Physics 106b — Classical Mechanics

Michael Cross

California Institute of Technology

Winter Term, 2014

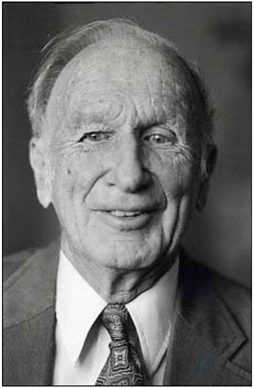
Dissipative Chaos

Outline

- Lorenz model
  - Motivation
  - Behavior
- Sensitive dependence on initial conditions:
  - Lyapunov exponents
- Strange attractors
  - Fractal dimensions
- One-dimensional maps

Lorenz Model

Deterministic Nonperiodic Flow, Journal of Atmospheric Sciences **20**, 130 (1963)

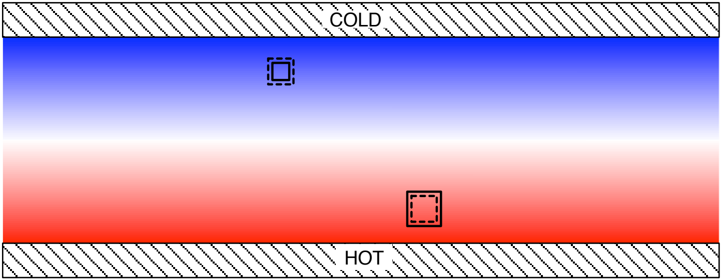


Edward Lorenz [1917-2008]

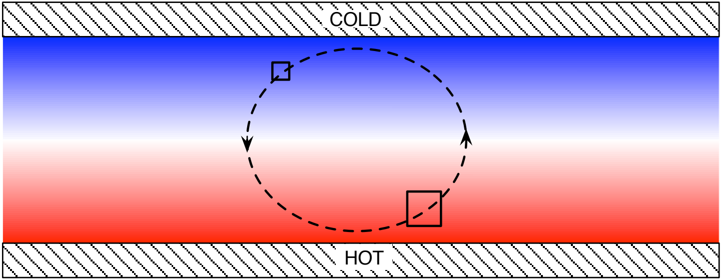
$$\begin{aligned} \dot{X} &= -\sigma(X - Y) \\ \dot{Y} &= rX - Y - XZ \\ \dot{Z} &= -bZ + XY \end{aligned}$$

"The feasibility of very long-range weather prediction is examined in the light of these results"

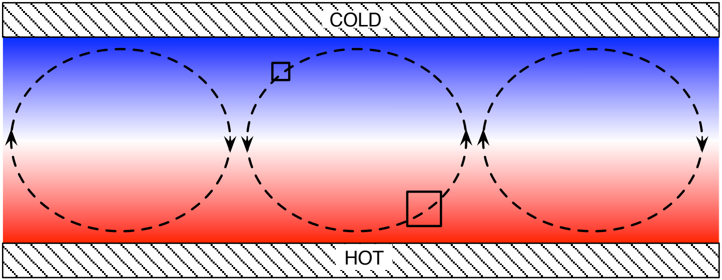
Rayleigh-Bénard Convection



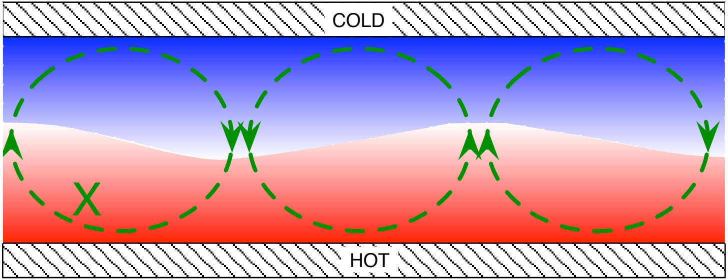
Rayleigh-Bénard Convection



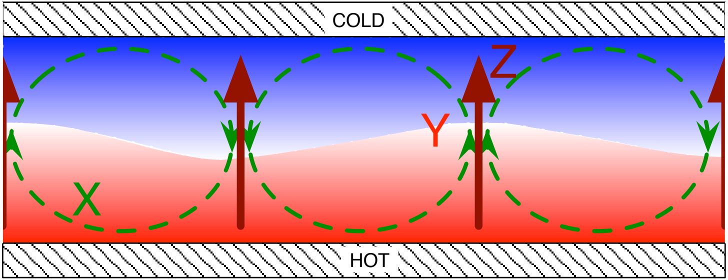
Rayleigh-Bénard Convection



Rayleigh-Bénard Convection



Lorenz Model (1963)



Lorenz Equations

$$\begin{aligned}\dot{X} &= -\sigma(X - Y)\\ \dot{Y} &= rX - Y - XZ\\ \dot{Z} &= -bZ + XY\end{aligned}$$

(where $\dot{X} = dX/dt$, etc.).

