We will take some practice problems from last year's exam.

We have two parallel plates separated by large D, but the bottom plate has a hemispherical bump of radius a in it. The bottom plate is held at $V = 0$ and the top plate obeys boundary condition $\vec{E} = E_0\hat{z}$. Find V everywhere.

Let's define the z axis normal to the bottom plate going through the center of the "boss" as the bump is called. Then we can define our bottom BC as

$$V(r = a, \theta) = 0 \tag{1}$$
$$V(r > a, \theta = \pi/2) = 0 \tag{2}$$

The top BC is then just $V \to -E_0 z$ as $z \to \infty$; note that the top plate doesn't actually affect our solution, just a physical construct. Then $V(z \to \infty) \to V_0 r P_1(\cos\theta)$. We then boundary match with our general solution for V in azimuthal symmetric problem

$$V(\vec{r}) = \sum_l \left(A_l r^l + \frac{B_l}{r^{l+1}}\right) P_l(\cos\theta) \tag{3}$$
$$V(z \to \infty) = \sum_l A_l r^l P_l(\cos\theta) = -E_0 r P_1(\cos\theta) \tag{4}$$
$$A_1 = -E_0 \tag{5}$$

where we drop the vanishing solution as $r \to \infty$ because we expect the potential to diverge.

Let's then look at the other BC $0 = V(r > a, \theta = \pi/2)$. We already know the behavior of the divergent terms so we can plug this in

$$0 = V(r > a, \theta = \pi/2) \tag{6}$$
$$= -E_0 r \cos\frac{\pi}{2} + \sum_l \frac{B_l}{r^{l+1}} P_l(0) \tag{7}$$

Then since $P_l(0)$ vanishes for only odd l, we find that $B_l = 0$ for even l (only way to make the sum over even l vanish). Then we impose final BC

$$0 = V(r = a, \theta) \tag{8}$$
$$= -E_0 a \cos\theta + \sum_l{}' \frac{B_l}{a^{l+1}} P_l(\cos\theta) \tag{9}$$

where we denote the sum over only odd l since the even B_l vanish. Here we are in a slight rut; The P_l aren't orthogonal over $[0, \pi/2]$! WE are saved by a hint he didn't mention earlier that was given: the P_l for odd l are orthogonal over $[0, \pi/2]$ by

$$\frac{2\delta_{ll'}}{2l+1} = 2\int_0^1 d\cos\theta\ P_l(\cos\theta) P_{l'} \cos\theta \tag{10}$$

Then we find that only the $l = 1$ term survives the integral obviously and so we find $0 = -E_0 a + \frac{B_1}{a^2}$ giving $B_1 = E_0 a^3$ and final solution

$$V(\vec{r}) = \left[-E_0 r + \frac{E_0 a^3}{r^2}\right]\cos\theta = E_0 r\left[\frac{a^3}{r^3} - 1\right]\cos\theta \tag{11}$$

Key point here is that we could only integrate over half our usual domain, which makes us have to apply orthonormality a bit more carefully.

Next part was to compute surface charge density on both the boss and the rest of the plate. Then

$$\sigma(r=a,\theta) = -\epsilon_0 \hat{n}\cdot\vec{\nabla}V \tag{12}$$
$$= -\epsilon_0 \frac{dV}{dr}\bigg|_{r=a} \tag{13}$$
$$= -\epsilon_0 E_0 \left\{\left[-1+\frac{a^3}{r^2}\right] + r\left[-\frac{3a^3}{r^4}\right]\right\}\cos\theta\bigg|_{r=a} \tag{14}$$
$$= 3\epsilon_0 E_0 \cos\theta \tag{15}$$
$$\sigma(r>a,\theta=\pi/2) = -\epsilon_0 \hat{z}\cdot\vec{\nabla}V \tag{16}$$
$$= (-\epsilon_0)(-E_0)\frac{d}{dz}\left[z\left(1-\frac{a^3}{r^3}\right)\right]_{\theta=\pi/2} \tag{17}$$
$$= \epsilon_0 E_0 \left\{\left(1-\frac{a^3}{r^3}\right) + z\left(-\frac{3a^3}{r^4}\frac{dr}{dz}\right)\right\}_{\theta=\pi/2} \tag{18}$$
$$= \epsilon_0 E\left(1-\frac{a^3}{r^3}\right) \tag{19}$$

where we recognize in (17) that $z=r\cos\theta$ and in (18) that we will evaluate the zecond term at $z=0$ which vanishes anyways so we don't need to compute the derivative to know that it will vanish.

