ACM 100b

Convergence of series solutions for ODEs

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From last section

- We discussed series for solving linear ODE's
- We can then expand term by term,
- Then match like powers of $(x - x_0)^n$
- And finally derive a recursion relation.
- We can write this in general as follows for the ODE

$$y'' + p(x)y' + q(x)y = 0$$

$$(n+1)(n+2)a_{n+2} + \sum_{k=0}^{n}(n-k+1)p_k a_{n-k+1} + \sum_{k=0}^{n} q_k a_{n-k} = 0. \qquad n = 0, 1, 2, \ldots$$

- Recall a_0 and a_1 come from the IVP
- We can determine the remaining a_n and develop a series solution.
- But are these series useful? We discuss this next.

Convergence of series solutions

- In order for a series solution to be useful it must converge in some neighborhood about $x = x_0$.
- Consider the recursion relation we derived:

$$(n+1)(n+2)a_{n+2} + \sum_{k=0}^{n}(n-k+1)p_k a_{n-k+1} + \sum_{k=0}^{n} q_k a_{n-k} = 0. \qquad n = 0, 1, 2, \ldots$$

- This cannot be solved in closed form the way we did on the previous example for the Airy equation.
- So how can we tell if the series converges?

Ordinary points

- Suppose $p_0(x)$ and $q_0(x)$ are analytic in some region of the complex x-plane containing the point x_0.
- Then the series will converge in that region.
- If $p(z)$ and $q(z)$ are analytic about the point $z = x_0$ then we call x_0 an *ordinary point*
- Note the result is about the complex behavior of $p(z)$ and $q(z)$ even though x_0 is on the real axis.

Convergence of series at ordinary points

Theorem

Suppose $z = x_0$ *is an ordinary point of*

$$y'' + p(x)y' + q(x)y = 0,$$

Then the general solution can be represented in the form of a series of the form

$$y(x) = a_0 y_1(x) + a_1 y_2(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

The functions $y_1(x)$ *and* $y_2(x)$ *are linearly independent series solutions of the ODE. These solutions are themselves analytic about the point* $x = x_0$. *Most importantly, the radius of convergence of the series for* $y_1(x)$ *and* $y_2(x)$ *is at least as large as the minimum of the radii of convergence of the series that represent the coefficient functions* $p(x)$ *and* $q(x)$.

An example of series about an ordinary point

- For example, consider the ODE

$$y'' + \frac{y}{1+x^2} = 0 \qquad y(0) = y_0 \quad y'(0) = y_1.$$

- From the theorem above, the ODE has series solutions about the point $x = 0$ with a radius of convergence of at least 1.
- Now look at the the coefficient function

$$q(x) = \frac{1}{1+x^2}$$

- It actually has finite derivatives *at any point* of the real x-axis.
- But it has pole singularities in the complex plane at $x = \pm i$
- This tells us that the radius of convergence for a series solution about the point $x = 0$ is at least 1 in size.

Series solutions about ordinary points

- Note you actually don't need to do any work to infer this.
- But if you go ahead and compute the series you will indeed see the radius of convergence is 1.
- In contrast recall the Airy equation we analyzed above

$$y'' = xy,$$

- This ODE will have series solutions with infinite radii of convergence
- This is because the function $q(x) = x$ is entire in the complex plane.
- Indeed we confirmed this by computing the series.
- But with the theorem there is no need to do that.