Orientation Representations

Using a 3×3 rotation matrix is not easy to visualise the projections of the frame axis.
There are only 6 constraints when it comes to describing orientations, being:

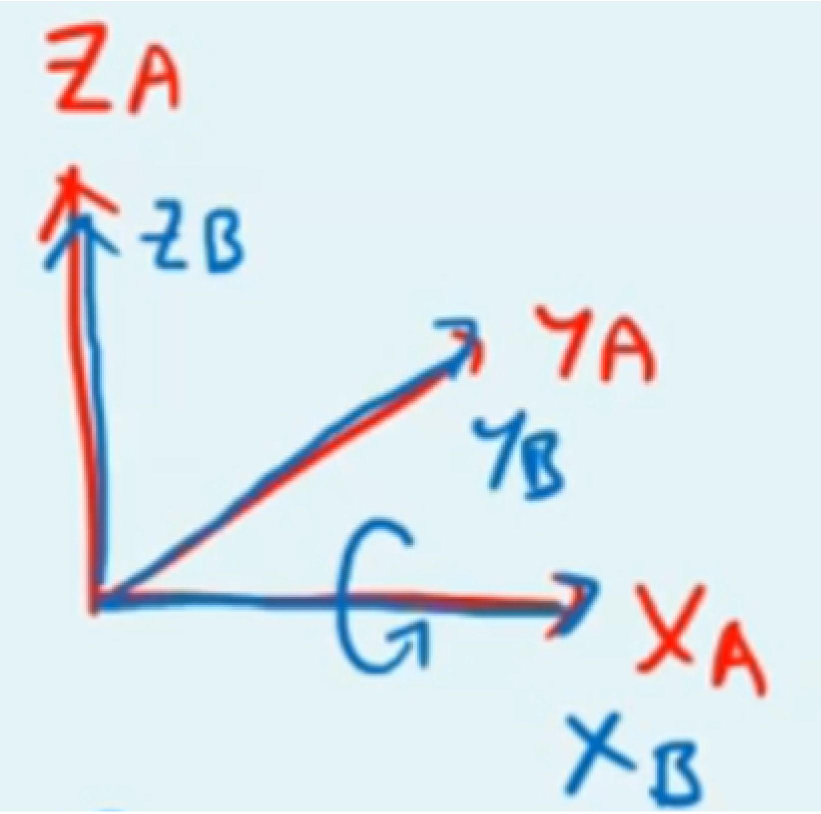
$$
\begin{aligned}
|\hat{x}| &= 1 \\
|\hat{y}| &= 1 \\
|\hat{z}| &= 1 \\
\hat{x} \cdot \hat{y} &= 0 \\
\hat{x} \cdot \hat{z} &= 0 \\
\hat{z} \cdot \hat{y} &= 0
\end{aligned}
$$

Meaning only three parameters are required to represent the rotation matrix.

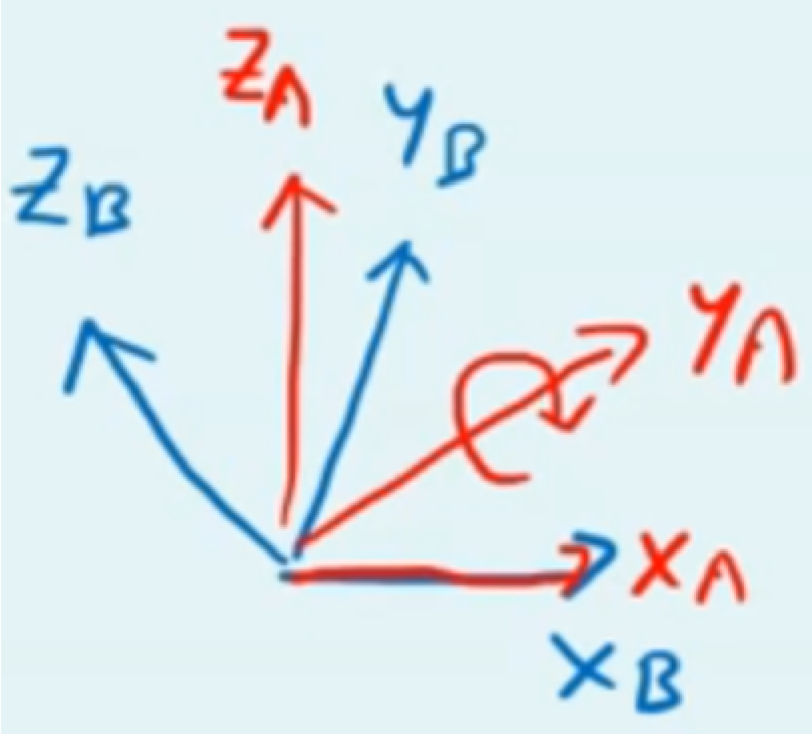
XYZ-Fixed Angles Representation

Also known as roll, pitch, and yaw.
ht hand rule.
To describe the orientation of B from A:

