Lecture 3: Fluid Statics

- Motivating problem
- Fluids and solids
- Forces acting on a fluid
- Static force balance for a fluid

Give me a firm place to stand and I will move the earth.
Archimedes *287-212 BCE*

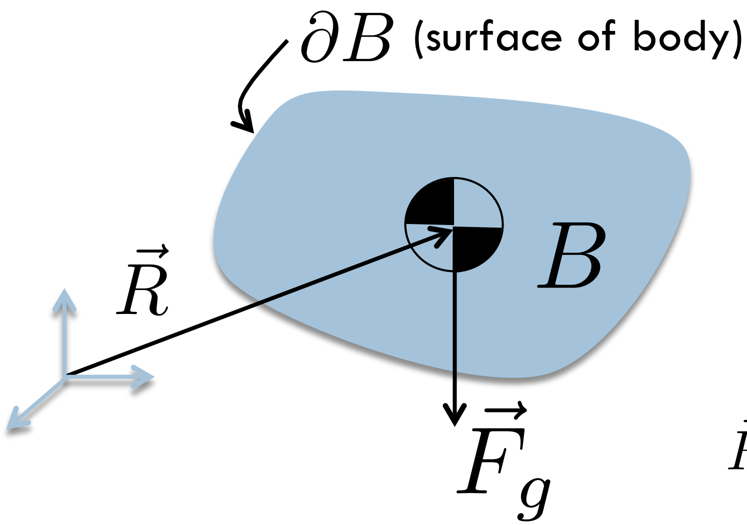
Motivating problem

- Archimedes: Consider a solid body submerged in a (still) fluid…in this case, air and water. What force balances the weight of the floating object?

- This problem is just the tip of the iceberg…

Motivating problem (cont'd)

- Free-body diagram (iceberg)



$$\vec{F}_g = \int_B \rho_b(x,y,z)\vec{g}dV$$
$$= M\vec{g}$$

$$\vec{R} = \frac{1}{M}\int_B \rho_b(x,y,z)\ \vec{x}dV$$

- The fluid (water below, air above) must somehow support this load with an equal-but-opposite force (called buoyancy force)

Body forces and surface forces

- Body force
  - Expressible as force/unit volume
  - Acts 'internally' at every point in a material
  - Forces that "act at a distance"
    - Gravity
    - Electromagnetic
    - Apparent forces like centrifugal force (really an acceleration)
- Surface (contact) force
  - Expressible as force/unit area
  - Acts on any surface (external surfaces separating two materials, or imaginary internal surfaces we draw through the material)
  - Pressure, viscous stresses

Statics

- Fluid statics are in many ways much easier than solid statics for a simple reason:
  - If the fluid is at rest, it can **only** support normal forces, and, more specifically, pressure forces
  - It suffices to determine the balance, at each point in the fluid, between gravity force and this pressure force
  - Once the pressure is known, we can determine the force on any immersed surface (real or imaginary) we wish to draw
  - Later we will generalize to fluid moving as a rigid body
- Solids: support both normal and shearing forces at rest.
  - Need to understand how the normal and shear forces are related to the deformed body

Pressure

- Scalar quantity
- Normal force (per unit area) due to molecular transport of momentum normal to a surface.
  - Pressure (positive normal force)
  - Tension (negative normal force)
- Pressure in a fluid is almost always a positive quantity.
  - Exceptions arise from intermolecular (van der Waals forces, hydrogen bonds) forces and are not important right now.
  - Example: very pure water can withstand O(1000) atmospheres of tension before 'breaking' (cavitating)
- Don't confuse with "gauge pressure"
  - Pressure compared with atmospheric pressure
    - Positive gauge pressure → "pressure"
    - Negative gauge pressure == "vacuum"

