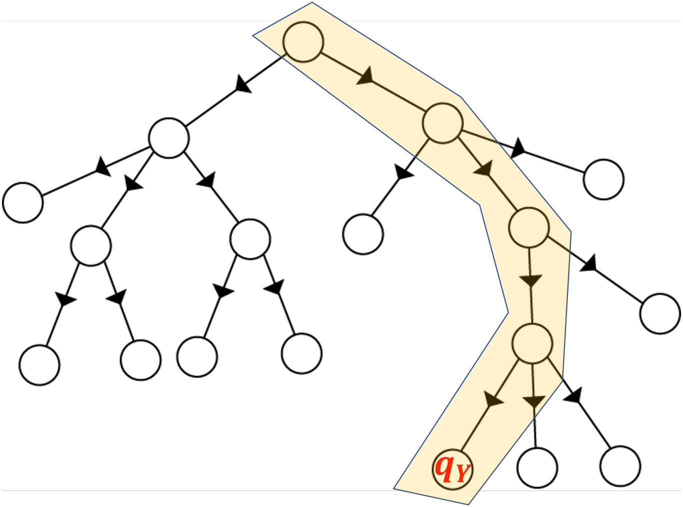
An NDTM configuration tree



An NDTM configuration tree



An NDTM configuration tree



A formula's satisfying assignment corresponds to an accepting path.

Proof (Cont.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

We define binary variable sets:

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

Goal: constructing a CNF formula ϕ that is satisfiable $\iff$ M accepts w.

Strategy: $\phi = \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4$, where:

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (Cont.)

Step	**State**	**Head**	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

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Strategy: $\phi = \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4$, where:

- ϕ_1 is true if the first row to correspond to the initial state.

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (Cont.)

Step	**State**	**Head**	$\boldsymbol{T_0}$	$\boldsymbol{T_1}$	$\boldsymbol{T_2}$	$\dots$	$\boldsymbol{T_{n-1}}$	$\boldsymbol{T_n}$	$\dots$	$\boldsymbol{T_N}$
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

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Strategy: $\phi = \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4$, where:

- ϕ_1 is true if the first row to correspond to the initial state.
- ϕ_2 is true if we reach q_Y.

Proof (Cont.)

Step	**State**	**Head**	$\boldsymbol{T_0}$	$\boldsymbol{T_1}$	$\boldsymbol{T_2}$	…	$\boldsymbol{T_{n-1}}$	$\boldsymbol{T_n}$	…	$\boldsymbol{T_N}$
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

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- ϕ_1 is true if the first row to correspond to the initial state.
- ϕ_2 is true if we reach q_Y.
- ϕ_3 is true if each cell has exactly one value.

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (Cont.)

Step	**State**	**Head**	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

We define binary variable sets:

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

Goal: constructing a CNF formula ϕ that is satisfiable $\iff$ M accepts w.

Strategy: $\phi = \phi_1 \wedge \phi_2 \wedge \phi_3 \wedge \phi_4$, where:

- ϕ_1 is true if the first row to correspond to the initial state.
- ϕ_2 is true if we reach q_Y.
- ϕ_3 is true if each cell has exactly one value.
- ϕ_4 is true if each line but the last can yield the next one.

Proof (ϕ_1 is true if the first row to correspond to the initial state.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_1 =$$

Proof (ϕ_1 is true if the first row to correspond to the initial state.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_1 = y_{0,q_0} \wedge$$

Proof (ϕ_1 is true if the first row to correspond to the initial state.)

Step	**State**	**Head**	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_1 = y_{0,q_0} \wedge z_{0,0} \wedge$$

Proof (ϕ_1 is true if the first row to correspond to the initial state.)

