Physics 106a — Classical Mechanics

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Lecture 5

Hamilton's Principle for Constrained Dynamics

Outline

- Hamilton's Principle with constraints
- Method of Lagrange multipliers
- Lagrange multipliers and constraint forces
- Application to nonholonomic constraints

Hamilton's principle

The physical path $q_k(t)$ is the one for which the action S is stationary

$$S = \int_{t_i}^{t_f} L(\{\dot{q}_k\}, \{q_k\}, t)\, dt$$

System with constraints

$3M$ elementary variables $\{q_k\}$, $k = 1, \ldots 3M$ define the state of system before taking into account constraints

N_c holonomic constraints

$$G_j(q_1, q_2, \ldots q_{3M}, t) = 0 \qquad j = 1 \ldots N_c$$

Action is stationary

$$\delta S = \int_{t_i}^{t_f} \sum_{k=1}^{3M} \frac{\delta L}{\delta q_k} \delta q_k \, dt = 0 \qquad \text{with} \qquad \frac{\delta L}{\delta q_k} = \frac{\partial L}{\partial q_k} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k}$$

for path changes $\{\delta q_k\}$ satisfying constraints.

For these changes the constraint forces are not needed to evaluate Lagrangian.

But the $3M$ coordinates δq_k not independent: cannot conclude $\delta L/\delta q_k = 0$

Method 1

Find some reduced number $N = 3M - N_c$ of generalized coordinates $\{\bar{q}_k, k = 1 \ldots N\}$ such that we can vary them independently and each variation is consistent with the constraints.

We might be able to simply choose $\{q_k\}, k = 1 \ldots N$ and vary $\{q_k\}, k = N + 1 \ldots 3M$ to maintain the constraints.

Evaluate the Lagrangian for the constrained motion in terms of $\{\dot{\bar{q}}_k\}$, $\{\bar{q}_k\}$, $k = 1 \ldots N$.

In the action variation, the $\{\delta\bar{q}_k\}, k = 1 \ldots N$ are independent, and so

$$\frac{\partial L}{\partial \bar{q}_k} - \frac{d}{dt}\frac{\partial L}{\partial \dot{\bar{q}}_k} = 0 \qquad k = 1 \ldots N$$

The derivatives of the Lagrangian are calculated with the constraints satisfied, and so the constraint forces are not involved.

This is what we did before.

Method 2: Lagrange multipliers

- Introduce the modified action

$$\bar{S} = \int_{t_i}^{t_f} \left[L + \sum_{j=1}^{N_c} \lambda_j(t) G_j(\{q_k\}, t) \right] dt$$

with arbitrary *Lagrange multipliers* $\lambda_j(t)$.

- Apply stationary condition to $\bar{S}$

$$\delta \bar{S} = \int_{t_i}^{t_f} \sum_{k=1}^{3M} \left[\frac{\delta L}{\delta q_k} + \sum_{j=1}^{N_c} \lambda_j \frac{\partial G_j}{\partial q_k} \right] \delta q_k \, dt = 0$$

- Treat all $3M$ coordinates δq_k as independent

$$\frac{\delta L}{\delta q_k} + \sum_{j=1}^{N_c} \lambda_j \frac{\partial G_j}{\partial q_k} = 0$$

- Solve together with constraints

$$G_j(\{q_k\}, t) = 0$$

Why does this work?

- $\bar{S} = S$ for paths satisfying the constraints ($G = 0$): making S stationary for path variations satisfying the constraints is certainly the same as making $\bar{S}$ stationary for such variations

- In

$$\delta \bar{S} = \int_{t_i}^{t_f} \sum_{k=1}^{3M} \left[\frac{\delta L}{\delta q_k} + \sum_{j=1}^{N_c} \lambda_j \frac{\partial G_j}{\partial q_k} \right] \delta q_k \, dt = 0$$

  choose N_c values of $\lambda_j(t)$ so that N_c of the $[] = 0$ (e.g. $k = N + 1 \ldots 3M$)

- For remaining N terms may take δq_k to be independent so that again $[] = 0$

- Hence

$$\frac{\delta L}{\delta q_k} + \sum_{j=1}^{N_c} \lambda_j \frac{\partial G_j}{\partial q_k} = 0 \qquad \text{for all } k$$

Nice result for formalists

Make the modified action

$$\bar{S} = \int_{t_i}^{t_f} \left[L + \sum_{j=1}^{N_c} \lambda_j(t) G_j(\{q_k\}, t) \right] dt$$

stationary with respect to path variations *and* variations of $\lambda_j(t)$

- path variations give

$$\frac{\delta L}{\delta q_k} + \sum_{j=1}^{N_c} \lambda_j \frac{\partial G_j}{\partial q_k} = 0, \qquad k = 1 \ldots 3M$$

- λ_j variations give

$$G_j(\{q_k\}, t) = 0, \qquad j = 1 \ldots N_c$$

Physical significance of λ_j

Return to generalized equation of motion in terms of $3M$ coordinates and *all* forces

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = \sum_i \vec{F}_i^{\mathrm{nc}} \cdot \frac{\partial \vec{r}_i}{\partial q_k} + \sum_i \vec{F}_i^{\mathrm{c}} \cdot \frac{\partial \vec{r}_i}{\partial q_k}$$

$\vec{F}_i^{\mathrm{nc}}$ derives from the "external" potential V we know, and is transferred to the left hand side to give the conventional Lagrangian L

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_k} - \frac{\partial L}{\partial q_k} = \sum_i \vec{F}_i^{\mathrm{c}} \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \mathcal{F}_k^c$$

Comparing with the Lagrange multiplier equation

$$\mathcal{F}_k^c = \sum_{j=1}^{N} \lambda_j \frac{\partial G_j}{\partial q_k}$$

$\lambda_j \Rightarrow$ strength of the constraint forces normal to the constraint surface G_j

Nonholonomic constraints

Only needed the constraints in differential form, and so Lagrange multiplier method works with nonintegrable differential nonholonomic constraints.

For changes satisfying constraints of the form

$$\sum_{k=1}^{3M} g_{jk}\delta q_k = 0 \qquad \text{or equivalently} \qquad \sum_{k=1}^{3M} g_{jk}\dot{q}_k = 0$$

(where g_{jk} may depend on $\{q_l\}, t$) we have

$$\delta\bar{S} = \int_{t_i}^{t_f} \sum_{k=1}^{3M} \left[\frac{\delta L}{\delta q_k} + \sum_{j=1}^{N_c} \lambda_j g_{jk} \right] \delta q_k \, dt = 0$$

and the argument proceeds as before.

Example: rolling wheel in two dimensions

Caution with nonholonomic constraints

Correct procedure:

1. Derive Euler-Lagrange equations using unconstrained variables and Lagrange multipliers
2. Solve together with constraint equations

For nonholonomic constraints, in general it is *incorrect* to use the constraints to eliminate variables from the Lagrangian (Hand and Finch do this in §2.8)

For a mathematical discussion of this rather subtle point see pp 274-6 of *A Mathematical Introduction to Robot Manipulation* by Murray, Li, and Shastry (pdf copy on Richard Murray's website)