Introduction to Quantum Information and Computing - Lecture 5

Arnav Negi, Kriti Gupta, Manav Shah, Mohammed Shamil,
Shiven Sinha ,Shrikara A, Swayam Agrawal, Vineeth Bhat, Yash Adivarekar

January 17, 2023

1 Measurements

The most general kind of measurements are called POVM's. POVM stands for positive operator valued measure.
They are denoted by $\{\Lambda^x\}_x$. It is a collection of positive semi-definite operators such that:

$\Lambda^x \geq 0$ and $\sum_x \Lambda^x = 1$.

1.1 Projective Measurements

They are denoted by $\{\mathrm{P}_x\}$. They are POVMs with additional properties:

- $\mathrm{P}_i{}^2 = P_i$
- $\mathrm{P}_i\ \mathrm{P}_j = \delta_{ij}\ \mathrm{P}_i = \delta_{ij}\ \mathrm{P}_j$

where x is the outcome of the measurement and P_i,$\mathrm{P}_j \in \{P_x\}$

2 Measurement Probability

2.1 Born's Rule

The Born's Rule is a postulate of Quantum Mechanics which helps determine the probability that a measurement of a quantum system will yield a given result.

The Born's rule states that if an observable corresponding to a self-adjoint operator A with discrete spectrum is measured in a system with normalized wave function $|\psi\rangle$ then :

1. The measured value will be one of the eigenvalues λ of A.

2. The probability of measuring a given eigenvalue λ_i will be equal to $\langle\psi|P_i|\psi\rangle$ where P_i is the projection onto the eigenspace of A corresponding to λ_i.
Equivalently, the probability can be written as $\left|\langle\lambda_i|\psi\rangle\right|^2$ where $|\lambda_i\rangle$ is the eigenvector associated with the eigenvalue λ_i.

2.1.1 POVM Version of Born's Rule

The POVM element F_i is associated with the measurement outcome i, such that the probability of obtaining it when making a measurement on the quantum state ρ is given by: $p(i) = \mathrm{tr}(\rho F_i)$,

The measurement of a quantum system in the state ρ according to the POVM M_x : x $\in$ X induces a probability distribution. This distribution takes values belonging to the set of all possible values of x, and is defined by the Born rule: p(x) = Tr[M_x ρ].

To determine the post-measurement states of the system being measured:
Taking measurement with a projective operator P_i. In this case let the post-measurement state be ρ^x.

$\rho \xrightarrow{P_i} \rho'$

$\rho' = \dfrac{P_i \rho P_i^\dagger}{Tr[P_i \rho P_i^\dagger]}$ ... eqn(1)

Set of Orthonormal Basis is a projective measurement because it satisifes POVM and also the projective measurement conditions.

After measuring once if we measure the same observable again , it will return the same outcome.

Proof: Assume we got outcome as i in our first measurement. From eqn(1) our new state is ρ'.
p(i) = $\mathrm{Tr}[P_x \rho'] = Tr[\mathrm{P}_x \dfrac{P_i \rho P_i^\dagger}{Tr[P_i \rho P_i^\dagger]}] = \mathrm{Tr}[\dfrac{\delta_{xi} P_i \rho P_i^\dagger}{Tr[P_i \rho]}]$.
This value clearly would be 1 in case x = i otherwise this will be equal to 0.
Hence Proved.

3 Transformation/Evolution of Quantum States

Quantum Communication necessarily involves the evolution
of quantum systems (such as the evolution of photons when travelling through an optical fiber). Mathematically, this evolution is described by a quantum channel.

A Quantum channel is a linear, completely positive, and trace-preserving map acting on the state of the system. Note that we are working with Open Quantum Systems while talking about Quantum Channels.

$$\mathcal{N}_{A\to B} : \mathcal{B}(\mathcal{H}_A) \to \mathcal{B}(\mathcal{H}_B)$$

Note : $\mathcal{B}(\mathcal{H}_A)$ denotes the set of operators. $dim(\mathcal{H}_A)$ need not be equal to $dim(\mathcal{H}_B.)$

Introduction to Trace Preserving and Completely Positive properties:
New density operator also has trace equal to 1 and the channel produces a semi-definite state as output always if the choi of the channel is ≥ 0.

Everything is a Quantum Channel with respect to time.