Introduction to Quantum Information and Computing - Notes

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1 Introduction and Motivation

1.1 Introduction

Quantum mechanics is a fundamental theory in physics that provides a description of the physical properties of nature at the scale of atoms and subatomic particles. It is a superset of classical mechanics and can explain behaviour in experiments that are not explained by only classical mechanics laws. One such experiment is the Stern Gerlach experiment.

The mathematical framework for quantum mechanics is linear algebra. States are described as vectors in a complex vector space, while measurements and operators are linear operators that act on the space.

1.2 Stern-Gerlach Experiment

Silver atoms are heated in an oven and projected at a screen through a hole. The stream of silver atoms are subjected to a homogeneous magnetic field. This causes them to bend in trajectory upwards or downwards.

The magnetic moment of the atom is proportional to the spin of the atom's unpaired electron: $\mu \propto S$

Because the interaction energy of the magnetic moment with the magnetic field is just $-\mu \cdot B$, the z-component of the force experienced by the atom is given by,

$$\mathbf{F}_z = \frac{\partial}{\partial z}(-\mu \cdot B) \simeq \mu_z \frac{\partial B_z}{\partial z}$$

The expected pattern is that of a band of silver, however only two spots are observed. This implies the spin of the atoms along the Z-axis is quantized to two values, S_z+ and S_z-. This goes against the classical prediction.

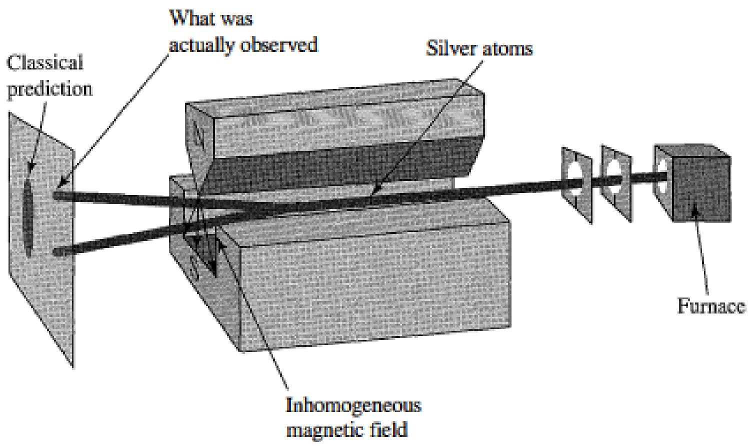

FIGURE 1.1 The Stern-Gerlach experiment.

Moreover, on sequential Stern-Gerlach experiments, more non classical behaviour was seen.

In the experiment, S_z was measured and the atoms with S_z- spin are blocked. Then the rest are sent through another Stern-Gerlach for measuring S_x. Again, the atoms with spin S_x- are blocked. The final leftover atoms are passed through an experiment measuring S_z again.

It is found that even though all atoms with S_z- were blocked, there still are atoms with S_z- spin after the third experiment. This phenomenon cannot be explained by classical mechanics. Measuring S_x destroys any information obtained about the S_z component of the atoms.

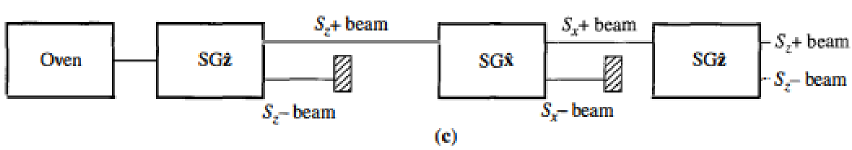

FIGURE 1.3 Sequential Stern-Gerlach experiments.

1.3 Shannon's Theory of Information and Entropy

Claude Shannon created the modern information theory. He justified use of entropy as measure of information about a system. The units for information were taken as bits. In terms of quantum information, the units are qubits.

Shannon defined entropy as the average measure of randomness or uncertainty in a system. The Shannon entropy is defined as entropy H (Greek capital letter eta) of a discrete random variable X, which takes values in the alphabet

$\mathcal{X}$ and is distributed according to

$$p : \mathcal{X} \longrightarrow [0, 1]$$

such that

$$p(x) := \mathbb{P}[X = x]$$

And H is given by

$$H(\mathcal{X}) = \mathbb{E}[-logp(X)] = -\sum_{x \in \mathbb{X}} p(x)logp(x)$$

1.4 No Cloning Theorem

In physics, the no-cloning theorem states that it is impossible to create an independent and identical copy of an arbitrary unknown quantum state. This has profound implications in quantum computation. The theorem follows from the fact that all quantum operations must be unitary linear transformation on the state.