Step	State	Head	T_0	T_1	T_2	…	T_{n-1}	T_n	…	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_1 = y_{0,q_0} \wedge z_{0,0} \wedge \left(\bigwedge_{0 \leq i \leq n-1} x_{0,i,w_i} \right) \wedge$$

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_1 is true if the first row to correspond to the initial state.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_1 = y_{0,q_0} \wedge z_{0,0} \wedge \left(\bigwedge_{0 \leq i \leq n-1} x_{0,i,w_i} \right) \wedge \left(\bigwedge_{n \leq i \leq N} x_{0,i,\sqcup} \right) .$$

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_1 is true if the first row to correspond to the initial state.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_1 = y_{0,q_0} \wedge z_{0,0} \wedge \left(\bigwedge_{0 \leq i \leq n-1} x_{0,i,w_i} \right) \wedge \left(\bigwedge_{n \leq i \leq N} x_{0,i,\sqcup} \right) .$$

How big is ϕ_1?

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_1 is true if the first row to correspond to the initial state.)

Step	**State**	**Head**	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_1 = y_{0,q_0} \wedge z_{0,0} \wedge \left(\bigwedge_{0 \leq i \leq n-1} x_{0,i,w_i} \right) \wedge \left(\bigwedge_{n \leq i \leq N} x_{0,i,\sqcup} \right) .$$

How big is ϕ_1? $O(N) = n^{O(1)}$.

Proof (ϕ_2 is true if we reach q_Y.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_2 =$$

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_2 is true if we reach q_Y.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_2 = \left(\bigvee_{0 \leq t \leq N} y_{t,q_Y} \right) \quad .$$

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_2 is true if we reach q_Y.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_2 = \left(\bigvee_{0 \leq t \leq N} y_{t,q_Y} \right) .$$

How big is ϕ_2?

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_2 is true if we reach q_Y.)

Step	**State**	**Head**	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_2 = \left(\bigvee_{0 \leq t \leq N} y_{t,q_Y} \right) .$$

How big is ϕ_2? $O(N) = n^{O(1)}$.

Proof (ϕ_3 is true if each cell has exactly one value.)

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_3 is true if each cell has exactly one value.)

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_3 = \bigwedge_{0 \leq t, i \leq N}$$

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_3 is true if each cell has exactly one value.)

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_3 = \bigwedge_{0 \leq t,i \leq N} \left(\left(\bigvee_{\gamma \in \Gamma} x_{t,i,\gamma} \right) \wedge \right.$$

Proof (ϕ_3 is true if each cell has exactly one value.)

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_3 = \bigwedge_{0 \leq t,i \leq N} \left(\left(\bigvee_{\gamma \in \Gamma} x_{t,i,\gamma} \right) \wedge \left(\bigwedge_{\gamma,\gamma' \in \Gamma,\ \gamma \neq \gamma'} \left(\overline{x_{t,i,\gamma}} \vee \overline{x_{t,i,\gamma'}} \right) \right) \right)$$

Proof (ϕ_3 is true if each cell has exactly one value.)

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\begin{aligned}\phi_3 = & \bigwedge_{0\leq t,i\leq N}\left(\left(\bigvee_{\gamma\in\Gamma} x_{t,i,\gamma}\right)\wedge\left(\bigwedge_{\gamma,\gamma'\in\Gamma,\ \gamma\neq\gamma'}\left(\overline{x_{t,i,\gamma}}\vee\overline{x_{t,i,\gamma'}}\right)\right)\right)\\ & \bigwedge_{0\leq t\leq N}\left(\left(\bigvee_{s\in Q} y_{t,s}\right)\wedge\left(\bigwedge_{s,s'\in Q,\ s\neq s'}\left(\overline{y_{t,s}}\vee\overline{y_{t,s'}}\right)\right)\right)\\ & \bigwedge_{0\leq t\leq N}\left(\left(\bigvee_{0\leq h\leq N} z_{t,h}\right)\wedge\left(\bigwedge_{0\leq h,h'\leq N,\ h\neq h'}\left(\overline{z_{t,h}}\vee\overline{z_{t,h'}}\right)\right)\right)\quad .\end{aligned}$$

Proof (ϕ_3 is true if each cell has exactly one value.)

