Contents

Contents

Contents

Contents

Lecture 1:
Introduction to Course
Basics of Electrostatics I: Electric Force and Field, Gauss's Law, Dirac Delta Function, $\vec{\nabla} \times \vec{E} = 0$

Date Revised: 2014/02/11 12:00

Date Given: 2014/02/11

Section 1

Introduction to Course

Course Material

This is a course on electrodynamics. It will review the basic material you learned in Ph1bc but will go beyond in both mathematical sophistication (boudary-value problems, multipole expansion) as well as in content (waves in matter, guided waves, radiation, electrodynamics and relativity).

The course will primarily use and follow *Introduction to Electrodynamics* by Griffiths. Supplementary material is drawn from Jackson and from Heald & Marion, both on reserve in the library, but the material presented here will be self-contained.

Course Material (cont.)

Prerequisites

Physics:

- Electricity and Magnetism: While Ph1bc is a formal prerequisite for the course, we will develop the material from scratch. However, review material will be covered quickly and a basic familiarity with the concepts will be assumed.
- Classical mechanics: Generally, mechanics at the level of Ph1a is sufficient for this course.

Course Material (cont.)

Mathematics:

- Chapter 1 of Griffiths except for Sections 1.1.5 ("How Vectors Transform") and 1.5 ("The Dirac Delta Function"). We will review some of prerequisite material as needed.
- Solutions to second-order linear ordinary differential equations with constant coefficients (*i.e.*, simple harmonic oscillator).
- Over the course, we will develop the following more sophisticated concepts:
  - Dirac Delta function.
  - Separation of variables to reduce second-order linear partial differential equations to ordinary differential equations.
  - Series solution to solve second-order linear ordinary differential equations with polynomical coefficients
  - Tensor formalism for relativity.
- Key point: Mathematics is the language of physics. You must be competent in above basic mathematical physics in order to understand the material in this course. Intuition is important, but few can succeed in physics without learning to formalize that intuition into mathematical concepts and calculate with it.

Course Material (cont.)

Topics to be covered:

- Review of basic electrostatics — Coulomb's Law; Gauss's Law; electric field, potential, and potential energy; conductors and capacitors.
- Advanced electrostatics — boundary value problems (BVP) for determining potentials and fields; multipole expansion of potential.
- Electrostatics in Matter — polarization, susceptibility, permittivity of matter; electrostatic BVP, energy, and forces in matter
- Magnetostatics — Lorentz force; Biot-Savart Law; Ampère's Law; vector potential; boundary conditions; and multipole expansion of potential.
- Magnetostatics in Matter — magnetization, susceptibility, and permeability of matter; magnetostatic boundary conditions; ferromagnetism.
- Electrodynamics — electromotive force and electromagnetic induction; inductance and energy in magnetic fields; Maxwell's equations in vacuum and in matter; boundary conditions for Maxwell's equations.
- Conservation Laws — Continuity equation; Poynting's Theorem; electrodynamic momentum and energy.
- Electromagnetic Waves — in vacuum, in polarizable/magnetizable matter, in wave guides.
- Potentials and Radiation — potential formulation; fields and potentials of moving point charges; radiated electromagnetic waves; antennas.
- Relativity and Electrodynamics — transformation of fields and field tensor, relativistic potentials.

Notational Deviations from Griffiths

- Griffiths uses boldface notation to indicate vectors and a script $\vec{r}$ to indicate the difference vector $\vec{r} - \vec{r}\,'$. In order to better match what can be written on a chalkboard, and also due to font unavailability, we use $\vec{\ }$ rather than boldface and $\vec{R}$ for the difference vector.
- Griffiths uses $\vec{r}$ to refer to the position of the test charge Q and $\vec{r}\,'$ to refer to the position of the source charge q. This seems unnecessarily confusing. We instead use q and $\vec{r}$ for the test charge and q' and $\vec{r}\,'$ for the source charge.
- Griffiths uses $\delta^3(\vec{r})$ to refer to the delta function in three spatial dimension. We use $\delta(\vec{r})$ for this for reasons that are explained after Equation 2.9.

Section 2
Review of Basics of Electrostatics

Study Guidelines

You have seen all the material in this section before in Ph1b. However, the derivations done there were not as rigorous as they could be because you were simultaneously learning vector calculus. Our goal in this section is to do more rigorous derivations to give you some practice in using the mathematical tools. We won't do any examples in lecture or the notes because they really would just be duplication of Ph1b. But you should be following the examples in Griffiths Chapter 2 and making sure you are comfortable with them.

The Assumed Conditions for Electrostatics

Electrostatics is the study of electric fields, potentials, and forces under two assumptions:

- All electric charges sourcing the electric field are stationary and have been so for a sufficiently long time that all fields are static and thus the electric field can written in terms of the source charges' current positions.
- The source charges are held fixed and cannot react to the fields from any test charges that may be stationary or moving relative to the source charges.

We will see later that, when charges are moving, it takes time for the information about the position to propagate and thus the fields at a given point depend on the configuration of the charges at earlier times.

Coulomb's Law and the Electric Field

Coulomb's Law, Electrostatic Forces, and Superposition

We begin with two empirical facts:

- **Coulomb's Law**: the empirical fact that the force on a test charge q at position $\vec{r}$ due to a source charge q' at $\vec{r}\,'$ is given by Coulomb's Law:

$$\vec{F} = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{q'\,q}{R^2}\,\widehat{R} \qquad \text{with} \quad \vec{R} \equiv \vec{r} - \vec{r}\,' \tag{2.1}$$

  where $\epsilon_0 = 8.85 \times 10^{-12}\ \mathrm{C}^2\ \mathrm{N}^{-1}\ \mathrm{m}^{-2}$. The force points along the line from q' to q as indicated by the sign of the definition of $\vec{R}$. The electric charge is in the units of *Coulombs* (C), which is a fundamental unit that cannot be written in terms of other fundamental units.

  Recall that we use $\vec{}$ rather than boldface to indicate vectors, R where Griffiths uses a script r, and a different convention from Griffiths's for the symbols for the two charges and their position vectors.

- **Superposition**: the empirical fact that Coulomb's Law obeys the principle of superposition: the force on a test charge q at $\vec{r}$ due to N charges $\{q_i'\}$ at positions $\{\vec{r}_i\,'\}$ is obtained by summing the individual vector forces:

$$\vec{F} = \sum_{i=1}^{N} \vec{F}_i = \sum_{i=1}^{N} \frac{1}{4\,\pi\,\epsilon_0}\,\frac{q_i'\,q}{R_i^2}\,\widehat{R}_i \qquad \text{with} \quad \vec{R}_i \equiv \vec{r} - \vec{r}_i\,' \tag{2.2}$$

The Electric Field

Given that any test charge q placed at the position $\vec{r}$ feels the same force, we are motivated to abstract away the test charge and define what we call the **electric field** at that position $\vec{r}$:

$$\vec{E}(\vec{r}) = \frac{\vec{F}}{q} = \begin{cases} \frac{1}{4\,\pi\,\epsilon_0}\,\frac{q'}{R^2}\,\widehat{R} & \text{for a single source charge } q' \text{ at } \vec{r}\,' \\ \sum_{i=1}^{N} \frac{1}{4\,\pi\,\epsilon_0}\,\frac{q_i'}{R_i^2}\,\widehat{R}_i & \text{for } N \text{ source charges } \{q_i'\} \text{ at positions } \{\vec{r}_i\,'\} \end{cases} \tag{2.3}$$

The electric field has units of N/C.

Coulomb's Law for Continuous Charge Distributions

If a charge distribution is continuous, then the natural extension of Coulomb's Law is to *integrate* the electric field or force over the contributions from the infinitesimal charge elements dq:

$$\vec{E}(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \int \frac{1}{R^2}\,\widehat{R}\,dq \tag{2.4}$$

where $\vec{R}$ varies with the location of dq as the integral is performed. dq is admittedly ill-defined. However, before worrying about that, let us note that the integrand is a *vector* and so this integral requires some care. What that means is that we must break up $\widehat{R}$ into its components and individually integrate each component. For example, if we use Cartesian coordinates, then $\widehat{R} = \widehat{x}\left(\widehat{R}\cdot\widehat{x}\right) + \widehat{y}\left(\widehat{R}\cdot\widehat{y}\right) + \widehat{z}\left(\widehat{R}\cdot\widehat{z}\right)$, and, since the Cartesian unit vectors do not depend on the location of the infinitesimal charge dq, we may write the integral out as follows:

$$\vec{E}(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\left[\widehat{x}\int \frac{1}{R^2}\left(\widehat{R}\cdot\widehat{x}\right)\,dq + \widehat{y}\int \frac{1}{R^2}\left(\widehat{R}\cdot\widehat{y}\right)\,dq + \widehat{z}\int \frac{1}{R^2}\left(\widehat{R}\cdot\widehat{z}\right)\,dq\right] \tag{2.5}$$

which is sum of three integrals with *scalar* integrands.

Coulomb's Law and the Electric Field (cont.)

Now, consider some specific charge distributions:

- **line charge distribution**:

$$\vec{E}(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\int_{\mathcal{C}} \frac{dl'\lambda(\vec{r}\,')}{R^2}\,\widehat{R} \qquad (2.6)$$

with $\lambda(\vec{r}\,')$ having units of C m^{-1}, $\vec{r}\,'$ running over all points in the line distribution $\mathcal{C}$, and dl' being the differential length element at $\vec{r}\,'$ for $\mathcal{C}$

- **surface charge distribution**:

$$\vec{E}(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\int_{\mathcal{S}} \frac{da'\sigma(\vec{r}\,')}{R^2}\,\widehat{R} \qquad (2.7)$$

with $\sigma(\vec{r}\,')$ having units of C m^{-2}, $\vec{r}\,'$ running over all points in the surface distribution $\mathcal{S}$, and da' being the differential area element at $\vec{r}\,'$ for $\mathcal{S}$

- **volume charge distribution**:

$$\vec{E}(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\int_{\mathcal{V}} \frac{d\tau'\rho(\vec{r}\,')}{R^2}\,\widehat{R} \qquad (2.8)$$

with $\rho(\vec{r}\,')$ having units of C m^{-3}, $\vec{r}\,'$ running over all points in the volume distribution $\mathcal{V}$, and $d\tau'$ being the differential volume element at $\vec{r}\,'$ for $\mathcal{V}$

Using the Dirac delta function we will define below, one can write the first two as special cases of the latter, using delta functions in the dimensions in which the charge distribution has no extent.

Aside: the Dirac Delta Function

Relating Equation 2.8 to Equation 2.2 offers us both the opportunity to rigorously connect them as well as a chance to introduce the *Dirac delta function*. The Dirac delta function at $\vec{r}_0$, $\delta(\vec{r} - \vec{r}_0)$, is defined by what it does when it is multiplied against an arbitrary function $f(\vec{r})$ and integrated: For any function $f(\vec{r})$ and any volume $\mathcal{V}$ containing the point $\vec{r}_0$, it holds that

$$\boxed{\int_{\mathcal{V}} f(\vec{r}\,')\,\delta(\vec{r}\,' - \vec{r}_0)\,d\tau' = f(\vec{r}_0)} \tag{2.9}$$

and, for any volume $\mathcal{V}$ not containing $\vec{r}_0$, the integral vanishes. In particular, if $f(\vec{r})$ is unity, then the right side of the above integral is unity.

Coulomb's Law and the Electric Field (cont.)

Two notes on dimensions and notation:

- In order for the units in the above equation to work out, the delta function above must have units of m^{-3}. The general rule is that the delta function's units are the inverse of the differential that its argument says it should be integrated over. In this case, the argument is a vector in 3D space, so the differnential is the differential volume element $d\tau$, and so the delta function has units of m^{-3}.
- Griffiths refers to the above delta function as $\delta^3(\vec{r} - \vec{r}_0)$. He does this because one can think of this delta function in terms of 1D delta function

$$\delta^3(\vec{r} - \vec{r}_0) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \quad \text{where} \quad \begin{aligned} \vec{r} &= x\,\hat{x} + y\,\hat{y} + z\,\hat{z} \\ \vec{r}_0 &= x_0\,\hat{x} + y_0\,\hat{y} + z_0\,\hat{z} \end{aligned} \tag{2.10}$$

We drop the ^{3} because it is unnecessary: the dimension of the delta function is implied by its argument. Moreover, the ^{3} notation is misleading and confusing because it suggests that δ^3 is the cube of something that has $\vec{r} - \vec{r}_0$ as its argument. It is not!

Coulomb's Law and the Electric Field (cont.)

With the above, we see that if we define the charge distribution for a set of point charges $\{q_i'\}$ at positions $\{\vec{r}_i{}'\}$ to be

$$\rho(\vec{r}) = \sum_{i=1}^{N} q_i' \, \delta(\vec{r} - \vec{r}_i{}') \tag{2.11}$$

then, when we do the integral in Equation 2.8 over any volume $\mathcal{V}$ containing all N charges, we obtain

$$\begin{aligned}
\vec{E}(\vec{r}) &= \frac{1}{4\,\pi\,\epsilon_0} \int_{\mathcal{V}} \sum_{i=1}^{N} \frac{d\tau' q_i' \, \delta(\vec{r}\,' - \vec{r}_i{}')}{|\vec{r} - \vec{r}\,'|^2} \, \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|} \\
&= \frac{1}{4\,\pi\,\epsilon_0} \sum_{i=1}^{N} \int_{\mathcal{V}} d\tau' q_i' \, \delta(\vec{r}\,' - \vec{r}_i{}') \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \\
&= \frac{1}{4\,\pi\,\epsilon_0} \sum_{i=1}^{N} q_i' \frac{\vec{r} - \vec{r}_i{}'}{|\vec{r} - \vec{r}_i{}'|^3} = \frac{1}{4\,\pi\,\epsilon_0} \sum_{i=1}^{N} \frac{q_i'}{R_i^2} \, \widehat{R}_i
\end{aligned} \tag{2.12}$$

which recovers Equation 2.2.

Statement of Gauss's Law

The *flux* of the electric field through a surface is simply the integral of the component of the electric field normal to the surface over the surface:

$$\mathcal{F}_{\mathcal{S}} = \int_{\mathcal{S}} \vec{E} \cdot \hat{n}(\vec{r})\, da \tag{2.13}$$

where $\vec{r}$ lies in the surface $\mathcal{S}$ and $\hat{n}(\vec{r})$ is the surface normal at that point $\vec{r}$. Note that the flux has a sign based on the choice of the direction of $\hat{n}$.

Gauss's Law relates the flux of the electric field through any *closed* surface to the total charge enclosed by that surface:

$$\boxed{\mathcal{F}_{\mathcal{S}} = \oint_{\mathcal{S}} \vec{E} \cdot \hat{n}(\vec{r})\, da = \frac{1}{\epsilon_0} \int_{\mathcal{V}(\mathcal{S})} d\tau\, \rho(\vec{r})} \tag{2.14}$$

where $\mathcal{V}(\mathcal{S})$ is the surface enclosed by $\mathcal{S}$ and $\oint$ indicates the integral over a closed surface.

Gauss's Law (cont.)

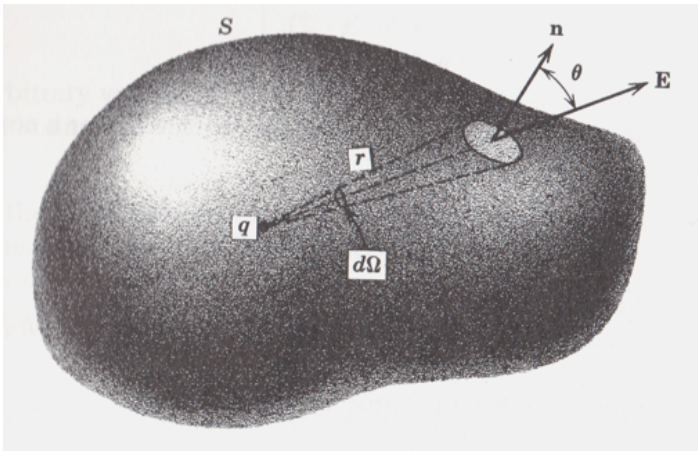
Utility of Gauss's Law

The reason we bother with Gauss's Law is that it provides a way to calculate the electric field for charge distributions having some amount of geometrical symmetry that is much easier than brute-force integration of Coulomb's Law. We will see that it will also enable us to relate the electric field's *boundary conditions* at an interface between two volumes (the conditions relating the electric field components on the two sides of the interface) to the amount of charge at that interface.

Gauss's Law (cont.)

Proof of Gauss's Law

The proof offered in Griffiths' is unnecessarily unrigorous; we follow Jackson.



© 1999 Jackson, *Classical Electrodynamics*

First consider a charge distribution $\rho(\vec{r})$ that lies completely inside an arbitrarily shaped closed surface $\mathcal{S}$. What is the infinitesimal flux through an infinitesimal portion $\mathcal{S}$ at a point $\vec{r}$ due to the infinitesimal amount of charge in the infinitesimal volume $d\tau'$ at the location $\vec{r}\,'$? It is

$$d^2\mathcal{F}_{\mathcal{S}}(\vec{r},\vec{r}\,') = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{d\tau'\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\,(\vec{r}-\vec{r}\,')\cdot\hat{n}(\vec{r})\,da \tag{2.15}$$

The left side is a double differential because the right side is. If one considers the geometry (see diagram above), one sees that the quantity $(\vec{r}-\vec{r}\,')\cdot\hat{n}(\vec{r})\,da/|\vec{r}-\vec{r}\,'|$ is the *projected area* of the area element da normal to the unit vector $(\vec{r}-\vec{r}\,')/|\vec{r}-\vec{r}\,'|$ from $\vec{r}\,'$ to $\vec{r}$. Since $|\vec{r}-\vec{r}\,'|^2$ is the square of the distance from $\vec{r}\,'$ to $\vec{r}$, then the quantity $(\vec{r}-\vec{r}\,')\cdot\hat{n}(\vec{r})\,da/|\vec{r}-\vec{r}\,'|^3$ is the *solid angle* $d\Omega(\vec{r},\vec{r}\,')$ subtended by da at $\vec{r}$ as viewed from $\vec{r}\,'$.

Gauss's Law (cont.)

The corresponding mathematical formula is

$$d^2\mathcal{F}_{\mathcal{S}}(\vec{r},\vec{r}\,') = \frac{1}{4\,\pi\,\epsilon_0}\,d\tau'\rho(\vec{r}\,')\,d\Omega(\vec{r},\vec{r}\,') \tag{2.16}$$

We know that if we integrate the solid angle over the entire closed surface $\mathcal{S}$ surrounding our source charge point $\vec{r}\,'$, we recover $4\,\pi$, so:

$$d\mathcal{F}_{\mathcal{S}}(\vec{r}\,') = \frac{1}{4\,\pi\,\epsilon_0}\oint_{\mathcal{S}} d\tau'\rho(\vec{r}\,')\,d\Omega(\vec{r},\vec{r}\,') = \frac{1}{\epsilon_0}\,d\tau'\rho(\vec{r}\,') \tag{2.17}$$

That is, for any infinitesimal volume element $d\tau'$ at $\vec{r}\,'$, the flux of the electric field due to that element through any surface $\mathcal{S}$ enclosing it is equal to the charge in that infinitesimal volume divided by ϵ_0.

Gauss's Law (cont.)

It stands to reason that, if the above is true for the flux due to an infinitesimal volume of charge, then it holds for the whole distribution of charge enclosed by $\mathcal{S}$. We can prove this by calculating the flux through $\mathcal{S}$ due to the entire charge distribution, using this fact that the distribution is fully contained inside $\mathcal{S}$:

$$\begin{aligned}\mathcal{F}_{\mathcal{S}} &= \oint_{\mathcal{S}} \vec{E}_{\mathcal{S}}(\vec{r}) \cdot \widehat{n}(\vec{r})\, da = \frac{1}{4\,\pi\,\epsilon_0} \oint_{\mathcal{S}} \int_{\mathcal{V}(\mathcal{S})} \frac{d\tau' \rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \left(\vec{r} - \vec{r}\,'\right) \cdot \widehat{n}(\vec{r})\, da \\ &= \frac{1}{4\,\pi\,\epsilon_0} \oint_{\mathcal{S}} \int_{\mathcal{V}(\mathcal{S})} d^2\mathcal{F}_{\mathcal{S}}(\vec{r}, \vec{r}\,') \end{aligned} \tag{2.18}$$

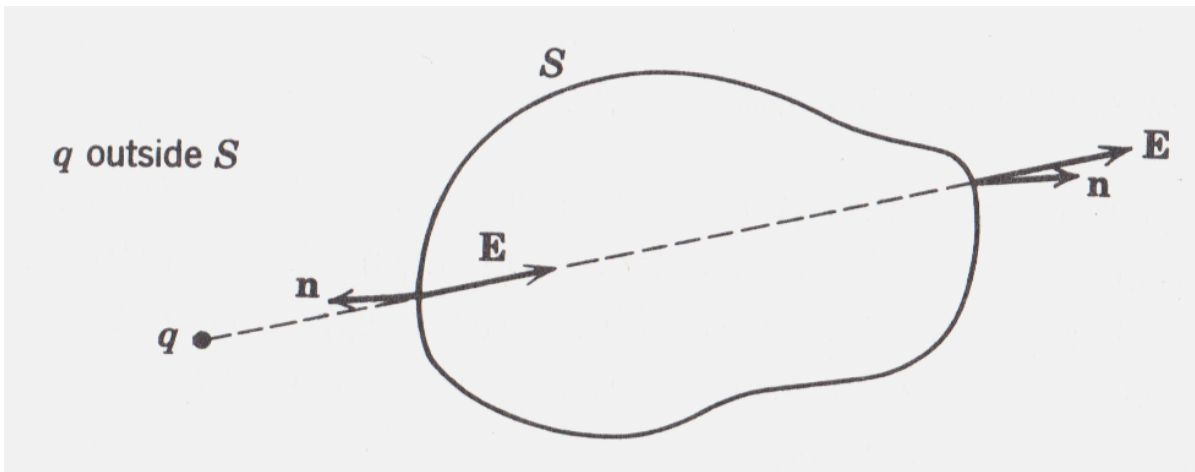
where $\vec{E}_{\mathcal{S}}(\vec{r})$ is the electric field at $\vec{r}$ due to all the charge contained by $\mathcal{S}$. Note that we implicitly used superposition in the above via the formula relating $\vec{E}_{\mathcal{S}}(\vec{r})$ to the charge distribution. Therefore, exchanging the order of integration,

$$\mathcal{F}_{\mathcal{S}} = \frac{1}{4\,\pi\,\epsilon_0} \int_{\mathcal{V}(\mathcal{S})} \oint_{\mathcal{S}} d^2\mathcal{F}_{\mathcal{S}}(\vec{r}, \vec{r}\,') = \frac{1}{\epsilon_0} \int_{\mathcal{V}(\mathcal{S})} d\mathcal{F}_{\mathcal{S}}(\vec{r}\,') = \frac{1}{\epsilon_0} \int_{\mathcal{V}(\mathcal{S})} d\tau' \rho(\vec{r}\,') \tag{2.19}$$

which is Gauss's Law. But we are not quite done yet, as we assumed in the above that the charge distribution vanished outside of $\mathcal{S}$. Does the result generalize? Yes, it does.

Gauss's Law (cont.)

Returning to $d^2\mathcal{F}_{\mathcal{S}}(\vec{r},\vec{r}\,')$ (Equation 2.16), suppose we consider a source charge at a point $\vec{r}\,'$ that lies outside of $\mathcal{S}$. (See diagram below.) Then, for a given point $\vec{r}$ on $\mathcal{S}$ and the solid angle it subtends $d\Omega(\vec{r},\vec{r}\,')$ as viewed from the source charge point $\vec{r}\,'$, there will be second point on $\mathcal{S}$ that has the same unit vector to the source charge point $\vec{r}\,'$ and subtends the same solid angle. But, because the direction of $\widehat{n}(\vec{r})$ enters the expression for $d^2\mathcal{F}_{\mathcal{S}}(\vec{r},\vec{r}\,')$, and the two points subtending the same solid angle will have opposite signs of $\widehat{n}$, their two contributions cancel. Thus, the integral over $\mathcal{S}$ that yields $d\mathcal{F}_{\mathcal{S}}(\vec{r}\,')$ vanishes for $\vec{r}\,'$ outside of $\mathcal{S}$. Thus, the charge distribution at points outside of $\mathcal{S}$ do not contribute to the flux through $\mathcal{S}$, and so our derivation remains valid.



Gauss's Law (cont.)

The Divergence of $\vec{E}$ and the Differential Version of Gauss's Law

You learned about the divergence theorem (Gauss's theorem) in Ma1abc. Applied to $\vec{E}$, the divergence theorem says

$$\int_{\mathcal{V}(\mathcal{S})} \vec{\nabla} \cdot \vec{E}(\vec{r})\, d\tau = \oint_{\mathcal{S}} \vec{E}(\vec{r}) \cdot \hat{n}(\vec{r})\, da \tag{2.20}$$

Gauss's Law tells us

$$\frac{1}{\epsilon_0} \int_{\mathcal{V}(\mathcal{S})} d\tau\, \rho(\vec{r}) = \oint_{\mathcal{S}} \vec{E}(\vec{r}) \cdot \hat{n}(\vec{r})\, da \tag{2.21}$$

Combining the two, we have

$$\int_{\mathcal{V}(\mathcal{S})} \vec{\nabla} \cdot \vec{E}(\vec{r})\, d\tau = \frac{1}{\epsilon_0} \int_{\mathcal{V}(\mathcal{S})} d\tau\, \rho(\vec{r}) \tag{2.22}$$

Since the above holds for any volume $\mathcal{V}$, the integrands must be equal, giving us the differential version of Gauss's Law:

$$\boxed{\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{\epsilon_0} \rho(\vec{r})} \tag{2.23}$$

Direct Proof of Differential Version of Gauss's Law

We can prove the above differential version by simply calculating the divergence of $\vec{E}$ using Coulomb's Law, also. This is of course not independent of Gauss's Law because Gauss's Law is proven using Coulomb's Law, but it provides some exercise in vector calculus and leads us to the Dirac delta function. We take the divergence of Coulomb's Law for $\vec{E}$:

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \vec{\nabla} \cdot \int_{\mathcal{V}'} \frac{1}{4\,\pi\,\epsilon_0} \frac{d\tau' \rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} (\vec{r} - \vec{r}\,') \tag{2.24}$$

Now, the integral is over $\vec{r}\,'$ over the volume $\mathcal{V}'$, but the divergence is calculated relative to the $\vec{r}$ coordinate, so we can bring the divergence inside the integral. Note that it does not act on ρ because it is a function of $\vec{r}\,'$, not $\vec{r}$. Thus, we have

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \int_{\mathcal{V}'} d\tau' \rho(\vec{r}\,') \vec{\nabla} \cdot \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \tag{2.25}$$

One could calculate the above divergence explicitly in any particular coordinate system. But it is both more rigorous and more instructive to do it using the divergence theorem.

Gauss's Law (cont.)

The divergence theorem lets us calculate the integral of the above divergence over some arbitrary volume $\mathcal{V}$ (with surface $\mathcal{S}$, with neither necessarily related to $\mathcal{V}'$ and $\mathcal{S}'$) by converting it to an easier-to-do surface integral:

$$\int_{\mathcal{V}} d\tau\, \vec{\nabla} \cdot \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} = \oint_{\mathcal{S}(\mathcal{V})} da\, \hat{n}(\vec{r}) \cdot \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \tag{2.26}$$

We can apply to the right side the same argument that we used in proving Gauss's Law regarding solid angles. The integrand above is just the solid angle subtended by the area element da at $\vec{r}$ as viewed from $\vec{r}\,'$. As before, if $\vec{r}\,'$ is inside $\mathcal{V}$, then the above integral yields the total solid angle, $4\,\pi$. If $\vec{r}\,'$ is not inside of $\mathcal{V}$, then, for every area element da at $\vec{r}$, there is an area element with an equal and opposite contribution, making the integral vanish. That is,

$$\int_{\mathcal{V}} d\tau\, \vec{\nabla} \cdot \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} = \begin{cases} 4\,\pi & \text{if } \vec{r}\,' \text{ is inside } \mathcal{V} \\ 0 & \text{if } \vec{r}\,' \text{ is outside } \mathcal{V} \end{cases} \tag{2.27}$$

You will note that the divergence in the integrand looks a lot like a delta function, a point we will return to later.

Gauss's Law (cont.)

Let's apply the above by integrating $\vec{\nabla} \cdot \vec{E}(\vec{r})$ over the same arbitrary volume $\mathcal{V}$:

$$\begin{aligned}\int_{\mathcal{V}} d\tau\, \vec{\nabla} \cdot \vec{E}(\vec{r}) &= \frac{1}{4\,\pi\,\epsilon_0} \int_{\mathcal{V}} d\tau \int_{\mathcal{V}'} d\tau' \rho(\vec{r}\,')\, \vec{\nabla} \cdot \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \\ &= \frac{1}{4\,\pi\,\epsilon_0} \int_{\mathcal{V}'} d\tau' \rho(\vec{r}\,') \int_{\mathcal{V}} d\tau\, \vec{\nabla} \cdot \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \qquad (2.28)\end{aligned}$$

where we have exchanged the order of integration (no prohibition on doing so because $\vec{\nabla}$ has not moved around). Now, we apply what we found out about the integral over $\mathcal{V}$, which says that the integral over $\mathcal{V}$ vanishes if $\vec{r}\,'$ is not inside $\mathcal{V}$ and yields $4\,\pi$ if it is inside $\mathcal{V}$. This turns the double integral over $\mathcal{V}'$ into a single integral over $\mathcal{V} \cap \mathcal{V}'$:

$$\int_{\mathcal{V}} d\tau\, \vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \int_{\mathcal{V}\cap\mathcal{V}'} d\tau' 4\,\pi\,\rho(\vec{r}\,') = \frac{1}{\epsilon_0} \int_{\mathcal{V}\cap\mathcal{V}'} d\tau' \rho(\vec{r}\,') \qquad (2.29)$$

Gauss's Law (cont.)

Now, consider points in $\mathcal{V}$ but outside $\mathcal{V} \cap \mathcal{V}'$. Because $\mathcal{V}'$ is the entire volume containing charge (by Coulomb's Law), the charge density vanishes in $\mathcal{V} - \mathcal{V} \cap \mathcal{V}'$. Since the right side is the integral of charge density, we can add the region $\mathcal{V} - \mathcal{V} \cap \mathcal{V}'$ without changing the integral: the contribution we have just added vanishes. This changes the volume of integration from $\mathcal{V} \cap \mathcal{V}'$ to $\mathcal{V}$. Therefore,

$$\int_{\mathcal{V}} d\tau \, \vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{\epsilon_0} \int_{\mathcal{V}} d\tau' \rho(\vec{r}\,') \tag{2.30}$$

The volume $\mathcal{V}$ is arbitrary, so the integrands must be equal:

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{\epsilon_0} \rho(\vec{r}\,') \tag{2.31}$$

which is again the differential version of Gauss's Law.

Gauss's Law (cont.)

Aside: Relation of the Dirac Delta Function to a Divergence, Invariance under Inversion of its Argument

We can use the above manipulations to prove another property of the Dirac delta function. Let's apply Gauss's Law to the left side of Equation 2.25, yielding

$$\rho(\vec{r}) = \frac{1}{4\pi}\int_{\mathcal{V}'} d\tau' \rho(\vec{r}\,') \vec{\nabla}\cdot\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \tag{2.32}$$

Now, $\rho(\vec{r})$ is an arbitrary function, so we see that the divergence we took acts like the δ function: it picks out $\rho(\vec{r}\,' = \vec{r})$. Thus, we have also proven

$$\boxed{\vec{\nabla}\cdot\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = 4\pi\,\delta(\vec{r}-\vec{r}\,')} \tag{2.33}$$

which we will find is a new, useful property of the delta function.

Since the delta function picks out the point where its argument vanishes, it doesn't matter what the sign of the argument is. One can prove this explicitly using change of variables: when the sign of the argument changes, the sign of the differential and of the limits of integration change also. Those two sign flips cancel each other. Thus

$$\boxed{\delta(\vec{r}-\vec{r}\,') = \delta(\vec{r}\,'-\vec{r})} \tag{2.34}$$

Gauss's Law (cont.)

It may seem that this last property is not true given the above relation between the delta function and a divergence. In particular, let's flip the sign on the function the divergence is acting on:

$$\frac{\delta(\vec{r}-\vec{r}\,')}{4\,\pi} = \vec{\nabla}\cdot\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = \vec{\nabla}\cdot-\frac{\vec{r}\,'-\vec{r}}{|\vec{r}\,'-\vec{r}|^3} = -\vec{\nabla}\cdot\frac{\vec{r}\,'-\vec{r}}{|\vec{r}\,'-\vec{r}|^3} \stackrel{?}{=} -\frac{\delta(\vec{r}\,'-\vec{r})}{4\,\pi} \quad (2.35)$$

Don't we have a problem? No, because we ignored the fact that $\vec{\nabla}$ takes derivatives with respect to $\vec{r}$. Since $\vec{r}-\vec{r}\,'$ just offsets $\vec{r}$, then the divergence with respect to $\vec{r}-\vec{r}\,'$ is the same as the divergence with respect to $\vec{r}$. So, when we flip the sign on $\vec{r}-\vec{r}\,'$, we should do the same for the divergence: the divergence should be taken with respect to $\vec{r}\,'-\vec{r}$. But that flips the sign of the divergence operator: $\vec{\nabla}_{\vec{r}-\vec{r}\,'} = -\vec{\nabla}_{\vec{r}\,'-\vec{r}}$. Finally, $\vec{r}$ acts like an offset for $\vec{r}\,'$, and so the divergence with respect to $\vec{r}\,'-\vec{r}$ is the same as with respect to $\vec{r}\,'$. Therefore, we have

$$\vec{\nabla}_{\vec{r}}\cdot\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = \vec{\nabla}_{\vec{r}-\vec{r}\,'}\cdot\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = \left(-\vec{\nabla}_{\vec{r}\,'-\vec{r}}\right)\cdot\left(-\frac{\vec{r}\,'-\vec{r}}{|\vec{r}\,'-\vec{r}|^3}\right) = \vec{\nabla}_{\vec{r}\,'}\cdot\frac{\vec{r}\,'-\vec{r}}{|\vec{r}\,'-\vec{r}|^3}$$

or, more concisely, $$\frac{\delta(\vec{r}-\vec{r}\,')}{4\,\pi} = \vec{\nabla}_{\vec{r}}\cdot\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = \vec{\nabla}_{\vec{r}\,'}\cdot\frac{\vec{r}\,'-\vec{r}}{|\vec{r}\,'-\vec{r}|^3} = \frac{\delta(\vec{r}\,'-\vec{r})}{4\,\pi} \quad (2.36)$$

The Electric Field has Vanishing Curl

Calculating the Curl of the Electric Field

The curl of $\vec{E}$ can be shown to vanish simply by calculating it for an arbitrary charge distribution:

$$\begin{aligned}\vec{\nabla} \times \vec{E}(\vec{r}) &= \frac{1}{4\,\pi\,\epsilon_0}\,\vec{\nabla} \times \int_{\mathcal{V}} d\tau'\,\rho(\vec{r}\,')\,\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \\ &= \frac{1}{4\,\pi\,\epsilon_0}\int_{\mathcal{V}} d\tau'\,\rho(\vec{r}\,')\,\vec{\nabla} \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \qquad (2.37)\end{aligned}$$

We could brute force calculate the curl in the integrand in Cartesian or spherical coordinates, but that would be painful because the function on which the curl is acting has no symmetry in the $\vec{r}$ coordinate system.

