1.

By definition, the eigenspace E_2 corresponding to the eigenvalue 2 is the null space of the matrix $A - 2I$. That is, we have

$$E_2 = \mathcal{N}(A - 2I).$$

We reduce the matrix $A - 2I$ by elementary row operations as follows.

$$A - 2I = \begin{bmatrix} -1 & 2 & 1 \\ -1 & 2 & 1 \\ 2 & -4 & -2 \end{bmatrix}$$

$$\xrightarrow[R_3+2R_1]{R_2-R_1} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{-R_1} \begin{bmatrix} 1 & -2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, the solutions $\mathbf{x}$ of $(A - 2I)\mathbf{x} = \mathbf{0}$ satisfy $x_1 = 2x_2 + x_3$.
Thus, the null space $\mathcal{N}(A - 2I)$ consists of vectors

$$\mathbf{x} = \begin{bmatrix} 2x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

for any scalars x_2, x_3.

Hence we have

$$E_2 = \mathcal{N}(A - 2I) = \operatorname{Span}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right).$$

It is straightforward to see that the vectors $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent, hence they form a basis of E_2.

Thus, a basis of E_2 is

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

2.

Since $\mathbf{u}$ is an eigenvector corresponding to the eigenvalue 2, we have

$$A\mathbf{u} = 2\mathbf{u}.$$

Similarly, we have

$$A\mathbf{v} = -\mathbf{v}.$$

From these, we have

$$A^5\mathbf{u} = 2^5\mathbf{u} \text{ and } A\mathbf{v} = (-1)^5\mathbf{v}.$$

To compute $A^5\mathbf{w}$, we first need to express $\mathbf{w}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$. Thus, we need to find scalars c_1, c_2 such that

$$\mathbf{w} = c_1\mathbf{u} + c_2\mathbf{v}.$$

By inspection, we have

$$\begin{bmatrix} 7 \\ 2 \\ -3 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + 2\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix},$$

and thus we obtain $c_1 = 3$ and $c_2 = 2$,

We compute $A^5\mathbf{w}$ as follows:

$$\begin{aligned} A^5\mathbf{w} &= A^5(3\mathbf{u} + 2\mathbf{v}) \\ &= 3A^5\mathbf{u} + 2A^5\mathbf{v} \\ &= 3 \cdot 2^5\mathbf{u} + 2 \cdot (-1)^5\mathbf{v} \\ &= 96\mathbf{u} - 2\mathbf{v} \\ &= 96\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \\ &= \begin{bmatrix} 92 \\ -2 \\ -96 \end{bmatrix}. \end{aligned}$$

Therefore, the result is

$$A^5\mathbf{w} = \begin{bmatrix} 92 \\ -2 \\ -96 \end{bmatrix}.$$

3.

To determine eigenvalues of A, we compute the determinant of $A - \lambda I$.
We have

$$\begin{aligned} \det(A - \lambda I) &= \begin{vmatrix} 3-\lambda & -2 \\ 6 & -4-\lambda \end{vmatrix} \\ &= (3-\lambda)(-4-\lambda) + 12 \\ &= \lambda^2 + \lambda = \lambda(\lambda + 1). \end{aligned}$$

The eigenvalues are solutions of $\det(A - \lambda I) = 0$, hence eigenvalues of A are $0, -1$.

Next, we find the eigenvector corresponding to the eigenvalue $\lambda = 0$.
Eigenvectors $\mathbf{x}$ are nonzero solutions of $(A - 0I)\mathbf{x} = \mathbf{0}$.
Thus, we solve $A\mathbf{x} = \mathbf{0}$. The augmented matrix of the system is

$$\left[\begin{array}{cc|c} 3 & -2 & 0 \\ 6 & -4 & 0 \end{array}\right] \xrightarrow{R_2-2R_1} \left[\begin{array}{cc|c} 3 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_1} \left[\begin{array}{cc|c} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

Thus, if $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is a solution, then $x_1 = \frac{2}{3}x_2$, hence

$$\mathbf{x} = x_2 \begin{bmatrix} 2/3 \\ 1 \end{bmatrix}$$

is an eigenvector corresponding to $\lambda = 0$ for any nonzero scalar x_2.

