Prove

$$\binom{n}{r} = \binom{n}{n-r},$$

$${}^nC_r = \frac{n!}{r!\,(n-r)!} \qquad {}^nC_{n-r} = \frac{n!}{(n-r)!\,(n-n+r)!} = \frac{n!}{(n-r)!\,r!}$$

Therefore, $\boxed{{}^nC_r = {}^nC_{n-r}}$

Hence Proved □

Prove

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}.$$

RHS → ${}^{n-1}C_{r-1} + {}^{n-1}C_r$

$$= \frac{(n-1)!}{(r-1)!\,(n-1-r+1)!} + \frac{(n-1)!}{r!\,(n-1-r)!}$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r)!} + \frac{(n-1)!}{r!\,(n-r-1)!}$$

$$= \frac{(n-1)!}{(r-1)!\,(n-r)(n-r-1)!} + \frac{(n-1)!}{r(r-1)!(n-r-1)!}$$

$$= \frac{(n-1)!\,(r+n-r)}{r\,(r-1)!\,(n-r)\,(n-r-1)!}$$

$$= \frac{n\,(n-1)!}{r!\,(n-r)!}$$

$$= \frac{n!}{r!\,(n-r)!} = {}^nC_r$$

Hence Proved □

Suppose, then, that we have n objects such that there are n_1 of one kind, n_2 of a second kind, . . . , n_k of a kth kind, where $n_1 + n_2 + \cdots + n_k = n$. Then the number of permutations of these n objects is given by

$$\frac{n!}{n_1!n_2! \cdots n_k!}.$$

Suppose that there are n type of balls.

$n_1 \rightarrow$ red balls
$n_2 \rightarrow$ white balls
$n_3 \rightarrow$ black balls

And, $n_1 + n_2 + n_3 = n$

Now, if we arrange n balls, the number of possible arrangements will be

$$\boxed{n!}$$

But, since we have n_1, red colour balls which are indistinguishable therefore it won't matter how we arrange those n_1 balls.
Similar is the case with black and white balls as well

Therefore, we need to reduce those arrangements

$$\boxed{\frac{n!}{n_1!\, n_2!\, n_3!}}$$

If we generalize it then we get

$$\boxed{\frac{n!}{n_1!\, n_2! \cdots n_k!}}$$ Hence Proved $\square$

2.1. The following group of persons is in a room: 5 men over 21, 4 men under 21, 6 women over 21, and 3 women under 21. One person is chosen at random. The following events are defined: A = {the person is over 21}; B = {the person is under 21}; C = {the person is male}; D = {the person is female}. Evaluate the following.
(a) $P(B \cup D)$
(b) $P(\overline{A} \cap \overline{C})$

A = The person is over 21 B = the person is under 21

C = the person is male $\qquad$ D = the person is female

$|A \cap C| = 5 \qquad |B \cap C| = 4 \qquad |A \cap D| = 6 \qquad |B \cap D| = 3$

Set A and B are mutually exclusive and
set C and D are also " " .

Therefore, $|A \cup B| = |A| + |B|$

$|C \cup D| = |C| + |D|$

$|A| = 5+6 = 11 \qquad |B| = 4+3 = 7 \qquad |C| = 5+4 = 9 \qquad |D| = 6+3 = 9$

$$|B \cup D| = |B| + |D| - |B \cap D| = 7 + 9 - 3 = 13$$

$$P(B \cup D) = \frac{13}{18}$$ Ans

$$P(\bar{A} \cap \bar{C}) = P(\overline{A \cup C}) \quad \text{(De Morgan's Law)}$$

$$= 1 - P(A \cup C)$$
$$= 1 - (P(A) + P(C) - P(A \cap C))$$
$$= 1 - \frac{(11 + 9 - 5)}{18}$$
$$= 1 - \frac{15}{18} = \frac{3}{18} = \frac{1}{6}$$ Ans

2.2. Ten persons in a room are wearing badges marked 1 through 10. Three persons are chosen at random, and asked to leave the room simultaneously. Their badge number is noted.

