Physics 106a **19 November, 2013**

Lecture 15: Rigid Body Rotation, Torque Free Motion

In this lecture I discuss the basic physics of rotating rigid bodies—angular velocity, kinetic energy, and angular momentum—introducing the moment of inertia tensor. I then solve a simple force-free dynamical system: the free symmetric top (e.g. a thrown football, or the Chandler wobble of the earth).

Equation of motion for a rigid body

A rigid body has six degrees of freedom corresponding to translational and rotational motion. The equations of motion are the equations for the total momentum and angular momentum.

$$\frac{d\vec{P}}{dt} = \vec{F}, \qquad \frac{d\vec{L}_{\text{tot}}}{dt} = \vec{N}, \tag{1}$$

with $\vec{P} = M\dot{\vec{R}}_{\text{cm}}$ the total momentum given by the mass times the center of mass velocity, $\vec{L}_{\text{tot}}$ the total angular momentum, $\vec{F}$ the force acting on the body and $\vec{N}$ the torque. The angular momentum and torque are calculated with respect to some fixed reference origin. The angular momentum equation was derived in Lecture 1 for a set of particles with *central* forces. In Assignment 7 you derive the result for a rigid body using d'Alembert's principle without this assumption (but assuming the constraint forces do not contribute to the virtual work).

Moment of inertia

For the *special case* of a body undergoing pure rotation, and no translation, about the stationary origin (e.g. a fixed pivot, or the instantaneous point of contact in a rolling problem), we have $\vec{v}_{s,i} = \vec{\omega} \times \vec{r}_{s,i}$ with $\vec{\omega}$ the angular velocity and $\vec{r}_{s,i}$ the displacement of the ith point from the stationary point, and so

$$\vec{L}_{\text{tot}} = \sum_i m_i \vec{r}_{s,i} \times (\vec{\omega} \times \vec{r}_{s,i}) = \overleftrightarrow{I_s} \cdot \vec{\omega}\,, \tag{2}$$

with $\overleftrightarrow{I_s}$ the *moment of inertia tensor* with respect to the fixed origin, with components

$$I_{s,\alpha\beta} = \sum_i m_i (r_{s,i}^2 \delta_{\alpha\beta} - r_{s,i,\alpha} r_{s,i,\beta}). \tag{3}$$

(sorry about the cumbersome subscript notation here: "s" denotes relative to a stationary origin (space frame), "i" the particle, and "α" the component).

Reference point moving with body

It is often convenient to change to a reference point moving with the body. Using $r_{s,i} = \vec{R} + \vec{r}_i$, $\vec{v}_{s,i} = \vec{V} + \vec{\omega} \times \vec{r}_i$ for some reference point K in the body with position with respect to the space frame origin $\vec{R}$ and velocity $\vec{V}$, gives

$$\vec{L}_{\text{tot}} = \sum_i m_i (\vec{R} + \vec{r}_i) \times (\vec{V} + \vec{\omega} \times \vec{r}_i). \tag{4}$$

The expression simplifies if we take the reference point to be the center of mass since then $\sum_i m_i \vec{r}_i = 0$. This gives

$$\vec{L}_{\text{tot}} = \vec{R}_{\text{cm}} \times \vec{P} + \vec{L}, \tag{5}$$

where $\vec{P}$ is the total linear momentum and $\vec{L}$ is the *intrinsic angular momentum* coming from the rotation of the body

$$\vec{L} = \sum_i m_i \vec{r}_i \times (\vec{\omega} \times \vec{r}_i) = \overleftrightarrow{I} \cdot \vec{\omega}, \tag{6}$$

where $\overleftrightarrow{I}$ is the moment of inertia tensor with respect to the center of mass, with components

$$I_{\alpha\beta} = \sum_i m_i (r_i^2 \delta_{\alpha\beta} - r_{i,\alpha} r_{i,\beta}), \tag{7}$$

where we have made some choice of the direction of the triad of unit vectors specifying the basis.

