Introduction to Quantum Information and Computing - Lecture 6

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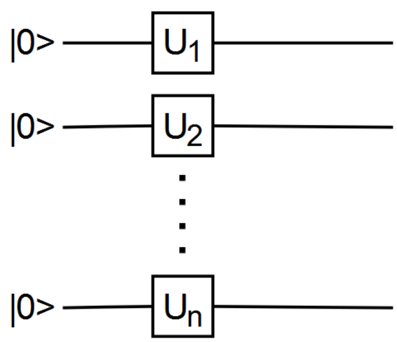
1 General Quantum Circuits

Elementary gates can be composed into bigger quantum circuits.
This can be done in two ways:

1. Tensor product
2. Ordinary matrix product

1.1 Tensor Product

Combination of gates applied to different registers.

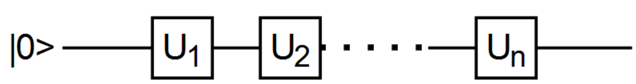


$$U_1 |0\rangle \otimes U_2 |0\rangle ... U_n |0\rangle = (U_1 \otimes U_2 ... \otimes U_n) |0...0\rangle$$
$$= U^{\otimes n} |0^n\rangle \text{ when } U_i = U \forall i$$

$(U_1 \otimes U_2 ... \otimes U_n)$ is a 2^n dimension unitary.

1.2 Ordinary Matrix Product

Combination of gates applied to same register.



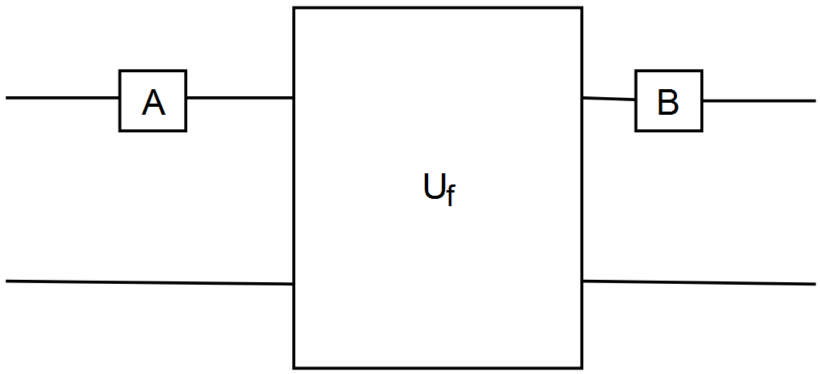
Output, $|\psi\rangle = (U_n ... U_1) |0\rangle$
Here, for all U, dim $= 2 \times 2$ (if all $U_i s$ are single qubit)

2 Cost of Quantum Circuits

1. Gate complexity
2. Depth complexity
3. Query complexity

2.1 Query Complexity

Query complexity is decided by number of queries made to a blackbox, U_f.



Gates added before and after the blackbox to prepare desired input/output state do not affect the complexity.

3 State produced by Hadamard Gate

3.1 Single Hadamard gate

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

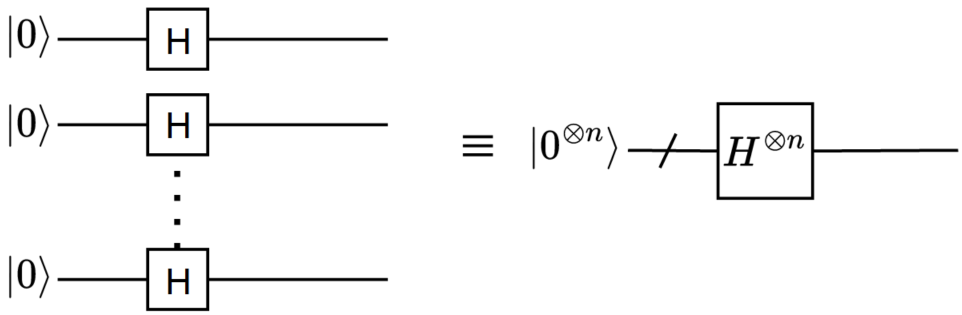
$$H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$$

$$H|1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

when $x \in \{0, 1\}$,

$$H|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x |1\rangle) = \frac{1}{\sqrt{2}} \sum_{z \in \{0,1\}} (-1)^{xz} |z\rangle$$

