Physics 106a — Classical Mechanics

Michael Cross

California Institute of Technology

Fall Term, 2013

Lecture 11

Canonical Transformations

Introduction

Rather than writing down equations of motion and trying to solve them, find transformations to simplify Hamiltonian to give easy equations of motion.

A transformation to new position and momentum variables ($\{Q_k\}$, $\{P_k\}$) that satisfy Hamilton's equations of motion (with possibly a new Hamiltonian $\bar{H}$) is called a canonical transformation.

"Contact" transformations mix q's and p's — more general than "point" transformations $q_k \to Q_k(\{q_j\}, t)$ [and then $p_k \to P_k(\{q_j\}, \{p_j\}, t)$]

Outline

- General method to find canonical transformations — generating functions
- Test for canonical transformation — Poisson bracket
- Action-angle variables for periodic motion
- Hamilton-Jacobi method (next lecture)

I will give results for two dimensional phase space (1 DOF)

See HF Chapter 6 Appendix or GPS §9.4-5 for general dimension

4 Classes of Generating Functions

$$
\begin{aligned}
F_1(q, Q, t) \quad &\Rightarrow \quad p = \frac{\partial F_1}{\partial q}, \quad P = -\frac{\partial F_1}{\partial Q} \\
F_2(q, P, t) \quad &\Rightarrow \quad p = \frac{\partial F_2}{\partial q}, \quad Q = \frac{\partial F_2}{\partial P} \\
F_3(p, Q, t) \quad &\Rightarrow \quad q = -\frac{\partial F_3}{\partial p}, \quad P = -\frac{\partial F_3}{\partial Q} \\
F_4(p, P, t) \quad &\Rightarrow \quad q = -\frac{\partial F_4}{\partial p}, \quad Q = \frac{\partial F_4}{\partial P}
\end{aligned}
$$

and in each case

$$\bar{H} = H + \frac{\partial F_j}{\partial t}$$

Canonical Perturbation Theory

- Slightly *anharmonic* oscillator $\epsilon \ll 1$

$$H(q,p) = \tfrac{1}{2}(p^2 + \omega^2 q^2) + \epsilon\beta q^4$$

- Look for a generating function with quadratic corrections of the form

$$F_2(q,P) = qP[1 + \epsilon(aq^2 + bP^2)]$$

- Calculate p and Q

$$\begin{aligned} q &= Q - 3\epsilon b Q P^2 - \epsilon a Q^3 + O(\epsilon^2) \\ p &= P + \epsilon b P^3 + 3\epsilon a Q^2 P + O(\epsilon^2) \end{aligned}$$

- The values of $a = 5\beta/8\omega^2$, $b = c = 3\beta/8\omega^4$ reduce the Hamiltonian to

$$H(Q,P) = \tfrac{1}{2}(P^2 + \omega^2 Q^2) + \epsilon c(P^2 + \omega^2 Q^2)^2 + O(\epsilon^2)$$

- Perform a canonical transformation to θ, I where $\tan\theta = \omega Q/P$

$$H(\theta, I) = \omega I + 4\epsilon c\omega^2 I^2 + O(\epsilon^2)$$

- To first order in ϵ the solution is

$$Q = A\sin(\Omega t + \delta), \quad P = \omega A\cos(\Omega t + \delta), \quad \Omega = \omega + 3\epsilon\beta A^2/2\omega$$

Time evolution is a canonical transformation

$$F_2(q, P) = q\,P + dt\; H(q, P)$$

Then

$$Q = \frac{\partial F_2}{\partial Q} = q + dt\,\frac{\partial H}{\partial P} \to Q \simeq q + dt\,\frac{\partial H}{\partial p} \simeq q(t + dt)$$

$$p = \frac{\partial F_2}{\partial q} = P + dt\,\frac{\partial H}{\partial q} \to P = p - dt\,\frac{\partial H}{\partial q} \simeq p(t + dt)$$

Series of infinitesimal canonical transformations $\Rightarrow$ a canonical transformation

The Hamiltonian is the generator of time translations

Angular momentum is the generator of rotations

Use $L_z = (\vec{r} \times \vec{p})_z = xp_y - yp_x$ in a type II double generating function

$$F_2(x, P_x; y, P_y) = xP_x + yP_y + d\theta\,[xP_y - yP_x]$$

Then

$$X = \frac{\partial F_2}{\partial P_x} = x - d\theta\, y; \qquad Y = \frac{\partial F_2}{\partial P_y} = y + d\theta\, x;$$

$$p_x = \frac{\partial F_2}{\partial x} = P_x + d\theta\, P_y; \qquad p_y = \frac{\partial F_2}{\partial y} = P_y - d\theta\, P_y;$$

$$X = x - d\theta\, y; \qquad Y = y + d\theta\, x;$$
$$P_x \simeq p_x - d\theta\, p_y; \qquad P_y \simeq p_y + d\theta\, p_y;$$

