Physics 106b: Lecture 3 – Relativistic Mechanics 14 January, 2014

Momentum and energy

Our discussion of 4-vectors in general led us to the energy-momentum 4-vector $\mathbf{p}$ with components in some inertial frame $(E, \vec{p}) = (m\gamma_u, m\gamma_u\vec{u})$ with $\vec{u}$ the velocity of the particle in that inertial frame and $\gamma_u = (1-u^2)^{-1/2}$.[1] The length-squared of the 4-vector is (since $\mathbf{u}^2 = 1$)

$$\mathbf{p}^2 = m^2 = E^2 - p^2 \tag{1}$$

where the last expression is the evaluation in some inertial frame and $p = |\vec{p}|$. In conventional units this is

$$E^2 = p^2c^2 + m^2c^4 \ . \tag{2}$$

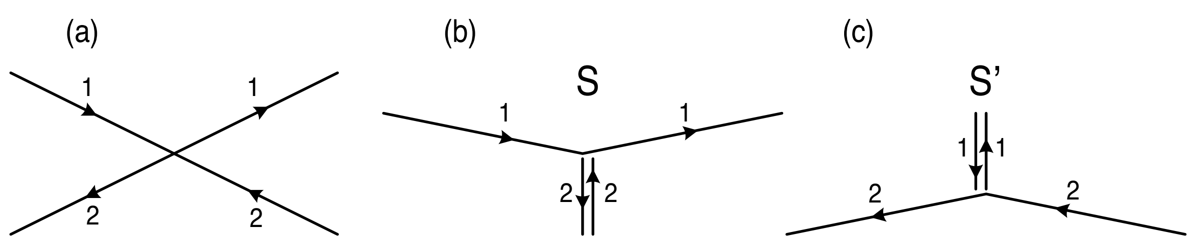
For a light particle (photon) $m = 0$ and so $E = p$ (going back to $c = 1$). Thus the energy-momentum 4-vector for a photon is $E(1, \hat{n})$ with $\hat{n}$ the direction of propagation. This is consistent with the quantum expressions $E = h\nu$, $\vec{p} = (h/\lambda)\hat{n}$ with ν the frequency and λ the wave length, and $\nu\lambda = 1$ (the speed of light)

We are led to the expression for the relativistic 3-momentum and energy

$$\vec{p} = \frac{m\vec{u}}{\sqrt{1-u^2}}, \quad E = \frac{m}{\sqrt{1-u^2}}. \tag{3}$$

Note that $\vec{u} = \vec{p}/E$. The frame independent statement that the energy-momentum 4-vector is conserved leads to the conservation of this 3-momentum and energy in each inertial frame, with the new implication that mass can be converted to and from energy.

I arrived at the expressions for relativistic energy and momentum by rather abstract arguments. A more concrete version of essentially the same argument, is the subject of of the following example you can work through.

> A simple symmetric collision in the xy plane between two identical particles in the center of mass frame looks like panel (a) in the figure below.
>
> 
>
>
> In two other frames S and S' moving along the x-axis (horizontal in the figure) relative to the center of mass frame the collision appears as in panels (b) and (c). The velocities of the two particles are $(u_x, \pm u_y)$ and $(0, \pm\bar{u}_y)$ in S, and $(0, \pm\bar{u}_y)$ and $(-u_x, \pm u_y)$ in S' (because of the symmetry of the collision) in the two frames.
>
> (a) Use the expressions for the transformation of velocities between inertial frames to relate u_y to $\bar{u}_y$.

[1] Remember I'm using bold for 4-vectors and units in which $c = 1$.

(b) Suppose that the relativistic 3-momentum of a particle of mass m with velocity $\vec{u}$ is $mf(u)\vec{u}$ with $f(u)$ an unknown function of the speed u. Use the conservation of the y-component of the momentum in the S frame of reference and the result of part (a) to find $f(u)$ by considering the case of a glancing collision where $u_y, \bar{u}_y$ (but not u_x) are small so that the Newtonian expression can be used for the momentum of particle 2 in S, and terms in $u_y^2, \bar{u}_y^2$ in various expressions can be neglected.

