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Lecture 6: Equilibria and Oscillations

We study *equilibria* and *linear oscillations* – one of the two types of behavior for dynamics near an equilibrium. You've probably seen most of this material before. Some new points (developed further in the notes below, or see Hand and Finch, chapter 3) might be:

- The Lagrangian description of equilibria, their stability, and the behavior for small displacements from the equilibria.
- The use of scaled variables, so that the equation for a damped simple harmonic oscillator is

$$\ddot{q} + \frac{1}{Q}\dot{q} + q = 0, \tag{1}$$

  with Q the quality factor that represents the dissipation in scaled units. The advantage of this approach is that *all* damped SHOs are reduced to the study of a single mathematical equation, with one parameter Q characterizing the different types of behavior (overdamped, underdamped etc.). Note that the scaled time represents $\omega_0 t$ in the original units, with ω_0 the "linear resonance frequency" of the oscillator, and a scaled frequency represents ω/ω_0 in the original units.

- The complex representation so that a physical solution (for $Q^{-1} = 0$) might be (HF Eq. (3.24))

$$q = \mathrm{Re}(\mathcal{A}_c e^{it}), \tag{2}$$

  where the complex amplitude $\mathcal{A}_c$describes both the magnitude and the phase of the oscillations. As we use this representation more, we tend to lazily write the solution as "$q = \mathcal{A}_c e^{it}$" with the implication to take the real part at the end of the calculation. An important point is that you must remember to form the real physical solution *before* calculating nonlinear quantities such as q^2.

- The solution for an undamped oscillator driven at the resonance frequency: it grows with an amplitude increasing linearly in time. This is important when we study nonlinear oscillators next term.

Equilibria

An *equilibrium point* or *fixed point* as a time independent solution is some reasonable coordinate system[1]. If we set the initial condition of coordinates $\{q_k\}$ to the equilibrium point and velocities $\{\dot{q}_k\} = 0$, then we require $\{\ddot{q}_k\} = 0$ for the particle to remain there. In the Euler-Lagrange equation which tells us $\{\ddot{q}_k\}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \tag{3}$$

expand out the first term in partial derivatives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) + \frac{\partial}{\partial q_k}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\dot{q}_k + \frac{\partial}{\partial \dot{q}_k}\left(\frac{\partial L}{\partial \dot{q}_k}\right)\ddot{q}_k. \tag{4}$$

[1] Of course for any dynamic solution $q = q_d(t)$ we can define — once we know the solution — a new coordinate $q - q_d(t)$ that is time independent. So by reasonable, I mean one that is chosen before you know a specific solution. However "time independent" does depend on the choice of coordinate system. We might often take a time independent solution in a rotating frame to be an equilibrium, even though it is then time dependent in a nonrotating frame.

At an equilibrium the second and third terms on the right hand side are zero. If we restrict our attention to time independent Lagrangians, the first term on the right hand side

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_k}\right) = \frac{\partial}{\partial \dot{q}_k}\left(\frac{\partial L}{\partial t}\right) \tag{5}$$

is also zero. I will make this restriction.[2] Then we see from Eq. (3) that the condition for an equilibrium is

$$\left.\frac{\partial L}{\partial q_k}\right|_{\dot{q}_k=0} = 0, \qquad \text{general.} \tag{6}$$

Solve this for the time independent q_k. For time independent holonomic constraints the kinetic energy T is a quadratic form in $\dot{q}_k$ and so at equilibrium $\partial T/\partial q_k = 0$ because $\dot{q}_k = 0$. In this case the equilibrium condition reduces to

$$\frac{\partial V}{\partial q_k} = 0, \qquad \text{for time independent constraints,} \tag{7}$$

so the potential is then a maximum, minimum, or (in greater than 1 dimension) a saddle.

