Physics 106a — Classical Mechanics

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Lecture 17: Rigid Body Rotation Examples

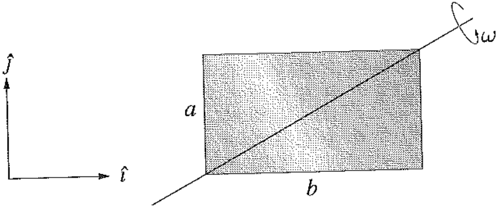
Examples

1. Rotating rectangle (Hand and Finch Problem 8-15)
2. Euler's disk
3. Ball on rotating turntable
4. Top on frictionless table

Rotating sheet

Hand and Finch Problem 8-15

A thin rectangular sheet of dimensions $a \times b$ and mass M is rotating about a diagonal with constant angular velocity $\vec{\omega}$. What is the torque?



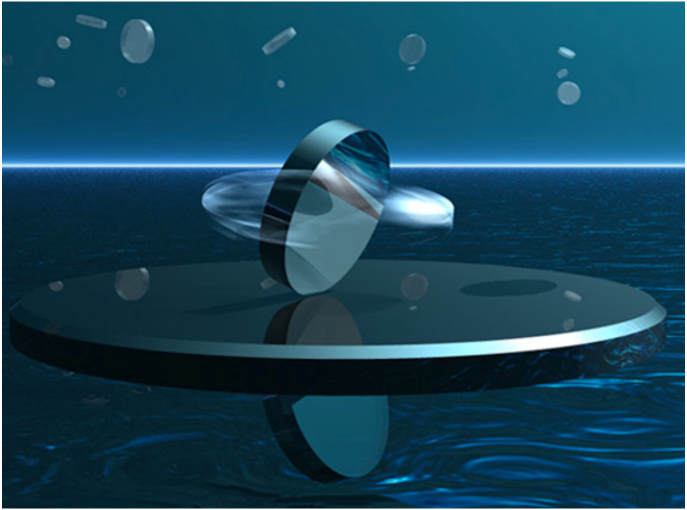
1. Use Euler's equations in the body frame
2. Use $d\vec{L}/dt = \vec{N}$ in space frame

Euler's equations

Equations of motion of *body frame* components of angular velocity along principal axes

$$
\begin{aligned}
I_1 \frac{d\omega_1}{dt} - \omega_2\omega_3(I_2 - I_3) &= N_1, \\
I_2 \frac{d\omega_2}{dt} - \omega_3\omega_1(I_3 - I_1) &= N_2, \\
I_3 \frac{d\omega_3}{dt} - \omega_1\omega_2(I_1 - I_2) &= N_3,
\end{aligned}
$$