Hand and Finch prefer to derive properties of energy and momentum using specific calculations for electromagnetism. This relies on the usual equations for E&M being consistent with relativity, which indeed we believe to be true.

Since $(E, \vec{p})$ form the components of a 4-vector they transform between inertial frames in the same way as $(t, \vec{x})$, i.e. in our standard configuration

$$p'_x = \gamma(p_x - vE) \tag{4}$$
$$p'_y = p_y \tag{5}$$
$$p'_z = p_z \tag{6}$$
$$E' = \gamma(E - vp_x) \tag{7}$$

and the inverse

$$p_x = \gamma(p'_x + vE') \tag{8}$$
$$p_y = p'_y \tag{9}$$
$$p_z = p'_z \tag{10}$$
$$E = \gamma(E' + vp'_x) \tag{11}$$

These reduce to the Galilean expressions for small particle speeds and small transformation speeds. The momentum expression is simply given by setting $\gamma \simeq 1$. For the energy we use $E \simeq m + T$ with T the Newtonian kinetic energy, and $\gamma \simeq 1 + \frac{1}{2}v^2$. These give to leading order the Galilean expressions

$$p_x = p'_x + mv \tag{12}$$
$$T = T' + vp'_x + \tfrac{1}{2}mv^2 \tag{13}$$

Since these are easy to intuit, they provide a good way of getting the sign right in the p_x and E equations.

Collisions

An important topic in relativistic mechanics is collisions, since colliding elementary particles at high energies is the main tool of particle physics. The physics principle is the conservation of 4-momentum.

Collisions can be classified as:

- **Elastic:** The masses of the particles are unchanged.
- **Inelastic:** The masses change — kinetic energy and mass are interchanged — and new particles may even be formed

Example on elastic collisions – Compton scattering: (This is Hand and Finch problem 12-17a.) A gamma ray photon scatters off a stationary electron. How does the outcoming energy/frequency of the photon depend on the incoming energy/frequency and the scattering angle θ'? See Hand and Finch for a sketch.

One scheme for solving the problem is the following. Four momentum conservation gives

$$\mathbf{p}_\gamma^{(in)} + \mathbf{p}_e^{(in)} = \mathbf{p}_\gamma^{(out)} + \mathbf{p}_e^{(out)}. \tag{14}$$

We are not asked for the details of $\mathbf{p}_e^{(out)}$ and so we rewrite this equation as

$$\mathbf{p}_\gamma^{(in)} - \mathbf{p}_\gamma^{(out)} + \mathbf{p}_e^{(in)} = \mathbf{p}_e^{(out)} \tag{15}$$

so that on taking the magnitude squared of both sides (in the 4-vector sense)

$$(\mathbf{p}_\gamma^{(in)} - \mathbf{p}_\gamma^{(out)})^2 + 2\mathbf{p}_e^{(in)} \cdot (\mathbf{p}_\gamma^{(in)} - \mathbf{p}_\gamma^{(out)}) + (\mathbf{p}_e^{(in)})^2 = (\mathbf{p}_e^{(out)})^2. \tag{16}$$

Now using $\mathbf{p}_e^2 = m_e^2$, $\mathbf{p}_\gamma^2 = 0$ this simplifies to

$$\mathbf{p}_e^{(in)} \cdot (\mathbf{p}_\gamma^{(in)} - \mathbf{p}_\gamma^{(out)}) = \mathbf{p}_\gamma^{(in)} \cdot \mathbf{p}_\gamma^{(out)} \tag{17}$$

and $\mathbf{p}_e$ drops out of the problem. Using $\mathbf{p}_e^{(in)} = (m_e, 0)$, $\mathbf{p}_\gamma^{(in)} = E_\gamma^{(in)}(1, \hat{x})$, $\mathbf{p}_\gamma^{(out)} = E_\gamma^{(in)}(1, \hat{n})$ with $\hat{n}$ the direction of propagation of the outgoing photon and $\cos\theta' = \hat{x} \cdot \hat{n}$ gives

$$m(E_\gamma^{(in)} - E_\gamma^{(out)}) = \mathbf{p}_\gamma^{(in)} \cdot \mathbf{p}_\gamma^{(out)} = E_\gamma^{(in)} E_\gamma^{(out)}(1 - \cos\theta'). \tag{18}$$