Pressure force on a surface

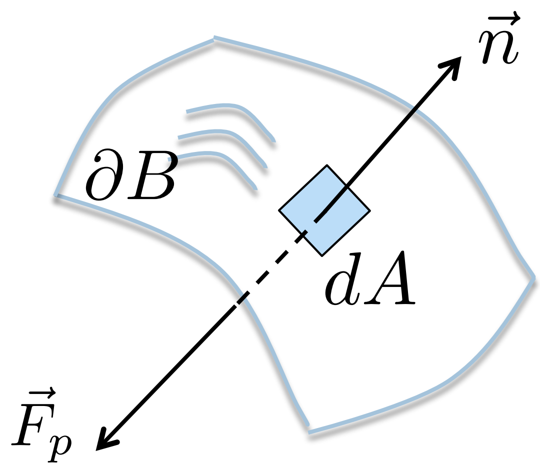
- Suppose that the pressure *field* is everywhere known within a fluid

$$p = p(x, y, z)$$

- The force acting over any surface (open/closed) is

$$p = p(x, y, z)$$

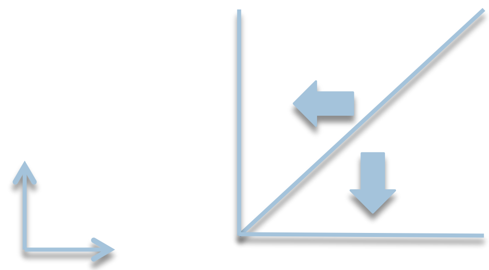
$$\vec{F}_p = -\int_{\partial B} p\vec{n}\,dA$$



This is the force acting on
The surface from the pressure
"above" the surface

Evaluating the integral: tricks

$$\vec{F}_p = -\int_{\partial B} p\vec{n}dA$$

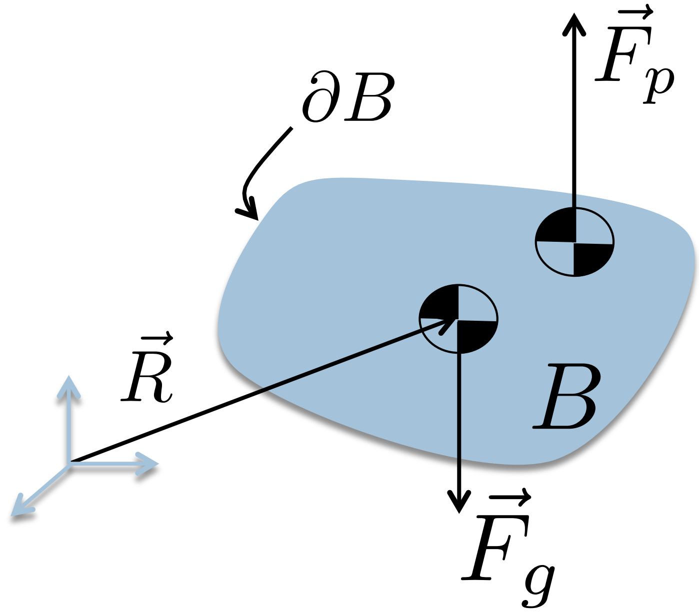


- Each component of $\vec{n}dA$ is the area projected in the i,j,k directions (be careful, though, when p is not constant)
- A constant pressure acting over an enclosed volume gives zero force
  - Projections must cancel in +/- directions
  - We can therefore always work with a relative pressure (e.g. measure pressure relative to a constant offset)

$$-\int_{\partial B} p\vec{n}dA = -\int_{\partial B} (p + \text{const})\vec{n}dA$$

Closed surface only

Force balance for iceberg (or any closed region)

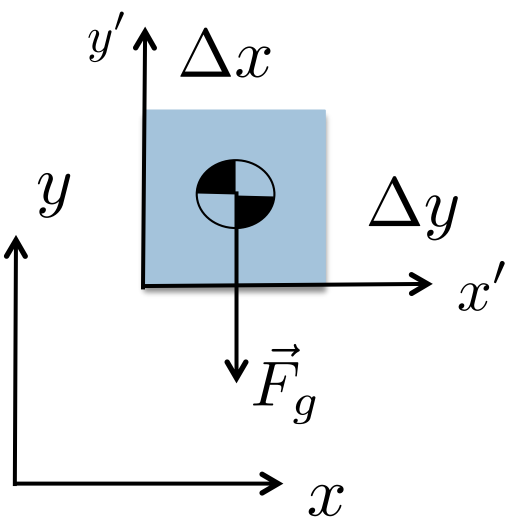


Note that unless otherwise supported, this iceberg as drawn is not in equilibrium, because the fluid cannot exert any more forces (we already accounted for the pressure force) …

$$\sum \vec{F} = 0 = \vec{F}_g + \vec{F}_p$$

How do we get the pressure internal to the fluid?