The equation of motion of the total angular momentum can be rewritten

$$\frac{d(\vec{R}_{\mathrm{cm}} \times \vec{P} + \vec{L})}{dt} = \sum_i (\vec{R}_{\mathrm{cm}} + \vec{r}_i) \times \vec{F}_i. \tag{8}$$

Using the equation of motion of the total momentum, and $\dot{\vec{R}}_{\mathrm{cm}} \times \vec{P} = 0$ gives

$$\frac{d\vec{L}}{dt} = \sum_i \vec{r}_i \times \vec{F}_i. \tag{9}$$

Thus we may use the equation for rate of change of angular momentum and the torque calculated about a *stationary point* or about the *center of mass*, even if moving (but not any other moving point).

Kinetic energy

The kinetic energy $T = \frac{1}{2} \sum_i m_i v_i^2$ with $\vec{v}_i = \vec{V} + \vec{\omega} \times r_i$ for solid body motion simplifies for two choices of reference point:

1. Reference point is the center of mass

$$T = \frac{1}{2} M \vec{V}_{\mathrm{cm}}^2 + \frac{1}{2} \sum_i m_i (\vec{\omega} \times \vec{r}_i)^2 \ , \tag{10}$$

2. Motion is pure rotation about a stationary reference point

$$T = \frac{1}{2} \sum_i m_i (\vec{\omega} \times \vec{r}_i)^2 \ . \tag{11}$$

In either case the rotational kinetic energy (which from now on I will call T) is

$$T = \frac{1}{2} \sum_i m_i (\vec{\omega} \times \vec{r}_i)^2 = \frac{1}{2} \vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega} = \frac{1}{2} I_{\alpha\beta} \omega_\alpha \omega_\beta \qquad \text{(Einstein summation)}, \tag{12}$$

with $\overleftrightarrow{I}$ the moment of inertia introduced before.

Moment of inertia tensor

The moment of inertia about some origin has components with respect to the coordinate basis

$$I_{\alpha\beta} = \sum_i m_i(r_i^2\delta_{\alpha\beta} - r_{i,\alpha}r_{i,\beta}), \tag{13}$$

with $\vec{r}_i$ the position of the ith mass relative to the origin. You should be able to evaluate the moment of inertia for simple mass distributions and solid bodies (see Hand and Finch problems 8-3 to 8-11 for practice examples).

The *displaced axis theorem* says that the moment of inertia for a reference point shifted by $\vec{a}$ from the center of mass is

$$I_{\vec{a},\alpha\beta} = I_{\mathrm{cm},\alpha\beta} + M(a^2\delta_{\alpha\beta} - a_\alpha a_\beta), \tag{14}$$

with M the total mass. This is often useful in the calculation of I.

The moment of inertia is a tensor. Under rotation of the coordinate axes the components of the moment of inertia tensor transforms as

$$I'_{\mu\nu} = U_{\mu\alpha}U_{\nu\beta}I_{\alpha\beta}, \tag{15}$$

(each index transforming like a vector) with U the rotation matrix relating the bases. Thinking of $I_{\alpha\beta}$ as a 3×3 matrix, this can be written

$$I' = UI\tilde{U} \qquad \text{or} \qquad I = \tilde{U}I'U\ . \tag{16}$$

Since I is a symmetric matrix, the theorems of linear algebra tell us that we can find some U such that I is diagonal

$$I = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}. \tag{17}$$

The particular choice of coordinate axes giving this diagonal form are called the *principal axes* of the solid body.

Symmetric top in free space

As a simple example consider a spinning body with axial symmetry so that $I_1 = I_2 \neq I_3$ with no applied torques. Then the physics is simply that the angular momentum is constant in an inertial frame, so that

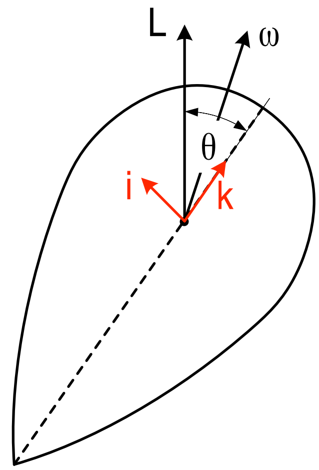
$$\overleftrightarrow{I} \cdot \vec{\omega} = \text{constant}. \tag{18}$$

The rest of the solution is "just geometry". It is tricky because the moment of inertia is most simply evaluated using the principal axes of the body, but these in turn are rotating relative to fixed axes (the space frame) at the angular velocity $\vec{\omega}$. Here is one way of getting the solution (see figure).

