Physics 106a **23 October, 2013**

Lecture 7: Central Forces – Bound States

The lecture discusses the *Kepler problem* of the orbital motion of planets and other objects in the gravitational field of the sun.

We first consider the more general problem of two point particles of mass M_1, M_2 interacting with a *central force* (one directed between the points or the centers of the spheres). There are 6 degrees of freedom, e.g. the position vectors $\vec{r}_1, \vec{r}_2$ described by the Lagrangian

$$L = \frac{1}{2}M_1\dot{\vec{r}}_1^2 + \frac{1}{2}M_2\dot{\vec{r}}_2^2 - V(|\vec{r}_1 - \vec{r}_2|) \tag{1}$$

with $V(r)$ giving the central force. The symmetries and consequent conservation laws allow us to reduce the problem to solving for 1 degree of freedom — an enormous simplification!

Translational symmetry: Translational symmetry gives the conservation of total momentum $\vec{P} = \vec{p}_1 + \vec{p}_2 = M_1\dot{\vec{r}}_1 + M_2\dot{\vec{r}}_2$ (using $\vec{p}_1 = \partial L/\partial\dot{\vec{r}}_1$ etc.). This can be shown using Noether's theorem (see Appendix) or directly from the equations of motion. We therefore introduce the center of mass coordinate $\vec{R}_{\rm cm} = (M_1\vec{r}_1 + M_2\vec{r}_2)/(M_1 + M_2)$, so that $\vec{P} = (M_1 + M_2)\dot{R}_{\rm cm}$. As the second coordinate we use the translationally invariant difference vector $\vec{r} = \vec{r}_1 - \vec{r}_2$. The Lagrangian becomes

$$L = \frac{1}{2}M\dot{\vec{R}}_{\rm cm}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r) \tag{2}$$

with $M = M_1 + M_2$ the total mass, and $\mu = M_1M_2/(M_1 + M_2)$ the reduced mass. Indeed we see that $\vec{R}_{\rm cm}$ is ignorable and $\vec{P}$ is the conserved conjugate momentum $\vec{P} = \partial L/\partial\dot{\vec{R}}_{\rm cm}$. The first term in L is constant, and thus we now only have to consider the relative motion described by the Lagrangian

$$L = \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r). \tag{3}$$

Rotational symmetry: The Lagrangian only involves scalars and is unchanged by any rotation of the whole system. This means the angular momentum $\vec{l} = (l_x, l_y, l_z)$ is a constant of the motion. Again this follows from Noether's theorem (see Appendix). Alternatively, in spherical polar coordinates

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2) - V(r). \tag{4}$$

Now ϕ is ignorable, and the conjugate momentum $l_z = \partial L/\partial\dot{\phi} = \mu r^2\sin^2\theta\,\dot{\phi}$ is a constant of the motion. The direction of z is arbitrary, so this means the full vector $\vec{l}$ is constant.

Planar motion: For constant $\vec{l} = \vec{r}(t) \times \vec{p}(t)$ the vectors $\vec{r}(t)$ and $\vec{p}(t)$ always lies in the fixed plane perpendicular to $\vec{l}$. Choosing the z-axis along the direction of $\vec{l}$ means that the orbital plane is $\theta = \pi/2$, and the Lagrangian for motion in the plane reduces to

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - V(r). \tag{5}$$

Now we see $\dot{\phi}$ is ignorable, so $l = \partial L/\partial\dot{\phi} = \mu r^2\dot{\phi}$ is constant. This immediately gives Keplers's second law, that the radius vector sweeps out area at a constant rate, since $\dot{A} = \frac{1}{2}r^2\dot{\phi} = l/2\mu$.

One dimensional reduction: The Euler-Lagrange equation for r is

$$\mu\ddot{r} - \mu r\dot{\phi}^2 + dV/dr = 0 \tag{6}$$

or

$$\mu\ddot{r} + dV_{\text{eff}}/dr = 0 \qquad \text{with} \qquad V_{\text{eff}}(r) = \frac{l^2}{2\mu r^2} + V(r) \tag{7}$$

where $\dot{\phi}$ has been written in terms of the constant l in V_{eff}[1] This is a one dimensional problem in an effective potential with an additional term from the rotational motion.