The Electric Field has Vanishing Curl (cont.)

Let's take a simpler approach. As we saw above, $\vec{r}\,'$ is just an offset to $\vec{r}$, thus

$$\vec{\nabla}_{\vec{r}} \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = \vec{\nabla}_{\vec{r}-\vec{r}\,'} \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \tag{2.38}$$

(Note that we are considering this expression alone, not the whole integral. We need not be concerned about suddenly making $\vec{\nabla}$, which did not originally depend on $\vec{r}\,'$ and thus could be brought inside the integral, now depend on $\vec{r}\,'$.) If we define $\vec{s} = \vec{r} - \vec{r}\,'$, then we have

$$\vec{\nabla}_{\vec{r}} \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} = \vec{\nabla}_{\vec{s}} \times \frac{\vec{s}}{s^3} \tag{2.39}$$

Now, the function on which the curl is acting has symmetry in the coordinate system in which the curl is acting, and hence the calculation will be simplified. You can probably see intuitively that the above curl vanishes, but let's prove it.

The Electric Field has Vanishing Curl (cont.)

With the above form, we can trivially apply the formula for the curl in spherical coordinates, which is listed in Griffiths. For the sake of being explicit, that formula is

$$\vec{\nabla} \times \vec{v} = \frac{1}{r \sin\theta}\left[\frac{\partial}{\partial\theta}\left(v_\phi \sin\theta\right) - \frac{\partial v_\theta}{\partial\phi}\right]\widehat{r} + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}\left(r\, v_\phi\right)\right]\widehat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\, v_\theta\right) - \frac{\partial v_r}{\partial\theta}\right]\widehat{\phi} \tag{2.40}$$

Don't get confused between $\vec{s}$ and $\vec{r}$; the r derivatives and subscripts refer to the radial coordinate of the coordinate system in which the curl is being taken. In our case, s is the radial variable and the radial component of $\vec{s}/s^3$ is $1/s^2$. Thus, $\vec{v}$ has only a radial component and that radial component depends only on the radial distance from the origin. All the derivatives involving the θ and ϕ components of $\vec{v}$ vanish because the components themselves vanish, and the derivatives involving the radial component vanish because those derivatives are with respect to θ and ϕ. (Don't be confused: $\vec{v}$ itself depends on θ and ϕ because the direction of $\vec{v}$ depends on them; but the curl formula takes care of that dependence.) Thus, we have $\vec{\nabla}_{\vec{s}} \times (\vec{s}/s^3) = 0$ and therefore the integrand in Equation 2.37 vanishes. So:

$$\boxed{\vec{\nabla} \times \vec{E}(\vec{r}) = 0} \tag{2.41}$$

The Electric Field has Vanishing Curl (cont.)

The Line Integral of the Electric Field

Stokes' Theorem (a mathematical theorem we will not prove here but that you saw in Ma1abc) then tells us that, for any surface $\mathcal{S}$ with boundary $\mathcal{C}(\mathcal{S})$,

$$\oint_{\mathcal{C}(\mathcal{S})} d\vec{\ell} \cdot \vec{E}(\vec{r}) = \int_{\mathcal{S}} da\, \hat{n}(\vec{r}) \cdot \vec{\nabla} \times \vec{E}(\vec{r}) = 0 \tag{2.42}$$

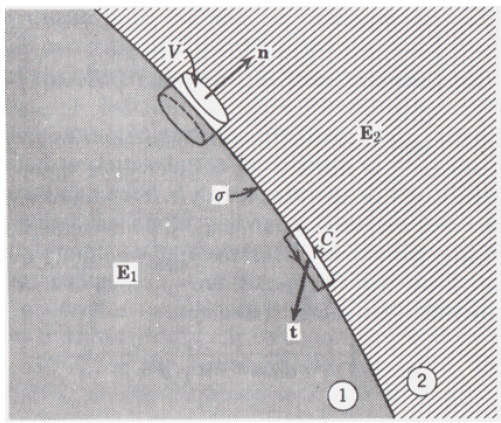
Lecture 2:
Basics of Electrostatics II:
Boundary Conditions, Electric Potential and Energy, Conductors

Date Revised: 2014/02/13 14:00
Date Given: 2013/02/13

Boundary Conditions on the Electric Field

While it makes it possible to determine the electric field for charge distributions with sufficient symmetry, the more important application of Gauss's Law and the vanishing of $\vec{\nabla} \times \vec{E}$ is to obtain generic information on the behavior of the electric field across an interface between two regions.

Relation of the normal component of the electric field to the surface charge density



© 1999 Jackson, *Classical Electrodynamics*

Construct a Gaussian cylinder of infinitesimal height dz whose axis is normal to the interface under question at the point of interest $\vec{r}$. Let $\hat{n}$ be the surface normal at $\vec{r}$, with orientation from region 1 to region 2. Let's calculate the flux through the cylinder's (non-infinitesimal) faces $\mathcal{S}_1$ and $\mathcal{S}_2$:

$$\mathcal{F}(\vec{r}) = \int_{\mathcal{S}_1} da' \left(-\hat{n}(\vec{r}\,')\right) \cdot \vec{E}_1(\vec{r}\,') + \int_{\mathcal{S}_2} da' \, \hat{n}(\vec{r}\,') \cdot \vec{E}_2(\vec{r}\,') \qquad (2.43)$$

where $\vec{E}$ is evaluated over the two faces.

Boundary Conditions on the Electric Field (cont.)

We neglect the flux through the cylindrical wall because we will let dz vanish in the end and so its area will vanish and it will contribute no flux. By Gauss's Law, this flux is related to the charge enclosed:

$$\mathcal{F}(\vec{r}) = \frac{1}{\epsilon_0} \int_{\mathcal{S}} da' \, \sigma(\vec{r}\,') \tag{2.44}$$

where $\mathcal{S}$ is the area at the interface intersected by the cylinder. We neglect any charge density in the half cylinders because those contributions to the charge enclosed will vanish as we let $dz \to 0$. Equating the two expressions for $\mathcal{F}(\vec{r})$, setting $dz \to 0$ consistently, and seeing that $\mathcal{S}_1, \mathcal{S}_2 \to \mathcal{S}$ yields

$$\int_{\mathcal{S}} da' \, \hat{n}(\vec{r}\,') \cdot \left[\vec{E}_2(\vec{r}\,') - \vec{E}_1(\vec{r}\,')\right] = \frac{1}{\epsilon_0} \int_{\mathcal{S}} da' \, \sigma(\vec{r}\,') \tag{2.45}$$

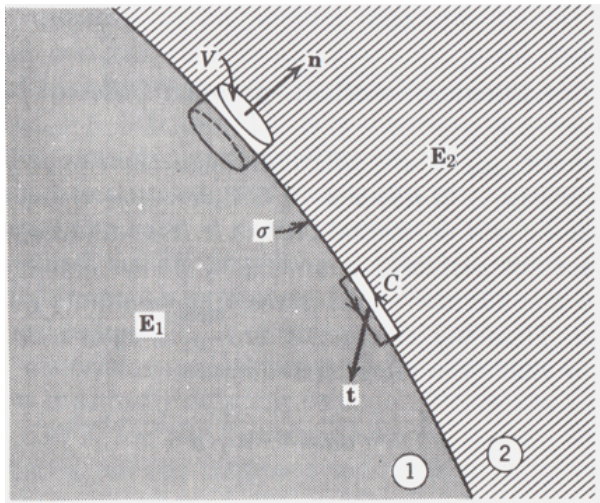
This holds for any choice of cylinder and thus any $\mathcal{S}$, and so the integrands must be equal:

$$\boxed{\hat{n}(\vec{r}) \cdot \left[\vec{E}_2(\vec{r}) - \vec{E}_1(\vec{r})\right] = \frac{1}{\epsilon_0} \sigma(\vec{r})} \tag{2.46}$$

That is, the change in the normal component of the electric field across the interface is equal to the surface charge density at the interface. If there is no surface charge at the interface, this component of the electric field must be continuous.

Boundary Conditions on the Electric Field (cont.)

Continuity of the tangential component of the electric field



© 1999 Jackson, *Classical Electrodynamics*

Construct a rectangular loop $\mathcal{C}$ with two legs normal to the interface of interest (*i.e.*, along $\widehat{n}(\vec{r})$) and with infinitesimal length dz and two (non-infinitesimal) legs parallel to the interface $\mathcal{C}_1$ and $\mathcal{C}_2$. Let $\widehat{t}(\vec{r})$ denote the normal to the loop area (so $\widehat{n}(\vec{r}) \cdot \widehat{t}(\vec{r}) = 0$). $\widehat{t}$ will set the orientation of the line integral we will do around the loop following the right-hand rule. The loop legs $\mathcal{C}_1$ and $\mathcal{C}_2$ parallel to the interface are along the vector $\widehat{s}(\vec{r}) = \widehat{t}(\vec{r}) \times \widehat{n}(\vec{r})$. Let's calculate the line integral of $\vec{E}$ along this loop:

$$\oint_{\mathcal{C}} d\vec{\ell}' \cdot \vec{E}(\vec{r}\,') = \int_{\mathcal{C}_1} \vec{E}_1(\vec{r}\,') \cdot (-\widehat{s}(\vec{r}\,'))\, ds + \int_{\mathcal{C}_2} \vec{E}_2(\vec{r}\,') \cdot \widehat{s}(\vec{r}\,')\, ds \qquad (2.47)$$

where we neglect the contributions from the infinitesimal legs because they will vanish as $dz \longrightarrow 0$.

Boundary Conditions on the Electric Field (cont.)

The fact that the curl of the electric field vanishes ensures the left side is zero. Also, $\mathcal{C}_1 \to \mathcal{C}_2$ as $dz \to 0$. Thus, we have

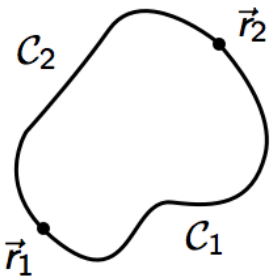
$$\int_{\mathcal{C}_2} \left[\vec{E}_2(\vec{r}\,') - \vec{E}_1(\vec{r}\,')\right] \cdot \widehat{s}(\vec{r}\,')\, ds = 0 \tag{2.48}$$

Since the contour $\mathcal{C}_2$ is arbitrary, the integrand must vanish, yielding

$$\boxed{\widehat{s}(\vec{r}) \cdot \left[\vec{E}_2(\vec{r}) - \vec{E}_1(\vec{r})\right] = 0} \tag{2.49}$$

This expression holds for any $\widehat{t}$ and thus $\widehat{s}$ parallel to the surface, so it tells us that the tangential component of the electric field is continuous across any boundary (regardless of whether there is surface charge present).

Definition using Line Integral



We saw above that the line integral of the electric field around any closed loop $\mathcal{C}$ vanishes. If we consider two points along the loop $\vec{r}_1$ and $\vec{r}_2$, there are two paths along the loop from $\vec{r}_1$ to $\vec{r}_2$, $\mathcal{C}_1$ and $\mathcal{C}_2$. The vanishing of the full line integral tells us the line integral along these two paths are equal:

$$\begin{aligned}\int_{\mathcal{C}_1,\vec{r}_1}^{\vec{r}_2} d\vec{\ell}' \cdot \vec{E}(\vec{r}') - \int_{\mathcal{C}_2,\vec{r}_1}^{\vec{r}_2} d\vec{\ell}' \cdot \vec{E}(\vec{r}') &= \int_{\mathcal{C}_1,\vec{r}_1}^{\vec{r}_2} d\vec{\ell}' \cdot \vec{E}(\vec{r}') + \int_{\mathcal{C}_2,\vec{r}_2}^{\vec{r}_1} d\vec{\ell}' \cdot \vec{E}(\vec{r}') \\ &= \oint_{\mathcal{C}} d\vec{\ell}' \cdot \vec{E}(\vec{r}') = 0 \end{aligned} \tag{2.50}$$

(Be careful with the sign flip and endpoint exchange on the second term! Think about it physically — because the line integral has a directionality, changing the order of the endpoints changes the directionality and yields a sign flip. You can prove this to yourself in a particular coordinate system by writing it out, but be careful with the multiple sign flips.) Therefore,

$$\int_{\mathcal{C}_1,\vec{r}_1}^{\vec{r}_2} d\vec{\ell}' \cdot \vec{E}(\vec{r}') = \int_{\mathcal{C}_2,\vec{r}_1}^{\vec{r}_2} d\vec{\ell}' \cdot \vec{E}(\vec{r}') \tag{2.51}$$

The Electric Potential (cont.)

The above relation tells us that the value of the above line integral depends only on the location of its endpoints, not on the path taken. Thus, we can construct a function, *the electric potential*, $V(\vec{r})$, defining it via its differences between points:

$$\boxed{V(\vec{r}_2) - V(\vec{r}_1) \equiv -\int_{\vec{r}_1}^{\vec{r}_2} d\vec{\ell}' \cdot \vec{E}(\vec{r}\,')} \tag{2.52}$$

The fundamental theorem of calculus for line integrals in multiple dimensions implies

$$V(\vec{r}_2) - V(\vec{r}_1) = \int_{\vec{r}_1}^{\vec{r}_2} d\vec{\ell}' \cdot \vec{\nabla}_{\vec{r}\,'} V(\vec{r}\,') \tag{2.53}$$

where $\vec{\nabla} V(\vec{r})$ is the *gradient* of the electric potential. The above two formulae hold regardless of choice of endpoints and path. Therefore, an alternate definition of the potential is

$$\boxed{\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})} \tag{2.54}$$

which makes it clear why the offset for V is not defined.

The electric potential has units of (N m/C), which we call the *volt*, V. The electric field is frequently written in units of V/m instead of N/C.

Relation of the Electric Potential to the Charge Distribution

We know two things now:

$$\vec{E}(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\int_{\mathcal{V}} d\tau'\rho(\vec{r}\,')\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \qquad \text{and} \qquad V(\vec{r}_2)-V(\vec{r}_1) \equiv -\int_{\vec{r}_1}^{\vec{r}_2} d\vec{\ell}' \cdot \vec{E}(\vec{r}\,')$$

We can use these to obtain an explicit expression for the potential in terms of the charge distribution. In practice, trying to explicitly do the line integral using the definition of $\vec{E}$ is tedious and not illuminating.

Instead, let us use $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$ to make an Ansatz. If we have a point charge at the origin, then the electric field points radially outward and falls off as $1/r^2$. What function's derivative gives that dependence? $V(\vec{r}) = 1/r$. This suggests to us

$$\boxed{V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\int_{\mathcal{V}} d\tau'\rho(\vec{r}\,')\,\frac{1}{|\vec{r}-\vec{r}\,'|}} \tag{2.55}$$

The Electric Potential (cont.)

We can prove this form by explicitly taking the gradient:

$$-\vec{\nabla} V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \int_{\mathcal{V}} d\tau' \rho(\vec{r}\,') \,\vec{\nabla} \left(\frac{1}{|\vec{r} - \vec{r}\,'|} \right) \tag{2.56}$$

As we did earlier when calculating $\vec{\nabla} \times \vec{E}$, we make the change of variables to $\vec{s} = \vec{r} - \vec{r}\,'$:

$$\vec{\nabla}_{\vec{r}} \left(\frac{1}{|\vec{r} - \vec{r}\,'|} \right) = \vec{\nabla}_{\vec{r} - \vec{r}\,'} \left(\frac{1}{|\vec{r} - \vec{r}\,'|} \right) = \vec{\nabla}_{\vec{s}} \frac{1}{s} = -\frac{\widehat{s}}{s^2} = -\frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} \tag{2.57}$$

where we used the formula for the gradient in spherical coordinates from Griffiths:

$$\vec{\nabla} T(\vec{r}) = \frac{\partial T}{\partial r} \widehat{r} + \frac{1}{r} \frac{\partial T}{\partial \theta} \widehat{\theta} + \frac{1}{r \sin\theta} \frac{\partial T}{\partial \phi} \widehat{\phi} \tag{2.58}$$

Hence, we see that our form for $V(\vec{r})$ yields the correct electric field:

$$-\vec{\nabla} V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \int_{\mathcal{V}} d\tau' \rho(\vec{r}\,') \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} = \vec{E}(\vec{r}) \tag{2.59}$$

The Electric Potential (cont.)

Comments on the Electric Potential

- **The electric potential obeys superposition**
  This is trivial consequence of superposition for the electric field: because the electric potential is a linear function of the electric field, superposition for the electric field transfers to superposition for the electric potential. One can also see it from Equation 2.55, where the potential is a linear function of the charge density. Of critical importance is that the offset value for the electric potentials being summed be the same (implicit when using the expression in terms of the charge distribution).
- **Definition of potential offset**
  There are two typical choices. When the charge distribution is confined to a finite volume, the electric field vanishes at infinity, which suggests one should define the electric potential to vanish at infinity too. When the charge distribution is not confined (*e.g.*, a uniform electric field over all of space), it is typical to choose the origin to be the point at which the potential vanishes. Any other point would work, too, but will generally make the explicit functional form of $V(\vec{r})$ unnecessarily complicated.
- **Utility of the electric potential**
  The electric potential is scalar, not a vector, function, and thus applying superposition to calculate the potential due to a charge distribution, followed by taking the gradient to find the electric field, is usually much simpler than explicitly calculating the electric field.

Boundary Conditions on the Electric Potential

From our definition of the electric potential as the line integral of the electric field, and the corollary $\vec{E} = \vec{\nabla} V$, we can derive boundary conditions on the electric potential:

- **Continuity of the electric potential**
  The electric potential is the line integral of the electric field. Therefore, the only way for it to be discontinuous aross a boundary is if the electric field can be a delta function at a boundary. (Think about calculating the change in V by integrating $\vec{E} \cdot \hat{n}\, ds$ across the boundary. As the length of the path goes to zero, how do you prevent the integral from vanishing? With a delta function.) But we have established that the electric field changes by a finite amount at any boundary, and thus it cannot be a delta function on a boundary. The only place the electric field is infinite is on a point charge, and we know that even this does not result in a discontinuity in the potential (though the potential does also become infinite).

The Electric Potential (cont.)

- **Change in the normal gradient**
  This is just a direct rewriting of the boundary condition on the normal component of the field, Equation 2.46:

$$\frac{1}{\epsilon_0}\sigma(\vec{r}) = \hat{n}(\vec{r}) \cdot \left[\vec{E}_2(\vec{r}) - \vec{E}_1(\vec{r})\right] = \hat{n}(\vec{r} \cdot \left[-\vec{\nabla} V_2(\vec{r}) + \vec{\nabla} V_1(\vec{r})\right]$$

$$\Longrightarrow \boxed{\hat{n}(\vec{r}) \cdot \left[\vec{\nabla} V_2(\vec{r}) - \vec{\nabla} V_1(\vec{r})\right] = -\frac{1}{\epsilon_0}\sigma(\vec{r})} \tag{2.60}$$

  Note the sign!

- **Continuity of the tangential gradient**
  Again, this follows directly from the continuity of the tangential component of the electric field, Equation 2.49:

$$0 = \hat{s}(\vec{r}) \cdot \left[\vec{E}_2(\vec{r}) - \vec{E}_1(\vec{r})\right] = \hat{s}(\vec{r} \cdot \left[-\vec{\nabla} V_2(\vec{r}) + \vec{\nabla} V_1(\vec{r})\right]$$

$$\Longrightarrow \boxed{\hat{s}(\vec{r}) \cdot \left[\vec{\nabla} V_2(\vec{r}) - \vec{\nabla} V_1(\vec{r})\right] = 0} \tag{2.61}$$

The Poisson and Laplace Equations

It is natural to rewrite Gauss's Law and the curl-freeness of the electric field in terms of the electric potential. Gauss's Law:

$$\frac{1}{\epsilon_0}\rho(\vec{r}) = \vec{\nabla}\cdot\vec{E}(\vec{r}) = -\nabla^2 V(\vec{r}) \tag{2.62}$$

Rewritten more cleanly:

$$\boxed{\nabla^2 V(\vec{r}) = -\frac{1}{\epsilon_0}\rho(\vec{r})} \tag{2.63}$$

This is known as *Poisson's Equation*.

Poisson's Equation is a partial differential equation. You know from basic calculus that a differential equation alone is not sufficient to obtain a full solution $V(\vec{r})$: constants of integration are required. For partial differential equations in multiple dimensions, the constants of integration are given by specifying *boundary conditions*, conditions for how the solution or its derivatives must behave on the boundary of the volume in which we are specifying $\rho(\vec{r})$ and would like to determine $V(\vec{r})$.

The Electric Potential (cont.)

Our expression for the potential in terms of the charge distribution, Equation 2.55, is the explicit solution to this equation *for a particular boundary condition*, $V(\vec{r}) \to 0$ as $r \to \infty$. In the next major section of the notes, will develop the concept of a Green function, which is the generic tool for solving the Poisson equation for arbitrary boundary conditions.

When there is no charge and the right side vanishes, Equation 2.63 is known as *Laplace's Equation*. The importance of this equation is that it implies that, in a region where there is no charge, the second derivative vanishes everywhere, which implies there can be no local maxima or minima (they would require a positive or negative second derivative).

As for the curl, there is a theorem that the curl of a gradient always vanishes:

$$\vec{\nabla} \times (-\vec{\nabla} V) = 0 \tag{2.64}$$

This is not surprising, as the vanishing of the curl of $\vec{E}$ is what allowed us to define the potential as a line integral, which then allowed us to write $\vec{E}$ as the gradient of the potential.

Electrostatic potential energy of a point charge in an electric field

Consider moving a point charge from $\vec{r}_1$ to $\vec{r}_2$ along the contour $\mathcal{C}$. The work done *on* the charge is given by doing the line integral of the negative of the electric force along the path (because that is the mechanical force that has to be exerted to move the charge against the electric force $\vec{F}_e$):

$$W_{12} = -\int_{\mathcal{C},\vec{r}_1}^{\vec{r}_2} d\vec{\ell}' \cdot \vec{F}_e(\vec{r}\,') \tag{2.65}$$

The force is related to the electric field, and so we have

$$W_{12} = -q \int_{\mathcal{C},\vec{r}_1}^{\vec{r}_2} d\vec{\ell}' \cdot \vec{E}(\vec{r}\,') = q\,[V(\vec{r}_2) - V(\vec{r}_1)] \tag{2.66}$$

That is, the work done on the charge by the mechanical force in going from $\vec{r}_1$ to $\vec{r}_2$ is given by the charge times the change in electric potential between the two positions. Note the sign: if the potential is higher at the end point, then the work done was positive.

Electrostatic Energy (cont.)

Of course, this lets us to define the *electric potential energy* by

$$\boxed{U(\vec{r}_2) - U(\vec{r}_1) = q\,[V(\vec{r}_2) - V(\vec{r}_1)]} \tag{2.67}$$

That is, the electric potential energy of the charge and the electric potential of the field are simply related. Since it was defined in terms of work done against a force, electric potential energy obviously has units of Joules (J). That is explicit in the above form, which is C (N m/C) = (N m) = J.

Note that the electric field can also do work on the charge. In this case, the sign in the above line integral for the work is flipped and work is done as the charge loses potential energy. In this case, the work done by the electric field is converted to the kinetic energy the charge has at the end.

Electrostatic Energy (cont.)

Electric potential energy of a charge distribution

How much work must be done to assemble a distribution of charge? This is most easily understood by first considering the assembly of a set of point charges one-by-one by bringing them in from infinity. When the ith charge is brought in, work must be done against the electric field of the first $i-1$ charges. Put another way, the ith charge starts with zero potential energy and ends with potential energy

$$U_i = \sum_{j=1}^{i-1} q_i \, \frac{1}{4\,\pi\,\epsilon_0} \, \frac{q_j}{|\vec{r}_i - \vec{r}_j|} \tag{2.68}$$

Thus, the total potential energy is

$$\boxed{U = \frac{1}{4\,\pi\,\epsilon_0} \sum_{i=1}^{N} \sum_{j=1}^{i-1} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{8\,\pi\,\epsilon_0} \sum_{i,j=1, i\neq j}^{N} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}} \tag{2.69}$$

where the factor of $1/2$ was introduced to allow i and j to both run from 1 to N. Generalizing this to a continuous charge distribution, we have

$$\boxed{U = \frac{1}{8\,\pi\,\epsilon_0} \int_{\mathcal{V}} d\tau \int_{\mathcal{V}} d\tau' \, \frac{\rho(\vec{r})\,\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|}} \tag{2.70}$$

Electrostatic Energy (cont.)

Electric potential energy in terms of the electric field

We can use the relations between potential, field, and charge density (Equations 2.8, 2.55, and 2.63) and the divergence theorem (Equation 2.20) to obtain an alternate expression for the electric potential energy in terms of the electric field as follows:

$$U = \frac{1}{8\,\pi\,\epsilon_0}\int_{\mathcal{V}} d\tau \int_{\mathcal{V}} d\tau'\,\frac{\rho(\vec{r})\,\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} = \frac{1}{2}\int_{\mathcal{V}} d\tau\rho(\vec{r})\,V(\vec{r}) = -\frac{\epsilon_0}{2}\int_{\mathcal{V}} d\tau\left[\nabla^2 V(\vec{r})\right]V(\vec{r})$$

$$\overset{ibp}{=} -\frac{\epsilon_0}{2}\int_{\mathcal{V}} d\tau\,\vec{\nabla}\cdot\left[V(\vec{r})\,\vec{\nabla}V(\vec{r})\right] + \frac{\epsilon_0}{2}\int_{\mathcal{V}}|\vec{\nabla}V(\vec{r})|^2 \quad \text{with } \overset{ibp}{=} \equiv \text{integration by parts}$$

$$\overset{\substack{divergence\\theorem}}{=} \frac{\epsilon_0}{2}\int_{\mathcal{S}(\mathcal{V})} da\,\hat{n}\cdot\left[V(\vec{r})\,\vec{E}(\vec{r})\right] + \frac{\epsilon_0}{2}\int_{\mathcal{V}}|\vec{\nabla}V(\vec{r})|^2 \tag{2.71}$$

In the last line, the first term is an integral of the product of the potential and the field at the surface of the volume. In order to get the full energy of the charge distribution, $\mathcal{V}$ must include all the charge. If we assume the charge distribution is restricted to some finite volume, then $\mathcal{V}$ is naturally the volume containing the charge distribution. But we can add volume that does not contain charge because it contributes nothing to the initial expression for the electric potential energy.

Electrostatic Energy (cont.)

Therefore, we replace $\mathcal{V}$ with all of space and let $\mathcal{S}$ go to infinity:

$$U = \frac{\epsilon_0}{2} \int_{r\to\infty} da\, \hat{n} \cdot \left[V(\vec{r})\, \vec{E}(\vec{r}) \right] + \frac{\epsilon_0}{2} \int_{all\ \ space} |\vec{\nabla} V(\vec{r})|^2 \tag{2.72}$$

Because the charge distribution is restricted to the finite volume $\mathcal{V}$ and thus looks like a point charge as $r \to \infty$, the field and potential fall off like $1/r^2$ and $1/r$. The surface area of $\mathcal{S}$ only grows as r^2, so the integral goes like $1/r$ and thus vanishes as $r \to \infty$. (If the charge distribution is not restricted to a finite volume, the surface term may not vanish and its potential energy could indeed be infinite!)

It may seem strange that we can make this choice of $\mathcal{S}$, as changing $\mathcal{V}$ and $\mathcal{S}$ affects both integrals in the last expression. The explanation is that the choice of $\mathcal{S}$ changes the two integrals but leaves their sum constant, and taking $\mathcal{S}$ to infinity simple zeros out the first integral, leaving only the contribution of the second integral.

We thus find

$$\boxed{U = \frac{\epsilon_0}{2} \int |\vec{E}(\vec{r})|^2} \tag{2.73}$$

where the integral is over all of space. Correspondingly, the quantity $u = \frac{\epsilon_0}{2} |\vec{E}|^2$ is an energy density. We interpret this form as indicating that the potential energy created by assembling the charge distribution is stored in the field: less charge implies a smaller field and therefore less potential energy.

Electrostatic Energy (cont.)

Superposition and Electric Potential Energy

Because the electric potential energy is a quadratic function of the electric field,

electric potential energy does not obey superposition

The energy of a sum of fields is more than just the sum of the energies of the individual fields because there is a cross term due to potential energy of the presence of the charges sourcing the second field in the first field (or vice versa, or half of both).

Electrostatic Energy (cont.)

Self-energy and Point Charges vs. Continuous Charge Distributions

We were slightly cavalier in going from Equation 2.69 to Equation 2.70 in that the "self-energy" term $i = j$ that was not included in the former did get included in the latter. In the point-charge version, this term is infinite because the denominator vanishes. In the continuous distribution version, $\rho(\vec{r})\,\rho(\vec{r}\,')\,d\tau \longrightarrow 0$ as $|\vec{r} - \vec{r}\,'| \longrightarrow 0$ as long as ρ remains finite over all space, and thus there is no infinite contribution. (If ρ included a delta function, as would be necessary to represent a point charge, then it would produce an infinite contribution because the integral would yield $\delta(\vec{0})/0$.) Thus, we must be careful and choose the appropriate formula depending on the situation.

The infinite self-energy of a point charge reflects the fact that we do not know how to assemble a point charge. In fundamental particle physics, the existence of point charges such as the electron is a given. In fact, there is scheme, called "renormalization," by which the infinite self-energy one calculates for such a charge is "subtracted off" in a self-consistent fashion across all situations. While this practice is accepted and applied carefully, it is not understood. String theory, which postulates that all particles are actually vibrating string-like objects with finite extent, may offer a solution, but string theory currently is not complete — it does not offer a way to calculate the Standard Model — and there is no explicit proof it is correct.

Definition and Behavior of a Conductor

We now talk about *electric conductors*, both because they are interesting and because they provide a first opportunity to use boundary conditions to determine properties of the charge distribution, field, and potential.

An *electric conductor* is defined to be a material in which charge is able to flow completely freely in response to an external electric field. It is assumed, *a priori*, to contain equal and opposite amounts of positive and negative electric charge that perfectly cancel everywhere in the absence of an electric field ($\rho = 0$) but that can separate in response to an electric field. One can add charge to a conductor explicitly.

Without any calculation, we know what the response of the conductor will be to an externally applied electric field: If there is any field present in the conductor, positive and negative charge densities will separate in response to the field. That separation results in an additional field whose direction is opposite the applied field because of the direction the two polarities of charge move in response to the applied field. This movement occurs until the sum field vanishes, at which point no more charge separation occurs.

Derived Properties of a Conductor

From the above behavior, we may derive the following conductor properties from the fact that $\vec{E} = 0$ inside a conductor everywhere:

- **ρ also vanishes inside a conductor**
  This follows directly from Gauss's Law: because $\vec{E} = 0$ everywhere in the interior, then $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ also vanishes.

  Another way of seeing this, at least for a conductor with no net charge, is that, if there were a nonzero ρ, then there must be an equal and opposite amount of charge elsewhere in the conductor because the conductor is neutral overall. That opposite charge will be attracted to the nonzero ρ and move to cancel it.

Electric Conductors and Capacitors (cont.)

- **Any net charge or induced charge resides on the surface**
  The picture we described above, of charge separation being induced by the external field, does imply that there may be such *induced charge* on the surface. This does not violate Gauss's Law because $\vec{E}$ may be nonzero outside the conductor and thus one has to be careful calculating $\vec{\nabla} \cdot \vec{E}$ at the conductor boundary (we must resort to the boundary conditions we derived, Equations 2.49 and 2.49).

  Also, if we intentionally add charge to a conductor, it must also move to the surface by the same Gauss's Law argument. The mechanical way of seeing this is that, if we take a neutral conductor in an external field, which has no electric field or charge density in its interior, and add charge to the interior, the added charge repels itself, pushing itself to the exterior (as far as it can go without leaving the conductor), or one can think of it as the added charge attracting charge from the surface to cancel it, leaving net charge on the boundary. Regardless, the added charge that now appears on the surface arranges itself so there is no net field in the interior.

  Aside: As Griffiths notes in a footnote, this property can be interepreted to be a consequence of the fact that the electric field obeys the Coulomb's Law $1/r^2$ dependence in three dimensions (from which we derived Gauss's Law, which we used above in the proof). In a different number of dimensions, or with a different dependence on r, we would not have been able to derive Gauss's Law!

Electric Conductors and Capacitors (cont.)

