ACM 100b

Inverting the Laplace transform

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At our last lecture

- We introduced the Laplace transform defined by

$$F(s) = \mathfrak{L}[f(t)] = \int_0^\infty \exp(-st) f(t) dt$$

- This integral defines a function that exists for $s > a$.
- Or more accurately $\mathrm{Re}(s) > a$, since in what follows we will treat s as a complex variable.
- We showed that it has useful properties when you transform derivatives

$$\mathfrak{L}[f^n(t)] = s^n \mathfrak{L}[f(t)] - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{(n-2)}(0) - f^{(n-1)}(0).$$

- Using this you can turn certain ODE's into algebraic problems

Inverting the Laplace transform

- Once you solve for the transform you have solved for $F(s)$ where
$$F(s)=\int_0^{\infty}\exp(-st)f(t)dt.$$
- To solve the problem you need to find out $f(t)$ given $F(s)$.
- This is known as inverting the transform.
- There are basically two ways to invert the transform.
- One is to create tables and then solve for a given transform and look it up.
- There are very extensive compilations for such pairs.
- The second is to apply the *inverse Laplace transform.*
- For a given Laplace transform $\mathfrak{L}[f(t)]=F(s)$ the inverse transform is given by
$$f(t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}F(s)\exp(+st)ds.$$

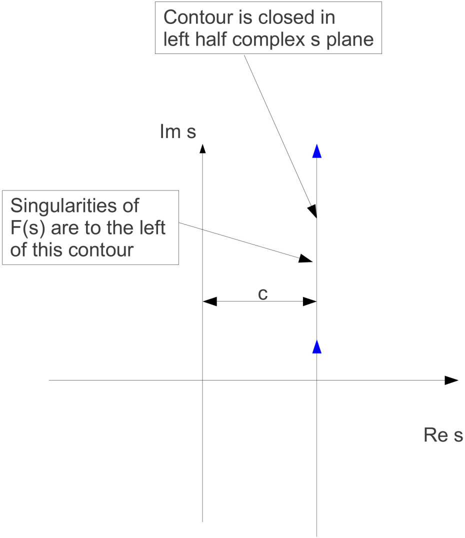
The inverse Laplace transform

- The notation above

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \exp(+st) ds.$$

  denotes a contour integral.
- The contour is called the *Bromwich contour*
- The contour originates for arbitrarily large negative imaginary values and proceeds to very large positive imaginary values.
- The number c is chosen to lie to the right of all singularities in the complex plane if $F(s)$ is viewed as a complex function.
- So the essential singularity of the exponential blowing up is on the right of the contour
- You close the contour on the left.

The Bromwich contour



Inverting the transform

- Note in the expression

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} F(s) \exp(+st) ds.$$

  that if $t < 0$, then we want the singularities to the right of c.
- That is so we can close the contour on the right.
- That there is such a value of c for all functions of exponential order can be seen by looking at

$$F(s) = \mathfrak{L}[f] = \int_0^{\infty} f(t) \exp(-st) dt.$$

- If $|f(t)| < K \exp(at)$ for some a, then $F(s)$ must be analytic for $\text{Re}(s) > a$.

Showing the inverse transform undoes the transform

- Here is a plausible (but not terribly rigorous) "verification" of the inverse transform.
- Consider taking the Laplace transform of the inverse expression.
- We should get back the original transform of the function if these expressions are really inverses of one another
- Consider then

$$\frac{1}{2\pi i}\int_0^\infty \exp(-st)dt \int_{c-i\infty}^{c+i\infty} F(z)\exp(zt)dz$$
$$= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(z)dz \int_0^\infty \exp[(z-s)t]dt$$

- And as long as $\mathrm{Re}(s) > c$ this reduces to

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \frac{F(z)}{s-z}dz.$$

Showing the inverse transform undoes the transform

- We now make the contour finite by going from $c - iR$ to $c + iR$.
- Then we closing it in the right half plane by adding a semicircle of radius R.
- We see then that for this contour there is only one singularity for the integrand at $z = s$.
- We selected c to lie to the right of all singularities of F and so F is analytic in the semicircular region.
- This leads to

$$\frac{1}{2\pi i}\int_{c-iR}^{c+iR}\frac{F(z)}{s-z}dz = F(s) + \frac{1}{2\pi i}\int_{C_R}\frac{F(z)}{s-z}dz.$$

Showing the inverse transform undoes the transform

- Now we want to do the contour integral

$$\frac{1}{2\pi i}\int_{C_R}\frac{F(z)}{s-z}dz.$$

- We can bound $F(z)$ as follows:

$$\begin{aligned}
F(z) &= \int_0^\infty f(t)\exp(-st)dt \\
&= \int_0^T f(t)\exp(-zt)dt + \int_T^\infty f(t)\exp(-zt)dt \\
&\leq \int_0^T M\exp(-zt)dt + \int_T^\infty K\exp(at)\exp(-zt)dt \\
&\leq \frac{M}{z}\exp(-zt)\Big|_0^T + \frac{1}{a-z}(-K)\exp((a-z)T).
\end{aligned}$$

- Clearly then the integrand vanishes over the semicircular part as $R\to\infty$

Showing the inverse transform undoes the transform

- So recall we had

$$\frac{1}{2\pi i}\int_{c-iR}^{c+iR}\frac{F(z)}{s-z}dz = F(s) + \frac{1}{2\pi i}\int_{C_R}\frac{F(z)}{s-z}dz.$$

- From our derivation above we now have

$$\frac{1}{2\pi i}\int_{c-iR}^{c+iR}\frac{F(z)}{s-z}dz = F(s) \qquad \text{as } R \to \infty$$

- So the two expressions are a transform pair:

$$\begin{aligned} F(s) &= \int_0^\infty \exp(-st)f(t)dt. \\ f(t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} F(s)\exp(st)ds \end{aligned}$$

- But where did these expressions come from?
- It turns out the Laplace transform originates from the well known Fourier transform which we will study later.