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_3 = \bigwedge_{0 \leq t,i \leq N} \left(\left(\bigvee_{\gamma \in \Gamma} x_{t,i,\gamma} \right) \wedge \left(\bigwedge_{\gamma,\gamma' \in \Gamma,\ \gamma \neq \gamma'} \left(\overline{x_{t,i,\gamma}} \vee \overline{x_{t,i,\gamma'}} \right) \right) \right)$$
$$\bigwedge_{0 \leq t \leq N} \left(\left(\bigvee_{s \in Q} y_{t,s} \right) \wedge \left(\bigwedge_{s,s' \in Q,\ s \neq s'} \left(\overline{y_{t,s}} \vee \overline{y_{t,s'}} \right) \right) \right)$$
$$\bigwedge_{0 \leq t \leq N} \left(\left(\bigvee_{0 \leq h \leq N} z_{t,h} \right) \wedge \left(\bigwedge_{0 \leq h,h' \leq N,\ h \neq h'} \left(\overline{z_{t,h}} \vee \overline{z_{t,h'}} \right) \right) \right) .$$

How big is ϕ_3?

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_3 is true if each cell has exactly one value.)

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_3 = \bigwedge_{0 \leq t,i \leq N} \left(\left(\bigvee_{\gamma \in \Gamma} x_{t,i,\gamma} \right) \wedge \left(\bigwedge_{\gamma,\gamma' \in \Gamma,\ \gamma \neq \gamma'} (\overline{x_{t,i,\gamma}} \vee \overline{x_{t,i,\gamma'}}) \right) \right)$$
$$\bigwedge_{0 \leq t \leq N} \left(\left(\bigvee_{s \in Q} y_{t,s} \right) \wedge \left(\bigwedge_{s,s' \in Q,\ s \neq s'} (\overline{y_{t,s}} \vee \overline{y_{t,s'}}) \right) \right)$$
$$\bigwedge_{0 \leq t \leq N} \left(\left(\bigvee_{0 \leq h \leq N} z_{t,h} \right) \wedge \left(\bigwedge_{0 \leq h,h' \leq N,\ h \neq h'} (\overline{z_{t,h}} \vee \overline{z_{t,h'}}) \right) \right) .$$

How big is ϕ_3? $O(N^3) = n^{O(1)}$.

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_4 is true if each line but the last can yield the next one.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_4 is true if each line but the last can yield the next one.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \ldots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$
  $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_4 is true if each line but the last can yield the next one.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \ldots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$
  $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

Let us look at time $t \in \{0, \ldots, N-1\}$, and break down $\phi_4 = \phi_5 \wedge \phi_6 \wedge \phi_7 \wedge \phi_8 \wedge \phi_9$ into

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_4 is true if each line but the last can yield the next one.)

Step	**State**	**Head**	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \dots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \dots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \dots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \dots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$ $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

Let us look at time $t \in \{0, \dots, N-1\}$, and break down $\phi_4 = \phi_5 \wedge \phi_6 \wedge \phi_7 \wedge \phi_8 \wedge \phi_9$ into

- ϕ_5 each transition is valid from its starting configuration.

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_4 is true if each line but the last can yield the next one.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \ldots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$
  $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

Let us look at time $t \in \{0, \ldots, N-1\}$, and break down $\phi_4 = \phi_5 \wedge \phi_6 \wedge \phi_7 \wedge \phi_8 \wedge \phi_9$ into

- ϕ_5 each transition is valid from its starting configuration.
- ϕ_6 ensures that all the tape is maintained, possibly expect location h_t.

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_4 is true if each line but the last can yield the next one.)

Step	**State**	**Head**	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \ldots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$ $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

Let us look at time $t \in \{0, \ldots, N-1\}$, and break down $\phi_4 = \phi_5 \wedge \phi_6 \wedge \phi_7 \wedge \phi_8 \wedge \phi_9$ into

- ϕ_5 each transition is valid from its starting configuration.
- ϕ_6 ensures that all the tape is maintained, possibly expect location h_t.
- ϕ_7 establishes that location h_t is updated correctly.

Proof (ϕ_4 is true if each line but the last can yield the next one.)

