Introduction to Quantum Information and Computing Half 2 Lecture 5

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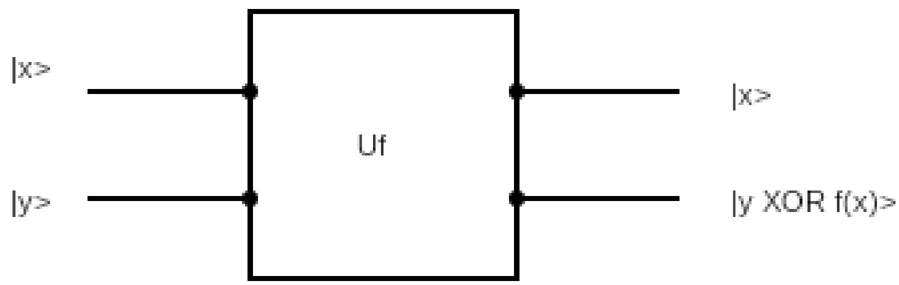
Some Convention:

- H denotes the Hadamard gate
- U_f and C_f have been used interchangeably.

Contents

1 Phase Kickback Oracle

Consider the CNOT gate where U_f denotes the unitary operation CNOT.



The action of U_f is given by:

$$
\begin{aligned}
|x\rangle|y\rangle \xrightarrow{U_f} & |x\rangle|y\rangle \text{ if } f(x) = 0 \\
& |x\rangle|\overline{y}\rangle \text{ if } f(x) = 1
\end{aligned}
$$

$|x\rangle, |y\rangle \in \{0, 1\}$

Consider the case when $|y\rangle = |-\rangle$

$$
\begin{aligned}
|x\rangle|-\rangle \xrightarrow{U_f} & U_f \frac{(|x\rangle|0\rangle + |x\rangle|1\rangle)}{\sqrt{2}} \\
= & \frac{|x\rangle(|0 \oplus f(x)\rangle + |1 \oplus f(x)\rangle)}{\sqrt{2}} \\
= & |x\rangle|-\rangle \text{ if } f(x) = 0 \\
& - |x\rangle|-\rangle \text{ if } f(x) = 1 \\
= & (-1)^{f(x)}|x\rangle|-\rangle
\end{aligned}
$$

If $x \in \{0, 1\}^n$ and before U_f, $H^{\otimes n}$ is applied on $|x\rangle$, then the output is:

$$
\begin{aligned}
& U_f \frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} |z\rangle|-\rangle && \textit{from the action of } H \\
= & \frac{1}{\sqrt{2^n}} \sum_{z \in \{0,1\}^n} (-1)^{f(x)}|z\rangle|-\rangle && \textit{from the action of } U_f
\end{aligned}
$$

Usually, the $|-\rangle$ in the second register is dropped when writing the phase kickback since it remains unchanged in output and is considered implicit when using the phase kickback oracle.

2 Deutsch Algorithm

2.1 The Problem

Suppose U_f is given as a black box for a boolean function $f : \{0,1\} \to \{0,1\}$, with the promise that either:

(i) $f(0) = f(1)$

(ii) $f(0) \neq f(1)$

How many queries do we need to make to U_f to determine which of the two is true?

2.2 Classical Reversible Computation

Classically, two queries to U_f are needed, one to determine the value of $f(0)$ and one to determine the value of $f(1)$. We can then compare the two and decide which promise is true.

images of classical circuit

2.3 Quantum Computation

Consider the following quantum circuit:

image of quantum circuit

2.3.1 Finding Final State

Finding the output:

$$
\begin{aligned}
|0\rangle|-\rangle &\xrightarrow{H\otimes\mathbb{I}} |+\rangle|-\rangle && \textit{Apply } H \textit{ to first register} \\
&\xrightarrow{U_f} \frac{1}{\sqrt{2}}((-1)^{f(0)}|0\rangle + (-1)^{f(1)}|-\rangle)|-\rangle && \textit{Apply phase kickback} \\
&\xrightarrow{H\otimes\mathbb{I}} \frac{1}{\sqrt{2}}((-1)^{f(0)}|+\rangle + (-1)^{f(1)}|-\rangle)|-\rangle && \textit{Apply } H \textit{ to first register} \\
&= \frac{1}{\sqrt{2}}\left(\frac{(-1^{f(0)})(|0\rangle + |1\rangle)}{\sqrt{2}} + \frac{(-1^{f(1)})(|0\rangle - |1\rangle)}{\sqrt{2}}\right)|-\rangle
\end{aligned}
$$

By rearranging terms containing $|0\rangle$ and $|1\rangle$, we obtain the final state $|\psi\rangle$ as

$$|\psi\rangle = \frac{1}{2}\Big(((-1)^{f(0)} + (-1)^{f(1)})|0\rangle + ((-1)^{f(0)} - (-1)^{f(1)})|1\rangle\Big)$$

Note that the $|-\rangle$ in the second register has been dropped since it is implicit for a phase kickback oracle.

