Physics 106a: Lecture 12 — 7 November, 2013

Starred sections are advanced topics for interest and future reference. The unstarred material will not be tested on the final of Ph106a but may be used in Ph106b.

Action-Angle Variables

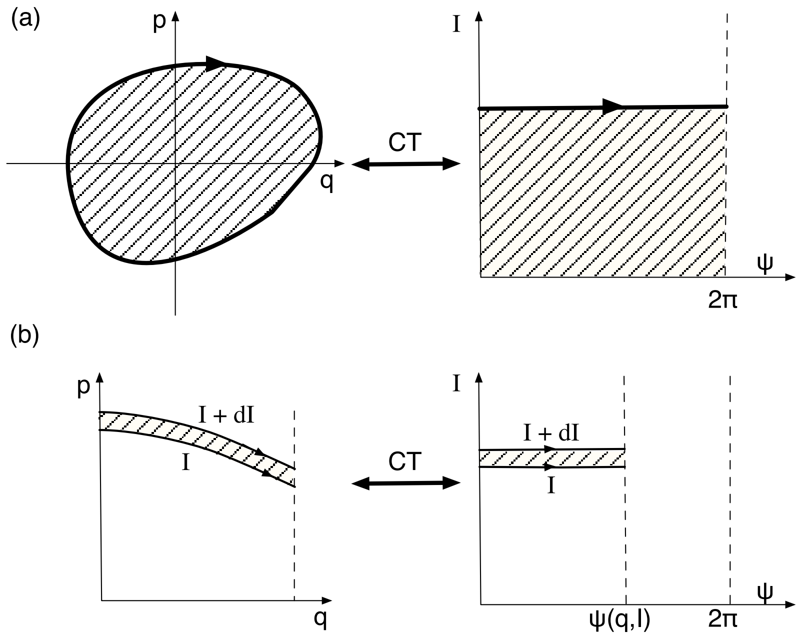


Figure 1: Canonical transformation to action angle variables: (a) Equality of areas gives expression for I; (b) Equality of areas gives $\psi(q, I)$

This approach extends the solution of the simple harmonic oscillator by canonical transformation to magnitude-angle variables to the case of a general periodic orbit. I will look at one degree of freedom (two dimensional phase space) and a time independent Hamiltonian so that the Hamiltonian is a constant, the energy E.

I seek a canonical transformation (I'll assume it is time independent, in which case the Hamiltonian is unchanged) $(q, p) \to (\psi, I)$ such that the Hamiltonian is just a function of the new momentum coordinate I, i.e. $H = H(I)$. Then

$$\dot{I} = 0, \quad \dot{\psi} = \partial H/\partial I = \text{constant}. \tag{1}$$

I choose ψ such that increasing ψ by 2π corresponds to one period of the motion (and so q, p return to the same values). Then the frequency of the motion is $\omega(I) = \partial E/\partial I$. ψ is called the angle variable and I the action variable.

It is simplest to proceed geometrically (Hand and Finch use dynamical equations to find the same results).

Since areas are preserved by the canonical transformation, equating the areas of the orbits in the two descriptions, Fig. 1a gives

$$2\pi I = \oint p\, dq, \tag{2}$$

with the integral taken around the periodic orbit. The orbit is a constant energy E curve, and we can think of this as evaluating $E(I)$. For one dimensional dynamics in a potential $V(q)$

$$I = \frac{1}{\pi}\int_{q_1}^{q_2} \sqrt{2m(E - V(q))}\, dq, \tag{3}$$

with q_1, q_2 the turning points, and the factor of 2 is from the to-and-fro motion. We can now calculate the frequency $\omega = dE/dI$ and then $\psi = \omega t + c$.

To find $q(t)$ we just need $q(\psi, I)$. One way to find this is to equate the areas between portions of orbits for I and $I + \delta I$ extending from $q = 0$ to some q. In the action angle representation the area is just $\psi(q, I)\delta I$ (taking $\psi = 0$ at $q = 0$). Hence from Fig. 1b

$$\psi(q, I) = \frac{\partial}{\partial I}\int_0^q p(q', I)\, dq'. \tag{4}$$

Inverting this to give $q(\psi, I)$ and using $\psi = \omega t + c$ we then have the solution for $q(t)$ (remember I is constant).