Step	**State**	**Head**	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \ldots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$
  $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

Let us look at time $t \in \{0, \ldots, N-1\}$, and break down $\phi_4 = \phi_5 \wedge \phi_6 \wedge \phi_7 \wedge \phi_8 \wedge \phi_9$ into

- ϕ_5 each transition is valid from its starting configuration.
- ϕ_6 ensures that all the tape is maintained, possibly expect location h_t.
- ϕ_7 establishes that location h_t is updated correctly.
- ϕ_8 verifies that the head is moving in the right direction.

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_4 is true if each line but the last can yield the next one.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \ldots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$
  $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

Let us look at time $t \in \{0, \ldots, N-1\}$, and break down $\phi_4 = \phi_5 \wedge \phi_6 \wedge \phi_7 \wedge \phi_8 \wedge \phi_9$ into

- ϕ_5 each transition is valid from its starting configuration.
- ϕ_6 ensures that all the tape is maintained, possibly expect location h_t.
- ϕ_7 establishes that location h_t is updated correctly.
- ϕ_8 verifies that the head is moving in the right direction.
- ϕ_9 checks that the state changes properly.

Proof (ϕ_5 each transition is valid from its starting configuration.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \ldots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$
  $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_5 each transition is valid from its starting configuration.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \ldots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$ $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

$$\phi_5 = \Bigg(\bigwedge_{\substack{0 \leq t, i \leq N-1 \\ \gamma, \gamma' \in \Gamma \\ s, s' \in Q \\ (s', \gamma', d) \in \delta(s, \gamma)}}$$

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_5 each transition is valid from its starting configuration.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \ldots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$
  $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

$$\phi_5 = \Bigg(\bigwedge_{\substack{0 \leq t,i \leq N-1 \\ \gamma,\gamma' \in \Gamma \\ s,s' \in Q \\ (s',\gamma',d) \in \delta(s,\gamma)}} \big(\overline{r_{t,i,s,s',\gamma,\gamma',d}} \vee$$

Proof (ϕ_5 each transition is valid from its starting configuration.)

Step	**State**	**Head**	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \ldots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$ $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

$$\phi_5 = \left(\bigwedge_{\substack{0 \leq t,i \leq N-1 \\ \gamma,\gamma' \in \Gamma \\ s,s' \in Q \\ (s',\gamma',d) \in \delta(s,\gamma)}} \left(\overline{r_{t,i,s,s',\gamma,\gamma',d}} \vee \left(x_{t,i,\gamma} \wedge y_{t,s} \wedge z_{t,i} \right) \right) \right)$$

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_5 each transition is valid from its starting configuration.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \ldots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$ $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

$$\phi_5 = \left(\bigwedge_{\substack{0 \leq t,i \leq N-1 \\ \gamma,\gamma' \in \Gamma \\ s,s' \in Q \\ (s',\gamma',d) \in \delta(s,\gamma)}} \left(\overline{r_{t,i,s,s',\gamma,\gamma',d}} \vee \left(x_{t,i,\gamma} \wedge y_{t,s} \wedge z_{t,i} \right) \right) \right) \wedge \left(\bigwedge_{0 \leq t \leq N-1} \text{Exactly one } r_{t,i,s,s',\gamma,\gamma',d} \text{ is true.} \right)$$

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_5 each transition is valid from its starting configuration.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \ldots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$ $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

$$\phi_5 = \left(\bigwedge_{\substack{0 \leq t,i \leq N-1 \\ \gamma,\gamma' \in \Gamma \\ s,s' \in Q \\ (s',\gamma',d) \in \delta(s,\gamma)}} \left(\overline{r_{t,i,s,s',\gamma,\gamma',d}} \vee \left(x_{t,i,\gamma} \wedge y_{t,s} \wedge z_{t,i}\right)\right) \right) \wedge \left(\bigwedge_{0 \leq t \leq N-1} \text{Exactly one } r_{t,i,s,s',\gamma,\gamma',d} \text{ is true.} \right)$$

This is not a CNF!