Finally, we find the eigenvectors corresponding to the eigenvalue $\lambda = -1$.
In this case we need to solve $(A - (-1)I)\mathbf{x} = \mathbf{0}$.
The augmented matrix is

$$\left[\begin{array}{cc|c} 4 & -2 & 0 \\ 6 & -3 & 0 \end{array}\right] \xrightarrow[\frac{1}{3}R_2]{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right] \xrightarrow{R_2-R_1} \left[\begin{array}{cc|c} 2 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

It follows that the eigenvectors associated to $\lambda = -1$ are

$$\mathbf{x} = x_2 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix}$$

for any nonzero scalar x_2.

4.

If λ is an eigenvalue of A, then λ satisfies

$$\begin{aligned} 0 = \det(A - \lambda I) &= \det\left(\begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}\right) \\ &= \begin{vmatrix} 2-\lambda & -3 & 0 \\ 2 & -5-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (3-\lambda)\begin{vmatrix} 2-\lambda & -3 \\ 2 & -5-\lambda \end{vmatrix} \\ &= (3-\lambda)\left[(2-\lambda)(-5-\lambda)+6\right] = (3-\lambda)(-10+5\lambda-2\lambda+\lambda^2+6) \\ &= (3-\lambda)(\lambda^2+3\lambda-4) = -(\lambda-3)(\lambda+4)(\lambda-1). \end{aligned}$$

In the above calculation, we calculated the determinant of the 3×3 matrix by expanding along the third row. From the above equation, we can conclude that the eigenvalues of A are $\lambda = 3, 1, -4$.

We will first find the eigenvectors corresponding to the eigenvalue $\lambda = 3$. Any such eigenvector $\mathbf{x}$ must satisfy

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (A - 3I)\mathbf{x} = \left(\begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix} - 3 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= \begin{bmatrix} -1 & -3 & 0 \\ 2 & -8 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The augmented matrix for this system is given by

$$\left[\begin{array}{ccc|c} -1 & -3 & 0 & 0 \\ 2 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_2 + 2R_1} \left[\begin{array}{ccc|c} -1 & -3 & 0 & 0 \\ 0 & -14 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow[-\frac{1}{14}R_2]{-R_1}$$
$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \xrightarrow{R_1 - 3R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Therefore, the solution is given by $x_1 = 0$ and $x_2 = 0$. Since no other restrictions are given on x_3, we can conclude that any eigenvector $\mathbf{x}$ of A corresponding to $\lambda = 3$ must be of the form

$$\mathbf{x} = x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where $x_3 \neq 0$.

Next, any eigenvector $\mathbf{x}$ corresponding to the eigenvalue $\lambda = 1$ must satisfy

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (A - 3I)\mathbf{x} = \left(\begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & -3 & 0 \\ 2 & -6 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

The augmented matrix for this system is given by

$$\left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 2 & -6 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array} \right] \xrightarrow[\frac{1}{2}R_3]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array} \right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & -3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right].$$

Therefore, the solution to this system is given by $x_1 - 3x_2 = 0$, i.e. $x_1 = 3x_2$, and $x_3 = 0$. Therefore, any eigenvector $\mathbf{x}$ of A corresponding to $\lambda = 1$ must be of the form

$$\mathbf{x} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$$

where $x_2 \neq 0$.

Finally, any eigenvector $\mathbf{x}$ corresponding to the eigenvalue $\lambda = -4$ must satisfy

$$\begin{aligned} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (A + 4I)\mathbf{x} &= \left(\begin{bmatrix} 2 & -3 & 0 \\ 2 & -5 & 0 \\ 0 & 0 & 3 \end{bmatrix} + 4 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -3 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}. \end{aligned}$$

The augmented matrix for this system is given by

$$\left[\begin{array}{ccc|c} 6 & -3 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right] \xrightarrow[\frac{1}{7}R_3]{R_2 - \frac{1}{6}R_1} \left[\begin{array}{ccc|c} 6 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow[R_2 \leftrightarrow R_3]{\frac{1}{6}R_1} \left[\begin{array}{ccc|c} 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Therefore, the solution to this system satifies $x_3 = 0$ and $x_1 - (1/2)x_2 = 0$, the latter of which reduces to $x_1 = (1/2)x_2$. Thus any eigenvalue $\mathbf{x}$ of A corresponding to $\lambda = -4$ must be of the form

$$\mathbf{x} = x_2 \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$$

where $x_2 \neq 0$.