The canonical transformation to the action-angle variables is given by the type-2 generating function

$$F_2(I, q) = \int_0^q p(q', I)\, dq'. \tag{5}$$

You can check that the appropriate derivatives give the right expressions for p, ψ. To get a type-1 generating function use the Legendre transformation

$$F_1(\psi, q) = F_2(I, q) - \psi I. \tag{6}$$

Adiabatic invariant

If a parameter of the Hamiltonian giving a periodic orbit is changed slowly (so that the change is small in one period) then the action variable remains approximately constant (the change in the action is zero as the time for a given change in the Hamiltonian is sent to infinity). The energy on the other hand will change in general. and for a ball bouncing off moving walls in problem 4 of Assignment 5. This is an important result with many applications. Here is a sketch of the proof.

The Hamiltonian depends on a parameter α which is varied slowly $\alpha \to \alpha(t)$. For each value of α, we first do a canonical transformation to action angle variables as if α were fixed at that value. For example we use an α-dependent generating function $F_1(q, \psi; \alpha)$. This will give a Hamiltonian that is just a function of the action variable $I(\alpha)$ and α, but not of ψ. Now if we include the time dependence of α, there is an additional term in the Hamiltonian due to the time dependence of the generating function

$$\bar{H} = H(I(\alpha), \alpha) + \frac{\partial F_1}{\partial \alpha}\dot{\alpha}. \tag{7}$$

The time dependence of the action is

$$\dot{I} = -\frac{\partial \bar{H}}{\partial \psi} = -\frac{\partial^2 F_1}{\partial \psi \partial \alpha}\dot{\alpha}. \tag{8}$$

Averaging over a period, and taking $\dot{\alpha}$ to be constant over this interval, since α is slowly varying, gives

$$\langle \dot{I} \rangle \simeq -\frac{\dot{\alpha}}{2\pi}\left[\frac{\partial F_1}{\partial \alpha}(q, \psi + 2\pi, \alpha(T)) - \frac{\partial F_1}{\partial \alpha}(q, \psi, \alpha(0))\right] \simeq -\frac{\dot{\alpha}^2 T}{2\pi}\frac{\partial^2 F_1}{\partial \alpha^2}, \tag{9}$$

using the periodicity with respect to ψ in the last approximation. Integrating to find the change in I for a change α gives a quantity that goes to zero as the rate of variation goes to zero. On the other hand $\langle \dot{E} \rangle \sim \dot{\alpha}$, and so the change in energy will be large if the change in α is large, no matter how slow the variation.

Consider the case of swinging pendulum undergoing small amplitude oscillations. What happens to the amplitude of oscillations if the string is slowly shortened? This is studied by direct manipulations in Hand and Finch Problem 3-12. Applying the result just derived to this case gives $I = E/\omega_0 \simeq$ constant since $\omega = \partial E/\partial I$ is constant for small amplitudes. For a slowly changing length $l(t)$ (this is D in the Hand and Finch problem) the constant action means $E \propto \omega_0 \propto l^{-1/2}$. If the angular amplitude of oscillation is θ_0, the energy is $E \propto l^2\omega_0^2\theta_0^2$ (moment of inertia times angular velocity squared). This gives the variation of the amplitude $\theta_0 \propto l^{-3/4}$, as found in that problem.

Hamilton-Jacobi Theory

Idea

Perform a time dependent canonical transformation to make the new Hamiltonian zero!

$$\bar{H}(\{Q_k\}, \{P_k\}, t) = 0. \tag{10}$$

Then $\dot{Q}_k = 0$, $\dot{P}_k = 0$, and the new position and momenta are constant.

Make the transformation using a type-2 generating function $S(\{q_k\}, \{P_k\}, t)$. Since the P_k are constant, they are conventionally written as α_k. The original momenta are given by $p_k = \partial S/\partial q_k$ so that $\bar{H} = 0$ gives

$$H\left(\{q_k\}, \left\{\frac{\partial S}{\partial q_k}\right\}, t\right) + \frac{\partial S}{\partial t} = 0 \tag{11}$$

(i.e. H evaluated replacing p_k by $\partial S/\partial q_k$). This is a first-order partial differential equation known as the *Hamilton-Jacobi* equation for *Hamilton's principal function* $S(\{q_k\}, \{\alpha_k\}, t)$.[1] The mechanics problem is "reduced" to solving a nonlinear PDE!

Usually the approach is only practical if the PDE is separable in the sense that S can be written as a sum of terms each one depending on just one of the q_k variables (see below for an example) [2]. For an N degree of freedom problem, there will then be $N + 1$ separation constants (one for the partial derivative in each q_k, one for time). Equation (11) provides one relationship between these, so there are N independent separation constants. These constants (or independent combinations of them) may be used as the new (constant) momenta α_k, completing the definition of the canonical transformation. There is considerable flexibility in this choice, corresponding to different transformations consistent with the desired properties, leading to different "flavors" of the Hamilton-Jacobi method. There is one overall additive constant to S, since only derivatives of S are involved in Eq. (11), but this is unimportant, since only derivatives of S lead to physical quantities.