(a) What is the probability that the smallest badge number is 5?
(b) What is the probability that the largest badge number is 5?

a) Out of 3, there must be the person with batch 5
And, remaining two are greater than 5 i.e $\{6, 7, 8, 9, 10\}$

$$\frac{^5C_2}{^{10}C_3} = \frac{5!}{\cancel{3!}\,2!} \times \frac{\cancel{3!}\,7!}{10!}$$

$$= \cancel{7} \times \cancel{6} \times \cancel{5}^{1} \times \cancel{4}^{1} \times \cancel{2} \times \cancel{2} \times 1 \quad 1$$

$$= \frac{\cancel{7}\times\cancel{6}\times\cancel{5}\times\cancel{4}\times\cancel{3}\times2\times1}{\cancel{10}\times\cancel{9}\times\cancel{8}\times\cancel{7}\times\cancel{6}} \times \frac{1}{2!}$$

$$= \frac{1}{12} \quad \text{Ans}$$

(b) Largest is 5, means that other two will be less than 5.

i.e, $\{1,2,3,4\}$

$$\frac{{}^4C_2}{{}^{10}C_3} = \frac{\cancel{4!}}{\cancel{2!\,2!}} \times \frac{7!\,3!}{10!}$$

$$= \frac{3\times\cancel{2}\times3!}{10\times\cancel{9}\times\cancel{8}} = \frac{1\times3!}{120} = \frac{6}{120}$$

$$= \frac{1}{20} \quad \text{Ans}$$

2.3. (a) Suppose that the three digits 1, 2, and 3 are written down in random order. What is the probability that at least one digit will occupy its proper place?

(b) Same as (a) with the digits 1, 2, 3, and 4.

(c) Same as (a) with the digits 1, 2, 3, . . . , *n*. [*Hint:* Use (1.7).]

(d) Discuss the answer to (c) if *n* is large.

a) $\underset{P(a)}{P(\text{Atleast one digit occupy proper place})} = 1 - P(\text{None occupy proper place})$
$= 1 - P(b)$

Total arrangements $|x| = 3!$
$= 6$

$|b| = 2\times1\times1$
$= 2$

$P(b) = \frac{2}{6} = \frac{1}{3}$

$P(a) = 1 - \frac{1}{3} = \frac{2}{3}$ Ans

3 3

(b) Let $P(x)$ = Probability of x being at correct position

All probabilty values will be divided by $4!$, which I am skipping for now.

$$P(1) = \boxed{1}\,\square\,\square\,\square = 6$$
$$3 \times 2 \times 1$$

$$P(2) = \square\,\boxed{2}\,\square\,\square = 6$$
$$3 \times \quad 2 \times 1$$

Similarly, $P(3) = 6$ & $P(4) = 6$

$$P(1 \cap 2) = \boxed{1}\,\boxed{2}\,\square\,\square = 2$$
$$2 \times 1$$

$$P(1 \cap 3) = \boxed{1}\,\square\,\boxed{3}\,\square = 2$$
$$2 \times 1$$

Similarly, $P(1 \cap 4) = P(2 \cap 3) = P(2 \cap 4) = P(3 \cap 4) = 2$

Now, $P(1 \cap 2 \cap 3) = \boxed{1}\,\boxed{2}\,\boxed{3}\,\square = 1$
$$1$$

Similarly, $P(1 \cap 3 \cap 4) = P(2 \cap 3 \cap 4) = P(1 \cap 2 \cap 4) = 1$

$$P(1 \cap 2 \cap 3 \cap 4) = 1$$

$$\begin{aligned} P(1 \cup 2 \cup 3 \cup 4) &= P(1) + P(2) + P(3) + P(4) \\ &\quad - P(1 \cap 2) - P(1 \cap 3) - P(1 \cap 4) - P(2 \cap 3) - P(2 \cap 4) - P(3 \cap 4) \\ &\quad + P(1 \cap 2 \cap 3) + P(1 \cap 3 \cap 4) + P(2 \cap 3 \cap 4) + P(1 \cap 2 \cap 4) \\ &\quad - P(1 \cap 2 \cap 3 \cap 4) \end{aligned}$$

$$\begin{aligned} &= 6 + 6 + 6 + 6 - 2 \times 6 + 4 - 1 \\ &= 24 - 12 + 2 \\ &= 15 \end{aligned}$$

$$P(1 \cup 2 \cup 3 \cup 4) = \frac{15}{24} = \frac{5}{8}$$ Ans

$$\frac{\ }{24} = \frac{\ }{8}$$ Ans

(c)

$$P(A_1 \cup A_2 \cup \dots A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j=2}^{n} P(A_i \cap A_j) + \sum_{i<j<r=3}^{n} P(A_i \cap A_j \cap A_r)$$
$$+ (-1)^{n-1} P(A_1 \cap A_2 \cap \dots A_n)$$

2.4. A shipment of 1500 washers contains 400 defective and 1100 nondefective items. Two-hundred washers are chosen at random (without replacement) and classified.

