Homework 1

Richeek Das – 66113700

September 1, 2025

Problem B1 (1-7 Lee)

(a) A line passing through points N and $x \in \mathbb{S}^n \backslash \{N\}$ can be parameterized as:

$$\begin{aligned} L(t) &= N + t(x - N) \text{ for } t \in \mathbb{R} \\ &= (0, 0, \ldots, 0, 1) + t((x^1, x^2, \ldots, x^{n+1}) - (0, 0, \ldots, 0, 1)) \\ &= (tx^1, tx^2, \ldots, tx^n, 1 + t(x^{n+1} - 1)) \end{aligned}$$

We find the intersection of this line with the hyperplane $H = \{x \in \mathbb{R}^{n+1} : x^{n+1} = 0\}$ by setting the last coordinate to zero and solving for t:

$$\begin{aligned} 0 &= 1 + t(x^{n+1} - 1) \\ \Rightarrow t &= \frac{1}{1 - x^{n+1}} \end{aligned}$$

The intersection point is then:

$$L\left(\frac{1}{1 - x^{n+1}}\right) = \left(\frac{x^1, x^2, \ldots, x^n}{1 - x^{n+1}}, 0\right) = (u, 0)$$

Therefore, the stereographic projection map σ from $\mathbb{S}^n \backslash \{N\}$ to $\mathbb{R}^n$ is given by:

$$\sigma(x) = u = \frac{(x^1, x^2, \ldots, x^n)}{1 - x^{n+1}}$$

Similarly, we can find the stereographic projection map from the south pole: $\tilde{\sigma} : \mathbb{S}^n \backslash \{S\} \to \mathbb{R}^n$ where $S = (0, 0, \ldots, 0, -1)$:

$$\tilde{\sigma}(x) = \frac{(x^1, x^2, \ldots, x^n)}{1 + x^{n+1}}$$

(b) We show that σ is bijective. Let us propose:

$$\sigma^{-1}(u) = \frac{\left(2u^1, 2u^2, \ldots, 2u^n, |u|^2 - 1\right)}{|u|^2 + 1}$$

We can show that the image of σ^{-1} lies on $\mathbb{S}^n \setminus \{N\}$:

$$\begin{aligned}\left|\sigma^{-1}(u)\right|^2 &= \frac{\Sigma_{i=1}^n (2u^i)^2 + (|u|^2 - 1)^2}{(|u|^2 + 1)^2} \\ &= \frac{4\,|u|^2 + |u|^4 - 2\,|u|^2 + 1}{|u|^4 + 2\,|u|^2 + 1} = 1\end{aligned}$$

So, $\sigma^{-1}(u) \in \mathbb{S}^n$. Also, $\sigma^{-1}(u) \neq N$ since the last coordinate is $\frac{|u|^2-1}{|u|^2+1} \neq 1$ for any $u \in \mathbb{R}^n$. Thus, $\sigma^{-1}(u) \in \mathbb{S}^n \setminus \{N\}$.

Next, we show that σ and σ^{-1} are inverses of each other:

$$\sigma(\sigma^{-1}(u)) = \left(\frac{2u^1, 2u^2, \ldots, 2u^n}{|u|^2 + 1}\right) \Big/ \left(1 - \frac{|u|^2 - 1}{|u|^2 + 1}\right) = \frac{(2u^1, 2u^2, \ldots, 2u^n)}{(|u|^2 + 1) \cdot \frac{2}{|u|^2+1}} = u$$

Let us calculate $|\sigma(x)|^2 = \frac{1+x^{n+1}}{1-x^{n+1}}$:

$$\begin{aligned}\sigma^{-1}(\sigma(x)) &= \frac{\left(\frac{2x^1}{1-x^{n+1}}, \frac{2x^2}{1-x^{n+1}}, \ldots, \frac{2x^n}{1-x^{n+1}}, \frac{1+x^{n+1}}{1-x^{n+1}} - 1\right)}{\frac{1+x^{n+1}}{1-x^{n+1}} + 1} \\ &= \frac{\left(\frac{2x^1}{1-x^{n+1}}, \frac{2x^2}{1-x^{n+1}}, \ldots, \frac{2x^n}{1-x^{n+1}}, \frac{2x^{n+1}}{1-x^{n+1}}\right)}{\frac{2}{1-x^{n+1}}} = (x^1, x^2, \ldots, x^n, x^{n+1}) = x\end{aligned}$$

Therefore, σ is bijective with inverse σ^{-1}.