For the rest of the lecture I simplify to systems with a single degree of freedom q. Let's choose the coordinate so that an equilibrium is at $q = 0$. Expand the Lagrangian for q near the equilibrium (small q). Ignoring terms that do not contribute to the equations of motion (e.g. time differentials of functions of q) gives

$$L = Dq^2 + F\dot{q}^2 + \cdots \tag{8}$$

where a possible term Bq is absent by the equilibrium condition which gives $B = 0$. The coefficient F is an effective mass for a kinetic energy, and so I assume it is positive. The Euler-Lagrange equation gives the equation of motion for small q, neglecting higher order terms beyond linear ones,

$$\ddot{q} - (D/F)q = 0. \tag{9}$$

This equation is easily solved. For $D/F > 0$ (L a minimum, or V a maximum – remember F is assumed to be positive) there is a solution that grows exponentially $q \propto e^{\sqrt{D/F}t}$, taking q away from the equilibrium point. This is an *unstable* equilibrium. For $D/F < 0$ (L a maximum or V a minimum) the solutions are oscillatory with frequency $\omega_0 = \sqrt{|D|/F}$, and q remains near the equilibrium point if the initial $q, \dot{q}$ are small.

We can eliminate the constants D, F from the analysis by introducing a scaled time variable $\bar{t} = \sqrt{|D|/F}t$ to give

$$\frac{d^2q}{d\bar{t}^2} \pm q = 0, \tag{10}$$

with the plus sign corresponding to $D < 0$ (stable) and the minus sign to $D > 0$ (unstable). This rescaling of time means that we need to mathematically investigate just *two equations* (one for the stable case, one for the unstable) to study the behavior near an equilibrium for *any* one dimensional system. Once we have found the mathematical solution (it will be a function of the scaled time $\bar{t}$, of course) we can return to physical units by setting $\bar{t} \to \omega_0 t$ with $\omega_0 = \sqrt{|D|/F}$. Rescaling variables to a dimensionless form to eliminate redundant constants introduced by our choice of units is a powerful technique for collapsing many physical problems onto a reduced family of mathematical ones. This is exploited a great deal in fluid mechanics, where there are often many parameters characterizing the physical system; hence the terms you will hear

[2] One case of a time dependent Lagrangian leading to equilibria of interest is the one for the dynamics inside a uniformly accelerating rail car (rocket ship, Volkswagen ...). As we saw, this leads to a constant effective force, and so to possible equilibria. For example: A child is holding a balloon in a Volkswagen moving with constant positive acceleration a. What is the angle of the string to the vertical in equilibrium? Is the balloon in front or behind of the child's hand?

in this field such as *Reynolds number*, *Rayleigh number* etc. Once the rescaling is done, the bar over the new variable is often dropped to save writing, and we have to remember that we are using the scaled time variable. The technique is not really necessary for the simple equation (9) and some students prefer to keep the original equation, perhaps writing $\sqrt{|D|/F} \to \omega_0$, to immediately see how this parameter affects the results derived. But you should remember it for more complicated problems, and I will use it here.

Using the scaled units, we can make a table of the two types of behavior

Stability	Lagrangian	EOM	Solutions	Hamiltonian	Potential
Stable	$\frac{1}{2}(\dot{q}^2 - q^2)$	$\ddot{q} + q = 0$	$e^{\pm it}$; $\cos t, \sin t$	$\frac{1}{2}(\dot{q}^2 + q^2)$	Minimum
Unstable	$\frac{1}{2}(\dot{q}^2 + q^2)$	$\ddot{q} - q = 0$	$e^{\pm t}$	$\frac{1}{2}(\dot{q}^2 - q^2)$	Maximum

In the unstable case, q grows and eventually reaching large values outside the the range of validity of the small q expansion, and little more can be said with any generality For the stable case, a small initial condition leads to $q(t)$ remaining small, and so we can study the general behavior of this important case further — linear oscillations near a stable equilibrium.