This gives the expression in Hand and Finch, but a more useful form is

$$\frac{1}{E_\gamma^{(out)}} - \frac{1}{E_\gamma^{(in)}} = \frac{1}{m_e}(1 - \cos\theta'), \tag{19}$$

or using $E_\gamma = h\nu = h/\lambda$ (remember $c = 1$)

$$\lambda^{(out)} - \lambda^{(in)} = \frac{h}{m_e}(1 - \cos\theta'). \tag{20}$$

Example on inelastic collisions – Colliding putty balls: Consider a ball of mass m_a moving with velocity $\vec{u}_a$ colliding with a stationary ball of mass m_b. After the collision the balls stick together and move with velocity $\vec{u}$. What is the mass m of the new ball, and what is $\vec{u}$?

It is easy enough to write down the total relativistic 3-momentum and energy, and equate the before and after values. However it is often simpler to manipulate the 4-momenta directly. With obvious notation, using the conservation law $\mathbf{P}_{\text{out}} = \mathbf{P}_{\text{in}}$ with $\mathbf{P}$ the total 4-momentum (sum of the individual particle 4-momenta), we have

$$\mathbf{P}_{\text{out}}^2 = m^2 = \mathbf{P}_{\text{in}}^2 = (\mathbf{p}_a + \mathbf{p}_b)^2 = \mathbf{p}_a^2 + \mathbf{p}_b^2 + 2\mathbf{p}_a \cdot \mathbf{p}_b = m_a^2 + m_b^2 + 2E_a m_b, \tag{21}$$

where in the last step we have used $\mathbf{p}_b = (m_b, 0)$ so that only the time-like component of $\mathbf{p}_a$, the energy E_a of particle a, appears in the scalar product. This gives

$$m = \sqrt{m_a^2 + m_b^2 + 2E_a m_b} \geqslant m_a + m_b. \tag{22}$$

The velocity is given by

$$\vec{u} = \frac{\vec{P}_{\text{out}}}{E_{\text{out}}} = \frac{\vec{P}_{\text{in}}}{E_{\text{in}}} = \frac{\gamma_a m_a \vec{u}_a}{\gamma_a m_a + m_b}, \tag{23}$$

here using the conservation of the total 3-momentum and of the total energy in the second step.

Center of momentum frame

An important general approach (not always necessary in simple cases) is to transform to the frame in which the total 3-momentum $\vec{P}$ is zero. This is analogous to the center of mass frame in Newtonian physics. The collision is particularly simple in this frame: in a binary collision, the incoming particles must have equal and opposite 3-momenta, and the same applies for the outgoing particles if there remain only two.

The general scheme is

- Choose the x-direction along the total 3-momentum so that $P_y = P_z = 0$, and we can use our standard configuration for Lorentz transforming from the lab frame S to the center of momentum frame S';
- Since $P'_x = 0$, Eq. (4) shows that the speed of S' relative to S is $v = P_x/E$;
- Transform the energies and 3-momenta of all the particles to S' using the Lorentz transformation with the speed v;
- Solve the collision in S' (outgoing energies and momenta etc.);
- Transform back to S.

This method is direct and always works. The results can also be found using the invariance of scalar products such as $\mathbf{p}_a \cdot \mathbf{p}_b = \mathbf{p}'_a \cdot \mathbf{p}'_b$ with $\mathbf{p}_a, \mathbf{p}_b$ and $\mathbf{p}'_a, \mathbf{p}'_b$ any two of the particle 4-momenta in the lab frame and in the CM frame, respectively. This approach leads to less algebra but requires more ingenuity.