(c) The transition map from the north pole chart to the south pole chart is given by $\tilde{\sigma} \circ \sigma^{-1}$. We can write this composition as:

$$\tilde{\sigma} \circ \sigma^{-1}(u) = \tilde{\sigma}\left(\frac{\left(2u^1, 2u^2, \ldots, 2u^n, |u|^2 - 1\right)}{|u|^2 + 1}\right) = \frac{\left(\frac{2u^1}{|u|^2+1}, \frac{2u^2}{|u|^2+1}, \ldots, \frac{2u^n}{|u|^2+1}\right)}{1 + \frac{|u|^2-1}{|u|^2+1}} = \frac{u}{|u|^2}$$

To verify that the atlas defines a smooth structure we check that the transition map is smooth C^∞. The map $\tilde{\sigma} \circ \sigma^{-1}$ is a rational function with denominator $|u|^2 = \Sigma_{i=1}^n (u^i)^2$ which is non-zero for all $u \in \mathbb{R}^n \setminus \{0\}$. Therefore, the transition map is smooth and the atlas defines a smooth structure on $\mathbb{S}^n$.

(d) We want to show that this smooth structure is the same as the standard smooth structure. This requires us to show that for each of the $2n + 2$ coordinate charts in the example, and each of the two stereographic projection charts, the transition maps are smooth.

Consider the standard chart domain: $U_i^+ = \{x \in \mathbb{S}^n : x^i > 0\}$ and the standard chart map: $\phi_i^+(x^1, x^2, \ldots, x^{n+1}) = (x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{n+1})$. Inverse of this map is given by $(\phi_i^+)^{-1}(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{n+1}) = (x^1, \ldots, x^{i-1}, \sqrt{1 - \Sigma_{j \neq i}(x^j)^2}, x^{i+1}, \ldots, x^{n+1})$.

Now consider the transformation from the stereographic projection chart at the north pole to the standard chart at U_i^+:

$$\begin{aligned}
\phi_i^+ \circ \sigma^{-1}(u) &= \phi_i^+ \left(\frac{\left(2u^1, 2u^2, \ldots, 2u^n, |u|^2 - 1\right)}{|u|^2 + 1} \right) \\
&= \left(\frac{2u^1}{|u|^2 + 1}, \ldots, \frac{2u^{i-1}}{|u|^2 + 1}, \frac{2u^{i+1}}{|u|^2 + 1}, \ldots, \frac{2u^n}{|u|^2 + 1}, \frac{|u|^2 - 1}{|u|^2 + 1} \right)
\end{aligned}$$

This is a rational function with denominator $|u|^2 + 1$ which is non-zero for all $u \in \mathbb{R}^n$. Therefore, the transition map is smooth. Similarly, consider the transformation from the standard chart at U_i^+ to the stereographic projection chart at the north pole:

$$\begin{aligned}
\sigma \circ (\phi_i^+)^{-1}(x^1, \ldots, x^{i-1}, x^{i+1}, \ldots, x^{n+1}) &= \sigma \left(x^1, \ldots, x^{i-1}, \sqrt{1 - \Sigma_{j \neq i}(x^j)^2}, x^{i+1}, \ldots, x^{n+1} \right) \\
&= \frac{\left(x^1, \ldots, x^{i-1}, \sqrt{1 - \Sigma_{j \neq i}(x^j)^2}, x^{i+1}, \ldots, x^n \right)}{1 - x^{n+1}}
\end{aligned}$$

This is a rational function with denominator $1 - x^{n+1}$ which is non-zero for all $x \in U_i^+$ since $x^{n+1} \leq 1$ on $\mathbb{S}^n$ and $x^{n+1} = 1$ only at the north pole which is not in U_i^+. Therefore, the transition map is smooth.

