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1 Recapitulation of Quantum Channels

A quantum channel $\mathcal{N}_{A\to B}$ taking operators from the set of bounded operators in A to the set of bounded operators in B is defined as

$$\mathcal{N}_{A\to B} : \mathcal{B}(A) \to \mathcal{B}(B)$$

has the following properties:

1.1 Trace Preserving

$$\forall X \in \mathcal{B}(A), \quad Tr(\mathcal{N}_{A\to B}(X)) = Tr(X)$$

This preserves the validity of the outcome of the operation by keeping the trace as 1.

1.2 Completely Positive

Taking the maximally entangled state Φ_{RA}, defined as

$$\Phi_{RA} = |\phi\rangle\langle\phi|_{RA}$$

where the system R is a space characteristic of the channel, and,

$$|\phi\rangle_{RA} = \frac{1}{\sqrt{d}}\, \Sigma_{i=0}^{d-1} |i\rangle_R |i\rangle_A$$

where

$$\{|i\rangle_R\}_i \quad \text{forms an orthonormal basis of R}$$
$$\{|i\rangle_A\}_i \quad \text{forms an orthonormal basis of A}$$
$$d = infinum\{dim(A), dim(R)\}$$

the channel produces a semi-positive definite state as output regardless of input if

$$\mathcal{N}_{A\to B}(\Phi_{RA}) \geq 0$$

Here, $\mathcal{N}_{A\to B}(\Phi_{RA})$ is referred to as the *Choi* of the channel.

2 Pure States

Pure states can be represented by $|\phi\rangle_{RA}$ such as:

$$|\phi\rangle_{RA} = \Sigma_{i=0}^{d-1}\sqrt{p_i}|i\rangle_R|i\rangle_A$$

where d, $|i\rangle_R$ and $|i\rangle_A$ remain the same as defined in the recapitulation.

p_i represents the probability of $|i\rangle_R|i\rangle_A$ in the superposition of states, and $\Sigma_{i=0}^{d-1}p_i = 1$.
Further, if the number of basis vectors in R exceeds d then we can choose any d arbitrary basis vectors for forming $\{|i\rangle_R\}_i$

3 Further comments

It must be noted that we are working in an *open quantum system* while working with quantum channels

Also,

$$\mathcal{N}_{A\to B}(\varphi_A) = Tr_{E'}[U_{AE\to BE'}(\varphi_A \otimes \omega_E)U^{\dagger}_{AE\to BE'}]$$

where $U_{AE\to BE'}$ is a unitary matrix. Now, unitary matrices are square matrices implying that their application on a matrix preserves the matrix's dimensions. So, we have

$$dim(AE) = dim(BE')$$

$Tr_{E'}(.)$ is the partial trace of (.) with respect to matrix E' defined as

$$Tr_B(X_{AB}) = \Sigma_i\langle i|_B X|i\rangle_B$$

where $\{|i\rangle_B\}_i$ forms the orthonormal basis of B.

The partial trace with respect to B trims the input X_{AB} to a matrix Y_A in A by removing all indication of B.

$M_A \otimes N_B(\varphi_{AB})$ is a *local operation* on φ_{AB} where M_A acts on A and N_B acts on B. Moreover,

$$[M_A \otimes \mathcal{I}_B, \mathcal{I}_A \otimes N_B] = 0$$

which indicates that the order of application of M_A and N_B on φ_{AB} doesn't matter.

We also note

$$Tr_A[M_A \otimes N_B(\varphi_{AB})] = N_B(\varphi_B)$$

where

$$\varphi_B = Tr_A[\varphi_{AB}]$$