ACM 100b

The Laplace transform as an application of Fourier transforms

Dan Meiron

Caltech

February 23, 2014

The Laplace transform

- The Laplace transform was introduced earlier in your courses as a way of solving initial value problems for linear ODE's.
- The transform is given by

$$\mathcal{L}f(t) \equiv F(s) = \int_0^\infty \exp(-st)f(t)dt,$$

and the inverse transform is given by a contour integral:

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\exp(st)ds.$$

- This result is often just quoted without giving any idea of where it comes from.
- We will derive it here from the Fourier transform.

The Laplace transform

- Suppose $f(x)$ is a function defined for $x > 0$
- And $f(x)$ is set to 0 for $x < 0$.
- Suppose also that $f(x)$ is of *exponential order*
- This means there exists some constant $c \geq 0$ such that $f(x)\exp(-cx) \to 0$ as $x \to \infty$.
- Next define

$$G_c(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\exp(-cx)\exp(-ikx)dx.$$

- All we're doing here is taking the Fourier transform of $\exp(-cx)f(x)$.
- Note that this means we can define $G_c(k)$ even for functions $f(x)$ that grow exponentially as long as we make c large enough.

The Laplace transform

- Now look at the integral defining $G_c(k)$

$$G_c(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-cx) \exp(-ikx) dx.$$

- We can write this as

$$G_c(k) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} f(x) \exp(-(c+ik)x) dx.$$

- The integral that defines $G_c(k)$ defines an analytic function of k in a region at least as large as $\mathrm{Im}(k) < 0$.
- This is simply because as long as $\mathrm{Im}(k) < 0$ we get faster exponential decay and so the integral converges.
- A theorem in complex analysis tells us the region of analyticity of the function defined by such an integral is at least as large as the region of k in the complex plane where the integral converges uniformly.
- And that region is clearly $\mathrm{Im}(k) < 0$

The Laplace transform

- Next define a complex variable $s = c + ik$.
- And define

$$F(s) = \sqrt{2\pi} G_c\left(-i(s-c)\right)$$
$$= \int_0^\infty f(x)\exp(-sx)dx$$

- This is the integral for the Laplace transform.
- Now recall that the function $G_c(k)$ is analytic for $\mathrm{Im}(k) < 0$
- So $F(s)$ must define an analytic function where $\mathrm{Re}(s-c) > 0$ or where $\mathrm{Re}(s) > c$

The Laplace transform

- Now since we got this from the Fourier transform we know how to recover the original function $f(x)\exp(-cx)$
- We do this by using the inverse Fourier transform:

$$\exp(-cx)f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G_c(k)\exp(ikx)dk.$$

- Now again let $s = c + ik$.
- And we also have $ds = idk$ or $dk = ds/i$
- Substituting this into the expression above we get

$$\exp(-cx)f(x) = \frac{1}{\sqrt{2\pi}i}\int_{c-i\infty}^{c+i\infty} G_c\left(\frac{s-c}{i}\right)\exp[(s-c)x]ds$$

- But now recall we defined

$$F(s) = \sqrt{2\pi}G_c\left(-i(s-c)\right)$$

The Laplace transform

- And the expression

$$\exp(-cx)f(x) = \frac{1}{\sqrt{2\pi}i}\int_{c-i\infty}^{c+i\infty} G_c\left(\frac{s-c}{i}\right)\exp[(s-c)x]ds$$

becomes

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\exp(sx)ds$$

- We recognize this as the *Bromwich contour* integral representation for the inverse Laplace transform
- This then gives us the Laplace transform pair

$$F(s) = \int_0^\infty f(x)\exp(-sx)dx$$

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\exp(sx)ds$$

The Laplace transform

- Recall that for the pair

$$F(s) = \int_0^\infty f(x)\exp(-sx)dx$$
$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\exp(sx)ds$$

  we took c so that $f(x)\exp(-cx)$ decays.
- In the Bromwich contour you are supposed to take c so that lies to the left of all singularities of $F(s)$ in the complex s plane.
- This approach using the Fourier transform explains why we choose c so that the contour lies to the right of all singularities of $F(s)$.
- It is this value of c which allowed us to create a suitably analytic function from $f(x)$ so we could transform it in the first place.

Convergence factors

- The above approach is also an example of how we can compute Fourier transforms for functions that don't decay fast enough as $|x| \to \infty$.
- We typically multiply such functions by a *convergence factor* like the factor of $\exp(-cx)$ we used above.
- We then work with these better behaving functions.
- After the result is obtained, we take the limit $c \to 0$ and make sure that limit provided reasonable results.