Problem B2 (1-9 Lee)

Let $\mathbb{CP}^n$ be the set of all 1-dimensional complex-linear subspaces of $\mathbb{C}^{n+1}$. This is an equivalence relation on $\mathbb{C}^{n+1} \setminus \{0\}$, where $z \sim w$ if $z = \lambda w$ for some $\lambda \in \mathbb{C} \setminus \{0\}$.

Manifold To show that $\mathbb{CP}^n$ is a topological manifold, we construct an atlas. For each $j \in \{0, \dots, n\}$, let $U_j = \{[z^0 : \cdots : z^n] \in \mathbb{CP}^n : z^j \neq 0\}$. Each U_j is open in the quotient topology because its preimage under the projection map $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{CP}^n$ is $\pi^{-1}(U_j) = \{z \in \mathbb{C}^{n+1} \setminus \{0\} : z^j \neq 0\}$, which is open in $\mathbb{C}^{n+1} \setminus \{0\}$. These sets form an open cover of $\mathbb{CP}^n$ (quotient topology). We define a chart map $\phi_j : U_j \to \mathbb{C}^n$ by

$$\phi_j([z^0 : \cdots : z^n]) = \left(\frac{z^0}{z^j}, \dots, \frac{z^{j-1}}{z^j}, \frac{z^{j+1}}{z^j}, \dots, \frac{z^n}{z^j}\right)$$

The map ϕ_j is a homeomorphism from U_j to $\mathbb{C}^n$. Since $\mathbb{C}^n$ is homeomorphic to $\mathbb{R}^{2n}$, this shows that $\mathbb{CP}^n$ is locally Euclidean of dimension $2n$.

The space $\mathbb{C}^{n+1} \setminus \{0\}$ is second-countable, and the quotient space of a second-countable space is second-countable. To show $\mathbb{CP}^n$ is Hausdorff, consider two distinct points $[z], [w] \in \mathbb{CP}^n$. This means z and w are not scalar multiples of each other. We can choose representatives such that $|z| = |w| = 1$. The angle between them is non-zero. We can find disjoint open neighborhoods of z and w in $\mathbb{C}^{n+1} \setminus \{0\}$ that are invariant under multiplication by complex numbers of modulus 1. Their projections to $\mathbb{CP}^n$ will be disjoint open neighborhoods of $[z]$ and $[w]$. Thus, $\mathbb{CP}^n$ is a $2n$-dimensional topological manifold.

Compactness Let $S^{2n+1} = \{z \in \mathbb{C}^{n+1} : |z|^2 = \sum_{i=0}^n |z^i|^2 = 1\}$. This is the unit sphere in $\mathbb{C}^{n+1} \cong \mathbb{R}^{2n+2}$, which is compact. The restriction of the projection map $\pi : S^{2n+1} \to \mathbb{CP}^n$ is surjective. Since the continuous image of a compact set is compact, $\mathbb{CP}^n$ is compact.

Smooth Structure We show that the atlas $\{(U_j, \phi_j)\}_{j=0}^n$ defines a smooth structure. Let $u = (u^0, \dots, u^{j-1}, u^{j+1}, \dots, u^n)$ be coordinates on $\mathbb{C}^n$ for the chart ϕ_j, and $v = (v^0, \dots, v^{k-1}, v^{k+1}, \dots, v^n)$ for the chart ϕ_k. The transition map $\phi_k \circ \phi_j^{-1}$ is defined on $\phi_j(U_j \cap U_k)$. For $u \in \phi_j(U_j \cap U_k)$, we have:

$$\phi_j^{-1}(u) = [u^0 : \cdots : u^{j-1} : 1 : u^{j+1} : \cdots : u^n].$$

In this equivalence class, the k-th component is u^k (if $k \neq j$) or 1 (if $k = j$). Since we are in U_k, this component is non-zero. Then $\phi_k \circ \phi_j^{-1}(u)$ is obtained by dividing all components by the k-th component. For $k < j$:

$$v^i = \begin{cases} u^i/u^k & i < k \\ u^{i+1}/u^k & k \leq i < j-1 \\ 1/u^k & i = j-1 \\ u^i/u^k & j-1 < i \end{cases}$$

