ACM 100b

First order linear ODE's

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First order linear ODE

- The list of exactly solvable ODE's is small
- We begin by examining the simplest first order linear ODE:

$$A_1(z)y' + A_0(z)y = f(z).$$

- We assume that the coefficients A_0 and A_1 are continuous functions.
- We can actually relax this, but will not do so for now.
- The first order scalar linear ODE can be solved completely in closed form.

Solving the first order linear ODE

- To begin, consider the homogeneous version:

$$A_1(z)y' + A_0(z)y = 0.$$

- We can rewrite this ODE as follows:

$$\frac{y'}{y} + \frac{A_0}{A_1} = 0 \quad \text{so} \quad \frac{y'}{y} = -\frac{A_0}{A_1}$$

- And then integrate both sides to get

$$y(z) = c_0 \exp\left[-\int_{z_0}^{z} \frac{A_0(t)}{A_1(t)} dt\right].$$

where c_0 is our arbitrary constant and z_0 is some arbitrary starting point for the integral.

Solving the first order linear ODE

- We note that if we evaluate the expression

$$y(z) = c_0 \exp\left[-\int_{z_0}^{z} \frac{A_0(t)}{A_1(t)}\,dt\right].$$

  at $z = z_0$ we get

$$y(z_0) = c_0,$$

- We made the lower limit of the integral z_0 for convenience.
- So we can solve the initial value problem of

$$A_1(z)y' + A_0(z)y = 0 \qquad y(z_0) = y_0$$

  by simply setting $c_0 = y_0$.

Solving the inhomogeneous first order linear ODE

- We can also solve the inhomogeneous problem given by

$$A_1(z)y' + A_0(z)y = f(z)$$

  as follows.
- We introduce the concept of the "adjoint" equation which is a first order homogeneous ODE that is derived from our original ODE:

$$\frac{dx}{dz} - \frac{A_0(z)}{A_1(z)}x(z) = 0 \qquad x(z = z_0) = 1.$$

- From our discussion above, we can solve this explicitly to get

$$x(z) = \exp\left[\int_{z_0}^{z} \frac{A_0(t)}{A_1(t)} dt\right].$$

Using the adjoint solution

- This special solution

$$x(z) = \exp\left[\int_{z_0}^{z} \frac{A_0(t)}{A_1(t)}dt\right].$$

  is useful because of the following property:

$$\begin{aligned}
\frac{d}{dz}\left[x(z)y(z)\right] &= xy' + yx' \\
&= -\frac{A_0}{A_1}xy + x\frac{f}{A_1} + \frac{A_0}{A_1}xy \\
&= x(z)\frac{f(z)}{A_1(z)}.
\end{aligned}$$

- Because we know $x(z)$ we can now integrate both sides of the equation above to get

$$y(z) = \frac{c_0}{x(z)} + \frac{1}{x(z)}\int_{z_0}^{z} \frac{x(t)f(t)}{A_1(t)}dt.$$

The solution of the inhomogeneous ODE

- Inserting the explicit solution

$$x(z) = \exp\left[\int_{z_0}^{z} \frac{A_0(t)}{A_1(t)} dt\right].$$

and using the initial condition $y(z_0) = y_0$ we get

$$y(z) = y_0 \exp\left[-\int_{z_0}^{z} \frac{A_0(t)}{A_1(t)} dt\right] + \exp\left[-\int_{z_0}^{z} \frac{A_0(t)}{A_1(t)} dt\right] \int_{z_0}^{z} \exp\left[\int_{z_0}^{t} \frac{A_0(t')}{A_1(t')} dt'\right] \frac{f(t)}{A_1(t)} dt.$$

Some observations on the solution

- Note the specific form of the solution:

$$y(z) = y_0 \exp\left[-\int_{z_0}^{z} \frac{A_0(t)}{A_1(t)}dt\right] +$$
$$\exp\left[-\int_{z_0}^{z} \frac{A_0(t)}{A_1(t)}dt\right] \int_{z_0}^{z} \exp\left[\int_{z_0}^{t} \frac{A_0(t')}{A_1(t')}dt'\right] \frac{f(t)}{A_1(t)}dt.$$

- This has the form

$$y(z) = y_0 \times [\text{homogeneous solution}] + [\text{particular solution}]\,.$$

- This exemplifies the roles of the particular and homogeneous solutions.
- A similar expression is valid for systems of linear equation and also for all n'th order linear ODE's as we will show later.
- Note also that the "adjoint" solution is nothing more than the "integrating factor" often used to solve linear ODEs.
- The idea of the adjoint equation is more general as we shall see.

Examples of solving linear first order ODE's

- As an example consider the initial value problem

$$(1+z^2)y' - zy = 0 \qquad y(0) = 1.$$

- We can solve this directly via separation of variables or we can simply use the solution we derived above.
- Setting

$$\begin{aligned} A_1 &= (1+z^2) \\ A_0 &= -z, \end{aligned}$$

the solution is

$$y(z) = y_0 \exp\left[-\int_{z_0}^{z} \frac{A_0(t)}{A_1(t)} dt\right].$$

Examples of solving linear first order ODE's

- So we have
$$y(z) = \exp\left[\int_0^z \frac{t}{1+t^2}dt\right].$$
- The integral inside the exponential is
$$\log\left[1+z^2\right]/2,$$
- And so the solution is
$$y(z) = \sqrt{1+z^2}.$$

Examples of solving linear first order ODE's

- As a second example, consider solving the inhomogeneous first order ODE
$$y' - zy = z^3 \qquad y(0) = 0.$$
- Here we'll use the adjoint approach.
- The adjoint equation is
$$x' + xz = 0 \qquad x(0) = 1.$$
- We solve this to get
$$x(z) = \exp(-z^2/2).$$
- We can then use the adjoint problem to rewrite the original ODE as
$$\frac{d}{dz}\left[\exp(-z^2/2)y(z)\right] = z^3 \exp(-z^2/2)$$

Examples of solving linear first order ODE's

- Integrating both sides of

$$\frac{d}{dz}\left[\exp(-z^2/2)y(z)\right] = z^3\exp(-z^2/2)$$

  and using the initial condition $y(0) = 0$ we get

$$y(z) = \exp(z^2/2)\int_0^z t^3\exp(-t^2/2)dt.$$

- Because of the t^3 factor the integral can be performed using integration by parts to give the following solution

$$y(z) = -2 - z^2 + 2\exp(z^2/2).$$

Sometimes the answer cannot be written in closed form

- It isn't always possible to perform the integral in the solution of the ODE in terms of elementary functions.
- But writing the solution in the form of an integral is considered a solution.
- One can then apply some numerical integration procedure to get further information.
- For example, if we wanted to solve

$$y' - zy = z^2 \qquad y(0) = 0,$$

  the solution is

$$y(z) = \exp(z^2/2) \int_0^z \exp(-t^2/2) t^2 dt.$$

- The latter integral can be evaluated in terms of error functions
- And since we have so much information about error functions we consider the problem essentially solved in closed form.