1.5 Outline of the Course

In the course we will be exploring the following topics:

1. Postulates
2. Everything is a quantum channel
3. Entanglement, Separability, Non-locality
4. Teleportation, No Cloning
5. Entropy, Trace Distance

2 Finite Dimensional Hilbert Spaces

A d-dimensional Hilbert space $\mathcal{H}$ ($1 \leq d < \infty$) is a complex vector space with an inner product defined on it. A vector in the Hilbert space $\mathcal{H}$ is denoted by $|\psi\rangle$. The inner product $\langle ., . \rangle : \mathcal{H} \times \mathcal{H} \rightarrow \mathbb{C}$ has the following properties:

- *Non negativity* - $\langle \psi, \psi \rangle \geq 0 \ \forall \ |\psi\rangle \in \mathcal{H}$. $\langle \psi, \psi \rangle = 0$ if and only if $\langle \psi \rangle = 0$.
- *Linearity in Second Argument* - $\langle \psi, \alpha\phi_1 + \beta\phi_2 \rangle = \alpha\langle \psi, \phi_1 \rangle + \beta\langle \psi, \phi_2 \rangle$
- *Conjugate Linearity in First Argument* - $\langle \alpha\psi_1 + \beta\psi_2, \phi \rangle = \bar{\alpha}\langle \psi_1, \phi \rangle + \bar{\beta}\langle \psi_2, \phi \rangle$
- *Conjugate Symmetry* - $\langle \psi, \phi \rangle = \overline{\langle \phi, \psi \rangle}$

3 Describing a Closed Physical System

The complete description of a closed physical system is given by its state $|\psi\rangle$ where $|\psi\rangle \in \mathcal{H}$ ($\mathcal{H}$ is a Hilbert Space) and norm of $|\psi\rangle$ is 1 ($\langle\psi,\psi\rangle = 1$). For every state $|\psi\rangle \in \mathcal{H}$, $\exists\ \langle\psi|$ in the dual vector space of $\mathcal{H}$. Also, $\langle\psi| = (|\psi\rangle)^\dagger$.

For $|\psi\rangle$ to represent a closed system, the Hilbert Space it belongs to must have dimension $d \geq 2$, $d \in \mathbb{N}$.

4 Axioms of Quantum Mechanics

4.1 Axiom 1 : Quantum Systems

A closed quantum system can be represented by its state $|\psi\rangle \in \mathcal{H}$ where $\langle\psi|\psi\rangle = 1$.

4.2 Axiom 2 : Observables

Observables are given by Hermitian operators, which take real eigenvalues. (Proven in assignment 1).

Observables are Linear Operators. $\hat{O} : \mathcal{H}_1 \to \mathcal{H}_2$ where $\hat{O}$ is the linear operator denoting the observable.

Observables are Hermitian. $\hat{O} = \hat{O}^\dagger$

Consider $\psi \in \mathcal{H}_1$, $\phi \in \mathcal{H}_2$.

$$\langle\psi, \hat{O}\phi\rangle = \langle\hat{O}^\dagger\psi, \phi\rangle$$

In particular, since $\hat{O}$ is Hermitian,

$$\langle\psi, \hat{O}\phi\rangle = \langle\hat{O}\psi, \phi\rangle \ \ \forall\psi, \phi$$

Since $\hat{O}$ is Hermitian, $\mathcal{H}_1$ and $\mathcal{H}_2$ are isomorphic. Also, its eigenvalues are real and distinct eigenvectors are orthogonal(Proved in assignment 1.)

Consider the sprectral decomposition of observable $\hat{O}$: $\hat{O} = \sum_{a_i} \lambda_i |a_i\rangle\langle a_i|$ where $|a_i\rangle$s are part of the orthonormal basis. Then

$$\begin{aligned} \hat{O}|a_j\rangle &= \sum_{a_i} \lambda_i |a_i\rangle\langle a_i||a_j\rangle \\ &= \sum_{a_i} \lambda_i |a_i\rangle\delta_{ij} \\ &= \lambda_j |a_j\rangle \end{aligned}$$

4.3 Aside

If $\dim \mathcal{H} = d$ and $|\alpha_1\rangle, |\alpha_2\rangle, \ldots, |\alpha_d\rangle$ is an orthonormal basis,

$$|\psi\rangle = \sum_1^d c_i|\alpha_i\rangle \quad , c_i \in \mathbb{C}$$

$$\sum_1^d |c_i|^2 = 1$$

WLOG, take $|\alpha_1\rangle = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, |\alpha_2\rangle = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \ldots$ as the standard orthonormal basis and denote them by $|0\rangle, |1\rangle, \ldots, |d-1\rangle$

For an operator $\hat{A}$, if $\hat{A}\hat{A}^\dagger = \hat{A}^\dagger\hat{A}$, $\hat{A}|a\rangle = a|a\rangle$, where a is the eigenvalue and $|a\rangle$ is the corresponding eigenvector.