Constant H, E: To get the first integral of this equation we can use the fact that the Lagrangian is time independent ($\partial L/\partial t = 0$), so that the Hamiltonian H is constant, and since there are no time dependent constraints the Hamiltonian is equal to the total energy $H = E = T + V$ so that E is constant. Explicitly

$$E = \tfrac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r) = \text{constant} \tag{8}$$

Qualitative solution: Now specialize to the gravitational potential $V(r) = -k/r$ with $k = GM_1M_2$ so that

$$V_{\text{eff}}(r) = -\frac{k}{r} + \frac{l^2}{2\mu r^2}. \tag{9}$$

Sketching $V_{\text{eff}}(r)$ gives the qualitative solution: for $E < 0$ (set by initial conditions) the r motion oscillates over a finite range between $r_{\text{min}}, r_{\text{max}}$, so that the motion is bound, whereas for $E > 0$ the separation r increases to ∞ (maybe after one "bounce") so that the motion is unbound.

Solution for $r(t)$: Equation (8) gives us $\dot{r}$, which we can formally integrate to get $t(r)$ starting from r_0 at $t = 0$ (which then implicitly gives $r(t)$)

$$t = \sqrt{\frac{\mu}{2}}\int_{r_0}^{r}\frac{dr'}{\sqrt{E + \frac{k}{r'} - \frac{l^2}{2\mu r'^2}}}\,. \tag{10}$$

See Hand and Finch pp148-9 for how to do this integral.

Period of radial motion for a bound orbit: The period τ_r for the radial motion of a bound orbit is given by integrating from r_{min} to r_{max} and back

$$\tau_r = \sqrt{2\mu}\int_{r_{\text{min}}}^{r_{\text{max}}}\frac{dr'}{\sqrt{E + \frac{k}{r'} - \frac{l^2}{2\mu r'^2}}}\,. \tag{11}$$

Explicit solution: It is actually easier to find the shape of the orbit $r(\phi)$ rather than the trajectory $r(t)$. Use $\dot{r} = \dot{\phi}\,dr/d\phi$ to write[2]

$$\dot{r}^2 = \dot{\phi}^2\left(\frac{dr}{d\phi}\right)^2 = \frac{l^2}{\mu^2}\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \frac{l^2}{\mu^2}\left(\frac{du}{d\phi}\right)^2 \qquad \text{with } u = 1/r\,. \tag{12}$$

Then a little algebra shows

$$E = \frac{l^2}{2\mu}\left[\left(\frac{du}{d\phi}\right)^2 + \left(u - \frac{1}{p}\right)^2 - \frac{1}{p^2}\right] \qquad \text{with } p = \frac{l^2}{\mu k}. \tag{13}$$

[1] Note that you may not eliminate $\dot{\phi}$ from the Lagrangian using the constant l – this gives the *wrong* answer (try it and see!). The Lagrangian must be expressed in terms of the velocities, *not* the momenta.

[2] In $1/r$ potentials, introducing $u = 1/r$ is often a useful trick. You will do this in solving the hydrogen atom in quantum mechanics.

This is the energy for simple harmonic motion $u(\phi)$ centered on $u = 1/p$. The period of $u(\phi)$ is 2π, so that the bound orbits are *closed*. This is true for r^{-1} and r^2 potentials (Bertrand's theorem). The explicit solution is

$$u = \frac{1}{r} = \frac{1}{p} + \frac{\epsilon}{p}\cos\phi \tag{14}$$

with an arbitrary amplitude of the simple harmonic motion (set by the free parameter ϵ, which may be chosen positive). The free additive phase has been set to zero by a choice of the origin of ϕ. Plugging Eq. (14) into Eq. (13) gives the energy for this orbit