- **A conductor has the same electric potential everywhere**
  That is, a conductor is an *equipotential*. This occurs because $\vec{E}$ vanishes everywhere in the conductor: any line integrals of $\vec{E}$ between two points must therefore also vanish. The conductor may have a nonzero electric potential, but the value is the same everywhere.

  One can see this using the gradient, too. If V were not constant in the conductor, there would be a nonzero $\vec{E} = -\vec{\nabla} V$, which we said above is not allowed.

- **The electric field just outside a conductor is always normal to its surface**
  This arises from the boundary conditions we derived, Equations 2.46 and 2.49. Since $\vec{E}$ vanishes inside the conductor, and the tangential component of $\vec{E}$ is continuous across any interface, the tangential component must vanish just outside the conductor, too. There is no such condition on the normal component because there may be an induced or net surface charge density σ on the surface.

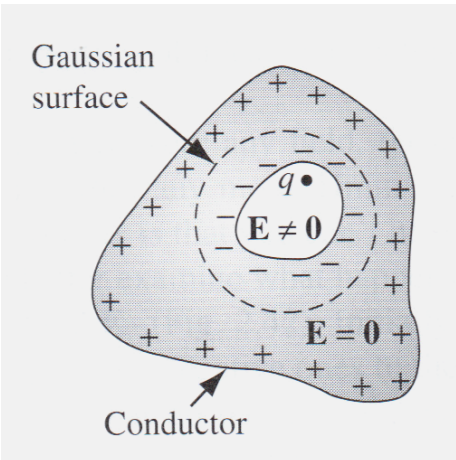
  Another way of looking at this is is that an electric field tangential to the surface would cause charge to move along the surface until that tangential component vanished. No such argument applies to the normal component because the charge is no longer free to move normal to the surface when it sits at the surface — it cannot leave the conductor.

Electric Conductors and Capacitors (cont.)

Conductors with Cavities

The mental image we have so far is of a conductor that has no cavities inside of it. What additional properties can we derive for a conductor with cavities?

- **A charge q inside a cavity in a conductor results in an equal induced charge q on the surface of the conductor**

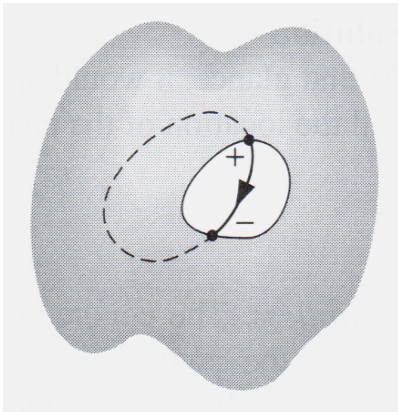


To see this, construct a surface $\mathcal{S}$ that lies inside the conductor but also contains the cavity. The electric field vanishes on $\mathcal{S}$ because it is in the conductor, so the net charge enclosed must vanish. Since a charge q is inside the cavity, there must be a canceling charge $-q$ induced at the inner surface of the cavity. If $-q$ is on the inner surface, then the canceling charge q must reside on the outer surface.

The exact distribution of q on the surface depends on the geometry. For a spherical outer surface, it will be uniformly distributed with surface charge density $\sigma = q/4\pi r^2$. Simply on the basis of symmetry, this is clearly also the solution if one adds charge q to a spherical conductor that has no cavity. For the case with the cavity, the charge q has no way of knowing that there is a cavity in the conductor because $\vec{E}$ vanishes inside the conductor, and thus the charge must rearrange itself in the same way as for a conductor without a cavity. A uniqueness theorem that we will prove later tells us that if we find one solution such as this one, it is the *only* solution.

Electric Conductors and Capacitors (cont.)

- **If there is no net charge inside a cavity in a conductor, the electric field inside the cavity vanishes, no matter what the external field applied to or net charge added to the conductor is**



© 2013 Griffiths, *Introduction to Electrodynamics*

This can be seen by assuming there is a nonzero electric field in the cavity. Since there is no charge in the cavity, the field lines must start and end on charges on the surface of the cavity. Therefore, there is a path through the cavity with $\int d\vec{\ell} \cdot \vec{E} \neq 0$. Now close the path with a segment inside the conductor. This portion of the now-closed loop $\mathcal{C}$ contributes nothing to the line integral $\oint_{\mathcal{C}} d\vec{\ell} \cdot \vec{E}$ over the entire loop because $\vec{E} = 0$ inside the conductor. To avoid violating $\oint_{\mathcal{C}} d\vec{\ell} \cdot \vec{E} = 0$, the first contribution from inside the cavity must vanish. Since this is true for any arbitrary path in the cavity, $\vec{E} = 0$ everywhere in the cavity.

Aside 1: Note the technique of proof we used, proof by contradiction, similar to what is done in mathematics. We will find that such techniques are frequently applied in electrodynamics.

Aside 2: This fact is used for electric shielding of experiments from external electric fields (and also electromagnetic waves) and is called a *Faraday cage*. Note that the conductor can have some net charge on it (and correspondingly sit at some nonzero electric potential with respect to infinity) and this property still holds. As we will show later, it also holds in the presence of external electromagnetic waves, which is actually the more typical and important application.

Lecture 3:
Force on a Conductor
Capacitors and Capacitance
Laplace's Equation
Uniqueness Theorems
Method of Images

Date Revised: 2014/02/18 17:30
Date Given: 2014/02/18
pp. 69–99 assigned as reading and reviewed in class

Surface Charge and the Force on the Surface of a Conductor

Our boundary condition for the normal component of the electric field combined with the fact that the electric field vanishes inside a conductor tells us that the electric field infinitesimally above the surface of the conductor is

$$\vec{E} = \frac{\sigma}{\epsilon_0}\,\hat{n} \tag{2.74}$$

where $\hat{n}$ points from the inside to the outside of the conductor.

There is a charge density σ at this point, and an electric field above it, so is there a force on the charge? Yes, but it is subtle to calculate. The thing to recognize is that the small element of charge $\sigma\, da$ in an infinitesimal area da cannot exert a force on itself. To find the field that this element of charge feels, we need to see what the field contribution of this charge itself is and subtract if off from the total field.

Electric Conductors and Capacitors (cont.)

We know (Griffiths Example 2.5) that the electric field of a charge sheet in the xy plane is $\vec{E} = \pm(\sigma/2\,\epsilon_0)\,\hat{z}$ where the sign applies depending on whether $z > 0$ or $z < 0$. While the small patch we are considering is not an infinite sheet, it looks like one if we are infinitesimally close to it. Thus, we may write the equations

$$\vec{E}_{outside} = \vec{E}_{other} + \frac{\sigma}{2\,\epsilon_0}\,\hat{n} \qquad \vec{E}_{inside} = \vec{E}_{other} - \frac{\sigma}{2\,\epsilon_0}\,\hat{n} \tag{2.75}$$

where $\vec{E}_{other}$ is the field due to the rest of the charge distribution excepting da. Using $\vec{E}_{outside} = (\sigma/\epsilon_0)\,\hat{n}$ and $\vec{E}_{inside} = 0$, we find $\vec{E}_{other} = (\sigma/2\,\epsilon_0)\,\hat{n}$. Therefore, the force per unit area is

$$\vec{f} = \frac{\vec{F}}{da} = \frac{\sigma\,da\,\vec{E}_{other}}{da} = \sigma\,\frac{\sigma}{2\,\epsilon_0}\,\hat{n} = \frac{\sigma^2}{2\,\epsilon_0}\,\hat{n} \tag{2.76}$$

Writing the force per unit area in terms of the field at the surface $\vec{E} = (\sigma/\epsilon_0)\,\hat{n}$, we have

$$\boxed{\vec{f} = \frac{\sigma^2}{2\,\epsilon_0}\,\hat{n} = \frac{\epsilon_0}{2}\,E^2\,\hat{n}} \tag{2.77}$$

That is, the surface of a conductor always feels an outward force. Consider what would happen if you put charge on a balloon with a metallized surface.

Capacitance

Consider two conductors (of arbitrary shapes) and suppose we put equal and opposite charges Q and $-Q$ on them. The potential difference ΔV between the two is of course given by the line integral of the electric field from any point on the surface of one to any point on the surface of the other. How does V scale with the charges?

The linear dependence of $\vec{E}$ on the charge density ρ ensures that ΔV is linear in Q. Therefore, we may define the *capacitance*

$$\boxed{C = \frac{Q}{\Delta V}} \tag{2.78}$$

Capacitance is a purely *geometric quantity*: it does not depend on the amount of charge on the two conductors (as long as equal and opposite charges are given to each). It does depend on the shapes of the conductors and their *relative* position and orientation because those determine the shape of the electric field (while Q varies its normalization). The unit of capacitance is Coulombs/volt, which we define to be the Farad, F.

One can talk about the capacitance of a single conductor with charge Q by implicitly assuming there is another conductor at infinity that has charge $-Q$ and is defined to be at $V = 0$.

Electric Conductors and Capacitors (cont.)

We can generalize the concept of capacitance to include multiple conductors by simply assuming a generalized linear relationship between voltages and charges as we argued must be true above:

$$V_i = \sum_{j=1}^{N} D_{ij} Q_j \qquad \text{or} \qquad \underline{V} = \underline{\underline{D}}\,\underline{Q} \tag{2.79}$$

where $\underline{V}$ and $\underline{Q}$ are N-element column matrices for the voltages and charges on the N conductors and $\underline{\underline{D}}$ is a $N \times N$ matrix that connects the two. It is explicit that any voltage depends linearly on all the charges. The capacitance matrix is $\underline{\underline{C}} = \underline{\underline{D}}^{-1}$, with

$$\boxed{Q_i = \sum_{j=1}^{N} C_{ij} V_j \qquad \text{or} \qquad \underline{Q} = \underline{\underline{C}}\,\underline{V}} \tag{2.80}$$

This form serves to make it clear that the capacitance is not just a single quantity between two conductors, but is more general.

To calculate the capacitance or capacitance matrix, one clearly needs to find out, given a set of charges $\{Q_i\}$, what the voltages $\{V_i\}$ are. To do this analytically, there typically must be a symmetry or approximation that allows one to guess what the charge distributions on the conductors are (*e.g.*, uniform as for an infinite parallel plate capacitor) and to calculate the field using Gauss's Law. The total charge on each electrode normalizes the voltage.

For the simple case of two electrodes with equal and opposite charges $\pm Q$ and voltages $\pm V$, we can relate the elements of the capacitance matrix to what we usually call the capacitance. First, let's write the usual scalar capacitance equation in terms of the individual electrode voltages $\pm V$ instead of the voltage difference $\Delta V = 2\,V$:

$$Q = C\,\Delta V = C\,(V - (-V)) = 2\,C\,V \tag{2.81}$$

Therefore

$$Q_1 = Q = 2\,C\,V = C\,V - (-C)\,(-V) = C\,V_1 - (-C)\,V_2 \tag{2.82}$$

$$Q_2 = -Q = -2\,C\,V = (-C)\,V + C\,(-V) = -C\,V_1 + C\,V_2 \tag{2.83}$$

and so we may conclude

$$\underline{\underline{C}} = \begin{bmatrix} C & -C \\ -C & C \end{bmatrix} \tag{2.84}$$

where C is the scalar capacitance we usually think of. We can see the matrix is symmetric, which we will prove generically later.

Electric Potential Energy of a Capacitor

In a simple two-electrode capacitor with charges $\pm q$ on the electrodes and a voltage difference $\Delta V = q/C$ between the two electrodes, the amount of work required to change the charge from q to $q + dq$ is given by the amount of work required to move a charge dq from the negative electrode (which has charge $-q$ and voltage $-\Delta V(q)/2$) to the positive electrode (which has charge $+q$ and voltage $+\Delta V(q)/2$):

$$dU = \Delta V(q)\, dq = \frac{q}{C}\, dq \tag{2.85}$$

Note that ΔV is a function of q here: the voltage is not held fixed while the charge is moved; rather, the voltage and charge increase together.

We integrate this expression from 0 to the final charge Q to find

$$U = \frac{1}{C}\int_0^Q q\, dq = \frac{1}{2}\,\frac{Q^2}{C} \tag{2.86}$$

Alternatively, using $Q = C\,\Delta V$,

$$\boxed{U = \frac{1}{2}\,\frac{Q^2}{C} = \frac{1}{2}\,C\,(\Delta V)^2} \tag{2.87}$$

Electric Conductors and Capacitors (cont.)

What happens for a multi-electrode system? Rather than modeling the process using transferral of charge from one electrode to the other, we need to instead model it as bringing charge in from infinity. If we put equal and opposite charges on two electrodes, then we bring equal and opposite charges in from infinity to put on the two electrodes, which is the equivalent of transferring the charge from one electrode to another: instead of bringing negative charge in from infinity to put on a given electrode, take an equal amount of positive charge off of it and take it out to infinity, then bring that charge back in to put on the complementary positive electrode. (We implicitly define $V = 0$ at infinity.)

So, we consider bringing charge dq_i in from infinity and adding it to electrode i. The change in the electric potential energy of the system due to adding this charge is

$$dU_i = V_i \, dq_i = \sum_{j=1}^{N} C_{ij}^{-1} q_j \, dq_i \tag{2.88}$$

There are two possible double-countings we must avoid: 1) This infinitesimal element of charge dq_i is moved from $V = 0$ at infinity to $V = V_i$ on the ith electrode, so the voltages of the other electrodes are irrelevant during this infinitesimal charge transfer and we should not bring them into the equation; 2) The charges on all the electrodes are fixed as we bring this infinitesimal element of charge dq_i in, so we do not need to somehow explicitly account for the change of the voltages on the other electrodes as this charge is brought in.

Electric Conductors and Capacitors (cont.)

Now, let's integrate over dq_i. We will later do a sum over i. The ordering of the two steps does not matter because we proved earlier that the electric potential energy does not depend on the order of assembly. But we do need to worry about what order the charges have been accumulated in that we don't want to calculate cross-terms for charges that don't yet exist. Let's assume that, if we are integrating the ith charge, then the first $i-1$ charges have already been integrated to their full value. Thus, when we integrate, we obtain

$$U_i = \frac{1}{2} C_{ii}^{-1} Q_i^2 + \sum_{j=1, j\neq i}^{i-1} C_{ij}^{-1} Q_i Q_j \tag{2.89}$$

where the first time has a $1/2$ because we integrate $q_i\, dq_i$ while the second term does not because we integrate $Q_j\, dq_i$. Next, we need to sum over i:

$$U = \frac{1}{2} \sum_{i=1}^{N} C_{ii}^{-1} Q_i^2 + \sum_{i=1}^{N} \sum_{j=1}^{i-1} C_{ij}^{-1} Q_i Q_j = \frac{1}{2} \sum_{i=1}^{N} C_{ii}^{-1} Q_i^2 + \frac{1}{2} \sum_{i,j=1, i\neq j}^{N} C_{ij}^{-1} Q_i Q_j \tag{2.90}$$

$$= \frac{1}{2} \sum_{i,j=1}^{N} C_{ij}^{-1} Q_i Q_j \tag{2.91}$$

where we modified the second sum to be symmetric (assuming $\underline{\underline{C}}^{-1}$ is symmetric, which we will prove below) and included a factor of $1/2$ to correct for double-counting.

Electric Conductors and Capacitors (cont.)

We can write this more succinctly as

$$\boxed{U = \frac{1}{2}\,\underline{Q}^T\,\underline{\underline{C}}^{-1}\,\underline{Q}} \tag{2.92}$$

Using $\underline{Q} = \underline{\underline{C}}\,\underline{V}$, we can rewrite as

$$\boxed{U = \frac{1}{2}\,\underline{V}^T\,\underline{\underline{C}}\,\underline{V}} \tag{2.93}$$

Let's check that this gives the correct result for an elementary capacitor with the two electrodes having equal and opposite charges $\pm Q$ and voltages $\pm V$. Using the capacitance matrix we derived earlier,

$$\begin{aligned} U &= \frac{1}{2}\left[C_{11}(+V)^2 + C_{22}(-V)^2 + C_{12}(+V)(-V) + C_{21}(-V)(+V)\right] \\ &= \frac{1}{2}\,C\,V^2\left[1 + (-1)^2 + (-1)^2 + (-1)^2\right] = 2\,C\,V^2 = \frac{1}{2}\,C(\Delta V)^2 \end{aligned} \tag{2.94}$$

(recall, $C_{11} = C_{22} = C$ and $C_{12} = C_{21} = -C$) as expected.

Symmetry of the Capacitance Matrix

In the above discussion, we implicitly assumed symmetry of the capacitance matrix when we converted the asymmetric cross-term sum to a symmetric cross-term sum. Why was it ok to assume this?

Let's consider two electrodes, i and j with $i \neq j$. From Equation 2.89, their contribution to the potential energy, assuming j has been charged up before i, is

$$U = \frac{1}{2}\left(C_{ii}^{-1} Q_i^2 + C_{jj}^{-1} Q_j^2\right) + C_{ij}^{-1} Q_i\, Q_j \tag{2.95}$$

If we reverse the charging order, which, in our initial discussion of the electric potential energy, we argued can have no effect on the energy, then the same contribution to the electric potential energy is

$$U = \frac{1}{2}\left(C_{ii}^{-1} Q_i^2 + C_{jj}^{-1} Q_j^2\right) + C_{ji}^{-1} Q_i\, Q_j \tag{2.96}$$

Equating the two and recognizing that Q_i and Q_j are arbitrary tells us

$$C_{ij}^{-1} = C_{ji}^{-1} \qquad \text{and, therefore,} \qquad C_{ij} = C_{ji} \tag{2.97}$$

Section 3
Advanced Electrostatics

Intuitive Approach to Laplace's Equation

As we mentioned earlier, the integral forms for the electric field or the potential

$$\vec{E}(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\int_{\mathcal{V}} d\tau'\rho(\vec{r}\,')\frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} \qquad \text{and} \qquad V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\int_{\mathcal{V}} d\tau'\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \quad (3.1)$$

are always correct but can be problematic to deal with in practice. Most systems will not have symmetries that make the integrals easily doable (or avoidable via Gauss's Law). Moreover, and this is the greater problem, it is rare that one completely specifies $\rho(\vec{r})$ in setting up a problem. Experimentally, what we can control are the shapes, positions, and potentials (voltages) of conductors. We don't control how the charge arranges itself on the conductors. Thus, we need to seek alternate ways to solve for the potential and field over all of space. Laplace's and Poisson's Equations are the key.

Intuitive Approach to Laplace's Equation (cont.)

Laplace's Equation in One Dimension

In one dimension, Laplace's Equation takes the simple form

$$\frac{d^2V}{dx^2} = 0 \tag{3.2}$$

We can solve this by direct integration to obtain

$$V(x) = mx + b \tag{3.3}$$

where m and b are two constants of integration. We determine m and b by *boundary conditions*: specification of V or dV/dx at specific point(s). In the one dimensional case, there are two options for how to specify the boundary conditions:

- Specify V at two points.
- Specify V at one point and dV/dx at one point (possibly the same point).

Note that these are the only choices in one dimension. Specifying dV/dx at two points either yields a contradiction (if two different values of dV/dx are given) or insufficient information (if the same value is given). There are no other quantities to specify: all higher derivatives vanish thanks to Laplace's Equation.

Intuitive Approach to Laplace's Equation (cont.)

Let us note two important characteristics of the solutions of Laplace's Equation:

- $V(x)$ is related to the average of any pair of points $V(x+a)$ and $V(x-a)$ for any a such that $x \pm a$ belong to the region being considered:

$$\begin{aligned}\frac{1}{2}\left[V(x+a)+V(x-a)\right] &= \frac{1}{2}\left[(m(x+a)+b)+(m(x-a)+b)\right] \\ &= mx+b = V(x) \end{aligned} \tag{3.4}$$

  Solutions to Laplace's Equation have this intrinsic *averaging* property.

- $V(x)$ has no *nontrivial* local maxima or minima. We already mentioned this property for the three-dimensional Laplace's Equation. The proof is straightforward in one dimension. Suppose x_0 is a local maximum or minimum. Then we have $dV/dx = 0$ at this point x_0. Then, for any other point x_1:

$$\left.\frac{dV}{dx}\right|_{x_1} = \left.\frac{dV}{dx}\right|_{x_0} + \int_{x_0}^{x_1} \frac{d^2V}{dx^2}\,dx = 0 \tag{3.5}$$

  Therefore, if dV/dx vanishes anywhere, $V(x)$ is a constant. This is a trivial local maximum/minimum. If dV/dx vanishes nowhere, then the endpoints of the region give the maximum and minimum of $V(x)$ or, if there are no endpoints, there are no maxima or minima at all. Consider, for example, a uniform electric field $\vec{E}_0$ over all of space.

Intuitive Approach to Laplace's Equation (cont.)

Laplace's Equation in Two or Three Dimensions

We quote the analogues of the above two properties for arbitrary numbers of dimensions and prove them for three dimensions:

- The value $V(\vec{r})$ of a solution to Laplace's Equation at any point is equal to the average of its value on any sphere centered on that point in the region of interest:

$$V(\vec{r}) = \frac{\int_{\mathcal{S}_a(\vec{r})} da\, V(\vec{r}\,')}{\int_{\mathcal{S}_a(\vec{r})} da} \tag{3.6}$$

where $\mathcal{S}_a(\vec{r})$ is the sphere of radius a centered on $\vec{r}$. This is straightforward to show (Griffiths Problem 3.37). Let's integrate Laplace's Equation over the volume enclosed by $\mathcal{S}_a(\vec{r})$, $\mathcal{V}_a(\vec{r})$, and use the divergence theorem:

$$\begin{aligned} 0 = \int_{\mathcal{V}_a(\vec{r})} d\tau'\, \nabla^2_{\vec{r}\,'} V(\vec{r}\,') &= \int_{\mathcal{S}_a(\vec{r})} da'\, \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} V(\vec{r}\,') \\ &= \int_{\mathcal{S}_a(\vec{r})} da'\, \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'-\vec{r}} V(\vec{r}\,') \end{aligned} \tag{3.7}$$

In the last step, we have used the fact that $\vec{\nabla}$ does not care about the location of the origin (since it is just an offset).

Intuitive Approach to Laplace's Equation (cont.)

Now, we can define $\vec{s} = \vec{r}\,' - \vec{r}$. In this coordinate system, where $\vec{r}$ is at the origin, $\widehat{n}(\vec{r}\,') = \widehat{s}$, the radial unit vector in the $\vec{s}$ coordinate system. So, we have (inserting a factor $1/4\,\pi\,a^2$):

$$0 = \frac{1}{4\,\pi\,a^2} \int_{\mathcal{S}_a(\vec{s}=0)} a^2 d\Omega_s \left.\frac{\partial V}{\partial s}\right|_{\vec{s}} \tag{3.8}$$

where $\mathcal{S}_a(\vec{s}=0)$ is the sphere of radius a centered on the origin of the $\vec{s}$ system (*i.e.*, the same as the sphere of radius a centered on $\vec{r}$ in the $\vec{r}\,'$ coordinate system). If we pull the factor a^2 outside of the integral, the integral is now over the spherical angles in the $\vec{s}$ coordinate system, while the derivative is in the radial direction in this coordinate system. Thus, we can pull the derivative outside the integral too, yielding

$$0 = \frac{1}{4\,\pi\,a^2}\, a^2 \int_{\mathcal{S}_a(\vec{s}=0)} d\Omega_s \left.\frac{\partial V}{\partial s}\right|_{\vec{s}} = \frac{1}{4\,\pi}\,\frac{\partial}{\partial s} \int_{\mathcal{S}_a(\vec{s}=0)} d\Omega_s\, V(\vec{s}) \tag{3.9}$$

(Note that, though V is a function of $\vec{s}$, only the angular coordinates of $\vec{s}$ are being varied in the integration because the integral is over these angular coordinates.)

Intuitive Approach to Laplace's Equation (cont.)

Thus, the integral must be a constant

$$C = \frac{1}{4\,\pi} \int_{\mathcal{S}_a(\vec{s}=0)} d\Omega_s\, V(\vec{s}) = \frac{1}{4\,\pi\,a^2} \int_{\mathcal{S}_a(\vec{r})} da'\, V(\vec{r}\,') \tag{3.10}$$

where we switched the variable of integration back to $\vec{r}\,'$ and we reinserted a^2. The right side is just the average of V over the sphere of radius a centered at $\vec{r}$. Since this holds for any a, it must hold as $a \longrightarrow 0$, which tells us $C = V(\vec{r})$. So, we have

$$V(\vec{r}) = \frac{1}{4\,\pi\,a^2} \int_{\mathcal{S}_a(\vec{r})} da'\, V(\vec{r}\,') \tag{3.11}$$

- As a consequence of Laplace's Equation and the above property, V can have no local maxima or minima in the region of interest. The proof of this property is trivial: if there were such a candidate maximum (minimum), simply draw a sphere around it. Because the point is a maximum (minimum) there must be some radius of the sphere for which the values of all the points on the sphere are less than (greater than) the value at the candidate maximum (minimum). The average over this sphere is therefore less than (greater than) the value at the candidate maximum (minimum). This contradicts the above averaging property.

  One could also prove this by calculating the gradient at any point $\vec{r}\,'$ in the region by doing a line integral of Laplace's Equation from the candidate extremum $\vec{r}$ to that point. Since the gradient vanishes at the candidate extremum, and the line integral vanishes by Laplace's Equation, the gradient vanishes at $\vec{r}\,'$.

Uniqueness Theorems

Before obtaining a solution of Laplace's and Poisson's Equations, we prove some uniqueness theorems we will need. This section draws from Jackson §1.8 and §1.9.

Green's Identities and Theorem

First, some mathematical preliminaries. Let us apply the divergence theorem to the function $\phi\vec{\nabla}\psi$ where $\phi(\vec{r})$ and $\psi(\vec{r})$ are arbitrary functions:

$$\oint_{\mathcal{S}} da\,\hat{n}\cdot\left(\phi\vec{\nabla}\psi\right) = \int_{\mathcal{V}(\mathcal{S})} d\tau\,\vec{\nabla}\cdot\left(\phi\vec{\nabla}\psi\right)$$

This yields *Green's First Identity*:

$$\boxed{\oint_{\mathcal{S}} da\,\phi\,\hat{n}\cdot\vec{\nabla}\psi = \int_{\mathcal{V}(\mathcal{S})} d\tau\left[\phi\nabla^2\psi + \vec{\nabla}\phi\cdot\vec{\nabla}\psi\right]} \tag{3.12}$$

The function $\hat{n}\cdot\vec{\nabla}\psi$ is the *normal derivative* of ψ because it is the projection of the gradient of ψ along the direction normal to the surface. If we exchange ϕ and ψ and then difference the two versions, we have *Green's Second Identity* or *Green's Theorem*:

$$\boxed{\oint_{\mathcal{S}} da\left[\phi\,\hat{n}\cdot\vec{\nabla}\psi - \psi\,\hat{n}\cdot\vec{\nabla}\phi\right] = \int_{\mathcal{V}(\mathcal{S})} d\tau\left[\phi\nabla^2\psi - \psi\nabla^2\phi\right]} \tag{3.13}$$

Generic Uniqueness Proof

Let us use the above Green's identities to prove the uniqueness of solutions to Poisson's Equation for three types of boundary conditions:

- *Dirichlet boundary condition*
  In this case, the value of the potential $V(\vec{r})$ is specified on all bounding surfaces. This is the most typical experimentally realized situation, where we attach a number of conductors to voltage sources to set their voltages.
- *Neumann boundary condition*
  In this case, the value of the normal derivative of the voltage, $\hat{n} \cdot \vec{\nabla} V(\vec{r})$, is specified on the boundary. An example of such a condition is specification of the electric field (or, equivalently, the surface charge density) at the surfaces of a set of conductors; since the tangential electric field vanishes at these surfaces, the normal electric field fully defines the electric field at the conductors.
- *Mixed boundary conditions*
  Dirichlet in some places, Neumann in others, is allowed as long as both are not specified at the same place.

If the volume under consideration is not bounded by a surface on which we specify the boundary condition, then we must also specify a boundary condition at infinity.

Uniqueness Theorems (cont.)

Suppose we have specified one of the above three types of boundary conditions. Assume that, for a particular given charge distribution $\rho(\vec{r})$, there are two independent solutions $V_1(\vec{r})$ and $V_2(\vec{r})$ of Poisson's Equation that satisfy this boundary condition. Let $V_3 = V_1 - V_2$. Since the charge distribution is the same, $\nabla^2 V_1 = -\rho/\epsilon_0 = \nabla^2 V_2$ and thus $\nabla^2 V_3 = 0$: V_3 satisfies Laplace's Equation. By a similar differencing argument, V_3 either satisfies the Dirichlet boundary condition $V_3(\vec{r} \in \mathcal{S}) = 0$, the Neumann boundary condition $\hat{n} \cdot \vec{\nabla} V_3(\vec{r} \in \mathcal{S}) = 0$, or a mixed boundary condition of these types. If we apply Green's first identity with $\phi = \psi = V_3$, we have

$$\oint_{\mathcal{S}} da\, V_3\, \hat{n} \cdot \vec{\nabla} V_3 = \int_{\mathcal{V}(\mathcal{S})} d\tau \left(V_3 \nabla^2 V_3 + \vec{\nabla} V_3 \cdot \vec{\nabla} V_3 \right) \tag{3.14}$$

The left side vanishes because of the boundary condition (any type). The first term on the right side vanishes by Laplace's Equation. Thus, we have

$$\int_{\mathcal{V}(\mathcal{S})} d\tau\, |\vec{\nabla} V_3|^2 = 0 \tag{3.15}$$

Since the integrand is nonnegative, we may conclude

$$\vec{\nabla} V_3(\vec{r}) = 0 \qquad \Longleftrightarrow \qquad V_3 = \text{constant} \tag{3.16}$$

which implies that our two candidate solutions $V_1(\vec{r})$ and $V_2(\vec{r})$ differ by at most a constant. Hence, uniqueness is proven.

Special Cases of Uniqueness Theorem

Given the above, we may state/prove three special cases of the uniqueness theorem, the ones given in Griffiths:

- *The solution to Laplace's Equation in some volume $\mathcal{V}$ is uniquely specified if V is specified on the boundary surface $\mathcal{S}(\mathcal{V})$.*
  This is the above uniqueness theorem with $\rho = 0$ in $\mathcal{V}$ and a Dirichlet boundary condition on $\mathcal{S}(\mathcal{V})$.
- *The solution to Poisson's Equation in some volume $\mathcal{V}$ is uniquely specified if $\rho(\vec{r})$ is specified throughout the region and V is specified on the boundary surface $\mathcal{S}(\mathcal{V})$.*
  This is the above uniqueness theorem with arbitrary $\rho(\vec{r})$ in $\mathcal{V}$ and a Dirichlet boundary condition on $\mathcal{S}(\mathcal{V})$.

Uniqueness Theorems (cont.)

- *In a volume $\mathcal{V}$ surrounded by conductors at the surface(s) $\mathcal{S}(\mathcal{V})$ and containing a specified charge density $\rho(\vec{r})$, the electric field is uniquely determined if the total charge on each conductor is specified.*

  This one is not as obvious, but is still relatively straightforward thanks to the generic uniquness theorem. A solution is uniquely specified if we have a Dirichlet boundary condition. Since $\mathcal{S}(\mathcal{V})$ consists only of conductors, and conductors are equipotentials, specifying the Dirichlet boundary condition in this case consists only of specifying the N voltages $\{V_i\}$ on the N bounding surfaces. But it is the charges that are given, so how do we know the voltages? Suppose $\rho = 0$ in $\mathcal{V}$. Then, we know that the capacitance matrix relates the charges $\{Q_i\}$ and the voltages in the absence of ρ, which we will call $\{V_i^0\}$: $\underline{V}^0 = \underline{\underline{C}}^{-1}\underline{Q}$. (We don't need to know what $\underline{\underline{C}}$ is, we only need to know it exists, which we argued when we defined it.) Finally, when we add back $\rho(\vec{r})$, we know that, by superposition, the voltages $\{V_i\}$ are

$$V_i = V_i^0 + \int_{\mathcal{V}} d\tau \, \frac{\rho(\vec{r})}{|\vec{r} - \vec{r}\,'|} \tag{3.17}$$

  Thus, we are assured that, if the $\{Q_i\}$ and $\rho(\vec{r})$ are given, they uniquely determine the $\{V_i\}$, and then we have a Dirichlet boundary condition and so the generic uniqueness theorem applies. Note how this proof relied on the boundary surfaces being conductors! Knowing the total charges on nonconducting boundary surfaces would not be sufficient.

Method of Images

The Basic Idea

The method of images uses the concept of uniqueness of solutions to Poisson's Equation. Basically, given a physical setup involving a true charge distribution $\rho(\vec{r})$ and a boundary condition for some volume $\mathcal{V}$, one tries to replace the region outside of $\mathcal{V}$ with an *image charge* distribution $\rho_{image}(\vec{r})$ such that, when the image charge's potential is summed with that of $\rho(\vec{r})$, the boundary condition is met.

The technique works because of the uniqueness theorem: since the potential due to the image and original charges matches the boundary condition and satisfies Poisson's Equation with the same source term inside $\mathcal{V}$, it is *the* solution to Poisson's Equation for that source term and choice of boundary condition.

The imagined charge distribution is called image charge because, at least in the example of the boundary condition being imposed by the presence of a conductor, the image charges appear to be a mirror image of the original charges through the boundary. "Image charge" is also used (somewhat erroneously) to refer to the surface charge induced on a conducting boundary that sources the potential that one imagines is due to the image charge.

We will see later how the potential due to the image charge distribution (the induced surface charge) is a component of the particular problem's Green Function.

Method of Images (cont.)