- Let's turn our iceberg argument in upon itself…
- Suppose our body is comprised of a small element of still fluid (why not, surely Newton's laws must still apply?)
- Apply force balance to small element



$$\vec{F}_g(x,y) = \int_x^{x+\Delta x} \int_y^{y+\Delta y} \rho\left(x', y'\right) \vec{g} dx' dy'$$

Expand argument in Taylor series about the point x,y

$$\rho(x',y') = \rho(x,y) + (x'-x) \left.\frac{\partial \rho}{\partial x}\right|_{x,y} + (y'-y) \left.\frac{\partial \rho}{\partial y}\right|_{x,y} + \dots$$

Continuing algebra…

$$\rho(x',y') = \rho(x,y) + (x'-x)\left.\frac{\partial \rho}{\partial x}\right|_{x,y} + (y'-y)\left.\frac{\partial \rho}{\partial y}\right|_{x,y} + \ldots$$

$$\begin{aligned}
\vec{F}_g = \vec{g}\int_x^{x+\Delta x}\int_y^{y+\Delta y}\rho(x',y')dx'dy' &= \rho(x,y)\Delta x\Delta y \\
&+ \frac{1}{2}(x'-x)^2\Big|_{x'=x}^{x'=x+\Delta x}\Delta y\left.\frac{\partial\rho}{\partial x}\right|_{x,y} \\
&+ \frac{1}{2}(y'-y)^2\Big|_{y'=y}^{y'=y+\Delta y}\Delta x\left.\frac{\partial\rho}{\partial y}\right|_{x,y} + \ldots \\
&= \rho(x,y)\Delta x\Delta y \\
&+ \frac{1}{2}\Delta x^2\Delta y\frac{\partial\rho}{\partial x} \\
&+ \frac{1}{2}\Delta y^2\Delta x\frac{\partial\rho}{\partial y} + \ldots
\end{aligned}$$

Write

$$\frac{\vec{F}_g}{V} = \frac{\vec{F}_g}{\Delta x \Delta y} = \vec{g}\left[\rho(x,y) + O(\Delta x) + O(\Delta y)\right]$$

If we let the element shrink to a point, we have, the very obvious:

$$\frac{\vec{F}_g}{V} = \lim_{\Delta x, \Delta y \to 0} \frac{\vec{F}_g}{\Delta x \Delta y} = \vec{g}\ \rho(x,y)$$

Generalization

- In 3D, the above immediately generalizes to

$$\frac{\vec{F}_g}{V} = \lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\vec{F}_g}{\Delta x \Delta y \Delta z} = \vec{g}\, \rho(x, y)$$

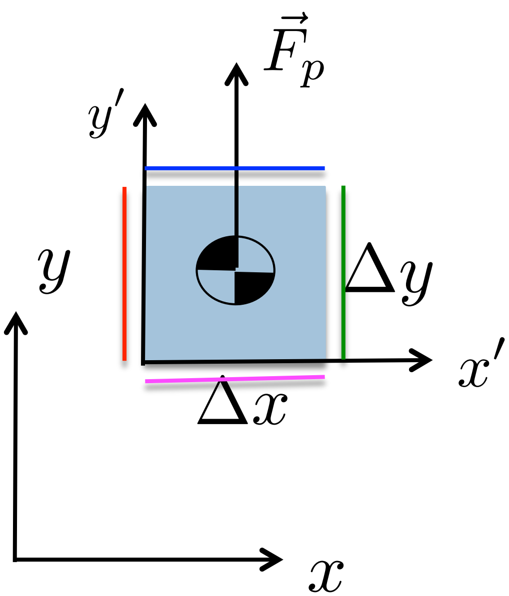
What about the force due to pressure?

