Number of Sites	Cycle Length
10	2
100	10
1000	50
10000	250
100000	1250

Table 8.2: Equilibrium cycle length for a single, roving agent.

Ott, 1993). In the limit, the cycles become infinitely long, implying that our agent will forever roam the lattice.

In the continuous case defined over the open interval, the tent map has a unique, fixed-point equilibrium at $x = 2/3$. In the discrete examples that we considered above, N was not divisible by 3, so this equilibrium did not exist. We can fix this problem by letting $N = 102$, in which case a fixed point arises at location 68 (since $68 = 2(102 - 68)$). Suppose that an agent at the fixed point makes a slight error and lands on location 67. From this location, the agent would find itself in the following locations: 70, 64, 76, 52, 100, 4, 8, 16, 32, 64, ..., ending up in a cycle of length eight. Thus, this equilibrium point is unstable. This instability is tied to the steep slope of the relocation function. Namely, small mistakes lead to larger corrections—after moving one location too low, the agent moves two locations too high, then four locations too low, and so on.

While the lack of stability of the fixed-point equilibrium above is problematic, of even more concern is its relatively small basin of attraction. Assuming no mistakes, an agent will end up at location 68 only if it starts at locations 17, 34, 85, or 68. Similarly, for the fixed point located at 0, only locations 51, 102, and 0, will get you to that equilibrium. Thus, only a small fraction of the total locations will get you to the fixed-point, implying that such equilibria may be hard to acquire and easy to lose.

In the model above the traditional notions of fixed-point equilibrium analysis may not be that useful. While the model is guaranteed to fall into an equilibrium, the cyclic equilibria have much larger basins of attraction than the fixed-points ones. The combination of multiple equilibria, small basins of attraction for the fixed points, and instability, suggest that traditional equilibrium analysis may have little predictive value.

8.2 Tipping

A classic "computational" social model is Schelling's (1978) work on racial segregation. In this model, the world consists of agents living on the squares of a checkerboard. Initially, agents are randomly sprinkled across the board (with one agent per square and some squares left unoccupied). There are two types of agents and each agent has a threshold for living with those of the opposite type. When the number of opposite-type neighbors exceeds an agent's threshold, it relocates to a more congenial spot on the board. One of models most striking results is that even when agents are tolerant of the opposite type, segregation is still likely to emerge. Segregation arises due to the phenomenon of tipping, whereby the early movements of even a few agents can create the incentive for other agents to move. This cascade of movement dies out only when the system becomes highly segregated.

We can create a one-dimensional version of Schelling's segregation model. Assume that two types of agents live on a circular lattice and that each agent's neighborhood consists of all locations k steps away. Each site on the lattice is initialized with either an agent of one of the two types or left empty (with equal probability among the three possibilities). Following Schelling, each type of agent has the identical threshold value, but these values may differ between the two types. When an agent is given a chance to act, it first calculates the percentage of its neighbors that are of the same type. If this value is less than its threshold, the agent moves to a random, vacant location, otherwise it remains in place. We assume a location-based, asynchronous updating mechanism—in each period we loop through the sites in spatial order and allow the occupant of the site, if any, to act.

To analyze our model we consider both tipping and segregation. We define tipping as the process in which the movement of agents causes cascades of further movement. We operationalize tipping by comparing the number of agents that move before the model reaches an equilibrium to a measure of the potential for agent movement (either the number of agents who want to move initially or the number that actually move during the first iteration[1]). Increased tipping is associated with higher values of the ratio of the number that actually move to the potential number of movers. Segregation occurs when the two types of agents fail to associate with one another. We measure

[1]Since agents are acting asynchronously, the number that actually move during the first iteration may differ from the number that initially wishes to move.

	40/40	**30/50**	**31/49**
Agents Initially Wanting to Move	19.2	26.0	19.8
Agents Who Move During First Iteration	21.9	30.1	23.2
Agents Who Move During First 100 Iterations	42.5	110.8	58.8
Final Blocks	5.8	5.0	5.8

Table 8.3: Results from a one-dimension tipping model. With $N = 100$, $k = 4$, and 500 trials.

segregation by counting the number of times two adjacent neighbors (ignoring any intervening spaces) are of different types. Lower values of this measure are associated with greater segregation in the system.

We run the model on a lattice of size 100 with a neighborhood size of four. The data is averaged over 500 separate trials.

In the initial model we give each type of agent a threshold of 40% (the 40/40 column in Table 8.3), whereby the agent will move if fewer than 40% of its neighbors are the same type. Under this condition there is not that much tipping—the number of agents who eventually move is only about twice that of those who wanted to move initially. Despite the low amount of tipping, segregation does increase a lot, with the number of opposite-type adjoining neighbors decreasing from around 30 to 5.8.