A Point Charge near a Grounded Infinite Conducting Plane

For a system with the point charge q at $d\,\hat{z}$ and the conducting plane at $z = 0$ with $V = 0$, the appropriate image charge is $-q$ at $-d\,\hat{z}$. By symmetry, the (Dirichlet) boundary condition $V = 0$ at $z = 0$ is met. Thus, the solution for $V(\vec{r})$ for $\vec{r}$ belonging to the $z > 0$ half-space is

$$V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\left[\frac{q}{\sqrt{x^2+y^2+(z-d)^2}} - \frac{q}{\sqrt{x^2+y^2+(z+d)^2}}\right] \tag{3.18}$$

The potential clearly satisfies $V(z = 0) = 0$. Let's use this solution to do some other calculations:

- *Induced surface charge*
  This we can calculate by recognizing that it is given by the change in the normal component of the electric field at the conducting boundary. Since $\vec{E} = -\vec{\nabla}V$,

$$\begin{aligned}\sigma &= -\,\epsilon_0\,\frac{\partial V}{\partial z}\bigg|_{z=0} = \frac{q}{4\,\pi}\left[\frac{z-d}{(x^2+y^2+(z-d)^2)^{3/2}} - \frac{z+d}{(x^2+y^2+(z+d)^2)^{3/2}}\right]\bigg|_{z=0} \\ &= -\frac{q}{2\,\pi}\,\frac{d}{(x^2+y^2+d^2)^{3/2}}\end{aligned} \tag{3.19}$$

Method of Images (cont.)

We can calculate the total induced surface charge:

$$Q_{ind} = \int_0^\infty r\,dr \int_0^{2\pi} d\phi \frac{-q\,d}{2\pi} \frac{1}{(r^2+d^2)^{3/2}} = q\,d \left. \frac{1}{\sqrt{r^2+d^2}} \right|_0^\infty = -q \quad (3.20)$$

This is an example of an important general rule: *The equality of the induced surface charge and the image charge holds for any method of images situation because of Gauss's Law*: the image charge produces the same potential in $\mathcal{V}$ as the induced surface charge, so the normal gradient of the potential at the surface is the same for the induced surface charge and the image charge, so the flux of the electric fields are the same, so the enclosed charge is the same.

- *Force on the point charge*
  The induced charge is equal and opposite in sign to the point charge, so the two are attracted to each other. We can calculate the force by taking the gradient of the potential due to the *image charge only* (because the original point charge does not feel a force due to its own potential). Since the image charge's potential is just that of a point charge, calculating the force is straightforward:

$$\vec{F} = q\,\vec{E}_{image\ charge}(z=d) = -\frac{1}{4\pi\epsilon_0}\frac{q^2}{(2d)^2}\hat{z} \quad (3.21)$$

We could have calculated this more directly by just assuming the image charge and the real charge act like two normal point charges, but it's good to see the two techniques give the same result.

Method of Images (cont.)

- *Electric potential energy*
  Here we have to be more careful because potential energy is not linear in charge, and, moreover, because the induced charge depends on the original point charge. Let's figure this out by calculating the work one would have to do against the attractive force to bring q from $z = d$ to $z = \infty$.

$$U = -\int_d^\infty (-F(z))\, dz = -\frac{1}{4\,\pi\,\epsilon_0}\frac{q^2}{4}\int_d^\infty \frac{dz}{z^2} = -\frac{1}{4\,\pi\,\epsilon_0}\frac{q^2}{4\,d} \tag{3.22}$$

Note that this result is half what one would get for the potential energy of two equal and opposite point charges separated by a distance $2d$:

$$U_{alt} = -\frac{1}{4\,\pi\,\epsilon_0}\frac{q^2}{2d} \tag{3.23}$$

There are two ways to understand this. The first is to recognize that, unlike in the case of two point charges, no energy is gained or lost in moving the negative charge because it is in the conductor, where $V = 0$ and thus $q\,V = 0$ everywhere. The second is to recognize that the above expression is the energy stored in all of space in the field of two point charges, but, in this case, the field is only real in the $z > 0$ half-space and so the integrated energy is reduced by a factor of 2.

A Point Charge near a Conducting Sphere

Consider a conducting sphere of radius R centered on the origin and held at $V = 0$. Place a point charge at $a\,\hat{z}$.

By symmetry, the appropriate image charge must be on the z axis. Let its value be q' and its position be $b\,\hat{z}$, where b may be positive or negative. We can find q' and b by requiring that $V = 0$ at $\vec{r} = \pm R\,\hat{z}$:

$$V(z = \pm R) = \frac{1}{4\,\pi\,\epsilon_0}\left[\frac{q}{a \mp R} + \frac{q'}{R \mp b}\right] \tag{3.24}$$

Setting both equations equal to zero yields

$$q' = -q\,\frac{R}{a} \qquad\qquad b = \frac{R^2}{a} \tag{3.25}$$

We see that both values are always physically reasonable because $R < a$. In particular, $b < R$ so the image charge remains outside the region for which the boundary condition is specified.

Method of Images (cont.)

The potential at a point $(r \geq R, \theta, \phi)$ is

$$V(r \geq R, \theta, \phi) = \frac{q}{4\,\pi\,\epsilon_0}\left[\frac{1}{|\vec{r} - a\,\widehat{z}|} - \frac{R/a}{|\vec{r} - \frac{R^2}{a}\,\widehat{z}|}\right] \tag{3.26}$$

$$= \frac{q}{4\,\pi\,\epsilon_0}\left[\frac{1}{\sqrt{r^2 sin^2\theta + (a - r\,\cos\theta)^2}} - \frac{R/a}{\sqrt{r^2 sin^2\theta + (\frac{R^2}{a} - r\,\cos\theta)^2}}\right] \tag{3.27}$$

Let's check that the boundary condition $V(r = R) = 0$ is satisfied:

$$V(r = R, \theta, \phi) = \frac{q}{4\,\pi\,\epsilon_0}\left[\frac{1}{\sqrt{R^2 sin^2\theta + (a - R\,\cos\theta)^2}} - \frac{R/a}{\sqrt{R^2 sin^2\theta + (\frac{R^2}{a} - R\,\cos\theta)^2}}\right]$$

$$= \frac{q}{4\,\pi\,\epsilon_0}\left[\frac{1}{\sqrt{R^2\,\sin^2\theta + (a - R\cos\theta)^2}} - \frac{1}{\sqrt{a^2\,\sin^2\theta + (R - a\cos\theta)^2}}\right]$$

$$= 0 \tag{3.28}$$

Method of Images (cont.)

Let's calculate the induced surface charge from $\hat{n} \cdot \vec{\nabla} V = \partial V / \partial r$:

$$\sigma = -\epsilon_0 \left. \frac{\partial V}{\partial r} \right|_{r=R} \tag{3.29}$$

$$= \frac{q}{4\pi} \left[\frac{R \sin^2\theta - (a - R\cos\theta)\cos\theta}{(R^2 sin^2\theta + (a - R\cos\theta)^2)^{3/2}} - \frac{R}{a} \frac{R\sin^2\theta - (\frac{R^2}{a} - R\cos\theta)\cos\theta}{\left(R^2 sin^2\theta + (\frac{R^2}{a} - R\cos\theta)^2\right)^{3/2}} \right]$$

$$= \frac{q}{4\pi} \left[\frac{R - a\cos\theta}{(R^2 + a^2 - 2aR\cos\theta)^{3/2}} - \frac{a^2}{R^2} \frac{R - \frac{R^2}{a}\cos\theta}{(a^2 + R^2 - 2aR\cos\theta)^{3/2}} \right]$$

$$= \frac{q}{4\pi} \frac{R(1 - \frac{a^2}{R^2})}{(R^2 + a^2 - 2aR\cos\theta)^{3/2}}$$

$$= -\frac{q}{4\pi R^2} \frac{R}{a} \frac{1 - \frac{R^2}{a^2}}{\left(1 + \frac{R^2}{a^2} - 2\frac{R}{a}\cos\theta\right)^{3/2}}$$

One can show by integration that the total induced charge is q', matching the image charge as one would expect from Gauss's Law. The force on the point charge and the electric potential energy can be calculated in a manner similar to that used for the conducting plane.

Some Related Examples

These are drawn from Jackson Chapter 2.

- *Point charge in the presence of a charged, insulated, conducting sphere*

  We can solve this problem by superposition. Suppose we want to have a charge Q on the sphere. This is the same as first bringing the point charge q in while the sphere is grounded, disconnecting the grounding wire, and then adding $Q - q'$ ($> Q$ for $q > 0$). Since the induced surface charge q' is in full equilibrium with q before $Q - q'$ is added — *i.e.*, the net electric field is normal to the surface of the conductor, so the induced charge can move nowhere — the additional charge $Q - q'$ just spreads out over the surface evenly. This maintains the boundary condition that the surface of the sphere is an equipotential (= the field must be normal to the surface). Since we have found a configuration that respects the boundary conditions, it is the only valid configuration by the uniqueness theorem. The potential is therefore

$$V(r \geq R, \theta, \phi) = \frac{q}{4\,\pi\,\epsilon_0}\left[\frac{1}{|\vec{r} - a\,\widehat{z}|} - \frac{R/a}{|\vec{r} - \frac{R^2}{a}\,\widehat{z}|}\right] + \frac{Q + q\frac{R}{a}}{4\,\pi\,\epsilon_0\,|\vec{r}|} \qquad (3.30)$$

Method of Images (cont.)

By Gauss's Law, the field due to $Q - q'$ is the same as that of a point charge $Q - q'$ at the origin. Therefore, the electric potential at the surface of the sphere is $V(r = R) = (Q - q')/(4\,\pi\,\epsilon_0\,R)$.

The surface charge density is the original induced charge for the grounded sphere with the addition of $Q - q'$ distributed uniformly, which we explained above.

The force and potential energy can be calculated in a manner similar to that for the grounded sphere.

- *Point charge in the presence of a conducting sphere at fixed potential*

We can solve this problem by applying the last problem. In the last case, the conductor acquired a voltage $(Q - q')/(4\,\pi\,\epsilon_0\,R)$. So, if we want the conductor to sit at a voltage V_0, we just have to apply the last problem with $Q = 4\,\pi\,\epsilon_0\,V_0\,R + q'$.

Lecture 4:
Green Functions
Obtaining Green Functions from the Method of Images

Date Revised: 2014/02/21 6:00
Date Given: 2014/02/20

Integral Equation for the Electric Potential

We apply Green's Theorem with $\phi(\vec{r}\,') = V(\vec{r}\,')$ and $\psi(\vec{r}\,') = |\vec{r} - \vec{r}\,'|^{-1}$. Note that $\vec{r}\,'$ is the variable we integrate over; $\vec{r}$ is considered a constant for the purposes of the Green's Theorem integrals.

$$\int_{\mathcal{V}(\mathcal{S})} d\tau' \left[V(\vec{r}\,') \nabla^2_{\vec{r}\,'} \frac{1}{|\vec{r} - \vec{r}\,'|} - \frac{1}{|\vec{r} - \vec{r}\,'|} \nabla^2_{\vec{r}\,'} V(\vec{r}\,') \right]$$
$$= \oint_{\mathcal{S}} da \left[V(\vec{r}\,') \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} \frac{1}{|\vec{r} - \vec{r}\,'|} - \frac{1}{|\vec{r} - \vec{r}\,'|} \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} V(\vec{r}\,') \right] \quad (3.31)$$

Next, note that

$$\nabla^2_{\vec{r}\,'} \frac{1}{|\vec{r} - \vec{r}\,'|} = -4\pi\,\delta(\vec{r} - \vec{r}\,') \quad (3.32)$$

because (Equations 2.33 and 2.57)

$$\vec{\nabla}_{\vec{r}\,'} \cdot \frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3} = 4\pi\,\delta(\vec{r} - \vec{r}\,') \quad \text{and} \quad \vec{\nabla}_{\vec{r}\,'} \frac{1}{|\vec{r} - \vec{r}\,'|} = -\frac{\vec{r} - \vec{r}\,'}{|\vec{r} - \vec{r}\,'|^3}$$

Formal Solution to Poisson's Equation: The Green Function (cont.)

Using the above expression for the Laplacian of $|\vec{r} - \vec{r}\,'|^{-1}$, doing the integral over the delta function, applying Poisson's Equation, moving the second term on the right side to the left side, and multiplying everything by $-\frac{1}{4\,\pi}$ yields, now only for $\vec{r} \in \mathcal{V}(\mathcal{S})$:

$$V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\int_{\mathcal{V}(\mathcal{S})} d\tau'\,\frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} + \frac{1}{4\,\pi}\oint_{\mathcal{S}} da\,\left[\frac{1}{|\vec{r} - \vec{r}\,'|}\,\widehat{n}(\vec{r}\,')\cdot\vec{\nabla}_{\vec{r}\,'}\,V(\vec{r}\,') - V(\vec{r}\,')\,\widehat{n}(\vec{r}\,')\cdot\vec{\nabla}_{\vec{r}}\frac{1}{|\vec{r} - \vec{r}\,'|}\right] \tag{3.33}$$

(The left side vanishes for $\vec{r} \notin \mathcal{V}(\mathcal{S})$). This is a formal equation for the electric potential. The boundary conditions are present on the right side: in the case of Dirichlet, we specify $V(\vec{r}\,')$ for $\vec{r}\,' \in \mathcal{S}$, while in the case of Neumann, we specify $\widehat{n}(\vec{r}\,')\cdot\vec{\nabla}_{\vec{r}\,'}\,V(\vec{r}\,')$ for $\vec{r}\,' \in \mathcal{S}$. Since one or the other is left unspecified, this is not a closed-form solution but rather an integral equation for $V(\vec{r}\,')$ for $\vec{r}\,' \in \mathcal{V}(\mathcal{S})$ given the unspecified boundary condition for $\vec{r}\,' \in \mathcal{S}$.

Note that, in the limit of $\mathcal{S} \to \infty$, the integrand of the surface integral falls off as r^{-3} and so the surface term vanishes and we recover the standard equation for $V(\vec{r})$, Equation 2.55.

The Concept of Green Functions

Suppose we have the generalized linear partial differential equation

$$O_{\vec{r}} f(\vec{r}) = g(\vec{r}) \tag{3.34}$$

where O is a linear partial differential operator, f is a generalized potential, and g is a generalized source function. Poisson's Equation is an example, with $O_{\vec{r}} = -\epsilon_0 \nabla^2$, $f(\vec{r}) = V(\vec{r})$, and $g(\vec{r}) = \rho(\vec{r})$. Is there a general approach for finding f given g?

Yes, there is, it is called the Green Function approach. The basic idea is to find the "impulse" response function for the differential equation: the generalized potential one gets if one has a point-like source. Given that impulse response function, one can convolve the impulse response function with an arbitrary source function to get the generalized potential for that source function.

Formal Solution to Poisson's Equation: The Green Function (cont.)

Mathematically, the impulse response function, or *Green Function*, is the function $G(\vec{r}, \vec{r}\,')$ that solves the equation

$$O_{\vec{r}} G(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,') \tag{3.35}$$

meaning that $G(\vec{r}, \vec{r}\,')$ calculates the generalized potential at the point $\vec{r}$ for a point source of size 1 at the position $\vec{r}\,'$ (*i.e.*, the total source charge recovered by integrating over the source function is 1). If such a G exists, then, for an arbitrary source function $g(\vec{r})$, G gives us the solution $f(\vec{r})$ to the generalized linear partial differential equation:

$$f(\vec{r}) = \int d\tau' \, G(\vec{r}, \vec{r}\,') \, g(\vec{r}\,') \tag{3.36}$$

We can check that the above differential equation is satisfied by the above solution by applying the operator:

$$O_{\vec{r}} f(\vec{r}) = O_{\vec{r}} \int d\tau' \, G(\vec{r}, \vec{r}\,') \, g(\vec{r}\,') = \int d\tau' \left[O_{\vec{r}} G(\vec{r}, \vec{r}\,') \right] g(\vec{r}\,') \tag{3.37}$$

$$= \int d\tau' \, \delta(\vec{r} - \vec{r}\,') \, g(\vec{r}\,') = g(\vec{r}) \tag{3.38}$$

Assuming solutions to the generalized linear partial differential equation are unique (true for Poisson's Equation), the Green Function is the *only* solution to be found.

General Discussion of Green Functions for Poisson's Equation

We have already seen that, in the absence of a bounding surface, the Green Function for Poisson's Equation is

$$G(\vec{r}, \vec{r}\,') = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{1}{|\vec{r} - \vec{r}\,'|} \tag{3.39}$$

We can see this explicitly by rewriting our usual expression for the potential for this boundary condition, Equation 2.55, in terms of G:

$$V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\int_{\mathcal{V}} d\tau'\,\frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} = \int_{\mathcal{V}} d\tau'\,G(\vec{r}, \vec{r}\,')\,\rho(\vec{r}\,') \tag{3.40}$$

Formal Solution to Poisson's Equation: The Green Function (cont.)

More generally — *i.e.*, for a more complex boundary condition — the Green Function for Poisson's Equation must satisfy

$$G(\vec{r},\vec{r}\,') = \frac{1}{4\,\pi\,\epsilon_0}\frac{1}{|\vec{r}-\vec{r}\,'|} + F(\vec{r},\vec{r}\,') \qquad \text{with} \qquad \nabla^2_{\vec{r}} F(\vec{r},\vec{r}\,') = 0 \tag{3.41}$$

with the first term taking care of the charge placed explicitly in the volume and the second term being added to replace the integration of the first term over the charge distribution on the boundary (which we may not know) with an integral over the given boundary conditions. We may add F because it solves Laplace's Equation in the volume and thus does not affect the source side of Poisson's Equation in the volume. Finding G thus consists of finding F.

What is F? We'll get into this a bit later when we talk about the conditions on Green Functions for specific Dirichlet or Neumann boundary conditions, as the interpretation depends on which type is imposed.

We note that both G and F are symmetric in their arguments, $G(\vec{r}\,',\vec{r}) = G(\vec{r},\vec{r}\,')$ and $F(\vec{r}\,',\vec{r}) = F(\vec{r},\vec{r}\,')$, for reasons we will explain later.

The Green Functions for Poisson's Equation with Dirichlet or Neumann Boundary Conditions

We can apply the above concept by taking $\phi(\vec{r}\,') = V(\vec{r}\,')$ and $\psi(\vec{r}\,') = -\epsilon_0 G(\vec{r}, \vec{r}\,')$ in Green's Theorem (Equation 3.13), giving

$$V(\vec{r}) = \int_{\mathcal{V}} d\tau'\, \rho(\vec{r}\,')\, G(\vec{r}, \vec{r}\,') + \epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da' \left[G(\vec{r}, \vec{r}\,')\, \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} V(\vec{r}\,') - V(\vec{r}\,')\, \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} G(\vec{r}, \vec{r}\,') \right] \tag{3.42}$$

We see that, if we can find the appropriate G for a particular boundary condition and force the term involving the other boundary condition to vanish, our integral equation for $V(\vec{r})$ reduces to an integration over the source distribution and boundary condition. We can be more specific about this by picking a boundary condition:

Formal Solution to Poisson's Equation: The Green Function (cont.)

- *Dirichlet boundary condition*
  In this case, $V(\vec{r})$ is specified for $\vec{r} \in \mathcal{S}$. Therefore, $\hat{n}(\vec{r}) \cdot \vec{\nabla}_{\vec{r}} V(\vec{r})$ should be left unspecified — it should be determined by the solution itself — so we need for it to not appear in the integral equation. We can eliminate the term containing this normal derivative if we require the Dirichlet Green Function to satisfy

$$G_D(\vec{r}, \vec{r}\,') = 0 \qquad \text{for} \quad \vec{r}\,' \in \mathcal{S}, \vec{r} \in \mathcal{V}, \mathcal{S} \tag{3.43}$$

Note that we want the above condition to hold for not just $\vec{r} \in \mathcal{V}$ but also for $\vec{r} \in \mathcal{S}$ so the expression is usable for calculating the potential on the boundary to ensure the boundary condition remains satisfied (*i.e.*, the expression for $V(\vec{r})$ is self-consistent).

With this condition, the solution for $V(\vec{r})$ reduces to

$$\boxed{V(\vec{r}) = \int_{\mathcal{V}} d\tau'\, \rho(\vec{r}\,')\, G_D(\vec{r}, \vec{r}\,') - \epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da'\, V(\vec{r}\,')\, \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} G_D(\vec{r}, \vec{r}\,')} \tag{3.44}$$

where it is now clear how the Dirichlet boundary condition applies — it simply needs to be explicitly inserted into the second term, and then the integration with $\hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} G_D(\vec{r}, \vec{r}\,')$ calculates its contribution (or, rather, the contribution of the corresponding induced surface charge) to the overall potential.

Formal Solution to Poisson's Equation: The Green Function (cont.)

For a Dirichlet boundary condition, the symmetry of G_D in its arguments can be proven by applying Green's Theorem with $\phi = G_D(\vec{r}, \vec{x})$ and $\psi = G_D(\vec{r}\,', \vec{x})$ where $\vec{x}$ is the variable that is integrated over and using the defining condition $G_D(\vec{r}, \vec{x}) = 0$ for $\vec{x}$ on the boundary and $\vec{r}$ in the volume and on the boundary (which also implies the same for $G_D(\vec{r}\,', \vec{x})$). Symmetry of G_D implies symmetry of F_D given that their difference is symmetric in $\vec{r}$ and $\vec{r}\,'$.

The symmetry requirement makes it easier to interpret the condition $G_D(\vec{r}, \vec{r}\,') = 0$. As the Green Function is originally defined, where $\vec{r}\,'$ is the location of the unit charge and $\vec{r}$ is the point at which the potential is given, the condition states that the potential at some point $\vec{r}$ in $\mathcal{V}$ or on $\mathcal{S}$ is not directly affected by the charge induced on the boundary at $\vec{r}\,'$. The interpretation of this is not obvious.

But when the symmetry of the Green Function is used, now one can think of the unit charge as being at $\vec{r}$ and the potential being calculated at $\vec{r}\,'$. Thus, our condition on G_D requires that G_D yields zero contribution to the potential on the *boundary* from charges *in the volume or on the surface*. If this did not hold, then both the given charge distribution in the volume, as well as the induced charges on the boundary, would modify the Dirichlet boundary condition. (The way this is enforced physically is by the appearance of induced charge on the boundary surface due to the charges in the volume. The induced charge is drawn from the voltage source(s) that enforces the Dirichlet boundary condition.)

Formal Solution to Poisson's Equation: The Green Function (cont.)

We can also now provide an interpretation of $F(\vec{r}, \vec{r}\,')$ in the Dirichlet case. Because 1) $F(\vec{r}, \vec{r}\,')$ satisfies Laplace's Equation in the volume $\mathcal{V}$, and 2) when added to the potential of a unit point charge at $\vec{r}\,'$ (the first term in our expression relating G and F, Equation 3.41), the sum satisifies the specified boundary condition on $\mathcal{S}$, $F(\vec{r}, \vec{r}\,')$ can be interpreted as the potential function in the volume due to the surface charge induced on the boundary by the real charges in the volume in order for the boundary condition to be satisfied.

Finally, we can now go back and reinterpret what is meant by $G_D(\vec{r}, \vec{r}\,') = 0$ when $\vec{r}\,'$ is considered to be the location of the unit charge. It implies that the induced charge on the boundary does not modify the potential in $\mathcal{V}$ or on $\mathcal{S}$ calculated by the first term on our expression above for $V(\vec{r})$, the one that convolves G_D with ρ in $\mathcal{V}$. The first part of this statement, the fact that the induced charge on $\mathcal{S}$ does not affect the potential in $\mathcal{V}$, simply avoids double-counting the effect of that surface charge — it is already accounted for by the first term. The second part of this statement, the fact that the induced charge on $\mathcal{S}$ does not affect the potential on $\mathcal{S}$, is the requirement that the Dirichlet boundary condition not be modified by the induced surface charge.

Formal Solution to Poisson's Equation: The Green Function (cont.)

- *Neumann boundary condition*
  In this case, $\hat{n} \cdot \vec{\nabla} V(\vec{r})$ is specified for $\vec{r} \in \mathcal{S}$, so we need to render irrelevant the term containing $V(\vec{r})$ because we should not have to simultaneously specify it. While we might be inclined to require $\hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} G_N(\vec{r}, \vec{r}\,') = 0$ for $\vec{r}\,' \in \mathcal{S}$ to make this happen, this requirement is not consistent with Poisson's Equation defining G: if one integrates Poisson's Equation for G_N over $\mathcal{V}(\mathcal{S})$ and turns it into a surface integral using the divergence theorem, one obtains the requirement

$$-\epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da'\, \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} G_N(\vec{r}, \vec{r}\,') = 1 \tag{3.45}$$

  Thus, the simplest condition we can impose on G_N is

$$\hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} G_N(\vec{r}, \vec{r}\,') = -\left[\epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da'\right]^{-1} \tag{3.46}$$

  The physical interpretation of this condition is that the normal electric field at a boundary point $\vec{r}\,'$ due to a unit point charge at $\vec{r}$ is independent of $\vec{r}$ and $\vec{r}\,'$. The physical meaning of this is not yet clear, but we will see its meaning below.

Formal Solution to Poisson's Equation: The Green Function (cont.)

Applying this condition, the solution for $V(\vec{r})$ reduces to

$$\boxed{V(\vec{r}) = \int_{\mathcal{V}} d\tau'\, \rho(\vec{r}\,')\, G_N(\vec{r}, \vec{r}\,') + \epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da'\, G_N(\vec{r}, \vec{r}\,')\, \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} V(\vec{r}\,') + \langle V(\vec{r}) \rangle_{\mathcal{S}(\mathcal{V})}}$$

$$\text{with} \quad \langle V(\vec{r}) \rangle_{\mathcal{S}(\mathcal{V})} \equiv \frac{\oint_{\mathcal{S}(\mathcal{V})} da'\, V(\vec{r}\,')}{\oint_{\mathcal{S}(\mathcal{V})} da'} \tag{3.47}$$

While $V(\vec{r})$ on the boundary has not been completely eliminated, its only appearance is via its average value on the boundary. This makes sense, as the Neumann boundary condition does not specify the potential offset since it only specifies derivatives of the potential. The appearance of this term reflects the freedom we have to set the potential offset for problems with Neumann boundary conditions.

What is the interpretation of a Neumann Green Function? Since $\hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} V(\vec{r}\,')$ specifies the surface charge density on the boundary, $G_N(\vec{r}, \vec{r}\,')$ simply calculates the potential at a point $\vec{r}$ in the volume due to the surface charge density at $\vec{r}\,'$ on the boundary. This is because we see in the above equation that G_N is convolved with the volume charge density and the surface charge density in the same way. A Neumann Green Function thus has a simpler interpretation than a Dirichlet Green Function.

Formal Solution to Poisson's Equation: The Green Function (cont.)

For a Neumann boundary condition, the symmetry of G_N and F_N is not a result of the boundary condition, but it may be assumed without loss of generality; see K.-J. Kim and J. D. Jackson, *Am. J. Phys.* **61**:1144 (1993)).

To make further progress in obtaining a functional form for the Green Function, we must specify the boundary conditions in more detail. We will consider examples of this later.

Obtaining the Green Function from the Method of Images

We discussed earlier how the component $F(\vec{r}, \vec{r}\,')$ of the full Green Function $G(\vec{r}, \vec{r}\,')$ is determined by the method of images. Let's see how this works for the two cases we have considered:

- *Point charge near grounded conducting plane*
  The full potential at a point $\vec{r}$ for the point charge at $d\,\hat{z}$ is

$$V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\left[\frac{q}{|\vec{r} - d\,\hat{z}|} - \frac{q}{|\vec{r} + d\,\hat{z}|}\right] \tag{3.48}$$

  Thus, the Dirichlet Green Function is given by letting $\vec{r}\,' = d\,\hat{z}$ and removing q:

$$G_D(\vec{r}, \vec{r}\,') = \frac{1}{4\,\pi\,\epsilon_0}\left[\frac{1}{|\vec{r} - \vec{r}\,'|} - \frac{1}{|\vec{r} + \vec{r}\,'|}\right] \tag{3.49}$$

  That is, the second term accounts for the fact that induced charge appears on the grounded conducting plane and calculate the contribution to the potential due to it; it is the $F(\vec{r}, \vec{r}\,')$ term while the first term is the usual Coulomb's Law term. The first term solves Poisson's Equation while the second term solves Laplace's Equation. One can see how the second term's contribution depends on the position of the point charge at $\vec{r}\,'$. One can also see how $G(z = 0, \vec{r}\,') = 0$ always, and how G_D is manifestly symmetric in $\vec{r}$ and $\vec{r}\,'$.

Obtaining Green Functions from the Method of Images (cont.)

It is also important to notice that the surface integral term went away. In the Dirichlet case, we expect a surface term from Equation 3.44

$$-\epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da' V(\vec{r}\,') \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} G_D(\vec{r}, \vec{r}\,') \tag{3.50}$$

In this case, the Dirichlet boundary condition is $V(z = 0) = 0$. Thus, this integral vanishes and we indeed only have the volume integral term from Equation 3.44 convolving the original charge distribution with G_D.

- *Point charge near conducting plane held at V_0*
  Suppose our boundary condition had instead been $V(z = 0) = V_0$, a constant (and also $V(r \to \infty) = V_0$ for consistency). Is the above Green Function still valid, and what is the impact of the new boundary condition? We expect it should have no impact because it just adds an offset to the potential. The charge distribution in the bulk and the type of boundary condition are unchanged, so we expect G_D is still valid. We can check that G_D provides the result we expect, an offset in the potential.

  First, because $V(\vec{r}\,') = V_0$ for $\vec{r}\,' \in \mathcal{S}(\mathcal{V})$, we can pull it outside the integral, so we just have the surface integral of the normal gradient of G_D over the surface:

$$-\epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da' V(\vec{r}\,') \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} G_D(\vec{r}, \vec{r}\,') = -\epsilon_0 V_0 \oint_{\mathcal{S}(\mathcal{V})} da' \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} G_D(\vec{r}, \vec{r}\,')$$

Obtaining Green Functions from the Method of Images (cont.)

We recall that, by definition, $G_D(\vec{r}, \vec{r}\,')$ is the potential at the point $\vec{r}$ due to a point charge of unit magnitude ($q = 1$) at $\vec{r}\,'$. By the symmetry of its arguments, it is also the potential at the point $\vec{r}\,'$ due to a unit point charge at $\vec{r}$. Earlier, when we did the method of images solution for the grounded conducting plane, we calculated the surface charge density at $\vec{r}$ due to the point charge at $d\,\hat{z}$ from $-\epsilon_0 \vec{\nabla}_{\vec{r}} V(\vec{r}, d\,\hat{z})$. So, in this case, $-\epsilon_0 \vec{\nabla}_{\vec{r}\,'} G_D(\vec{r}, \vec{r}\,')$ is the surface charge density at $\vec{r}\,'$ due to a unit charge at $\vec{r}$. Since V_0 has come outside the integral, our surface integral is now just the integral of this surface charge density over the boundary, or the total induced charge on the boundary. We proved earlier the general rule (via Gauss's Law) that the total induced surface charge is equal to the image charge. In this case, the image charge is equal and opposite to the real charge so the integral gives -1. That is:

$$\begin{aligned} -\epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da'\, V(\vec{r}\,')\, \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} G_D(\vec{r}, \vec{r}\,') &= -V_0 \oint_{\mathcal{S}(\mathcal{V})} da'\, \sigma_{ind}(\vec{r}\,', q = 1) \\ &= -V_0 q_{image} = V_0 \end{aligned} \tag{3.51}$$

So, we see that the surface term serves to add the potential offset that the boundary condition $V(z = 0) = V_0$ requires.

Obtaining Green Functions from the Method of Images (cont.)

- *Point charge near grounded conducting sphere*
  The full potential at a point $\vec{r}$ for the point charge at $a\,\hat{z}$ was (Equation 3.27):

$$V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\left[\frac{q}{|\vec{r} - a\,\hat{z}|} - \frac{q\,\frac{R}{a}}{|\vec{r} - \frac{R^2}{a}\,\hat{z}|}\right] \tag{3.52}$$

  Thus, the Dirichlet Green Function is given by letting $\vec{r}\,' = a\,\hat{z}$ and removing q:

$$G_D(\vec{r}, \vec{r}\,') = \frac{1}{4\,\pi\,\epsilon_0}\left[\frac{1}{|\vec{r} - \vec{r}\,'|} - \frac{R/r'}{\left|\vec{r} - \vec{r}\,'\frac{R^2}{(r')^2}\right|}\right] \tag{3.53}$$

  Again, the second term accounts for the potential due to the charge induced on the surface of the sphere and is the term that solves Laplace's Equation in this situation (the $F(\vec{r}, \vec{r}\,')$ term).

  It is perhaps not so obvious that the second term in this Green Function is symmetric in its arguments. Let's rewrite it:

$$\frac{R/r'}{|\vec{r} - \vec{r}\,'\frac{R^2}{(r')^2}|} = \frac{R}{|\hat{r}\,r\,r' - R^2\hat{r}'|} = \frac{R}{\sqrt{(r\,r')^2 + R^4 - 2\,r\,r'\,R^2\hat{r}\cdot\hat{r}'}} \tag{3.54}$$

  Now the symmetry is manifest.

  The same point about the surface integral term as for the conducting plane holds here: that term vanishes because $V(\vec{r}\,') = 0$ for $\vec{r}\,' \in \mathcal{S}$.

Obtaining Green Functions from the Method of Images (cont.)

- *Point charge near conducting sphere held at fixed potential*

  In this case, we can see the effect of the surface integral term in Equation 3.44 because $V(\vec{r})$ on the boundary does not vanish. The integral term is, from Equation 3.44:

$$-\epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da'\, V(\vec{r}\,')\, \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} G_D(\vec{r}, \vec{r}\,') \tag{3.55}$$

  When we encountered this nonvanishing surface term for the prior case of a point charge near a conducting plane, we recognized that $V(\vec{r}\,') = V_0$ could be pulled outside the integral and that the integral of the normal gradient of the Green Function gives the total charge induced on the boundary for a unit charge at $\vec{r}$. That argument carries through unchanged here, and, again, the total charge induced on the boundary is equal to the image charge by Gauss's Law, so we have

$$-\epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da'\, V(\vec{r}\,')\, \hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} G_D(\vec{r}, \vec{r}\,') = -V_0 q_{image} = V_0 \frac{R}{r} \tag{3.56}$$

  This is just the potential due to a point charge at the origin whose magnitude is such that the potential at radius R is V_0.