$$\vec{F}_p = -\int_{\partial B} p\vec{n}dA = -\int_y^{y+\Delta y} p(x,y')(-\hat{i})dy' \quad \text{LEFT}$$

$$-\int_y^{y+\Delta y} p(x+\Delta x,y')(+\hat{i})dy' \quad \text{RIGHT}$$

$$-\int_x^{x+\Delta x} p(x',y)(-\hat{j})dx' \quad \text{BOTTOM}$$

$$-\int_x^{x+\Delta x} p(x',y+\Delta y)(+\hat{j})dx' \quad \text{TOP}$$



Continuing algebra…

T.S. of integrand

$$= -\int_y^{y+\Delta y} p(x,y')(-\hat{i})dy' \longrightarrow +\hat{i}\left[p(x,y)\Delta y + O(\Delta y)^2\right]$$

$$-\int_y^{y+\Delta y} p(x+\Delta x,y')(+\hat{i})dy' \longrightarrow -\hat{i}\left[p(x+\Delta x,y)\Delta y + O(\Delta y)^2\right]$$

$$-\int_x^{x+\Delta x} p(x',y)(-\hat{j})dx' \longrightarrow +\hat{j}\left[p(x,y)\Delta x + O(\Delta x)^2\right]$$

$$-\int_x^{x+\Delta x} p(x',y+\Delta y)(+\hat{j})dx' \longrightarrow -\hat{j}\left[p(x,y+\Delta y)\Delta x + O(\Delta x)^2\right]$$

$$\frac{\vec{F}_p}{\Delta x \Delta y} = -\,\hat{i}\left[\frac{p(x+\Delta x, y) - p(x,y)}{\Delta x}\right]$$
$$-\,\hat{j}\left[\frac{p(x, y+\Delta y) - p(x,y)}{\Delta y}\right] + O(\Delta x) + O(\Delta y)$$

$$\frac{\vec{F}_p}{V} = \lim_{\Delta x, \Delta y \to 0} \frac{\vec{F}_p}{\Delta x \Delta y} = -\frac{\partial p}{\partial x}\,\hat{i} - \frac{\partial p}{\partial y}\,\hat{j}$$

Vector Calculus is useful!

- Gradient of a scalar field (back to 3D)

$$\nabla p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$$

- Gradient of a scalar is a vector that points in the direction of increasing scalar, and whose magnitude is the derivative of the scalar in that direction
- In 3D, result on last page therefore generalizes to:

$$\boxed{\frac{\vec{F}_p}{V} = -\nabla p}$$

Force balance for fluid parcel is now…

$$\frac{\sum \vec{F}}{V} = \frac{\vec{F}_g}{V} + \frac{\vec{F}_p}{V} = \boxed{0 = \vec{g}\,\rho - \nabla p}$$

- If we can integrate this equation for p, we will be able to determine the pressure at each point in the fluid, given the density

Don't let PDEs intimidate you!

- Both formulations express exactly the same physics

$$-\nabla p + \rho \vec{g} = 0$$

Infinitesimal parcel of fluid

Surface force due to pressure

Body force due to gravity

$$-\int_{\partial B} p\vec{n}\,dA + \int_B \rho\vec{g}\,dV = 0$$

Finite parcel

Let's work backwards

- Start with differential equation

$$-\nabla p + \rho\vec{g} = 0$$

Must be true for all small parcels comprising the body

$$-\int_B \nabla p dV + \int_B \rho\vec{g} dV = 0$$

Integrate over the body

Already very close…look at the first term more carefully

Recall divergence theorem

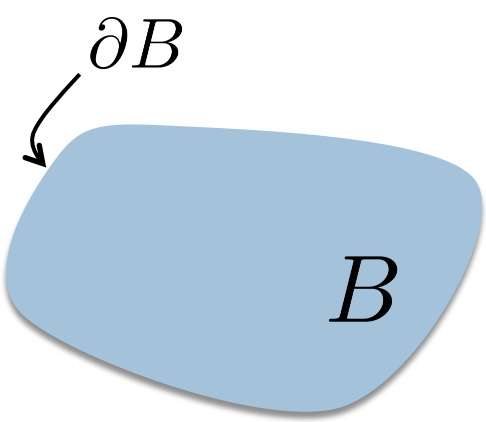
- For any sufficiently smooth vector field, g:

