Lecture 2: Computation and Polynomial Time Complexity

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- **Reading material:** "Computers and Intractability", sec. 1.3.: Polynomial Time Algorithms and Intractable Problems.

In order terms, rank from highest to lowest

- n^{10}
- $\sqrt{n}$
- 2^n
- $n!$
- 2^{n^2}
- $2^{\sqrt{log_2 n}}$
- n

Answer on Mentimeter: .

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Stirling's approximation:

$$n! = \Theta\left(\sqrt{n}\,(n/e)^n\right) = 2^{n\log n - O(n)} \leq 2^{n^2}$$

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- Let M be a TM. Let $T : \mathbb{N} \to \mathbb{N}$ be a function such that if input tape has n symbols then M must halt in $\leq T(n)$ steps.
- $T(n)$ is upper bound for time complexity of M.
  If M does not halt on some inputs then time complexity is undefined (or infinite).

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- The time complexity of an algorithm is the time complexity of the best implementation of it.
- Time complexity depends on the computation model (e.g., TM, multi-headed TM, RAM machine).
- It turns out that the choice of computing model does not make much difference. [1]

[1] That is, most deterministic models are polynomially equivalent.

Polynomials

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In general

$$p(n) = a_k n^k + a_{k-1} n^{k-1} + \ldots + a_2 n^2 + a_1 n + a_0$$

But not

$$\sqrt{n}, \quad \frac{1}{n}, \quad 2^n$$

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- Checking if an array has duplicates.
- Determining the longest increasing subsequence in an array.
- Finding a minimal spanning tree.

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- TMs that run in times $\log n$, $n \log n$, or $2^{\sqrt{\log n}}$, which are not polynomial, are referred to as polynomial-time!
- Some problems are outside **P**, in that they have a super-polynomial time complexity of, e.g., $2^{(\log n)^2}$.

Super

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$\rightarrow$

- Super-polynomial?

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Sub- and Super

- A super-polynomial function $f(n)$ is one that satisfies $f(n) = n^{\omega(1)}$.
- A sub-polynomial function $f(n)$ satisfies $f(n) = n^{o(1)}$, e.g., $f(n) = 2^{\sqrt{\log n}}$.
- Can be used for other time classes, e.g., super-linear or sub-quadratic.

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- Some exponential time algorithms are quite useful in practice! An example is the Simplex algorithm of linear programming. This algorithm has an exp time complexity, but in practice runs quite quickly.
- Some problems are provably intractable and the first one of these is Turing's halting problem.

Quote of the day

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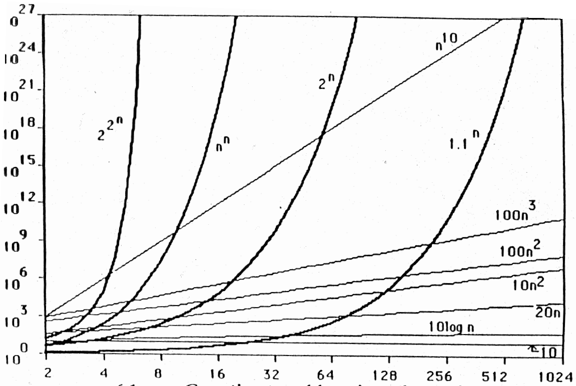
"The greatest shortcoming of the human race is our inability to understand the exponential function."

Prof. Albert Bartlett, 1969.

Illustration



6.1 Growth rates of functions (log scales)

Faster computers

3 computers (or TMs) C_1 (ordinary) , C_2 (2^{12} times as fast as C_1), and C_3 (10^{12} times as fast as C_1).

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	C_1	C_2	C_3
$O(n)$	α_0		
$O(n^2)$	α_1		
$O(n^3)$	α_2		
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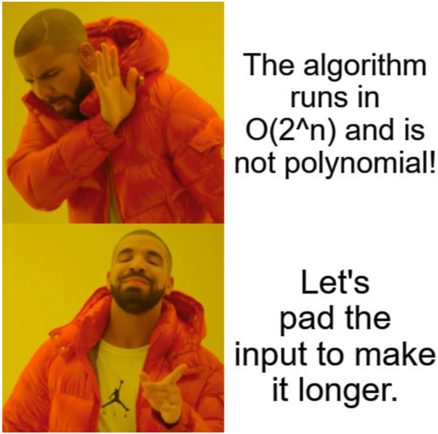
[2] The complexity is correct for a RAM machine.


The algorithm runs in O(2^n) and is not polynomial!
Let's pad the input to make it longer.

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- Most often, we will ask if the most succinct representation of the inputs allow polynomial-time recognizability.

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- Discussed what is polynomial time and its importance.
- Pondered about the affect the input representation has on the notion of polynomiality.