These results are a bit puzzling, as the amount of tipping is well below that observed by Schelling. There are a number of differences in our model that might account for this discrepancy: we use only one versus two dimensions, our updating rule is slightly different, and threshold choices may be important. Additional experiments suggest that while the dimensionality and updating can make a difference, the more interesting area of investigation is in the choice of thresholds.

One of the major differences between the parameters of this model and Schelling's is that we used symmetric thresholds. Schelling relied on asymmetric thresholds for the two types of agents.[2] The 30/50 column in Table 8.3 gives the result when the two types of agents have 30% and 50% thresholds respectively. Under these thresholds, tipping becomes much more pronounced, with around four times the number of agents who initially wished to move

[2]In Schelling's model, dimes needed one half of their neighbors to be dimes and pennies needed one third of their neighbors to be pennies to be content.

eventually moving. Although the amount of tipping is much higher, the final segregation level is roughly equal to that seen under the symmetric threshold.

Given the discrete nature of the neighborhood, there is the potential for discontinuities to arise in the the model. These discontinuities can become particularly important under asymmetric thresholds. Since agents base their actions on the *percentage* of neighbors, the critical thresholds depend on the number of neighbors that surround the agent. With only two neighbors, you can have only 0%, 50%, or 100% of the same neighbor type; with three neighbors, you can have 0%, 33%, 67%, or 100%; with four, you can have 0%, 25%, 50%, 75%, and 100%; and so on. These discontinuities imply that the behavioral differences induced by various thresholds may be slight. For example, with anywhere between two and eight neighbors, a 40% and 50% threshold only imply different behavior in two out of the forty-two possible configurations of the world. To demonstrate such effects, we ran the model with asymmetric thresholds of 31% and 49%. As shown in Table 8.3 these slight changes in threshold values cause the system to resemble the symmetric 40% thresholds case rather than the parametrically much closer asymmetric case.

While the data indicate that the final segregation level is similar across the parameters, the dynamics are very different. With symmetric thresholds, the system rapidly converges to a segregated outcome as agents of both types find acceptable neighborhoods. With asymmetric thresholds, segregation takes much longer. In these systems, almost all of the movement is by the agents with the higher (less tolerant) threshold. On average, 23 of the 29 agents that move in the first period have the less-tolerant threshold. This behavioral discrepancy continues on into later periods as well. This differential movement explains the puzzle of why there is so much more movement, yet less segregation, in the asymmetric system: with only one type of agent moving, it takes longer for the system to segregate.

Finally, we can further refine our analysis of movement by tracing the impact of a given move on subsequent moves. One possibility here is that when an agent moves, they are still unhappy with their new location and move again. This type of movement is a common occurrence with a 50% threshold. Another possibility is that the agent's move impacts either their old neighborhood (where neighbors of the same type now want to move) or their new neighborhood (where agents of the opposite type now want to move). We find that in the asymmetric case, there tend to be far more of the former type of tips, that is, agents in the old neighborhood decide to move when like-type neighbors leave.

8.3 City Formation

In the prior models, each location could hold at most one agent. Here we relax this assumption and allow multiple agents to reside simultaneously at the same site. A nice application of such a model is to the formation of cities. The interrelationship between economics and geography has been of interests to scholars since von Thunen's work in the early 1800's. More recently, the impact of cites on economic growth and development has become a central topic prompted by the writings of Jacobs (1984) and others.

Most theorizing about city formation has emphasized two key aspects of the problem: space and agglomeration. A city's spatial location determines many of its resource possibilities, ranging from the price of natural resources to the ease of transportation. Furthermore, as agents agglomerate within a city, both positive and negative externalities accrue. Agglomeration effects are often subject to nonlinear feedbacks and chance—small events can have big impacts on the eventual outcome.[3] Given the emphasis of these models on spatial and nonlinear feedback mechanisms, computational tools are a natural means by which to investigate this topic.

To model the formation of cities we construct a one-dimensional world in which each site is capable of holding an unlimited number of agents. We begin by assuming that these locations are arranged in a line. A city in such a model will be a site that is occupied by a relatively large number of agents. Agents move in response to economic and social variables associated with each site, such as wage rates, living conditions, and commodity prices. Rather than separately model each of these variables, we collapse them into two generic categories that indirectly serve as proxies for key elements driving locational choice. The two proxies we use are the agent's home population, that is the number of other agents located at the agent's current site, and the average distance of the agent to all of the other agents, given by $\sum_i p_i d_i / \sum_i p_i$, where p_i is the population at location i and d_i is the distance from the agent's current location to site i.

[3]While examples abound, the first president of the Santa Fe Institute, George Cowan, notes that Albuquerque bankers rejected a loan application from a young Bill Gates during the early years of the company that eventually became Microsoft. Gates received an alternative loan from his father contingent on him returning to Seattle.