Last part is to compute the total charge on the boss. Then

$$Q_b = \int_0^{2\pi}\int_0^{\pi/2} \sigma\ da \tag{20}$$
$$= 2\pi\int_0^1 dx\ 3\epsilon_0 E_0 x \tag{21}$$
$$= 6\pi a^2\epsilon_0 E_0 \frac{x^2}{2}\bigg|_0^1 = 3\pi\epsilon_0 E_0 a^2 \tag{22}$$

We then were supposed to plot the charge density; not going to copy that, but key point is that the charge density vanishes at the sharp junction. This was the hardest problem last year, just to give an idea; it takes about 30 minutes to do if you know what you're doing.

There were a total of 3 problems last year (so I guess this year too), so we'll go over one more. Let's put a dipole inside a grounded conducting sphere of radius R but slightly offset from the center by distance a. Let dipole and offset both point along $\hat{z}$. Hint: a single image dipole won't be enough.

The natural way to go about this is to start with two charges with nonzero separation $2l$ and constant dipole $p=2ql$. However, the image charges are not separated by $2l$ nor are the image charges even the same charge! The actual numbers are

$$q_1 = -\frac{qR}{a+1},\ \text{at}\ \left(0,0,\frac{R^2}{a+l}\right) \tag{23}$$
$$q_2 = \frac{qR}{a-1},\ \text{at}\ \left(0,0,\frac{R^2}{a-l}\right) \tag{24}$$

Let's Taylor expand our two charges and obtain

$$q_1 = -\frac{qR}{a}\left(\frac{1}{1+\frac{l}{a}}\right) = -\frac{qR}{a}\left(1-\frac{l}{a}\right) \tag{25}$$
$$= -\frac{qR}{a} + \frac{1}{2}\frac{pR}{a^2} \tag{26}$$
$$q_2 = \frac{qR}{a} + \frac{1}{2}\frac{pR}{a^2} \tag{27}$$

where we recall $ql = \frac{p}{2}$. Thus it is clear that our image dipole will have a net charge. In any case though, our image dipole will then be

$$p' = \frac{2qR}{a}\frac{R^2 l}{2a} = \frac{pR^3}{a^3} \tag{28}$$

$$q' = \frac{1}{2}\frac{pR}{a^2}\cdot 2 = \frac{pR}{a^2} \tag{29}$$

with q' our extra image charge. We would need to take $l \to 0$ limit here but everything has already been reexpressed in terms of non-limit quantities. Note then that our image charge and image dipole sit at the same place.

We then want to find the total potential. Just to check signs, we can thnk about this a bit and realize that the image dipole points *in the same direction* as the original dipole (farther charge maps to closer image charge but sign flip). Then we can just put togteher the cumulative potential

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0}\left[\frac{p\hat{z}\cdot(\vec{r}-a\hat{z})}{|\vec{r}-a\hat{z}|^3} + \frac{p'\hat{z}\left(\vec{r}-\frac{R^2}{a}\hat{z}\right)}{\left|\vec{r}-\frac{R^2}{a}\hat{z}\right|^3} + \frac{\frac{pR}{a^2}}{\left|\vec{r}-\frac{R^2}{a}\hat{z}\right|}\right] \tag{30}$$

Intuitively, we aren't too surprised that the extra image charge pops up; we are much more concerned with the integral over the surface having equal charge as the image charge, and it's not surprising that the grounded sphere acquires an induced charge.

Note that we can also solve the last problem either using BC matching or Greens Functions as well. We will do it again via Greens Functions by popular request. Let's first get our Greens Function (a Dirichlet GF)

$$G_D(\vec{r},\vec{r}') = \frac{1}{\epsilon_0}\sum_{l=0}^{\infty}\sum_{m=-l}^{l} r_<^l\left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right)\frac{Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)}{2l+1} \tag{31}$$

$$V(\vec{r}) = \int_{\mathcal{V}} d\tau'\ \rho(\vec{r}')G_D(\vec{r},\vec{r}') - \underbrace{\epsilon_0\int_{\mathcal{S}} da'\hat{n}(\vec{r}')\cdot\vec{\nabla}_{r'}G_D(\vec{r},\vec{r}')V(\vec{r}')}_{=0} \tag{32}$$

since $V(\mathcal{S}) = 0$ our BC. Let's then write our charge distribution for our dipole

$$\rho(\vec{r}) = \lim_{2ql\to p} q\left[\delta(\vec{r}-(a+l)\hat{z}) - \delta(\vec{r}-(a-l)\hat{z})\right] \tag{33}$$

(omitted $l \to 0$ limit, implied from here on) We will need to turn this into spherical coordinates to use our spherical harmonics GF, so

$$\rho(\vec{r}) = \lim_{2ql\to p}\frac{q}{2\pi}\left[\frac{\delta(r-(a+l)}{(a+l)^2} - \frac{\delta(r-(a-l)}{(a-l)^2}\right]\delta(\cos\theta - 1) \tag{34}$$