A state $|\psi\rangle$ is of norm 1, i.e. if $|\psi\rangle = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{bmatrix}$, $\sqrt{\sum_i |a_i|^2} = 1$

4.4 Axiom 3 : Measurement

Measurement $\hat{M}$ corresponding to an observable $\hat{O}$ for any state $|\psi\rangle$ is such that $\hat{M}|\psi\rangle \rightarrow |a_i\rangle$ with outcome λ_i. After measurement, the final state of the system is one of the eigenstates. Further measurement will produce the same eigenstate and eigenvalue (as long as the repeated measurement is not preceded by the measurement of some other observable.)

A projection matrix $\mathbb{P}$ is a matrix such that $\mathbb{P}^2 = \mathbb{P}$. $\hat{M}$ is a projection matrix.

4.5 Axiom 4 : Evolution

The evolution of quantum states is given by a unitary transformation, $U : \mathcal{H} \rightarrow \mathcal{H}$ where $U^\dagger U = UU^\dagger = \mathbb{I}$

5 Multiple Systems

Let H_A and H_B be two system. To visualize multiple systems as a single system we use the idea of tensor product.
$H_A \otimes H_B$ denotes the tensor product of H_A and H_B
For e.g. consider the following example :

$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ A = $\begin{bmatrix} a_{11}A & a_{12}A \\ a_{21}A & a_{22}A \end{bmatrix}$

We can easily see that

$$dim(H_A \otimes H_B) = dim(H_A) \times dim(H_B)$$

6 Noisy Quantum Theory

Expectation value of an observable A is

$$< A >_\psi = < \psi|A|\psi >$$

If ψ is not normalized then normalize it by dividing with $< \psi|\psi >$.
Now even if I change $|\psi >$ to $e^{\iota\phi}|\psi >$ expectation value does not changes because

$$< \psi|e^{-\iota\phi}Ae^{\iota\phi}|\psi > = < \psi|A|\psi > \qquad (e^{\iota\phi} \text{ is just a scalar})$$

7 Density Operator

A quantum state is represented by a density operator defined on a
Hilbert space H.

$$\rho : H \to H$$

7.1 Requirement of Density operator

The question arises that what's the need of representing a state in terms of density operator.
State vectors or wave functions ($|\psi >$) can only represent pure states. Density operator can also represent mixed state.
Density operator also allows for the calculation of the probabilities of the outcomes of any measurement performed upon system, using the Born rule.

7.2 Properties of Density operator

- $\rho \geq 0$
  This means that all the eigenvalues of ρ are ≥ 0
- $\rho = \rho^\dagger$
  ρ should be a Hermitian-matrix
- $Tr(\rho) = 1$
  The summation of the diagonal elements of $\rho = 1$

So summarising, Density operator for a system is a positive semi-definite, Hermitian operator of trace one acting on the Hilbert space of the system

For any state $|\psi>$, its density operator is $|\psi><\psi|$.
Let's prove that $|\psi><\psi|$ follows all the property of the density operator :

We know that $(|\psi><\psi|)^2 = |\psi><\psi|\psi><\psi| = |\psi><\psi|$
So it's clear that $|\psi><\psi|$ is a pure state $\implies$ all eigenvalues are ≥ 0.
So first property is satisfied.

2

$$\begin{aligned}(|\psi><\psi|)^\dagger &= (<\psi|)^\dagger(|\psi>)^\dagger \\ &= |\psi><\psi|\end{aligned}$$

Hence $<\psi|\psi>$ is a hermitian operator.

3.

$$\begin{aligned}Tr(|\psi><\psi|) &= Tr(<\psi|\psi>) \\ &= 1\end{aligned} \qquad \text{(Trace Theorem)}$$

Hence, the trace of $|\psi><\psi| = 1$.
All the properties are satisfied by $|\psi><\psi|$.