- Define $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ as the principal axes with the symmetry axis along $\hat{k}$.
- Since the body is axially symmetric, at each time we can choose $\hat{\imath}$ to lie in any direction perpendicular to $\hat{k}$: choose it to lie in the plane of $\vec{L}$ (constant) and $\hat{k}$ (at that time). Then $\hat{\jmath}$ is perpendicular to this plane and

  $$\omega_2 = \frac{L_2}{I_2} \tag{19}$$

  (I is diagonal for this basis) and this gives $\omega_2 = 0$ since $L_2 = 0$ ($\hat{\jmath}$ is perpendicular to $\vec{L}$). Hence $\vec{\omega}$ also lies in the $\vec{L}$, $\hat{k}$ plane.



- Since $\vec{\omega}$ lies in the $\vec{L}$, $\hat{k}$ plane, the motion $d\hat{k}/dt$ of $\hat{k}$ (coming from the rotation at rate $\vec{\omega}$) is perpendicular to the plane. This means that the angle θ between $\hat{k}$ and $\vec{L}$ is constant $\dot{\theta} = 0$.

- This means that at the next time instant we can draw the same picture (with a new choice of $\hat{\imath}$). $\omega_1 = L\sin\theta/I_1$ and $\omega_3 = L\cos\theta/I_3$ are the same, and so the same argument applies. Thus the motion of $\vec{\omega}$, $\hat{k}$ is *steady precession* about the direction of $\vec{L}$.

- The motion of the body axis is given by

$$\frac{d\hat{k}}{dt} = \vec{\omega} \times \hat{k} = \omega_1 \hat{\imath} \times \hat{k} = \frac{L\sin\theta}{I_1}\hat{\imath} \times \hat{k} = \frac{\vec{L}}{I_1} \times \hat{k}. \tag{20}$$

  Since $\vec{L}$ is constant this corresponds to steady precession at the angular velocity $\vec{\omega}_P = \vec{L}/I_1$ which is a rotation rate L/I_1 along $\vec{L}$.

This precession is the wobble you see if a non-expert throws a football with the angular velocity and the angular momentum not aligned.

My treatment is different from the one in Hand and Finch. They define the "spin" Ω of the body subtracting off from ω_3 (the component of $\vec{\omega}$ along the symmetry axis) the part coming from the precession

$$\Omega = \omega_3 - \omega_P \cos\theta = L\left(\frac{1}{I_3} - \frac{1}{I_1}\right)\cos\theta. \tag{21}$$

This is a strange notion of the spin, since it goes to zero for a spinning sphere $I_1 = I_3$. They start off by assuming that Ω is constant, whereas this should be deduced from the analysis based on constant $\vec{L}$ (or the Euler equation method, see below), and they talk about the "missing piece" of the angular velocity, which I find confusing. They also use the conservation of energy to argue θ is constant, which is fine.

Euler equations

A second approach is to evaluate the terms in the torque equation in terms of time derivatives in the body frame. This gives the *Euler equations*. For the top problem this approach is particularly convenient to understand the motion from the perspective of an observer on the top.

In the inertial (space) frame we have

$$\left.\frac{d\vec{L}}{dt}\right|_s = \vec{N}, \tag{22}$$

with $\vec{N}$ the torque. This gives in the body frame rotating at angular velocity $\vec{\omega}$ relative to the space frame

$$\left.\frac{d\vec{L}}{dt}\right|_b = \vec{N} - \vec{\omega} \times \vec{L}. \tag{23}$$

Now write $\vec{L}$ in terms of components along the principle axes $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ so that $\vec{L} = I_1\omega_1\hat{\imath} + I_2\omega_2\hat{\jmath} + I_3\omega_3\hat{k}$. In the body frame $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$ are independent of time, so that in component form (the components are with respect to axes fixed in the body frame)

$$I_1\frac{d\omega_1}{dt} - \omega_2\omega_3(I_2 - I_3) = N_1, \tag{24}$$

$$I_2\frac{d\omega_2}{dt} - \omega_3\omega_1(I_3 - I_1) = N_2, \tag{25}$$

$$I_3\frac{d\omega_3}{dt} - \omega_1\omega_2(I_1 - I_2) = N_3, \tag{26}$$

where the equations are related by cyclic permutations of the indices.

These are equations of motion for the components of $\vec{\omega}$ with respect to the principal axes of the body, which are themselves rotating relative to the space frame with the angular velocity ω. Translating back to the space frame is "only" geometry, but it can be quite complex.