$$\int_B \nabla \cdot \vec{g} dV = \int_{\partial B} \vec{g} \cdot \vec{n} dA$$

In Cartesian components:

$$\nabla \cdot \vec{g} \equiv \frac{\partial g_1}{\partial x_1} + \frac{\partial g_2}{\partial x_2} + \frac{\partial g_3}{\partial x_3}$$

- Nothing more than the fundamental theorem of calculus applied to a multidimensional vector field



  - 1D version

$$\int_{x_1}^{x_1} \frac{dg}{dx} dx = +g(x_2) - g(x_1)$$

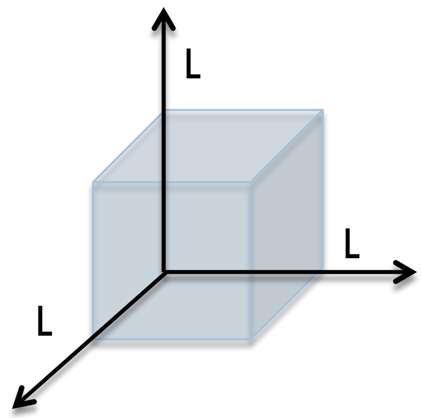
The plus/ minus here are the 1D equivalent of a unit normal

Verify the divergence theorem for a particular vector field

- Take the vector field

$$\vec{g} = \vec{x}/3$$

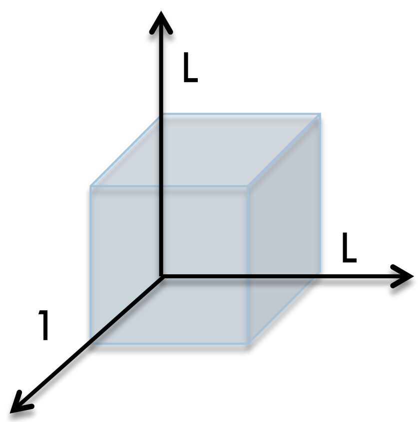
- Take the volume as cube with side L



- Evaluate the LHS

$$\nabla \cdot \vec{x} = \frac{1}{3}\left(\frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z}\right) = 1$$

$$\int \nabla \cdot \vec{x} dV = \int dV = V = L^3$$



- Evaluate the RHS $\frac{1}{3}\int_{\partial B} \vec{x} \cdot \vec{n} dA$
  - 6 faces, evaluate integrand on each face
  - Right $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = x = L$
  - Left $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix} = -x = 0$
  - Back $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} = -z = 0$
  - Etc.

- RHS, cont'd
- Each integrand is a constant on the faces.
- The left, bottom, back are all 0 and do not contribute
- On the other 3 faces, the integrands are the constant L, and we integrate over the face to get L^3.
- So the total of all 6 faces is 3 L^3. With factor of 1/3, Integrand is L^3

- Note that this formula allows us to find the volume of a arbitrary region in terms of an integral over the surface of the region

Divergence theorem: another interpretation

$$\int \nabla \cdot \vec{g} dV = \int \vec{g} \cdot \vec{n} dA$$

Derivation on board

Simple corollary of div. thm.

$$\int_B \nabla \cdot \vec{g} dV = \int_{\partial B} \vec{g} \cdot \vec{n} dA$$

Let $\vec{g} = f(x,y,z)\hat{i} \qquad \rightarrow \qquad \int \frac{\partial f}{\partial x} dV = \int f\hat{i} \cdot \vec{n} dA.$