Euler's disk/spinning coin



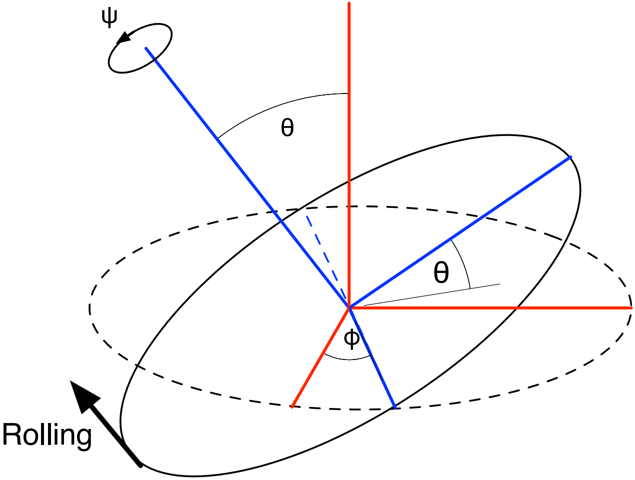
http://www.eulersdisk.com

Euler's disk/spinning coin

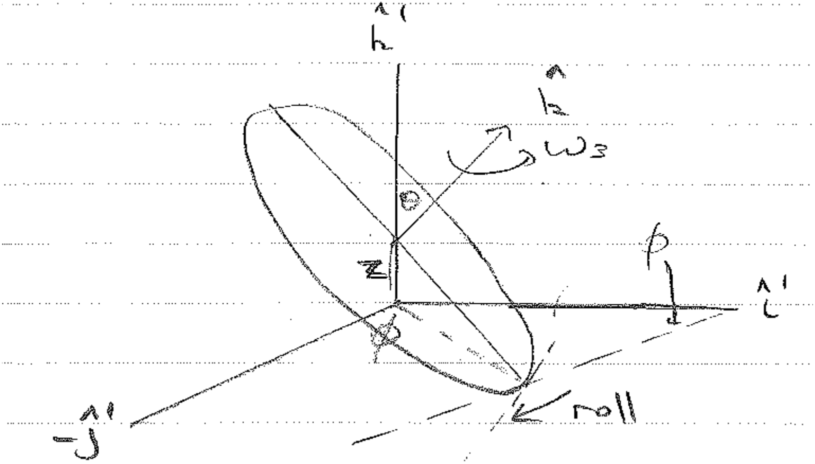
Observations

1. Wobble frequency depends on height of center of mass but not initial spin rate
2. Wobble frequency increases with time
3. Wobble frequency may diverge and motion stops in finite time
4. Rotation rate of face on coin decreases in time

Euler's disk



Euler's disk



Euler angle expressions

Angular velocity components with respect to principal (body frame) axes

$$
\begin{aligned}
\omega_1 \equiv \omega_x &= \dot{\phi} \sin\psi \sin\theta + \dot{\theta} \cos\psi \\
\omega_2 \equiv \omega_y &= \dot{\phi} \cos\psi \sin\theta - \dot{\theta} \sin\psi \\
\omega_3 \equiv \omega_z &= \dot{\psi} + \dot{\phi} \cos\theta
\end{aligned}
$$

Kinetic energy for axially-symmetric body $I_1 = I_2 = I_\perp$

$$
T = \frac{1}{2} I_\perp (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2} I_3 (\dot{\psi} + \dot{\phi} \cos\theta)^2
$$

Angular velocity components with respect to inertial (space frame) axes

$$
\begin{aligned}
\omega_{x'} &= \dot{\psi} \sin\phi \sin\theta + \dot{\theta} \cos\phi \\
\omega_{y'} &= -\dot{\psi} \cos\phi \sin\theta + \dot{\theta} \sin\phi \\
\omega_{z'} &= \dot{\psi} \cos\theta + \dot{\phi}
\end{aligned}
$$

Euler's disk - full equations

Lagrangian

$$L = \tfrac{1}{2} I_\perp (\dot\theta^2 + \sin^2\theta\, \dot\phi^2) + \tfrac{1}{2} I_3 (\dot\psi + \dot\phi\cos\theta)^2 + \tfrac{1}{2} M (\dot X^2 + \dot Y^2 + R^2\cos^2\theta\,\dot\theta^2) - MgR\sin\theta$$

Differential constraints

$$\delta X + R\cos\phi(\cos\theta\,\delta\phi + \delta\psi) - R\sin\theta\sin\phi\,\delta\theta = 0$$
$$\delta Y + R\sin\phi(\cos\theta\,\delta\phi + \delta\psi) + R\sin\theta\cos\phi\,\delta\theta = 0$$

Euler-Lagange equations

$$\frac{d}{dt}\left(I_\perp\dot\theta + MR^2\cos^2\theta\dot\theta\right) - I_\perp\dot\phi^2\sin\theta\cos\theta + I_3\omega_3\cos\theta\dot\phi + MgR\cos\theta$$
$$- \lambda_X R\sin\theta\sin\phi + \lambda_Y R\sin\theta\cos\phi = 0$$
$$\frac{d}{dt}\left(I_\perp\dot\phi\sin^2\theta + I_3\omega_3\cos\theta\right) + \lambda_X R\cos\phi\cos\theta + \lambda_Y R\sin\phi\cos\theta = 0$$
$$\frac{d}{dt}(I_3\omega_3) + \lambda_X R\cos\phi + \lambda_Y R\sin\phi = 0$$
$$M\ddot X + \lambda_X = 0$$
$$M\ddot Y + \lambda_Y = 0$$

Euler's disk - footnotes

1. Discussion by H. K. Moffat [Nature **404**, 833 (2000)] for
   - Dissipation due to air viscosity and the finite time singularity $\Omega \sim (t_0 - t)^{-1/6}$ with $t_0 \sim 100$ secs
   - Elimination of the singularity when the vertical acceleration of the rim exceeds g
2. Long paper on the Newtonian approach by Alexander J. McDonald and Kirk T. McDonald (see website)

Ball rolling on rotating turntable

What is the motion of a solid ball rolling, without slipping, on a rotating turntable?

- Motion is a circle and the period is always 7/2 times that of the turntable!
- Analogies with charged particle in magnetic field
- For more details see the paper by J. A. Burns, Am. J. Phys. **49**, 56 (1981).