The equations give the velocity $\mathbf{V}_{\text{ph}} = (\dot{X}, \dot{Y}, \dot{Z})$ of the point $\mathbf{X} = (X, Y, Z)$ in the *phase space*

$r = R/R_c$, $b = 8/3$, and σ is the Prandtl number (a fluid property)

Properties of the Lorenz Equations

- Autonomous—time does not explicitly appear on the right hand side;
- Involve only first order time derivatives so that the evolution depends only on the instantaneous value of (X, Y, Z);
- Non-linear—the quadratic terms XZ and XY in the second and third equations;
- Dissipative—crudely the diagonal terms such as $\dot{X} = -\sigma X$ correspond to decaying motion. More systematically, *volumes in phase space contract* under the flow

$$\begin{aligned}\vec{\nabla}_{\mathrm{ph}} \cdot \mathbf{v}_{\mathrm{ph}} &= \frac{\partial}{\partial X}[-\sigma(X-Y)] + \frac{\partial}{\partial Y}[rX - Y - XZ] + \frac{\partial}{\partial Z}[-bZ + XY] \\ &= -\sigma - 1 - b < 0\end{aligned}$$

- Solutions are bounded—trajectories eventually enter and stay within an ellipsoidal region

Lorenz investigated the equations with $b = 8/3$, $\sigma = 10$ and $r = 27$ and uncovered chaos!

Butterfly Effect

The *sensitive dependence on initial conditions* found by Lorenz is often called the *butterfly effect*, and is the essential feature of chaos.

In fact Lorenz first said (Transactions of the New York Academy of Sciences, 1963):

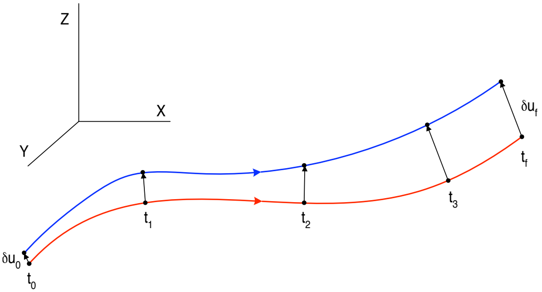
One meteorologist remarked that if the theory were correct, one flap of the sea gull's wings would be enough to alter the course of the weather forever.

By the time of Lorenz's talk at the December 1972 meeting of the American Association for the Advancement of Science in Washington D.C., the sea gull had evolved into the more poetic butterfly — the title of his talk was:

Predictability: Does the Flap of a Butterfly's Wings in Brazil set off a Tornado in Texas?

Sensitive Dependence on Initial Conditions

Trajectories diverge exponentially



Lyapunov exponent:

$$\lambda = \lim_{t_f \to \infty} \lim_{|\delta \mathbf{u}_0| \to 0} \left[\frac{1}{t_f - t_0} \ln \left| \frac{\delta \mathbf{u}_f}{\delta \mathbf{u}_0} \right| \right]$$

Sensitive Dependence on Initial Conditions

More generally for an N-dimensional phase space there are N Lyapunov exponents $\lambda_1 \geq \lambda_2 \ldots \lambda_N$ and

- λ_1 gives the rate of growth of a line of initial conditions
- $\lambda_1 + \lambda_2$ gives the rate of growth of an area of initial conditions
- ...
- $\sum_{i=1}^{N} \lambda_i = \langle \boldsymbol{\nabla}_{\text{ph}} \cdot \vec{V}_{\text{ph}} \rangle \Rightarrow$ contraction of phase space volume
- For a continuous time system with dynamics not approaching a fixed point there is always one zero Lyapunov exponent
- For the Lorenz equations I simulated $\lambda_1 = 0.906$, $\lambda_2 = 0$, $\lambda_3 = -14.572$

