XV. ANGULAR MOMENTUM CONTINUED

A. Ladder operators

Our next job is to construct what in the jargon is known as the 'representation' of the angular momentum operators. By this we mean - understand how we can write the angular momentum operators as finite-size matrices.

The first thing to do is to again realize that the total angular momentum does not change due to space rotations. This implies that it commutes with all the angular momentum operators:

$$[L^2, L_\alpha] \tag{405}$$

to think about the commutator, we may well think of the question - how does L^2 change if we rotate the system about the α axis. L^2 is a scalar, so the answer is - not at all. Therefore:

$$[L^2, L_\alpha] = 0 \tag{406}$$

But this means that we can choose one axis, which we quickly take advantage of to define α as the z exis , and simultaneaously find the eigenstates of both operators (Matrices that commute can be simultaneously diagonalized). Let us denote the states which are eigenstates of both $\hat{L}_z$ and $\hat{\vec{L}}^2$ as $\left|\Lambda^2, m\right\rangle$. It is implied by this labeling that:

$$\hat{L}_z \left|\Lambda^2, m\right\rangle = \hbar m \left|\Lambda^2, m\right\rangle, \text{ and} \hat{\vec{L}}^2 \left|\Lambda^2, m\right\rangle = \hbar^2 \Lambda \left|\Lambda^2, m\right\rangle. \tag{407}$$

It is indeed suggestive that $\hbar\Lambda$ is roughly the magnitude of the *total* angular momentum. Also, at least classically, we know that $\hat{L}_z \sim \cos\theta |\hat{\vec{L}}|$. This strongly suggests that

$$m \leq \Lambda. \tag{408}$$

We know from our analogy to the particle on a ring problem that m should be an integer. Can we construct operators that go between these different m states?

It is again time to get back to the harmonic oscillator problem in search for inspiration. In the case of the harmonic oscillators we constructed the whole Hilbert space in terms of eigenstates of $\hat{\mathcal{H}}$ by figuring out ladder operators:

$$[\hat{\mathcal{H}}, a^\dagger] = \hbar\omega a^\dagger \tag{409}$$

and

$$[\hat{\mathcal{H}}, a] = -\hbar\omega a \tag{410}$$

Can we do something like that here, but for L_Z?

Sure thing. check out:

$$L_- = L_x + iL_y \tag{411}$$

consider:

$$[L_z, L_x + iL_y] = i\hbar(L_y - iL_x) = \hbar(L_x + iL_y) \tag{412}$$

Indeed a ladder operator! It raises the angular momentum by - wait - just an $\hbar$. What about the step down operator? Easy, just the adjoint:

$$L_- = (L_x + iL_y)^\dagger = L_x - iL_y. \tag{413}$$

and

$$[L_z, L_x - iL_y] = -[L_z, L_x + iL_y]^\dagger = i\hbar(L_y + iL_x) = -\hbar(L_x - iL_y). \tag{414}$$

Now let's think a bit. We're thinking about the angular momentum of some particle. What could the total angular momentum be classically? Anything! just like regular momentum. We could have a particle not moving - angular momentum zero. we can have a particle orbiting a nuclei - possibly a lot of angular momentum!

We can, again, in a manner very similar to what we did with the harmonic oscillator. Start with:

$$L^2 = L_x^2 + L_y^2 + L_z^2 \tag{415}$$

the L_z squared we have down - this is the so-called axis of quantization, and we know that $L_z = m\hbar$. Does the rest look familiar? Doesn't it tickle you to think about it as an harmonic oscillator? There we tried:

$$L_x^2 + L_y^2 = (L_x - iL_y)(L_x + iL_y) - i[L_x, L_y] \tag{416}$$

Which coincides with our ladder operators:

$$= L_- L_+ + L_z \tag{417}$$

cool. This means that:

$$L^2 = L_- L_+ + L_z + L_z^2. \tag{418}$$

Now comes the only complicated part. Consider a particle, or a top, in a state with all angular momentum in the L_z direction: $\langle L_z \rangle = \ell\hbar$ with ℓ a positive integer. What would be the result of applying the raising operator?