Each component of the transition map is a rational function of the complex variables u^i. Since we identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$ by $(x^1 + iy^1, \dots) \mapsto (x^1, y^1, \dots)$, these maps are smooth where the denominator is non-zero, which is true on $U_j \cap U_k$. The atlas defines a smooth structure on $\mathbb{CP}^n$.

Problem D

We want to show that the volume of the unit 3-sphere $\mathbb{S}^3$ in $\mathbb{R}^4$ is $2\pi^2$. The equation for $\mathbb{S}^3$ is $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$.

Let's group the coordinates into two pairs: (x_1, x_2) and (x_3, x_4). Let $r_1 = \sqrt{x_1^2 + x_2^2}$ be the radius in the x_1x_2-plane and $r_2 = \sqrt{x_3^2 + x_4^2}$ be the radius in the x_3x_4-plane. The equation for the 3-sphere becomes $r_1^2 + r_2^2 = 1$.

Since r_1 and r_2 are non-negative, this equation describes a quarter circle in the (r_1, r_2)-plane. We can parameterize this quarter circle with an angle $\psi \in [0, \pi/2]$:

$$r_1 = \cos\psi, \quad r_2 = \sin\psi$$

For any fixed value of ψ between 0 and $\pi/2$, we have two separate circles:

- $x_1^2 + x_2^2 = \cos^2\psi$: A circle in the x_1x_2-plane with radius $\cos\psi$. Its circumference is $2\pi\cos\psi$.
- $x_3^2 + x_4^2 = \sin^2\psi$: A circle in the x_3x_4-plane with radius $\sin\psi$. Its circumference is $2\pi\sin\psi$.

The set of points in $\mathbb{R}^4$ for a fixed ψ is the product of these two circles, which forms a torus. The surface area of this torus is the product of the two circumferences: $(2\pi\cos\psi)(2\pi\sin\psi) = 4\pi^2\cos\psi\sin\psi$. The 3-sphere can be seen as a union of these tori parameterized by ψ. To find the total volume of the 3-sphere, we can integrate the surface area of these tori.

We integrate along the arc of the quarter circle in the (r_1, r_2)-plane. The arc length element ds for the curve $(r_1(\psi), r_2(\psi)) = (\cos\psi, \sin\psi)$ is $ds = d\psi$. The volume of the 3-sphere is the integral of the torus surface area along this arc:

$$\begin{aligned}
\mathrm{Vol}(\mathbb{S}^3) &= 4\pi^2 \int_0^{\pi/2} \frac{1}{2}\sin(2\psi)\, d\psi \\
&= 2\pi^2 \left[-\frac{1}{2}\cos(2\psi)\right]_0^{\pi/2} \\
&= 2\pi^2 \left(-\frac{1}{2}\cos(\pi) - \left(-\frac{1}{2}\cos(0)\right)\right) \\
&= 2\pi^2 \left(\frac{1}{2} + \frac{1}{2}\right) = 2\pi^2
\end{aligned}$$

Thus, the volume of the unit 3-sphere is $2\pi^2$.

Problem E

We want to show that the Grassmann manifold $G_k(\mathbb{R}^n)$ of oriented k-planes in $\mathbb{R}^n$ is a smooth manifold. We will do this by constructing an atlas of coordinate charts and showing that the transition maps are smooth.

An oriented k-plane can be represented by an ordered, orthonormal basis $\{v_1, \ldots, v_k\}$. We can represent this basis as an $n \times k$ matrix $V = [v_1 | \cdots | v_k]$ such that $V^T V = I_k$.

Let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be an ordered multi-index with $1 \leq \alpha_1 < \cdots < \alpha_k \leq n$. There are $\binom{n}{k}$ such multi-indices. For each α, let V_α be the $k \times k$ submatrix of V formed by the rows indexed by α.

