ACM 100b

Review of ODE's - Basic concepts

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ODE's - some terminology

- We begin with some fundamental definitions for ordinary differential equations (ODEs).
- The most general type of ODE of order n is a general relation
  - among an independent variable (call it z)
  - a dependent scalar variable (call it $y(z)$)
  - and up to n derivatives of $y(z)$:

$$F(z, y, y', y'', \ldots, y^{(n)}) = 0.$$

- Most of our discussions will deal with real values of z.
- But it will be useful in certain cases to think about complex values of z.
- The theory of ODEs is actually set in the complex plane.

The solution $y(z)$ will have n constants of integration

- Our goal is to find all the functions $y(z)$ which satisfy

$$F(z, y, y', y'', \ldots, y^{(n)}) = 0.$$

- In general we expect that our solution will involve n arbitrary constants.
- This is because the integration of up to n derivatives of y will allow for this many arbitrary constants.
- So we could write formally

$$y(z) = G(z, c_1, c_2, \ldots, c_n).$$

where G is our solution and the c_i are the constants

Linear ODE's

- We say the ODE given by

$$F(z, y, y', y'', \ldots, y^{(n)}) = 0$$

  is *linear* if F is a linear relation in y and its derivatives.
- This just means

$$F(z) = \sum_{j=0}^{n} A_j(z) \frac{d^j y(z)}{dz^j} - f(z)$$

  so the ODE is

$$\sum_{j=0}^{n} A_j(z) \frac{d^j y}{dz^j} = f(z)$$

- If $f(z) \equiv 0$ we call the linear ODE *homogeneous*.
- Otherwise we call it *inhomogeneous*.

General solution of linear ODE's

- For a linear ODE the solution can be written in the form

$$y(z) = \sum_{i=1}^{n} c_i y_i(z) + y_{part}(z).$$

- The *homogeneous solutions* $y_1, y_2, \ldots, y_n$ satisfy

$$A_n(z)y_i^{(n)} + A_{n-1}(z)y_i^{(n-1)} + \ldots + A_1(z)y_i' + A_0(z)y_i(z) = 0.$$

  for $i = 1, 2, \ldots, n$
- The *inhomogeneous* or *particular* solution $y_{part}(z)$ satisfies

$$A_n(z)y_{part}^{(n)} + A_{n-1}(z)y_{part}^{(n-1)} + \ldots + A_1(z)y_{part}' + A_n(z)y_{part} = f(z).$$

- Note that the full solution is a linear superposition of the homogeneous solutions
- Note too that the n constants of integration appear in a very simple linear way.

Nonlinear ODE's are much more complicated

- In contrast, suppose the ODE

$$F(z, y, y', y'', \ldots, y^{(n)}) = 0$$

  is nonlinear – that is F is not linear in y and/or its derivatives.
- Then the solution still has n arbitrary constants.
- But one can't talk about homogeneous or particular solutions.
- Instead the solution is still in the form

$$y(z) = G(c_1, c_2, \ldots, c_n, z),$$

- But the constants appear in a generally nonlinear way in the solution.
- In addition, it is sometimes possible to have solutions that are not connected in any simple way to the set of solutions gotten by varying the c_n.

Usually we are interested in specific solutions

- One is usually not interested in the general solution of an ODE.
- Usually there are additional conditions associated with the problem at hand that fix the constants c_j
- We do care if the solution exists and whether it's unique.
- For example in mechanics problems Newton's laws of motion are expressed as second order ODE's
- We often know the initial position of a body or particle as well as its velocity.
- These two pieces of information would be used to compute the subsequent motion.

Initial value vs boundary value problems

- In general, for a linear problem we need to provide n pieces of information to determine the c_n uniquely in the solution

$$y(z) = G(z, c_1, c_2, \ldots, c_n).$$

- There are many ways to do this.
- But two common approaches are as follows:
  1. Initial value problem
  2. Boundary value problem

Initial value problem

- To determine the c_i we give the n values of the function $y(z)$ and its derivatives $y^{(i)}$ at some point $z = z_0$.
- For example, in a mechanics problem governed by Newton's laws we give position and velocity at some initial time.
- The ODE's are second order so have two arbitrary constants and so this should be enough to specify a unique solution.
- This is an example of an initial value problem.

Boundary value problem

- We still give n values and derivatives of $y(z)$.
- But these are given at different points z_0, z_1, etc.
- For example for a second order ODE we might give $y(z_0)$ and $y(z_1)$ for a total of two conditions.
- For a third order problem we might give $y(z_0), y'(z_0)$ and $y(z_1)$ again for a total of three conditions.

Mathematical issues for ODE solutions

- The mathematical theory of ODEs concentrates on several questions:
  1. Does a solution exist?
  2. Is it unique?
  3. How does it behave as we make small changes to the initial or boundary data?
- We'll focus on the first two questions mostly.
- The third question is important if we want to understand how "smooth" our solution is as we vary the initial or boundary conditions.
- This is important in applications because we don't want solutions to physical problems that behave in some strange singular way as we change the initial or boundary conditions.