Example - Threshold energy to produce new particles: This is a very important problem in particle physics. Consider the production of proton-antiproton pairs by colliding a beam of protons with a target of stationary protons $p + p \to p + p + \bar{p} + p$. What is the minimum kinetic energy T of the beam particles for the production to occur? You might guess $T = 2m_p$ with just enough kinetic energy to convert to two proton masses. In fact the answer is significantly larger.

Call the incoming beam proton a and the target proton b so that $\mathbf{p}_b = (m_p, 0)$. In the center of momentum frame the threshold is when all the outgoing particles are at rest (of course this is consistent with zero total 3-momentum). This means that the outgoing 4-momentum is $\mathbf{p}'_{\text{out}} = (4m_p, 0)$ and so the incoming 4-momenta are $\mathbf{p}'_a = (2m_p, \vec{p}^*)$ and $\mathbf{p}'_b = (2m_p, -\vec{p}^*)$ with $\vec{p}^*$ unknown. Now use $\mathbf{p}_a \cdot \mathbf{p}_b = \mathbf{p}'_a \cdot \mathbf{p}'_b$

$$E_a m_p = 4m_p^2 + (\vec{p}^*)^2 = 4m_p^2 + (4m_p^2 - m_p^2) = 7m_p^2 \tag{24}$$

using Eq. (1) in the second step. Thus $E_a = 7m_p$ and the threshold kinetic energy is $6m_p$, three times larger than the naive guess.

The general expression for the threshold energy for the binary collision between a beam of particles of mass m_a colliding with a target of stationary particles of mass m_b to produce outgoing particles with total mass $\sum m_{\text{out}}$ is

$$E_a|_{\text{threshold}} = \frac{(\sum m_{\text{out}})^2 - m_a^2 - m_b^2}{2m_b}. \tag{25}$$

The result is even more dramatic for the production of the ψ (mass 3100 MeV) particle by colliding electrons and positrons (mass 0.5 MeV), $e^- + e^+ \to \psi$. The threshold energy for the beam (e.g. of positrons) and a stationary electron target is 10^7 MeV. The solution is to use *two colliding beams* - which puts us in the center of momentum frame for symmetric collisions. For the proton-antiproton example, each beam would just need a kinetic energy of m_p.

Relativistic Force

The 4-force or Mikowski force takes the form

$$\mathbf{f} = \frac{d\mathbf{p}}{d\tau} \tag{26}$$

(4-vector $d\mathbf{p}$ times scalar $d\tau^{-1}$). In some inertial frame where the particle velocity is $\vec{u}$ this gives

$$\mathbf{f} \to (\gamma_u \frac{dE}{dt}, \gamma_u \frac{d\vec{p}}{dt}) \ . \tag{27}$$

There are some nice formal expressions involving the 4-force, such as the expression for the electromagnetic 4-force

$$\mathbf{f} = q\mathbf{F} \cdot \mathbf{u} \tag{28}$$

with $\mathbf{F}$ the electromagnetic field tensor and $\mathbf{u}$ the particle 4-velocity, but more often we focus on the relativistic 3-force.

It turns out that the most physical definition of the relativistic 3-force is

$$\vec{f} = \frac{d\vec{p}}{dt}. \tag{29}$$

Using $\mathbf{p}^2 = m^2$ gives

$$\frac{d\mathbf{p}^2}{d\tau} = 0 = 2\mathbf{p} \cdot \frac{d\mathbf{p}}{d\tau} = 2m\mathbf{u} \cdot \mathbf{f} \tag{30}$$

so that $\mathbf{u} \cdot \mathbf{f} = 0$ (the 4-force is always "perpendicular" to the 4-velocity). Evaluating this in an inertial frame where $\mathbf{u} = \gamma_u(1, \vec{u})$ and $\mathbf{f} = \gamma_u(dE/dt, \vec{f})$ gives

$$\frac{dE}{dt} = \vec{f} \cdot \vec{u} \tag{31}$$

so that the 3-force defined by Eq. (29) has the desired connection with the energy. We will see that for a charged particle in electric and magnetic fields, the expression for $\vec{f}$ is the usual Lorentz force. Thus Eq. (29) is the most useful definition of the 3-force. Note that $\gamma_u \vec{f}$ is the spacelike component of the force 4-vector: the transformation behavior of $\vec{f}$ between inertial frames can be obtained from this.