3.2 N Hadamard gates



$H^{\otimes n} = H \otimes H ... \otimes H$

$$
\begin{aligned}
Output &= \left|+^{\otimes n}\right\rangle \\
&= \frac{1}{\sqrt{2}^n}(|0\rangle + |1\rangle) \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) ... \otimes \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
&= \frac{1}{\sqrt{2}^n}(|0...000\rangle + |0...001\rangle + |0...010\rangle ...) \\
&= \frac{1}{\sqrt{2}^n} \sum (\text{basis of n qubits})
\end{aligned}
$$

$$H^{\otimes n}\left|0^{\otimes n}\right\rangle = \frac{1}{\sqrt{2}^n} \sum_{z \in 0,1^n} \otimes |z\rangle$$

$$H^{\otimes n}|x\rangle = \frac{1}{\sqrt{2}^n} \sum_{z \in 0,1^n} \otimes (-1)^{x \cdot z} |z\rangle$$

4 Universality of Quantum Gates

Can any U be decomposed into a combination of unitaries/gates from some finite set?

1. CNOT, all single qubit gates $\equiv$ universal set for QC $\rightarrow$ but, not finite
2. CNOT, H, T = G } Good approximation for any quantum gate.

4.1 Solovay-Kitaev Theorem

- Uses G, and each of the gates' inverse

4.1.1 1-2 Qubit

It is possible to approximate any unitary (gate) in one or two qubits upto an error ϵ by using only $O(polylog(\frac{1}{\epsilon})$ gates from G.

$\|U\,|\psi\rangle - U_m U_{m-1} ... U_1\,|\psi\rangle\| \leq \epsilon$

Where U is the matrix being approximated and each $U_i \in G$ and $m \in polylog(\frac{1}{\epsilon})$

4.1.2 t-Gate

Any 't' gate quantum circuit can be ϵ-approximated by only $O(t \cdot polylog(\frac{1}{\epsilon})$ gates from G

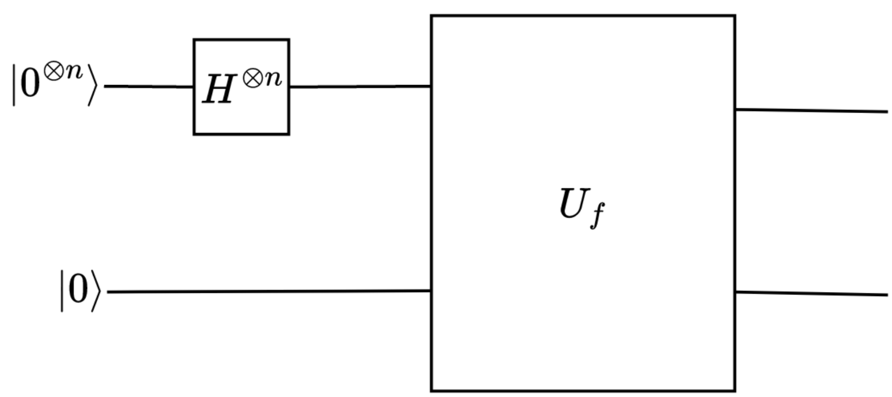
5 Quantum Parallelism

- Quantum circuits can be applied to multiple/all states in one query.
- But, output would still be of no use until we apply appropriate transformations to separate the outputs.

E.g., Consider a classical circuit that computes

$$f : \{0,1\}^n \rightarrow \{0,1\}$$

We need to run this 2^n times (once for each input) to find output for each input. Converting it to a quantum circuit but replacing gates with reversible unitaries,



$Output, |\psi\rangle = \frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} |z\rangle\,|f(|z\rangle)\rangle$