Physics 106a 24 October, 2013

Lecture 8: Central Forces – Scattering States

In this lecture I discuss scattering problems, particularly *Rutherford scattering* in a 1/r potential.

Hyperbolic orbits

These exist for attractive potentials $V(r) = -k/r$, and also for repulsive ones $V(r) = k/r$. Note in both cases I take k to be positive and I define $p = l^2/\mu k > 0$. The equations are

$$\begin{array}{lll} \text{Attractive, } k > 0 & \frac{1}{r} = \frac{1}{p}\left[1 + \epsilon\cos\phi\right] & \text{for } \cos\phi > -\frac{1}{\epsilon} \\ \text{Repulsive, } k < 0 & \frac{1}{r} = \frac{1}{p}\left[-1 + \epsilon\cos\phi\right] & \text{for } \cos\phi > \frac{1}{\epsilon} \end{array} \tag{1}$$

with $\epsilon > 1$, $E > 0$ (solutions for the repulsive case only exist for this range). Physical solutions $r > 0$ only exist for the range of angles specified. The geometry of the orbits is shown in Fig. 1 (using the same value of k in the two cases).

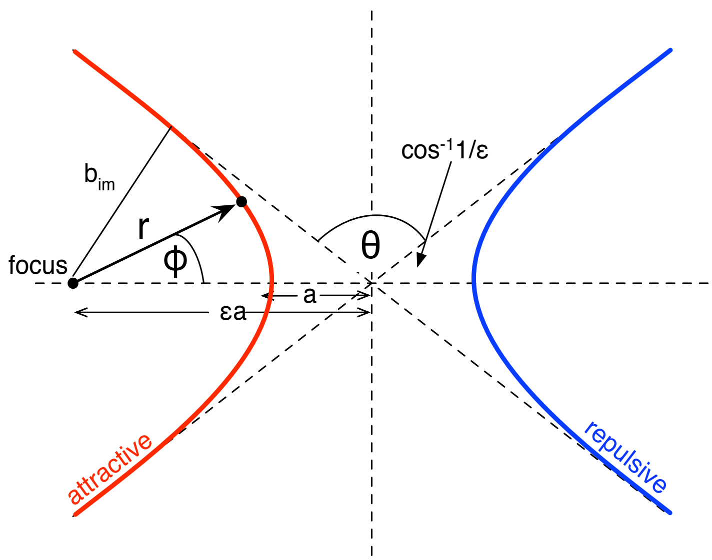

Figure 1: Hyperbolic orbits for attractive and repulsive $1/r$ potentials.

Rutherford scattering

The hyperbolic orbits for the repulsive case allow us to investigate the scattering of alpha particles (charge $2|e|$) off a nucleus (charge $Z|e|$). We will suppose the target nucleus is heavy, so that its recoil is negligible and we can consider scattering off a fixed potential. In a scattering problem we have a parallel beam of incident particles all with the same energy. The scattering angle θ is determined by how close the alpha particle comes to the nucleus, which depends on the *impact parameter* $b_{\rm im}$ for that particle (see Fig. 2). We are usually interested in the rate of scattering of particles into a scattering angle between θ and $\theta + d\theta$, which corresponds to a solid angle $d\Omega = 2\pi \sin\theta d\theta$. These particles are ones with an impact parameter between $b_{\rm im}$ and $b_{\rm im} + db_{\rm im}$, i.e. the particles hitting an area $d\sigma = 2\pi b_{\rm im} db_{\rm im}$. The scattering is described

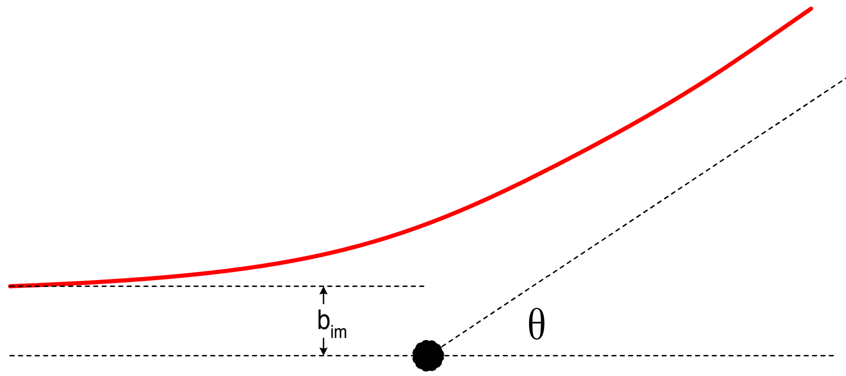

Figure 2: Scattering geometry

by the *differential scattering cross-section*

$$\frac{d\sigma}{d\Omega} = \left| \frac{2\pi b_{\rm im} db_{\rm im}}{2\pi \sin\theta d\theta} \right| = \frac{b_{\rm im}}{\sin\theta} \left| \frac{db_{\rm im}}{d\theta} \right| \tag{2}$$

(we put the mod in since $d\sigma/d\Omega$ is defined to be postive) so that all we need to know from the hyperbolic orbit calculation is $\theta(b_{\rm im})$. This is the topic of Hand and Finch problem 4-28. Here is the solution.