Proof (ϕ_5 each transition is valid from its starting configuration.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \ldots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$ $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

$$\phi_5 = \left(\bigwedge_{\substack{0 \leq t, i \leq N-1 \\ \gamma, \gamma' \in \Gamma \\ s, s' \in Q \\ (s', \gamma', d) \in \delta(s, \gamma)}} \left(\overline{r_{t,i,s,s',\gamma,\gamma',d}} \vee \left(x_{t,i,\gamma} \wedge y_{t,s} \wedge z_{t,i} \right) \right) \right) \wedge \left(\bigwedge_{0 \leq t \leq N-1} \text{Exactly one } r_{t,i,s,s',\gamma,\gamma',d} \text{ is true.} \right)$$

This is not a CNF! But we can use the fact that $a \vee (b \wedge c \wedge d) = (a \vee b) \wedge (a \vee c) \wedge (a \vee d)$.

Proof (ϕ_5 each transition is valid from its starting configuration.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \ldots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$ $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

$$\phi_5 = \left(\bigwedge_{\substack{0 \leq t,i \leq N-1 \\ \gamma,\gamma' \in \Gamma \\ s,s' \in Q \\ (s',\gamma',d) \in \delta(s,\gamma)}} \left(\overline{r_{t,i,s,s',\gamma,\gamma',d}} \vee \left(x_{t,i,\gamma} \wedge y_{t,s} \wedge z_{t,i}\right)\right) \right) \wedge \left(\bigwedge_{0 \leq t \leq N-1} \text{Exactly one } r_{t,i,s,s',\gamma,\gamma',d} \text{ is true.} \right)$$

This is not a CNF! But we can use the fact that $a \vee (b \wedge c \wedge d) = (a \vee b) \wedge (a \vee c) \wedge (a \vee d)$.

How big is ϕ_5?

Proof (ϕ_5 each transition is valid from its starting configuration.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \ldots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$ $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

$$\phi_5 = \left(\bigwedge_{\substack{0 \leq t,i \leq N-1 \\ \gamma,\gamma' \in \Gamma \\ s,s' \in Q \\ (s',\gamma',d) \in \delta(s,\gamma)}} \left(\overline{r_{t,i,s,s',\gamma,\gamma',d}} \vee \left(x_{t,i,\gamma} \wedge y_{t,s} \wedge z_{t,i}\right)\right) \right) \wedge \left(\bigwedge_{0 \leq t \leq N-1} \text{Exactly one } r_{t,i,s,s',\gamma,\gamma',d} \text{ is true.} \right)$$

This is not a CNF! But we can use the fact that $a \vee (b \wedge c \wedge d) = (a \vee b) \wedge (a \vee c) \wedge (a \vee d)$.

How big is ϕ_5? $O(N^3) = n^{O(1)}$.

Proof (ϕ_6 ensures that all the tape is maintained, possibly expect location h_t.)

Step	State	Head	T_0	T_1	T_2	…	T_{n-1}	T_n	…	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \dots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \dots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \dots, N\}$: $z_{t,h}$ is true if $h_t = h$.

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_6 ensures that all the tape is maintained, possibly expect location h_t.)

Step	**State**	**Head**	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_6 = \bigwedge_{\substack{0 \leq t, i \leq N-1 \\ \gamma \in \Gamma}} \left(\overline{z_{t,i}} \vee \overline{x_{t,i,\gamma}} \vee x_{t+1,i,\gamma} \right)$$

How big is ϕ_6?

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_6 ensures that all the tape is maintained, possibly expect location h_t.)

Step	State	Head	T_0	T_1	T_2	…	T_{n-1}	T_n	…	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.

$$\phi_6 = \bigwedge_{\substack{0 \leq t,i \leq N-1 \\ \gamma \in \Gamma}} \left(\overline{z_{t,i}} \vee \overline{x_{t,i,\gamma}} \vee x_{t+1,i,\gamma} \right)$$

How big is ϕ_6? $O(N^2) = n^{O(1)}$.

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_7 establishes that location h_t is updated correctly.)