Obtaining Green Functions from the Method of Images (cont.)

With this integral evaluated, we can write down the full solution for $V(\vec{r})$:

$$V(\vec{r}) = \int_{\mathcal{V}(\mathcal{S})} d\tau' q\,\delta(\vec{r}\,' - a\,\hat{z})\,G_D(\vec{r},\vec{r}\,') + V_0\frac{R}{r} \tag{3.57}$$

$$= \frac{1}{4\,\pi\,\epsilon_0}\int_{\mathcal{V}(\mathcal{S})} d\tau' q\,\delta(\vec{r}\,' - a\,\hat{z})\left[\frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{R/r'}{|\vec{r}-\vec{r}\,'\frac{R^2}{(r')^2}|}\right] + V_0\frac{R}{r}$$

$$= \frac{q}{4\,\pi\,\epsilon_0}\left[\frac{1}{|\vec{r}-a\hat{z}|} - \frac{R/a}{|\vec{r}-\frac{R^2}{a}\hat{z}|}\right] + V_0\frac{R}{r}$$

We thus see that, in this case, the solution consists of the superposition of the grounded-conductor solution — the potential of the point charge and its image charge — and the potential of a point charge at the origin whose magnitude would give a potential V_0 at radius R. Explicitly, that point charge has magnitude $Q = 4\,\pi\,\epsilon_0\,R\,V_0$. This charge is what we found earlier when we discussed the same problem using the method of images.

- *Point charge in the presence of a charged, insulated, conducting sphere*

The prior situation is identical to this one: specifying the charge on a conductor is the same as specifying its potential. So the result for $V(\vec{r})$ is the same, where we must take $V_0 = (Q + (R/a)q)/(4\,\pi\,\epsilon_0\,R)$.

Lecture 5:
Separation of Variables in Cartesian Coordinates
Separation of Variables in Spherical Coordinates: General Theory
Separation of Variables in Spherical Coordinates with Azimuthal Symmetry

Date Revised: 2014/02/26 14:30
Date Given: 2014/02/25
pp. 122–132 assigned to be read ahead of time

Separation of Variables in Cartesian Coordinates

General Points

Griffiths makes this seem harder than it is. In separation of variables, we assume that the solution of Laplace's Equation factors into functions of single coordinates. This allows us to reduce the partial differential equation to a set of ordinary differential equations, which can be solved by standard techniques. Constants of integration appear that help to define the solutions. We apply the boundary conditions as defined by the voltages and/or the charge densities (normal derivative of voltage) at the boundaries. Once we find a set of solutions, we know from Sturm-Liouville theory that they form a *complete* set, so we are assured that we can write any solution to Laplace's Equation for the given boundary conditions in terms of these solutions.

One interesting point is that we only will seek to solve Laplace's Equation. We can do this because of the argument we made earlier: the Green Function for Poisson's Equation is of the form

$$G(\vec{r},\vec{r}\,') = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{1}{\vec{r}-\vec{r}\,'} + F(\vec{r},\vec{r}\,') \tag{3.58}$$

where $F(\vec{r},\vec{r}\,')$ solves Laplace's Equation. So we only need to develop separation of variables for Laplace's Equation.

Separation of Variables in Cartesian Coordinates (cont.)

Cartesian Coordinates

We assume that the function $V(\vec{r})$ can be factorized as

$$V(\vec{r}) = X(x)\,Y(y)\,Z(z) \tag{3.59}$$

Plugging this into Laplace's Equation, we obtain

$$\begin{aligned} Y(y)\,Z(z)\,\frac{d^2X}{dx^2} + X(x)\,Z(z)\,\frac{d^2Y}{dY^2} + X(x)\,Y(y)\,\frac{d^2Z}{dz^2} &= 0 \\ \frac{1}{X(x)}\,\frac{d^2X}{dx^2} + \frac{1}{Y(y)}\,\frac{d^2Y}{dY^2} + \frac{1}{Z(z)}\,\frac{d^2Z}{dz^2} &= \end{aligned} \tag{3.60}$$

We have three terms, the first a function of x, the second of y, and the third of z. Given these mismatched dependences, the only way the equation can hold is if each term is a constant. That is, it must hold that

$$\frac{1}{X(x)}\,\frac{d^2X}{dx^2} = C_1 \qquad \frac{1}{Y(y)}\,\frac{d^2Y}{dY^2} = C_2 \qquad \frac{1}{Z(z)}\,\frac{d^2Z}{dz^2} = C_3 \tag{3.61}$$

with $C_1 + C_2 + C_3 = 0$.

Separation of Variables in Cartesian Coordinates (cont.)

We know that the solution to these ordinary differential equations are exponentials,

$$X(x) = A\,\exp(x\sqrt{C_1}) + B\,\exp(-x\sqrt{C_1}) \tag{3.62}$$

$$Y(y) = C\,\exp(y\sqrt{C_2}) + C\,\exp(-y\sqrt{C_2}) \tag{3.63}$$

$$Z(z) = E\,\exp(z\sqrt{-(C_1+C_2)}) + F\,\exp(-z\sqrt{-(C_1+C_2)}) \tag{3.64}$$

We have not specified which of C_1, C_2, and C_3 are positive and which are negative (clearly, they cannot all be the same sign). That will be determined by the boundary conditions.

At this point, we cannot make further generic progress, we need to apply a set of boundary conditions. These will place constraints on the allowed values of the exponents and coefficients and restrict the family of solutions. There are a number of examples in Griffiths. To avoid duplication, we use a different one here from Jackson §2.9.

Separation of Variables in Cartesian Coordinates (cont.)

Box with five walls grounded and one held at a potential

Consider a box with one corner at the origin and of side lengths a, b, and c in the x, y, and z dimensions. The boundary conditions are

$$V(x=0)=0 \qquad V(y=0)=0 \qquad V(z=0)=0 \tag{3.65}$$

$$V(x=a)=0 \qquad V(y=b)=0 \qquad V(z=c)=\phi(x,y) \tag{3.66}$$

where $\phi(x,y)$ is a function that is given. Let's apply the boundary conditions in x and y:

$$A+B=0 \qquad A\exp(a\sqrt{C_1})+B\exp(-a\sqrt{C_1})=0 \tag{3.67}$$

$$C+D=0 \qquad C\exp(b\sqrt{C_2})+D\exp(-b\sqrt{C_1})=0 \tag{3.68}$$

Reducing,

$$A\left[\exp(a\sqrt{C_1})-\exp(-a\sqrt{C_1})\right]=0 \tag{3.69}$$

$$C\left[\exp(b\sqrt{C_2})-\exp(-b\sqrt{C_2})\right]=0 \tag{3.70}$$

Separation of Variables in Cartesian Coordinates (cont.)

There is no solution to these equations for $C_1 > 0$ and $C_2 > 0$: the unit-normalized decaying and rising exponentials are only equal when their arguments both vanish, which they don't. Therefore, let's take $C_1 = -\alpha^2$ and $C_2 = -\beta^2$ so these become oscillating exponentials. We thus obtain the conditions

$$\sin(\alpha\, a) = 0 \qquad \sin(\beta\, b) = 0 \tag{3.71}$$

This places conditions on the allowed values of α and β:

$$\alpha_n = \frac{n\,\pi}{a} \qquad \beta_m = \frac{m\,\pi}{b} \qquad m,\ n \text{ integers} \tag{3.72}$$

Thus, we have

$$X(x) = \sum_{n=1}^{\infty} A_n \sin \alpha_n x \qquad Y(y) = \sum_{m=1}^{\infty} C_m \sin \beta_m y \tag{3.73}$$

where n and m may only be positive integers (since the negative values are redundant with the positive ones and $n = 0$ and $m = 0$ contribute nothing) and the $\{A_n\}$ and $\{C_m\}$ are constants to be determined. No offset in V is allowed for these boundary conditions because $V = 0$ on some boundaries.

Separation of Variables in Cartesian Coordinates (cont.)

Now, let's apply the boundary conditions to $Z(z)$. At $z = 0$, we have

$$E + F = 0 \qquad \Longrightarrow \qquad F = -E \tag{3.74}$$

Therefore, $Z(z)$ is of the form

$$Z(z) = E\left[\exp(z\sqrt{\alpha_n^2 + \beta_m^2}) - \exp(-z\sqrt{\alpha_n^2 + \beta_m^2})\right] \tag{3.75}$$

$$= E' \sinh(\gamma_{nm} z) \qquad \text{with} \qquad \gamma_{nm} = \sqrt{\alpha_n^2 + \beta_m^2} \tag{3.76}$$

Thus, prior to applying the boundary condition at $z = c$, we have

$$V(\vec{r}) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z) \tag{3.77}$$

where $A_{nm} = A_n C_m E'_{nm}$ are constants to find. The boundary condition at $z = c$ yields

$$\phi(x, y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c) \tag{3.78}$$

This condition will let us determine the A_{mn}, but how, and why are we certain they exist? For this, we must briefly talk about orthonormal functions and completeness.

Digression on Orthonormal Functions

The general topic of the properties of solutions to second-order linear differential equations is beyond the scope of this course; it falls under the name *Sturm-Liouville theory*, and it is covered in ACM95/100. We will simply quote some results that are important for this course.

Sturm-Liouville theory consists of recognizing that the second-order linear ordinary differential equations we encounter in many places in this course are self-adjoint (Hermitian) operators on the Hilbert space of functions that satisfy the differential equation. You know from linear algebra that Hermitian operators are guaranteed to have a set of eigenvalues and eigenvectors (in this case, eigenfunctions), and that the eigenvectors form an orthonormal basis for the space under consideration (here, again, the space of functions that satisfy the differential equation). The same results apply here. What this means is that, for such equations, there are a set of solution functions $\{f_p(w)\}$ that are the eigenfunctions of the operator, and there are corresponding eigenvalues $\{\lambda_n\}$. These eigenfunctions form a *complete, orthonormal set*.

Separation of Variables in Cartesian Coordinates (cont.)

Orthonormality is written mathematically as

$$\int_s^t dw\, f_p^*(w)\, f_q(w) = \delta_{pq} \tag{3.79}$$

where integration over the interval of interest $[s, t]$ is the Hilbert space inner product. Completeness means

$$\sum_p f_p^*(w')\, f_p(w) = \delta(w' - w) \tag{3.80}$$

where the sum is over all eigenfunctions of the differential equation. Completeness implies that any solution of the differential equation is expandable in terms of this orthonormal set of functions:

$$g(w) = \sum_p A_p f_p(w) \tag{3.81}$$

Separation of Variables in Cartesian Coordinates (cont.)

Orthonormality tells us how to find the A_p: multiply the above equation by $f_q^*(w)$ and integrate:

$$\int_s^t dw\, f_q^*(w)\, g(w) = \int_s^t dw \sum_p A_p f_q^*(w)\, f_p(w) = \sum_p A_p \delta_{pq} = A_q \tag{3.82}$$

It is instructive to see how completeness can be used:

$$\begin{aligned} g(w) &= \int_s^t dw'\, g(w')\, \delta(w' - w) = \int_s^t dw'\, g(w') \sum_p f_p^*(w') f_p(w) \\ &= \sum_p f_p(w) \int_s^t dw'\, f_p^*(w') g(w') \end{aligned} \tag{3.83}$$

If we use Equation 3.81 and apply the orthonormality condition by multiplying by $f_q^*(w)$ and integrating over w, we explicitly recover

$$A_q = \int_s^t dw'\, f_q^*(w')\, g(w') \tag{3.84}$$

Separation of Variables in Cartesian Coordinates (cont.)

Completing the solution to the box with five walls grounded and one held at a potential

We ended with

$$\phi(x,y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c) \tag{3.85}$$

Now, it turns out that the functions $\{\sqrt{2/a}\sin(\alpha_n x)\}$ for $n \geq 1$ form an orthonormal set on the $x \in [0,a]$ interval, as do $\{\sqrt{2/b}\sin(\beta_n y)\}$ for $m \geq 1$ on $y \in [0,b]$, so we may recover the A_{nm} by multiplying by them and integrating:

$$\begin{aligned}
&\int_0^a dx \int_0^b dy\, \phi(x,y) \sqrt{\frac{2}{a}} \sin(\alpha_p x) \sqrt{\frac{2}{b}} \sin(\beta_q y) \\
&\quad = \int_0^a dx \int_0^b dy \sum_{n,m=1}^{\infty} A_{nm} \sinh(\gamma_{nm} c)\, \sin(\alpha_p x) \sqrt{\frac{2}{a}} \sin(\alpha_n x) \sin(\beta_m y) \sqrt{\frac{2}{b}} \sin(\beta_q y) \\
&\quad = \sum_{n,m=1}^{\infty} A_{nm} \sinh(\gamma_{nm} c) \sqrt{\frac{a}{2}} \delta_{pn} \sqrt{\frac{b}{2}} \delta_{qm} = \frac{\sqrt{a\,b}}{2} A_{pq}\, \sinh(\gamma_{pq} c)
\end{aligned} \tag{3.86}$$

Separation of Variables in Cartesian Coordinates (cont.)

That is,

$$A_{nm} = \frac{1}{\sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b dy\, \phi(x,y)\, \frac{2}{a} \sin(\alpha_n x)\, \frac{2}{b} \sin(\beta_m y) \tag{3.87}$$

Thus, our full solution is

$$V(\vec{r}) = \frac{4}{a\,b} \sum_{n,m=1}^{\infty} \sin(\alpha_n x) \sin(\beta_m y)\, \frac{\sinh(\gamma_{nm} z)}{\sinh(\gamma_{nm} c)} \times \int_0^a dx' \int_0^b dy' \phi(x',y') \sin(\alpha_n x') \sin(\beta_m y') \tag{3.88}$$

If we had used a more general boundary condition, specifying V to be nonzero on all six sides of the box, then we could solve the similar problem for each of the six faces independently (*i.e.*, let V be nonzero and arbitrary on that face and zero on all the other faces) and then sum the solutions since each individual solution does not affect the capability of the other solutions to satisfy their boundary conditions. (Of course, the boundary conditions themselves must be consistent with each other at the edges and corners where they meet.) *In fact, it is likely we would have to do this; it is not clear that the boundary conditions could all be satisfied by a single separated solution.*

Separation of Variables in Cartesian Coordinates (cont.)

Referring back to our discussion of Green Functions, the above solution is the surface term in Equation 3.44 for the particular boundary condition we have applied. By comparison of the two expressions, we infer (not derive!)

$$
\begin{aligned}
-\epsilon_0 \vec{\nabla}_{\vec{r}\,'} F_D(\vec{r}, \vec{r}\,' = x'\hat{x} + y'\hat{y} + c\hat{z}) \qquad (3.89)\\
= \frac{4\hat{z}}{a\,b} \sum_{n,m=1}^{\infty} \sin(\alpha_n x)\, \sin(\beta_m y)\, \frac{\sinh(\gamma_{nm} z)}{\sinh(\gamma_{nm} c)}\, \sin(\alpha_n x')\, \sin(\beta_m y')
\end{aligned}
$$

Note that this expression does not fully specify F! The above information is sufficient for the particular physical situation we have set up, which consists of no physical charge in the volume and the above boundary condition, because:

- The term consisting of the integral of the charge density in the volume convolved with G_D (and with F_D) is zero in this case because the charge density vanishes *in the volume*.
- The above surface term is the only one needed because $V = 0$ on the other boundaries.

Separation of Variables in Cartesian Coordinates (cont.)

But, for the more general problem of an arbitrary charge distribution in the volume and arbitrary Dirichlet boundary conditions on the surfaces, we need to know the full G_D by finding the full F_D term. The natural way to find this would be the method of images with the condition $V = 0$ on all the surfaces. It is left as an exercise for the reader to think about what set of image charges is appropriate.

Certainly, from the resulting G_D, we could compute the normal gradient of G_D on any surface and thus obtain the general solution for V in the volume for any Dirichlet boundary conditions. We should find that the normal gradient of F_D (not G_D, which includes the potential directly due to the charges in the volume) on the $z = c$ surface is what is given above.

It may seem like separation of variables is unsatisfactory for this reason — it does not give you the full Green Function, while the method of images does. But, as we have seen, the method of images is not a systematic procedure — one has to guess the correct image charge distribution. On the other hand, separation of variables is an entirely algorithmic procedure to give you a solution if a separable one exists for the particular boundary condition you are applying. It is less general but more reliable.

There is however, no guarantee that there will be a separable solution; this depends on the nature of the boundary conditions. The boundary conditions need to respect the separability assumed. For example, a boundary condition on a spherical boundary would not likely yield a solution via separation of variables in Cartesian coordinates!

Note also that the above method of images solution is not appropriate for a Neumann boundary condition because the method of images solution always solves the $V = 0$ on the boundaries situation, a Dirichlet boundary condition.

Separation of Variables in Spherical Coordinates

We do this in a slightly more general manner than Griffiths, dropping the assumption of azimuthal symmetry until it is time to solve the separated differential equations. We then return to the azimuthally dependent case later.

Laplace's Equation in spherical coordinates is:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial V}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial\phi^2}=0 \tag{3.90}$$

If we assume a separable form

$$V(r,\theta,\phi)=R(r)\,\Theta(\theta)\,\Phi(\phi) \tag{3.91}$$

Then, after dividing through by $V(r,\theta,\phi)$ and multiplying by $r^2\sin^2\theta$, we have

$$\sin^2\theta\left[\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta(\theta)}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right]+\frac{1}{\Phi(\phi)}\frac{d^2\Phi}{d\phi^2}=0 \tag{3.92}$$

Separation of Variables in Spherical Coordinates: General Theory (cont.)

We see the that first term depends only on r and θ while the second term depends only on ϕ, so we can immediately assume they are each equal to a constant:

$$\frac{1}{\Phi(\phi)}\frac{d^2\Phi}{d\phi^2} = -m^2 \tag{3.93}$$

The choice of the form of the constant is motivated by what will come next, but we can see why it needs to be of this form. As we saw in Cartesian coordinates, the above differential equation is solved either by growing/decaying exponentials (right side positive) or oscillating exponentials (right side negative). Since ϕ is a coordinate that repeats on itself ($\phi = 2\,n\,\pi$ are the same physical coordinate) the solutions $\Phi(\phi)$ must also be periodic, forcing the choice of the oscillating exponential. We saw from before that the coefficient in the argument appears squared, hence m^2 instead of m.

The solutions of this equation are straightforward:

$$\Phi(\phi) = A\,\exp(i\,m\,\phi) + B\,\exp(-i\,m\phi) \tag{3.94}$$

Periodicity in ϕ with period $2\,\pi$ requires m be an integer. One can either require $m \geq 0$ and keep the $\{A_m\}$ and $\{B_m\}$ or allow m to be any integer and drop the $\{B_m\}$ (which would be redundant with the $\{A_m\}$ for $m < 0$). In either case, only A_0 or B_0 is required.

Separation of Variables in Spherical Coordinates: General Theory (cont.)

Returning to the other term, we now have

$$\sin^2\theta\left[\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\frac{1}{\Theta(\theta)}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)\right]=m^2 \tag{3.95}$$

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)+\left[\frac{1}{\Theta(\theta)}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{m^2}{\sin^2\theta}\right]=0 \tag{3.96}$$

Now, we see that the first term depends only on r and the second only on θ, so we can separate again by setting the two terms equal to a constant. Here, we rely an prior knowledge of the result to choose the constant to be $\ell(\ell+1)$ so that

$$\frac{1}{R(r)}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right)=\ell(\ell+1) \tag{3.97}$$

$$\frac{1}{\Theta(\theta)}\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)-\frac{m^2}{\sin^2\theta}=-\ell(\ell+1) \tag{3.98}$$

Separation of Variables in Spherical Coordinates: General Theory (cont.)

Solving the Radial Equation

Here, we pull something out of our "bag of tricks" and define $U(r)$ by $R(r) = U(r)/r$ and plug in. We find

$$\frac{d^2U}{dr^2} - \frac{\ell(\ell+1)}{r^2}\,U(r) = 0 \tag{3.99}$$

Since the two derivatives would reduce the exponent of a power-law solution by 2, and the second term does the same by dividing by r^2, the above equation suggests $U(r)$ is a power law in r. If we plug in such a form $U(r) = r^a$, we find

$$a(a-1)r^{a-2} - \ell(\ell+1)r^{a-2} = 0 \qquad \Longrightarrow \qquad a = \ell+1 \quad \text{or} \quad a = -\ell \tag{3.100}$$

$$\Longrightarrow \qquad \boxed{R(r) = A\,r^{\ell} + \frac{B}{r^{\ell+1}}} \tag{3.101}$$

There is no constraint on ℓ yet.

Separation of Variables in Spherical Coordinates: General Theory (cont.)

The Polar Equation and the Generalized Legendre Equation

We may rewrite the polar angle equation as

$$\frac{1}{\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right)+\left[\ell(\ell+1)-\frac{m^2}{\sin^2\theta}\right]\Theta(\theta)=0 \tag{3.102}$$

If we use another one of our bag of tricks to write

$$x=\cos\theta \qquad \Theta(\theta)=P(\cos\theta)=P(x) \qquad 1-x^2=\sin^2\theta \tag{3.103}$$

then we may rewrite the polar differential equation as

$$\boxed{\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right]+\left[\ell(\ell+1)-\frac{m^2}{1-x^2}\right]P(x)=0} \tag{3.104}$$

This is called the *generalized Legendre equation*.

Separation of Variables in Spherical Coordinates: General Theory (cont.)

The generalized Legendre equation can be solved by assuming the solution is a polynomial in x. You have probably learned this technique in ACM95/100. One assumes

$$P_\ell^m(x) = \sum_{k=1}^{\infty} a_k x^k \tag{3.105}$$

and then, plugging the above form into the differential equation, one requires the series to terminate ($a_k = 0$ for some k). This condition requires ℓ to be a nonnegative integer and $-\ell \leq m \leq \ell$. These polynomials are the *associated Legendre polynomials.* We will come back to these shortly.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry

The Polar Equation Solution with Azimuthal Symmetry: the Legendre Equation and Legendre Polynomials

Consider the special case of azimuthal symmetry, for which $m = 0$ and $\Phi(\phi) =$ constant. The generalized Legendre Equation reduces to the *Legendre Equation*:

$$\boxed{\frac{d}{dx}\left[(1-x^2)\frac{dP}{dx}\right] + \ell(\ell+1)\,P(x) = 0} \tag{3.106}$$

One can apply the same series solution technique to this equation. Termination requires ℓ be a nonnegative integer. These solutions are the *Legendre Polynomials*. One can show they obey *Rodrigues' Formula*:

$$\boxed{P_\ell(x) = \frac{1}{2^\ell \ell!}\left(\frac{d}{dx}\right)^\ell (x^2-1)^\ell} \tag{3.107}$$

Mathematically, there should be a second solution for a given ℓ because the equation is second order. These are the solutions one finds by not requiring termination but simply convergence for $-1 < x < 1$ (corresponding to $0 < \theta < \pi$). If one has a geometry that excludes the z-axis (where the divergence occurs), these solutions must be considered. If the z-axis is in the space, then these solutions are unphysical and can be discarded.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Properties of the Legendre Polynomials

One can see by inspection or prove the following properties:

- $P_\ell(x)$ is a ℓth-order polynomial in x.
- $P_\ell(x)$ has only even powers of x if ℓ is even and only odd powers if ℓ is odd. Therefore, $P_\ell(x)$ is an even function of x for ℓ even and an odd function for ℓ odd.
- The Legendre polynomials are a complete orthonormal set: any function on the interval $[0, 1]$ can be written in terms of them. Their orthonormality relation is

$$\boxed{\int_{-1}^{1} dx \sqrt{\frac{2\ell+1}{2}} P_\ell(x) \sqrt{\frac{2\ell'+1}{2}} P_{\ell'}(x) = \delta_{\ell\,\ell'}} \tag{3.108}$$

and their completeness relation is

$$\boxed{\sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} P_\ell(x) P_\ell(x') = \delta(x - x')} \tag{3.109}$$

- $P_\ell(1) = 1$ and $P_\ell(-1) = (-1)^\ell$.
- $P_\ell(0) = [(-1)^n\,(2n-1)!!]/2^n\,n!$ for even $\ell = 2\,n$ and vanishes for odd ℓ.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Full Solution to Laplace's Equation with Azimuthal Symmetry

Combining our radial and polar equation solutions, we have that, for any problem with azimuthal symmetry and in which the z-axis is included, the potential must have the form

$$\boxed{V(r,\theta) = \sum_{\ell=0}^{\infty} \left(A_\ell \, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta)} \tag{3.110}$$

The coefficients $\{A_\ell\}$ and $\{B_\ell\}$ are set by the boundary conditions. If the volume includes the origin and the boundary conditions imply the potential must be finite there, the $\{B_\ell\}$ may be eliminated, and, if the volume includes infinity and requires the potential be finite (usually zero) there, the $\{A_\ell\}$ may be eliminated. Usually, application of the boundary conditions on V will require use of the orthonormality relations for the Legendre polynomials.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

We note that, in the process of doing separation of variables, we have proven that the components of the solution satisfy the *eigenvalue-eigenfunction equations*

$$\boxed{\nabla^2 A_\ell r^\ell = \frac{\ell(\ell+1)}{r^2} A_\ell r^\ell \qquad \nabla^2 B_\ell \frac{1}{r^{\ell+1}} = \frac{\ell(\ell+1)}{r^2} B_\ell \frac{1}{r^{\ell+1}}} \tag{3.111}$$

$$\boxed{\nabla^2 P_\ell(\cos\theta) = -\frac{\ell(\ell+1)}{r^2} P_\ell(\cos\theta)} \tag{3.112}$$

The fact that there is a r^2 on the right side of the radial equations reflects the fact that it is really $U(r) = R(r)/r$ that satisfies the eigenvalue-eigenfunction equation. For the angular equation, r is like a constant and so naturally appears in the eigenvalue.

Lecture 6:
Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)
Separation of Variables in Spherical Coordinates without Azimuthal Symmetry

Date Revised: 2014/03/13 6:00
(correction to Equation 3.181 and page that follows)
Date Given: 2014/02/27

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry

Dirichlet Boundary Condition on a Spherical Boundary with Azimuthal Symmetry

Suppose $V(R,\theta)$, the potential as a function of θ on a sphere of radius R, is specified, where the sphere is either the outer boundary or inner boundary of the space. What is the explicit form for the resulting potential?

Let's consider the two cases together. If the space is $r < R$, then we require the $\{B_\ell\}$ to vanish to ensure a finite potential at the origin. (There is no charge in the volume, so we are assured that the potential cannot be infinite there.) If the space is $r > R$, then we require the $\{A_\ell\}$ to vanish so the potential goes to zero at infinity. That is:

$$V(r,\theta) = \sum_{\ell=0}^{\infty} A_\ell\, r^\ell\, P_\ell(\cos\theta) \qquad \text{or} \qquad V(r,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}}\, P_\ell(\cos\theta) \tag{3.113}$$

Since we know $V(R,\theta)$, we evaluate the above equations at that value:

$$V(R,\theta) = \sum_{\ell=0}^{\infty} A_\ell\, R^\ell\, P_\ell(\cos\theta) \qquad \text{or} \qquad V(R,\theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}}\, P_\ell(\cos\theta) \tag{3.114}$$

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Then, to find the coefficients, we apply orthornomality to both sides. For the case of $r < R$, we have:

$$\frac{2\,\ell+1}{2}\int_0^\pi \sin\theta\, d\theta\, V(R,\theta)\, P_\ell(\cos\theta) \tag{3.115}$$

$$= \sum_{\ell=0}^{\infty} A'_\ell\, R^{\ell'} \int_0^\pi \sin\theta\, d\theta\, \frac{2\,\ell+1}{2}\, P_\ell(\cos\theta)\, P_{\ell'}(\cos\theta) \tag{3.116}$$

$$= \sum_{\ell=0}^{\infty} A'_\ell\, R^{\ell'}\, \delta_{\ell\ell'} = A_\ell\, R^\ell \tag{3.117}$$

Therefore,

$$A_\ell = \frac{2\,\ell+1}{2}\,\frac{1}{R^\ell}\int_0^\pi \sin\theta\, d\theta\, V(R,\theta)\, P_\ell(\cos\theta) \tag{3.118}$$

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Similarly, for the case $r > R$,

$$B_\ell = \frac{2\,\ell + 1}{2}\, R^{\ell+1} \int_0^\pi \sin\theta\, d\theta\, V(R,\theta)\, P_\ell(\cos\theta) \tag{3.119}$$

Therefore, the solutions are

$$V(r < R, \theta) = \sum_{\ell=0}^{\infty} \frac{2\,\ell + 1}{2}\, \frac{r^\ell}{R^\ell} \int_0^\pi \sin\theta\, d\theta\, V(R,\theta)\, P_\ell(\cos\theta) \tag{3.120}$$

$$V(r > R, \theta) = \sum_{\ell=0}^{\infty} \frac{2\,\ell + 1}{2}\, \frac{R^{\ell+1}}{r^{\ell+1}} \int_0^\pi \sin\theta\, d\theta\, V(R,\theta)\, P_\ell(\cos\theta) \tag{3.121}$$

Notice how the units of the coefficients have come out to cancel the powers of r.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Other Examples

- Griffiths does an example in which a surface charge density is specified at $r = R$ and the potential has to be found over all of space. This is almost a Neumann boundary condition, but not quite, since the surface charge density specifies the change in the normal derivative of V at r, not the normal derivative of V itself. By solving for V over all of space, one effectively turns it into a Neumann boundary condition by using the solution in one region to specify the condition on the normal derivative as one approaches the surface from the other side. One writes down different solutions for the two regions: the $\{B_\ell\}$ vanish for the $r < R$ solution to avoid a divergence at the origin, and the $\{A_\ell\}$ vanish for the $r > R$ solution to ensure the potential vanishes at infinity (as we saw above). Then, one applies the conditions that the potential must be continuous at R and that the normal derivative must change by the surface charge density (divided by $-\epsilon_0$). The first condition is effectively the specification of $\langle V \rangle_R$, which we recall from our generic discussion of Green Functions for Neumann boundary conditions. The second condition is the actual Neumann boundary condition. This first condition relates the $\{A_\ell\}$ and $\{B_\ell\}$ at each ℓ. With now just a single set of coefficients to determine, the Neumann boundary condition can be used with the orthonormality relation to find a formula for the coefficient for each ℓ.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

- Griffiths does the example of an uncharged metal sphere in a uniform electric field in the z direction, $\vec{E} = E_0\hat{z}$. The boundary condition is a bit mixed again. Because the sphere is metal, it is an equipotential. But that doesn't specify the value of V on the sphere. Since the field is uniform, we cannot set V to vanish at infinity. Instead, $V(z = 0) = 0$ is chosen. With that choice, symmetry tells us the sphere satisfies $V = 0$. But now V at infinity is not specified, so we don't yet have a Dirichlet boundary condition. The sensible thing to do is to require the potential approach $V(\vec{r}) = -E_0 z$ at infinity: whatever induced charge the sphere picks up, its contribution to the potential must fall off at infinity, leaving only the uniform field. Now we have a Dirichlet boundary condition. Because the potential is allowed to diverge at infinity, we cannot eliminate the $\{A_\ell\}$ in this case. But it is easy to see that only A_1 is nonzero: for $\ell > 1$, the behavior goes like r^ℓ, and since the potential must go like $z = r\cos\theta$ at large r, all the $\ell > 1$ terms must vanish. $A_0 = 0$ because the potential has no offset. The large r behavior sets $A_1 = -E_0$. That leaves the $\{B_\ell\}$ to be determined. Applying the boundary condition $V = 0$ at $r = R$ gives:

$$0 = A_1 R\cos\theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) \tag{3.122}$$

$$-A_1 R\cos\theta = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos\theta) \tag{3.123}$$

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Since the left side has a $\ell = 1$ term, and the Legendre polynomials are orthonormal, there can also be only a $\ell = 1$ term on the right side, implying $B_1/R^2 = -A_1 R$ or $B_1 = E_0 R^3$. Thus, the solution is

$$V(\vec{r}) = E_0 \left(r - \frac{R^3}{r^2} \right) \cos\theta \tag{3.124}$$

Generally speaking, one can see that the boundary condition is not always very obvious. One has to use whatever information one has and turn it into a boundary condition of some type.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

A Useful Expansion in Legendre Polynomials

We can easily show

$$\boxed{\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)} \tag{3.125}$$

with $\quad r_< = \min(|\vec{r}|, |\vec{r}\,'|) \quad r_> = \max(|\vec{r}|, |\vec{r}\,'|) \quad \cos\gamma = \hat{r}\cdot\hat{r}'$

This will have obvious utility.

To prove this, orient the coordinate system so $\vec{r}\,' = r'\,\hat{z}$. The function is the potential of a point charge $q = 4\,\pi\,\epsilon_0$ in magnitude at r' along the z-axis. It satisfies azimuthal symmetry and thus is expandable in terms of the above solutions of Laplace's Equation in spherical coordinates with azimuthal symmetry:

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty} \left(A_\ell\, r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta) \tag{3.126}$$

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Consider two cases separately:

- $r < r'$
  We must eliminate the B_ℓ coefficients to keep the function finite as $r \to 0$. To find the A_ℓ, let's consider the point $\vec{r} = r\,\hat{z}$ (*i.e.*, $\cos\gamma = 1$), which implies

$$\frac{1}{r' - r} = \sum_{\ell=0}^{\infty} A_\ell\, r^\ell \tag{3.127}$$

  (Recall, $P_\ell(1) = 1$.) Thus, the A_ℓ are just the coefficients of the power series expansion of the left side, which we know (recall: $(1-x)^{-1} = 1 + x + x^2 + \cdots$ for $0 < x < 1$) is

$$\frac{1}{r' - r} = \frac{1}{r'}\,\frac{1}{1 - \frac{r}{r'}} = \frac{1}{r'}\sum_{\ell=0}^{\infty}\left(\frac{r}{r'}\right)^\ell \tag{3.128}$$

  Thus, $A_\ell = 1/(r')^{\ell+1}$. This now sets the $\{A_\ell\}$ for arbitrary $\vec{r}$ (*i.e.*, arbitrary $\cos\gamma$ rather than the special case $\cos\gamma = 1$ we have considered).

