ACM 100b

Using Greens functions to solve inhomogeneous linear ODEs

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Recap

- In the previous lecture we introduced the concept of the Green's function for inhomogeneous ODE's with homogeneous boundary conditions
- We looked at the S-L ODE
$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda r(x)y = r(x)f(x), \qquad a < x < b,$$
with boundary conditions $y(a) = 0$, $y(b) = 0$.
- Rather than solve this problem we considered
$$\frac{d}{dx}\left(p(x)\frac{dG}{dx}\right) - q(x)G(x) + \lambda r(x)G = \delta(x - x'), \qquad a < x < b,$$
where $G(x, x')$ is the Green's function and $\delta(x)$ is the delta function.
- The Green's function satisfies homogeneous boundary condition (BC's)

Recap

- We also introduced the δ function:

$$\delta(t) = \begin{cases} 0 & t < 0 \\ \infty & t = 0 \\ 0 & t > 0 \end{cases} .$$

- But $\delta(t)$ has a finite integral:

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

- In fact the integral is the same as long as the interval of integration includes $t = 0$.

Using Greens functions to solve inhomogeneous ODE's

- Greens functions can be used to solve inhomogeneous linear ODE's with homogeneous boundary conditions
- This is independent of their relationship to the Sturm-Liouville problem
- But there is an important relationship with Sturm-Liouville theory that we will discuss later.
- Suppose we are given an inhomogeneous ODE of the form

$$y'' + p(x)y' + q(x)y = f(x) \qquad a \leq x \leq b, \qquad y(a) = y(b) = 0.$$

- From the sifting property derived above we can write

$$f(x) = \int_a^b f(x')\delta(x - x')dx',$$

and so our ODE above becomes

$$y'' + p(x)y' + q(x)y = \int_a^b f(x')\delta(x - x')dx', \qquad a < x < b$$

Using Greens functions to solve inhomogeneous ODE's

- Now suppose we could compute a special solution $G(x, x')$ defined by the inhomogeneous problem

$$\frac{d^2G}{dx^2} + p(x)\frac{dG}{dx} + q(x)G = \delta(x - x'), \qquad a \le x \le b,$$
$$G(a) = G(b) = 0$$

- Then, by superposition, the solution to the inhomogeneous ODE

$$y'' + p(x)y' + q(x)y = \int_a^b f(x')\delta(x - x')dx', \qquad a \le x \le b,$$
$$y(a) = y(b) = 0,$$

is

$$y(x) = \int_a^b G(x, x')f(x')dx'.$$

Using Greens functions to solve inhomogeneous ODE's

- You can see that $G(x, x')$ satisfies the (homogeneous) boundary conditions and that

$$\frac{d^2 G}{dx^2} + p(x)\frac{dG}{dx} + q(x)G = \delta(x - x'),$$

- If we then perform the operations associated with computing the left hand side of the ODE above on the integral

$$y(x) = \int_a^b G(x, x')f(x')dx'$$

we get back the right hand side $f(x)$.

Computing the Greens function

- Let's do a simple example to illustrate how to compute a Greens function for an ODE
- Consider the ODE

$$y'' = f(x) \qquad a \le x \le b \qquad y(a) = y(b) = 0$$

- Of course there are lots of ways to solve this but we will use the Greens function approach.

Computing the Greens function

- To start we define the Greens function by

$$\frac{d^2G}{dx^2} = \delta(x - x') \qquad G(a) = 0 \quad G(b) = 0$$

- Two linearly independent homogeneous solutions of this ODE are just

$$y_1 = 1 \qquad y_2 = x$$

- Now we know the Greens function satisfies the homogeneous boundary conditions at $x = a$ and $x = b$
- We also know the derivative of the Green's function jumps when you cross $x = x'$
- We see that

$$\int_{x'-\epsilon}^{x'+\epsilon} \frac{d^2G(x,x')}{dx^2}dx = \int_{x'-\epsilon}^{x'+\epsilon} \delta(x - x')dx = 1$$