$$E = \frac{l^2}{2\mu p^2}(\epsilon^2 - 1)\,. \tag{15}$$

Shape of orbit: Equation (14) is the equation in polar coordinates r, ϕ for a conic section with focus at the origin $r = 0$ and ϵ the *eccentricity*. For $\epsilon < 1$ the orbit is an ellipse, for $\epsilon > 1$ a hyperbola, and $\epsilon = 1$ a parabola. You can see this from geometrical definitions of conic sections, or by changing to Cartesian coordinates $r = \sqrt{x^2 + y^2}$, $r\cos\phi = x$ which gives the equation [3]

$$\frac{(x - x_c)^2}{a^2} \pm \frac{y^2}{b^2} = 1 \tag{16}$$

with the $+$ for $\epsilon < 1$ and the $-$ for $\epsilon > 1$, and

$$a = \frac{p}{|1-\epsilon^2|}, \qquad b = \frac{p}{\sqrt{|1-\epsilon^2|}}, \qquad x_c = -\frac{\epsilon p}{1-\epsilon^2}\,. \tag{17}$$

Keppler's third law: The area of an elliptical orbit is $A = \pi ab$ and we know $\dot{A} = l/2\mu$. This gives the period

$$\tau = \frac{A}{\dot{A}} = 2\pi\sqrt{\frac{\mu}{k}}a^{3/2} = \frac{2\pi}{\sqrt{G(M_1 + M_2)}}a^{3/2}\,. \tag{18}$$

For a planet with mass much smaller than the sun (good for the earth, less so for Jupiter) $M_1 + M_2 \simeq M_{\text{sun}}$, in which case $\tau^2 \propto a^3$ for all such planets. This is Kepler's third law.

Elliptical orbit: For bound orbits, the shape is an ellipse with the origin of $\vec{r}$ (i.e. the position of the center of mass, which is the position of the sun for light planets) at the focus. You should be aware of the following geometrical properties:

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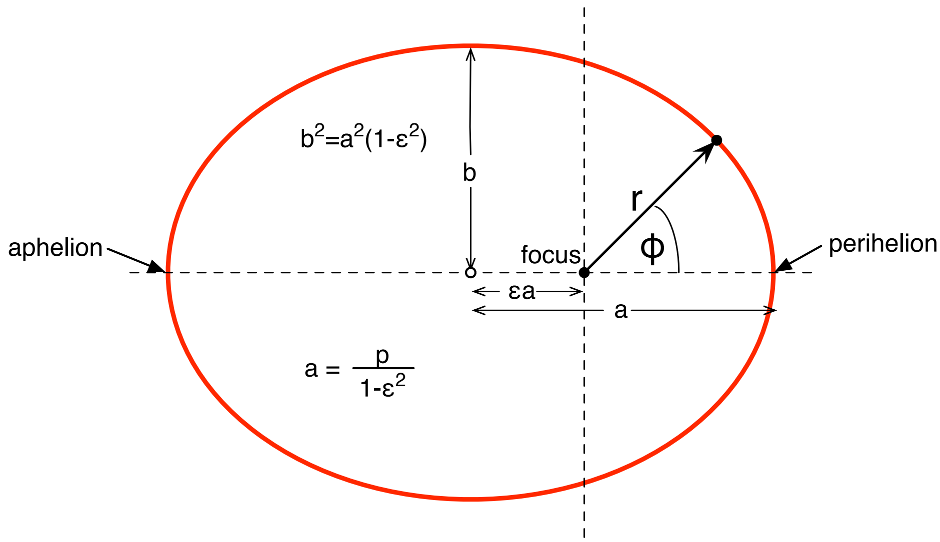
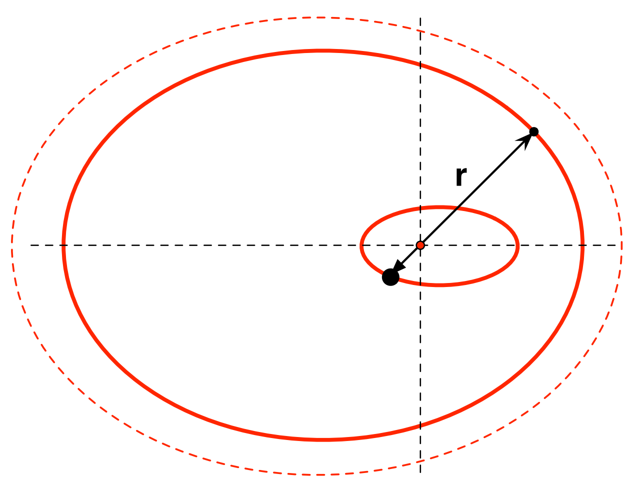
[3] The origin of the Cartesian coordinates is at the focus. A more natural origin might be at the center of the ellipse: with such coordinates the x_c in the $(x - x_c)^2$ would be absent.



