ACM 100b

Structure of the Sturm-Liouville eigenfunctions

Dan Meiron

Caltech

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Zeroes of the eigenfunctions

- We next look at proving results about the zeroes of the eigenfunctions.
- Recall we showed in two examples that the eigenfunction corresponding to the smallest eigenvalue had no internal zeroes
- Then the eigenfunction corresponding to the next eigenvalue has one internal zero - and so forth.
- We will next see how the Lagrange identity strongly constrains how the eigenfunctions behave.
- In particular, it's possible to get some fairly strong results on where the zeroes of the eigenfunctions have to be.
- To get at this we need some preliminary results.

The Sturm comparison theorem

- Let $u(x)$ and $v(x)$ be two functions and recall the Sturm-Liouville ODE:

$$L[y] = \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y(x) = -r(x)\lambda y(x) \qquad a \leq x \leq b.$$

- Note that L is the negative of what we usually use.
- Define the auxiliary function $w(x)$ by

$$w(x) = u\frac{dv}{dx} - v\frac{du}{dx}.$$

- We will show that if

$$uL[v] \geq vL[u] \quad \text{and} \quad p(a)w(a) \geq 0 \geq p(b)w(b)$$

then

$$w(x) = 0.$$

The Sturm comparison theorem

- We'll then use this result to compare eigenfunctions with varying values of λ.
- To show why this result holds, assume that we do have

$$uL[v] \geq vL[u].$$

- Using the ODE this implies that

$$u(x)\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) + u(x)q(x)v(x) \geq v(x)\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + v(x)q(x)u(x)$$

Differential form of Lagrange's identity

- Previously we showed

$$\int_a^b \{L[u(x)]v(x) - u(x)L[(v(x)]\}\, dx =$$
$$-\left\{p(x)\left[\frac{du}{dx}v - u\frac{dv}{dx}\right]\right\}\Big|_a^b .$$

- This is the integral form of Lagrange's identity over an interval $a \leq x \leq b$
- But you can see the result is still true if $a \leq x' \leq x$ where the interval is now variable in x:

$$\int_a^x \{L[u(x')]v(x') - u(x')L[(v(x')]\}\, dx' =$$
$$-\left\{p(x')\left[\frac{du}{dx'}v - u\frac{dv}{dx'}\right]\right\}\Big|_a^x .$$

Differential form of Lagrange's identity

- Now differentiate both sides of

$$\int_a^x \left\{L[u(x')]v(x') - u(x')L[(v(x'))]\right\} dx' = -\left\{p(x')\left[\frac{du}{dx'}v - u\frac{dv}{dx'}\right]\right\}\Big|_a^x .$$

  with respect to x.
- We get the differential form of Lagrange's identity:

$$uL[v] - vL[u] = \frac{d}{dx}(p(x)w(x))$$

  where $w(x)$ was introduced earlier

$$w(x) = u\frac{dv}{dx} - v\frac{du}{dx}.$$

The Sturm comparison theorem

- Now recall the inequality we had

$$u(x)\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right)+u(x)q(x)v(x)\geq$$
$$v(x)\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+v(x)q(x)u(x)$$

- And look at the differential form of the Lagrange identity:

$$uL[v]-vL[u]=\frac{d}{dx}(p(x)w(x))$$

- Put these results together to get

$$\frac{d}{dx}(p(x)w(x))\geq 0.$$

The Sturm comparison theorem

- This result
$$\frac{d}{dx}(p(x)w(x)) \geq 0.$$
means that $p(x)w(x)$ must be a non-decreasing function of x.
- Now look at the end point $x = a$ where we assumed
$$p(a)w(a) \geq 0$$
- We just showed $p(x)w(x)$ is a non-decreasing function so
$$0 \leq p(a)w(a) \leq p(x)w(x) \leq p(b)w(b).$$
- But earlier the assumption is made that
$$p(a)w(a) \geq 0 \geq p(b)w(b).$$
- But if both these are true then
$$0 \leq p(a)w(a) \leq p(x)w(x) \leq p(b)w(b) \leq 0.$$
- This means $w(x)$ is both less than or equal and greater than or equal to 0, so $w(x) = 0$

The Sturm comparison theorem

Theorem (Sturm comparison theorem)

Suppose $u(x)$ and $v(x)$ are solutions of the S-L ODE but for any value of λ. Let α, β be two consecutive zeroes of $v(x)$ Suppose also on the interval $\alpha < x < \beta$ we have

$$\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u(x) + P(x)u(x) = 0$$
$$\frac{d}{dx}\left(p(x)\frac{dv}{dx}\right) + q(x)v(x) + Q(x)v(x) = 0$$

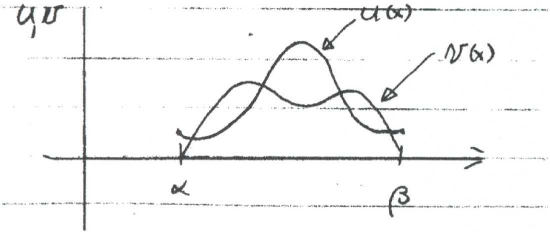
with

$$P(x) \geq Q(x) \qquad \alpha < x < \beta,$$

then either $u(x)$ has a zero on $\alpha < x < \beta$ or $u(x) = cv(x)$ where c is a constant.