Undamped simple harmonic oscillator (SHO)

Equation of motion

The equation of motion is

$$\ddot{q} + |D/F|q = 0. \tag{11}$$

Solution

The general analysis has therefore led us to study an equation of the form

$$\ddot{q} + q = 0\,. \tag{12}$$

This is a 2nd order (two time derivatives), linear (single power of $q, \dot{q}$ etc.), homogeneous (no constant or $f(t)$ term) ODE (ordinary differential equation) with constant coefficients. For linear, homogeneous equations the principle of the superposition applies: if $q_1(t)$ and $q_2(t)$ are solutions then $q = aq_1(t) + bq_2(t)$ is also a solution. A second order ODE solved as an initial value problem (solution defined by conditions at some initial time we set to $t = 0$) requires two initial conditions $q(0)$ and $\dot{q}(0)$, as we expect for a mechanics problem. We can write the general solution as the superposition form where $q_1(t), q_2(t)$ are *any* choice of two linearly independent solutions, and the constant a and b are fixed by the initial conditions.

The constant coefficients mean that an exponential satisfies the equation: $q \propto e^{\lambda t}$ where, substituting in Eq. (12)

$$\lambda^2 + 1 = 0 \qquad \text{giving} \qquad \lambda = \pm i\,. \tag{13}$$

The general solution is therefore

$$q(t) = ae^{it} + be^{-it}. \tag{14}$$

The (complex) constants a and b are fixed by the initial conditions $a = [q(0) - i\dot{q}(0)]/2$, $b = [q(0) + i\dot{q}(0)]/2$, so that the solution becomes

$$q(t) = q(0)\frac{1}{2}(e^{it} + e^{-it}) + \dot{q}(0)\frac{1}{2i}(e^{it} - e^{-it}). \tag{15}$$

Since $q(0)$ and $\dot{q}(0)$ are real, this shows that the solution is real, as must be true physically, although we are used complex notation to arrive at the result. Of course the combination of exponentials are just $\cos t$ and $\sin t$ and another way to solve Eq. (12) is to remember that a sinusoidal function satisfies this equation, but this approach often leads to more complicated algebra.

An alternative scheme is to note that since $q(t)$ is real, the be^{-it} must be the complex conjugate of ae^{it} so we can write the solution as

$$q = \mathrm{Re}\left[A_c e^{it}\right] \qquad \text{with } A_c = a/2. \tag{16}$$

Writing $A_c = Ae^{i\phi}$ with A the amplitude and ϕ the phase (both real) gives

$$q = A\cos(t + \phi), \tag{17}$$

with A, ϕ set by the initial conditions. You will often see the solution written as Eq. (16) as

$$\text{“ } q(t) = A_c e^{it} \text{ ”} \tag{18}$$

(without the quotes, but they should really be there!) with the understanding that the physical $q(t)$ is given by taking the real part at the end of the calculation. (Hand and Finch write this q_{complex} but then drop the “complex”.) This is a *dangerous practice*. For example if we want the quantity $q^2(t)$ such as in the energy, this is *not given* by squaring the complex form $A_c e^{it}$ and then taking the real part: in nonlinear terms you *must* form the correct (physical) function first $\mathrm{Re}[A_c e^{it}]$, and then calculate the nonlinear term (e.g. the square). Check this for yourself.

Driven Oscillator

Now we turn to the *driven linear harmonic oscillator* described by the equation (after scaling out constants)

$$\ddot{q} + q = F(t)\,. \tag{19}$$

This equation is *inhomogeneous* but still *linear* in q. This means that for a forcing function that is a linear combination of two forcing functions, the solution is the same linear superposition of the individual solutions

$$F_1(t) \Rightarrow q_1(t),\ F_2(t) \Rightarrow q_2(t) \qquad \text{then} \qquad F(t) = \alpha F_1(t) + \beta F_2(t) \Rightarrow q(t) = \alpha q_1(t) + \beta q_2(t). \tag{20}$$

I look at step, impulse, and oscillatory forcing functions. Superposition means that the solution of any one of these can be derived from the others, and that each one can be used to construct the solution to an arbitrary $F(t)$.