Elliptical orbit for attractive $1/r$ potential

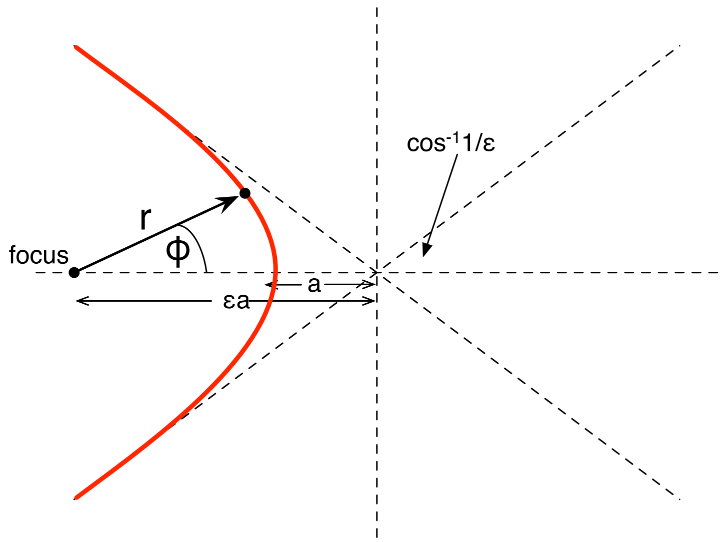
Planet and Sun: This discussion is for the separation vector. The individual planet and sun position vectors from the center of mass are

$$\vec{r}_1 = \vec{R}_{\rm cm} + \frac{M_2}{M_2 + M_1}\vec{r}, \qquad \vec{r}_2 = \vec{R}_{\rm cm} - \frac{M_1}{M_2 + M_1}\vec{r} \tag{19}$$



Planet and sun orbits: both rotate in elliptical orbits about the center of mass

Hyperbolic orbit: For unbound orbits, the shape is a hyperbola with the origin of $\vec{r}$ at the focus. The corresponding geometrical properties are:



Hyperbolic orbit for attractive $1/r$ potential

Virial theorem*

The virial theorem discusses averages of the motion for interacting bound particles. It is most interesting for particles interacting with a power-law pair potential $U(r) \propto r^{\alpha}$, when the theorem takes the form

$$\langle T \rangle = \frac{\alpha}{2} \langle V \rangle \tag{20}$$

where the $\langle\rangle$ denotes an average over time. The average is taken over a period for periodic motion, or otherwise over a very long time in which case the result relies on the coordinates and momenta remaining bounded. In either case the method works only for bound states, not for scattering states which are nonperiodic and unbounded. See the Appendix Slides or Goldstein, Poole and Safko §3.4 for the derivation.

An important application of the virial theorem was by Zwicky who analyzed the Coma nebulae cluster, and found that the average kinetic energy implied more potential energy than could be supplied by the visible sources. Using this, he argued for the existence of dark matter. The work done in 1933, although a more accessible reference is Astrophysical Journal **86**, 217 (1937).

