ACM 100b

Frobenius expansions for second order ODEs

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Computing the Frobenius expansion

- So far we have shown that a Taylor series can be used to expand the solution of an ODE at an ordinary point.
- At a regular singular point, a Taylor series solution (in general) won't work.
- Consider for example
$$y'' + \frac{y}{4x^2} = 0$$
which has a regular singular point at $x = 0$.
- We can try a Taylor series of the form
$$y = \sum_{n=0}^{\infty} a_n x^n.$$
- Plugging this in we get
$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} x^n = -\sum_{n=0}^{\infty} a_n x^{n-2}/4.$$

Computing the Frobenius expansion

- But now look at what happens when we try to match powers in

$$\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n = -\sum_{n=0}^{\infty} a_n x^{n-2}/4.$$

- We see immediately there is a problem.
- The first two powers that need to be matched are x^{-2} and x^{-1} on the right hand side of the equation.
- They do not match anything coming from the left hand side.
- To satisfy them we would have to set $a_0 = 0$ and $a_1 = 0$ meaning all other coefficients are zero too.

Computing the Frobenius expansion

- But we see $x = 0$ is a regular singular point of

$$y'' + \frac{y}{4x^2} = 0$$

- So instead we use the Frobenius form

$$y(x) = x^\alpha \sum_{n=0}^{\infty} a_n x^n.$$

- This case is a bit trivial because we get

$$\sum_{n=0}^{\infty} (n+\alpha)(n+\alpha-1) a_n x^{n+\alpha-2} = -(1/4) \sum_{n=0}^{\infty} a_n x^{n+\alpha-2}.$$

- We see here that at $n = 0$ we get an equation

$$[\alpha(\alpha-1) + 1/4]\, a_0 = 0.$$

Computing the Frobenius expansion

- We can get a nontrivial solution and leave a_0 arbitrary if we choose α to satisfy the equation

$$\alpha(\alpha - 1) + 1/4 = 0.$$

- There will be at least one root and then a_0 can be arbitrary.
- For the remaining powers of x there is no recursion relation.
- In order to make the equation work term by term we have to set the coefficients $a_n = 0$ for $n > 0$ and that completes the solution.
- So we got an exact solution $y = a_0 x^{1/2}$
- In fact, here we see that there is a double root $\alpha = 1/2$ and so the two solutions are

$$x^{1/2} \text{ and } x^{1/2} \ln x.$$

- The second solution is gotten from reduction of order.
- In this case we got the exact solution because our ODE is of Euler form.

A less trivial example

- Consider the modified Bessel equation

$$y'' + \frac{y'}{x} - \left(1 + \frac{\nu^2}{x^2}\right) y = 0$$

- Assume here that the parameter ν is not an integer or a half-integer (e.g. 1/2, 3/2, etc.)
- This ODE has a regular singular point at $x = 0$
- Now substitute the Frobenius series

$$y = x^\alpha \sum_{n=0}^{\infty} a_n x^n$$

- Now equating powers gives us the following

$$\begin{aligned} x^{\alpha-2} &: \quad (\alpha^2 - \nu^2) a_0 &&= 0 \\ x^{\alpha-1} &: \quad \left[(\alpha+1)^2 - \nu^2\right] a_1 &&= 0 \\ x^{\alpha+n-2} &: \quad \left[(\alpha+n)^2 - \nu^2\right] a_n &&= a_{n-2} \end{aligned}$$

A less trivial example

- We see that in order not to have $a_0 = 0$ in the first equation we need to pick α so that it solves the indicial equation

$$\alpha^2 - \nu^2 = 0$$

- This tells us the allowable roots are

$$\alpha_1 = +\nu \qquad \alpha_2 = -\nu$$

- We see that with either of these values the coefficient multiplying a_n in the recursion relation

$$\left[(\alpha + n)^2 - \nu^2\right] a_n = a_{n-2}$$

will never vanish.

A less trivial example

- Note that the second equation in the recursion relation

$$\left[(\alpha+1)^2-\nu^2\right] a_1 = 0$$

  tells us that $a_1 = 0$ for both solutions of the ODE.
- In fact for both solutions $a_1 = a_3 = a_5 = \ldots = 0$
- The second solution comes from picking $\alpha = -\nu$
- The recursion relation then gives us the terms in the series for that solution
- The two series are given by

$$y_\pm = a_\pm x^{\pm\nu} \sum_{n=0}^{\infty} \frac{x^{2n}}{2^{2n} n! (\pm\nu+n)(\pm\nu+n-1)\cdots(\pm\nu+1)}$$

A less trivial example

- The new functions defined by these series are called modified Bessel functions.
- Note that the solution associated with the positive root does not blow up at the singular point $x = 0$
- But the one associated with the negative root does.
- Note too that the series modifying $x^{\pm\nu}$ actually has an infinite radius of convergence (from the ratio test)
- This is to be expected because the only finite singularity of the coefficient functions $p(x)$ and $q(x)$ is at $x = 0$.

The general Frobenius expansion for 2'nd order ODEs

- In the general case we would have for a second order ODE

$$y'' + \frac{p(x)}{x-x_0}y' + \frac{q(x)}{(x-x_0)^2}y = 0.$$

- We assume that

$$\begin{aligned} p(x) &= \sum_{n=0}^{\infty} p_n (x-x_0)^n \\ q(x) &= \sum_{n=0}^{\infty} q_n (x-x_0)^n, \end{aligned}$$

because $p(x)$ and $q(x)$ are analytic at $x = x_0$.

- Now we substitute a Frobenius expansion of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^{n+\alpha},$$

The general Frobenius expansion for 2'nd order ODEs

- We get the relations

$$\left[(\alpha+n)^2+(p_0-1)(\alpha+n)+q_0\right]a_n = -\sum_{k=0}^{n-1}\left[(\alpha+k)p_{n-k}+q_{n-k}\right]a_k \qquad n=0,1,2,\ldots.$$

- We'll write this as

$$P(\alpha+n)a_n = -\sum_{k=0}^{n-1}\left[(\alpha+k)p_{n-k}+q_{n-k}\right]a_k \qquad n=0,1,2,\ldots.$$

- If we look at the recursion relation for $n=0$ we get the equation

$$P(\alpha)a_0 = \left[\alpha^2+(p_0-1)\alpha+q_0\right]a_0 = 0.$$

- By assumption $a_0 \neq 0$ so we must have

$$\alpha^2+(p_0-1)\alpha+q_0 = 0.$$

The general Frobenius expansion for 2'nd order ODE's

- Assume this indicial equation

$$P(\alpha)a_0 = \left[\alpha^2 + (p_0 - 1)\alpha + q_0\right] a_0 = 0.$$

  has two distinct roots α_1 and α_2.
- We can now solve for the other coefficients a_n

$$P(\alpha + n)a_n = -\sum_{k=0}^{n-1} \left[(\alpha + k)p_{n-k} + q_{n-k}\right] a_k \qquad n = 0, 1, 2, \ldots.$$

- It seems we can use the recursion relation and get two solutions corresponding to the two roots of the indicial equation.
- This is the usual case
- But there are two complications.