Physics 106a — Classical Mechanics

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Lecture 8

Unbound Orbits in Central Potentials: Scattering

Unbound Orbits in Central Potentials: Scattering

- Review of planetary orbits
- Repulsive $1/r$ potential: Rutherford scattering
- Scattering problems: what do we want to know?
- Particle trajectory for repulsive $1/r$ potential
- Rutherford scattering cross section
- Effect of finite target mass: kinematics and dynamics

- Eliminate center of mass motion
- Angular momentum $\vec{l}$ is conserved
- Orbit lies in plane perpendicular to $\vec{l}$: specify by (r, ϕ)
- Lagrangian for 2d motion

$$\mathcal{L} = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$

- Constant of the motion

$$l = \mu r^2 \dot{\phi}$$

— Kepler's second law (constant rate of sweeping out area)

- Hamiltonian is total energy and is constant

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r) \quad \text{with} \quad V_{\text{eff}}(r) = \frac{l^2}{2\mu r^2} + V(r)$$

- For $V(r) = -\frac{k}{r}$, introduce $u = \frac{1}{r}$, and use then $\dot{r} = \dot{\phi}\frac{dr}{d\phi} = -\frac{l}{\mu}\frac{du}{d\phi}$

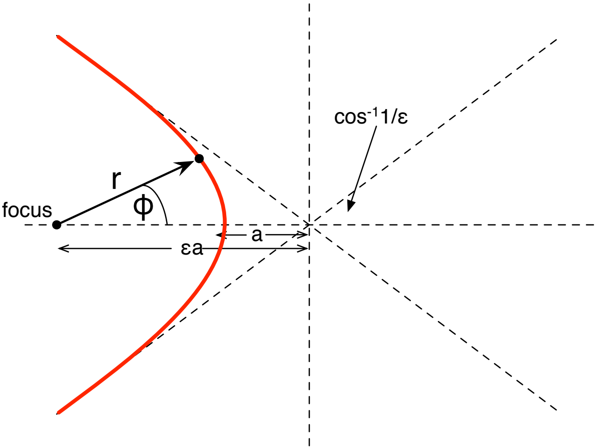
$$E = \frac{l^2}{2\mu}\left[\left(\frac{du}{d\phi}\right)^2 + \left(u - \frac{1}{p}\right)^2 - \frac{1}{p^2}\right] \quad \text{with } p = \frac{l^2}{\mu k}.$$

- $u(\phi)$ is sinusoidal about 1/p

$$\frac{1}{r} = \frac{1}{p} + \frac{\epsilon}{p}\cos\phi$$

- Conic section with eccentricity ϵ and $E = \frac{l^2}{2\mu p^2}(\epsilon^2 - 1)$

For $\epsilon > 1$ the orbits are hyperbolas $$\frac{1}{r} = \frac{1}{p} + \frac{\epsilon}{p}\cos\phi$$



Scattering of alpha particles off thin gold film [Ernest Rutherford]

> *It was quite the most incredible event that has ever happened to me in my life. It was almost as incredible as if you fired a 15-inch shell at a piece of tissue paper and it came back and hit you.*
>
> *On consideration, I realized that this scattering backward must be the result of a single collision, and when I made calculations I saw that it was impossible to get anything of that order of magnitude unless you took a system in which the greater part of the mass of the atom was concentrated in a minute nucleus. It was then that I had the idea of an atom with a minute massive center, carrying a charge.*

What do we want to know?

Repulsive k/r potential

- Hamiltonian is total energy and is constant

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r) \quad \text{with} \quad V_{\text{eff}}(r) = \frac{l^2}{2\mu r^2} + V(r)$$

- For $V(r) = \frac{k}{r}$ (with $k > 0$), introduce $u = \frac{1}{r}$ so that $\dot{r} = \dot{\phi}\frac{dr}{d\phi} = -\frac{l}{\mu}\frac{du}{d\phi}$

$$E = \frac{l^2}{2\mu}\left[\left(\frac{du}{d\phi}\right)^2 + \left(u + \frac{1}{p}\right)^2 - \frac{1}{p^2}\right] \quad \text{with } p = \frac{l^2}{\mu k} > 0$$

- $u(\phi)$ is sinusoidal about $-1/p$

$$\frac{1}{r} = -\frac{1}{p} + \frac{\epsilon}{p}\cos\phi$$

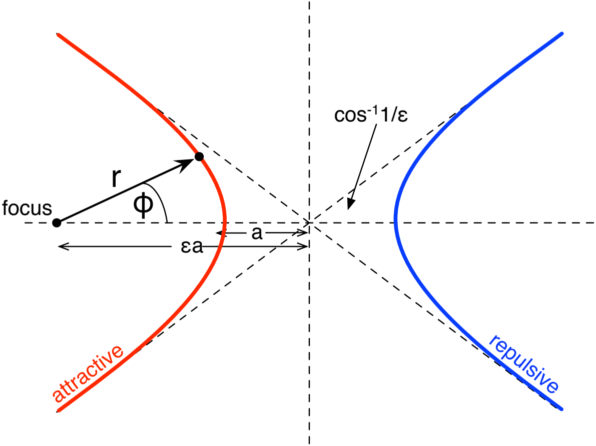
  Physical solutions $r > 0$ for $\cos\phi > 1/\epsilon$

- Conic section with eccentricity $\epsilon > 1$ (hyperbola) and $E = \frac{l^2}{2\mu p^2}(\epsilon^2 - 1)$

