Physics 106b: Lecture 7 **28 January 2014**

Hamiltonian Chaos: Introduction

Preliminaries

Hamiltonian dynamics of a single degree of freedom with a time independent Hamiltonian cannot give chaos, since trajectories cannot cross and this restricts the possible dynamics on the two dimensional phase space. The simplest examples of Hamiltonian chaos are

- 2 degrees of freedom (two positions, two conjugate momenta) with a time independent Hamiltonian. The constraint of constant energy confines the dynamics to a three dimensional surface in the four dimensional phase space.
- 1 degree of freedom (one position, one conjugate momentum) with a periodic Hamiltonian.

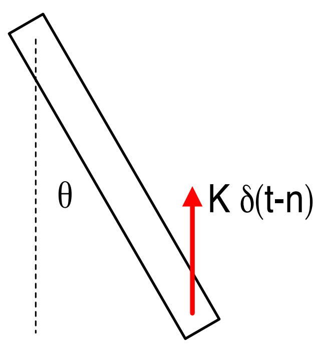
I first discuss two examples, and then a formal approach to the issues.

Double pendulum

Hand and Finch look at this example of the first type, and I also demonstrated it and discussed it in class. You will fill in some of the details in Assignment 5.

Periodically Kicked Rotor

This is a very simple example of the second class.



Consider a rigid pendulum but with no gravity. The dynamics is given by the angle θ and the conjugate momentum $p = \dot{\theta}$ (taking the moment of inertia to be 1). The pendulum is given a periodic impulsive vertical kick of strength K with period 1 (i.e. a kick at times $t = n$ with n an integer).

The dynamics can be reduced to the discrete map for θ_n, p_n which are the angle and momentum just *after* the nth kick. At each kick the (angular) momentum changes by the impulsive moment, but the angle doesn't change. Between kicks the momentum is constant and angle increases at the constant rate $\dot{\theta} = p$. This gives the equations

$$p_{n+1} - p_n = K \sin\theta_{n+1} \ , \tag{1}$$
$$\theta_{n+1} - \theta_n = p_n \ . \tag{2}$$

The equations are usually transformed by introducing

$$x_n = \frac{\theta_n}{2\pi} \bmod 1 \ , \tag{3}$$

$$y_n = p_n \ , \tag{4}$$

to give the *standard map*

$$x_{n+1} = x_n + \frac{y_n}{2\pi} \ \bmod \ 1 \ , \tag{5}$$

$$y_{n+1} = y_n + K \sin(2\pi x_{n+1}) \ . \tag{6}$$

Liouville's theorems tells us the map is *area preserving* — the key property of the map linked to the Hamiltonian nature of the dynamics [1]. This also follows from direct calculation of the Jacobean

$$J = \det \begin{bmatrix} \frac{\partial x_{n+1}}{\partial x_n} & \frac{\partial x_{n+1}}{\partial y_n} \\ \frac{\partial y_{n+1}}{\partial x_n} & \frac{\partial y_{n+1}}{\partial y_n} \end{bmatrix} = 1 \tag{7}$$

Dynamics of the Standard Map

For $K = 0$ (no drive) the system is integrable — the rotor rotates at a constant rate determined by the initial conditions. In the iterated standard map this translates into a series of points at fixed y. The appearance after many iterations depends on the *winding number* Ω which is defined as the average (here constant) advance in x per iteration (or the ratio of the average rotor frequency to the frequency of the periodic kicks). For a *rational* winding number $\Omega = p/q$, with p, q integers, we see a discrete set of p points. For *irrational* winding numbers (cannot be expressed in this form) the points will eventually fill in the line $0 \le x \le 1$. The winding number for $K = 0$ is just the value of y, so that almost all[2] initial conditions will give an irrational winding number. Even for rational $y = y_r$ giving rational winding numbers the line $0 \le x \le 1$, $y = y_r$ is *invariant* under the dynamics (any point on the line is mapped into another point on the line.) Since x is a periodic variable the lines correspond to closed circles, or "one dimensional tori". These are analogous to the circles in phase space describing the normal modes of simple harmonic oscillators and other integrable systems.

The study of Hamiltonian chaos begins by seeing how these tori break down on adding the periodic kicks. See the numerical demonstrations to illustrate the result that tori with irrational winding numbers survive adding small kicks, whereas those with rational winding numbers break down to a set of elliptical and hyperbolic fixed points, with chaotic motion in regions surrounding the hyperbolic fixed points. The break down occurs because the rotor motion is resonant with the drive for rational winding numbers.

Since the system is periodically driven, an interesting question is whether the kinetic energy ($p^2/2 \propto y^2$) of the rotor continuously grows. Because the map is area preserving, the initial tori spanning the whole range of x act as barriers to the growth of energy. Thus the break down of the last surviving of these tori is an important event. It happens at $K = 1$ for the standard map. For larger values of K the energy can increase to large values. It does so diffusively (i.e. the mean over a long time t remains zero, but the variance grows as t. This is known as *Arnold diffusion*.