$$L_+ \left|\Lambda^2, \ell\right\rangle = \left|\Lambda^2, \ell + 1\right\rangle ??? \tag{419}$$

But the L_+ consists of generators of rotation. They cannot change the total angular momentum. The putative state $|\ell + 1\rangle$ must have more angular momentum in the z-direction than its antecedent, but this would contradict the above result! So we must conclude:

$$L_+ |\ell\rangle = 0 \tag{420}$$

Looking back at our equation for the total angular momentum squared:

$$L^2 = L_- L_+ + L_z(\hbar I + L_z) \tag{421}$$

implies:

$$L^2 \left|\Lambda^2, \ell\right\rangle = \hbar^2 \ell(\ell + 1) \left|\Lambda^2, \ell\right\rangle \tag{422}$$

An eigenstate of L^2.

This requires us to modify our notation a bit. Let's call this state $|\ell, \ell\rangle$ why two entries? One for the integer that determines L^2, and the other for the integer that describes L_Z.

Note that the total angular momentum squared is a bit more then $\ell^2\hbar^2$. It has an extra $\ell\hbar^2$. This implaies that even when the entire angular momentum is pointing in the z direction ($L_z = \ell\hbar$, $L_z^2 = \ell^2\hbar^2$) there is some extra angular momentum in the x-y plane, $L_x^2 + L_y^2 = \hbar^2\ell$. This is the result of uncertainty. We know that when two operators do not commute, they can not both have zero uncertainty. Here it is L_x and L_y that have uncertainty.

B. Angular momentum matrices

What did we find? We found that the Hilbert space of angular wave functions is characterized by two numbers, ℓ and m. The first is the total angular momentum:

$$L^2 |\ell, m\rangle = \hbar^2 \ell(\ell + 1) |\ell, m\rangle \tag{423}$$

and the second is the eigenvalue of the z-component of the angular momentum:

$$L_z |\ell, m\rangle = \hbar m |\ell, m\rangle \tag{424}$$

m can go between values smaller or equal the total angular momentum integer:

$$-\ell \leq m \leq \ell. \tag{425}$$

This implies something interesting about degeneracy. The energy of the Hydrogen atom depends only on L^2. Therefore, for each ℓ, there are actually $2\ell + 1$ degenerate states.

So far we found the possible states, and we know they are connected by ladder operators. In fact, we can generally write:

$$L_+ |\ell, m\rangle = c_{\ell,m} |\ell, m+1\rangle \tag{426}$$

or equivalently:

$$L_+ = c_{\ell,m} |\ell, m+1\rangle \langle \ell, m| \tag{427}$$

and by taking the adjoint:

$$L_- |\ell, m\rangle = c^*_{\ell,m} |\ell, m\rangle \langle \ell, m+1| \tag{428}$$

But we don't know the matrix elements in the Ladder operators. Furthermore, we need to find how L_x and L_y operate on all these state. But the formula for the total angular momentum is so pretty, we should have another look at i, but write it as:

$$L_- L_+ = L^2 - L_z(I + L_z) \tag{429}$$

This representation though allows us to write:

$$|c_{\ell,m}|^2 |\ell, m\rangle = L_- L_+ |\ell, m\rangle = \left(L^2 - L_z(I + L_z)\right) |\ell, m\rangle = \hbar^2 \left(\ell(\ell+1) - m(m+1)\right) |\ell, m\rangle \tag{430}$$

and:

$$\frac{1}{\hbar} c_{\ell,m} = \sqrt{\ell(\ell+1) - m(m+1)}. \tag{431}$$

Let's take spin-1 as an example. Each wave function can be written as a superposition of three states:

$$|\psi\rangle = \psi_1 |1, 1\rangle + \psi_0 |1, 0\rangle + \psi_{-1} |1, -1\rangle \tag{432}$$

and could be arranged in vector form:

$$\begin{pmatrix} \psi_1 \\ \psi_0 \\ \psi_{-1} \end{pmatrix} \tag{433}$$

In this subspace of the angular Hilbert space, L_x and L_y are 3X3 matrices. We don't know them directly, but we do know the action of L_+ and L_-. L_+ for instance can only raise L_z by one, so it has to look like this:

$$L_+ = \begin{pmatrix} 0 & c_{1,0} & 0 \\ 0 & 0 & c_{1,-1} \\ 0 & 0 & 0 \end{pmatrix} \tag{434}$$