Calculating the Lyapunov Exponemts

For a mathematical model

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$$

linearize about the solution $\mathbf{x}_s(t)$, i.e. look at $\mathbf{x}(t) = \mathbf{x}_s(t) + \delta\mathbf{x}(t)$

$$\dot{\delta\mathbf{x}} = \mathbf{J}(\mathbf{x}_s) \cdot \delta\mathbf{x}, \quad \text{with} \quad J_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{x}=\mathbf{x}_s(t)}$$

For the Lorenz model

$$\mathbf{J} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - X & -1 & -X \\ Y & X & -b \end{bmatrix}$$

where we substitute in the numerically evolved $(X(t), Y(t), Z(t))$

Integrate (numerically) to give

$$\delta\mathbf{x}(t) = \mathbf{M}(t, t_0) \cdot \delta\mathbf{x}(t_0)$$

Calculating the Lyapunov Exponemts

- The largest exponent is given by

$$\lambda = \lim_{t\to\infty}\left[\frac{1}{t-t_0}\ln\left|\frac{\delta\mathbf{x}(t)}{\delta\mathbf{x}(t_0)}\right|\right]$$

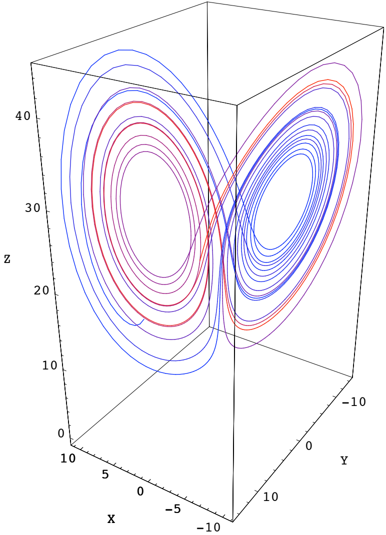
- The full set of exponents is given by one of the following:
  - It is possible to find a particular orthonormal set of initial displacements $\delta\mathbf{x}(t_0) = \mathbf{V}_i$ such that

$$\lambda_i = \lim_{t\to\infty}\frac{1}{t-t_0}\ln|\mathbf{M}(t,t_0)\cdot\mathbf{V}_i|$$

  - The λ_i are given via the eigenvalues of $\tilde{\mathbf{M}}\mathbf{M}$

$$\lambda_i = \lim_{t\to\infty}\frac{1}{2(t-t_0)}\mathrm{Ev}_i[\tilde{\mathbf{M}}(t,t_0)\mathbf{M}(\mathbf{t},\mathbf{t_0})]$$

Lorenz Butterfly



Strange Attractor

Trajectories settle onto a strange attractor:

Definition: *strange attractor* — an attractor that exhibits sensitive dependence on initial conditions (Ruelle and Takens).

The Lorenz attractor has no volume but is not a sheet: it is a *fractal* of noninteger dimension ($D = 2.06$ for the parameters I used)

Pendulum

- Equation of motion:

$$\ddot{\theta} + \frac{1}{Q}\dot{\theta} + \sin\theta = g\,\cos(\omega t)$$

- Phase space form:

$$\begin{aligned}
\dot{\theta} &= v \\
\dot{v} &= -\frac{1}{Q}v - \sin\theta + g\cos\Phi \\
\dot{\Phi} &= \omega
\end{aligned}$$

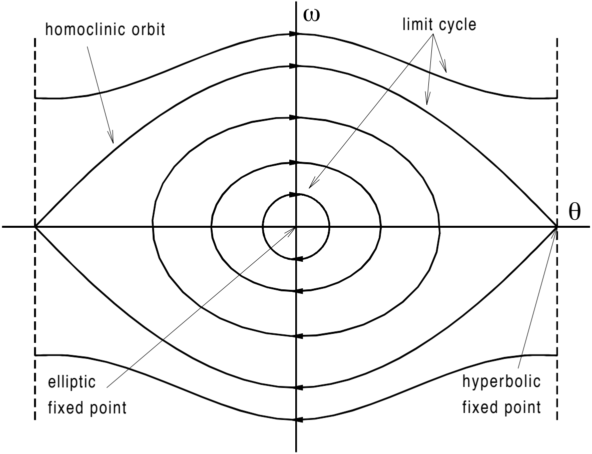
- Contraction of phase space volumes $\boldsymbol{\nabla}_{\text{ph}} \cdot \mathbf{v}_{\text{ph}}$ with $\mathbf{v}_{\text{ph}} = (\dot{\theta}, \dot{v}, \dot{\Phi})$