[1] The two dimensional map given by the β, l_β Poincaré section of the double pendulum considered by Hand and Finch is also area preserving, but this no longer can be deduced simply from Liouville's thoerem. Instead the *symplectic* property of Hamiltonian dynamics must be used. This is discussed in the appendix to Chapter 6 of Hand and Finch. Actually the form they discuss is not sufficient to prove the map is area conserving because the Poincaré section is not a fixed time slice. Instead the *Poincaré-Cartan* invariant $\oint (\mathbf{p} \cdot d\mathbf{q} - H dt)$ must be used. This reduces to $\oint \mathbf{p} \cdot d\mathbf{q}$ for a time independent H even when the integral is taken around a loop on the Poincaré section — see Hand and Finch for the rest of the discussion.

[2] "Almost all" is used as a precise term to mean "all except a set of measure zero".

Integrable Systems

Action-angle variables are a choice of variables that simplify the discussion of periodic orbits. We discussed these in Lecture 12 of Ph106a.

One degree of freedom

First consider a system with one degree of freedom (q, p) governed by the Hamiltonian $H(p, q)$, which can be solved to give periodic motion. An example would be a simple harmonic oscillator, but more generally the frequency of the motion might depend on the amplitude of motion.

We seek a time independent canonical transformation $(q, p) \to (\theta, I)$ such that the Hamiltonian H is a function of only I

$$H = H(I) \; . \tag{8}$$

Then θ is an ignorable coordinate, and the conjugate momentum I is a constant of the motion $\dot{I} = 0$. The angle θ advances at a constant rate giving the frequency, which will in general depend on the action variable

$$\dot{\theta} = \frac{\partial H}{\partial I} = \omega(I) \; . \tag{9}$$

The action variable I can be evaluated as an integral over the orbit in the (q, p) plane

$$I = \frac{1}{2\pi} \oint p \, dq \tag{10}$$

(see Hand and Finch §6.5).

The canonical transformation can be expressed in terms of a generating function. We choose the type-2 generating function that depends on the *new* momentum and the *old* coordinate. Thus we introduce the generating function $S(I, q)$ with the properties (cf. Hand and Finch §6.2 and Table 6.1)

$$\text{new coordinate } \theta = \frac{\partial S}{\partial I} \; , \tag{11}$$

$$\text{old momentum } p = \frac{\partial S}{\partial q} \; . \tag{12}$$

We want to find S such that the new Hamiltonian is a function just of I. This is given by the Hamilton-Jacobi equation (cf. Hand and Finch pp 222-5) [3]

$$H\left(\frac{\partial S}{\partial q}, q\right) = H(I) = \text{ constant} \tag{13}$$

where we have evaluated the Hamiltonian using the momentum from Eq. (12). This is a pde in q for $S(I, q)$ for each I and solves the problem.

[3] In Lecture 12 of Ph106a and in Hand and Finch, the symbol W was used for the generating function (Hamiltonian's Characteristic Function) for a time independent Hamiltonian leading to a constant Hamiltonian. Here I will use the symbol S which is more common in the chaos literature, although we used this symbol before for the case of time dependent generating functions. Also this is a slightly different form of Hamilton-Jacobi theory than discussed there: we are now using the action variables as the constant momenta rather than the separation variables.

Many degrees of freedom

For N degrees of freedom a system for which the same action angle procedure works is called *integrable*. These are very special systems — most Hamiltonians will not be integrable. The integrable system will be the starting point for the perturbation theory, and so I will call the Hamiltonian $H_0(\mathbf{p}, \mathbf{q})$ with $\mathbf{q}$ and $\mathbf{p}$ N-dimensional vectors of the coordinates and conjugate momenta. Integrability implies we can find a canonical transformation $(\mathbf{q}, \mathbf{p}) \to (\boldsymbol{\theta}, \mathbf{I})$ such that the new Hamiltonian is a function of just $\mathbf{I}$. The angles $\boldsymbol{\theta} = (\theta_1, \theta_2, \theta_N)$ are ignorable, and the action variables $\mathbf{I} = (I_1, I_2, \ldots I_N)$ are constants of the motion. Thus a necessary condition for a system with N degrees of freedom to be integrable is that there are N constants of the motion. This is not sufficient—the constants must also be *in involution*. (This is analogous to the idea of commuting operators in quantum mechanics, that are then simultaneously diagonalizable.)

