ACM 100c

Properties of the Sturm-Liouville eigenfunctions and eigenvalues

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Properties of the solutions to the S-L problem

- The Lagrange identity remains true for this scalar product as well provided we use separable boundary conditions

$$\int_a^b \overline{u} L v dx = \int_a^b v L \overline{u} dx$$

- So far we have shown that the Sturm-Liouville operator is self adjoint
- We have also shown that for specific choices of $p(x)$, $q(x)$, and $r(x)$ solutions exist for special values of λ:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda r(x)y = 0, \qquad a < x < b,$$

- We will call the special values of λ at which nontrivial solutions exist *eigenvalues*
- Correspondingly we will call the associated solutions $y(x)$ the *eigenfunctions*

Analogy with the linear algebra eigenvalue problem

- These names make sense because we can write the S-L problem in the form

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right)+q(x)y(x)=\lambda r(x)y, \qquad a<x<b,$$

which looks formally like the algebraic eigenvalue problem:

$$Ly(x)=\lambda r(x)y(x)$$

- Here L plays the role of the "matrix", $y(x)$ plays the role of the eigenvector, and λ plays the role of the eigenvalue.

Properties of the solutions to the S-L problem

- We will show that if there are eigenfunctions for other choices of $p(x)$, $q(x)$, and $r(x)$, then they all have certain common properties.
- It is actually remarkable how much we can infer about these solutions even though we cannot solve the ODE for general functions $p(x)$, $q(x)$, and $r(x)$.
- We assume for now that the coefficient functions obey all the rules we discussed above

$$p(x) > 0 \qquad r(x) > 0$$

- We also assume that the boundary conditions are also of the linear separable form discussed above.

$$c_1 u(a) + c_2 u'(a) = 0,$$
$$d_1 u(b) + d_2 u'(b) = 0$$

Properties of the solutions to the S-L problem

- We will show the following results:
- All the eigenvalues of the Sturm-Liouville ODE are real.
- If $\phi_1(x)$ and $\phi_2(x)$ are two eigenfunctions corresponding to different eigenvalues λ_1 and λ_2 then the eigenfunctions are *orthogonal* in the following sense:

$$\int_a^b r(x)\phi_1(x)\phi_2(x)dx = 0.$$

- The eigenvalues of the Sturm-Liouville problem are all simple - that is there are no “multiple roots”

More properties

- The sequence of eigenvalues λ_1, λ_2 etc. can be ordered according to increasing magnitude.
- If this is done it is seen that $\lambda_n \to +\infty$ as $n \to \infty$.
- In other words the eigenvalues have no point of accumulation (except at ∞)
- If an eigenfunction ϕ_1 has an eigenvalue λ_1 and an eigenfunction ϕ_2 has an eigenvalue λ_2 with $\lambda_2 > \lambda_1$ then there is at least one zero of the eigenfunction ϕ_2 that lies between the zeroes of the eigenfunction of ϕ_1.

Expansions of arbitrary functions as series of eigenfunctions

- We say that any piecewise continuous function $f(x)$ is "square integrable" if
$$\int_a^b |f(x)|^2 r(x)dx \text{ is finite}$$
- For all such functions there exists a series expansion in terms of Sturm-Liouville eigenfunctions that converges to the function in the sense of *mean-square*
- This means
$$\int_a^b \left| f(x) - \sum_{n=1}^{N} B_n \phi_n(x) \right|^2 r(x)dx \to 0 \text{ as } N \to \infty$$
- The coefficients of the series expansion are unique and can be obtained using the orthogonality properties of the eigenfunctions.

Showing the eigenvalues are real

- We'll now show that all the eigenvalues are real.
- Suppose λ is an eigenvalue (possibly complex)
- In this case the corresponding eigenfunction is complex.
- Let the eigenvalue be expressed in terms of real and imaginary parts

$$\lambda = \mu + i\nu,$$

- Let the corresponding eigenfunction also be expressed in terms of real and imaginary parts:

$$\phi(x) = U(x) + iV(x).$$

Showing the eigenvalues are real

- In Lagrange's identity set $u = \phi$ and $v = \phi$.
- Then the complex form of Lagrange's identity is still

$$(L\phi, \phi) = (\phi, L\phi)$$

- Or in terms of integrals

$$\int_a^b dx \overline{\phi} L\phi = \int_a^b dx \phi L\overline{\phi}.$$

- But recall that the Sturm-Liouville ODE implies

$$L\phi = \lambda r(x)\phi,$$

- So

$$(\lambda r\phi, \phi) = (\phi, \lambda r\phi),$$

Showing the eigenvalues are real

- So we have
$$\int_a^b \lambda r(x)\phi\overline{\phi}dx = \int_a^b \phi\overline{\lambda}r(x)\overline{\phi}dx,$$
- Another way to write this is
$$(\lambda - \overline{\lambda}) \int_a^b r(x)\phi\overline{\phi}dx = 0.$$
- But from complex numbers we know
$$\phi\overline{\phi} \geq 0$$
and can have the value zero only at isolated points

Showing the eigenvalues are real

- So we must have

$$\int_a^b r(x)\phi\overline{\phi}dx > 0,$$

- So it must be that

$$\lambda = \overline{\lambda},$$

- This means that λ is real.
- This is the same kind of proof we used to show reality of eigenvalues for symmetric matrices.
- This also implies that the eigenfunction $\phi(x)$ is real (up to some complex multiplicative constant which we could take to be real).
- We also see that we need only use the real version of the scalar product for these eigenfunctions because we've shown everything is real

Showing the eigenvectors are orthogonal

- We next show that if $\phi_1(x)$ and $\phi_2(x)$ are two eigenfunctions corresponding to *differing eigenvalues*, that is

$$\lambda_1 \neq \lambda_2$$

  then

$$\int_a^b \phi_1(x)\phi_2(x)r(x)dx = 0,$$

- if you think about this in terms of scalar products this means ϕ_1 and ϕ_2 are orthogonal relative to the weighting function $r(x)$.
- To show this, recall

$$L\phi_1 = \lambda r(x)\phi_1,$$
$$L\phi_2 = \lambda r(x)\phi_2.$$

Showing the eigenvectors are orthogonal

- Now recall the Lagrange identity and set $u = \phi_1$ and $v = \phi_2$.
- Then

$$(L\phi_1, \phi_2) - (\phi_1, L\phi_2) = 0,$$

- So this means

$$\lambda_1 \int_a^b \phi_1(x)\phi_2(x)r(x)dx - \lambda_2 \int_a^b \phi_1(x)\phi_2(x)r(x)dx = 0,$$

- This implies

$$(\lambda_1 - \lambda_2) \int_a^b \phi_1(x)\phi_2(x)r(x)dx = 0.$$

Showing the eigenvectors are orthogonal

- But since $\lambda_1 \neq \lambda_2$ it must be that

$$\int_a^b \phi_1(x)\phi_2(x)r(x)dx = 0.$$

  and so the eigenvectors are orthogonal relative to the weighting factor $r(x)$.
- Note the scalar product for orthogonality is different from that used in the Lagrange identity unless $r(x) = 1$
- It is always possible to redefine the S-L problem slightly (as well as the eigenfunctions) so that the weighting function is always 1 provided $r(x) > 0$.
- Note the proof above is really the same as that used to show orthogonality of eigenvectors of symmetric matrices.