We use the results for scattering off a 1/r potential. The trick is to relate the parameters of the orbit calculation, in particular the energy E and angular momentum l, to those of the scattering problem. Fig. 2:

$$E = \tfrac{1}{2}\mu v_\infty^2, \qquad l = \mu v_\infty b_{\rm im}, \tag{3}$$

with v_∞ the speed of the particles in the incident beam. Now use Eqs. (1) for the repulsive case with $p = l^2/\mu k$, $k = Z_{Au} Z_\alpha e^2$. Also, from Lecture 7, the energy is given in terms of the angular momentum l and the eccentricity ϵ by

$$E = \frac{l^2}{2\mu p^2}(\epsilon^2 - 1) = \frac{1}{2}\frac{k^2}{l^2/\mu}(\epsilon^2 - 1) \tag{4}$$

where the second expression is given by substituting for p. The scattering angle θ is given in terms of ϵ by

$$\theta = \pi - 2\phi_\infty \quad \text{where} \quad \phi_\infty = \phi(r \to \infty) = \cos^{-1}\left(\frac{1}{\epsilon}\right) \tag{5}$$

again using Eq. (1) (see also Fig. 1). This gives $\sin(\theta/2) = 1/\epsilon$, so that

$$E = \frac{1}{2}\frac{k^2}{l^2/\mu}[\operatorname{cosec}^2(\theta/2) - 1] = \frac{1}{2}\frac{k^2}{l^2/\mu}\cot^2(\theta/2). \tag{6}$$

Equations (3) give

$$l^2/\mu = \mu v_\infty^2 b_{\rm im}^2 = 2E b_{\rm im}^2. \tag{7}$$

Substituting into Eq. (6), rearranging, and taking the square root gives

$$b_{\rm im} = \frac{|k|}{2E}\cot\left(\frac{\theta}{2}\right), \tag{8}$$

the relationship $b_{\rm im}(\theta)$ we need. Using Eq. (2) gives the Rutherford differential scattering cross-section

$$\frac{d\sigma}{d\Omega} = \left(\frac{k}{4E}\right)^2 \frac{1}{\sin^4\frac{\theta}{2}}. \tag{9}$$

Finite mass target particle

If the target particle is not infinitely heavy, it will recoil in the scattering, and the scattering problem is more complicated: for example the energy and speed of the outgoing scattered particle (in the lab frame) will not be the same as the incoming values. The scattering problem separates into two parts: the *dynamics* – the probability of scattering at some angle which depends on solving for the particle trajectories in the interaction potential as we have just done; and the *kinematics* – how the outgoing energy depends on the scattering angle, what is the momentum of the outgoing target particle etc., which are determined simply by conservation of energy and momentum. The scattering with a finite mass target is most easily addressed by transforming to the center of mass frame, solving for the dynamics there, and then transforming back to the original "laboratory" frame.

Kinematics

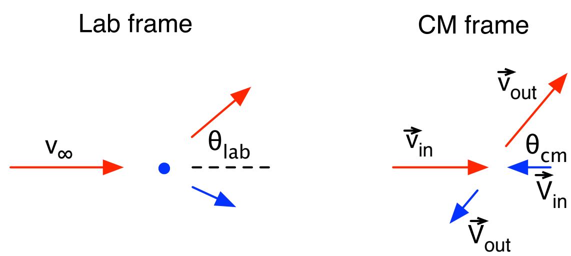

Figure 3: Scattering geometry in the lab and center of mass frames. Only the incoming and outgoing velocity vectors (not the details of the trajectory etc.) are depicted. In the center of mass frame the incoming velocities are colinear, as are the outgoing velocities. Also the outgoing speed is the same as the incoming speed for that particle.

In the center of mass frame the scattering geometry is simple. Calling the velocity of the mass m particle $\vec{v}$ and that of the mass M particle $\vec{V}$ for elastic scattering $|\vec{v}_{\text{out}}| = |\vec{v}_{\text{in}}|$ and $|\vec{V}_{\text{out}}| = |\vec{V}_{\text{in}}|$. Also $m\vec{v}_{\text{in,out}} = -M\vec{V}_{\text{in,out}}$ since the total momentum is zero. If in the laboratory frame the incoming velocity of the mass m particles is $v_\infty \hat{x}$, the center of mass velocity is

$$\vec{V}_{\text{cm}} = \frac{m}{M+m} v_\infty \hat{x}, \tag{10}$$

and the incoming velocity of the mass m in the center of mass frame is

$$\vec{v}_{\text{in}}|_{\text{cm}} = v_\infty \hat{x} - \vec{V}_{\text{cm}} = \frac{M}{M+m} v_\infty \hat{x}. \tag{11}$$