8 Observables

- Each observable can be represented by Hermitian operators.
- Expectation value of an observable $\hat{O}$ for a quantum state ρ is $\mathrm{Tr}[\hat{O}\rho]$
- If ρ is pure state $\{|\psi\rangle\langle\psi|\}$, then:

  $\mathrm{Tr}[\hat{O}\rho]$=$\mathrm{Tr}[\hat{O}|\psi\rangle\langle\psi|]$=$\mathrm{Tr}[\langle\psi|\hat{O}|\psi\rangle]$=$\langle\psi|\hat{O}|\psi\rangle$
- If ρ is not a pure state, i.e., ρ=$\sum\limits_i p_i|\psi_i\rangle\langle\psi_i|$, then:

  $\mathrm{Tr}[\hat{O}\rho]$=$\mathrm{Tr}[\hat{O}\sum\limits_i p_i|\psi_i\rangle\langle\psi_i|]$=$\sum\limits_i p_i\mathrm{Tr}[\hat{O}|\psi_i\rangle\langle\psi_i|]$=$\sum\limits_i p_i\langle\psi|\hat{O}|\psi_i\rangle$

9 Mixed State

Mixed state is said to be the mixture of the different pure states.
Mixed states can be written as: $\sigma = \sum\limits_i p_i\rho_i, where \sum\limits_i p_i = 1$
Here σ: is the mixed state , ρ_i:pure state and p_i: probability of state ρ_i to be present in the mixed state.

- A state is pure if $\rho^2 = \rho$

Example1:- Let us consider a system which has 5 qubits of $|0\rangle$ and 5 qubits of $|1\rangle$ where $\{|0\rangle, |1\rangle\}$ are orthonormal basis. Find the mixed state of the system.
Answer1:- $\sigma = \frac{1}{2}\{|0\rangle\langle 0| + |1\rangle\langle 1|\}$

Example2:- Let us consider a system which has 5 qubits of $|+\rangle$ and 5 qubits of $|-\rangle$ where $\{|+\rangle, |-\rangle\}$ are orthonormal basis. Find the mixed state of the system.
Answer2:- $\sigma = \frac{1}{2}\{|+\rangle\langle +| + |-\rangle\langle -|\}$

Note:- Both σ obtained in the above two examples are one and the same.
Proof:-
$\sigma = \frac{1}{2}\{|0\rangle\langle 0| + |1\rangle\langle 1|\} = \frac{1}{2}\{|+\rangle\langle +| + |-\rangle\langle -|\}$

$\sigma=\frac{1}{2}(\sum\limits_i |i\rangle\langle i|)=\frac{1}{2}(\sum\limits_i |i\rangle\langle i|)$

$\sigma = \frac{1}{2}\mathbb{1} = \frac{1}{2}\mathbb{1}$

10 Superposition

The linear combination of states is called superposition.
A superpositioned state can be represented as:
$|\psi\rangle = \sum\limits_i c_i|\psi_i\rangle$,
where $\sum\limits_i c_i^2 = 1$ and $|\psi_i\rangle$ are the states that are being superimposed.

11 Measurements

The most general kind of measurements are called POVM's. POVM stands for positive operator valued measure.
They are denoted by $\{\Lambda^x\}_x$. It is a collection of positive semi-definite operators such that:

$\Lambda^x \geq 0$ and $\sum\limits_x \Lambda^x = 1$.

11.1 Projective Measurements

They are denoted by $\{\mathrm{P}_x\}$. They are POVMs with additional properties:

- ${\mathrm{P}_i}^2 = \mathbb{P}_i$
- $\mathrm{P}_i\ \mathrm{P}_j = \delta_{ij}\ \mathrm{P}_i = \delta_{ij}\ \mathrm{P}_j$

where x is the outcome of the measurement and $P_i, P_j \in \{\mathbb{P}_x\}$

12 Measurement Probability

12.1 Born's Rule

The Born's Rule is a postulate of Quantum Mechanics which helps determine the probability that a measurement of a quantum system will yield a given result.

The Born's rule states that if an observable corresponding to a self-adjoint operator A with discrete spectrum is measured in a system with normalized wave function $|\psi\rangle$ then :

1. The measured value will be one of the eigenvalues λ of A.

2. The probability of measuring a given eigenvalue λ_i will be equal to $\langle\psi|P_i|\psi\rangle$ where P_i is the projection onto the eigenspace of A corresponding to λ_i.
Equivalently, the probability can be written as $\left|\langle\lambda_i|\psi\rangle\right|^2$ where $|\lambda_i\rangle$ is the eigenvector associated with the eigenvalue λ_i.

