ACM 100b

Solving constant coefficient linear systems

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Solving systems with constant matrices

- The simplest case to consider is if the coefficient matrix A is constant.
- We will demonstrate that for an $n \times n$ constant ODE system there are precisely n linearly independent solutions.
- First we'll make a simplifying assumption.
- Suppose the matrix A has n *distinct* eigenvalues.
- That is there are n distinct solutions of the homogeneous linear system

$$A\boldsymbol{x}_i = \lambda_i \boldsymbol{x}_i \qquad i = 1, \dots, n$$

- Recall the λ_i are called the eigenvalues of A and the vectors $\boldsymbol{x}_i$ are called the eigenvectors.
- Because of our assumption we never have $\lambda_i = \lambda_j$ if $i \neq j$.

Using the eigenvectors and eigenvalues to solve the system

- We recall from linear algebra that a matrix with distinct eigenvalues can be diagonalized.
- This means an $n \times n$ matrix T can be found such that

$$T^{-1}AT = D$$

where

$$D = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

and where the λ_i are the distinct eigenvalues of A.
- This matrix T has as its columns the eigenvectors $\boldsymbol{x}_i$

Using the eigenvectors and eigenvalues to solve the system

- Now define a new set of dependent variables as follows:

$$\boldsymbol{y} = T^{-1}\boldsymbol{x}$$

- The new variables $\boldsymbol{y}$ satisfy the following system:

$$\boldsymbol{y}' = T^{-1}\boldsymbol{x}' = T^{-1}AT\boldsymbol{y} = D\boldsymbol{y}$$

- In terms of these new variables the system of ODE's is decoupled because D is diagonal.
- In scalar notation we have

$$y_i' = \lambda_i y_i, \quad i = 1, 2, ..., n.$$

- And we see immediately that

$$y_i = c_i \exp(\lambda_i z) \qquad i = 1, \ldots, n$$

Verifying linear independence

- Given that
$$y_i = c_i \exp(\lambda_i z) \qquad i = 1, \ldots, n$$
it's very easy to get n linearly independent solutions for the system satisfied by $\boldsymbol{y}$:
$$\boldsymbol{y}_1 = \begin{pmatrix} e^{\lambda_i z} \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}, \quad \boldsymbol{y}_2 = \begin{pmatrix} 0 \\ e^{\lambda_2 z} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad ..., \quad \boldsymbol{y}_n = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ e^{\lambda_n z} \end{pmatrix}.$$
- It's easy to verify these vectors are linearly independent.
- The determinant formed from these column vectors is easy to compute and is given by
$$\exp(\lambda_1 z + \lambda_2 z + ... + \lambda_n z)$$
which can never vanish for finite z.

Recovering the solution for $\boldsymbol{x}$

- We want to get back to our original variable x.
- So we want to map this basis back to the original ODE.
- Undoing the transformation we have linearly independent solutions

$$
\begin{aligned}
\boldsymbol{x}_1 &= T\boldsymbol{y}_1, \\
\boldsymbol{x}_2 &= T\boldsymbol{y}_2, \\
&\vdots \\
\boldsymbol{x}_n &= T\boldsymbol{y}_n.
\end{aligned}
$$

- So every homogeneous solution $\boldsymbol{x}_i$ can be written as a linear combination (superposition) of these vectors $\boldsymbol{y}_i$ once you know T

Repeated eigenvalues

- You might have noticed the above analysis works if A has n distinct eigenvalues.
- But you may remember from linear algebra that such a decomposition is not guaranteed in general.
- If the matrix has n distinct eigenvalues then all is fine and you can get n linearly independent eigenvectors.
- If there are repeated roots it still may happen that you get n linearly independent eigenvectors.
- The classic example of this happening is the identity matrix.
- But what if there are less than n distinct eigenvectors?
- This can happen - such matrices are called defective.
- In this case we have to get the remaining solutions via reduction of order.