Although Eq. (29) is an expression analogous to Newton's law, force=mass $\times$ acceleration is no longer true. In general the acceleration $\vec{a} = d\vec{u}/dt$ is not even parallel to the force:

$$\begin{aligned} \vec{f} &= \frac{d}{dt}(m\gamma_u\vec{u}) && (32) \\ &= m\gamma_u^3(\vec{u} \cdot \vec{a})\vec{u} + \gamma_u m\vec{a} && (33) \end{aligned}$$

For forces parallel and perpendicular to $\vec{u}$ the result is simpler

$$\begin{aligned} f_\parallel &= \gamma^3 m a_\parallel && (34) \\ \vec{f}_\perp &= \gamma m\vec{a}_\perp && (35) \end{aligned}$$

but with different proportionality constants (sometimes called longitudinal and transverse masses). For other directions $\vec{f}$ and $\vec{a}$ are not parallel.

Example – motion under constant force: Consider a constant force $\vec{f} = f\hat{x}$. The equation of motion is

$$\frac{dp_x}{dt} = f \qquad \text{so that } p_x = ft\ . \tag{36}$$

The velocity in the x-direction is most easily calculated from

$$u_x = \frac{p_x}{E} = \frac{p_x}{\sqrt{p_x^2 + m^2}} \tag{37}$$

to give

$$u_x = \frac{ft}{\sqrt{m^2 + (ft)^2}}. \tag{38}$$

The velocity approaches 1 (the speed of light) in the long time limit.

Lagrangian Formalism

I discussed the relativistic Lagrangian of a free particle and a particle in an electromagnetic field in Lecture 2 of Ph106a. The argument might make more sense now, so I will outline it again here.

The Lagrangian approach is given by minimizing the action. We make the bold guess that the action is a Lorentz invariant (same in all inertial frames). For a single particle of mass m we can write it as the integral along the worldline parameterized by the proper time between fixed beginning and ending events $\mathcal{P}_1$ and $\mathcal{P}_2$

$$S = \int_{\mathcal{P}_1}^{\mathcal{P}_2} \mathcal{L}\, d\tau\ . \tag{39}$$

For different physical situations we must choose a physically sensible Lorentz invariant expression for $\mathcal{L}$: this is a strong constraint and the appropriate expression (up to unimportant multiplicative and additive constants) is often unique.

Evaluating in our inertial frame gives

$$S = \int_{t_1}^{t_2} L\, dt = \int_{t_1}^{t_2} (\gamma L)\, d\tau \tag{40}$$

where the factor $\gamma \equiv \gamma_u$ with $\vec{u}$ the instantaneous speed of the particle comes from the $dt \to d\tau$ transformation for our time to proper time. Thus γL is Lorentz invariant.

Free particle

The only invariant that we can use is a constant for the particle. (Check that others don't make sense, e.g. $\mathbf{x}^2$ would give an origin dependent result, $\mathbf{u}^2 = 1$ and so does not give anything nontrivial, ...). Matching to the Newtonian expression for small speeds gives $\gamma L = -m$ with m the mass so that

$$L = -m\sqrt{1 - u^2}\ . \tag{41}$$

This gives for small u

$$L \to -m + \frac{1}{2}mu^2 \tag{42}$$

the correct Newtonian result (additive constants do not affect the results derived from the Lagranigan). Also

$$p_x = \frac{\partial L}{\partial u_x} = \frac{mu_x}{\sqrt{1 - u^2}} \tag{43}$$

and we recover the 3-momentum expression.

The Hamiltonian is

$$H = \vec{p} \cdot \vec{u} - L = \gamma m = E, \tag{44}$$

and is equal to the energy even though the kinetic energy is not a quadratic form in $\vec{u}$.