Step	**State**	**Head**	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \dots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \dots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \dots, N\}$: $z_{t,h}$ is true if $h_t = h$.

Proof (ϕ_7 establishes that location h_t is updated correctly.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \dots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \dots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \dots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \dots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$ $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_7 establishes that location h_t is updated correctly.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
$\vdots$									$\ddots$	
N	q_{i_N}	h_N	$w_{N,0}$	$w_{N,1}$	$w_{N,2}$		$w_{N,n-1}$	$w_{N,n}$		$w_{N,N}$

- For all $t, i \in \{0, \ldots, N\}, \gamma \in \Gamma$: $x_{t,i,\gamma}$ is true if $w_{t,i} = \gamma$.
- For all $t \in \{0, \ldots, N\}, s \in Q$: $y_{t,s}$ is true if $q_{i_t} = s$.
- For all $t, h \in \{0, \ldots, N\}$: $z_{t,h}$ is true if $h_t = h$.
- For all $t, h \in \{0, \ldots, N-1\}$ and $s, s', \gamma, \gamma', d$ such that $(s', \gamma', d) \in \delta(s, \gamma)$ $r_{t,h,s,s',\gamma,\gamma',d}$ is true if the t^{th} transition follows this rule.

$$\phi_7 = \bigwedge_{\substack{0 \leq t,i \leq N-1 \\ \gamma,\gamma' \in \Gamma \\ s,s' \in Q \\ (s',\gamma',d) \in \delta(s,\gamma)}} \left(\overline{r_{t,i,s,s',\gamma,\gamma',d}} \vee x_{t+1,i,\gamma'}\right)$$

How big is ϕ_7?

The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

Proof (ϕ_7 establishes that location h_t is updated correctly.)

Step	State	Head	T_0	T_1	T_2	...	T_{n-1}	T_n	...	T_N
0	q_0	0	w_0	w_1	w_2		w_n	$\sqcup$		$\sqcup$
0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
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How big is ϕ_7? $O(N^2) = n^{O(1)}$.

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Proof (ϕ_8 verifies that the head is moving in the right direction.)

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1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
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0	q_{i_0}	h_0	$w_{0,0}$	$w_{0,1}$	$w_{0,2}$		$w_{0,n-1}$	$w_{0,n}$		$w_{0,N}$
1	q_{i_1}	h_1	$w_{1,0}$	$w_{1,1}$	$w_{1,2}$		$w_{1,n-1}$	$w_{1,n}$		$w_{1,N}$
2	q_{i_2}	h_2	$w_{2,0}$	$w_{2,1}$	$w_{2,2}$		$w_{2,n-1}$	$w_{2,n}$		$w_{2,N}$
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The Cook-Levin theorem part II: $L \in NP \implies L \leq_p SAT$

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All in all, we received a $n^{O(1)}$-sized formula that is satisfiable if and only if the underlying NDTM accepts the input.

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All in all, we received a $n^{O(1)}$-sized formula that is satisfiable if and only if the underlying NDTM accepts the input. Namely, a satisfying assignment corresponds to an accepting run. But did we make any mistake along the way?

Hint: What if we reach q_Y sooner than time N?

NP-completeness by p-time reduction

We know SAT $\in$ **NPC**.

Theorem

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Proof.

$A \in \mathbf{NP}$ by (1). For NP-hardness, let $B \in \mathbf{NP}$ be an arbitrary decision problem. We need to show that $B \leq_p A$.

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then A is NP-*complete.*

Proof.

$A \in \mathbf{NP}$ by (1). For NP-hardness, let $B \in \mathbf{NP}$ be an arbitrary decision problem. We need to show that $B \leq_p A$.

$$B \quad \leq \quad \mathbf{SAT} \quad \text{since } \mathbf{SAT} \in \mathbf{NPC}$$

NP-completeness by p-time reduction

We know SAT $\in$ **NPC**.

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$\square$

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Answer on Mentimeter: `https://www.menti.com/8abbpwt5py`, enter `1427 4147` on `www.menti.com`, or scan: .

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Next lecture: Many NP-hard problems.