Computing the Green's function

- So

$$\left.\frac{dG}{dx}\right|_{x'+\epsilon} - \left.\frac{dG}{dx}\right|_{x'-\epsilon} = 1$$

- Also two integrals of the delta function give a continuous function (a ramp function) so G is continuous at $x = x'$
- We can then write

$$G = \begin{cases} A(x-a)(x'-b) & x < x' \\ A(x'-a)(x-b) & x > x' \end{cases}$$

- This satisfies the boundary conditions at $x = a$ and $x = b$ and is continuous at $x = x'$
- Note how we did this using homogeneous solutions that satisfy the boundary conditions at one of the end points at a time.

Computing the Greens function

- We just need to figure out the constant A
- At $x = x'$ we know the derivative must jump so compute the derivative from the right and left and take the difference:

$$A(x' - a) - A(x' - b) = 1 \quad \text{so } A = \frac{1}{b - a}$$

- Finally we can write our solution:

$$\begin{aligned} y(x) &= \int_a^b G(x, x')f(x')dx' \\ &= \int_a^x G(x, x')dx' f(x')dx' + \int_x^b G(x, x')dx' f(x') \end{aligned}$$

- You can write this integral out to see it gives the same result as standard approaches to solve this problem.

Computing the Green's function

- We will consider solving for the Green's function more generally and provide some formulas for it.
- Consider the ODE for the Green's function

$$G'' + P(x)G' + Q(x)G = \delta(x - x')$$

- Suppose we have the two homogeneous solutions to the ODE

$$y'' + P(x)y' + Q(x)y = 0$$

- Call them $y_1(x)$ and $y_2(x)$
- Assume also that $y_1(a) = 0$ and $y_2(b) = 0$
- That is, $y_1(x)$ satisfies the homogeneous BC at $x = a$ but not at $x = b$
- And $y_2(x)$ satisfies the homogeneous BC at $x = b$ but not at $x = a$
- It is not hard to set this up by taking appropriate linear combinations

Computing the Green's function

- Then the Green's function can be written as

$$G(x,x') = \begin{cases} Ay_1(x)y_2(x') & x < x' \\ Ay_2(x)y_1(x') & x > x' \end{cases}$$

- You can see the Green's function already satisfies the BC's
- You can also see the Green's function is continuous at $x = x'$
- To determine the constant A we integrate both sides of the ODE

$$G'' + P(x)G' + Q(x)G = \delta(x - x')$$

to find that

$$\lim_{x\to x'+} \frac{dG(x,x')}{dx} - \lim_{x\to x'-} \frac{dG(x,x')}{dx} = 1$$

Computing the Green's function

- Computing the indicated limits on the expression

$$G(x, x') = \begin{cases} Ay_1(x)y_2(x') & x < x' \\ Ay_2(x)y_1(x') & x > x' \end{cases}$$

  we see

$$Ay_2'(x')y_1(x') - Ay_1'(x')y_2(x') = 1$$

- This tells us

$$A = \frac{1}{y_2'(x')y_1(x') - y_1'(x')y_2(x')} = \frac{1}{W(x')}$$

  where $W(x')$ is the Wronskian at $x = x'$

Computing the Green's function

- This then gives us

$$G(x,x') = \begin{cases} \dfrac{y_1(x)y_2(x')}{W(x')} & x < x' \\ \dfrac{y_2(x)y_1(x')}{W(x')} & x > x' \end{cases}$$

- One does however have to make sure the ODE

$$G'' + P(x)G' + Q(x)G = \delta(x - x')$$

  with homogeneous BC's actually has a solution.
- If it turns out

$$G'' + P(x)G' + Q(x)G = 0$$

  has a nontrivial solution with homogeneous BC's then the Green's function problem

$$G'' + P(x)G' + Q(x)G = \delta(x - x')$$

  has no solution and a modified procedure is required.

