ACM 100b

Properties of the Fourier transform

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February 23, 2014

Recap

- In the previous lecture we introduced the Fourier transform pair:

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ikx) dx,$$
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \exp(ikx) dk$$

- We can transform any function which is integrable over the interval $-\infty < x < \infty$ to get the transform $F(k)$
- By thinking of F as a function of a complex variable k we can extend the transform to a wider class of functions.
- The original function can then be recovered by performing the second integral given $F(k)$.

Recap

- We saw that the Fourier transform is like an extension of Fourier series except for the singular situation of an infinite interval $-\infty < x < \infty$
- There is a completeness relation that bears some similarity to the completeness relation for discrete sets of complete functions:

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx) \exp(-ikx_0) dk$$

- In this lecture we will introduce some additional properties of the Fourier transform
- We will also see how the transform can be used to help solve various ODE's in the interval $-\infty < x < +\infty$
- We will then introduce some new transforms, the sine and cosine transforms appropriate for the interval $0 \leq x < \infty$

Linearity of the Fourier transform

- Fourier transforms are very useful for solving constant coefficient ODE's and PDE's on infinite domains.
- The most important property is linearity.
- Let $\mathcal{F}$ denote the operation of performing a Fourier transform:

$$\mathcal{F}[f(x)] \equiv \int_{-\infty}^{\infty} f(x) \exp(-ikx) dx.$$

- Then it is easy to see that

$$\mathcal{F}[f(x) + g(x)] = \mathcal{F}[f(x)] + \mathcal{F}[g(x)].$$

- It is this property plus the way the transform behaves under differentiation that makes the transform useful for linear problems.

Transform of a derivative

- Another important property is that derivatives of functions transform simply.
- It is customary to denote the transform of a function $y(x)$ by $Y(k)$.
- Assume also that the function under consideration $y(x)$ has the property that

$$y(x) \to 0 \text{ as } |x| \to \infty.$$

- Actually, this requirement can be relaxed by considering complex values of k and thinking of the transform as a contour integral.
- Now consider the transform of the derivative:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} y'(x) \exp(-ikx) dx.$$

Transform of a derivative

- Using integration by parts we have

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} y'(x)\exp(-ikx)dx = \frac{1}{\sqrt{2\pi}}\, y(x)\exp(-ikx)|_{-\infty}^{\infty} + ik\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-ikx)y(x)dx.$$

- As long as we have

$$y(x) \to 0 \text{ as } |x| \to \infty,$$

we see that

$$\mathcal{F}[y'(x)] = ikY(k).$$

Transform of a derivative

- And repeating this analysis for the second derivative

$$\mathcal{F}[y''(x)] = -k^2 Y(k),$$

as long as

$$y'(x) \to 0 \text{ as } |x| \to \infty.$$

- This property is very important in that it can be used to transform differential equations into algebraic equations.
- More about this later.
- One thing that is sometimes forgotten is that if you write

$$\mathcal{F}[y'(x)] = ikY(k).$$

you have actually *implicitly assumed*

$$y(x) \to 0 \text{ as } |x| \to \infty,$$

- This is a useful way to impose boundary conditions but you need to be aware that you are doing this.

The convolution theorem

- Let $\mathcal{F}[f(x)]$ be the Fourier transform for $f(x)$ and written as $F(k)$
- Correspondingly let $\mathcal{F}[g(x)]$ be given by $G(k)$.
- Define the *convolution* of f and g by

$$f \star g \equiv \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\zeta) g(x-\zeta) d\zeta.$$

- We assume f and g behave such that this integral exists.
- Then the *convolution theorem* for Fourier transforms states that

$$\mathcal{F}[f \star g] = F(k)G(k).$$

The convolution theorem

- To show this consider the integral

$$\int_{-\infty}^{\infty} F(k)G(k)\exp(ikx)dk,$$

- This integral can be manipulated as follows:

$$\begin{aligned}
\int_{-\infty}^{\infty} F(k)G(k)e^{ikx}dk &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(k)e^{ikx}dk\int_{-\infty}^{\infty} f(\zeta)e^{-ik\zeta}d\zeta \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\zeta)\int_{-\infty}^{\infty} G(k)e^{(ik(x-\zeta))}dk\,d\zeta \\
&= \int_{-\infty}^{\infty} f(\zeta)g(x-\zeta)d\zeta,
\end{aligned}$$

- Multiply both sides of this relation by $1/\sqrt{2\pi}$ and this shows the result.

The convolution theorem

- Consider the following convolution:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\zeta)\delta(x-\zeta)d\zeta.$$

- In other words we're convolving some function $f(x)$ with a δ function.
- We can apply the convolution theorem to take the Fourier transform of this integral.
- The Fourier transform of $f(x)$ can simply be labeled $F(k)$.
- The Fourier transform of $\delta(x)$ has been computed already:

$$\mathcal{F}[\delta(x)] = \frac{1}{\sqrt{2\pi}}.$$

The convolution theorem

- The convolution theorem tells us that

$$\frac{1}{\sqrt{2\pi}}\mathcal{F}\left[\int_{-\infty}^{\infty} f(\zeta)\delta(x-\zeta)d\zeta\right] = \frac{F(k)}{\sqrt{2\pi}}$$

- Now consider taking the inverse Fourier transform of both sides of this relationship
- From this we infer immediately that it must be that

$$\int_{-\infty}^{\infty} f(\zeta)\delta(x-\zeta)d\zeta = f(x),$$

- This is just the sifting property for the δ function we had derived earlier.

Parseval's theorem again

- The convolution theorem says

$$\int_{-\infty}^{\infty} F(k)G(k)\exp(ikx)dk = \int_{-\infty}^{\infty} f(\zeta)g(x-\zeta)d\zeta.$$

- Suppose we set $x = 0$ in the convolution theorem.
- We then get

$$\int_{-\infty}^{\infty} F(k)G(k)dk = \int_{-\infty}^{\infty} f(\zeta)g(-\zeta)d\zeta.$$

- Now set

$$g(-\zeta) = \overline{f(\zeta)},$$

  where we now want to think of f and g as complex.
- So this means

$$g(\zeta) = \overline{f(-\zeta)}$$

Parseval's theorem again

- The Fourier transform of g is given by

$$G(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \overline{f(-\zeta)} \exp(-ik\zeta) d\zeta$$

- From this definition of g and the definition of the Fourier transform we can infer that

$$G(k) = \overline{F(k)}.$$

- Substitute this into the result

$$\int_{-\infty}^{\infty} F(k)G(k)dk = \int_{-\infty}^{\infty} f(\zeta)g(-\zeta)d\zeta,$$

- We get

$$\int_{-\infty}^{\infty} |F(k)|^2 dk = \int_{-\infty}^{\infty} |f(x)|^2 dx.$$

Parseval's theorem again

- Recall when we were dealing with Fourier series we derived the Parseval relation:

$$\int_{-L}^{L} f(x)^2 dx = L\left[2B_0^2 + \sum_{n=1}^{\infty}\left(A_n^2 + B_n^2\right)\right]$$

- In this relation the A_n and B_n are the Fourier coefficients over the interval $-L \leq x \leq L$.
- The Parseval relation for the Fourier transform is the extension of this result to the fully infinite interval $-\infty < x < \infty$.