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

- $r > r'$
  We must eliminate the A_ℓ coefficients to keep the function finite as $r \to \infty$.
  Again, consider $\vec{r} = r\,\hat{z}$, which implies

$$\frac{1}{r - r'} = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} \tag{3.129}$$

  Note that here we consider an expansion in r'/r rather than r/r' because now $0 < r'/r < 1$ while, above, $0 < r/r' < 1$. Again, the B_ℓ are just the coefficients of the power series expansion of the left side, which we know is

$$\frac{1}{r - r'} = \frac{1}{r}\,\frac{1}{1 - \frac{r'}{r}} = \frac{1}{r}\sum_{\ell=0}^{\infty}\left(\frac{r'}{r}\right)^{\ell} \tag{3.130}$$

  Thus, $B_\ell = (r')^\ell$.

Combining the above two cases yields Equation 3.125.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Separation of Variables for a Point Charge near a Grounded Conducting Sphere

We can use the identity we just proved to obtain the potential of a point charge near a grounded, conducting sphere. The setup is as before, with the point charge at $a\hat{z}$ and the sphere centered on the origin with radius R and $V = 0$ on its surface.

One difficulty is that the presence of the point charge implies Laplace's equation is not satisfied in the full volume! However, it is satisfied separately in the regions $R < r < a$ and $a < r < \infty$, and we have the charge density at $r = a$, so we should somehow solve separately in the two regions and then join the solutions together.

The charge density at $r = a$ is

$$\sigma(\theta, \phi) = \frac{q}{2\,\pi\,a^2 \sin\theta}\,\delta(\theta) \tag{3.131}$$

where one sees that this is sensible because integration returns q:

$$\int_0^{\pi}\int_0^{2\,\pi} da'\,\sigma(\theta,\phi) = \int_0^{\pi}\sin\theta d\theta\int_0^{2\,\pi} d\phi\,a^2\,\frac{q}{2\,\pi\,a^2\sin\theta}\,\delta(\theta) \tag{3.132}$$

$$= \frac{1}{2\,\pi}\int_0^{\pi} d\theta\int_0^{2\,\pi} d\phi\,q\,\delta(\theta) = q \tag{3.133}$$

Notice that no $\delta(\phi)$ is required.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

We can largely apply what we did in the case of a specified charge density above, except we cannot eliminate the $\{B_\ell\}$ for $r < a$ because the boundary is at $r = R$. But let's apply the boundary condition $V(r = R) = 0$:

$$0 = \sum_{\ell=0}^{\infty} \left(A_\ell \, R^\ell + \frac{B_\ell}{R^{\ell+1}} \right) P_\ell(\cos\theta) \tag{3.134}$$

Since the Legendre polynomials are orthonormal, the coefficent of P_ℓ at each ℓ must vanish independently, giving

$$A_\ell R^\ell = -\frac{B_\ell}{R^{\ell+1}} \quad \Longrightarrow \quad V(r < a, \theta) = \sum_{\ell=0}^{\infty} A_\ell \left(r^\ell - \frac{R^{2\ell+1}}{r^{\ell+1}} \right) P_\ell(\cos\theta) \tag{3.135}$$

For $r > a$, the $\{A_\ell\}$ must all vanish so the potential vanishes at infinity. So we have

$$V(r > a, \theta) = \sum_{\ell=0}^{\infty} \frac{C_\ell}{r^{\ell+1}} \, P_\ell(\cos\theta) \tag{3.136}$$

where we use C_ℓ to avoid confusion with B_ℓ above.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Now, matching requires that V be continuous at $r = a$ and that $\frac{\partial V}{\partial r}$ be continuous except at $\theta = 0$. Instead, it has a discontinuity there specified by $\sigma(0)$. We apply the first condition, term-by-term because of the orthonormality of the P_ℓ: matching:

$$A_\ell \left(a^\ell - \frac{R^{2\ell+1}}{a^{\ell+1}} \right) = \frac{C_\ell}{a^{\ell+1}} \qquad \Longrightarrow \qquad C_\ell = A_\ell \left(a^{2\ell+1} - R^{2\ell+1} \right) \tag{3.137}$$

Let's do some manipulations to put the potentials into a useful form:

$$V(r < a, \theta) = \sum_{\ell=0}^{\infty} A_\ell a^{\ell+1} \left(\frac{r^\ell}{a^{\ell+1}} - \frac{\frac{R}{a}\left(\frac{R^2}{a}\right)^\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta) \tag{3.138}$$

$$V(r > a, \theta) = \sum_{\ell=0}^{\infty} A_\ell a^{\ell+1} \left(\frac{a^\ell}{r^{\ell+1}} - \frac{\frac{R}{a}\left(\frac{R^2}{a}\right)^\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta) \tag{3.139}$$

Next, we apply the derivative matching condition, which is

$$\left(\frac{\partial V_>}{\partial r} - \frac{\partial V_<}{\partial r} \right) \Bigg|_{r=a} = -\frac{\sigma(\theta)}{\epsilon_0} \tag{3.140}$$

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

The derivatives are

$$\frac{\partial V_<}{\partial r} = \sum_{\ell=0}^{\infty} A_\ell a^{\ell+1} \left(\frac{\ell\, r^{\ell-1}}{a^{\ell+1}} + (\ell+1)\, \frac{\frac{R}{a}\left(\frac{R^2}{a}\right)^\ell}{r^{\ell+2}} \right) P_\ell(\cos\theta) \tag{3.141}$$

$$\frac{\partial V_>}{\partial r} = \sum_{\ell=0}^{\infty} A_\ell a^{\ell+1} (\ell+1) \left(-\frac{a^\ell}{r^{\ell+2}} + \frac{\frac{R}{a}\left(\frac{R^2}{a}\right)^\ell}{r^{\ell+2}} \right) P_\ell(\cos\theta) \tag{3.142}$$

Evaluating at $r = a$ gives

$$\left.\frac{\partial V_<}{\partial r}\right|_{r=a} = \sum_{\ell=0}^{\infty} A_\ell a^{\ell+1} \left(\frac{\ell}{a^2} + (\ell+1)\, \frac{\frac{R}{a}\left(\frac{R^2}{a}\right)^\ell}{a^{\ell+2}} \right) P_\ell(\cos\theta) \tag{3.143}$$

$$\left.\frac{\partial V_>}{\partial r}\right|_{r=a} = \sum_{\ell=0}^{\infty} A_\ell a^{\ell+1} (\ell+1) \left(-\frac{1}{a^2} + \frac{\frac{R}{a}\left(\frac{R^2}{a}\right)^\ell}{a^{\ell+2}} \right) P_\ell(\cos\theta) \tag{3.144}$$

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

When we difference the two, the second term in the expressions cancel, leaving

$$-\sum_{\ell=0}^{\infty}(2\,\ell+1)A_\ell\,a^{\ell-1}P_\ell(\cos\theta) = -\frac{q\,\delta(\theta)}{2\,\pi\,a^2\epsilon_0\,\sin\theta} \tag{3.145}$$

Now, we use orthonormality, multiplying by $P_{\ell'}(\cos\theta)$, multiplying by $\sin\theta$, and integrating over θ and ϕ. (Recall $\int_0^\pi \sin\theta\,d\theta\,P_\ell(\cos\theta)\,P_{\ell'}(\cos\theta) = 2\,\delta_{\ell\ell'}/(2\ell+1)$). This pulls out the $A_{\ell'}$ term:

$$-2\,(2\,\pi)\,A_{\ell'}\,a^{\ell'-1} = -\frac{q}{2\,\pi\,a^2\epsilon_0}\int_0^\pi \sin\theta\,d\theta\int_0^{2\,\pi} d\phi\,\frac{\delta(\theta)\,P_{\ell'}(\cos\theta)}{\sin\theta} \tag{3.146}$$

$$= -\frac{q}{\epsilon_0}\,P_{\ell'}(\cos\theta=1) = -\frac{q}{\epsilon_0} \tag{3.147}$$

$$A_{\ell'} = \frac{1}{a^{\ell'+1}}\,\frac{q}{4\,\pi\,\epsilon_0} \tag{3.148}$$

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Writing the full solution, we have

$$V(r<a,\theta)=\frac{q}{4\,\pi\,\epsilon_0}\sum_{\ell=0}^{\infty}\left(\frac{r^\ell}{a^{\ell+1}}-\frac{\frac{R}{a}\left(\frac{R^2}{a}\right)^\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta) \tag{3.149}$$

$$V(r>a,\theta)=\frac{q}{4\,\pi\,\epsilon_0}\sum_{\ell=0}^{\infty}\left(\frac{a^\ell}{r^{\ell+1}}-\frac{\frac{R}{a}\left(\frac{R^2}{a}\right)^\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta) \tag{3.150}$$

Comparing to Equation 3.125, we see that all four terms are of that form. The first term of the first equation has $r_<=r$ and $r_>=a$ as appropriate for $r<a$, while the first term of the second equation has $r_<=a$ and $r_>=r$ as needed for $r>a$. The second terms of both equations are of the same form with $r_<=R^2/a$, $r_>=r$, and the charge multiplied by $-R/a$. Thus, we recover

$$V(\vec{r})=\frac{q}{4\,\pi\,\epsilon_0}\left[\frac{1}{|\vec{r}-a\,\widehat{z}|}-\frac{R/a}{\left|\vec{r}-\frac{R^2}{a}\,\widehat{z}\right|}\right] \tag{3.151}$$

which matches Equation 3.27.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry

The Full Polar Equation Solution: the Associated Legendre Polynomials

The associated Legendre polynomials can be derived from the Legendre polynomials for $m \geq 0$:

$$P_\ell^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x) \tag{3.152}$$

which, using Rodrigues' Formula (Equation 3.107), implies

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \, \ell!} (1-x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2-1)^\ell \tag{3.153}$$

which is now valid for all m. It should be clear that $P_\ell^0 = P_\ell$. It should also be clear that parity in x (evenness/oddness) of the associated Legendre functions is given by $(-1)^{\ell+m}$ (where -1 implies oddness): the parity of P_ℓ is given by $(-1)^\ell$, and each derivative changes the parity by a factor of -1 (note that the powers of $(1-x^2)$ have no effect on the parity because it is an even function). There are a number of other properties of these functions, but it is more useful to consider them with the ϕ solutions.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

The Full Solution to the Angular Piece of Laplace's Equation: the Spherical Harmonics

When one combines the $P_\ell^m(\cos\theta)$ and the $e^{im\phi}$ solutions of the polar and azimuthal equations, one obtains the *Spherical Harmonics*

$$\boxed{Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\,\ell+1}{4\,\pi}\,\frac{(\ell-m)!}{(\ell+m)!}}\,P_\ell^m(\cos\theta)\,e^{im\phi}} \tag{3.154}$$

They are an orthonormal, complete basis for functions on the sphere (θ,ϕ). They satisfy numerous important and useful conditions:

- Conjugation:

$$Y_{\ell(-m)}(\theta,\phi) = (-1)^m\,Y_{\ell m}^*(\theta,\phi) \tag{3.155}$$

- Orthonormality:

$$\int_0^{2\,\pi} d\phi \int_0^{\pi} \sin\theta\,d\theta\,Y_{\ell' m'}^*(\theta,\phi)\,Y_{\ell m}(\theta,\phi) = \delta_{\ell\ell'}\,\delta_{mm'} \tag{3.156}$$

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

- Completeness:

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi) = \delta(\phi - \phi') \, \delta(\cos\theta - \cos\theta) \tag{3.157}$$

- $m = 0$ devolves to Legendre polynomials:

$$Y_{\ell\, 0}(\theta, \phi) = \sqrt{\frac{2\,\ell + 1}{4\,\pi}} \, P_\ell(\cos\theta) \tag{3.158}$$

This should be obvious from Equation 3.152, the relation between the Legendre and the associated Legendre polynomials.

- The $\theta = 0$ behavior is simple given Equation 3.152 (the $(1 - x^2)$ factor):

$$P_\ell^{m \neq 0}(\pm 1) = 0 \quad \Longrightarrow \quad Y_{\ell\, m \neq 0}(\theta = 0, \phi) = Y_{\ell\, m \neq 0}(\theta = \pi, \phi) = 0 \tag{3.159}$$

Recall that we also stated $P_\ell(1) = 1$, $P_\ell(-1) = (-1)^\ell$, which implies

$$Y_{\ell\, 0}(\theta = 0, \phi) = \sqrt{\frac{2\,\ell + 1}{4\,\pi}} \qquad Y_{\ell\, 0}(\theta = \pi, \phi) = (-1)^\ell \sqrt{\frac{2\,\ell + 1}{4\,\pi}} \tag{3.160}$$

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

- The above implies that any expansion in terms of $Y_{\ell m}$ simplifies at $\theta = 0,\ \pi$:

$$\text{given} \quad g(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m}\, Y_{\ell m}(\theta,\phi) \tag{3.161}$$

$$\text{then} \quad g(\theta=0,\phi) = \sum_{\ell=0}^{\infty} \sqrt{\frac{2\,\ell+1}{4\,\pi}}\, A_{\ell\,0} \tag{3.162}$$

$$\text{and} \quad g(\theta=\pi,\phi) = \sum_{\ell=0}^{\infty} (-1)^{\ell} \sqrt{\frac{2\,\ell+1}{4\,\pi}}\, A_{\ell\,0} \tag{3.163}$$

- The *Addition Theorem for Spherical Harmonics*: Given $\widehat{r}$ and $\widehat{r}'$ pointing in the directions (θ,ϕ) and (θ',ϕ'), respectively, then

$$\boxed{P_{\ell}(\widehat{r}\cdot\widehat{r}') = \frac{4\,\pi}{2\,\ell+1} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^{*}_{\ell m}(\theta',\phi')\, Y_{\ell m}(\theta,\phi)} \tag{3.164}$$

where $\widehat{r}\cdot\widehat{r}' = \cos\gamma = \cos\theta\,\cos\theta' + \sin\theta\,\sin\theta'\cos(\phi-\phi')$. The proof of this can be found in Jackson §3.6.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

- An important corollary of the Addition Theorem can be obtained by combining the above with Equation 3.125, the formula for the inverse of the relative distance between two points in terms of the Legendre polynomials:

$$\frac{1}{|\vec{r}-\vec{r}\,'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)$$

Plugging in the addition theorem gives us

$$\boxed{\frac{1}{|\vec{r}-\vec{r}\,'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta',\phi') Y_{\ell m}(\theta,\phi)} \qquad (3.165)$$

The utility of this relation is even more obvious that of Equation 3.125, especially for doing integrals over charge distributions with the relative distance function (*i.e.*, calculating the potential due to Coulomb's Law): decompose the charge distribution in terms of spherical harmonics, integrate up the charge distribution in a particular spherical harmonic $Y_{\ell m}$ over r' with weighting by $(r')^\ell$, and it gives the component of the potential at a distance r from the origin with spatial dependence $Y_{\ell m}(\theta,\phi)/r^{\ell+1}$.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

The Full Solution of Laplace's Equation in Spherical Coordinates

Putting it all together, we see that the most general solution to Laplace's Equation in spherical coordinates is

$$\boxed{V(r,\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m}\, r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta,\phi)} \qquad (3.166)$$

Again, the coefficients $\{A_{\ell m}\}$ and $\{B_{\ell m}\}$ are set by the volume under consideration and one or the other entire set may vanish. As well, application of the boundary conditions will require the orthonormality relations for the spherical harmonics.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

As with the case of azimuthal symmetry, we note that, in the process of doing separation of variables, we have proven that the components of the solution satisfy the *eigenvalue-eigenfunction equations*

$$\boxed{\nabla^2 A_{\ell m} r^\ell = \frac{\ell(\ell+1)}{r^2} A_{\ell m} r^\ell \qquad \nabla^2 B_{\ell m} \frac{1}{r^{\ell+1}} = \frac{\ell(\ell+1)}{r^2} B_{\ell m} \frac{1}{r^{\ell+1}}} \tag{3.167}$$

$$\boxed{\nabla^2 Y_{\ell m}(\theta,\phi) = -\frac{\ell(\ell+1)}{r^2} Y_{\ell m}(\theta,\phi)} \tag{3.168}$$

As before, the appearance of r^2 on the right side is not surprising. Note also that m does not appear in the angular equation. This is because Laplace's Equation itself is spherically (and therefore azimuthally) symmetric. The charge distribution and boundary conditions are what may break the spherically symmetry.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

Expansion of the Green Function in Spherical Coordinates in Terms of the Spherical Harmonics

The fact that the spherical harmonics combined with the usual power laws in radius solve the Laplace Equation for problems that are separable in spherical coordinates coordinates implies that the Green Function for such problems will have a convenient expansion in terms of spherical harmonics. This just follows from the fact that the Green Function is the potential due to a point charge of unit magnitude. Let's see this explicitly for a couple geometries:

- *Free space*
  The corollary of the addition theorem above is the desired expansion of the Green Function for charge in free space with no boundaries at finite radius and with the condition $V \to 0$ as $r \to \infty$.
- *Point charge near a grounded, conducting sphere*
  For this geometry, we saw that the Green Function can be written as sum of the Coulomb potential of two point charges, the original one at $r' \hat{z}$ and the image charge $q' = -q\,R/r'$ at $\hat{z}\,R^2/a$:

$$G(\vec{r},\vec{r}\,') = \frac{1}{4\,\pi\,\epsilon_0}\left[\frac{1}{|\vec{r}-\vec{r}\,'|} - \frac{R/r'}{\left|\vec{r}-\vec{r}\,'\left(\frac{R}{r'}\right)^2\right|}\right] \tag{3.169}$$

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

Using the same corollary, we can then immediately write

$$G(\vec{r},\vec{r}\,') = \frac{1}{\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{R}{r'}\frac{\left[r'\left(\frac{R}{r'}\right)^2\right]^\ell}{r^{\ell+1}}\right]\frac{Y_{\ell m}^*(\theta',\phi')\,Y_{\ell m}(\theta,\phi)}{2\,\ell+1} \tag{3.170}$$

$$= \frac{1}{\epsilon_0}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{1}{R}\left(\frac{R^2}{r\,r'}\right)^{\ell+1}\right]\frac{Y_{\ell m}^*(\theta',\phi')\,Y_{\ell m}(\theta,\phi)}{2\,\ell+1} \tag{3.171}$$

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

The general approach to the problem for an arbitrary (spherical) geometry is to go back to the definition of the Green Function:

$$-\epsilon_0 \nabla^2 G(\vec{r}, \vec{r}\,') = \delta(\vec{r} - \vec{r}\,') \tag{3.172}$$

and decompose both sides in terms of spherical harmonics. We don't know the Green Function yet, so its expansion is the arbitrary general form, which here we write

$$G(\vec{r}, \vec{r}\,') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m}(r|\vec{r}\,')\, Y_{\ell m}(\theta, \phi) \tag{3.173}$$

where the coefficients in the expansion $A_{\ell m}$ depend on r, as usual, and they also depend parametrically on $\vec{r}\,'$ because it is a parameter in the differential equation. (We don't know the solutions for the radial dependence of the $A_{\ell m}$ yet for the general case we are trying to solve, so we cannot assume they are the power laws we saw for the cases we have considered so far.)

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

The right side can be rewritten using the breakdown of the delta function into delta functions in each spherical coordinate followed by completeness of the spherical harmonics. The breakdown of the delta function is:

$$\delta(\vec{r}-\vec{r}\,') = \frac{\delta(r-r')}{r^2}\,\delta(\phi-\phi')\,\delta(\cos\theta-\cos\theta') \tag{3.174}$$

The $1/r^2$ on the radial component is required to cancel the r^2 in the volume element in spherical coordinates. The fact that the delta function in θ is a function of $\cos\theta$ and $\cos\theta'$ is because the volume element contains $\sin\theta\,d\theta = d(\cos\theta)$. One could have instead written $\delta(\theta-\theta')/\sin\theta$. Using completeness of the spherical harmonics, we have

$$\delta(\vec{r}-\vec{r}\,') = \frac{\delta(r-r')}{r^2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta',\phi')\,Y_{\ell m}(\theta,\phi) \tag{3.175}$$

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

Thus, our differential equation for the Green Function becomes

$$-\epsilon_0 \nabla^2 \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m}(r|\vec{r}\,') \, Y_{\ell m}(\theta, \phi) = \frac{\delta(r - r')}{r^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \phi') \, Y_{\ell m}(\theta, \phi) \tag{3.176}$$

Note that the Laplacian acts on the unprimed coordinates only. We wrote down above the eigenvalue-eigenfunction equation that the angular solutions of Laplace's Equation satisfies (Equation 3.168), which we use here to determine the action of ∇^2 on the angular coordinates:

$$\begin{aligned} -\epsilon_0 \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[\left(\nabla^2 - \frac{\ell(\ell+1)}{r^2} \right) A_{\ell m}(r|\vec{r}\,') \right] Y_{\ell m}(\theta, \phi) \qquad (3.177) \\ = \frac{\delta(r - r')}{r^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \phi') \, Y_{\ell m}(\theta, \phi) \end{aligned}$$

Now the Laplacian on the left side is only acting with its radial derivatives on $A_{\ell m}$; its action on the spherical harmonics has yielded the $\ell(\ell+1)/r^2$ term.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

The coefficients of the individual $Y_{\ell m}(\theta, \phi)$ on the two sides must be equal because of the orthonormality relation for the spherical harmonics, implying

$$-\epsilon_0 \left[\left(\nabla^2 - \frac{\ell(\ell+1)}{r^2} \right) A_{\ell m}(r|\vec{r}\,') \right] = \frac{\delta(r-r')}{r^2} Y^*_{\ell m}(\theta', \phi') \tag{3.178}$$

Now, given that we have $Y^*_{\ell m}(\theta', \phi')$ on the right side, and again the spherical harmonics are orthogonal polynomials, the dependence of $A_{\ell m}(r|\vec{r}\,')$ on its $\vec{r}\,'$ angular coordinates must be proportional to $Y^*_{\ell m}(\theta', \phi')$. Therefore, we may write

$$A_{\ell m}(r|r', \theta', \phi') = g_\ell(r, r')\, Y^*_{\ell m}(\theta', \phi') \tag{3.179}$$

Plugging in this form to the reduced version of Laplace's Equation, we get:

$$-\epsilon_0 \left(\nabla^2 - \frac{\ell(\ell+1)}{r^2} \right) g_\ell(r, r') = \frac{\delta(r-r')}{r^2} \tag{3.180}$$

$$\implies \quad \frac{d}{dr}\left[r^2 \frac{d}{dr} g_\ell(r, r') \right] - \ell(\ell+1)\, g_\ell(r, r') = -\frac{\delta(r-r')}{\epsilon_0} \tag{3.181}$$

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

We see that $g_\ell(r, r')$ satisfies the usual radial equation in r when $r \neq r'$ (r' is a parameter, not a variable, here). Therefore, for $r < r'$ and $r > r'$,

$$g_\ell(r, r') = \begin{cases} A_\ell(r')\, r^\ell + B_\ell(r')\, r^{-(\ell+1)} & r < r' \\ C_\ell(r')\, r^\ell + D_\ell(r')\, r^{-(\ell+1)} & r > r' \end{cases} \tag{3.182}$$

Because r' is a parameter of the differential equation, the coefficients and therefore the solutions depend on it parametrically. Integration of Equation 3.181 implies $r^2 dg_\ell(r, r')/dr$ must change by $-1/\epsilon_0$ at $r = r'$. Furthermore, the function $g_\ell(r, r')$ itself must be continuous at $r = r'$. These conditions, along with the Dirichlet or Neumann boundary conditions on any boundaries in the problem, should fully specify the $A_\ell(r')$, $B_\ell(r')$, $C_\ell(r')$, and $D_\ell(r')$. In particular, we can see how our solution for the point charge near a conducting sphere has this form. Recalling Equations 3.149 and 3.150, we simply identify the terms with each other and we see they imply

$$A_\ell(r') = \frac{1}{\epsilon_0}\left(\frac{1}{r'}\right)^{\ell+1} \qquad B_\ell(r') = -\frac{1}{\epsilon_0}\frac{1}{R}\left(\frac{R^2}{r'}\right)^{\ell+1} \tag{3.183}$$

$$C_\ell(r') = 0 \qquad D_\ell(r') = \frac{1}{\epsilon_0}\left[(r')^\ell - \frac{1}{R}\left(\frac{R^2}{r'}\right)^{\ell+1}\right] \tag{3.184}$$

which is also consistent with Equation 3.171, where we used the addition theorem to rewrite the Green Function for this system in terms of the spherical harmonics.

Lecture 7:
Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

Date Revised: 2014/03/05 7:00
Date Given: 2014/03/04

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry

Examples of Using the Expansion of the Green Function in Terms of the Spherical Harmonics

We did a lot of gymnastics to get the expansion of the Green Function in terms of spherical harmonics. Let's see how it can be used. For each of the examples we will consider, it would be possible to solve for the potential by separation of variables and application of boundary conditions without explicitly using our expansion. The advantages of using the Green Function, and in particular of using its spherical harmonic expansion, will be:

- As with any Green Function, whether expanded in spherical harmonics or not, the Green Function obviates re-solving the same kind of problem many times by simply providing integrals that need to be done.
- The breakdown of the Green Function in terms of spherical harmonics makes these integrals over the charge distribution easier to do, especially when the charge distribution has azimuthal symmetry.
- More importantly, the Green function connects a particular spherical harmonic mode of the charge distribution to the corresponding spherical harmonic mode of the potential. This correspondence makes the structure of the solution much easier to understand. The effect of a spherical harmonic mode in the charge distribution at one radius r' on the potential at another radius r is just a function of the two radii, the $g(r, r')$ function.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

For our examples, we will consider charge distributions inside a conducting sphere. We quote the general result from Jackson for the Green Function expansion in spherical harmonics for a geometry consisting of the volume between two conducting spheres at $r = a$ and $r = b$

$$G_D(\vec{r}, \vec{r}\,') \tag{3.185}$$
$$= \frac{1}{\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[r_<^\ell - \frac{1}{a} \left(\frac{a^2}{r_<} \right)^{\ell+1} \right] \left[\frac{1}{r_>^{\ell+1}} - \frac{1}{b} \left(\frac{r_>}{b^2} \right)^\ell \right] \frac{Y_{\ell m}^*(\theta', \phi')\, Y_{\ell m}(\theta, \phi)}{\left[1 - \left(\frac{a}{b} \right)^{2\ell+1} \right] (2\ell + 1)}$$

where, as usual, $r_< = \min\{r, r'\}$ and $r_> = \max\{r, r'\}$. Obtaining this more general result is a matter of doing the same thing as we did to obtain the result for a spherical conducting boundary at $r = R$, except now the C_ℓ term cannot be assumed to vanish. Now, taking the limit $a \to 0$, we get the result we will need for our work below:

$$G_D(\vec{r}, \vec{r}\,') = \frac{1}{\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} r_<^\ell \left[\frac{1}{r_>^{\ell+1}} - \frac{1}{b} \left(\frac{r_>}{b^2} \right)^\ell \right] \frac{Y_{\ell m}^*(\theta', \phi')\, Y_{\ell m}(\theta, \phi)}{2\ell + 1} \tag{3.186}$$

You will also be able to read off this simpler result from a method of images problem you will do in homework.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

On to our examples:

- *Potential inside the conducting sphere due to an arbitrary Dirichlet boundary condition potential but no charge in the volume*

  With no charge in the volume, we just need to calculate the surface term in Equation 3.44, for which we need the normal gradient of G_D at the surface:

$$\begin{aligned}
&\hat{n}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} G_D(\vec{r}, \vec{r}\,') \\
&= \frac{1}{\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Y^*_{\ell m}(\theta', \phi')\, Y_{\ell m}(\theta, \phi)}{2\ell + 1}\, r^\ell \frac{d}{dr'} \left[\frac{1}{(r')^{\ell+1}} - \frac{1}{b} \left(\frac{r'}{b^2} \right)^\ell \right] \Bigg|_{r'=b} \\
&= -\frac{1}{\epsilon_0} \frac{1}{b^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{r}{b} \right)^\ell Y^*_{\ell m}(\theta', \phi')\, Y_{\ell m}(\theta, \phi) \qquad (3.187)
\end{aligned}$$

  Therefore, the potential in the volume for the Dirichlet B.C. $V(b, \theta, \phi)$ is

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{r}{b} \right)^\ell Y_{\ell m}(\theta, \phi) \int d\Omega'\, Y^*_{\ell m}(\theta', \phi')\, V(b, \theta', \phi') \qquad (3.188)$$

  We see that the spherical harmonic component ℓm of the potential at r is determined by the spherical harmonic component ℓm of the potential on the boundary — very simple.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

- *Potential inside a grounded spherical conductor with a ring of charge of radius a in the xy plane.*

  This time, we do the volume integral but there is no integral over the surface. The charge density due to the ring is

$$\rho(\vec{r}\,') = \frac{Q}{2\,\pi\,a^2}\,\delta(r'-a)\delta(\cos\theta') \tag{3.189}$$

  Again, one can check that the charge density is correct by integrating it: the a^{-2} cancels the $(r')^2$ factor in the volume element and the argument of the θ' delta function is $\cos\theta'$ because the volume element contains $d(\cos\theta')$.

  The potential is then, as usual using Equation 3.44,

$$\begin{aligned} V(\vec{r}) &= \int_{\mathcal{V}} d\tau'\rho(\vec{r}\,')\,G_D(\vec{r},\vec{r}\,') \\ &= \frac{Q}{2\,\pi\,\epsilon_0\,a^2}\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta,\phi) \\ &\qquad \times \int_{\mathcal{V}} d\tau'\delta(r'-a)\,\delta(\cos\theta')\,r_<^{\ell}\left[\frac{1}{r_>^{\ell+1}} - \frac{1}{b}\left(\frac{r_>}{b^2}\right)^{\ell}\right]\frac{Y^*_{\ell m}(\theta',\phi')}{2\,\ell+1} \end{aligned} \tag{3.190}$$

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

Because the charge density has no azimuthal dependence, the ϕ' integral picks out the $m = 0$ term. Recall that $Y_{\ell 0} = \sqrt{(2\ell+1)/4\,\pi} P_\ell$, so we may rewrite as

$$V(\vec{r}) = \frac{Q}{4\,\pi\,\epsilon_0\,a^2} \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \int_{-1}^{1} d(\cos\theta')\,\delta(\cos\theta')\,P_\ell(\cos\theta') \qquad (3.191)$$

$$\times \int_0^b (r')^2 dr' \delta(r'-a)\, r_<^\ell \left[\frac{1}{r_>^{\ell+1}} - \frac{1}{b}\left(\frac{r_>}{b^2}\right)^\ell\right]$$

$$= \frac{Q}{4\,\pi\,\epsilon_0} \sum_{\ell=0}^{\infty} P_\ell(\cos\theta)\,P_\ell(0)\, r_<^\ell \left[\frac{1}{r_>^{\ell+1}} - \frac{1}{b}\left(\frac{r_>}{b^2}\right)^\ell\right] \qquad (3.192)$$

where now $r_< = \min\{r, a\}$ and $r_> = \max\{r, a\}$. Now, recall $P_\ell(0) = 0$ for odd ℓ and $P_\ell(0) = [(-1)^n\,(2n-1)!!]/2^n\,n!$ for even $\ell = 2\,n$, so we may reduce the above further to (replacing ℓ with $2\,n$ so n runs over all nonnegative integers rather than ℓ runing over all nonnegative even integers):

$$V(\vec{r}) = \frac{Q}{4\,\pi\,\epsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n-1)!!}{2^n\,n!}\, r_<^{2n} \left[\frac{1}{r_>^{2n+1}} - \frac{1}{b}\left(\frac{r_>}{b^2}\right)^{2n}\right] P_{2n}(\cos\theta) \qquad (3.193)$$

This is now a complete solution.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

The induced surface charge density at $r = b$ is obtained by the normal gradient of V. Since the normal gradient is just d/dr, it does not act at all on P_{2n}. In calculating this gradient, $r_< = a$ and $r_> = r$ since we will in the end evaluate at $r = b$. Therefore:

$$\begin{aligned}
\sigma(\vec{r}) &= \epsilon_0 \left.\frac{dV}{dr}\right|_{r=b} \\
&= -\frac{Q}{4\pi b^2} \sum_{n=0}^{\infty} \frac{(4n+1)(-1)^n (2n-1)!!}{2^n n!} \left(\frac{a}{b}\right)^{2n} P_{2n}(\cos\theta) \\
&= -\frac{Q}{4\pi b^2} \left[1 + \sum_{n=1}^{\infty} \frac{(4n+1)(-1)^n (2n-1)!!}{2^n n!} \left(\frac{a}{b}\right)^{2n} P_{2n}(\cos\theta)\right]
\end{aligned} \tag{3.194}$$

The expression is written in the above suggestive form on the last line so that it is easy to obtain the total induced surface charge. Since $P_0(\cos\theta) = 1$, the integral of the $n > 0$ terms over $\cos\theta$ can be viewed as integrating P_{2n} with P_0; by orthonormality of the Legendre polynomials, these terms all yield zero. The first term yields $-Q$ when integrated over the sphere.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

We have seen in this example how the integration of the charge density with the Green Function breaks the charge density down into its spherical harmonic components, calculates the potential due to each component individually (and fairly trivially, just multiplying by a function of the radius at which the source charge is and the radius at which the potential is desired) and then sums up those components. The same kind of correspondence clearly holds for the induced surface charge density.

Note that the additional $4n + 1$ factor implies the θ dependence of the induced surface charge density is different from that of the original ring charge; *i.e.*, the induced surface charge is not just a ring.

- *Potential inside a grounded spherical conductor with a line charge density along the z axis*

  This is done in Jackson Section 3.10, you may look it up there if you would like to see another example. We also did it somewhat interactively in class (no notes provided).