The canonical transformation is given by the generating function $S_0(\mathbf{I}, \mathbf{q})$ with the Hamilton-Jacobi equation for S_0

$$H_0\left(\frac{\partial S_0}{\partial \mathbf{q}}, \mathbf{q}\right) = H_0(\mathbf{I}) \qquad \text{independent of } \boldsymbol{\theta} \text{ .} \tag{14}$$

The θ_j evolve at constant rates $\omega_{0,j}$ with

$$\boldsymbol{\omega}_0 = \frac{\partial H_0}{\partial \mathbf{I}} \qquad \text{or in component form} \qquad \omega_{0,j} = \frac{\partial H_0}{\partial I_j} \text{ .} \tag{15}$$

The motion is on the product of N limit cycles, i.e. motion on an N-torus. This is of dimension N, compared with $2N - 1$ for the constant energy surface, so that the integrable dynamics sample a tiny fraction of the constant energy surface. We assume that this part of the problem is solved (e.g. for N balls with linear springs).

Perturbation Theory

Formulation

Now add a perturbation εH_1 and ask whether the new Hamiltonian can be solved in the same way so that the N-torus dynamics survives. For small ε we try to solve the problem by a perturbation expansion. We express the perturbation in terms of the action angle variables of H_0, so that

$$H(\mathbf{I}, \boldsymbol{\theta}) = H_0(\mathbf{I}) + \varepsilon H_1(\mathbf{I}, \boldsymbol{\theta}) \tag{16}$$

with ε small. We again try to solve by a canonical transformation $(\boldsymbol{\theta},\mathbf{I}) \to (\boldsymbol{\theta}', \mathbf{I}')$ to new action $\mathbf{I}'$ and angle $\boldsymbol{\theta}'$ variables so that the new Hamiltonian H' does not depend on $\boldsymbol{\theta}'$. *If* we can do this, the dynamics remain N-frequency motion on an N-torus. The canonical transformation is given by the generating function $S(\mathbf{I}', \boldsymbol{\theta})$ satisfying the Hamilton-Jacobi equation

$$H\left(\frac{\partial S}{\partial \boldsymbol{\theta}}, \boldsymbol{\theta}\right) = H'(\mathbf{I}'). \tag{17}$$

The generating function is expanded in a power series in ε

$$S(\mathbf{I}', \boldsymbol{\theta}) = \mathbf{I}' \cdot \boldsymbol{\theta} + \varepsilon S_1(\mathbf{I}', \boldsymbol{\theta}) + \cdots \text{ .} \tag{18}$$

Note the zeroth order generating function $S_0(\mathbf{I}', \boldsymbol{\theta}) = \mathbf{I}' \cdot \boldsymbol{\theta}$ is the identity transformation since

$$\mathbf{I} = \frac{\partial S_0}{\partial \boldsymbol{\theta}} = \mathbf{I}' \qquad \text{and} \qquad \boldsymbol{\theta}' = \frac{\partial S_0}{\partial \mathbf{I}'} = \boldsymbol{\theta} \text{ .} \tag{19}$$

Substitute Eqs. (16,18) into Eq. (17) and expand to first order in ε. In the H_0 term we will need to keep the order ε term in Eq. (18) and we expand H_0 in a Taylor expansion. The H_1 term is already first order in ε and so there we can use the zeroth order approximation to S. This gives

$$H_0(\mathbf{I}) + \varepsilon \left[\boldsymbol{\omega}_0(\mathbf{I}) \cdot \frac{\partial S_1}{\partial \boldsymbol{\theta}} + H_1(\mathbf{I}, \boldsymbol{\theta}) \right] + \cdots = H'(\mathbf{I}) \tag{20}$$

using $\partial H_0/\partial \mathbf{I} = \boldsymbol{\omega}_0(\mathbf{I})$ (and I have replaced $\mathbf{I}' \to \mathbf{I}$ since this variable is a dummy variable of the functional dependence of each term). Each component of the variable $\boldsymbol{\theta}$ is periodic, and so H_1 and S_1 must be periodic functions of $\boldsymbol{\theta}$ and they can be expanded in a Fourier series

$$H_1 = \sum_{\mathbf{m} \neq \mathbf{0}} H_{1,\mathbf{m}}(\mathbf{I}) e^{i\mathbf{m}\cdot\boldsymbol{\theta}} \tag{21}$$

$$S_1 = \sum_{\mathbf{m} \neq \mathbf{0}} S_{1,\mathbf{m}}(\mathbf{I}) e^{i\mathbf{m}\cdot\boldsymbol{\theta}} \tag{22}$$

where the sum is over vectors of integers $(m_1, m_2 \ldots m_N)$. We can absorb the θ-independent part of the Hamiltonian into H_0 making $H'(\vec{I}) = H_0(\vec{I})$ and so I have left out the $\mathbf{m} = \mathbf{0}$ term which can be absorbed into the integrable part H_0. Substituting into Eq. (20) gives

$$\varepsilon \sum_{\mathbf{m} \neq \mathbf{0}} \left[i\mathbf{m} \cdot \boldsymbol{\omega}_0(\mathbf{I}) S_{1,\mathbf{m}}(\mathbf{I}) + H_{1,\mathbf{m}}(\mathbf{I}, \boldsymbol{\theta}) \right] e^{i\mathbf{m}\cdot\boldsymbol{\theta}} + \cdots = H'(\mathbf{I}) - H_0(\mathbf{I}) = 0. \tag{23}$$

