ACM 100b

Solving linear systems with the adjoint matrix

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At our last lecture...

- We looked at the problem of solving inhomgeneous systems:

$$\boldsymbol{x}' = A(z)\boldsymbol{x} + \boldsymbol{f}(z), \quad \boldsymbol{x}(z_0) = \boldsymbol{x}_0$$

- We introduced the adjoint system

$$\boldsymbol{y}' = -A^T(z)\boldsymbol{y}$$

  where A^T is the transpose of $A(z)$.
- We showed that the adjoint system can be used to reduce the order of the original system
- We also showed that if we knew all the solutions of the adjoint system we could write a formal solution to the inhomogeneous system:

$$\boldsymbol{x} = \Phi^{-1}(z)\Phi(z_0)\boldsymbol{x}_0 + \Phi^{-1}(z)\int_{z_0}^{z} \Phi(t)\boldsymbol{f}(t)dt.$$

  where Φ satisfies the matrix system

$$\Phi' = -\Phi A.$$

The adjoint system for constant coefficients

- Now let's return to the adjoint system which was

$$\Phi' = -\Phi A.$$

  where now A is a constant matrix.

- If A is a constant matrix then we have

$$\Phi = \Phi_0 \exp(-Az)$$

  where Φ_0 is some arbitrary non-singular matrix.

- We also have

$$\Phi^{-1} = \exp(Az)\Phi_0^{-1},$$

Solving the constant system using the adjoint

- Now recall we can use the adjoint to solve the original system:

$$\boldsymbol{x} = \Phi^{-1}(z)\Phi(z_0)\boldsymbol{x}_0 + \Phi^{-1}(z)\int_{z_0}^{z}\Phi(t)\boldsymbol{f}(t)dt.$$

- And so the solution for $\boldsymbol{x}$ becomes

$$\begin{aligned}\boldsymbol{x} &= \exp(Az)\Phi_0^{-1}\Phi(z_0)\boldsymbol{x}_0 + \exp(Az)\Phi_0^{-1}\int_0^z \Phi_0 \exp(-At)\boldsymbol{f}(t)dt \\ &= \exp(Az)\boldsymbol{x}_0 + \int_0^z \exp(A(z-t))\boldsymbol{f}(t)dt\end{aligned}$$

- We see that the solution looks very much like the solution for a first order ODE with constant coefficients except exponentials become matrix exponentials.
- Note too we got the particular solution as well this way.

Example of using the adjoint method

- As an example of using the adjoint equation method, we will solve a 2 by 2 system.
- Recall if we can find any solution to a 2'nd order problem then the other solution can be found by these methods.
- The system we consider is

$$\boldsymbol{x}' = \begin{pmatrix} 0 & \dfrac{4z^2 - 2z + 2}{2z - 1} \\ -1 & -\dfrac{4z^2 + 1}{2z - 1} \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} \exp(-z) \\ 0 \end{pmatrix}, \quad \boldsymbol{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

- Creating the adjoint system, we get

$$\boldsymbol{y}' = \begin{pmatrix} 0 & 1 \\ -\dfrac{4z^2 - 2z + 2}{2z - 1} & \dfrac{4z^2 + 1}{2z - 1} \end{pmatrix} \boldsymbol{y}.$$

Example of using the adjoint method

- By inspection, a solution to this system is

$$\boldsymbol{y}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \exp(z).$$

- Now employ the relationship between the adjoint solution and the true solution:

$$\begin{aligned}
(\boldsymbol{y}_1^T \boldsymbol{x})' &= \boldsymbol{y}_1^T \boldsymbol{x}' + \boldsymbol{y}_1^{T'} \boldsymbol{x} \\
&= \boldsymbol{y}_1^T (A\boldsymbol{x} + \boldsymbol{f}) - \boldsymbol{y}_1^T A\boldsymbol{x} \\
&= \boldsymbol{y}_1^T \boldsymbol{f} \\
&= \boldsymbol{y}_1^T \begin{pmatrix} \exp(-z) \\ 0 \end{pmatrix} \\
&= 1.
\end{aligned}$$

Example of using the adjoint method

- Integrating the result

$$(\boldsymbol{y}_1^T \boldsymbol{x})' = 1.$$

  we get

$$\boldsymbol{y}_1^T \boldsymbol{x} = z$$

- Or in scalar form,

$$x_1 + x_2 = z \exp(-z).$$

- Take this scalar relation and use it in the first equation of the system

$$\boldsymbol{x}' = \begin{pmatrix} 0 & \dfrac{4z^2 - 2z + 2}{2z - 1} \\ -1 & -\dfrac{4z^2 + 1}{2z - 1} \end{pmatrix} \boldsymbol{x} + \begin{pmatrix} \exp(-z) \\ 0 \end{pmatrix}, \quad \boldsymbol{x}(0) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Example of using the adjoint method

- In particular use the relation

$$x_1 + x_2 = z\exp(-z).$$

  to eliminate x_1, i.e.

$$x_2' = \frac{4z^2 - 2z + 2}{2z - 1} x_2 - z\exp(-z).$$

- This is now first order for x_2 and can be solved directly to get

$$x_2 = \frac{\exp(-z^2)}{2z-1} \int_0^z \exp(t^2 - t)(2t^2 - t)dt$$

  and so

$$x_1 = z\exp(-z) + \frac{\exp(-z^2)}{2z-1} \int_0^z \exp(t^2 - t)(2t^2 - t)dt.$$