Since the speed is the same after the scattering

$$\vec{v}_{\text{out}}|_{\text{cm}} = \frac{M}{M+m} v_\infty (\cos\theta_{\text{cm}} \hat{x} + \sin\theta_{\text{cm}} \hat{y}). \tag{12}$$

Transforming back to the laboratory frame by adding $\vec{V}_{\text{cm}}$ to all the velocities

$$\vec{v}_{\text{out}}|_{\text{lab}} = \frac{M}{M+m} v_\infty [(\cos\theta_{\text{cm}} + m/M)\hat{x} + \sin\theta_{\text{cm}} \hat{y}]. \tag{13}$$

This gives the scattering angle θ_{lab} in the laboratory frame

$$\tan\theta_{\text{lab}} = \frac{\sin\theta_{\text{cm}}}{\cos\theta_{\text{cm}} + m/M}. \tag{14}$$

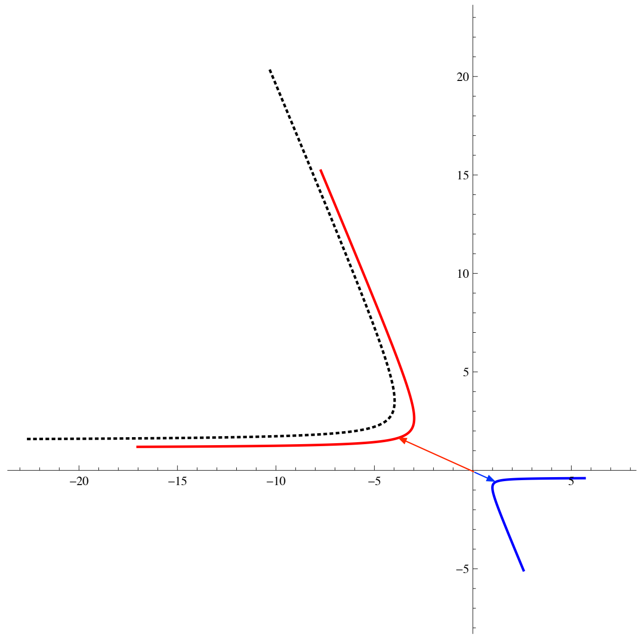

Figure 4: Scattering hyperbolas for a particle of mass m off one with mass M with $M/m = 3$: the dashed curve is the hyperbola traced out by the difference coordinate $\vec{r} = \vec{r}_m - \vec{r}_M$ given by the orbit calculation. I used $p = 1$ and an eccentricity $\epsilon = 1.2$ which fixes the scattering angle as 146°. The red curve is the trajectory of particle m given by $\vec{r}_m(t)$ and the blue curve is the trajectory of particle M given by $\vec{r}_M(t)$, both in the center of mass frame. The foci of the hyperbolas are all at the origin. The curves are plotted over the same time interval, and the heavy mass moves a smaller distance than the light mass. The red and blue arrows are $\vec{r}_m, \vec{r}_M$ at some particular time.

Now I calculate the scattering dynamics in the center of mass frame. The particle trajectories in the center of mass frame are given by

$$\vec{r}_m(t) = \frac{M}{M+m}\vec{r}(t), \quad \vec{r}_M(t) = -\frac{m}{M+m}\vec{r}(t), \tag{15}$$

where $\vec{r} = \vec{r}_m - \vec{r}_M$ is the difference coordinate used in the orbit calculation. These are just scaled (and flipped in the later case) versions of the hyperbola traced out by $\vec{r}(t)$, as shown in Fig. 4. This means that the scattering angle $\theta_{\rm cm}$ of particle m in the center of mass frame is the same as the scattering angle of the reduced mass μ scattering off a stationary center calculated in the previous section.

To calculate the differential scattering cross-section in the laboratory frame from the differential scattering cross section calculated for the reduced mass particle in the previous section, use the fact that the incoming fluxes are the same (both incoming speeds are $\vec{v}_\infty$) and $2\pi \sin\theta \, d\sigma/d\Omega \, d\theta$ counts the same outcoming particles when evaluated in the two frames. Thus

$$\left.\frac{d\sigma}{d\Omega}\right|_{\rm lab} = \frac{\sin\theta_{\rm cm}}{\sin\theta_{\rm lab}}\frac{d\theta_{\rm cm}}{d\theta_{\rm lab}} \left.\frac{d\sigma}{d\Omega}\right|_{\rm cm}, \tag{16}$$

with $\theta_{\rm cm}, \theta_{\rm lab}$ related by Eq. (14).

See the discussion in §3.11 of Goldstein, Poole and Safko for more details.

Michael Cross, October 21, 2013