The general solution of Eq. (19) is

$$q(t) = q_s(t) + q_t(t) \tag{21}$$

where

- $q_s(t)$ is *a* solution of the inhomogeneous equation (19) — the *particular integral*, or the “steady state” solution;
- $q_t(t)$ is the general solution to the homogeneous equation (Eq. (19) with $F(t) = 0$) with two integration constants) — the *complementary function*, or the “transient” solution[3]. The solution is

  $$q_t(t) = a\cos t + b\sin t, \quad \text{or} \quad Ae^{it} + Be^{-it}, \tag{22}$$

  with a, b or A, B the two constants to be fixed from the initial conditions. The solution could also be written $q_t(t) = a\cos(t + \phi)$ with a, ϕ the two constants.

[3] If we added dissipation to the equation, $q_t(t)$ would decay exponentially.

Thus Eq. (21) has two unknown constants that are set by initial conditions. I assume that the forcing is zero for $t < 0$ and that this implies $q(t) = 0$ for $t < 0$ (certainly the case even if only infinitesimal damping is present), so that the initial conditions are $q(0) = \dot{q}(0) = 0$.

You should make sure you can arrive at the following solutions:

Unit step function: $F(t) = \Theta(t)$ with $\Theta(t)$ the Heaviside function

$$\Theta(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 & \text{for } t > 0 \end{cases} \tag{23}$$

The solution for $t > 0$ is

$$q = 1 - \cos t. \tag{24}$$

Check that $q(0) = \dot{q}(0) = 0$, as required to match to $q(t < 0)$.

Unit impulse: $F(t) = \delta(t)$ with $\delta(t)$ the Dirac delta function defined by its integral properties

$$\int_{t'-a}^{t'+b} g(t)\delta(t - t')dt = g(t') \tag{25}$$

for any smooth $g(t)$ and any $a, b > 0$. The function $\delta(t - t')$ (actually a *generalized function*) is infinitely high and infinitely narrow, but with a unit integral. The solution for $t > 0$ is (remember I am assuming $q(t) = 0$ for $t \leq 0$)

$$q(t) = \sin t. \tag{26}$$

We can argue this physically: a unit impulse gives unit momentum change, so the speed $\dot{q}$ for our unit mass "particle" after the impulse is $\dot{q}(t = 0^+) = 1$, whilst $q(0^+)$ remains zero. Or we can integrate the equation of motion formally from $t = -\epsilon$ to $t = \epsilon$ with $\epsilon \to 0$.

The solution for a unit impulse at time t' (i.e., $F(t) = \delta(t - t')$) is called the *Green's function*, and is (by a simple time translation)

$$G(t, t') = \begin{cases} \sin(t - t') & \text{for } t > t' \\ 0 & \text{for } t < t' \end{cases} \tag{27}$$

assuming no response before the impulse is applied (the casual Green's function).

By superposition, the solution for a general forcing $F(t)$ (which can trivially be written as $F(t) = \int_{-\infty}^{\infty} \delta(t - t')F(t')\,dt'$) is

$$q(t) = \int_{-\infty}^{\infty} G(t, t')F(t')dt' = \int_{-\infty}^{t} \sin(t - t')F(t')dt'. \tag{28}$$

Oscillatory forcing (switched on at $t = 0$):

$$F(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ \cos \omega t & \text{for } t > 0 \end{cases} \tag{29}$$

The solution for $t > 0$ is

$$q(t) = \frac{1}{1 - \omega^2} \cos \omega t - \frac{1}{1 - \omega^2} \cos t. \tag{30}$$

(The first term oscillating at the drive frequency is the particular integral, the second term oscillating at the natural frequency is the complementary function, with the amplitude and phase obtained from the initial conditions.)

Driving on resonance $\omega = 1$, this expression just gives infinity. We can instead use the Green's function to calculate the response to a $\cos t$ forcing switched on at $t = 0$

$$q(t) = \int_0^t \sin(t - t') \cos t' \, dt'. \tag{31}$$

The result is

$$q(t) = \tfrac{1}{2} t \sin t \tag{32}$$

showing oscillations with an amplitude growing linearly in time. Such a growth is called *secular*. Secular growth of oscillations for an undamped system driven at resonance is an important physical process, and is responsible for many interesting phenomena such as the rings of Saturn. We will also find it to be important when we investigate perturbation methods in mechanics next term.