Lecture 8:
Multipoles
Electrostatics in Matter:
Polarizability and Polarization, Bound Charges and their Potential

Date Revised: 2014/03/11 10:25

Date Given: 2014/03/06

Dipoles: Quick Review

Recall from Ph1b the idea of an *electric dipole*: two charges of equal and opposite size $\pm q$ spaced very close together at $\vec{r}_+$ and $\vec{r}_-$. The net charge cancels almost perfectly, so, rather than the potential falling off like $1/r$ at large radius, it falls off as $1/r^2$ with functional form

$$V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\frac{\vec{p}\cdot\hat{r}}{r^2} \qquad \text{as} \quad \frac{r}{|\vec{r}_+ - \vec{r}_-|} \to \infty \tag{3.195}$$

where $\vec{p} = q(\vec{r}_+ - \vec{r}_-)$ is the dipole moment.

This idea generalizes. When one has a charge distribution with vanishing net charge, but inside of which there is a variation in the charge density, that variation is still noticeable at large distance as a set of potentials that fall off more quickly than $1/r$. The first additional term is the *dipole*, falling as $1/r^2$, the second is the *quadrupole*, falling as $1/r^3$, the third is the *octupole*, falling as $1/r^4$, and so on. The nomenclature comes from the minimum number of different source charges one must have to obtain that moment: one for monopole, two for dipole, four for quadrupole, etc.

Multipole Expansions (cont.)

Multipoles: Full Derivation

We derive the full form by considering the potential due to a charge distribution near the origin as viewed at a point $\vec{r}$ such that r is much larger than the extent of the charge distribution. We begin with

$$V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\int_{\mathcal{V}} d\tau'\,\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \tag{3.196}$$

We now use Equation 3.125, taking $r_< = r'$ and $r_> = r$ because r is outside the charge distribution. Thus,

$$V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\int_{\mathcal{V}} d\tau'\,\rho(\vec{r}\,')\sum_{\ell=0}^{\infty}\frac{(r')^{\ell}}{r^{\ell+1}}\,P_{\ell}(\cos\gamma) \tag{3.197}$$

where $\cos\gamma = \widehat{r}\cdot\widehat{r}'$ is the angle between the two vectors. There is common $1/r$ we can factor out, leaving

$$\boxed{V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{1}{r}\sum_{\ell=0}^{\infty}\frac{1}{r^{\ell}}\int_{\mathcal{V}} d\tau'\rho(\vec{r}\,')\,(r')^{\ell}\,P_{\ell}(\cos\gamma)} \tag{3.198}$$

This is the *multipole expansion* of the potential of the charge distribution. One can see that the successive terms fall off as successively higher powers of $1/r$.

The Monopole, Dipole, and Quadrupole Terms

Let's write out the first three terms more explicitly to get some physical intuition:

- *the monopole term*
  The first term is

$$V_1(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{1}{r}\int_{\mathcal{V}} d\tau'\rho(\vec{r}\,') = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{Q}{r} \tag{3.199}$$

  This is the standard Coulomb's Law term due to the total charge. Far enough away, all charge distributions look pointlike. But, if $Q = 0$, this term vanishes identically and the next-order terms must be considered. Even if $Q \neq 0$, if one is close enough to the charge distribution to see its non-pointlike nature, the next-order terms will be important corrections to the monopole term.

Multipole Expansions (cont.)

- *the dipole term*
  The second term is

$$V_2(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{1}{r^2}\int_{\mathcal{V}} d\tau'\rho(\vec{r}\,')\,r'\cos\gamma = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{1}{r^2}\int_{\mathcal{V}} d\tau'\rho(\vec{r}\,')\,r'\,\widehat{r}'\cdot\widehat{r}$$
$$= \frac{1}{4\,\pi\,\epsilon_0}\,\frac{1}{r^2}\,\widehat{r}\cdot\int_{\mathcal{V}} d\tau'\rho(\vec{r}\,')\,\vec{r}\,' \qquad (3.200)$$

or
$$\boxed{V_2(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{1}{r^2}\,\widehat{r}\cdot\vec{p} \qquad \text{where} \qquad \vec{p} = \int_{\mathcal{V}} d\tau'\rho(\vec{r}\,')\,\vec{r}\,'} \qquad (3.201)$$

is the dipole moment vector. It is the generalization of $\vec{p} = q\,(\vec{r}_+ - \vec{r}_-)$.

- *the quadrupole term*
  The third term is

$$V_3(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{1}{r^3}\int_{\mathcal{V}} d\tau'\rho(\vec{r}\,')\,(r')^2\,\frac{1}{2}\,(3\,\cos^2\gamma - 1)$$
$$= \frac{1}{4\,\pi\,\epsilon_0}\,\frac{1}{r^3}\,\widehat{r}\cdot\left[\int_{\mathcal{V}} d\tau'\rho(\vec{r}\,')\,(r')^2\,\frac{1}{2}\,(3\,\widehat{r}'\,\widehat{r}' - 1)\right]\cdot\widehat{r} \qquad (3.202)$$

$$\text{or} \quad \boxed{V_3(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{1}{r^3}\,\frac{1}{2}\,\hat{r}\cdot\underline{\underline{Q}}\cdot\hat{r} \quad \text{where} \quad \underline{\underline{Q}} = \int_{\mathcal{V}} d\tau'\rho(\vec{r}\,')\left[3\,\vec{r}\,'\,\vec{r}\,' - (r')^2\underline{\underline{1}}\right]} \tag{3.203}$$

is the quadrupole moment, where $\underline{\underline{1}} = \text{diag}(1,1,1)$ is the identity tensor with ones along the diagonal. Because it is composed of $\vec{r}\,'\,\vec{r}\,'$ and $\underline{\underline{1}}$, $\underline{\underline{Q}}$ is a tensor, implying that one can take a dot product with a vector on each side.

Origin Dependence of the Dipole Moment

Suppose we take a charge distribution and shift the origin by a vector $\vec{a}$ such that the charge distribution is now centered around $\vec{a}$. Then the new dipole moment is

$$\vec{p}\,' = \int d\tau'\rho'(\vec{r}\,')\,\vec{r}\,' = \int d\tau\,\rho(\vec{r})\,(\vec{a}+\vec{r}) = \vec{a}\,Q + \vec{p} \tag{3.204}$$

where we define the charge distribution in the new coordinate system $\rho'(\vec{r}\,')$ in terms of the original charge distribution $\rho(\vec{r})$ to be such that $\rho'(\vec{r}\,') = \rho(\vec{r} = \vec{r}\,' - \vec{a})$ when $\vec{r}\,' = \vec{r} + \vec{a}$. Thus, an origin shift can induce an artificial dipole moment for a charge distribution that has a monopole moment.

Field of an Electric Dipole

This is simply a matter of taking the gradient. If we let $\vec{p} = p\,\hat{z}$, then this is easy:

$$V_2(\vec{r}) = \frac{p\,\cos\theta}{4\,\pi\,\epsilon_0\,r^2} \tag{3.205}$$

$$\Longrightarrow \quad E_r(\vec{r}) = -\frac{\partial V_2}{\partial r} = -\frac{2\,p\,\cos\theta}{4\,\pi\,\epsilon_0\,r^3} \tag{3.206}$$

$$E_\theta(\vec{r}) = -\frac{1}{r}\,\frac{\partial V_2}{\partial \theta} = \frac{p\,\sin\theta}{4\,\pi\,\epsilon_0\,r^3} \tag{3.207}$$

$$E_\phi(\vec{r}) = -\frac{1}{r\,\sin\theta}\,\frac{\partial V_2}{\partial \phi} = 0 \tag{3.208}$$

$$\text{or} \quad \vec{E}(\vec{r}) = \frac{p}{4\,\pi\,\epsilon_0\,r^3}\left(2\,\hat{r}\,\cos\theta + \hat{\theta}\,\sin\theta\right) \tag{3.209}$$

Multipole Expansions (cont.)

To generalize this result for an arbitrary orientation of $\vec{p}$ requires some vector algebra. We have Equation 3.201 for the dipole potential in generic form, which we write out as

$$V_2(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\,\frac{1}{r^3}\sum_i r_i\,p_i \tag{3.210}$$

Now, we take the gradient, first noting

$$\frac{\partial}{\partial r_j}\,\frac{r_i}{r^3} = \frac{\partial}{\partial r_j}\,\frac{r_i}{(r^2)^{3/2}} = -\frac{3}{2}\,\frac{r_i}{(r^2)^{5/2}}\,\frac{\partial r^2}{\partial r_j} + \frac{\delta_{ij}}{r^3} = -\frac{3}{2}\,\frac{r_i}{r^5}\,(2\,r_j) + \frac{\delta_{ij}}{r^3} \tag{3.211}$$

Where we used $r^3 = \left(r^2\right)^{3/2}$ to more easily calculate the partial derivative. Therefore, with r_j and $\widehat{r}_j$ being the jth Cartesian coordinate and unit vector,

$$\begin{aligned}\vec{E}_2(\vec{r}) = -\vec{\nabla}\,V_2(\vec{r}) &= -\sum_j \widehat{r}_j\,\frac{\partial\,V_2}{\partial r_j} = \frac{1}{4\,\pi\,\epsilon_0\,r^5}\sum_{ij}\widehat{r}_j\,p_i\left[3\,r_i\,r_j - \delta_{ij}\,r^2\right] \\ &= \frac{1}{4\,\pi\,\epsilon_0\,r^5}\sum_j\left[r_j\,\widehat{r}_j\left(3\sum_i p_i\,r_i\right) - p_j\,\widehat{r}_j\right] \\ &\Longrightarrow \quad \boxed{\vec{E}_2(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0\,r^3}\left[3\,(\vec{p}\cdot\widehat{r})\,\widehat{r} - \vec{p}\right]}\end{aligned} \tag{3.212}$$

Electrostatic Potential Energy of a Multipole Distribution in an External Potential

The general expression for the potential energy of a charge distribution in an external potential is

$$U = \int_{\mathcal{V}} \rho(\vec{r}\,')\, V(\vec{r}\,') \tag{3.213}$$

Now, we want to expand $V(\vec{r})$ about some point in the distribution. Without loss of generality, assume the charge distribution is centered around the origin, which we will expand around. We use the multidimensional Taylor expansion of $V(\vec{r})$:

$$V(\vec{r}\,') = V(\vec{0}) + \sum_{j=1}^{3} r_j' \left.\frac{\partial V}{\partial r_j}\right|_{\vec{0}} + \frac{1}{2}\sum_{j,k=1} r_j' r_k' \left.\frac{\partial^2 V}{\partial r_j\, \partial r_k}\right|_{\vec{0}} + \cdots \tag{3.214}$$

Multipole Expansions (cont.)

Using $E_j = -\frac{\partial V}{\partial r_j}$, we may simplify

$$V(\vec{r}\,') = V(\vec{0}) - \vec{r}\,' \cdot \vec{E}(\vec{0}) - \frac{1}{6}\sum_{j,k=1}^{3} 3\, r_j' r_k' \left.\frac{\partial E_j}{\partial r_k}\right|_{\vec{0}} + \cdots \tag{3.215}$$

$$= V(\vec{0}) - \vec{r}\,' \cdot \vec{E}(\vec{0}) - \frac{1}{6}\sum_{j,k=1}^{3} \left(3\, r_j' r_k' - \delta_{jk}(r')^2\right) \left.\frac{\partial E_j}{\partial r_k}\right|_{\vec{0}} + \cdots \tag{3.216}$$

where we were able to add the $\delta_{jk}(r')^2$ term because

$$\sum_{j,k} r^2 \delta_{jk} \frac{\partial E_j}{\partial r_k} = r^2 \vec{\nabla} \cdot \vec{E} = 0 \tag{3.217}$$

because the charge distribution sourcing V and $\vec{E}$ is not present near the origin.

Multipole Expansions (cont.)

With the above expansion, the electrostatic potential energy is now

$$U = V(\vec{0}) \int_{\mathcal{V}} d\tau' \rho(\vec{r}\,') - \vec{E}(\vec{0}) \cdot \int_{\mathcal{V}} d\tau' \, \rho(\vec{r}\,') \, \vec{r}\,' \tag{3.218}$$

$$- \frac{1}{6} \sum_{j,k=1}^{3} \left. \frac{\partial E_j}{\partial r_k} \right|_{\vec{0}} \int_{\mathcal{V}} d\tau' \rho(\vec{r}\,') \left[3 \, r'_k r'_k - \delta_{jk} (r')^2 \right] + \cdots$$

$$= Q \, V(\vec{0}) - \vec{p} \cdot \vec{E}(\vec{0}) - \frac{1}{6} \sum_{j,k=1}^{3} Q_{jk} \left. \frac{\partial E_j}{\partial r_k} \right|_{\vec{0}} + \cdots \tag{3.219}$$

or, more generally, if the charge distribution is centered around $\vec{r}$,

$$U = Q \, V(\vec{r}) - \vec{p} \cdot \vec{E}(\vec{r}) - \frac{1}{6} \sum_{j,k=1}^{3} Q_{jk} \left. \frac{\partial E_j}{\partial r_k} \right|_{\vec{r}} + \cdots \tag{3.220}$$

$$= Q \, V(\vec{r}) - \vec{p} \cdot \vec{E}(\vec{r}) - \frac{1}{6} \vec{\nabla}_{\vec{r}} \cdot \underline{\underline{Q}} \cdot \vec{E}(\vec{r}) + \cdots \tag{3.221}$$

where we have written the last term in tensor dot product form. There are now contributions to the potential energy from the relative alignment of $\vec{p}$ and $\vec{E}$ and from the alignment of $\underline{\underline{Q}}$'s principal axes relative to the principal axes of the potential's curvature matrix (the field's derivatives). Note that $\vec{\nabla}_{\vec{r}}$ acts on the spatial dependence of $\vec{E}$; $\vec{r}\,'$ has already been integrated over to obtain $\underline{\underline{Q}}$.

Force on a Multipole Distribution in an External Field

We can calculate the force on the charge distribution by taking the derivative of U with respect to the charge distribution's nominal position $\vec{r}$:

$$\begin{aligned}\vec{F}(\vec{r}) &= -\vec{\nabla}U(\vec{r}) = Q\left(-\vec{\nabla}V(\vec{r})\right) + \vec{\nabla}\left(\vec{p}\cdot\vec{E}(\vec{r})\right) + \frac{1}{6}\sum_{j,k,m=1}^{3} Q_{jk}\frac{\partial^2 E_j}{\partial r_k\,\partial r_m}\,\hat{r}_m + \cdots \\ &= Q\,\vec{E}(\vec{r}) + \left(\vec{p}\cdot\vec{\nabla}\right)\vec{E}(\vec{r}) + \frac{1}{6}\sum_{j,k,m=1}^{3} Q_{jk}\frac{\partial^2 E_j}{\partial r_k\,\partial r_m}\,\hat{r}_m + \cdots \\ &= Q\,\vec{E}(\vec{r}) + \left(\vec{p}\cdot\vec{\nabla}\right)\vec{E}(\vec{r}) + \frac{1}{6}\vec{\nabla}\left[\vec{\nabla}\cdot\left(\underline{\underline{Q}}\cdot\vec{E}(\vec{r})\right)\right] + \cdots \qquad (3.222)\end{aligned}$$

In going from the second to the third row, we used the vector identity $\vec{\nabla}\left(\vec{a}\cdot\vec{f}(\vec{r})\right) = \left(\vec{a}\cdot\vec{\nabla}\right)\vec{f}(\vec{r})$ when $\vec{a}$ is a constant vector and $\vec{f}(\vec{r})$ has no curl. Note that all $\vec{\nabla}$ are with respect to $\vec{r}$ (since $\vec{r}\,'$ has been integrated over already).

We see that, in the absence of a net charge, the dipole moment responds to gradients in the electric field and, in the absence of a dipole moment, the quadrupole moment responds to the local curvature (second derivatives) of the electric field.

Torque on a Multipole in an External Field

Let's also calculate the torque. To calculate a torque, we need to take the gradient of the potential energy in spherical coordinates with respect to the orientation of the charge distribution relative to the electric field.

The monopole term yields no torque because there is no orientation angle involve: Q and $V(\vec{r})$ are scalars.

Considering the dipole term, we understand that there are only two vectors involved, $\vec{p}$ and $\vec{E}$, and the potential energy only depends on the angle between them. So the torque will be given by the derivative with respect to this angle, which we call θ_p to differentiate it from the θ coordinate of the system in which we consider $\vec{E}$. This angle will be measured from $\vec{E}$ to $\vec{p}$. Then,

$$\vec{N}_{elec} = -\frac{\partial}{\partial p_\theta}\left(-\vec{p}\cdot\vec{E}(\vec{r})\right) \tag{3.223}$$

$$= \frac{\partial}{\partial p_\theta} p\left|\vec{E}(\vec{r})\right|\cos\theta_p = -p\left|\vec{E}(\vec{r})\right|\sin\theta_p$$

$$= \vec{p}\times\vec{E}(\vec{r}) \tag{3.224}$$

This is a result you are familiar with from Ph1b.

Multipole Expansions (cont.)

Moving on to the quadrupole term, we recognize from Ph106ab that any symmetric tensor can be diagonalized via a rotation. Let's write

$$\underline{\underline{Q}} = \mathcal{R}(\phi_Q, \theta_Q, \psi_Q)\, \underline{\underline{\mathcal{Q}}}\, [\mathcal{R}(\phi_Q, \theta_Q, \phi_Q)]^T \qquad \text{with} \quad \underline{\underline{\mathcal{Q}}} = \text{diag}(Q_1, Q_2, Q_3) \quad (3.225)$$

where the Q_i are quadrupole moments along the principal axes of the quadrupole tensor and $\mathcal{R}(\phi_Q, \theta_Q, \psi_Q)$ is the rotation matrix that rotates from the frame in which the coordinate axes align with the quadrupole tensor's principal axes to the arbitrary frame we started in, with the three Euler angles $(\phi_Q, \theta_Q, \psi_Q)$ defining the orientation of the principal axes of $\underline{\underline{\mathcal{Q}}}$ relative to the this arbitrary frame. This kind of diagonalization should be familiar to you from Ph106ab, with $\mathcal{R}$ rotating from the "body" frame (the one fixed to the charge distribution's quadrupole principal axes) to the "space" frame. The quadrupole potential energy term is then

$$U_3 = -\frac{1}{6}\, \vec{\nabla}_{\vec{r}} \cdot \left\{ \mathcal{R}(\phi_Q, \theta_Q, \psi_Q)\, \underline{\underline{\mathcal{Q}}}\, [\mathcal{R}(\phi_Q, \theta_Q, \phi_Q)]^T \right\} \cdot \vec{E}(\vec{r}) \qquad (3.226)$$

To calculate the torque, we need to take the gradient of U_3 with respect to the orientation of the quadrupole. This amounts to taking gradients of $\mathcal{R}$ and $\mathcal{R}^T$ with respect to this orientation. As you know from the case of the symmetric top, the Euler angles are particularly useful angles with respect to which these derivatives can be taken. $\partial/\partial\phi_Q$ gives the torque about the z-axis of the space frame. $\partial/\partial\theta_Q$ gives the torque that causes motion in the polar angle direction with respect to the space frame's $\widehat{z}$. And $\partial/\partial\psi_Q$ calculates the torque about one particular principal axis of the quadrupole, chosen at will. You are familiar with symmetric tops, with $I_1 = I_2$. Here, we can have symmetric quadrupoles, with $Q_1 = Q_2$. In this case, the ψ_Q angle is the angle about the 3 axis of the quadrupole (the principal axis that aligns with the z-axis in the body frame). We do not take this further because, as you know from the study of tops in Ph106ab, the phenomenology can be quite rich.

Section 4
Electrostatics in Matter

Review of Polarizability of Materials

Griffiths §4.1 does a good job of providing physical motivation for the study of the polarizability of materials, and also reviews material you saw in Ph1b, so we only summarize the basics here.

- Atoms and molecules are *polarizable*, meaning that they can acquire a dipole moment when an external electric field is applied because of the separation of the positive and negative charge in response to the applied field. The charge distribution that results is such that its field cancels the applied field at the location of the atom or molecule.
- We assume that this polarizability is *a linear process*, so that the induced dipole moment is linear in the applied electric field, though the response may be anisotropic. The *polarizability tensor* $\underline{\underline{\alpha}}$ relates the induced dipole moment to the applied field:

$$\vec{p} = \underline{\underline{\alpha}} \cdot \vec{E} \tag{4.1}$$

Polarizability and Polarization (cont.)

- As we showed in our discussion of multipoles, dipoles can experience torques and forces in an electric field. If a dipole is placed in an electric field, it feels a torque (Equation 3.224)

$$\vec{N} = \vec{p} \times \vec{E} \tag{4.2}$$

If the electric field is nonuniform, the dipole feels a force (Equation 3.222)

$$\vec{F} = \left(\vec{p} \cdot \vec{\nabla}\right) \vec{E} \tag{4.3}$$

- If a medium consists of polarizable atoms or molecules, then that medium can become polarized under the application of an electric field. The *polarization* (or *polarization density*) of the medium is

$$\vec{P} = n\,\vec{p} \tag{4.4}$$

where n is the density of polarization atoms or molecules and $\vec{p}$ is the induced dipole per atom or molecule.

Bound Charges and the Potential of a Polarizable Material

When a medium is polarized and acquires a polarization vector $\vec{P}$, then it can generate its own electric field. This comes from the superposition of the dipole fields of the individual polarized atoms or molecules. In Ph1b, you saw how the polarization could be interpreted as yielding *bound charge* densities: when the medium polarizes, the positive components of some dipoles are cancelled by the negative components of nearby dipoles, but there can appear a net effective charge: on the boundaries, where the cancellation fails, and in the bulk if the dipole density is not uniform, also causing the cancellation to fail. This argument was made in Purcell in Ph1b to derive the bound charge densities, and Griffiths makes it in §4.2.2. Here we derive the relationship between the polarization vector and the bound charge density in rigorous fashion.

Polarizability and Polarization (cont.)

The total electric potential generated by a polarizable medium is found by summing up the dipole potentials of the individual dipoles:

$$V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \int_{\mathcal{V}} d\tau'\, \frac{\vec{P}(\vec{r}\,') \cdot (\vec{r} - \vec{r}\,')}{|\vec{r} - \vec{r}\,'|^3} \tag{4.5}$$

We use the identity $(\vec{r} - \vec{r}\,')/|\vec{r} - \vec{r}\,'|^3 = \vec{\nabla}_{\vec{r}\,'}(1/|\vec{r} - \vec{r}\,'|)$ (note: no minus sign because this is $\vec{\nabla}_{\vec{r}\,'}$, not $\vec{\nabla}_{\vec{r}}$ and we have $\vec{r} - \vec{r}\,'$ in the numerator, not $\vec{r}\,' - \vec{r}$) to rewrite this as

$$V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0} \int_{\mathcal{V}} d\tau'\, \vec{P}(\vec{r}\,') \cdot \vec{\nabla}_{\vec{r}\,'} \left(\frac{1}{|\vec{r} - \vec{r}\,'|} \right) \tag{4.6}$$

Polarizability and Polarization (cont.)

We can integrate by parts to obtain

$$V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\left[\int_{\mathcal{V}} d\tau' \vec{\nabla}_{\vec{r}'} \cdot \left(\frac{\vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|}\right) - \int_{\mathcal{V}} d\tau' \, \frac{1}{|\vec{r}-\vec{r}'|}\left(\vec{\nabla}_{\vec{r}'} \cdot \vec{P}\right)\right] \tag{4.7}$$

The first term can be converted to a surface integral via the divergence theorem:

$$V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\left[\int_{\mathcal{S}(\mathcal{V})} da' \, \frac{\widehat{n}(\vec{r}') \cdot \vec{P}(\vec{r}')}{|\vec{r}-\vec{r}'|} - \int_{\mathcal{V}} d\tau' \, \frac{1}{|\vec{r}-\vec{r}'|}\left(\vec{\nabla}_{\vec{r}'} \cdot \vec{P}(\vec{r}')\right)\right] \tag{4.8}$$

We thus see that the potential appears to be that of a surface charge density $\sigma_b(\vec{r}')$ on $\mathcal{S}(\mathcal{V})$ and a volume charge density $\rho_b(\vec{r}')$ in $\mathcal{V}$ with ($\widehat{n}$ is the outward normal from the polarizable material):

$$\sigma_b(\vec{r}') = \widehat{n}(\vec{r}') \cdot \vec{P}(\vec{r}') \qquad \rho_b(\vec{r}') = -\vec{\nabla}_{\vec{r}'} \cdot \vec{P}(\vec{r}') \tag{4.9}$$

$$V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\left[\int_{\mathcal{S}(\mathcal{V})} da' \, \frac{\sigma(\vec{r}')}{|\vec{r}-\vec{r}'|} + \int_{\mathcal{V}} d\tau' \, \frac{\rho_b(\vec{r}')}{|\vec{r}-\vec{r}'|}\right] \tag{4.10}$$

These charges are called "bound charges" because they are bound to the polarizable medium.

Lecture 9:
Electrostatics in Matter:
The Displacement Field, Linear Dielectrics,
Boundary Value Problems with Dielectrics,
Electrostatic Energy in and Forces on Dielectrics

Date Revised: 2014/03/13 6:00
(corrections to Equations 4.47 and 4.48)
Date Given: 2014/03/11

Potential and Field of a Uniformly Polarized Sphere

This problem from Ph1b is much easier to solve with our knowledge of solutions to Laplace's Equation. The polarization density is a constant $\vec{P} = P\,\hat{z}$. The bound volume charge density vanishes because it is a constant and the bound surface charge density at radius R is

$$\sigma_b = \hat{n}(\vec{r}) \cdot \vec{P} = \hat{r} \cdot P\,\hat{z} = P\,\cos\theta \tag{4.11}$$

This is a problem Griffiths solves in Example 3.9 for a generic $\sigma(\theta)$, and we talked through the solution earlier. The generic solution was

$$V(r < R, \theta) = \sum_{\ell=0}^{\infty} A_\ell\, r^\ell\, P_\ell(\cos\theta) \qquad V(r > R, \theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}}\, P_\ell(\cos\theta) \tag{4.12}$$

$$\text{with} \quad A_\ell = \frac{1}{2\,\epsilon_0\, R^{\ell-1}} \int_0^\pi d\theta'\, \sin\theta'\, \sigma(\theta')\, P_\ell(\cos\theta') \qquad B_\ell = A_\ell\, R^{2\,\ell+1} \tag{4.13}$$

Since $\sigma(\theta) = P\,P_1(\cos\theta) = P\,\cos\theta$, the orthonormal polynomials do their work and we get (making sure to include the normalization factor $2/(2\,\ell+1) = 2/3$):

$$V(r < R, \theta) = \frac{P\,r\,\cos\theta}{3\,\epsilon_0} \qquad V(r > R, \theta) = \frac{P\,R^3\,\cos\theta}{3\,\epsilon_0\,r^2} \tag{4.14}$$

Polarizability and Polarization (cont.)

We can write these more simply. We recognize $z = r\cos\theta$ and that the total dipole moment of the sphere is $\vec{p} = 4\pi R^3 P\,\hat{z}/3$, yielding

$$V(r<R,\theta) = \frac{P\,z}{3\,\epsilon_0} \qquad V(r>R,\theta) = \frac{\vec{p}\cdot\hat{r}}{4\pi\,\epsilon_0\,r^2} \tag{4.15}$$

Thus, the field inside the sphere is *uniform*, $\vec{E} = -\vec{P}/3\,\epsilon_0$, and the field outside the sphere is that of a dipole $\vec{p}$. Note that the field outside the sphere is a perfect dipole field all the way to $r = R$; this is not an approximation.

We remind the reader from Ph1b that we could have obtained this same result by treating the sphere as two spheres of uniform charge density $\rho = q/(4\pi R^3/3)$ with their centers displaced by $\vec{d} = \vec{p}/q$. The field inside a uniform sphere of charge is proportional to the radial vector outward from its center, so the two vectors $\vec{r} - \vec{d}/2$ and $\vec{r} + \vec{d}/2$ end up differencing (because the two spheres have opposite charge) to yield $\vec{d}$, yielding the uniform internal field. Outside the spheres, they look like point charges, so the system looks like a point dipole $\vec{p}$.

One could also use this argument to figure out that the charge density on the surface is $\sigma = P\cos\theta$ and evaluate the potential and field of that charge distribution.

The Electric Displacement Field

We proved earlier that the potential due to a polarization density $\vec{P}(\vec{r})$ is

$$V(\vec{r}) = \frac{1}{4\,\pi\,\epsilon_0}\left[\int_{\mathcal{S}(\mathcal{V})} da'\,\frac{\widehat{n}(\vec{r}\,')\cdot\vec{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} + \int_{\mathcal{V}} d\tau'\,\frac{-\vec{\nabla}_{\vec{r}\,'}\cdot\vec{P}(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\right] \tag{4.16}$$

These are analogues of Coulomb's law for ρ_b, so the potential and field due to the polarization density satisfy

$$\nabla^2 V_b = -\frac{1}{\epsilon_0}\rho_b \qquad \vec{\nabla}\cdot\vec{E}_b = \frac{1}{\epsilon_0}\,\rho_b = -\frac{1}{\epsilon_0}\,\vec{\nabla}\cdot\vec{P} \tag{4.17}$$

If there is a *free* charge density ρ_f, then we see that the total potential and field satisfy

$$\nabla^2 V = -\frac{1}{\epsilon_0}\,(\rho_f + \rho_b) \qquad \vec{\nabla}\cdot\vec{E} = \frac{1}{\epsilon_0}\left(\rho_f - \vec{\nabla}\cdot\vec{P}\right) \tag{4.18}$$

The Electric Displacement Field (cont.)

We will see later that it will be convenient to have a field that depends only on the free charge density. Thus, we define the *electric displacement field* by

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \tag{4.19}$$

We immediately see that Gauss's Law can be written as

$$\vec{\nabla} \cdot \vec{D} = \rho_f \qquad \Longleftrightarrow \qquad \oint_{\mathcal{S}} da\, \hat{n} \cdot \vec{D} = Q_{free,encl} \tag{4.20}$$

To fully understand $\vec{D}$, we also need to determine its curl:

$$\vec{\nabla} \times \vec{D} = \epsilon_0 \vec{\nabla} \times \vec{E} + \vec{\nabla} \times \vec{P} = \vec{\nabla} \times \vec{P} \tag{4.21}$$

The issue here is that the right side may not vanish, so the left side may not vanish. Thus, while $\vec{D}$ satisfies Gauss's Law in the free charge density, $\vec{D}$ *may not have vanishing curl, so it may not satisfy Coulomb's Law in the free charge density. Also, one may not be able to apply standard Gauss's Law arguments because of the possibly nonvanishing curl (and the corresponding more complex boundary condition on the tangential component of* $\vec{D}$*) — the symmetries we usually assume may not apply.*

The Electric Displacement Field (cont.)

However, if one knows that, due to symmetry or some other consideration, $\vec{\nabla} \times \vec{P} = 0$, then $\vec{D}$ *satisfies Gauss's Law and Coulomb's Law in* ρ_f *for the special case* $\vec{\nabla} \times \vec{P} = 0$. ($\vec{\nabla} \times \vec{P} = 0$ should be interpreted as also requiring that any boundaries be normal to $\vec{P}$, as we will see below that, unlike for $\vec{E}$, the tangential component of $\vec{D}$ is not continuous if $\vec{P}$ has a tangential component.)

When the above is true, $\vec{D}$ provides a calculational convenience: if a free charge density ρ_f and a polarization field $\vec{P}$ are specified, then we should calculate $\vec{D}$ from the free charge density using Gauss's Law and then obtain the electric field from $\vec{E} = (\vec{D} - \vec{P})/\epsilon_0$. This simplification is possible only because of the particular form of the bound charge density, $\rho_b = -\vec{\nabla} \cdot \vec{P}$, which parallels the mathematical form of Gauss's Law, along with the condition $\vec{\nabla} \times \vec{P} = 0$.

Note the extra condition $\vec{\nabla} \times \vec{P} = 0$ that has to be specified; this reflects the fact that $\vec{P}$ has more degrees of freedom than a scalar field ρ_b, so those extra degrees of freedom need to be specified (via the curl-free condition) for ρ_b to tell the whole story (and thus for $\vec{D}$ to be derivable from ρ_f).

The situation will simplify somewhat when we consider linear, uniform dielectrics where $\vec{P} \propto \vec{E}$; then $\vec{\nabla} \times \vec{P} = 0$ is guaranteed, though the requirement that $\vec{P}$ be normal to any boundaries may still create complications.

The Electric Displacement Field (cont.)

Boundary Conditions on the Displacement Field

We derived boundary conditions on $\vec{E}$ earlier, Equations 2.46 and 2.49:

$$\hat{n} \cdot \left(\vec{E}_2 - \vec{E}_1 \right) = \frac{1}{\epsilon_0} \sigma \qquad \hat{s} \cdot \left(\vec{E}_2 - \vec{E}_1 \right) = 0 \tag{4.22}$$

where $\hat{n}$ is the normal vector pointing from region 1 into region 2 and $\hat{s}$ is any tangential vector (*i.e.*, $\hat{s} \cdot \hat{n} = 0$). We derived the equation for the normal component using the divergence of $\vec{E}$. So, here, we can use the fact that $\vec{\nabla} \cdot \vec{D} = \rho_f$, which yields

$$\boxed{\hat{n} \cdot \left(\vec{D}_2 - \vec{D}_1 \right) = \sigma_f} \tag{4.23}$$

Note that, by definition, we have $\sigma_b = \hat{n} \cdot \vec{P}$ where $\hat{n}$ is the outward normal going from a region with a polarization density to vacuum. Therefore, by superposition,

$$\boxed{\hat{n} \cdot \left(\vec{P}_2 - \vec{P}_1 \right) = -\sigma_b} \tag{4.24}$$

We could also have used $\rho_b = -\vec{\nabla} \cdot \vec{P}$ and followed the same type of derivation as used for $\vec{E}$ and $\vec{D}$. The sign on the right side of the boundary condition enters because of the sign in $\vec{\nabla} \cdot \vec{P} = -\rho_b$.