$(\vec{R}, \vec{P})$ is infinitesimal rotation of $(\vec{r}, \vec{p})$

Canonical transformations are area preserving

$$dF_1 = p\,dq - P\,dQ + (\bar{H} - H)dt$$

Integrate around a closed loop in phase space at fixed time

$$\oint dF_1 = 0 = \oint p\,dq - \oint P\,dQ$$

so that

$$\oint p\,dq = \oint P\,dQ \quad \Rightarrow \qquad \sum_k \oint p_k\,dq_k = \sum_k \oint P_k\,dQ_k$$

Canonical transformations

- preserve areas in a two dimensional phase space
- preserve sum of projected areas on q, p planes in higher dimensional phase space

Connection with Poisson Brackets

$$\text{Area preserving} \quad \Longleftrightarrow \quad \text{Poisson bracket}$$

Poisson bracket:

- simple test for a transformation to be canonical
- generalizes to higher dimensional phase spaces
- formal approach to Hamiltonian mechanics
- connection with quantum mechanics

Jacobians

When transforming variables $q, p \to Q, P$ the element of area transforms as

$$dq\,dp \to \left\| \frac{\partial(q,p)}{\partial(Q,P)} \right\| dQ\,dP$$

with the Jacobian (determinant)

$$\left| \frac{\partial(q,p)}{\partial(Q,P)} \right| = \frac{\partial q}{\partial Q}\frac{\partial p}{\partial P} - \frac{\partial q}{\partial P}\frac{\partial p}{\partial Q} = \begin{vmatrix} \frac{\partial q}{\partial Q} & \frac{\partial q}{\partial P} \\ \frac{\partial p}{\partial Q} & \frac{\partial p}{\partial P} \end{vmatrix}$$

Jacobians have the following properties

Inverse rule $$\left| \frac{\partial(q,p)}{\partial(Q,P)} \right| = \left(\left| \frac{\partial(Q,P)}{\partial(q,p)} \right| \right)^{-1}$$

Product rule $$\left| \frac{\partial(q,p)}{\partial(Q',P')} \right| = \left| \frac{\partial(q,p)}{\partial(Q,P)} \right| \left| \frac{\partial(Q,P)}{\partial(Q',P')} \right|$$

Simplification rule $$\left| \frac{\partial(q,p)}{\partial(Q,p)} \right| = \left(\frac{\partial q}{\partial Q} \right)_p$$

2d Canonical transformations preserve areas

$$\left|\frac{\partial(Q,P)}{\partial(q,p)}\right| = \left|\frac{\partial(Q,P)}{\partial(q,Q)}\right|\left|\frac{\partial(q,Q)}{\partial(q,p)}\right| = \left|\frac{\partial(Q,P)}{\partial(q,Q)}\right|\left[\left|\frac{\partial(q,p)}{\partial(q,Q)}\right|\right]^{-1}$$

$$= -\left(\frac{\partial P}{\partial q}\right)_Q \Big/ \left(\frac{\partial p}{\partial Q}\right)_q$$

Each of the partials in the last expression is $\partial^2 F_1/\partial q \partial Q$ and so

$$\left|\frac{\partial(Q,P)}{\partial(q,p)}\right| = 1 \qquad \text{for canonical transformation}$$

Poisson Brackets

Definition

For functions $A(\{q_k\}, \{p_k\}, t)$, $B(\{q_k\}, \{p_k\}, t)$, the Poisson bracket is defined as

$$[A, B]_{q,p} = \sum_{k=1}^{N} \left(\frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial q_k} \right)$$

For a 2d phase space area preserving condition is

$$\frac{\partial Q}{\partial q}\frac{\partial P}{\partial p}-\frac{\partial Q}{\partial p}\frac{\partial P}{\partial q}=[Q,P]_{q,p}=1$$

For a canonical transformation $\{q_k, p_k\} \to \{Q_k, P_k\}$ in higher dimensional phase space the condition generalizes to

$$\left[Q_j, P_k\right]_{q,p}=\delta_{jk}, \quad \left[Q_j, Q_k\right]_{q,p}=0, \quad \left[P_j, P_k\right]_{q,p}=0.$$

Both necessary and sufficient $\Rightarrow$ test for a canonical transformation

The Poisson bracket is unchanged by canonical transformation

$$[A, B]_{q,p} = [A, B]_{Q,P}$$

In 2d this follows directly from the product rule for Jacobians.

We can leave off the q, p subscript.