Straightforwardly,

$$L_- = (L_+)^\dagger = \begin{pmatrix} 0 & 0 & 0 \\ c^*_{1,0} & 0 & 0 \\ 0 & c^*_{1,-1} & 0 \end{pmatrix} \tag{435}$$

If we know $c_{1,-1}$ and $c_{1,0}$, we can also find L_x and L_y. These numbers are now easy to find - we have the identity that connects $L_+ L_-$ with L^2 and L_z:

$$L_- L_+ = L^2 - L_z(I + L_z) \tag{436}$$

Now, the LHS is expressible in terms of our two unknowns:

$$\begin{pmatrix} 0 & c_{1,0} & 0 \\ 0 & 0 & c_{1,-1} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ c^*_{1,0} & 0 & 0 \\ 0 & c^*_{1,-1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & |c_{1,0}|^2 & 0 \\ 0 & 0 & |c_{1,-1}|^2 \end{pmatrix} \tag{437}$$

The RHS is also diagonal in our language, since we are working with eigenstates of the angular momentum in the z direction:

$$L^2 - L_z(I + L_z) = \hbar^2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} \tag{438}$$

So:

$$c_{1,0} = c_{1,-1} = \sqrt{2} \tag{439}$$

And L_x and L_y are easily expressed:

$$L_x = \frac{1}{2}(L_+ + L_-) = \frac{1}{2}\hbar \begin{pmatrix} 0 & \sqrt{2} & 0 \\ \sqrt{2} & 0 & \sqrt{2} \\ 0 & \sqrt{2} & 0 \end{pmatrix} \tag{440}$$

and

$$L_y = \frac{1}{2i}(L_+ - L_-) = \frac{1}{2}\hbar \begin{pmatrix} 0 & -i\sqrt{2} & 0 \\ i\sqrt{2} & 0 & -i\sqrt{2} \\ 0 & i\sqrt{2} & 0 \end{pmatrix}. \tag{441}$$

This way we can get the representations for all integer ℓ's. As we will see in the last problem set, and as you can see in the book, they are associated with spatial representations as well.

BTW, what is the representation for L_z? Clearly:

$$L_z = \hbar \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{442}$$

and in general, the integer $-\ell \leq m \leq \ell$ on the diagonal. knowing that $L_z = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$, what are the spatial representations of a state with z-angular momentum m? Has to be:

$$\psi(\theta, \phi) = P_{\ell m}(\theta) \cdot e^{im\phi}. \tag{443}$$

The θ part is describe by associated Legendre polynomials which I would leave aside for now.

XVI. SPIN 1/2

A. revisiting the ladder operators

We have the ladder operators, so we know that eigenvalues of L_z must be of the form:

$$L_z = m\hbar + m_0\hbar \tag{444}$$

with m being integer, and this way the algebra connects all states to each other. But what could m_0 be? Could it be any number? Let's think for a bit about the possible answer. First, in the midterm, we had a particle in a ring. The solutions for the particle in the ring were:

$$\psi(x) = e^{ikx} \tag{445}$$

such that $2\pi rk = 2\pi m$. This implies:

$$rp = m\hbar \tag{446}$$

so clearly $m_0 = 0$ is allowed, and seems to make sense here. Could it be other things? When we turn our system upside down, since the angular momentum is a vector, it seems that this makes:

$$L_z \to -L_z \tag{447}$$

and $m_0 \to -m_0$. If we want to stay within the same Hilbert subspace, what must m_0 be? Zero works, since $-0 = 0$. So $L_z = m\hbar$ seems to be okay, with m being an integer. We would mape integers to integers. Any other values? $m_0 = 1/2$ is a tempting possibility.