$$\boldsymbol{\nabla}_{\text{ph}} \cdot \mathbf{v}_{\text{ph}} = -\frac{1}{Q}$$

  so phase space volumes decrease proportional to $e^{-t/Q}$

- Poincaré section: intersections with $\Phi =$ constant plane

Ideal Pendulum – Phase Space



Now add dissipation and drive

Dimensions of Strange Attractors

- Chaotic attractors are fractals with noninteger dimensions.
- This is most easily seen for the models we considered by looking at the Poincare section, e.g. for the driven damped pendulum
- To illustrate the procedure I will use the Henon 2d map $(x_n, y_n) \to (x_{n+1}, y_{n+1})$

$$\begin{aligned} x_{n+1} &= y_n + 1 - a x_n^2 \\ y_{n+1} &= b x_n \end{aligned}$$

- The capacity dimension is defined by counting the number $N(\varepsilon)$ of M-dimensional cubes of side ε needed to cover the set ($M =$ dimension of phase space)

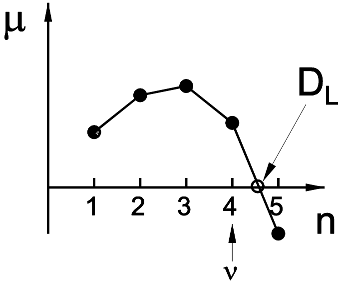
$$D_C = \lim_{\varepsilon \to 0} \frac{\log N(\varepsilon)}{\log(1/\varepsilon)}$$

- The dimension of the attractor of the ODEs would be $D_{PC} + 1$ with D_{PC} the capacity measured on the Poincaré section

Lyapunov Dimension

- Dimension D_L of a ball of initial conditions that neither grows nor decays under evolution

- Estimated from the sum of the first n Lyapunov exponents $\mu(n) = \sum_1^n \lambda_n$, and interpolating to a noninteger value D_L where $\mu(D_L) = 0$

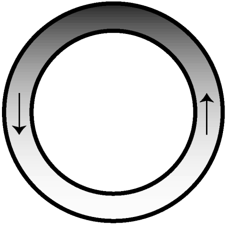


Summary

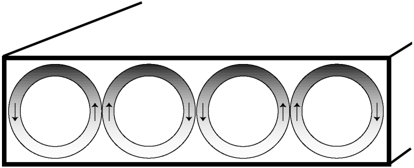
- Very simple dynamical systems may show complex dynamics with random/unpredictable behavior
- Sensitivity to initial conditions quantified by Lyapunov exponent
- Strange attractor with nonintegral dimension — a fractal
- Poincaré section gives 2d map (for 3d phase space)
- For strong contraction can reduce description to 1d map (return map)
- One route to chaos — through an infinite number of period doubling bifurcations — shows universal properties that can be understood from the *quadratic map* (not part of course: see Hand and Finch, Appendix to Ch11 if you are interested)
- Much more to discuss…

Lorenz model does not describe Rayleigh-Bénard convection

Thermosyphon

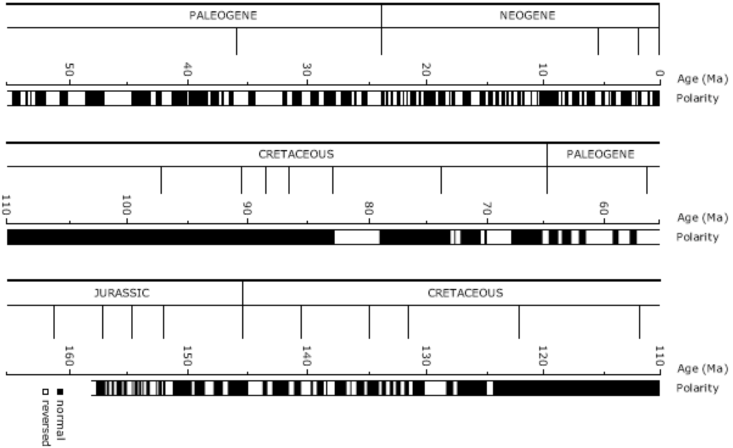


Rayleigh-Benard Convection



However the ideas of low dimensional models *do* apply to fluids and other continuum systems.

Reversal of Earth's Magnetic Field



For a simulation of the earth's dynamo showing reversal see
http://es.ucsc.edu/ glatz/geodynamo.html