ACM 100c

Fundamental solutions for linear second order ODE's

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Fundamental sets of solutions

- As we stated earlier because we're considering linear second order ODE we know the solution appears in the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x),$$

- Here $y_1(x)$ and $y_2(x)$ are homogeneous solutions satisfying

$$\begin{aligned} y_1'' + p(x)y_1' + q(x)y_1 &= 0 \\ y_2'' + p(x)y_2' + q(x)y_2 &= 0 \end{aligned}$$

- And $y_P(x)$ is the particular solution which satisfies

$$y_P'' + p(x)y_P' + q(x)y_P = r(x).$$

- So we have a two parameter set of solutions but the key question is whether this is enough to solve the general IVP?

Fundamental sets of solutions

- We focus first on the homogeneous IVP problem

$$y'' + p(x)y' + q(x)y = 0 \qquad y(x_0) = y_0, \quad y'(x_0) = y'_0.$$

Definition

Two solutions of the homogeneous problem $y_1(x)$ and $y_2(x)$ are called a *fundamental set* if every solution of the initial value problem can be expressed as a linear combination of $y_1(x)$ and $y_2(x)$.

Fundamental sets of solutions

- To determine if two solutions form a fundamental set, we need to show that for every solution $y = \phi(x)$ we can find two constants c_1 and c_2 such that

$$\phi(x) = c_1 y_1(x) + c_2 y_2(x).$$

- Both y_1 and y_2 satisfy the homogeneous ODE but we also have to satisfy the conditions of the IVP so we also want

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= y_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= y_0'. \end{aligned}$$

- If we can solve this 2×2 system we can get c_1 and c_2 and thus our solution $\phi(x)$.
- This is the analog to the condition for existence of the solutions for IVP's in the first order case.

The Wronskian

- If we can solve this 2×2 system

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= y_0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= y_0'. \end{aligned}$$

  we can get c_1 and c_2 and thus our solution $\phi(x)$.
- We know from linear algebra that the above equations have unique solutions provided the determinant

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0) \neq 0$$

  for all x_0 in the interval $\alpha < x < \beta$.
- This determinant is called the *Wronskian*

Example

- Consider the ODE

$$y'' + y = 0 \qquad -\infty < x < \infty$$

- This ODE has two solutions

$$y_1(x) = \sin(x) \qquad y_2(x) = \cos(x)$$

- If we calculate the determinant

$$W(y_1, y_2) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix}$$

we see that this is given by

$$\begin{vmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{vmatrix} = -1$$

- This determinant never vanishes and so these two solutions form a fundamental set for the ODE over any finite interval.

Abel's theorem

Theorem

If the functions $p(x)$ and $q(x)$ in

$$y'' + p(x)y' + q(x)y = 0 \qquad y(x_0) = y_0, \quad y'(x_0) = y'_0.$$

are continuous on the interval $\alpha < x < \beta$ and if $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0,$$

then the Wronskian $W(y_1, y_2)$ is either zero everywhere in the interval or else is never zero in the interval.

Implications of Abel's theorem

- This means the following
- If the coefficient functions $p(x)$ and $q(x)$ are continuous on the interval $\alpha < x < \beta$
- And $y_1(x)$ and $y_2(x)$ are solutions of the homogeneous ODE

$$y'' + p(x)y' + q(x)y = 0,$$

- And if there exists at least one point where $W(y_1, y_2) \neq 0$ then any solution of the ODE can be expressed as

$$y = c_1 y_1(x) + c_2 y_2(x).$$

Building a fundamental set of solutions

- This result also lets us build fundamental sets of solutions at any point x_0 in the interval $\alpha < x < \beta$.
- There are lots of ways to do this but one way is to solve the two IVPs

$$
\begin{array}{rclccc}
y_1'' + p(x)y_1' + q(x)y_1 & = & 0 & \qquad y_1(x_0) = 1 & \qquad y_1'(x_0) = 0 \\
y_2'' + p(x)y_2' + q(x)y_2 & = & 0 & \qquad y_2(x_0) = 0 & \qquad y_2'(x_0) = 1
\end{array}
$$

- By construction $W(y_1, y_2) = 1$ at $x = x_0$.
- So as long as the coefficient functions are smooth in the interval of interest the Wronskian cannot vanish there so y_1 and y_2 form a fundamental set.

Proof of Abel's theorem

- The proof of Abel's theorem is based on the fact that even if we don't know y_1 and y_2 we can explicitly solve for $W(y_1, y_2)$.
- To see this consider the two equations

$$\begin{aligned} y_1'' + py_1' + qy_1 &= 0 \\ y_2'' + py_2' + qy_2 &= 0 \end{aligned}$$

and multiply the first by y_2 and the second by y_1.

- Subtract one from the other to get

$$(y_1 y_2'' - y_2 y_1'') + p(x)(y_1 y_2' - y_2 y_1') = 0$$

Proof of Abel's theorem

- Now note that

$$\begin{aligned} W(x) &= y_1 y_2' - y_2 y_1' \\ \frac{dW(x)}{dx} &= y_1 y_2'' - y_2 y_1''. \end{aligned}$$

- So the equation

$$(y_1 y_2'' - y_2 y_1'') + p(x)(y_1 y_2' - y_2 y_1') = 0$$

becomes

$$\frac{dW}{dx} + p(x)W = 0,$$

Proof of Abel's theorem

- But this is a first order ODE we can solve:

$$W(y_1, y_2) = c \exp \left[-\int^x p(t) dt \right],$$

  where c is some arbitrary constant.
- Since the integral is in the exponential the only way the Wronskian could go to zero from this part of the expression is if the integral blows up somehow.
- But that would mean $p(x)$ certainly is not continuous in the interval of interest and we had assumed it was continuous.
- We can clearly see that the Wronskian can never vanish unless $c = 0$ and in that case it's zero everywhere.