The values of $\{\alpha_k\}$ are set by initial conditions. In particular $p_k(0) = (\partial S/\partial q_k)_{t=0}$ is a function of $\{\alpha_k\}$ and $\{q_k(0)\}$, and so given $\{q_k(0)\}$, $\{p_k(0)\}$ the values $\{\alpha_k\}$ can be found. Also, the new coordinates, again relabeled $Q_k \to \beta_k$ to remind us that they are constants, are given by $\beta_k = \partial S/\partial \alpha_k$. At $t = 0$ this gives

[1] The reason for using the symbol S is because the function is actually the action as a function of the endpoints $\{q_k\}, t$, given by integrating the Lagrangian along the actual dynamical path from the initial conditions, cf. before we had the action as a function of the path for fixed endpoints. We will not be using this fact. See Hand and Finch p220 for more details.

[2] This is a little different from the notion of separability in solving linear PDEs such as Laplace's equation where the solution is written as a product of functions depending on each variable separately.

$\beta_k(\{q_k(0)\}, \{p_k(0)\})$ fixing the values of the constants. Finally at general time we have

$$\beta_k = \frac{\partial S}{\partial \alpha_k} \quad \text{invert for } q_k(t), \tag{12}$$

$$p_k(t) = \frac{\partial S}{\partial q_k}, \tag{13}$$

and the problem is solved.

For a time independent Hamiltonian we can separate out the time dependence of S

$$S = W(\{q_k\}, \{P_k\}) - Et, \tag{14}$$

with E the constant value of H in this case. The function $W(\{q_k\}, \{\alpha_k\})$ is called *Hamilton's characteristic function*. In fact we could arrive at the same equations by performing a *time independent* canonical transformation with a generating function $W(\{q_k\}, \{\alpha_k\})$ to make the Hamiltonian the constant value E rather than zero

$$\bar{H} = H\left(\{q_k\}, \left\{\frac{\partial W}{\partial q_k}\right\}\right) = E, \tag{15}$$

and then could choose E as one of the new momenta.

Example: Mass in gravitational potential

Since the procedure is rather intricate in the abstract, I will illustrate it using a problem you solved in freshman physics: a particle of mass m in a vertical gravitational acceleration g. I just want to illustrate the method, and so to simplify the writing I will set $m = 1$, $g = 1$. (Hand and Finch discuss the problem on pp220-5, including m, g and the y variable.) The The Hamilton-Jacobi equation is

$$\frac{1}{2}\left(\frac{\partial S}{\partial x}\right)^2 + \left[\frac{1}{2}\left(\frac{\partial S}{\partial z}\right)^2 + z\right] + \left(\frac{\partial S}{\partial t}\right) = 0 \tag{16}$$

where the terms are grouped so it is clear the equation can be solved by the method of separation of variables[3], when the solution can be written in the form

$$S = W_1(x) + W_3(z) + W_0(t), \tag{17}$$

with

$$\frac{\partial W_0}{\partial t} = \alpha_0 \tag{18}$$

$$\frac{1}{2}\left(\frac{dW_1}{dx}\right)^2 = \alpha_1 \tag{19}$$

$$\frac{1}{2}\left(\frac{dW_3}{dz}\right)^2 + z = \alpha_3 \tag{20}$$

and

$$\alpha_1 + \alpha_3 + \alpha_0 = 0. \tag{21}$$

[3] Separability does not require that the equation can be written in this way: see for example the Kepler solution in Goldstein et al. §10.5

The solutions are

$$W_0 = \alpha_0 t, \quad \text{with} \quad \alpha_0 = -E \tag{22}$$

$$W_1 = \pm\sqrt{2\alpha_1}x + \text{constant}, \tag{23}$$

$$W_3 = \pm\sqrt{\tfrac{8}{9}}(\alpha_3 - z)^{3/2} + \text{constant}. \tag{24}$$

where the $\pm$ are independent, and come from the two possible signs on taking the square roots.

Using Eq. (21) to eliminate E gives

$$S = \pm\sqrt{2\alpha_1}x \pm \sqrt{\tfrac{8}{9}}(\alpha_3 - z)^{3/2} - (\alpha_1 + \alpha_3)t. \tag{25}$$

I've dropped an additive constant (the sum of the integration constants of the W solutions) that is unimportant since only derivatives of S are used.

