Physics 106a — Classical Mechanics

Michael Cross

California Institute of Technology

Fall Term, 2013

Lecture 19

Modes in Solids, Continuum Mechanics

Outline

- Normal mode coordinates diagonalize L and H
- Modes in a crystal
- Continuum mechanics
  - Stretched string
  - General (scalar) field theory
  - Elasticity theory
  - Conservation laws

Transformation matrix

Define $\boldsymbol{R}$ as the matrix with *columns* given by the normal mode vectors

$$R_{i\alpha} = \Phi_i^{(\alpha)}$$

Then

$$\boldsymbol{q}(t) = \boldsymbol{R} \cdot \boldsymbol{\rho}\ (t)$$

Congruence transformation with $\boldsymbol{R}$ diagonalizes $\boldsymbol{T}$ and $\boldsymbol{V}$

$$\tilde{\boldsymbol{R}} \cdot \boldsymbol{T} \cdot \boldsymbol{R} = \boldsymbol{I}$$

$$\tilde{\boldsymbol{R}} \cdot \boldsymbol{V} \cdot \boldsymbol{R} = \boldsymbol{\Omega}$$

where $\boldsymbol{\Omega}$ is the diagonal matrix with entries ω_α^2

Lagrangian and Hamiltonian

Substitute $\boldsymbol{q} = \boldsymbol{R} \cdot \boldsymbol{\rho}$ into expressions for T, V

$$T = \tfrac{1}{2}\tilde{\dot{\boldsymbol{q}}} \cdot \boldsymbol{T} \cdot \dot{\boldsymbol{q}} = \tfrac{1}{2}\tilde{\dot{\boldsymbol{\rho}}} \cdot \dot{\boldsymbol{\rho}}$$
$$V = \tfrac{1}{2}\tilde{\boldsymbol{q}} \cdot \boldsymbol{V} \cdot \boldsymbol{q} = \tfrac{1}{2}\tilde{\boldsymbol{\rho}} \cdot \boldsymbol{\Omega} \cdot \boldsymbol{\rho}$$

so that the Lagrangian is

$$L = \tfrac{1}{2}\sum_{\alpha}(\dot{\rho}_{\alpha}^{2} - \omega_{\alpha}^{2}\rho_{\alpha}^{2})$$

Defining the momentum conjugate to the normal mode coordinate

$$p_{\rho,\alpha} = \frac{\partial L}{\partial \dot{\rho}_{\alpha}} = \dot{\rho}_{\alpha}$$

gives the Hamiltonian

$$H = \tfrac{1}{2}\sum_{\alpha}(p_{\rho,\alpha}^{2} + \omega_{\alpha}^{2}\rho_{\alpha}^{2})$$

Strain tensor

$$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$$

$$e_{ii} = \vec{\nabla} \cdot \vec{u} \Rightarrow \text{ dilation}$$

$$e_{ij} - \tfrac{1}{3} e_{kk}\delta_{ij} \Rightarrow \text{ shear}$$

Potential energy of distortions

$$V = \tfrac{1}{2}\boldsymbol{e} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{e} \quad \Rightarrow \quad \mathcal{L} = \tfrac{1}{2}[\rho \dot{u}^2 - \boldsymbol{e} \cdot \boldsymbol{\Lambda} \cdot \boldsymbol{e}]$$

Elasticity tensor $\boldsymbol{\Lambda}$ components Λ_{ijkl} (symmetric under $i \leftrightarrow j, k \leftrightarrow l, ij \leftrightarrow kl$)

Stress tensor

$$\sigma_{ij} = \frac{\partial V}{\partial e_{ij}} = \Lambda_{ijkl} e_{kl}$$

gives the force in the i direction per unit area with normal in the j direction

Elasticity tensor must be combination of 4th rank isotropic tensors with correct index symmetries

$$\lambda_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) \quad \Rightarrow \quad V = \tfrac{1}{2}\lambda e_{ii}^2 + \mu e_{ij}^2$$

Two independent elastic coefficients λ, μ called the *Lamé coefficients*.
In terms of the dilation and shear

$$V = \tfrac{1}{2}K e_{ii}^2 + \mu(e_{ij} - \tfrac{1}{3}\delta_{ij}e_{kk})^2$$

with $K = \lambda + \frac{2}{3}\mu$ the bulk modulus, and μ is now called the shear modulus.
Young's modulus E and Poisson ratio σ give behavior of rod under tension

$$E = \frac{9K\mu}{3K+\mu} \quad \text{ratio of tension/area to extension}$$

$$\sigma = \frac{3K-2\mu}{2(3K+\mu)} \quad \text{transverse compression/longitudinal extension}$$

$$\mathcal{L} = \frac{1}{2}\rho\dot{u}^2 - \frac{1}{2}[Ke_{ii}^2 + \mu(e_{ij} - \frac{1}{3}\delta_{ij}e_{kk})^2]$$

Euler-Lagrange equations gives two types of propagating waves

- Transverse waves with $\vec{\nabla} \cdot \vec{u} = 0$ propagating with the speed $\sqrt{\mu/\rho}$
- Longitudinal waves with $\vec{\nabla} \times \vec{u} = 0$ propagating with speed $\sqrt{(\lambda + 2\mu)/\rho}$

A good references on elasticity theory is volume 7 of the Landau and Lifshitz series, e.g. chapter I for the basic physics and chapter III for the waves.