2.3.2 Probability Distribution

The probabilities of the final states being $|0\rangle$ and $|1\rangle$ can be calculated from the square of the corresponding amplitudes of $|0\rangle$ and $1\rangle$, giving

$$\mathbb{P}(|0\rangle) = \frac{1}{4}\Big((-1)^{f(0)} + (-1)^{f(1)}\Big)^2$$
$$\mathbb{P}(|1\rangle) = \frac{1}{4}\Big((-1)^{f(0)} - (-1)^{f(1)}\Big)^2$$

2.3.3 Resolving a Promise

To resolve which one of the two promises are true, we measure the final state. If $f(0) = f(1)$

$$\langle 0|\psi\rangle = 1$$
$$\langle 1|\psi\rangle = 0$$

If $f(0) \neq f(1)$

$$\langle 0|\psi\rangle = 0$$
$$\langle 1|\psi\rangle = 1$$

Thus, if the final state is orthogonal to $|1\rangle$, then $f(0) = f(1)$ and if it is orthogonal to $|0\rangle$, then $f(0) \neq f(1)$.

2.3.4 Comparison

We observe that the quantum computer needs only 1 query while the classical reversible computer needed 2 queries to U_f.

3 Deutsch-Jozsa Algorithm

3.1 The Problem

This is a generalisation of the Deutsch algorithm that we previously saw. In this algorithm, the boolean function f is from n-bit strings to a bit, i.e.

$f : \{0,1\}^n \rightarrow \{0,1\}$.

Promises:

(i) f is constant, i.e. $f(x) = 0\ \forall x \in \{0,1\}^n$ or $f(x) = 1\ \forall x \in \{0,1\}^n$

(ii) f is balanced, i.e.

$$f(x) = 0 \textit{ for } \frac{2^n}{2} \textit{ values of } x$$
$$f(x) = 1 \textit{ for the other } \frac{2^n}{2} \textit{ values of } x$$

3.2 Classical Reversible Computing

Classically, to resolve a promise with probability 1, the worst case number of queries needed to U_f is $\frac{2^n}{2}+1$. This is when out of the 2^n possible n-bit strings to evaluate, the first half, i.e. $\frac{2^n}{2}$ strings all give the same output, either 0 or 1. Now, we need one additional query to resolve a promise. If it is the same as the result of the first half of bit strings, then f is constant, else f is balanced.

3.3 Quantum Computing

image of circuit

3.3.1 Finding the Final state

:

$$\begin{aligned}
|0\rangle^{\otimes n}|-\rangle \xrightarrow{H^{\otimes n}\otimes\mathbb{I}} & \frac{1}{\sqrt{2^n}}\left(\sum_{x\in\{0,1\}^n}|x\rangle\right)|-\rangle && \textit{Applying } H^{\otimes n} onfirstregisterset \\
\xrightarrow{U_f} & \frac{1}{\sqrt{2^n}}\left(\sum_{x\in\{0,1\}^n}(-1)^{f(x)}|x\rangle\right)|-\rangle && \textit{Applying phase kickback} \\
\xrightarrow{H^{\otimes n}\otimes\mathbb{I}} & \frac{1}{\sqrt{2^n}}\left(\sum_{x\in\{0,1\}^n}(-1)^{f(x)}\frac{1}{\sqrt{2}}\left(\sum_{z\in\{0,1\}^n}(-1)^{x\cdot z}|z\rangle\right)\right) && \textit{Applying } H^{\otimes n} onfirstregisterset \\
|\psi\rangle = & \frac{1}{2^n}\sum\sum_{x,z\in\{0,1\}^n}\left((-1)^{f(x)+x\cdot z}\ |z\rangle\right) && \textit{Final state}
\end{aligned}$$

Checking the inner product of the final state with an n-bit string of 0s,

$$\begin{aligned}\langle 00\ldots 0|\psi\rangle = \frac{1}{2^n}\sum_{x\in\{0,1\}^n}(-1)^{f(x)} & \qquad f(x)\in\{0,1\} \\ =1 & \qquad \textit{if } f(x)=0,\ f(x) \textit{ is constant} \\ -1 & \qquad \textit{if } f(x)=1,\ f(x) \textit{ is constant} \\ 0 & \qquad \textit{if } f(x) \textit{ is balanced}\end{aligned}$$

3.3.2 Probabilities of Final State

Finding the probabilities of the final state using the squares of amplitudes,

$$\begin{aligned}\mathbb{P}(|00\ldots 0\rangle) =& (\pm 1)^2 = 1 \qquad \textit{if } f(x) \textit{ is constant} \\ & 0^2 = 0 \qquad \textit{if } f(x) \textit{ is balanced}\end{aligned}$$

3.3.3 Resolving a Promise

If $f(x)$ is constant, then the measured final state $|\psi\rangle$ will be $|00\ldots 0\rangle$ with probability 1.
If $f(x)$ is balanced, then the measured final state $|\psi\rangle$ is a state other than $|00\ldots 0\rangle$ with probability 1.

3.3.4 Comparison

Compared to the classical reversible computer, which needed a worst case of $\frac{2^n}{2}+1$ queries to C_f, the quantum computer needs only 1 query to U_f to resolve a promise. This is an exponential speedup. Note, however, that this exponential speedup is when we must resolve the correct promise with probability 1. If we allow for an ε uncertainty to both, the speedup offered by the quantum computer will reduce to $\mathcal{O}(\log \frac{1}{\varepsilon})$, as seen in Assignment 1.