The Electric Displacement Field (cont.)

In general, we know nothing about $\vec{\nabla} \times \vec{P}$, so the boundary condition on the tangential component of $\vec{D}$ just reflects the fact that its curl is the curl of the polarization field. We obtain this condition by inserting the relation between $\vec{E}$, $\vec{D}$, and $\vec{P}$ into the above tangential condition:

$$\hat{s} \cdot \left(\vec{D}_2 - \vec{D}_1 \right) = \hat{s} \cdot \left(\vec{P}_2 - \vec{P}_1 \right) \tag{4.25}$$

Note that, even in the case of linear dielectrics, the right side can be nonzero, as we will see below.

Susceptibility, Permittivity, and Dielectric Constant

So far, we have considered situations where $\vec{P}$ has been specified for us. But, it is usually caused by an external field, and so what we really want to do is figure out what observed potential and field arise by summing the externally applied potential/field and that due to the polarization of the dielectric in response to that external potential/field. For most substances, at least at low fields, the relation between the two is linear:

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \tag{4.26}$$

where χ_e is the *electric susceptibility*. Such materials are called *linear dielectrics*. An immediate implication of the above is:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 (1 + \chi_e) \vec{E} \equiv \epsilon \vec{E} \tag{4.27}$$

where $\epsilon \equiv \epsilon_0 (1 + \chi_e)$ is the *permittivity* of the material and $\epsilon_r \equiv 1 + \chi_e$ is the *relative permittivity* or *dielectric constant* of the material.

A very important point is that $\vec{E}$ above is the *total* field, not just the externally applied field. You can think of polarization as an iterative process: an applied field $\vec{E}_0$ causes polarization $\vec{P}_0$, which creates its own field $\vec{E}_1$, which the polarization responds to by adding a contribution $\vec{P}_1$, which creates its own field $\vec{E}_2$, and so on. The process converges to the final total electric field $\vec{E}$ and polarization $\vec{P}$.

Conducting sphere with dielectric shell around it

Consider a conducting sphere of radius a with (free) charge Q on it surrounded by a shell of dielectric ϵ with inner and outer radii a and b. Because the system is spherically symmetric and contains a linear dielectric, we know that $\vec{E}$, $\vec{D}$, and $\vec{P}$ all have the form

$$\vec{E} = E(r)\hat{r} \qquad \vec{D} = D(r)\hat{r} \qquad \vec{P} = P(r)\hat{r} \tag{4.28}$$

This ensures that the curl of all three vanish and that, at the boundaries, we have no tangential components of $\vec{D}$ and $\vec{P}$. We have now satisfied all the conditions required for us to be able to derive $\vec{D}$ directly from the free charge by Gauss's Law, which yields

$$\vec{D}(\vec{r}) = \frac{Q}{4\,\pi\,r^2}\,\hat{r} \qquad r > a \tag{4.29}$$

($\vec{D} = \vec{E} = \vec{P} = 0$ for $r < a$.) Then we just apply the relation between $\vec{D}$ and $\vec{E}$:

$$\vec{E}(\vec{r}) = \frac{Q}{4\,\pi\,\epsilon(r)\,r^2}\,\hat{r} = \begin{cases} \left(Q/4\,\pi\,\epsilon\,r^2\right)\hat{r} & a < r < b \\ \left(Q/4\,\pi\,\epsilon_0\,r^2\right)\hat{r} & b < r \end{cases} \tag{4.30}$$

The electric field is *screened* (reduced) inside the dielectric and unchanged outside.

Linear Dielectrics (cont.)

Let's calculate the polarization vector and bound charge density:

$$\vec{P}(\vec{r}) = \epsilon_0\,\chi_e(r)\,\vec{E}(\vec{r}) = (\epsilon(r)-\epsilon_0)\,\vec{E}(\vec{r}) = \frac{\epsilon(r)-\epsilon_0}{\epsilon(r)}\,\frac{Q}{4\,\pi\,r^2}\,\hat{r}$$

$$= \begin{cases} \frac{\epsilon-\epsilon_0}{\epsilon}\,\frac{Q}{4\,\pi\,r^2}\,\hat{r} & a<r<b \\ 0 & b<r \end{cases} \tag{4.31}$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0 \tag{4.32}$$

$$\sigma_b = \begin{cases} -\hat{r}\cdot\vec{P}(r=a) = -\frac{\epsilon-\epsilon_0}{\epsilon}\,\frac{Q}{4\,\pi\,a^2} & r=a \\ \hat{r}\cdot\vec{P}(r=b) = \frac{\epsilon-\epsilon_0}{\epsilon}\,\frac{Q}{4\,\pi\,b^2} & r=b \end{cases} \tag{4.33}$$

Note the ϵ in the denominator! We see that $\vec{P}$ is radially outward and decreasing with r like $1/r^2$ as $\vec{E}$ does. Note that, even though, $\vec{P}$ is position-dependent, its divergence vanishes, so there is no bound charge density. There is surface charge density, negative at $r=a$ and positive at $r=b$. This is to be expected, as the dielectric polarizes so the negative ends of the dipoles are attracted to Q on the conducting sphere and the positive ends are repelled, leaving uncancelled layers of negative charge on the inner boundary and positive charge on the outer boundary.

Note that the total surface charge on the outer boundary cancels that on the inner boundary, so the net charge enclosed inside a sphere of radius $r>b$ is just Q. This explains why, outside the dielectric, no screening effect is present!

Finally, let's calculate the electric potential from $\vec{E}$:

$$V(\vec{r}) = -\int_{\infty}^{\vec{r}} d\vec{s}\,' \cdot \vec{E}(\vec{r}\,') = -\int_{\infty}^{r} dr' E(r')$$

$$V(r > b) = -\frac{Q}{4\pi}\left[\int_{\infty}^{r} dr' \frac{1}{\epsilon_0 r^2}\right] = \frac{Q}{4\pi}\frac{1}{\epsilon_0 r} \tag{4.34}$$

$$\begin{aligned} V(a < r < b) &= -\frac{Q}{4\pi}\left[\int_{\infty}^{b} dr' \frac{1}{\epsilon_0 r^2} + \int_{b}^{r} dr' \frac{1}{\epsilon r^2}\right] \\ &= \frac{Q}{4\pi}\left[\frac{1}{\epsilon_0 r}\Big|_{\infty}^{b} + \frac{1}{\epsilon r}\Big|_{b}^{r}\right] = \frac{Q}{4\pi}\left[\frac{1}{b}\left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon}\right) + \frac{1}{\epsilon r}\right] \end{aligned} \tag{4.35}$$

$$V(r < a) = V(r = a) = \frac{Q}{4\pi}\left[\frac{1}{b}\left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon}\right) + \frac{1}{\epsilon a}\right] \tag{4.36}$$

where V is constant for $r < a$ because $r < a$ is occupied by a conductor.

Dielectrics and Capacitors

You all know from Ph1b that filling the volume between the plates of a parallel-plate capacitor increases the capacitance to $C = \epsilon_r\, C_{vac}$ where C_{vac} is the capacitance with vacuum between the plates. We remind you why this is true.

Let the capacitor plates lie parallel to the xy-plane at $z = 0$ (negative plate) and $z = a$ (positive plate) so $\widehat{z}$ is the unit vector pointing from the negative plate to the positive one. In such a geometry, we know from symmetry that $\vec{E}$, $\vec{D}$, and $\vec{P}$ are all parallel to $\widehat{z}$ and independent of xy, assuming we ignore the capacitor edges. Thus, at the interfaces at $z = 0$ and $z = a$, all these vectors are normal to the interface and so no tangential components are present. These features of the fields imply that we can apply Gauss's Law to the free charge density to find $\vec{D}$.

The free charge density is $\sigma_f = \pm Q/A$ where Q is the charge on the plates ($+Q$ at $z = a$ and $-Q$ at $z = 0$) and A is the plate area. Gauss's Law for an infinite sheet of charge (Griffiths Example 2.5) tells us that the field of a single sheet is $E = \sigma/2\,\epsilon_0$. Therefore, we have for this case

$$\vec{D} = \begin{cases} -\frac{Q}{A}\,\widehat{z} & 0 < z < a \\ 0 & z < 0,\ z > a \end{cases} \tag{4.37}$$

because the fields of the two plates cancel for $z < 0$ and $z > a$ but add for $0 < z < a$, and there is no ϵ_0 because we are calculating $\vec{D}$, not $\vec{E}$.

Linear Dielectrics (cont.)

This implies:

$$\vec{E} = \begin{cases} -\frac{1}{\epsilon}\frac{Q}{A}\hat{z} & 0 < z < a \\ 0 & z < 0,\ z > a \end{cases} \qquad \vec{P} = \begin{cases} -\frac{\epsilon-\epsilon_0}{\epsilon}\frac{Q}{A}\hat{z} & 0 < z < a \\ 0 & z < 0,\ z > a \end{cases} \tag{4.38}$$

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = 0 \tag{4.39}$$

$$\sigma_b = \hat{n}\cdot\vec{P} = \begin{cases} \hat{z}\cdot\vec{P}(z=a) & z = a \\ -\hat{z}\cdot\vec{P}(z=0) & z = 0 \end{cases} = \begin{cases} -\frac{\epsilon-\epsilon_0}{\epsilon}\frac{Q}{A} & z = a \\ \frac{\epsilon-\epsilon_0}{\epsilon}\frac{Q}{A} & z = 0 \end{cases} \tag{4.40}$$

We have negative surface charge near the positive plate and positive surface charge near the negative plate. Finally, the voltage is

$$V(0 < z < a) = -\int_0^z d\vec{s}\,'\cdot\vec{E}(\vec{r}\,') = -\int_0^z dz'\left(-\frac{1}{\epsilon}\frac{Q}{A}\right) = \frac{1}{\epsilon}\frac{Q}{A}z \tag{4.41}$$

From this, we can calculate the capacitance, which comes out as expected:

$$C = \frac{Q}{\Delta V} = \frac{Q}{(1/\epsilon)\,(Q/A)\,a} = \epsilon\frac{A}{a} = \epsilon_r C_{vac} \tag{4.42}$$

C is increased because ΔV is reduced because the surface charge densities screen the electric field inside the dielectric.

Linear Dielectrics (cont.)

Capacitor with two layer dielectric

Let's repeat, but now with a capacitor that has two slabs of dielectric with different ϵ: ϵ_1 for $0 < z < a$ and ϵ_2 for $a < z < b$, where the top plate is now at $z = b$. Because the interface is normal to $\vec{P}$, we can apply Gauss's Law for $\vec{D}$ as we did before, yielding no change in $\vec{D}$, but now the ϵ quantities in $\vec{E}$ and $\vec{P}$ depend on z.

The volume bound charge density vanishes again. The surface charge density at the top and bottom has the same expression, but again with ϵ being evaluated for the particular value of z. The surface bound charge density at the $z = a$ interface is

$$\sigma_b(z = a) = \hat{n}_1 \cdot \vec{P}_1 + \hat{n}_2 \cdot \vec{P}_2 = \hat{z} \cdot \vec{P}_1 - \hat{z} \cdot \vec{P}_2 = \frac{Q}{A}\left(-\frac{\epsilon_1 - \epsilon_0}{\epsilon_1} + \frac{\epsilon_2 - \epsilon_0}{\epsilon_2}\right) \quad (4.43)$$

Depending on which dielectric constant is greater, this can be positive or negative. Of course, it vanishes if $\epsilon_1 = \epsilon_2$. The potential and capacitance are

$$V(0 < z < a) = \frac{1}{\epsilon_1}\frac{Q}{A}z \qquad V(a < z < b) = \frac{1}{\epsilon_1}\frac{Q}{A}a + \frac{1}{\epsilon_2}\frac{Q}{A}(z - a) \quad (4.44)$$

$$C = \frac{Q}{\Delta V} = \left(\frac{a}{\epsilon_1} + \frac{b-a}{\epsilon_2}\right)^{-1} A = \epsilon_{eff}\frac{A}{b} = \epsilon_{eff,r}\, C_{vac} \quad (4.45)$$

where $1/\epsilon_{eff} = [a/\epsilon_1 + (b-a)/\epsilon_2]/b$ is the thickness-weighted inverse mean of the dielectric constant and $\epsilon_{eff,r} = \epsilon_{eff}/\epsilon_0$.

Linear Dielectrics (cont.)

Capacitor with two side-by-side (parallel) dielectrics

Now, allow the capacitor to have plate spacing a but with two different dielectrics side-by-side, with ϵ_1 occupying A_1 and $\mathcal{V}_1$ and ϵ_2 occupying A_2 and $\mathcal{V}_2$. It is a reasonable guess that one should treat this as two capacitors in parallel so that

$$C = C_1 + C_2 = \frac{1}{a}\left(\epsilon_1 A_1 + \epsilon_2 A_2\right) \tag{4.46}$$

But let's derive this from scratch, appreciating the subtlety at the interface.

Because the voltage difference between the two plates is independent of ϵ (they are equipotentials), it is reasonable to guess that $\vec{E}$ is the same in ϵ_1 and ϵ_2. Because the dielectrics are uniform in z, it is also reasonable to assume it is independent of z as one would have in the single-dielectric case. So, our guess for the form of the fields is:

$$\vec{E} = -E_0\,\hat{z} \quad \vec{D} = \begin{cases} -\epsilon_1 E_0\,\hat{z} & \text{in } \mathcal{V}_1 \\ -\epsilon_2 E_0\,\hat{z} & \text{in } \mathcal{V}_2 \end{cases} \qquad \vec{P} = \begin{cases} -\left(\epsilon_1 - \epsilon_0\right) E_0\,\hat{z} & \text{in } \mathcal{V}_1 \\ -\left(\epsilon_2 - \epsilon_0\right) E_0\,\hat{z} & \text{in } \mathcal{V}_2 \end{cases} \tag{4.47}$$

We see this form respects the tangential boundary conditions at the interface between the two dielectrics:

$$\hat{z}\cdot\left(\vec{E}_2 - \vec{E}_1\right) = 0 \qquad \hat{z}\cdot\left(\vec{D}_2 - \vec{D}_1\right) = \left(\epsilon_1 - \epsilon_2\right) E_0 = \hat{z}\cdot\left(\vec{P}_2 - \vec{P}_1\right) \tag{4.48}$$

Linear Dielectrics (cont.)

Because $\vec{D}$ is different in the two volumes, we must allow the free (and bound) charge densities to be different. This provides us a set of equations to solve for E_0:

$$\epsilon_1 E_0 = \sigma_{f,1} \qquad \epsilon_2 E_0 = \sigma_{f,2} \qquad A_1 \sigma_{f,1} + A_2 \sigma_{f,2} = Q \tag{4.49}$$

$$\Longrightarrow \qquad E_0 = \frac{1}{\epsilon_{eff}} \frac{Q}{A} \qquad \epsilon_{eff} = \frac{\epsilon_1 A_1 + \epsilon_2 A_2}{A_1 + A_2} \qquad A = A_1 + A_2 \tag{4.50}$$

$$C = \frac{Q}{\Delta V} = \frac{Q}{a E_0} = \epsilon_{eff} \frac{A}{a} = \epsilon_{eff,r} C_{vac} \tag{4.51}$$

which matches our parallel-capacitor expectation. The polarizations and free bound charge densities are

$$\vec{P} = \begin{cases} -\frac{\epsilon_1 - \epsilon_0}{\epsilon_{eff}} \frac{Q}{A} \hat{z} & \text{in } \mathcal{V}_1 \\ -\frac{\epsilon_2 - \epsilon_0}{\epsilon_{eff}} \frac{Q}{A} \hat{z} & \text{in } \mathcal{V}_2 \end{cases} \qquad |\sigma_f| = \begin{cases} \frac{\epsilon_1}{\epsilon_{eff}} \frac{Q}{A} & \text{in } \mathcal{V}_1 \\ \frac{\epsilon_2}{\epsilon_{eff}} \frac{Q}{A} & \text{in } \mathcal{V}_2 \end{cases} \qquad \sigma_b = \begin{cases} -|\vec{P}| & z = 0 \\ |\vec{P}| & z = a \end{cases} \tag{4.52}$$

where the sign for σ_b assumes $Q > 0$. The sign of σ_f is positive at $z = a$ and negative at $z = 0$ under this assumption.

Finally, if one calculates the total charge density $\sigma_f + \sigma_b$ at $z = 0$ or $z = a$, one gets

$$\sigma_{t,1} = \sigma_{f,1} + \sigma_b = \left(\frac{\epsilon_1}{\epsilon_{eff}} - \frac{\epsilon_1 - \epsilon_0}{\epsilon_{eff}} \right) \frac{Q}{A} = \frac{\epsilon_0}{\epsilon_{eff}} \frac{Q}{A} = \sigma_{t,2} \tag{4.53}$$

This makes sense: since the electric field is the same in $\mathcal{V}_1$ and $\mathcal{V}_2$, the total (free + bound) surface charge density sourcing it should be the same. The total charge density is a factor $\epsilon_0/\epsilon_{eff}$ smaller than would be present in the absence of dielectrics because the bound charge density screens the free charge density. The free charge density is different in the two regions because the opposite-sign bound charge density is different because of the different dielectric constants.

General Conditions for Linear, Homogeneous Dielectrics

In linear, homogeneous dielectrics,

$$\rho_b = -\vec{\nabla}\cdot\vec{P} = -\vec{\nabla}\cdot\left(\frac{\epsilon-\epsilon_0}{\epsilon}\,\vec{D}\right) = -\left(\frac{\epsilon-\epsilon_0}{\epsilon}\right)\vec{\nabla}\cdot\vec{D} = -\left(\frac{\epsilon-\epsilon_0}{\epsilon}\right)\rho_f \qquad (4.54)$$

(Homogeneity is requires so the gradient does not act on ϵ.) Therefore, if there is no free charge density in a linear, homogeneous dielectric, there is no bound charge density either. Thus, the dielectric volume satisfies Laplace's Equation. All our machinery for solving Laplace's Equation applies here.

We always need boundary conditions, though, and we can use the ones we derived earlier:

$$\hat{n}\cdot\left[\vec{D}_2-\vec{D}_1\right] = \sigma_f \qquad (4.55)$$

Writing this in terms of the potential, we have

$$\hat{n}\cdot\left[\epsilon_2\vec{\nabla}V_2-\epsilon_1\vec{\nabla}V_1\right] = -\sigma_f \qquad (4.56)$$

And, we always require $V_1 = V_2$: the potential must be continuous.

Spherical cavity in a dielectric medium with uniform field applied

Let's apply the above to a spherical cavity of radius R in a medium with dielectric constant ϵ with a uniform field $\vec{E} = E_0 \hat{z}$ applied. There is no free charge anywhere. Our boundary conditions therefore are

$$V(r \to \infty) = -E_0\, z = -E_0\, r\, P_1(\cos\theta) \tag{4.57}$$

and

$$\epsilon_0 \left.\frac{\partial V_<}{\partial r}\right|_{r=R} = \epsilon \left.\frac{\partial V_>}{\partial r}\right|_{r=R} \qquad \text{and} \qquad V_<(r=R) = V_>(r=R) \tag{4.58}$$

where $V_<(r) = V(r < R)$ and $V_>(r) = V(r > R)$. We also choose the zero of the potential to be at $z = 0$, $V(z = 0) = 0$.

Boundary Value Problems with Dielectrics (cont.)

As usual, we begin by writing our generic solutions to Laplace's Equation in spherical coordinates:

$$V(r<R)=\sum_{\ell=0}^{\infty}C_\ell r^\ell P_\ell(\cos\theta) \qquad V(r>R)=\sum_{\ell=0}^{\infty}\left(A_\ell r^\ell+\frac{B_\ell}{r^{\ell+1}}\right)P_\ell(\cos\theta) \tag{4.59}$$

where we have applied the requirement that V be finite at the origin to eliminate the $1/r^{\ell+1}$ terms for $V(r<R)$. Recall that we cannot eliminate the r^ℓ terms for $V(r>R)$ because the potential does not vanish at infinity.

Let's first apply the $r\to\infty$ condition. We did this before in the case of a metal sphere in a uniform field, and we found

$$A_1=-E_0 \qquad A_{\ell\neq 1}=0 \tag{4.60}$$

Next, we apply the continuity condition at $r=R$, making use of orthonormality of the P_ℓ:

$$C_1R=-E_0R+\frac{B_1}{R^2} \qquad C_{\ell\neq 1}R^\ell=\frac{B_{\ell\neq 1}}{R^{\ell+1}} \tag{4.61}$$

Boundary Value Problems with Dielectrics (cont.)

Finally, let's take the radial derivative and apply the matching condition on it, again using orthonormality:

$$\epsilon_0 C_1 = -\epsilon \left(E_0 + \frac{2}{R^3} B_1 \right) \qquad \epsilon_0 C_{\ell \neq 1} \ell R^{\ell-1} = -\epsilon \frac{B_{\ell \neq 1}}{R^{\ell+2}} (\ell + 1) \tag{4.62}$$

Doing the algebra, we find

$$C_{\ell \neq 1} = B_{\ell \neq 1} = 0 \qquad B_1 = -\frac{\epsilon - \epsilon_0}{2\epsilon - \epsilon_0} E_0 R^3 \qquad C_1 = -\frac{3\epsilon}{2\epsilon + \epsilon_0} E_0 \tag{4.63}$$

Thus, the potential is

$$V(r < R) = -\frac{3\epsilon}{2\epsilon + \epsilon_0} E_0 r \cos\theta = -\frac{3\epsilon}{2\epsilon + \epsilon_0} E_0 z \tag{4.64}$$

$$V(r > R) = -E_0 r \cos\theta - \frac{\epsilon - \epsilon_0}{2\epsilon + \epsilon_0} E_0 \frac{R^3}{r^2} \cos\theta \tag{4.65}$$

$$= -E_0 z + \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2} \qquad \text{with} \quad \vec{p} = -\frac{4\pi}{3} R^3 E_0 \frac{3\epsilon_0}{2\epsilon + \epsilon_0} (\epsilon - \epsilon_0) \hat{z}$$

Boundary Value Problems with Dielectrics (cont.)

The potential inside the cavity is that of a uniform electric field in the same direction as the applied field but multiplied by the factor $3\,\epsilon/(2\,\epsilon+\epsilon_0) > 1$, while the potential outside is that of the uniform field plus that of a dipole whose orientation is *opposite* the uniform field and whose magnitude is given above. It is as if the *cavity* acquired a polarization density in the negative z direction, though of course that cannot happen because $\chi_e(r<R)=0$ there and thus $\vec{P}(r<R)=\epsilon_0\chi_e(r<R)\vec{E}(r<R)=0$. The polarization density outside the cavity is just the field times $\epsilon-\epsilon_0$ (which is not particularly illuminating).

The (bound) surface charge density is

$$\begin{aligned}\sigma_b &= \widehat{n}\cdot\vec{P}(r=R) = \widehat{n}\cdot(\epsilon-\epsilon_0)\,\vec{E}(r=R)\\ &= (-\widehat{r})\cdot(\epsilon-\epsilon_0)\,\frac{3\,\epsilon}{2\,\epsilon+\epsilon_0}\,E_0\widehat{z} = -3\,\epsilon\,\frac{\epsilon-\epsilon_0}{2\,\epsilon+\epsilon_0}\,E_0\cos\theta \qquad (4.66)\end{aligned}$$

(Notice that $\widehat{n}=-\widehat{r}$ because $\widehat{n}$ is taken to point *out of* the dielectric medium in the definition of σ_b.) We see the boundary of the cavity acquires a surface charge density with the same cosine dependence as the bound charge on the surface of a uniformly polarized sphere, though with *opposite* sign (so there is negative charge at the $+z$ end and positive charge at the $-z$ end). The sign follows naturally from our arguments about cancellation of dipole charge.

The field is enhanced in the cavity for two reasons: first, there is no polarizable material to screen the electric field, and, second there is surface charge density on the cavity's boundary that creates an additional field in the direction of the uniform field.

Boundary Value Problems with Dielectrics (cont.)

For reference, we note that the solution for a dielectric sphere (Griffiths Example 4.7) in a uniform field looks very similar:

$$V(r<R) = -\frac{3\,\epsilon_0}{2\,\epsilon_0+\epsilon}\,E_0 z \qquad V(r>R) = -E_0 z + \frac{\vec{p}\cdot\hat{r}}{4\,\pi\,\epsilon_0\,r^2} \tag{4.67}$$

$$\text{with} \quad \vec{p} = \frac{4\,\pi}{3}\,R^3\,E_0\,\frac{3\,\epsilon_0}{2\,\epsilon_0+\epsilon}\,(\epsilon-\epsilon_0)\,\hat{z} \equiv \frac{4\,\pi}{3}\,R^3\,\vec{P}(r<R) \tag{4.68}$$

$$\sigma_b = 3\,\epsilon_0\,\frac{\epsilon-\epsilon_0}{2\,\epsilon_0+\epsilon}\,E_0\cos\theta \tag{4.69}$$

Basically, exchange ϵ_0 and ϵ everywhere to go between the two results. In this case, the sphere acquires a polarization density $3\,\epsilon_0(\epsilon-\epsilon_0)/(2\,\epsilon_0+\epsilon)$, now in the direction of the applied field. The surface charge density is also of same form as the cavity case with the $\epsilon \leftrightarrow \epsilon_0$ exchange. That exchange flips the sign so that the $+z$ end acquires a positive charge, again as expected from the dipole charge cancellation argument.

Electrostatic Potential Energy due to Assembly of Free Charge

It turns out that electrostatic potential energy in dielectrics is a subtle topic because of the existence of the charges forming the dielectric. There are different kinds of electrostatic potential energy: that needed to assemble the free and bound charge distribution versus that needed to assemble the free charge distribution and polarize the preexisting dielectric. It is generally the latter we are interested in, so we consider that case.

Suppose we have a system in which an electric field $\vec{E}(\vec{r})$ and its potential $V(\vec{r})$ have already been set up and we want to bring in additional *free charge* $\delta\rho_f$ from infinity (assuming the potential vanishes at infinity). In this case, the change in potential energy is

$$\delta U = \int_{\mathcal{V}} d\tau' \left(\delta\rho_f(\vec{r}\,')\right) V(\vec{r}\,') \tag{4.70}$$

The free charge density is related to the displacement field by $\vec{\nabla}\cdot\vec{D} = \rho_f$, so a change $\delta\rho_f$ corresponds to a change in the divergence of the displacement field $\delta\left(\vec{\nabla}\cdot\vec{D}\right)$. Linearity of the divergence lets us rewrite this as $\delta\rho_f = \vec{\nabla}\cdot\delta\vec{D}$.

Electrostatic Energy in and Forces on Dielectrics (cont.)

Then, we may integrate by parts and apply the divergence theorem:

$$\begin{aligned}\delta U &= \int_{\mathcal{V}} d\tau' \left(\vec{\nabla}\cdot\delta\vec{D}(\vec{r}\,')\right) V(\vec{r}\,') \\ &= \int_{\mathcal{V}} d\tau' \vec{\nabla}\cdot\left(V(\vec{r}\,')\,\delta\vec{D}(\vec{r}\,')\right) - \int_{\mathcal{V}} d\tau' \left(\delta\vec{D}(\vec{r}\,')\right)\cdot\vec{\nabla}V(\vec{r}\,') \\ &= \oint_{\mathcal{S}(\mathcal{V})} da'\,\hat{n}(\vec{r}\,')\cdot\left(V(\vec{r}\,')\,\delta\vec{D}(\vec{r}\,')\right) + \int_{\mathcal{V}} d\tau' \left(\delta\vec{D}(\vec{r}\,')\right)\cdot\vec{E}(\vec{r}\,') \qquad (4.71)\end{aligned}$$

Assuming the potential falls off at infinity, the surface term can be taken out to infinity to vanish. So, we are then left with

$$U = \int_0^{\vec{D}} \int_{\mathcal{V}} d\tau'\,\vec{E}(\vec{r}\,')\cdot d\vec{D}(\vec{r}\,') \qquad (4.72)$$

There are two integrals here, one over volume and one over the value of $\vec{D}$ from zero to its final value. $\vec{E}$ is of course tied to $\vec{D}$ and they vary together.

Electrostatic Energy in and Forces on Dielectrics (cont.)

For the case of a linear dielectric, we may use $\vec{D} = \epsilon \vec{E}$ and therefore

$$\begin{aligned} U &= \epsilon \int_0^{\vec{E}} \int_{\mathcal{V}} d\tau' \, \vec{E}(\vec{r}\,') \cdot d\vec{E}(\vec{r}\,') \\ &= \frac{\epsilon}{2} \int_0^{\vec{E}} \int_{\mathcal{V}} d\tau' \, d\left(\vec{E}(\vec{r}\,') \cdot \vec{E}(\vec{r}\,')\right) \\ &= \frac{\epsilon}{2} \int_{\mathcal{V}} d\tau' \, E^2(\vec{r}\,') = \frac{1}{2} \int_{\mathcal{V}} d\tau' \, \vec{E}(\vec{r}\,') \cdot \vec{D}(\vec{r}\,') \end{aligned} \tag{4.73}$$

By contrast, if we wanted to know the total electrostatic potential energy stored in the assembly of the free and bound charge, we would just do the usual volume integral of E^2 with ϵ_0 instead of ϵ. That energy is smaller because $\epsilon > \epsilon_0$. The reason for this difference is that, assembling that medium in the first place, which consists of bringing positive and negative charges together, creates a system with negative potential energy, and thus the total potential energy of the system would be lower if we accounted for the energy of assembling the medium. But we will never pull the dielectric apart, so it is natural to treat that component of the potential energy as an offset that is inaccessible and neglect it in the potential energy.

Energy of a Dielectric in an External Field

A topic naturally related to the above is the electrostatic energy of a polarizable material in an external field.

Suppose we start with a system with a free charge distribution ρ_f that sources a field $\vec{E}_1$ in a dielectric medium ϵ_1, yielding a displacement $\vec{D}_1 = \epsilon_1 \vec{E}_1$. The initial energy is

$$U_1 = \frac{1}{2} \int d\tau' \vec{E}_1 \cdot \vec{D}_1 \tag{4.74}$$

Now, *with the charge sourcing* $\vec{E}_1$ *held fixed*, let's introduce a piece of dielectric occupying the volume $\mathcal{V}_2$ and having dielectric constant ϵ_2, replacing the dielectric of dielectric constant ϵ_1 there. The remainder of space outside $\mathcal{V}_2$ is occupied by ϵ_1 in both configurations. The electric field and displacement field everywhere change to $\vec{E}_2$ and $\vec{D}_2$, where $\vec{D}_2(\vec{r}) = \epsilon(\vec{r})\, \vec{E}_2(\vec{r})$. The energy is now

$$U_2 = \frac{1}{2} \int d\tau' \vec{E}_2 \cdot \vec{D}_2 \tag{4.75}$$

Electrostatic Energy in and Forces on Dielectrics (cont.)

The difference in energy between the two configurations is

$$U_2 - U_1 = \frac{1}{2}\int d\tau' \left[\vec{E}_2 \cdot \vec{D}_2 - \vec{E}_1 \cdot \vec{D}_1\right] \tag{4.76}$$

which we rewrite as

$$U_2 - U_1 = \frac{1}{2}\int d\tau' \left[\vec{E}_2 \cdot \vec{D}_1 - \vec{E}_1 \cdot \vec{D}_2\right] + \frac{1}{2}\int d\tau' \left[\vec{E}_2 + \vec{E}_1\right] \cdot \left[\vec{D}_2 - \vec{D}_1\right] \tag{4.77}$$

It holds that $\vec{\nabla} \times \left[\vec{E}_2 + \vec{E}_1\right] = 0$, so it can be derived from a potential V, so the second integral becomes

$$-\frac{1}{2}\int d\tau' \left(\vec{\nabla} V\right) \cdot \left[\vec{D}_2 - \vec{D}_1\right] \tag{4.78}$$

We integrate by parts (taking the boundary term to infinity so it vanishes) to obtain

$$\frac{1}{2}\int d\tau' V \vec{\nabla} \cdot \left[\vec{D}_2 - \vec{D}_1\right] \tag{4.79}$$

This divergence vanishes because the free charge has not changed between the two configurations (recall, $\vec{\nabla} \cdot \vec{D} = \rho_f$).

Electrostatic Energy in and Forces on Dielectrics (cont.)

So the second term in the energy vanishes, leaving

$$U_2 - U_1 = \frac{1}{2}\int d\tau' \left[\vec{E}_2 \cdot \vec{D}_1 - \vec{E}_1 \cdot \vec{D}_2\right] \tag{4.80}$$

Now, outside $\mathcal{V}_2$, it holds that $\vec{D}_2 = \epsilon_1 \vec{E}_2$ (remember, ϵ only changed inside $\mathcal{V}_2$), so the two terms cancel each other outside $\mathcal{V}_2$. So the integrand vanishes outside $\mathcal{V}_2$ and we can restrict the integral to $\mathcal{V}_2$:

$$U_2 - U_1 = -\frac{1}{2}\int_{\mathcal{V}_2} d\tau' \left(\epsilon_2 - \epsilon_1\right) \vec{E}_2 \cdot \vec{E}_1 \tag{4.81}$$

If $\epsilon_1 = \epsilon_0$ (vacuum outside $\mathcal{V}_2$ and in $\mathcal{V}_2$ before the introduction of ϵ_2), then we can use $\vec{P} = (\epsilon - \epsilon_0)\,\vec{E}$ to rewrite as

$$U_2 - U_1 = -\frac{1}{2}\int_{\mathcal{V}_2} d\tau' \vec{P}_2 \cdot \vec{E}_1 \tag{4.82}$$

This is just like the energy of a dipole in an external electric field, except that the factor of $1/2$ accounts for the fact that the dipole polarizes in response to the external field (it accounts for the integration from zero field to actual field). *Note that $\vec{E}_1$ is the external field in the absence of the dielectric!*.