12.1.1 POVM Version of Born's Rule

The POVM element F_i is associated with the measurement outcome i, such that the probability of obtaining it when making a measurement on the quantum state ρ is given by: $p(i) = \mathrm{tr}(\rho F_i)$,

The measurement of a quantum system in the state ρ according to the POVM M_x : x $\in$ X induces a probability distribution. This distribution takes values belonging to the set of all possible values of x, and is defined by the Born rule: p(x) = Tr[M_x ρ].

To determine the post-measurement states of the system being measured:
Taking measurement with a projective operator $\mathbb{P}_\beth$. In this case let the post-measurement state be ρ^x.

$\rho \xrightarrow{\mathbb{P}_\beth} \rho'$

$$\rho' = \frac{P_i \rho P_i^\dagger}{Tr[P_i \rho P_i^\dagger]} \text{ ... eqn(1)}$$

Set of Orthonormal Basis is a projective measurement because it satisifes POVM and also the projective measurement conditions.

After measuring once if we measure the same observable again , it will return the same outcome.

Proof: Assume we got outcome as i in our first measurement. From eqn(1) our new state is ρ'.
p(i) = Tr$[P_x \rho'] = Tr[\mathrm{P}_x \frac{P_i \rho P_i^\dagger}{Tr[P_i \rho P_i^\dagger]}] = \mathrm{Tr}[\frac{\delta_{xi} P_i \rho P_i^\dagger}{Tr[P_i \rho]}]$.
This value clearly would be 1 in case x = i otherwise this will be equal to 0.
Hence Proved.

13 Transformation/Evolution of Quantum States

Quantum Communication necessarily involves the evolution
of quantum systems (such as the evolution of photons when travelling through an optical fiber). Mathematically, this evolution is described by a quantum channel.

A Quantum channel is a linear, completely positive, and trace-preserving map acting on the state of the system. Note that we are working with Open Quantum Systems while talking about Quantum Channels.

$$\mathcal{N}_{A \to B} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$$

Note : $\mathcal{B}(\mathcal{H}_A)$ denotes the set of operators. $dim(\mathcal{H}_A)$ need not be equal to $dim(\mathcal{H}_B.)$

Introduction to Trace Preserving and Completely Positive properties:
New density operator also has trace equal to 1 and the channel produces a semi-definite state as output always if the choi of the channel is ≥ 0.

Everything is a Quantum Channel with respect to time.

14 Quantum Channel

$$\mathcal{N}_{A \to B} : \mathcal{B}(\mathcal{H}_\mathcal{A}) \to \ B(\mathcal{H}_\mathcal{B})$$

where

$$(\mathcal{H}_\mathcal{A}) = \text{Hilbert space of input system,}$$
$$(\mathcal{B}) = \text{Set of bounded operators}$$

Quantum channel is trace-preserving since it maps trace-class operators to trace-class operators.

A matrix with a finite trace represents a trace-class operator. A finite-dimensional matrix is a trace-class operator if it is bounded.

A **Super Operator** acts on an operator to give an operator.

14.1 Properties of Quantum Channel

For a super operator to be a quantum channel, it has to follow the following properties:

1. Completely positive

$$id_B \otimes \mathcal{N}_{A\to C}(\Phi_{AB}) \geq 0$$

   where $\Phi_{AB} \geq 0$ and id = identity super operator, and ϕ_{AB} is a maximally entangled state.

   Positivity: $X_A \geq 0 \Rightarrow \mathcal{N}_{A\to C}(\mathcal{X}_A) = \mathcal{Y}_C \geq 0$

2. Trace-preserving map

$$Tr[\rho_A] \Rightarrow Tr[\mathcal{N}_{A\to C}(\rho_A)]$$

15 Joint States

15.1 Product State

The density operator of the product state of A and B is the tensor product of the density operators of A and B.

$$\rho_{AB} = \rho_A \otimes \sigma_B$$

In a product state, there is no correlation between the two states. A and B are independent.

15.2 Separable States

The density operator of a separable state is obtained as:

$$\rho_{AB} = \sum_x p_x \rho_A^{(x)} \otimes \sigma_B^{(x)}$$

where p_x is the probability of each state such that

$$\sum p_x = 1, p_x \geq 0$$

Here, $\rho_A^{(x)}$ and $\sigma_B^{(x)}$ may be pure or mixed states.