Computing the Green's function

- For example, suppose the ODE is the Sturm-Liouville ODE:

$$\frac{d}{dx}\left(p(x)\frac{dG}{dx}\right) - q(x)G + \lambda G = \delta(x - x')$$

- Then we know if $\lambda = \lambda_n$ where λ_n are the eigenvalues, we do have a nontrivial solution to the homogeneous problem
- So for these special values of λ in the ODE

$$\frac{d}{dx}\left(\frac{dG}{dx}\right) - q(x)G + \lambda_n G = \delta(x - x')$$

we have no solution to this problem.

- There is a way to rescue the Green's function concept in this case which we will discuss later
- Basically the idea is to build in the Fredholm alternative

The Green's function as a kind of inverse matrix

- The expression

$$y(x) = \int_a^b G(x, x')f(x')dx'$$

  resembles formally the solution to the problem of solving a linear system of equations.
- Suppose we have a $n \times n$ system of equations

$$A\boldsymbol{x} = \boldsymbol{b}.$$

- One way to formally write the solution is

$$\boldsymbol{x} = A^{-1}\boldsymbol{b},$$

- Here A^{-1} is the matrix inverse for A.

The Green's function as a kind of inverse matrix

- We generally never use this expression when we compute things in linear algebra
- This is because it turns out the matrix inverse is a more expensive way to compute a solution than Gaussian elimination.
- But the form is suggestive.
- It says that if we knew A^{-1} then we would have solved the system not just for this particular right hand side $\boldsymbol{b}$ but for any $\boldsymbol{b}$.
- In this sense, the solution

$$y(x) = \int_a^b G(x, x')f(x')dx',$$

  is reminiscent of this approach.
- But here summation is represented by integration and the Green's function is a two variable function rather than a discrete matrix.

The Green's function as a kind of inverse matrix

- There is another similarity if you think about what the inverse matrix is.
- Suppose you solve the set of n linear systems

$$A\boldsymbol{x}_m = \begin{bmatrix} 0 \\ \vdots \\ \text{1 in the } m\text{'th location} \\ \vdots \\ 0 \end{bmatrix} \qquad m = 1, \ldots, n,$$

- Now form a matrix

$$B = \begin{bmatrix} \boldsymbol{x}_1 & \boldsymbol{x}_2 & \ldots & \boldsymbol{x}_n \end{bmatrix}$$

- The columns of B are the solutions $\boldsymbol{x}_m$
- Then you can see that $B = A^{-1}$.

The Green's function as a kind of inverse matrix

- This is exactly what is happening when you solve

$$\frac{d^2G}{dx^2}+P(x)\frac{dG}{dx}+Q(x)G=\delta(x-x')$$

- The δ function is playing the role of an identity matrix, but for functions of the real variable x.
- And the Green's function is like the inverse matrix for the operator

$$L=\frac{d^2}{dx^2}+P(x)\frac{d}{dx}+Q(x)$$

- So you can see that the Green's function is really just a fancy application of the method of superposition in linear algebra
- Except we have to deal with pathological functions like $\delta(x)$ because we are working on the dense real line.

The Green's function for the Sturm-Liouville ODE

- In previous lectures we showed that a solution of the inhomogeneous S-L ODE

$$y'' + \lambda y = f(x) \qquad 0 \le x \le \pi$$

with homogeneous BC's was

$$y(x) = \int_0^\pi G(x, x'; \lambda) f(x') dx',$$

where

$$G(x, x'; \lambda) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)\sin(nx')}{\lambda - n^2}.$$

- We identified $G(x, x'; \lambda)$ as the Green's function - the solution to

$$G'' + \lambda G = \delta(x - x') \qquad G(0) = 0 \quad G(\pi) = 0$$

The Green's function for the Sturm-Liouville ODE

- There are several things that are striking about the expression

$$G(x, x'; \lambda) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(nx)\sin(nx')}{\lambda - n^2}.$$

- First, viewed as a function of λ in the complex λ-plane, G has poles at $\lambda = n^2$
- These are the locations of the eigenvalues
- This is not surprising
- There is no solution of the Green's function ODE when $\lambda = n^2$ so our expression for G reflects this.
- Second, if we think again of G in the complex λ-plane, the residues at the poles $\lambda = n^2$ are

$$\frac{2}{\pi} \sin(nx') \sin(nx) \quad n = 1, 2, 3, \ldots$$

- These residues are the eigenfunctions $\sin(nx)$ multiplied by a constant (in this case $\sin(nx')$)