Symmetric top in free space from the body perspective: Chandler wobble

What would we see if we were sitting on the wobbling football? Well, we are! Ignoring the small effects of gravitational forces acting on the aspherical mass distribution, the Earth is effectively a symmetric top in free space and it is oblate $I_3 > I_1 = I_2$.

We could transfer the results we have calculated to the body frame (see below), but let's instead redo the calculation from scratch. In the body frame we can conveniently use Euler's equations, and since the top is free, $\vec{N} = 0$. The equality $I_1 = I_2$ (lets call this $I_\perp$) gives $\omega_3 =$ constant from the third equation. Then the other two equations become

$$\frac{d\omega_1}{dt} = -\left[\left(\frac{I_3}{I_\perp} - 1\right)\omega_3\right]\omega_2, \tag{27}$$

$$\frac{d\omega_2}{dt} = \left[\left(\frac{I_3}{I_\perp} - 1\right)\omega_3\right]\omega_1. \tag{28}$$

The quantity inside the [ ] is constant, and so these equations can be solved to give

$$\omega_1 = A\cos(\Omega_p t + \phi),\ \omega_2 = A\sin(\Omega_p t + \phi) \quad \text{with} \quad \Omega_p = \left(\frac{I_3}{I_\perp} - 1\right)\omega_3, \tag{29}$$

with A, ϕ integration constants. Thus the angular velocity vector steadity precess about the symmetry axis at the rate Ω_p. Note that Ω_p is positive for the oblate earth, negative for the football.

For the Earth the axis of rotation is slightly misaligned from the north-south symmetry axis and $(I_3/I_\perp - 1) \simeq 0.0033$. The calculation predicts we should see the rotation axis precess about the geometric NS axis with a period of about 306 days. Such an effect is observed, and is called Chandler wobble, but the period is about 435 days, apparently due to the dynamic deformation of the earth due to the tides. Hand and Finch discuss this more in §8.6[1]. For observations of the Chandler Wobble see http://hpiers.obspm.fr/eop-pc/.

[1] Note that their Fig. 8.8 is a little misleading in that the wobble is only about 10 meters, nothing like as large as implied by the sketch. The Z' axis in their picture is the direction of the constant angular momentum, and so this too is very close to the symmetry axis and is not perpendicular to the plane of the orbit, as you might guess from the figure and your knowledge of the tilt of the symmetry axis.

The motion calculated in this section is the same as discussed for the wobbling football, but the results look rather different. For example, for $I_\perp \simeq I_3$ the Chandler frequency Ω_p gets small, but the football wobble frequency L/I is just the frequency of the spin of the football and remains large. Here is one way to relate the discussion in the two frames.

In the body frame we have

$$\left.\frac{d\vec{\omega}}{dt}\right|_b = \Omega_p \hat{k} \times \omega, \quad \left.\frac{d\hat{k}}{dt}\right|_b = 0. \tag{30}$$

Transforming to the time dependence in the space frame in the usual way gives

$$\left.\frac{d\vec{\omega}}{dt}\right|_s = \Omega_p \hat{k} \times \omega, \quad \left.\frac{d\hat{k}}{dt}\right|_s = \vec{\omega} \times \hat{k}, \tag{31}$$

(the equation for $\vec{\omega}$ is unchanged of course since $\vec{\omega} \times \vec{\omega} = 0$). These equations are hard to interpret, since $\vec{\omega}$ rotates about $\hat{k}$, and $\hat{k}$ rotates about $\vec{\omega}$. However, introducing

$$\vec{\omega}_p = \Omega_p \hat{k} + \vec{\omega}, \tag{32}$$

the equations can be written

$$\left.\frac{d\vec{\omega}}{dt}\right|_s = \vec{\omega}_p \times \omega, \quad \left.\frac{d\hat{k}}{dt}\right|_s = \vec{\omega}_p \times \hat{k}, \tag{33}$$

(the extra pieces give zero on taking the cross products) so that both vectors rotate at $\vec{\omega}_p$. Furthermore, evaluating the components of ω in terms of $\vec{L}$,

$$\vec{\omega}_p = \left(\frac{1}{I_1} - \frac{1}{I_3}\right) L\cos\theta\hat{k} + \left(\frac{L\sin\theta}{I_1}\hat{\imath} + \frac{L\cos\theta}{I_3}\hat{k}\right) = \frac{\vec{L}}{I_1}, \tag{34}$$

so that the motion of both vectors is steady precession at angular frequency L/I_1 about the (constant) angular momentum in the space frame. The algebra $\vec{\omega} = \vec{\omega}_p + (-\Omega_p)\hat{k}$ is described pictorially in Fig. 8.7 of Hand and Finch (cf. the decomposition of the angular velocity of the cone rolling on a plane you investigated in Assignment 7).

