Physics 106a — Classical Mechanics

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Lecture 15

Rigid Body Dynamics

Outline

- Equations of motion for a rigid body
- Rotational motion
- Moment of inertia tensor
- Torque free motion of symmetric top
  - Football
  - Euler equations
  - Chandler wobble

Angular momentum and moment of inertia: summary

Starting point: Conservation of angular momentum

$$\frac{d\vec{L}_{\text{tot}}}{dt} = \vec{N}, \qquad \text{with } \vec{L}_{\text{tot}}, \vec{N} \text{calculated with respect to fixed point}$$

We will derive the equations for rotational motion

$$\frac{d\vec{L}}{dt} = \vec{N} \text{ with } \vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} \quad \text{and} \quad T_{\text{rot}} = \frac{1}{2}\vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega}$$

1. for pure rotation and torque about a stationary point
2. for rotation and torque about the center of mass

$\overleftrightarrow{I}$ is the moment of inertia tensor calculated about the appropriate point

$$I_{\alpha\beta} = \sum_i m_i (r_i^2 \delta_{\alpha\beta} - r_{i,\alpha} r_{i,\beta})$$

Pure rotation about stationary point

$$\frac{d\vec{L}_{\text{tot}}}{dt} = \vec{N} \quad \text{with} \quad \vec{L}_{\text{tot}} = \sum_i m_i \vec{r}_{s,i} \times \vec{v}_{s,i}$$

For pure rotation about the origin we have $\vec{v}_{s,i} = \vec{\omega} \times \vec{r}_{s,i}$, so that

$$\vec{L}_{\text{tot}} = \sum_i m_i \vec{r}_{s,i} \times (\vec{\omega} \times \vec{r}_{s,i}) = \overleftrightarrow{I_s} \cdot \vec{\omega} \quad \text{or} \quad L_{\text{tot},\alpha} = I_{s,\alpha\beta}\, \omega_\beta$$

$\overleftrightarrow{I_s}$ — moment of inertia tensor with respect to the origin, with components

$$I_{s,\alpha\beta} = \sum_i m_i (r_{s,i}^2 \delta_{\alpha\beta} - r_{s,i,\alpha} r_{s,i,\beta})$$

Angular momentum about center of mass

For reference point K in rigid body so that $\vec{r}_{s,i} = \vec{R} + \vec{r}_i$ and $\vec{v}_{s,i} = \vec{V} + \vec{\omega} \times \vec{r}_i$

$$\vec{L}_{\text{tot}} = \sum_i m_i \vec{r}_{s,i} \times \vec{v}_{s,i} = \sum_i m_i (\vec{R} + \vec{r}_i) \times (\vec{V} + \vec{\omega} \times \vec{r}_i)$$

This simplifies if we take reference point at center of mass so that $\sum_i m_i \vec{r}_i = 0$

$$\vec{L}_{\text{tot}} = \vec{R}_{\text{cm}} \times \vec{P} + \vec{L}$$

where $\vec{P}$ is the total linear momentum and $\vec{L}$ is the *intrinsic angular momentum*

$$\vec{L} = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \overleftrightarrow{I} \cdot \vec{\omega}$$

with $\overleftrightarrow{I}$ the moment of inertia with respect to the center of mass

$$I_{\alpha\beta} = \sum_i m_i (r_i^2 \delta_{\alpha\beta} - r_{i,\alpha} r_{i,\beta})$$

Rotation about center of mass + translation

The equation of motion of the angular momentum

$$\frac{d\vec{L}_{\text{tot}}}{dt} = \sum_i \vec{r}_{s,i} \times \vec{F}_i$$

can be rewritten

$$\frac{d(\vec{R}_{\text{cm}} \times \vec{P} + \vec{L})}{dt} = \sum_i (\vec{R}_{\text{cm}} + \vec{r}_i) \times \vec{F}_i$$

giving

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i$$

We may use the equation *rate of change of angular momentum equals torque* calculated about the *center of mass*, even if moving.