(a) What is the probability that exactly 90 defective items are found?

(b) What is the probability that at least 2 defective items are found?

Total = 1500

Defective = 400 Nondefective = 1100

(a) $$\boxed{\frac{{}^{400}C_{90} \times {}^{1100}C_{110}}{{}^{1500}C_{200}}}$$ Ans

(b) P(Atleast 2 defective) = 1 − P(less than 2 defective)

$$= 1 - \frac{{}^{1100}C_{200} + {}^{1100}C_{199}\,{}^{400}C_1}{{}^{1500}C_{200}}$$

Ans

2.5. Ten chips numbered 1 through 10 are mixed in a bowl. Two chips numbered (X, Y) are drawn from the bowl, successively and without replacement. What is the probability that $X + Y = 10$?

No of ways choosing two chips = ${}^{10}P_2$ (Order matters here)

Possibilities of $X+Y=10$ => 1+9 9+1

2+8 8+2

3+7 7+3

4+6 6+4

$$P(x+y=10) = \frac{8}{^{10}P_2} = 8 \times \frac{8!}{10!} = \frac{8}{10\times 9} = \frac{4}{45}$$ Ans

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2.6. A lot consists of 10 good articles, 4 with minor defects, and 2 with major defects. One article is chosen at random. Find the probability that:

(a) it has no defects,
(b) it has no major defects,
(c) it is either good or has major defects.

Good (G) = 10
Minor Deffective (d) = 4
Major Deffective (D) = 2

Total = 16

(a) $P(\text{no defect}) = \frac{10}{16} = \frac{5}{8}$

(b) $P(\text{no major defects}) = \frac{14}{16} = \frac{7}{8}$

(c) $P(\text{good} \cup \text{major}) = \frac{10}{16} + \frac{2}{16} = \frac{3}{4}$

2.7. If from the lot of articles described in Problem 2.6 two articles are chosen (without replacement), find the probability that:

(a) both are good, (b) both have major defects,
(c) at least one is good, (d) at most one is good,
(e) exactly one is good, (f) neither has major defects, (g) neither is good.

Good (G) = 10
Minor Deffective (d) = 4
Major Deffective (D) = 2

(a) Both Good $= \frac{^{10}C_2}{^{16}C_2} = \frac{10!}{8!\,2!} \times \frac{2!\,14!}{16!} = \frac{\cancel{10}\times\cancel{9}}{\cancel{15}\times\cancel{16}} = \frac{3}{8}$

(b) Both have major defects $= \frac{^{2}C_2}{^{16}C_2} = 1 \times \frac{2}{15\times 16} = \frac{1}{120}$

(c) P(Atleast one is good) = 1 − P(None is good)

$$= 1 - \frac{{}^6C_2}{{}^{16}C_2} = 1 - \frac{\cancel{6}^{\cancel{2}\,1} \times \cancel{5}^{1}}{\cancel{16}_{8} \times \cancel{15}_{\cancel{3},1}}$$

$$= \frac{7}{8}$$

(d) P(Atmost one is good) = 1 − P(Both are good)

$$= 1 - \frac{3}{8} = \frac{5}{8}$$

(e) Exactly one is good $= \frac{{}^{10}C_1 \times {}^6C_1}{{}^{16}C_2} = \frac{10!}{9!} \times \frac{6!}{5!} \times \frac{2!}{15 \times 16}$

$$= \frac{\cancel{10}^{2} \times \cancel{6}^{\cancel{2}\,1} \times \cancel{2}}{\cancel{15}_{\cancel{3}\,1} \times \cancel{16}_{\cancel{4}\,2}} = \frac{1}{2}$$

(f) neither has major defects $= \frac{{}^{14}C_2}{{}^{16}C_2} = \frac{\cancel{14}^{7} \times 13}{\cancel{16}_{8} \times 15}$

$$= \frac{91}{120}$$

(g) Neither is good $= \frac{{}^6C_2}{{}^{16}C_2} = \frac{\cancel{6}^{\cancel{2}\,1} \times \cancel{5}^{1}}{\cancel{16}_{8} \times \cancel{15}_{\cancel{3},1}} = \frac{1}{8}$

2.8. A product is assembled in three stages. At the first stage there are 5 assembly lines, at the second stage there are 4 assembly lines, and at the third stage there are 6 assembly lines. In how many different ways may the product be routed through the assembly process?

$$5 \times 4 \times 6 = 120$$

2.9. An inspector visits 6 different machines during the day. In order to prevent operators from knowing when he will inspect he varies the order of his visits. In how many ways may this be done?

