ACM 100b

Existence and uniqueness for nonlinear first order ODE

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Existence and uniqueness for nonlinear ODE

- For nonlinear equations the situation regarding existence and uniqueness is more complicated.
- Consider the nonlinear ODE

$$y' = f(x, y) \qquad y(x_0) = y_0,$$

  where x is real and f is *not* a linear function of y.
- We again would like to know under what conditions the solution exists and is unique.
- And we also want to know over what interval about the initial point is it defined.
- In contrast to linear equations we don't have an explicit solution here.
- Instead we have to rely on qualitative results about the solution.

Nonlinear ODE

A relevant result is the following:

Theorem

Let $f(x, y)$ *and* $\dfrac{\partial f}{\partial y}$ *be continuous on some rectangle defined by* $\alpha < x < \beta$, $\gamma < y < \delta$ *containing the initial point* (x_0, y_0).

- *Then there exists a unique solution of the ODE that satisfies the initial conditions.*
- *The solution will be defined over an interval* $x_0 - h < x < x_0 + h$ *contained in* $\alpha < x < \beta$.

Nonlinear ODE

- These conditions are again sufficient but not necessary.
- For example the continuity of $\dfrac{\partial f}{\partial y}$ can be weakened.
- If all you care about is existence then you only need continuity of $f(x, y)$.
- Finally, the size of h must be determined as part of the problem.
- The interval of definition can change with the initial value
- This is because the solutions to nonlinear problems can develop spontaneous singularities.

Example of lack of uniqueness

- For example, consider the ODE

$$y' = y^{1/3} \qquad y(0) = 0.$$

- In this case we have

$$f(z, y) = y^{1/3} \qquad \frac{\partial f}{\partial y} = \frac{y^{-2/3}}{3}$$

- This equation can be solved in closed form because it's separable:

$$\begin{aligned} \frac{dy}{y^{1/3}} &= dz \\ \frac{3}{2} y^{2/3} &= z + c \\ \text{so } y &= \left[\frac{2(z+c)}{3}\right]^{3/2}. \end{aligned}$$

Example of lack of uniqueness

- We apply the initial value $y(0) = 0$ to get

$$y = (2z/3)^{3/2} \quad z \geq 0.$$

- So we got a solution - what's the problem?
- Notice that the solution $y = 0$ will also work here so our solution is not unique.
- This happens because while $f(z, y)$ is continuous at $y = 0$, $\frac{\partial f}{\partial y}$ is not.
- So the local existence and uniqueness theorem doesn't apply here.
- If our initial value were given at $z = z_0 \neq 0$ then we would have had a unique solution as guaranteed by the theorem.

The interval of existence is not known ahead of time for nonlinear ODE's

- For nonlinear ODE, how far the solution exists past the initial point is not known until the problem is solved.
- This is again in stark contrast with linear problems.
- Recall that for the linear equation

$$\frac{dy}{dz} + p(z)y = g(z),$$

  the solution will exist about the initial point $z = z_0$ in any interval in which the coefficient functions $p(z)$ and $g(z)$ are continuous.

Spontaneous singularities

- As an example of what can happen consider the simple ODE

$$y' = y^2 \qquad y(0) = 1.$$

- This has the solution

$$y = \frac{1}{1-x}.$$

- As $x \to 1$ the solution blows up and so we don't have existence beyond this point.
- However if you look at the original initial value problem one could not infer ahead of time that there would be a problem at $x = 1$.

Spontaneous singularities

- In fact the location of the singularity is dependent on the initial condition.
- For example suppose you change the initial value to

$$y(0) = 2,$$

- Then the solution is

$$y = \frac{2}{1 - 2x},$$

- And now the singularity occurs at $x = 1/2$.
- The existence of spontaneous singularities in nonlinear problems makes their analysis much more challenging.

Form of the solution

- Finally the general solution of nonlinear and linear problems differs markedly in form.
- For a linear equation, the solution is a linear function of the undetermined constants.
- If the equation is inhomogeneous then we know the solution is the sum of two parts, a homogeneous solution and a particular solution.
- But for a nonlinear equation the dependence on the constants of integration is in general nonlinear.
- There is no simple partition between homogeneous solutions and particular solutions.
- And to make things more complicated, there may exist additional solutions to nonlinear ODEs with no arbitrary constants which are completely disconnected from the family of solutions described by those solutions obtained from all possible values of the arbitrary constants.