We may choose α_1, α_3 as the new constant momenta. The corresponding new coordinates are then

$$\beta_1 = \frac{\partial S}{\partial \alpha_1} = \pm\frac{1}{\sqrt{2\alpha_1}}x - t, \tag{26}$$

$$\beta_3 = \frac{\partial S}{\partial \alpha_3} = \pm\sqrt{2(\alpha_3 - z)} - t. \tag{27}$$

Also, the expressions for the original momenta are

$$p_x = \frac{\partial S}{\partial x} = \frac{dW_1}{dx} = \pm\sqrt{2\alpha_1}, \tag{28}$$

$$p_z = \frac{\partial S}{\partial z} = \frac{dW_3}{dz} = \mp\sqrt{2(\alpha_3 - z)}. \tag{29}$$

Now let's fix constants from the initial conditions for a particular problem: shoot the cannonball from $x = z = 0$ at $t = 0$ at 45° to the horizontal with a speed 2, so that $p_x(0) = p_z(0) = \sqrt{2}$. These give

$$\alpha_1 = 1 \quad \text{and use upper choice of sign,} \tag{30}$$

$$\alpha_3 = 1 \quad \text{and use lower choice of sign,} \tag{31}$$

$$\beta_1 = 0, \tag{32}$$

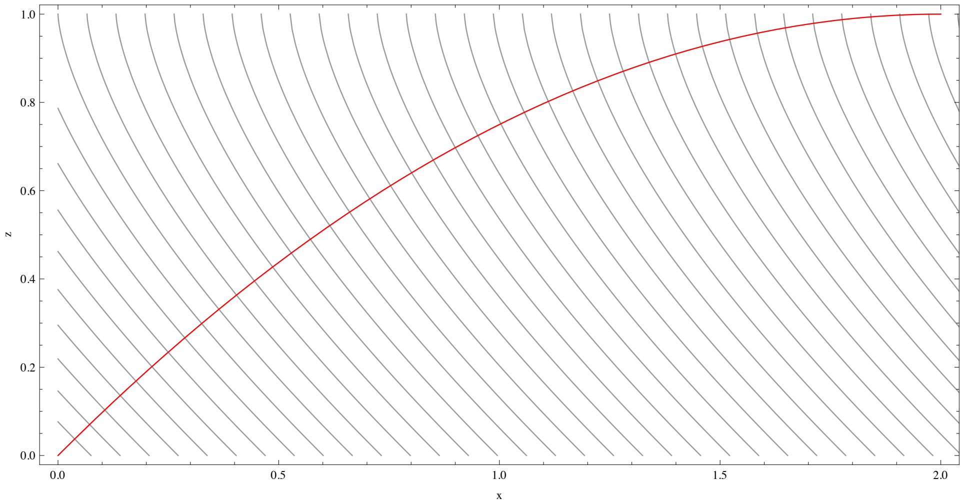
$$\beta_3 = -\sqrt{2}. \tag{33}$$

Read off the solutions: from Eqs. (28,29)

$$p_x = \sqrt{2}, \quad p_z = \sqrt{2(1 - z)}; \tag{34}$$

and inverting Eqs. (26,27)

$$x(t) = \sqrt{2}t, \quad z(t) = \sqrt{2}t - \tfrac{1}{2}t^2. \tag{35}$$



The figure shows contours of constant S at some time (gray lines) and the particle trajectory (red line) for $\alpha_1 = 1, \alpha_3 = 1, E = 2$. The particle trajectory is perpendicular to the constant S contours $\vec{p} = \vec{\nabla} S$. This is analogous to constructing the *ray* perpendicular to the wave fronts corresponding to the *geometric optics* approximation. The surfaces in x, z of constant S propagate with the *phase speed* $E/|\nabla W| = E/|p|$. The actual particle velocity is the *group speed*.

An alternative way of manipulating the constants We could instead use α_1 and $E = -\alpha_0$ as the new constant momenta and write

$$S = W(x, z, \alpha_1, E) - Et, \tag{36}$$

with

$$W = \pm\sqrt{2\alpha_1}x \pm \sqrt{\tfrac{8}{9}}(E - \alpha_1 - z)^{3/2} + C \tag{37}$$

so that

$$\beta_1 = \frac{\partial S}{\partial \alpha_1} = \frac{\partial W}{\partial \alpha_1} = \pm\frac{1}{\sqrt{2\alpha_1}}x \mp \sqrt{2}(E - \alpha_1 - z)^{1/2}, \tag{38}$$

$$p_x = \frac{\partial S}{\partial x} = \frac{\partial W}{\partial x} = \pm\sqrt{2\alpha_1}, \tag{39}$$

$$p_z = \frac{\partial S}{\partial z} = \frac{\partial W}{\partial z} = \mp\sqrt{2(E - \alpha_1 - z)}. \tag{40}$$