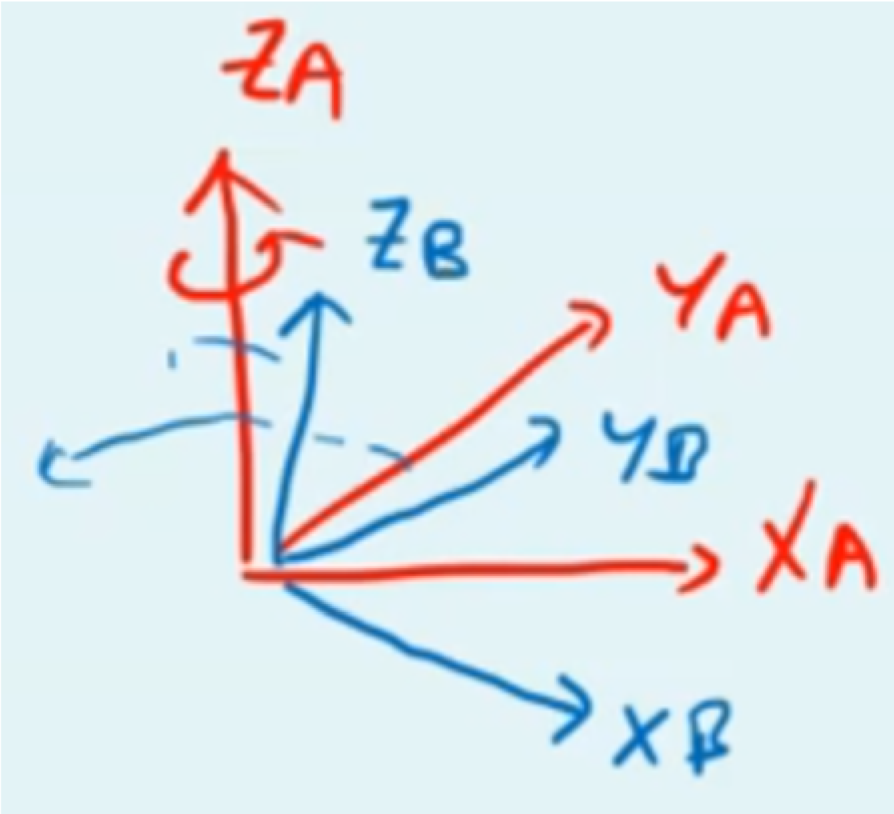
- Align the B frame with the A frame.