Charge in electromagnetic field

We look for a Lagrangian linear in the fields and depending on the particle velocity. It turns out to be convenient to use the scalar and vector potentials Φ, $\vec{A}$, since $(\Phi, \vec{A})$ form a 4-vector, rather than $\vec{E}$, $\vec{B}$. The only new Lorentz invariant (with the right symmetry properties, linear in the field strength etc.) is a constant times the scalar product $\mathbf{u} \cdot \mathbf{A}$ where $\mathbf{u}$ is the *velocity 4-vector* and $\mathbf{A}$ is the *electromagnetic potential 4-vector*. Thus for a particle in an electromagnetic field

$$\mathcal{L} = -m - q\mathbf{u} \cdot \mathbf{A}$$

where again the constant is chosen to match onto Newtonian physics at small velocities: q is then the charge of the particle. In our inertial frame, the components of the 4-vectors are $\mathbf{u} = \gamma(1, \vec{u})$, $\mathbf{A} = (\Phi, \vec{A})$ with Φ the electric scalar potential and $\vec{A}$ the magnetic vector potential. Thus the Lagrangian in our frame of reference is

$$L = -m\sqrt{1 - u^2} - q\Phi(\vec{x}, t) + q\vec{u} \cdot \vec{A}(\vec{x}, t).$$

Note how the combination $\Phi - \vec{u}.\vec{A}$ is forced on us by the requirement of Lorentz invariance.

From this Lagrangian we can derive the Lorentz force. The canonical 3-momentum is

$$\vec{p} = \frac{\partial L}{\partial \vec{u}} = m\gamma\vec{u} + q\vec{A}, \tag{45}$$

so that the *kinetic momentum* $\vec{\pi} = m\gamma\vec{u} = (\vec{p} - q\vec{A})$. The Euler-Lagrange equation is

$$\frac{d\vec{p}}{dt} = \frac{\partial L}{\partial \vec{x}}. \tag{46}$$

This gives

$$\frac{d\vec{\pi}}{dt} = q\left[-\vec{\nabla}\Phi + \vec{\nabla}(\vec{u} \cdot \vec{A}) - \frac{d\vec{A}}{dt}\right], \tag{47}$$

$$= q\left[-\vec{\nabla}\Phi + \vec{u} \times (\vec{\nabla} \times \vec{A}) + (\vec{u} \cdot \vec{\nabla})\vec{A} - \frac{d\vec{A}}{dt}\right], \tag{48}$$

using a standard vector equality in the last identity. Now $d\vec{A}/dt$ is the *total* time derivative of the vector potential moving with the particle

$$\frac{d\vec{A}}{dt} = \frac{\partial \vec{A}}{\partial t} + (\vec{u} \cdot \vec{\nabla})\vec{A}, \tag{49}$$

and from the definitions of the scalar and vector potential

$$\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}, \qquad \vec{B} = \vec{\nabla} \times \vec{A}, \tag{50}$$

so that the Lorenz force $\vec{f} = d\vec{\pi}/dt = d(m\gamma\vec{u})/dt$ is

$$\vec{f} = q(\vec{E} + \vec{u} \times \vec{B}) . \tag{51}$$

Example – Charge in uniform $\vec{B}$: The equations of motion are

$$\frac{d\vec{\pi}}{dt} = q\vec{u} \times \vec{B}\,, \qquad \frac{dW}{dt} = 0 \tag{52}$$

(here I am using W for the energy to remove confusion with the electric field magnitude E). The second equation gives u, γ constant, and then the first equation is

$$\frac{d\vec{u}}{dt} = \vec{u} \times \vec{\omega}_B \qquad \text{with} \qquad \vec{\omega}_B = \frac{q\vec{B}}{\gamma m}. \tag{53}$$

This is circular motion in the plane perpendicular to $\vec{B}$ at the cyclotron frequency ω_B, together with constant velocity along $\vec{B}$.