$6! = 720$

2.10. A complex mechanism may fail at 15 stages. If it fails at 3 stages, in how many ways may this happen?

$$^{15}C_3 = \frac{\cancel{15}^{5} \times \cancel{14}^{7} \times 13}{3 \times \cancel{2}}$$
$$= 35 \times 13$$
$$= 455$$

2.11. There are 12 ways in which a manufactured item can be a minor defective and 10 ways in which it can be a major defective. In how many ways can 1 minor and 1 major defective occur? 2 minor and 2 major defectives?

1 minor & 1 major $= 10 \times 12 = 120$

2 minor & 2 major $= {}^{10}C_2 \times {}^{12}C_2 = \frac{\cancel{10}^{5} \times 9}{\cancel{2}} \times \frac{\cancel{12}^{6} \times 11}{\cancel{2}} = 45 \times 66$

$= 2970$

2.12. A mechanism may be set at any one of four positions, say a, b, c, and d. There are 8 such mechanisms which are inserted into a system.

(a) In how many ways may this system be set?

(b) Assume that these mechanisms are installed in some preassigned (linear) order. How many ways of setting the system are available if no two adjacent mechanisms are in the same position?

(c) How many ways are available if only positions a and b are used, and these are used equally often?

(d) How many ways are available if only two different positions are used and one of these positions appears three times as often as the other?

(a) 4^8

(b) □ □ □ □ □ □ □ □

4 3 3 3 3 3 3 3

4×3^7

(c) There are 8 mechanisms and 2 positions a and b.
We need to spread 4 a and 4 b in 8 positions.

Arrange 4 a in 8 mechanisms $= {}^8P_4 = \frac{8!}{4!} = 8 \times 7 \times 6 \times 5$

But all are a, so order doesnot matter $= \frac{8 \times 7 \times 6 \times 5}{4!}$

$= \frac{\cancel{8}^{2} \times 7 \times \cancel{6} \times 5}{\cancel{4}_{1} \times \cancel{3 \times 2} \times 1} = 70$

Ans $\rightarrow$ 70

(d) Positions possible = 2 (x and y) Let x & y be two positions

count of x = 3 X count of y

count of x + count of y = 8

$\Rightarrow$ $3y + y = 8$

$4y = 8$

$y = 2$ $x = 6$

So, 6 x and 2 y are possible according to the given condition

No of ways to arrange two positions according To given condition

${}^8C_6 = 28$

No of ways to choose two positions from given 4 positions

${}^4C_2 = 6$

No of ways to arrange the two chosen positions

$$2! = 2$$

$$\text{Ans} = 28 \times 6 \times 2 = \underline{\underline{336}}$$

2.13. Suppose that from N objects we choose n at random, *with* replacement. What is the probability that no object is chosen more than once? (Suppose that $n < N$.)

Total number of ways to choose n out of N objects
Every time we pick an object, there are N possibilities

$$\text{Total} = N^n$$

$$P(\text{No object is chosen more than once}) = \frac{N(N-1)(N-2)\cdots(N-n-1)}{N^n}$$

$$= \frac{N!}{(N-n)!\,N^n}$$

$$= \frac{N(N-1)!}{(N-n)!\,N^n} = \underline{\underline{\frac{(N-1)!}{(N-n)!\,N^{n-1}}}}$$

Ans

2.14. From the letters a, b, c, d, e, and f how many 4-letter code words may be formed if,

(a) no letter may be repeated?
(b) any letter may be repeated any number of times?

(a) $^6P_4 = 360$ (b) $6^4 = 1296$

2.15. Suppose that $\binom{99}{5} = a$ and $\binom{99}{4} = b$. Express $\binom{100}{95}$ in terms of a and b. [*Hint:* Do *not* evaluate the above expressions to solve this problem.]