Proof of the Sturm comparison theorem

- To show this result we show that assuming otherwise leads to a contradiction.
- Suppose $u(x)$ had no zero on $\alpha < x < \beta$.
- Then $u(x)$ is either all positive or all negative on that interval.
- Given that the Sturm-Liouville ODE is homogeneous we can adjust the overall sign of $u(x)$ and $v(x)$ so that their respective graphs on the interval $\alpha < x < \beta$ must look as follows



Proof of the Sturm comparison theorem

- In the figure the function $v(x)$ is zero at $x = \alpha$ and $x = \beta$ by assumption.
- We see that $u(x)$ can be above or below $v(x)$
- But $u(x)$ must start and end above $v(x)$ at the end points
- And it cannot cross the x-axis.
- In any case we must have

$$uL[v] - vL[u] = (P - Q)uv \geq 0 \qquad \alpha < x < \beta$$

  because we assumed that $P \geq Q$.
- But if we look at the figure we see that because of the assumptions we're also constrained to have

$$\begin{aligned} u(\alpha) &\geq 0 & \qquad v'(\alpha) &\geq 0 \\ u(\beta) &\geq 0 & \qquad v'(\beta) &\leq 0 \end{aligned}$$

Proof of the Sturm comparison theorem

- If we consider the function $w(x) = uv' - vu'$ this means

$$w(\alpha) \geq 0 \qquad w(\beta) \leq 0$$

- But from the the previous result which constrains $w(x)$ we must have

$$w(x) = 0,$$

which implies

$$uv' - vu' = 0 \text{ or } \frac{u'}{u} = \frac{v'}{v}$$

which then implies

$$u = cv$$

where c is a constant.
- This means u and v are essentially the same function.
- The only way to avoid this when $P > Q$ is to let there be some zeros in $u(x)$ between the consecutive zeros of $v(x)$.

Application of the comparison theorem

- We can now use the Sturm comparison theorem to show statements such as the following:

Theorem (Interlacing of roots for S-L eigenfunctions)

Let $u(x)$ be an eigenfunction with eigenvalue λ and $v(x)$ be an eigenfunction with eigenvalue μ. If $\lambda > \mu$ there is at least one zero of u between any two consecutive zeroes of v.

- To show this take $P = \lambda r$ and $Q = \mu r$ in the Sturm comparison theorem.
- Since $\lambda > \mu$ it must be that $P > Q$ and so u must have a zero between consecutive zeroes of v.
- The alternative is that $u = cv$ which means we don't have two eigenfunctions as u is really v
- So the two eigenvalues must correspond to different eigenfunctions.

Application of the comparison theorem

- Another result of this type is as follows

Theorem (Eigenfunctions and the number of zeroes)

Let $u(x)$ be an eigenfunction with eigenvalue λ and let $v(x)$ be an eigenfunction with eigenvalue μ. Suppose u and v have exactly n zeroes in the interval $a < x < b$. Then it must be that $\lambda = \mu$, and that u and v are linearly dependent and have the same zeroes.

- To show this consider the case with $n \geq 2$. The zeros x_j of $v(x)$ divide the interval $a < x < b$ into intervals

$$(a, x_1) \quad (x_1, x_2) \quad \cdots \quad (x_n, b)$$

- Now the separable boundary conditions make

$$p(a)w(a) = p(b)w(b) = 0$$

Application of the comparison theorem

- Suppose we have an eigenvalue $\lambda > \mu$ but we still have n zeros.
- Then the eigenfunction $u(x)$ has at least one zero on each interval made between the zeros of $v(x)$.
- That means $u(x)$ will have too many zeroes.
- If $\lambda < \mu$ then you can see $u(x)$ will end up with too few zeroes.
- This means the only way to have n zeroes for u is for $\lambda = \mu$ and so it must be that $u = v$.
- This exemplifies the remarkable range of results we can get from the special properties of the Sturm-Liouville ODE's.
- So far everything we said relies on the conditions that the boundary conditions are separable and that $p(x) > 0$.
- When we relax any of these assumptions we lose some of the guarantees.