Example - Relativistic corrections to planetary motion - precession of the perihelion: Using the Lagrangian given by adding a potential $V(r) = -k/r$ to the free particle relativistic Lagrangian for a particle of mass m (with r the radius from the heavy fixed sun and k a positive constant) show that the equation for a planetary orbit in polar coordinates is (with no further approximations)

$$\frac{1}{r} = \frac{1}{p} + \frac{\epsilon}{p}\cos[\beta(\phi - \phi_0)] \tag{54}$$

and find the constants p, ϵ, β in terms of k, m and the constant energy E and angular momentum l. You may assume that the motion takes place in a plane containing the origin.

The form of the equation for the orbit is the same as the Newtonian result except for the key difference that $\beta \neq 1$, so that the orbit is not closed. For β close to 1 the orbit is a precessing ellipse. What is the precession angle of the ellipse per period for a planet orbiting the sun? Write your answer in terms of the length $R_s = GM_\odot/c^2 = 1.477$km with $M_\odot$ the mass of the sun, and p which is the average radius of the orbit?

Find the precession rate for Mercury resulting from this effect in arcseconds per century. Hand and Finch Table 4.9 gives the necessary planetary data. Compare with the measured value of $40''$/century.

Solution: The relativistic Lagrangian is (leaving in the c factors for variety)

$$L = -m\sqrt{1 - \frac{v^2}{c^2}} + \frac{k}{r}, \tag{55}$$

with v the planet speed. It is straightforward to show that

$$\vec{p} = \frac{\partial L}{\partial \vec{v}} = \frac{m\vec{v}}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{56}$$

and then

$$H = \vec{p} \cdot \vec{v} - L = \gamma mc^2 - \frac{k}{r} = \sqrt{m^2c^4 + p^2c^2} - \frac{k}{r} \tag{57}$$

($\vec{p} = \gamma m\vec{v}$ gives $p^2/m^2 + 1 = \gamma^2$). However we want to use polar coordinates and the conjugate momenta, so we write

$$v^2 = \dot{r}^2 + r^2\dot{\phi}^2 \tag{58}$$

and then

$$p_r = \frac{\partial L}{\partial \dot{r}} = \frac{m\dot{r}}{\sqrt{1 - \frac{v^2}{c^2}}}, \quad p_\phi = \frac{\partial L}{\partial \dot{\phi}} = \frac{mr^2\dot{\phi}}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{59}$$

and then

$$H = p_r\dot{r} + p_\phi\dot{\phi} - L = \sqrt{m^2c^4 + (p_r^2 + p_\phi^2/r^2)c^2} - \frac{k}{r}\ , \tag{60}$$

which is the same as the general expression if I use $p^2 = p_r^2 + p_\phi^2/r^2$.

With these expressions we follow the path of the Newtonian calculation we did last term.

Since L is independent of ϕ the conjugate momentum $p_\phi = l$ is constant. And then

$$\frac{p_r}{l} = \frac{\dot{r}}{r^2\dot{\phi}} = \frac{1}{r^2}\frac{dr}{d\phi} = -\frac{du}{d\phi}\ , \tag{61}$$

introducing $u = 1/r$. The Hamiltonian is then

$$H = \sqrt{m^2c^4 + l^2c^2\left[\left(\frac{du}{d\phi}\right)^2 + u^2\right]} - ku\ . \tag{62}$$

We could find the Euler-Lagrange equations and integrate, but since the Lagrangian has no explicit time dependence, it is easier to use the fact that the Hamiltonian is a constant E. This means

$$(E + ku)^2 = m^2c^4 + l^2c^2\left[\left(\frac{du}{d\phi}\right)^2 + u^2\right]\ . \tag{63}$$