We can then plug into (32) slowly and painfully

$$V(\vec{r}) = \int d\tau'\rho(\vec{r}')G_D(\vec{r},\vec{r}') \tag{35}$$

$$= \frac{1}{2\pi\epsilon_0}\lim_{2ql\to p}\sum_{lm}\int_0^{2\pi} d\phi'\int_{-1}^{1} d\cos\theta'\int_0^b (r')^2\,dr'\left[\frac{\delta(r'-(a+l)}{(a+l)^2} - \frac{\delta(r'-(a-l)}{(a-l)^2}\right]\delta(\cos\theta'-1)\frac{Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)}{2l+1} \tag{36}$$

We can then immediately do the $d\theta'\cos\theta'$ and obtain $\theta' = 0$. Moreover, we can integrate over ϕ' and this forces $m, m' = 0$ and also takes care of the 2π as part of the normalization. Note $Y_{l0}^* = Y_{l0}$ since the only complex dependence comes from m. If we then only examine the angular part we can consolidate these changse (integrals over angular parts) and obtain

$$\delta(\cos\theta'-1)\frac{Y_{lm}^*(\theta',\phi')Y_{lm}(\theta,\phi)}{2l+1} = \frac{Y_{l0}^*(0,\phi')Y_{l0}(\theta,\phi}{2l+1} \tag{37}$$

$$= \frac{P_l(\cos\theta)}{4\pi} \tag{38}$$

We can then plug this back into (36) and we obtain

$$V(\vec{r}) = \lim_{2ql\to p} \frac{q}{4\pi\epsilon_0} \sum_l P_l(\cos\theta) \int_0^b (r')^2\, dr' r_<^l \left(\frac{1}{r_>^{l+1}} - \frac{r_>^l}{b^{2l+1}}\right) \left[\frac{\delta(r'-(a+l)}{(a+l)^2} - \frac{\delta(r'-(a-l)}{(a-l)^2}\right] \tag{39}$$

Now is when we need to distinguish between when we are closer or farther than the dipole radius. Let's first examine $V(r<a,0)$ where $r_< = r, r_> = r'$ which then gives

$$V(r<a,\theta) = \lim_{2ql\to p} \frac{q}{4\pi\epsilon_0} \sum_l P_l(\cos\theta) r^l \left[\frac{1}{(a+l)^{l+1}} - \frac{(a+l)^l}{b^{2l+1}} - \frac{1}{(a-l)^{l+1}} + \frac{(a-l)^l}{b^{2l+1}}\right] \tag{40}$$

where we've already computed all of the radial integrals.

If we want to see the equivalence to the image charge expansion, we should plug in our formula for $\frac{1}{|\vec{r}-\vec{r}'|}$ into our above expression and we will get

$$\frac{1}{|\vec{r}-\vec{r}'|} \propto \sum_l \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma) \tag{41}$$

$$r_> = \frac{b^2}{a+l} \tag{42}$$

$$\frac{1}{r_>^{l+1}} = \frac{(a+l)^l}{b^{2l+1}} \frac{a+l}{b} \tag{43}$$

$$V(r<a,0) = \lim_{2ql\to p} \frac{q}{4\pi\epsilon_0} \left[\frac{1}{|\vec{r}-(a+l)\hat{z}|} - \frac{1}{|\vec{r}-(a-l)\hat{z}|} - \frac{\frac{a+l}{b}}{\left|\vec{r}-\left(\frac{b^2}{a+l}\right)\hat{z}\right|} + \frac{\frac{a-l}{b}}{\left|\vec{r}-\left(\frac{b^2}{a-l}\right)\hat{z}\right|}\right] \tag{44}$$

We won't actually use this of course; going through the troubles of expanding in spherical harmonics means we don't really want to go back to this Cartesian representation.

Instead, what makes much more sense is to take (39) and Taylor expand the following terms (we define$\bar{l}$ since l refers to the separation; hopefully things have been clear up until here!)

$$\frac{1}{(a+l)^{\bar{l}+1}} - \frac{1}{(a-l)^{\bar{l}+1}} = \frac{1}{a^{\bar{l}+1}}\left[\left(1-(l+1)\frac{l}{a}\right) - \left(1+(l+1)\frac{l}{a}\right)\right] \tag{45}$$

$$= -2(l+1)\frac{l}{a^{l+2}} \tag{46}$$

$$\frac{(a+l)^{\bar{l}} - (a-l)^{\bar{l}}}{b^{2\bar{l}+1}} = \frac{a^{\bar{l}}}{b^{2\bar{l}+1}}\left[\left(1+\bar{l}\frac{l}{a}\right) - \left(1-\bar{l}\frac{l}{a}\right)\right] \tag{47}$$

$$= \frac{2a^{\bar{l}}}{b^{2\bar{l}+1}}\bar{l}\frac{l}{a} \tag{48}$$

which takes us to

$$V(r<a) = -\frac{p}{4\pi\epsilon_0} \sum_{\bar{l}} P_{\bar{l}}(\cos\theta) r^{\bar{l}} \left[\frac{\bar{l}+1}{a^{\bar{l}+2}} - \frac{\bar{l}a^{\bar{l}-1}}{b^{2\bar{l}+1}}\right] \tag{49}$$

