ACM 100b

Fourier series as a minimization problem

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Fourier series as a minimization problem

- We will next show that the Fourier series has another interpretation
- Combined with Parseval's theorem this will help us understand the sense in which Fourier series converge.
- Suppose you knew nothing about Fourier series.
- But you wanted to approximate a given function $f(x)$ on say the interval $-L < x < L$ by a finite series of trigonometric functions:

$$f(x) \approx \beta_0 + \sum_{n=1}^{N} \left[\beta_n \cos(n\pi x/L) + \alpha_n \sin(n\pi x/L)\right]$$

Fourier series as a minimization problem

- We now ask how can we determine the coefficients α_n and β_n so that the series is the “best possible” fit to $f(x)$
- There are many ways to define what “best possible” means.
- But one way to do it is to minimize the mean square deviation between the function and the series that fits the function.
- This is the basis for what we call a “least squares” fit.
- Stated mathematically, we want to minimize the integral

$$I = \int_{L}^{L} \left[f(x) - \beta_0 - \sum_{n=1}^{N} [\beta_n \cos(n\pi x/L) + \alpha_n \sin(n\pi x/L)] \right]^2 dx$$

Fourier series as a minimization problem

- The integral I can be rewritten in the following way using orthogonality of the sines and cosines

$$I = \int_{-L}^{L} f(x)^2 dx + L \sum_{k=1}^{N} \left(\alpha_k^2 + \beta_k^2 \right) +$$

$$2L\beta_0^2 - 2L \sum_{k=1}^{N} (\alpha_k A_k + \beta_k B_k) - 4L\beta_0 B_o$$

where A_k and B_k are the Fourier series coefficients:

$$A_k = \frac{1}{L} \int_{-L}^{L} f(x) \sin(k\pi x/L) dx$$

$$B_k = \frac{1}{L} \int_{-L}^{L} f(x) \cos(k\pi x/L) dx$$

$$B_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

Fourier series as a minimization problem

- Now notice that given the way we have defined I

$$I = \int_{-L}^{L} \left[f(x) - \frac{\beta_0}{2} + \sum_{n=1}^{N} \beta_n \cos(n\pi x/L) + \alpha_n \sin(n\pi x/L) \right]^2 dx$$

  there ought to be some set of coefficients that give us a minimum value for I .
- In the multidimensional space of the coefficients I has the shape of a bowl.
- We want to find the values of α_k and β_k that get us to the bottom of the bowl.
- At the bottom is where I is smallest and so the mean square error is smallest.

Fourier series as a minimization problem

- To get the values of α_k and β_k corresponding to the minimum I we search for where the derivatives of I with respect to the coefficients vanish:

$$\frac{\partial I}{\partial \alpha_k} = 0 \quad k = 1, \ldots N \qquad \frac{\partial I}{\partial \beta_k} = 0 \quad k = 0, \ldots N$$

- It is easy to check that this happens when

$$\alpha_k = A_k \qquad \beta_k = B_k$$

where

$$A_k = \frac{1}{L}\int_{-L}^{L} f(x)\sin(k\pi x/L)dx \qquad k \neq 0$$

$$B_k = \frac{1}{L}\int_{-L}^{L} f(x)\cos(k\pi x/L)dx \qquad k \neq 0$$

$$B_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx \qquad k = 0$$

Fourier series as a minimization problem

- This means that we have an alternative interpretation of the Fourier series over a periodic interval
- The Fourier series over a periodic interval is the series of sines and cosines that approximates $f(x)$ with the minimum mean square error.
- This statement is also not specific to Fourier series.
- You can see that the derivation really relies only on orthogonality
- So it actually can be applied to any family of Sturm-Liouville eigenfunctions coming from a regular Sturm-Liouville problem.
- More generally it can be applied to any family of S-L eigenfunctions that have discrete eigenvalues (which includes periodic and some singular S-L problems)