Physics 106a — Classical Mechanics

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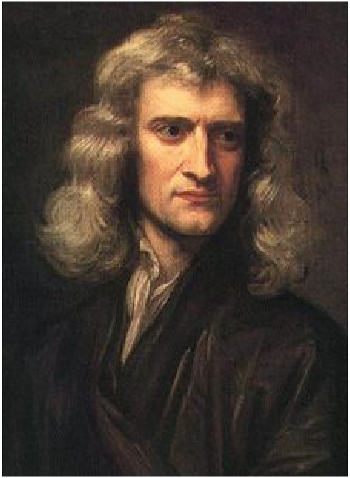
Lecture 2: Variational Approach

Outline

- Fictional approach
- Hamilton's principle
- Calculus of variations
- Particle in free space and particle with forces
- Advantages of Lagrangian approach
- Examples

Newton



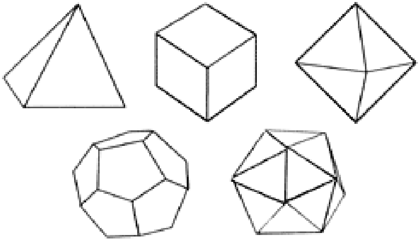
Isaac Newton [1643 - 1727]

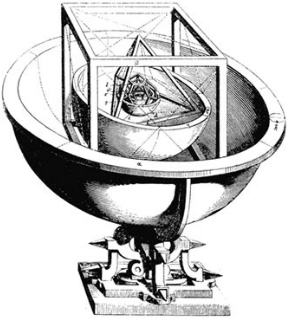
16th Century Cosmology

- 6 planets
- Planetary orbits are uniform motion on circles with sun at center
- Radii of planets predicted by nesting of platonic solids (Kepler)

Simple picture of geometric perfection

16th Century Cosmology

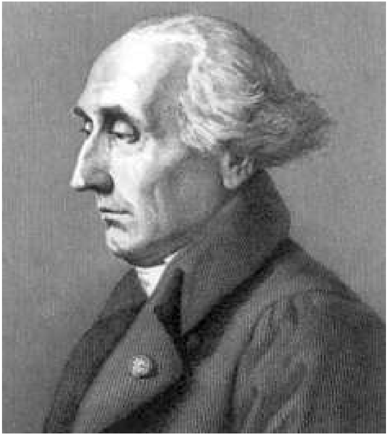




	octahedron	icosahedron	dodecahedron	tetrahedron	cube
Mercury	Venus	Earth	Mars	Jupiter	Saturn

Simple picture of geometric perfection, but does not quite work!

Lagrange



Joseph-Louis Lagrange [1736 - 1813]

Hamilton's Principle

The dynamics of the system $\{q_k(t)\}$ from time t_i to time t_f is such that the *action*

$$S = \int_{t_i}^{t_f} L(\{q_k\}, \{\dot{q}_k\}, t)\, dt$$

is stationary over all trial paths with fixed endpoints $\{q_k(t_i)\}$, $\{q_k(t_f)\}$.

Stationary (minimum, maximum, or saddle) means that for an infinitesimal change in path $\delta q_k(t)$ we have $\delta S = 0 + O(\delta q_k^2)$.

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For $\{q_k\}$ independent coordinates, this gives the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

Advantages of the Lagrangian formulation

- The action is a simple scalar quantity that reflects the symmetries of the physical system: often enough for us to guess the form it must take
- The principle is easy to derive as the $\hbar \to 0$ limit of quantum mechanics
- The path variation may be expressed in terms of any convenient coordinates defining the particle positions
- For dynamics problems with constraints we can eliminate the constraints from the formulation by using coordinates whose variations are consistent with the constraints
- It is often easy to evaluate the action, since the kinetic energy (scalar function of the scalar speed) is usually easy to evaluate even in complicated situations
- The approach generalizes quite naturally to relativistic mechanics, quantum mechanics, quantum field theory ....