The initial conditions give $\alpha_1 = 1$, $E = 2$, $\beta_1 = \sqrt{2}$ and the sign choices such that

$$W = \sqrt{2}x - \tfrac{8}{9}(1 - z)^{3/2}. \tag{41}$$

Finally Eq. (38) gives

$$z = x - \tfrac{1}{4}x^2. \tag{42}$$

This approach gives us the equation of the orbit, without having to solve for the time dependence, much as in our approach to the Kepler problem. If we wanted the time dependence we could use the equation for the constant coordinate β_0 conjugate to E

$$\beta_0 = \frac{\partial S}{\partial E} = -t. \tag{43}$$

General comments

The method seems to be useful mainly (only?) if the PDE can be solved by the method of the separation of variables. As in other PDE problems, whether the method work depends on the coordinate system chosen, as well as on the form of the Hamiltonian. For some less trivial examples, see Hand and Finch pp 226-8 for motion in the potential $V(r,z) = -k/r + gz$, and Goldstein, Poole and Safko pp. 450-1 and §10.8 for the solution of the Kepler problem in r, θ, Φ coordinates without restricting the dynamics to the $\theta = \pi/2$ plane. As the latter example shows, since the method proceeds through identifying $2N$ constants, constants of the motion appear naturally.

For separable systems with periodic motion, the Hamilton-Jacobi method connects with the action-angle approach. For a time independent Hamiltonian we have $p_k = dW_k/dq_k$ with W_k the separated components of W i.e. $W = \sum_k W_k(q_k)$, and the action variable is

$$I_k = \frac{1}{2\pi} \oint \frac{dW_k}{dq_k}\, dq_k \tag{44}$$

which can be evaluated in terms of the separation constants α_k. (The integral is not necessarily as trivial as it looks, since different branches of $p_k(q_k)$ may be involved over different portions of the loop, and q_k may be an angle variable increasing by 2π over the loop.) Thus $\{\alpha_k\} \Leftrightarrow \{I_k\}$ and we can rewrite $W = W(\{q_k\}, \{I_k\})$ and calculate $H(\{I_k\})$, $\omega_k = \partial H/\partial I_k$, $\psi_k = \partial W/\partial I_k$, etc. For an example, again see the discussion of the Kepler problem by Goldstein et al. Note that these authors use J for the action variable, and without the $(2\pi)^{-1}$ factor so that my I_k is their $J_k/2\pi$.

Connection with quantum mechanics *

As we have seen in the above example, for Hamilton's principal function of the form $S(\{q_k\}, t) = W(\{q_k\}) - Et$ the surfaces in q_k of constant S propagate with a speed $E/|\nabla W| = E/|p|$: particle dynamics can be understood in terms of wave propagation (the propagation of wavefronts of constant S)! Miraculously, these "waves" are the waves of quantum mechanics in the limit $\hbar \to 0$.[4] I'll briefly sketch the argument for your interest, although it will probably only make sense after you have done a fair amount of quantum mechanics.

Schrödinger's equation for a particle with classical Hamiltonian $H = p^2/2m + V(\vec{r}, t)$ is

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi = -\frac{\hbar^2}{2m}\nabla^2 \Psi + V(\vec{r}, t)\Psi. \tag{45}$$

The semiclassical limit is given by considering $\hbar$ small and looking for a solution in the WKB form

$$\Psi = \sqrt{\rho(\vec{r}, t)}\, e^{iS(\vec{r},t)/\hbar} \tag{46}$$

where I assume gradients and time dependence of ρ, S are $O(1)$ and do not bring in any powers of $\hbar$ to leading order. For the moment S is unknown. We find it by substituting Ψ into Schrödinger's equation, and collecting leading order terms, those in $\hbar^0$. Inspection shows these will involve derivatives of the phase, bringing inverse powers of $\hbar$ that cancel the $\hbar$ factors in the equation, and also the potential term, to give

$$\frac{1}{2m}(\vec{\nabla} S)^2 + V(\vec{r}, t) + \frac{\partial S}{\partial t} = 0. \tag{47}$$

This is exactly the Hamilton-Jacobi equation for the Hamiltonian H, so we see that Hamilton's principal function $S(\vec{r}, t)$ is equal to $\hbar$ times the quantum phase.

Michael Cross: November 5, 2013

[4] Of course $\hbar$ is a constant, and cannot be sent to zero. What I mean is that $\hbar$ is small compared with something appropriate in the dynamics, presumably the action since this has the same dimensions. More practically, we require the potential to vary little over a wavelength of the quantum wave given by $2\pi\hbar/p$ with p the momentum.