Alternatively, let's start with the equation for the steady precession of $\vec{\omega}$ in the space frame

$$\left.\frac{d\vec{\omega}}{dt}\right|_s = \frac{\vec{L}}{I_1} \times \vec{\omega}, \tag{35}$$

and go to the body frame. The time derivative of $\vec{\omega}$ is the same in the body frame, but Eq. (35) is less useful there, since $\vec{L}$ is not constant in that frame. Instead $\hat{k}$ is constant and we would like to express the motion of $\vec{\omega}$ in terms of rotation about this direction. From $\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$ we can write

$$\vec{L} = I_1\vec{\omega} + (I_3 - I_1)(\hat{k}\cdot\vec{\omega})\,\hat{k}. \tag{36}$$

The first term does not contribute to the cross product in Eq. (35) and so we get

$$\left.\frac{d\vec{\omega}}{dt}\right|_b = \left(\frac{I_3}{I_1} - 1\right)\omega_3\,\hat{k} \times \vec{\omega}, \tag{37}$$

giving the uniform precession of $\vec{\omega}$ about $\hat{k}$ at the rate Ω_p (note that Eq. (37) shows that ω_3 is constant).

Or even more simply: the pair of vectors $\Omega, \hat{k}$ steadily rotate with angular velocity $\Omega_p\hat{k}$ in the body frame, and $\vec{\omega}_p = \Omega_p\hat{k} + \vec{\omega}$ in the space frame.

Appendix: Torque-free motion of an asymmetric top – Poinsot construction*

(Just for fun: not for examination) For an asymmetric top, the moment of inertia components with respect to the principal axes are I_1, I_2, I_3, all different. The torque free motion in the space frame can be constructed as follows (see Hand and Finch §8.9 for more details): $\vec{L}$ is constant, and $\vec{\omega}$ evolves so that

- the kinetic energy T is a constant

$$\tfrac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) = T. \tag{38}$$

  This is the equation for an ellipsoid in ω space with semi major axes $\sqrt{2T/I_1}$ etc.

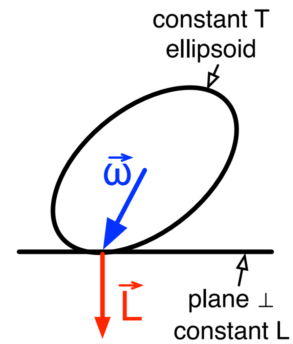
- Use the component form $L_\alpha = I_{\alpha\beta}\omega_\beta$ and $T = \frac{1}{2}I_{\alpha\beta}\omega_\alpha\omega_\beta$ to show

$$\vec{L} = \vec{\nabla}_{\vec{\omega}} T, \tag{39}$$

  with $\vec{\nabla}_{\vec{\omega}}$ the gradient with respect to $\vec{\omega}$. This means that $\vec{L}$ is perpendicular to the constant T ellipsoid at each time. Since $\vec{L}$ is constant in the torque free motion, this gives a constraint on the time evolution of the ellipsoid.

- The component form also shows

$$\vec{\omega} \cdot \vec{L} = 2T = \text{constant}. \tag{40}$$



The T ellipsoid rotates with angular velocity $\vec{\omega}$ about the contact point with the plane perpendicular to $\vec{L}$, maintaining the height of the center constant.

These equations show that the motion of the principle axes (and therefore the body) is given by rolling the constant T ellipsoid about the point of contact with the plane perpendicular to the constant vector $\vec{L}$, maintaining the height of the center fixed (Eq. (40)), at a rate given by the angular velocity vector, which is the vector joining the center of the ellipsoid to the contact point.

These arguments would, of course, be valid for the symmetric top, and so provide a different approach to that problem too.

Michael Cross November 19, 2013