Appendix: Noether's Theorem

This formalizes the idea that a continuous or differential symmetry leads to conserved quantities (constants of the motion).

We parameterize the symmetry operation by describing how the generalized coordinates q_k change with a continuous parameter s (e.g. this would be a rotation angle for rotational symmetry). Introduce $Q_k(s)$ which gives the different versions of q_k as the symmetry operation is performed. We choose the definition so that $Q_k(s=0)=q_k$. The transformation represented by s is a *symmetry operation* if the Lagrangian is unchanged

$$\frac{d}{ds}L(\{Q_k\},\{\dot{Q}_k\},t)=0. \tag{21}$$

Expressing the total derivative in terms of the partials

$$\begin{aligned}
\frac{dL}{ds} &= \sum_k\left[\frac{\partial L}{\partial Q_k}\frac{\partial Q_k}{\partial s}+\frac{\partial L}{\partial \dot{Q}_k}\frac{\partial \dot{Q}_k}{\partial s}\right] && (22)\\
&= \sum_k\left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{Q}_k}\right)\frac{\partial Q_k}{\partial s}+\frac{\partial L}{\partial \dot{Q}_k}\frac{\partial \dot{Q}_k}{\partial s}\right] && (23)\\
&= \frac{d}{dt}\left[\sum_k\left(\frac{\partial L}{\partial \dot{Q}_k}\right)\frac{\partial Q_k}{\partial s}\right]=\frac{d}{dt}\sum_k p_k\frac{\partial Q_k}{\partial s}, && (24)
\end{aligned}$$

introducing the usual momentum $p_k=\partial L/\partial \dot{Q}_k$. Putting $dL/ds=0$ for a symmetry operation gives us the conserved quantity (which we choose to evaluate at $s=0$)

$$I(\{q_k\},\{\dot{q}_k\},t)=\sum_k p_k\left.(\partial Q_k/\partial S)\right|_{s=0}. \tag{25}$$

Note that the Euler-Lagrange equation is used to get the second line Eq. (23): it is not correct to say "symmetry implies a conserved quantity"; rather symmetry *together with Lagrangian equations of motion* imply a conserved quantity. For example, adding dissipation usually destroys conservations laws, e.g. momentum.

Example 1: Rotations about the z-axis

For Cartesian coordinates (x_i,y_i) of the particles we introduce the rotated coordinates depending on the rotation angle ϕ about the z-axis (i.e. $s\equiv\phi$ in this case)

$$X_i(\phi)=x_i\cos\phi-y_i\sin\phi, \tag{26}$$
$$Y_i(\phi)=x_i\sin\phi+y_i\cos\phi. \tag{27}$$

If the Lagrangian is unchanged under rotation about the z-axis gives us the conserved quantity

$$\begin{aligned}
L_z &= \sum_i p_{xi}\left.\frac{\partial X_i}{\partial\phi}\right|_{\phi=0}+p_{yi}\left.\frac{\partial Y_i}{\partial\phi}\right|_{\phi=0} && (28)\\
&= \sum_i p_{xi}(-y_i)+p_{yi}x_i=\sum_i(\vec{r}_i\times\vec{p}_i)_z. && (29)
\end{aligned}$$

Thus rotational symmetry about the z-axis implies the conservation of the z-component of the angular momentum. Full rotational symmetry, about any axis, implies the conservation of the vector angular momentum $\vec{L}$.

Example 2: Translational symmetry for M particles

Define the extended coordinates $\vec{R}_i(\vec{d}) = \vec{r}_i + \vec{d}$ with $\vec{d}$ a translation. For a Lagrangian that is translationally invariant for displacements in the x-direction the conserved quantity is

$$P_x = \sum_i \vec{p}_i \cdot (\partial \vec{R}_i / \partial d_x)_{d=0} = \sum_i p_{ix}, \tag{30}$$

which is the total x-momentum. For full translational invariance, the vector total momentum $\vec{P}$ is conserved.

Michael Cross, October 22, 2013