15.3 Entangled State

State which cannot be written as separable states or product of states is in an entangled state.

$$\mathcal{N}_{A\to C}(\Phi_{AB})$$

Dimension, d $= min(dim(\mathcal{H}_A), dim(\mathcal{H}_B))$

Maximally entangled state: $\Phi_{AB} = \frac{1}{d}\sum_{i,j=0}^{d-1} i_A \otimes i_B j_A \otimes j_B$
$= \frac{1}{d}\sum_{i,j=0}^{d-1} ij_A \otimes ij_B$

16 Kraus Operators: K_i

$$\mathcal{N}_{A\to C}(\cdot) = \sum_i K_i(\cdot){K_i}^\dagger$$

where

$$\sum {K_i}^\dagger K_i = \not\Vdash$$
$$K_i : \mathcal{H}_A \to \mathcal{H}_C$$

For any completely positive, trace-preserving quantum channel, $\exists Krausoperators$.
If an N has Kraus operators $\Rightarrow$ it is completely positive and trace-preserving.

Projective measurement is a special case of quantum channel where Ps are Kraus operators.

17 Recapitulation of Quantum Channels

A quantum channel $\mathcal{N}_{A\to B}$ taking operators from the set of bounded operators in A to the set of bounded operators in B is defined as

$$\mathcal{N}_{A\to B} : \mathcal{B}(A) \to \mathcal{B}(B)$$

has the following properties:

17.1 Trace Preserving

$$\forall X \in \mathcal{B}(A), \quad Tr(\mathcal{N}_{A\to B}(X)) = Tr(X)$$

This preserves the validity of the outcome of the operation by keeping the trace as 1.

17.2 Completely Positive

Taking the maximally entangled state Φ_{RA}, defined as

$$\Phi_{RA} = |\phi\rangle\langle\phi|_{RA}$$

where the system R is a space characteristic of the channel, and,

$$|\phi\rangle_{RA} = \frac{1}{\sqrt{d}}\ \Sigma_{i=0}^{d-1}|i\rangle_R|i\rangle_A$$

where

$$\{|i\rangle_R\}_i \quad \text{forms an orthonormal basis of R}$$
$$\{|i\rangle_A\}_i \quad \text{forms an orthonormal basis of A}$$
$$d = infinum\{dim(A), dim(R)\}$$

the channel produces a semi-positive definite state as output regardless of input if

$$\mathcal{N}_{A\to B}(\Phi_{RA}) \geq 0$$

Here, $\mathcal{N}_{A\to B}(\Phi_{RA})$ is referred to as the *Choi* of the channel.

18 Pure States

Pure states can be represented by $|\phi\rangle_{RA}$ such as:

$$|\phi\rangle_{RA} = \Sigma_{i=0}^{d-1}\sqrt{p_i}|i\rangle_R|i\rangle_A$$

where d, $|i\rangle_R$ and $|i\rangle_A$ remain the same as defined in the recapitulation.

p_i represents the probability of $|i\rangle_R|i\rangle_A$ in the superposition of states, and $\Sigma_{i=0}^{d-1}p_i = 1$.

Further, if the number of basis vectors in R exceeds d then we can choose any d arbitrary basis vectors for forming $\{|i\rangle_R\}_i$

19 Further comments

It must be noted that we are working in an *open quantum system* while working with quantum channels

Also,

$$\mathcal{N}_{A\to B}(\varphi_A) = Tr_{E'}[U_{AE\to BE'}(\varphi_A \otimes \omega_E)U^{\dagger}_{AE\to BE'}]$$

where $U_{AE\to BE'}$ is a unitary matrix. Now, unitary matrices are square matrices implying that their application on a matrix preserves the matrix's dimensions. So, we have

$$dim(AE) = dim(BE')$$

$Tr_{E'}(.)$ is the partial trace of (.) with respect to matrix E' defined as

$$Tr_B(X_{AB}) = \Sigma_i \langle i|_B X |i\rangle_B$$

where $\{|i\rangle_B\}_i$ forms the orthonormal basis of B.

The partial trace with respect to B trims the input X_{AB} to a matrix Y_A in A by removing all indication of B.

$M_A \otimes N_B(\varphi_{AB})$ is a *local operation* on φ_{AB} where M_A acts on A and N_B acts on B. Moreover,

$$[M_A \otimes \mathcal{I}_B, \mathcal{I}_A \otimes N_B] = 0$$

which indicates that the order of application of M_A and N_B on φ_{AB} doesn't matter.

We also note

$$Tr_A[M_A \otimes N_B(\varphi_{AB})] = N_B(\varphi_B)$$

where

$$\varphi_B = Tr_A[\varphi_{AB}]$$