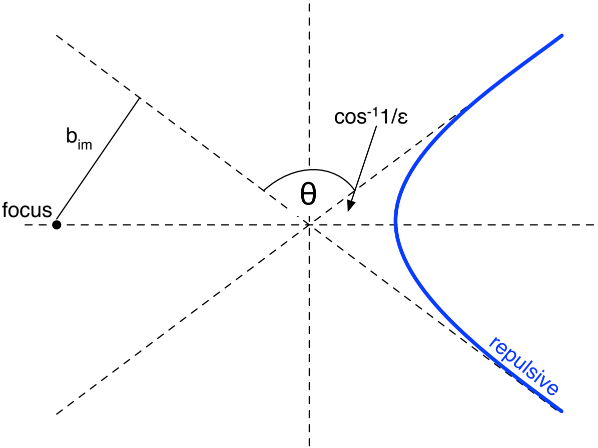
Hyperbolic orbits: $\epsilon > 1$

Attractive and repulsive interactions

$$\frac{1}{r} = \frac{1}{p} + \frac{\epsilon}{p}\cos\phi \qquad \frac{1}{r} = -\frac{1}{p} + \frac{\epsilon}{p}\cos\phi$$

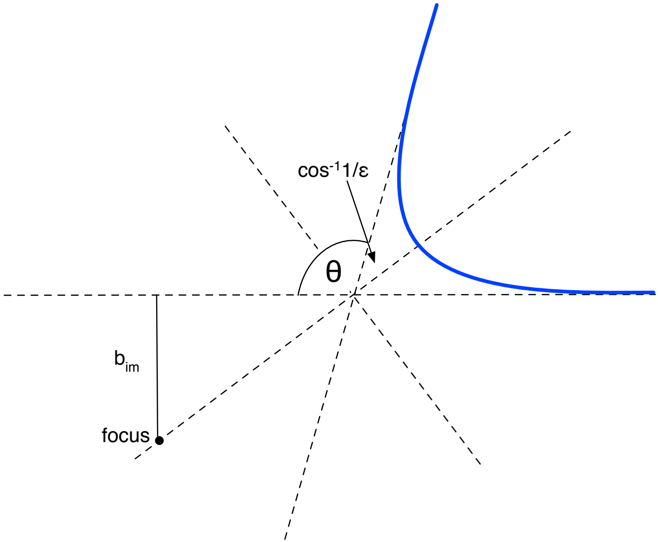


$$\frac{1}{r} = -\frac{1}{p} + \frac{\epsilon}{p}\cos\phi$$



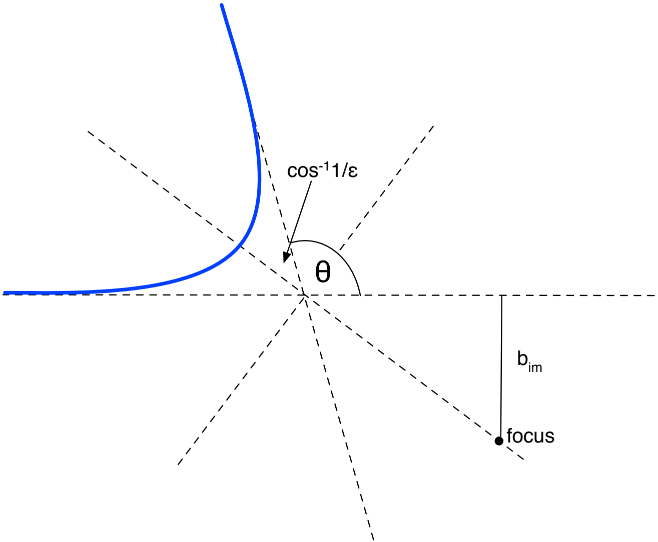
Hyperbolic orbits

Scattering in repulsive 1/r potential

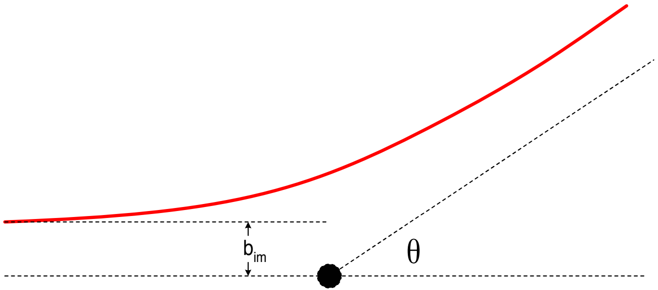


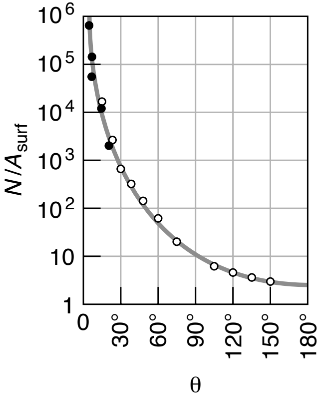
Hyperbolic orbits

Scattering in repulsive 1/r potential



Scattering Geometry





From *The Atomic Nucleus* by R. D. Evans

The closest distance of approach was 30 fm (3×10^{-14}m) for 7.78 MeV alpha particle at 150 deg scattering

Finite Target Mass

$p = 1, \epsilon = 1.2 \Rightarrow \theta_{\rm cm} = 146°$

Center of mass frame