$a = {}^{99}C_5 \qquad b = {}^{99}C_4$

${}^{100}C_{95} = {}^{100}C_5$ since ${}^{n}C_r = {}^{n}C_{n-r}$

C_{95} C_5 since C_{n-r}

and as we know that ${}^nC_r = {}^{n-1}C_{r-1} + {}^{n-1}C_r$

Therefore, ${}^{100}C_5 = {}^{99}C_5 + {}^{99}C_4$

$\Rightarrow {}^{100}C_{95} = {}^{100}C_5 = {}^{99}C_5 + {}^{99}C_4 = a + b$ Ans

---

2.16. A box contains tags marked 1, 2, . . . , n. Two tags are chosen at random. Find the probability that the numbers on the tags will be consecutive integers if

(a) the tags are chosen without replacement,
(b) the tags are chosen with replacement.

(a) No of pairs possible $= {}^nC_2$

No of consecutive pairs possible $= n-1$

2.17. How many subsets can be formed, containing at least one member, from a set of 100 elements?

C_2 $n!$

$$= \frac{\cancel{(n-1)} \times 2}{n\,\cancel{(n-1)}} = \frac{2}{n} \quad \text{Ans}$$

(b) No of consecutive pairs possible $= n^2$

No of pairs possible $= (n-1)\,2!$

$$\text{Probability} = \frac{(n-1)\,2!}{n^2} = \frac{2(n-1)}{n^2} \quad \text{Ans}$$

2.17. How many subsets can be formed, containing at least one member, from a set of 100 elements?

$$\boxed{2^{100} - 1}$$ Ans

2.18. One integer is chosen at random from the numbers 1, 2, . . . , 50. What is the probability that the chosen number is divisible by 6 or by 8 ?

$n(\text{Divisible by } 6) = 8$

$n(\text{Divisible by } 8) = 6$

$n(\text{Divisible by 6 and } 8) = 2$

$n(\text{Divisible by 6 or } 8) = 8 + 6 - 2$
$= 12$

$P(\text{Divisible by 6 or } 8) = \frac{12}{50} = \frac{6}{25}$ Ans

---

2.19. From 6 positive and 8 negative numbers, 4 numbers are chosen at random (without replacement) and multiplied. What is the probability that the product is a positive number ?

Positive $= 6$ Negative $= 4$

Total $= 10$

Product is positive when both numbers are either positive or negative

$$P = \frac{{}^{6}C_2 + {}^{4}C_2}{{}^{10}C_2} = \frac{\frac{6!}{4!2!} + \frac{4!}{2!2!}}{\frac{10!}{8!2!}}$$

$$= \frac{\frac{\cancel{6}^{3} \times 5}{\cancel{2}_{1}} + \frac{\cancel{2}}{\cancel{2}}}{\frac{5\,\cancel{10} \times 9}{\cancel{2}_{1}}} = \frac{15 + 1}{45}$$

$$= \frac{17}{45}$$ Ans

2.20. A certain chemical substance is made by mixing 5 separate liquids. It is proposed to pour one liquid into a tank, and then to add the other liquids in turn. All possible combinations must be tested to see which gives the best yield. How many tests must be performed?

$$5 \times (4 \times 3 \times 2 \times 1) = \underline{\underline{120}} \quad \text{Ans}$$

2.21. A lot contains n articles. If it is known that r of the articles are defective and the articles are inspected in a random order, what is the probability that the kth article ($k \geq r$) inspected will be the last defective one in the lot?

Total = n articles
Defective = r articles

Inspected = k articles

For k^{th} article to be last defective article i.e, r^{th} defective

$r-1$ defective articles should already present in $k-1$ elements

So, if $k-1$ elements are already selected

Remaining number of articles = $n-k+1$

Remaining number of defective articles = 1

Remaining number of non defective articles = $n-k+1-1$
$= n-k$

No of ways to choose $k-1$ articles $= {}^{n}C_{k-1}$

Probability of $r-1$ articles in $k-1$ articles $= \dfrac{{}^{r}C_{r-1} \times {}^{n-r}C_{k-r}}{{}^{n}C_{k-1}}$

Probability of k^{th} article to be r^{th} defective article $= \underline{\underline{\dfrac{{}^{r}C_{r-1} \times {}^{n-r}C_{k-r}}{{}^{n}C_{k-1}} \times \dfrac{1}{n-k+1}}}$ Ans

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