The Green's function for the S-L ODE

- Finally we note for this case

$$G(x, x'; \lambda) = G(x', x; \lambda)$$

- That is, this G is manifestly symmetric.
- This too is to be expected.
- Since the S-L operator is real and self-adjoint it is like a symmetric matrix
- And the inverse matrix of a symmetric matrix (which is the interpretation of the Green's function) is also symmetric.

The Green's function for the S-L ODE

- The "summed" version of the Green's function must also have all these properties
- Recall the summed version is

$$H(x, x'; \lambda) = \begin{cases} \dfrac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)} \sin(\sqrt{\lambda}x)\sin(\sqrt{\lambda}(x' - \pi)) & x < x' \\ \dfrac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)} \sin(\sqrt{\lambda}x')\sin(\sqrt{\lambda}(x - \pi)) & x > x' \end{cases}$$

- You can see that by interchanging x and x' that this function is symmetric
- You can also see that this function has poles at

$$\lambda_n = n^2 \qquad n = 1, 2, 3$$

- Note $\lambda = 0$ is *not* an eigenvalue - the Green's function is finite as $\lambda \to 0$
- And with a little more work it's possible to show the residues at the poles $\lambda = n$ are proportional to the eigenfunctions $\sin(nx)$

The Green's function for the S-L ODE

- These types of results hold for general S-L ODE problems.
- Recall for the ODE

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda r(x)y = r(x)f(x), \qquad a < x < b,$$

- The Green's function is given by

$$G(x, x'; \lambda) = \sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n(x')}{\lambda - \lambda_n}.$$

  if the eigenfunctions $\phi_n(x)$ are orthonormal.
- So we see that in general that the poles of the Green's function in the complex λ plane are the eigenvalues
- And the residues are proportional to the eigenfunctions
- Finally we see the symmetry of the Green's function

The completeness relation

- There is another identity that is also relevant.
- Recall that if we knew the Green's function $G(x, x'; \lambda)$, then it satisfies

$$\frac{d}{dx}\left(p(x)\frac{dG}{dx}\right) - q(x)G(x) + \lambda r(x)G = \delta(x - x'), \qquad a < x < b.$$

- Now apply the operator

$$\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) - q(x) + \lambda r(x)$$

to our series

$$\sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n(x')}{\lambda - \lambda_n},$$

The completeness relation

- We get

$$\sum_{n=0}^{\infty} \phi_n(x)\phi_n(x')r(x).$$

- By comparing with the right hand side of

$$\frac{d}{dx}\left(p(x)\frac{dG}{dx}\right) - q(x)G(x) + \lambda r(x)G = \delta(x - x'), \qquad a < x < b,$$

  we see that we must have

$$\sum_{n=0}^{\infty} \phi_n(x)\phi_n(x') = \frac{\delta(x - x')}{r(x)}.$$

- This is known as the *completeness relation*.

The completeness relation

- The series is not convergent but exists in a formal sense.
- In fact we can see the coefficients of the series don't go to zero at all.
- This means we have a very pathological function.
- And indeed the δ "function" is not really a function at all.
- This type of relationship holds for any complete set of Sturm-Liouville eigenfunctions.
- Note that this expression becomes very clean if we transform our S-L operator so that the S-L ODE is

$$y'' + [\lambda - q(z)]y = 0,$$

which we can always do

The completeness relation

- For this form of the operator $r(x) = 1$.
- Again lets call the eigenfunctions for this operator $\phi_n(x)$.
- Then the completeness relation becomes

$$\sum_{n=0}^{\infty} \phi_n(x)\phi_n(x') = \delta(x - x'),$$

- This is the way the completeness relation is traditionally written.
- But it's important to understand we use this type of relation in conjunction with integration over some smooth function
- In and of itself it's a formal expression.