ACM 100b

Further properties of the Laplace transform

Dan Meiron

Caltech

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Properties of the Laplace transform

- Before we use the Laplace transform in solving ODE's we first describe some useful properties
- We've already discussed the linearity of the transform
- And we've shown the transforms of derivatives.
- These are helpful in manipulating transforms into expressions that are familiar and so more easily inverted to give solutions.
- We will discuss
  - the shifting property
  - inverse Laplace transform of a derivative
  - The convolution theorem
  - Laplace transforms of discontinuous functions

The shifting property

- Suppose we notice that the transform is in the form of

$$F(s-a)$$

where we recognize F(s) but it's shifted by a parameter *a*

- The inverse Laplace transform of a shifted transform can be found from the following identity:

$$\int_0^\infty \exp(-st)\exp(at)f(t)dt = \int_0^\infty \exp(-(s-a)t)f(t)dt = F(s-a),$$

where

$$F(s) = \int_0^\infty \exp(-st)f(t)dt.$$

- So we see that the inverse transform of $F(s-a)$ is

$$\exp(at)f(t) \qquad t > 0$$

Inverse transform of a derivative (in s)

- Recall the definition of the Laplace transform:

$$F(s) = \int_0^\infty f(t)\exp(-st)dt.$$

- If we differentiate both sides with respect to s we get

$$\frac{\partial F}{\partial s} = \int_0^\infty (-t)f(t)\exp(-st)dt.$$

- From this we can see that the inverse transform of

$$\frac{\partial F}{\partial s} \quad \text{is} \quad -tf(t)$$

- And similarly the inverse transform of

$$\frac{\partial^2 F}{\partial s^2} \quad \text{is} \quad t^2 f(t).$$

The convolution theorem

- In general the transform of a product of functions is not the product of the transforms
- But there is an expression that does behave this way.
- If $f(t)$ and $g(t)$ are continuous for $t \geq 0$ and are both of exponential order, then the convolution of $f(t)$ with $g(t)$ is defined by

$$h(t) = f \star g = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau.$$

- The convolution of two functions appears in many contexts.
- There are actually several definitions of convolution depending on the type of transform being considered.
- The one above is the one appropriate for the Laplace transform.

The convolution theorem

Theorem

The convolution theorem for Laplace transforms is that the Laplace transform of $f \star g$ where

$$f \star g = \int_0^t f(\tau)g(t-\tau)d\tau = \int_0^t f(t-\tau)g(\tau)d\tau.$$

is simply

$$\mathcal{L}[f \star g] = F(s)G(s).$$

where F and G are the Laplace transforms of $f(t)$ and $g(t)$ respectively.

The convolution theorem

- It's not hard to see how this comes about.
- Consider taking directly the Laplace transform of the convolution:

$$
\begin{aligned}
H(s) &= \int_0^{\infty} \exp(-st)dt \int_0^{t} f(t-\tau)g(\tau)d\tau \\
&= \iint_A \exp(-st)f(t-\tau)g(\tau)dtd\tau \\
&= \int_0^{\infty} g(\tau)d\tau \int_{\tau}^{\infty} dtf(t-\tau)\exp(-st)dt \\
&= \int_0^{\infty} g(\tau)d\tau \int_0^{\infty} d\sigma f(\sigma)\exp(-s(\sigma+\tau))d\sigma \\
&= \int_0^{\infty} g(\tau)\exp(-s\tau)d\tau \int_0^{\infty} d\sigma f(\sigma)\exp(-s\sigma) \\
&= F(s)G(s).
\end{aligned}
$$

- And that shows the result.

An application of the convolution theorem

- We notice that the expression
$$\int_0^\infty f(\tau)d\tau,$$
is just the integral of $f(t)$.
- But it's also the convolution of $f(t)$ with the function $g = 1$.
- We recall that the Laplace transform of $g = 1$ is
$$G(s) = 1/s,$$
- And so the Laplace transform of
$$\int_0^\infty f(\tau)d\tau$$
is
$$\frac{F(s)}{s}.$$
- This result can of course also be gotten by integration by parts.

Laplace transform of discontinuous functions

- It is just as easy to transform discontinuous functions as it is to transform continuous ones.
- This is very helpful when we have to solve ODE's with discontinuous right hand sides.
- As an example consider the unit step function defined by

$$u_c(t) = \begin{cases} 0 & t < c \quad c \geq 0 \\ 1 & t \geq c. \end{cases}$$

- Using this function we can construct pretty much any function that has some sort of jump.

Laplace transform of discontinuous functions

- For example, consider the function

$$h(t) = u_\pi(t) - u_{2\pi}(t).$$

- This is the function defined by

$$h(t) = \begin{cases} 0 & t < \pi \\ 1 & \pi \le t < 2\pi \\ 0 & t \ge 2\pi. \end{cases}$$

- The Laplace transform of $u_c(t)$ is given by

$$\begin{aligned} \mathcal{L}[u_c(t)] &= \int_0^\infty \exp(-st)u_c(t)dt \\ &= \int_c^\infty \exp(-st)dt \\ &= \frac{\exp(-cs)}{s} \qquad t \ge 0. \end{aligned}$$

Laplace transforms of discontinuous functions

- We can use this result and the definition of $u_c(t)$ to calculate the transforms for functions shifted in time.
- For example consider the function

$$u_c(t)f(t-c) = \begin{cases} 0 & 0 \leq t < c \\ f(t-c) & t \geq c. \end{cases}$$

- The transform of this function is

$$\mathcal{L}[u_c(t)f(t-c)] = \exp(-cs\mathcal{L}[f(t)] = \exp(-cs)F(s).$$

- Compare this result to the shifting property:

$$\mathcal{L}[\exp(at)f(t)] = F(s-a).$$

Laplace transforms of discontinuous functions

- The results are similar
- But there important differences associated with the presence of the step function.
- This of course also sets up an important identity regarding Laplace transforms that can be used to invert certain transforms.
- That is, if

$$f(t) = \mathcal{L}^{-1}[F(s)],$$

then

$$u_c(t)f(t-c) = \mathcal{L}^{-1}[\exp(-cs)F(s)].$$