We won't compute it out for the $r>a$ term but it will be a bit easier since $r_< = a$. Conceptually, we just had to integrate over the charge density, be careful with our $r_>, r_<$ and solve out; this example was much more difficult because we had to be careful with our Taylor Expand.

Let's now do a dielectric problem (lol, unanimous vote for dielectrics over separation of variables for previous problem). This is Griffiths 4.39; consider a conductor of potential V_0 with radius R halfway immersed in a dielectric ϵ. Let's show that $V(\vec{r}) = V_{vac}(\vec{r})$.

The key thing to recognize is that the tangential component of $\vec{E}$ is the same across an interface, so given the radially-outwards $\vec{E}$ at the interface we can see that the $\vec{E}$ shouldn't change across the interface; think back to the first day of lecture, constructing a rectangular path across the interface with vanishing height, which shows equal $\vec{E}$ across an interface.

Let's see this through a capacitor first. Consider two parallel plates separated by a both with and without (two different configurations, not simultaneously) a dielectric ϵ. We know that in vacuum the electric field inside the capacitor is given $\vec{E}_0 = -\frac{Q}{aC}\hat{z}$. We note then that $\vec{D}$ is the same across both configurations because it only cares about the free charge, so *if* there were the same charge in both configurations we would find

$$\vec{D} = \epsilon_0 \vec{E}_0 \qquad\qquad \vec{D} = \epsilon \vec{E} \tag{50}$$

which results in a voltage drop across the capacitor. But then in our current problem with the sphere (and with the capacitor) the voltage is what's usually being held constant. So instead $\vec{E}$ is the one that should be the same!

If we then look back to our sphere problem, we note that we must hold the voltage constant which keeps $\vec{E}$ the same. The top half then must have the same charge density as in the vacuum case $\sigma_f = \sigma_0, \sigma_b = 0$. The bottom half instead must still have $\sigma_t = \sigma_0$ with $\sigma_t = \sigma_f + \sigma_b$, since it produces the same voltage, and then σ_f must increase and σ_b is negative.

Note in general that holding a system of conductors at constant voltage, we can introduce as much dielectric as we want without changing the $\vec{E}$ field or the potential.

Then we note that $\vec{E} = \vec{E}_0$ and $\vec{D} = \epsilon(\vec{r})\vec{E}$ which then yields the following flurry of formulae (the cases are with respect to the regions in which permittivity is ϵ or ϵ_0, or "epsilon land" - Prof. Golwala).

$$\vec{D} = \begin{cases} \epsilon_0 \vec{E}_0 & \epsilon_0 \\ \epsilon \vec{E}_0 & \epsilon \end{cases} \tag{51}$$

$$\vec{P} = (\epsilon(\vec{r}) - \epsilon_0)\vec{E} = \begin{cases} 0 & \epsilon_0 \\ (\epsilon - \epsilon_0)\vec{E}_0 & \epsilon \end{cases} \tag{52}$$

$$\sigma_f = \hat{n} \cdot \vec{D} = \begin{cases} \epsilon_0 E_0 = \sigma_0 & \epsilon_0 \\ \epsilon E_0 = \frac{\epsilon}{\epsilon_0}\sigma_0 & \epsilon \end{cases} \tag{53}$$

$$\sigma_b = -\hat{n} \cdot \vec{P} = \begin{cases} 0 & \epsilon_0 \\ -\left(\epsilon - \epsilon_0\right) E_0 = -\frac{\epsilon - \epsilon_0}{\epsilon_0}\sigma_0 & \epsilon \end{cases} \tag{54}$$

Note the careful distinction between holding the conductor at constant potential and holding it at constant charge; the setup of this problem is akin to keeping the sphere connected to some voltage source, rather than just naively dunking the sphere into a dielectric after disconnecting it.

Let's have some qualitative problems to think about before the final

- Consider the same setup as above but the sphere is submerged beyond the midpoint by some distance.
- Consider a sphere such that some amount of the solid angle is surrounded by the dielectric.

Note that the first example *will not* have nonchanging $\vec{E}$ by introduction of the dielectric! This is because the interface isn't parallel to the $\vec{E}$. Then the effective ϵ_{eff} is just the weighted sum over the solid angle.