Kinetic energy

Kinetic energy $T = \frac{1}{2}\sum_i m_i v_{s,i}^2$ with $\vec{v}_{s,i} = \vec{V} + \vec{\omega} \times r_i$ for solid body motion

Simplifies for two choices of reference point:

1. Reference point is a stationary point

$$T = \frac{1}{2}\sum_i m_i (\vec{\omega} \times \vec{r}_{s,i})^2$$

2. Reference point is the center of mass

$$T = \frac{1}{2} M \vec{V}_{\mathrm{cm}}^2 + \frac{1}{2}\sum_i m_i (\vec{\omega} \times \vec{r}_i)^2$$

The rotational kinetic energy is

$$T_{\mathrm{rot}} = \frac{1}{2}\sum_i m_i (\vec{\omega} \times \vec{r}_i)^2 = \frac{1}{2}\vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega} = \frac{1}{2} I_{\alpha\beta}\omega_\alpha\omega_\beta$$

Summary

Equations for rotational motion

$$\frac{d\vec{L}}{dt} = \vec{N} \text{ with } \vec{L} = \overleftrightarrow{I} \cdot \vec{\omega} \quad \text{and} \quad T_{\text{rot}} = \frac{1}{2}\vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega}$$

1. for pure rotation and torque about a stationary point
2. for rotation and torque about the center of mass

$\overleftrightarrow{I}$ is the moment of inertia tensor calculated about the appropriate point

$$I_{\alpha\beta} = \sum_i m_i (r_i^2 \delta_{\alpha\beta} - r_{i,\alpha} r_{i,\beta})$$

Moment of inertia

The moment of inertia is given by

$$I_{\alpha\beta} = \sum_i m_i (r_i^2 \delta_{\alpha\beta} - r_{i,\alpha} r_{i,\beta})$$

Matrix form

$$I = \begin{pmatrix} \sum_i m_i (y_i^2 + z_i^2) & -\sum_i m_i x_i y_i & -\sum_i m_i x_i z_i \\ -\sum_i m_i x_i y_i & \sum_i m_i (z_i^2 + x_i^2) & -\sum_i m_i y_i z_i \\ -\sum_i m_i x_i z_i & -\sum_i m_i y_i z_i & \sum_i m_i (x_i^2 + y_i^2) \end{pmatrix}$$

For example, the 33 component is

$$I_{33} = \sum_i m_i (x_i^2 + y_i^2) \Rightarrow \int_{\mathrm{Vol}} \rho(x, y, z)(x^2 + y^2)\, dx\, dy\, dz$$

Moment of inertia is: number $\times$ mass $\times$ length^2

Moment of inertia is a tensor

Under rotation U of the coordinate axes the components of I transform as

$$I'_{\mu\nu} = U_{\mu\alpha} U_{\nu\beta} I_{\alpha\beta}.$$

Thinking of $I_{\alpha\beta}$ as a 3×3 matrix, this can be written

$$I' = U I \tilde{U} \qquad \text{or} \qquad I = \tilde{U} I' U \ .$$

Since I is a symmetric matrix, we can find some U such that I is diagonal

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

The particular choice of coordinate axes giving this diagonal form are called the *principal axes* of the solid body.

Displaced axis theorem

For a reference point shifted by $\vec{a}$ from the center of mass

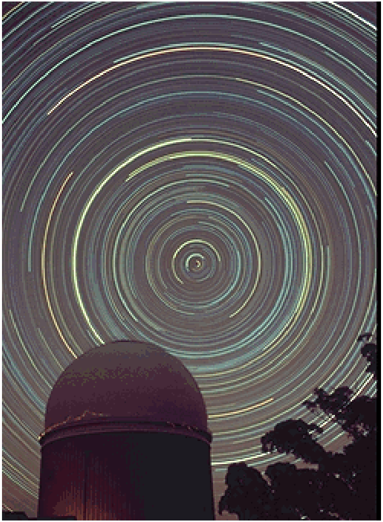
$$I_{\vec{a},\alpha\beta} = I_{\mathrm{cm},\alpha\beta} + M(a^2\delta_{\alpha\beta} - a_\alpha a_\beta)$$