Let $\vec{g} = f(x,y,z)\hat{j} \qquad \rightarrow \qquad \int \frac{\partial f}{\partial y} dV = \int f\hat{j} \cdot \vec{n} dA.$

Let $\vec{g} = f(x,y,z)\hat{k} \qquad \rightarrow \qquad \int \frac{\partial f}{\partial y} dV = \int f\hat{k} \cdot \vec{n} dA.$

$$\int_B \nabla f dV = \int_{\partial B} f \vec{n} dA$$

…back to our force balance

$$-\nabla p + \rho \vec{g} = 0$$

Must be true for all small (infinitesimal) parcels comprising the body

$$-\int_B \nabla p dV + \int_B \rho \vec{g} dV = 0$$

Integrate over the body

$$-\int_{\partial B} p \vec{n} dA + \int_B \rho \vec{g} dV = 0$$

Apply corollary

We arrive back at our force balance for the finite-sized parcel !

Simpler derivation of infinitesimal balance

$$-\int_{\partial B} p\vec{n}dA + \int_B \rho\vec{g}dV = 0$$

Finite force balance

$$-\int_B \nabla p dV + \int_B \rho\vec{g}dV = 0$$

Apply corollary

$$\int_B \left(-\nabla p + \rho\vec{g}\right) dV = 0$$

Re-write

$$-\nabla p + \rho\vec{g} = 0$$

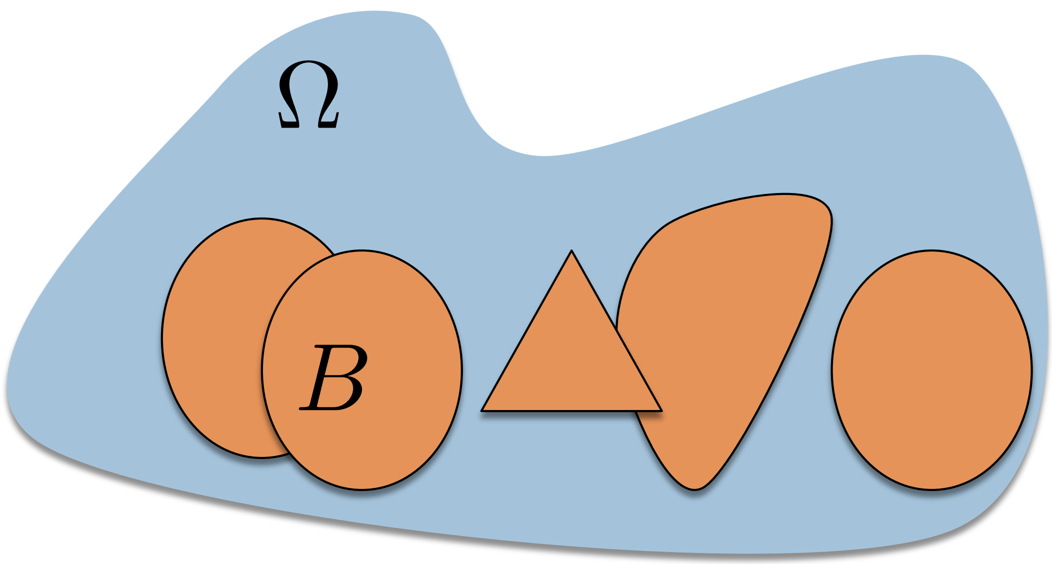
Since last integral must also be true for any and all possible subdivisions of original body

Localization thm.

- Last step in previous derivation is obvious, but can be stated as a formal theorem

$$\int_B g(x,y,z)dV = 0 \ \forall \ B \in \Omega \qquad \Longrightarrow \ g(x,y,z) = 0 \ \forall \ x,y,z \in \Omega$$

g is any
scalar,
vector field



Closing

If

$$-\int_{\partial B} p\vec{n}dA + \int_B \rho\vec{g}dV = 0$$

For all possible B within some contiguous region then

$$-\nabla p + \rho\vec{g} = 0$$

must hold at each point in that region

- The consequences of this simple equation are extensive and will be discussed in the next two lectures