Appendix: Damped simple harmonic oscillator

In the real world, mechanical systems usually have some dissipation or damping. In this Appendix I generalize the results discussed in class and above for no dissipation to the case where there is dissipation. A common form (*not* valid for solid on solid friction) is a force proportional to the negative of the velocity.

Undriven motion

For no drive this leads to the equation of motion

$$\ddot{q} + \frac{1}{Q}\dot{q} + q = 0 \tag{33}$$

with Q^{-1} proportional to the dissipation coefficient. Q is the *quality factor* of the oscillator: large Q is high quality (small dissipation); small Q is low quality (high dissipation).

Sometimes, this type of dissipation force proportional to velocity is shoe-horned into a Lagrangian type approach by introducing a dissipation function $\mathcal{D}(\{\dot{q}_k\})$ such that the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial \mathcal{D}}{\partial \dot{q}_k} = 0. \tag{34}$$

For the damped simple harmonic oscillator the dissipation function (using the scaled time) would be

$$\mathcal{D} = \tfrac{1}{2} Q^{-1} \dot{q}^2. \tag{35}$$

Note that $2\mathcal{D}$ is the rate of energy dissipation.

The general solution to Eq. (33) is

$$q(t) = A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t} \tag{36}$$

with $\lambda_\pm$ the roots of the polynomial $\lambda^2 + Q^{-1}\lambda + 1 = 0$

$$\lambda_\pm = -\frac{1}{2Q} \pm \sqrt{\frac{1}{4Q^2} - 1}. \tag{37}$$

The behavior depends on the sign inside the $\sqrt{\ }$.

$Q > \frac{1}{2}$	Underdamped	$q(t) = Ae^{-\frac{t}{2Q}}\cos(\omega' t + \phi)$ with $\omega' = \sqrt{1 - \frac{1}{4Q^2}}$
$Q < \frac{1}{2}$	Overdamped	$q(t) = A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t}$ with λ_+, λ_- both real and negative
$Q = \frac{1}{2}$	Critically damped	$q(t) = (A + Bt)e^{-t}$ (special case, solution form not pure exponential)

See the Mathematica notebook for plots of these solutions (there I chose $q(0) = 1, \dot{q}(0) = 0$).

The energy $E = \frac{1}{2}(q^2 + \dot{q}^2)$ decays as

$$\dot{E} = \dot{q}(\ddot{q} + q) = -\frac{1}{Q}\dot{q}^2 \tag{38}$$

where the second equality comes from the equation of motion. Note that the right hand side is always negative (or zero). For large Q oscillatory motion, the energy decays little in one cycle, and averaging over one cycle gives

$$\langle V \rangle = \langle \tfrac{1}{2} q^2 \rangle \simeq \langle T \rangle = \langle \tfrac{1}{2}\dot{q}^2 \rangle \simeq \tfrac{1}{2}\langle E \rangle \tag{39}$$

and so the average energy decays as

$$\langle \dot{E} \rangle \simeq -\frac{\langle E \rangle}{Q}. \tag{40}$$

This is exponential decay with a $1/e$ time of Q (dimensionless, because we have scaled time to units of ω_0^{-1}). There are $Q/2\pi$ oscillations in this decay time - a result independent of the time scaling. Another way of saying this is the the fractional energy loss per period (cycle) is $2\pi/Q$. These expressions are for large Q.