with M the total mass

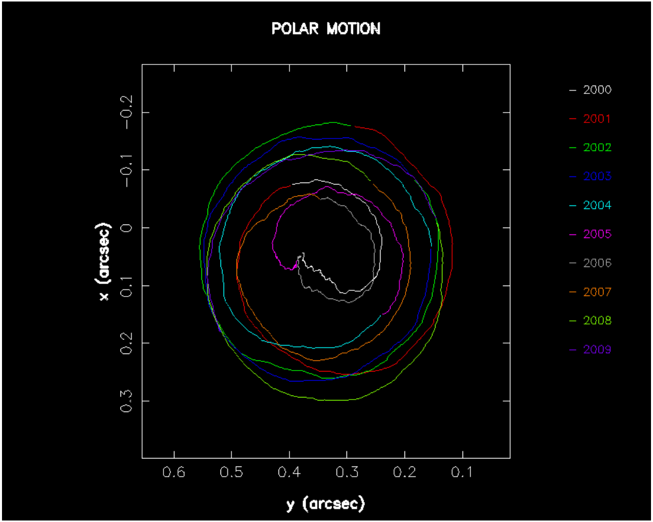
$$\text{e.g. } I_{\vec{a},33} = I_{\mathrm{cm},33} + M(a_x^2 + a_y^2)$$

This is often useful in the calculation of I, and relating the moment of inertia about a fixed pivot point to the moment of inertia about the center of mass.

Celestial pole

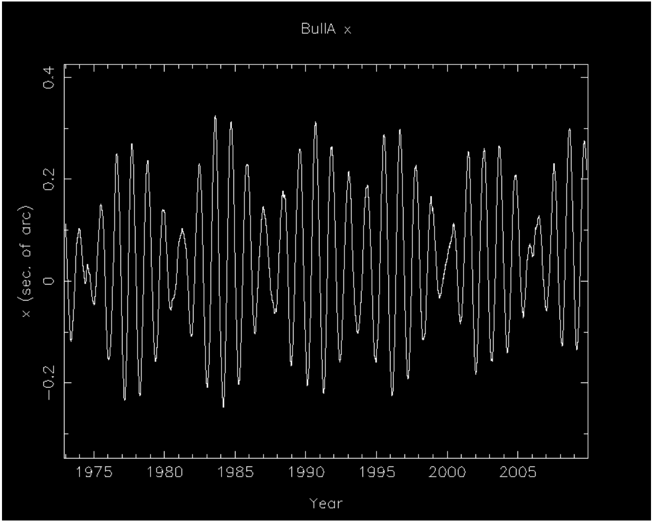


Chandler Wobble: x-y plot

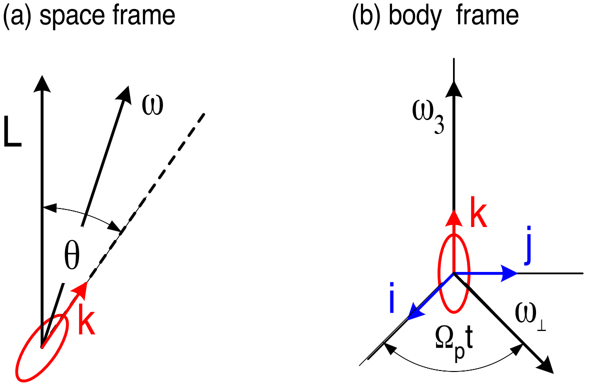


From: http://maia.usno.navy.mil/ (0.5 arcsec ≡ 15m)

Chandler Wobble: x(t) plot



Free symmetric top: space and body frames



$$\vec{\omega}_p = \frac{\vec{L}}{I_1}$$

$$\vec{\Omega}_p = \left(\frac{I_3}{I_1} - 1\right)\omega_3 \hat{k}$$

Free symmetric top: "rolling cones" picture

Hand and Finch Fig. 8.7