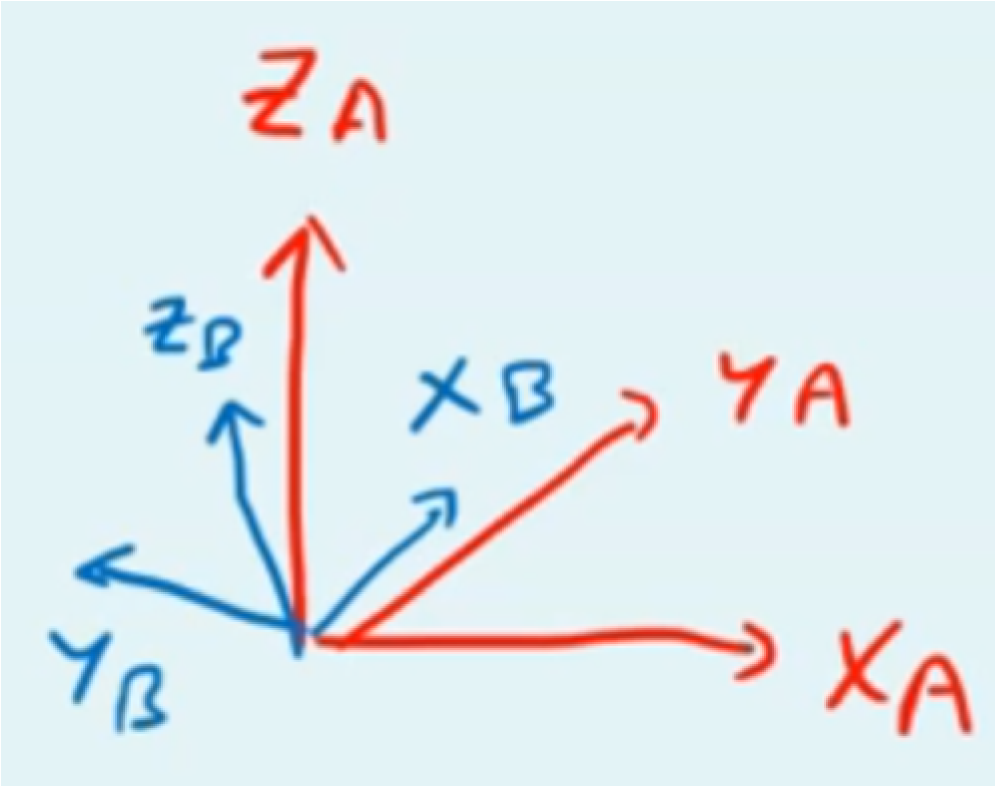
- Rotate B about X_A by angle γ.



- Next rotate B about Y_A by angle β.



- Finally Rotate B about Z_A by angle α.



The rotation matrix is obtained by multiplying each of the individual rotations from right to left, or in the reverse order of which the transformations were applied:

$$ {}^A_BR = R_Z(\alpha) \cdot R_Y(\beta) \cdot R_X(\gamma) $$

All the individual rotations are known:

$$ R_Z(\alpha) = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} $$
$$ R_Y(\beta) = \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} $$
$$ R_X(\gamma) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix} $$

Meaning:

$$ {}^A_BR = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix} $$

Which is equivalent to:

$$ \begin{bmatrix} c\alpha \cdot c\beta & c\alpha \cdot s\beta \cdot s\gamma - s\alpha \cdot c\gamma & c\alpha \cdot s\beta \cdot c\gamma + s\alpha \cdot s\gamma \\ s\alpha \cdot c\beta & s\alpha \cdot s\beta \cdot s\gamma + c\alpha \cdot c\gamma & s\alpha \cdot s\beta \cdot c\gamma - c\alpha \cdot s\gamma \\ -s\beta & c\beta \cdot s\gamma & c\beta \cdot c\gamma \end{bmatrix} $$

Given a rotation matrix, we can find α, β, γ by solving simultaneous equations.

For the rotation matrix:

$$
{}^{A}_{B}R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c\alpha c\beta & c\alpha s\beta s\gamma - s\alpha c\gamma & c\alpha s\beta c\gamma + s\alpha s\gamma \\ s\alpha c\beta & s\alpha s\beta s\gamma + c\alpha c\gamma & s\alpha s\beta c\gamma - c\alpha s\gamma \\ -s\beta & c\beta s\gamma & c\beta c\gamma \end{bmatrix}
$$

β can be found:

$$
\begin{aligned}
\cos^2\beta &= r_{11}^2 + r_{22}^2 \\
-\sin\beta &= r_{31} \\
\tan\beta &= \frac{\sin\beta}{\cos\beta} \\
&= \frac{-r_{31}}{\sqrt{r_{11}^2 + r_{21}^2}} \\
\beta &= arctan2(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2})
\end{aligned}
$$

α can then be found using β:

$$
\begin{aligned}
\tan\alpha &= \frac{\sin\alpha}{\cos\alpha} \\
&= \frac{r_{11} \div \cos\beta}{r_{21} \div \cos\beta} \\
\alpha &= arctan2(r_{11} \div \cos\beta, r_{21} \div \cos\beta)
\end{aligned}
$$

γ similarly can be found using β:

$$
\begin{aligned}
\tan\gamma &= \frac{\sin\gamma}{\cos\gamma} \\
&= \frac{r_{32} \div \cos\beta}{r_{33} \div \cos\beta} \\
\gamma &= arctan2(r_{32} \div \cos\beta, r_{33} \div \cos\beta)
\end{aligned}
$$

If $\cos\beta = 0$ then both α and γ cannot be solved for.