Driven motion

The *driven damped linear harmonic oscillator* is described by the equation (after scaling out constants)

$$\ddot{q} + \frac{1}{Q}\dot{q} + q = F(t)\,. \tag{41}$$

Using the same method as before gives the following solutions (for $q(t \le 0) = 0$):

Unit step function: $F(t) = \Theta(t)$ with $\Theta(t)$ the Heaviside function

$$\Theta(t) = \begin{cases} 0 & \text{for } t \le 0 \\ 1 & \text{for } t > 0 \end{cases} \tag{42}$$

The solution is[4]

$$q = 1 - e^{-t/2Q}(\cos\omega' t + \frac{1}{2\omega' Q}\sin\omega' t), \tag{43}$$

with $\omega' = \sqrt{1 - \frac{1}{4Q^2}}$. Check that $q(0) = \dot{q}(0) = 0$, as required to match to $q(t < 0)$.

[4] I write the expression in the form natural for the underdamped case $Q > \frac{1}{2}$.

Unit impulse: For a unit impulse at the origin $F(t) = \delta(t)$ the solution is

$$q(t) = \frac{1}{\omega'} e^{-t/2Q} \sin \omega' t \tag{44}$$

The solution for a unit impulse at time t' (i.e., $F(t) = \delta(t - t')$) gives the Green's function

$$G(t, t') = \begin{cases} \frac{1}{\omega'} e^{-(t-t')/2Q} \sin \omega'(t - t') & \text{for } t > t' \\ 0 & \text{for } t < t' \end{cases} \tag{45}$$

assuming no response before the impulse is applied (the casual Green's function).

Uisng superposition the solution to a general forcing $F(t)$ is

$$q(t) = \int_{-\infty}^{\infty} G(t, t') F(t') dt' = \int_{-\infty}^{t} \frac{1}{\omega'} e^{-(t-t')/2Q} \sin \omega'(t - t') F(t') dt'. \tag{46}$$

Oscillatory forcing: We are usually mainly interested in the steady state solution. Write the forcing $F = \cos \omega t$ as $F = \mathrm{Re}[e^{i\omega t}]$, calculate the response $q_c(t)$ to the forcing $F_c(t) = e^{i\omega t}$, and take the real part at the end. For $F_c(t)$ the steady state solution is

$$q_c(t) = A_c e^{i\omega t} \quad \text{with} \quad A_c = A e^{i\phi}, \; A, \phi \text{ real} \tag{47}$$

and

$$A_c = \frac{1}{(-\omega^2 + 1) + \frac{i\omega}{Q}}. \tag{48}$$

This gives the solution for $F(t) = \cos \omega t$ as $q(t) = A \cos(\omega t + \phi)$.

The amplitude is given by

$$A^2 = \frac{1}{(\omega^2 - 1)^2 + \omega^2/Q^2} \simeq \frac{1}{4} \frac{1}{(\omega - 1)^2 + 1/(4Q^2)} \tag{49}$$

where the second expression (a *Lorentzian*) is a good approximation for large Q and near the large response $\omega \simeq 1$. For large Q the response is a sharp *resonance peak* centered at $\omega_r \simeq 1$, with full width at half intensity (half height in A^2) $1/Q$. Note for large Q the large enhancement at resonance compared with the static response

$$A_{\max} = A(\omega_r) \simeq Q \, A(\omega = 0). \tag{50}$$

The phase is

$$\phi = -\tan^{-1} \left(\frac{\omega/Q}{1 - \omega^2} \right) \tag{51}$$

and varies from slightly negative for low frequencies, passing rapidly (for large Q) through $-\pi/2$ on resonance, and decreasing to $-\pi$ for large frequencies. You should plot $A(\omega), \phi(\omega)$ for various Q to become familiar with the shapes.

This *resonant response* – an enhanced amplitude or rapidly varying phase over a narrow frequency range – is very widely used in experimental science and technology to extract a desired signal from the background. Remember in these expression we are measuring time in units of ω_0^{-1} with ω_0 the frequency of the undamped oscillator: you should make sure you can translate all these results back into original, unscaled time and frequency units.

The response to an oscillatory forcing we have calculated can be used to calculate the response to an arbitrary $F(t)$ using the method of Fourier transforms.

Michael Cross October 17, 2013