As in the Newtonian calculation, this has the form of a constant sum of a "kinetic energy" proportional to $(du/d\phi)^2$ and a quadratic "potential energy" proportional to $(u - u_0)^2$ giving harmonic motion $u(\phi)$ about a mean u_0, with the constants to be calculated. Completing the square in u and manipulating the terms to give a form reminiscent of the Newtonian expression leads to

$$\frac{E^2 - (mc^2)^2}{2mc^2} = \frac{l^2}{2m}\left[\left(\frac{du}{d\phi}\right)^2 + \beta^2\left(u - \frac{1}{p}\right)^2 - \frac{\beta^2}{p^2}\right] \tag{64}$$

with

$$\beta = \sqrt{1 - \frac{k^2}{l^2c^2}} \quad \text{and} \quad p = \beta^2\frac{l^2}{(E/c^2)k}\ . \tag{65}$$

In the Newtonian limit $\beta \to 1$, $E/c^2 \to m$, $[E^2 - (mc^2)^2]/2mc^2 \to U = E - mc^2$, and the expression reduces to Eq. (10) of my notes to Lecture 8 last term with U playing the role of the Newtonian energy.

The solution giving constant E is

$$u = \frac{1}{p} + \frac{\epsilon}{p}\cos(\beta\phi) \tag{66}$$

with ϵ the amplitude of the harmonic motion $u(\phi)$ about the mean $u = 1/p$. The parameter ϵ which measures the fractional deviation from a circular orbit is related to the energy by plugging the solution into Eq. (64)

$$\frac{E^2 - (mc^2)^2}{2mc^2} = \frac{\beta^2 l^2}{2mp^2}(\epsilon^2 - 1) \tag{67}$$

or in terms of the original parameters, not trying to make the expression look "pretty"

$$\epsilon^2 - 1 = \frac{(E^2 - m^2c^4)(l^2c^2 - k^2)}{E^2k^2}\ . \tag{68}$$

The key point is that because $\beta \neq 1$ appears in the argument to the cosine in Eq. (66) u is not periodic in the angle ϕ and so the orbit is not closed. For $\beta - 1$ small, the solution will be almost an ellipse, but the ellipse will slowly precess.

The radial motion is periodic in $\phi = 2\pi/\beta$, and so the precession rate Ω of the ellipse for $\beta \approx 1$ is $2\pi(\beta^{-1} - 1)$ radians per period. In these units

$$\frac{\Omega}{2\pi} \simeq \frac{k^2}{2l^2c^2} \simeq \frac{k}{2pmc^2} \tag{69}$$

using Eq. (65) to lowest order to eliminate l in favor of p. Since $k = GM_{\odot}m$ this gives

$$\Omega \simeq \frac{\pi R_s}{p} \text{ radians/period,} \tag{70}$$

a nice dimensionless expression.

For mercury, the average radius p is about 0.39 AU with 1 AU = 1.5×10^{11}m, and the period is 88/365 (earth) years. Using $R_s = 1.477 \times 10^3$m and π radians = $180 \times 60 \times 60$ arcseconds, I find the precession rate 7 arcseconds per century. The measured result, after other larger known effects are taken into account, is about 40 arcseconds per century.

Is our formulation of the relativistic problem correct? We have used the relativistic Lagrangian for the free particle, and added the Newtonian potential. If this were the corresponding electrostatic problem, of a negative charge orbiting a positive one for example, we would know what we are doing. For the static positive source the vector potential is zero, and so adding the scalar potential is correct in the rest frame of the positive charge. We could calculate the problem in other frames of reference (with more difficulty!), in which case the moving positive charge would lead to a vector potential and magnetic field in addition. The answer for the orbit would be the same. We *would* be neglecting the electromagnetic potentials coming from the *negative* charge. Since this charge is accelerating, it actually generates electromagnetic waves, which carry energy and angular momentum away, so that the charge would eventually spiral in to the sun. For orbiting charges, we are usually interested in microscopic particles, such as the electron orbiting the proton in a Hydrogen atom. For such problems we also need to add quantum effects, which eliminate the radiation for the quantized energy levels.

Gravity does not work the same way as electromagnetism. A full treatment of the orbit problem requires the use of General Relativity which gives (see Hand and Finch Eq. (10.115) – they use the symbol m for R_s) a result involving the same form but 6 times as large, in agreement with the observations. This was an early triumph of General Relativity theory.

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