Since the $e^{i\mathbf{m}\cdot\boldsymbol{\theta}}$ are linearly independent functions of $\boldsymbol{\theta}$ each coefficient of $e^{i\mathbf{m}\cdot\boldsymbol{\theta}}$ must be set to zero, so that

$$S_{1,\mathbf{m}}(\mathbf{I}) = i \frac{H_{1,\mathbf{m}}(\mathbf{I}, \boldsymbol{\theta})}{\mathbf{m} \cdot \boldsymbol{\omega}_0(\mathbf{I})} \tag{24}$$

and we have the perturbation result

$$S(\mathbf{I}, \boldsymbol{\theta}) = \mathbf{I} \cdot \boldsymbol{\theta} + i\varepsilon \sum_{\mathbf{m} \neq \mathbf{0}} \frac{H_{1,\mathbf{m}}(\mathbf{I}, \boldsymbol{\theta})}{\mathbf{m} \cdot \boldsymbol{\omega}_0(\mathbf{I})} e^{i\mathbf{m}\cdot\boldsymbol{\theta}} + \cdots . \tag{25}$$

Break down

We can now study the convergence of the perturbation theory. Clearly there is a problem if any denominator in the sum is zero

$$\mathbf{m} \cdot \boldsymbol{\omega}_0(\mathbf{I}) = 0 \qquad \text{for any } \mathbf{m}, \tag{26}$$

i.e. for those values of the action for which there is a rational relationship between the unperturbed frequencies

$$m_1\omega_{0,1} + m_2\omega_{0,2} + \cdots + m_N\omega_{0,N} = 0 \tag{27}$$

with m_j integers, or in terms of the winding numbers $\Omega_{0,j} = \omega_{0,j}/\omega_{0,1}$

$$m_1 + m_2\Omega_{0,2} + \cdots + m_N\Omega_{0,N} = 0\ . \tag{28}$$

Thus the perturbation theory diverges for the tori with a rational relationship between the frequencies — the "rational tori", and these will be destroyed by an arbitrarily small perturbation.

There will also be problems if the denominator becomes too small, i.e. if the frequencies are such that the rational relationship is almost satisfied. Increasing $|\mathbf{m}|$ gives an increasing flexibility in choosing the m_j so that $\mathbf{m} \cdot \boldsymbol{\omega}_0(\mathbf{I})$ is close to zero. On the other hand for increasing $|\mathbf{m}|$ the numerator $H_{1,\mathbf{m}}(\mathbf{I}, \boldsymbol{\theta})$ becomes smaller (the size of very high frequency components decreases for a smooth function). Also if the first term in the expansion is becoming dangerously large, we must also look at higher order terms and check them too. This leads to a very hard mathematical problem — the problem of small divisors — corresponding physically to the difficulty or treating resonances in Hamiltonian (undamped) systems. The problem was eventually solved by Kolmogorov (1954), Arnold (1963), and Mosur (1967) (the numbers in brackets are the years of their individual contributions) in a theorem known as the KAM theorem. The result is delicate, and a precise statement depends on various mathematical restrictions placed on H_0 and H_1. The statement also depends on quantifying the idea of how close an irrational number is to a rational. Hand and Finch give one version of the result for the $N = 2$ case: The tori of H_0 survive a small enough smooth perturbation if the winding number Ω is sufficiently irrational, i.e. for any integers r, s

$$\left|\Omega - \frac{r}{s}\right| > \frac{C(\varepsilon)}{s^{2.5}} \qquad \text{with } C(\varepsilon) \to 0 \text{ as } \varepsilon \to 0\,. \tag{29}$$

Thus we end up with an intricate picture: Tori with winding numbers over a range $\propto s^{-2.5}$ about every rational r/s are destroyed by the perturbation. Although there are an infinite number of rationals in the unit interval (winding numbers can be restricted to the unit interval), the sum of all these ranges is finite, and goes to zero as $C(\varepsilon) \to 0$. This is because there are of order s rationals in the unit interval with denominator s (r runs from 1 to $s - 1$ but some have already been counted, e.g. $2/4 \equiv 1/2$) and the sum

$$\sum_{s=1}^{\infty} s \frac{1}{s^{2.5}} \tag{30}$$

converges. Tori with winding numbers outside these windows survive the perturbation. These consist of almost all of the tori as $\varepsilon \to 0$ — an infinite number of tori are destroyed but most survive!

Michael Cross January 25, 2014