We define a collection of open sets $\{U_\alpha\}$ that cover $G_k(\mathbb{R}^n)$. Let U_α be the set of all oriented k-planes whose representing matrix V has an invertible submatrix V_α.

$$U_\alpha = \{[V] \in G_k(\mathbb{R}^n) : \det(V_\alpha) \neq 0\}$$

These sets cover the manifold because for any k-plane, we can always find a basis such that the corresponding matrix V has at least one invertible $k \times k$ submatrix.

Now, we define the coordinate charts $\phi_\alpha : U_\alpha \to M_{k \times (n-k)}(\mathbb{R}) \cong \mathbb{R}^{k(n-k)}$. For any plane in U_α, we can choose a unique basis (and thus a unique representative matrix V) such that $V_\alpha = I_k$. This is our canonical representative for the plane in this chart. Let α^c be the multi-index of the $n-k$ rows not in α. Let V_{α^c} be the $(n-k) \times k$ submatrix of V with rows from α^c. The chart map is defined as:

$$\phi_\alpha([V]) = (V_{\alpha^c})^T$$

This map takes a plane in U_α and maps it to a $k \times (n-k)$ matrix, which we identify with a point in $\mathbb{R}^{k(n-k)}$. The map is well-defined and bijective.

To show this is a smooth manifold, we must check that the transition maps are smooth. Let's consider two overlapping charts, (U_α, ϕ_α) and (U_β, ϕ_β). Let $X = \phi_\alpha([V])$ and $Y = \phi_\beta([V])$ for some plane $[V] \in U_\alpha \cap U_\beta$.

From the definition of ϕ_α, the canonical matrix V for the chart U_α has $V_\alpha = I_k$ and $(V_{\alpha^c})^T = X$. We can reconstruct V from X. This matrix V represents our plane, but it is not the canonical representative for the chart U_β, because $V_\beta \neq I_k$. To find the coordinates in the β-chart, we must find a new matrix V' representing the same plane, such that $V'_\beta = I_k$. Since V and V' represent the same plane, they are related by $V' = VG$ for some $G \in GL(k, \mathbb{R})$. From $V'_\beta = I_k$, we have $(VG)_\beta = I_k$. The rows of VG are the rows of V multiplied by G. So, $V_\beta G = I_k$, which implies $G = (V_\beta)^{-1}$. The new coordinates are $Y = \phi_\beta([V]) = \phi_\beta([V']) = (V'_{\beta^c})^T$.

$$Y = ((VG)_{\beta^c})^T = (V_{\beta^c} G)^T = G^T (V_{\beta^c})^T = ((V_\beta)^{-1})^T (V_{\beta^c})^T$$

The matrices V_β and V_{β^c} are submatrices of V, which is constructed from X. The entries of V are either constants (0 or 1, from $V_\alpha = I_k$) or are the entries of X (from V_{α^c}). Therefore, the entries of V_β and V_{β^c} are smooth (in fact, linear) functions of the entries of X.

The transition map $\phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \to \phi_\beta(U_\alpha \cap U_\beta)$ is given by $X \mapsto Y = ((V_\beta(X))^{-1})^T (V_{\beta^c}(X))^T$.

The entries of $V_\beta(X)$ and $V_{\beta^c}(X)$ are smooth functions of the coordinates of X. The determinant of $V_\beta(X)$ is a polynomial in the entries of X, and it is non-zero on the domain of the transition map, $U_\alpha \cap U_\beta$. The entries of the inverse matrix $(V_\beta)^{-1}$ are rational functions of the entries of V_β, with the non-zero determinant in the denominator, making the inversion map smooth. Since matrix multiplication is also a smooth operation, the transition map $X \mapsto Y$ is a composition of smooth functions and is therefore smooth. This confirms that we have a covering family of charts with smooth transition maps, proving that $G_k(\mathbb{R}^n)$ is a smooth manifold of dimension $k(n-k)$.