PART IV: QUALITATIVE ANALYSIS OF 2D AUTONOMOUS SYSTEMS

32. Orbits of autonomous systems

32.1. Autonomous systems.

$$\dot{x} = v(x), \qquad x(t) \in D \subset \mathbb{R}^n.$$

The domain D (which can be the whole space $\mathbb{R}^n$) is the *phase space* of the system. We interpret the given vector-valued function

$$v : D \to \mathbb{R}^n$$

as a vector field in the phase space: think of $v(x)$ as a vector based at x. Thus

$$\text{autonomous systems} \equiv \text{vector fields}$$

(The concept of a vector field, and therefore the concept of an autonomous system, generalizes to arbitrary manifolds, such as spheres and tori, see Arnold. E.g., consider the velocity field of the wind on the surface of the earth.)

A solution is a vector-function

$$x = x(t) : \ (t_1, t_2) \to D,$$

which we interpret as a motion in the phase space. Geometrically, $x(t)$ is a parametrized curve in D. The set

$$x(t_1, t_2) = \{x(t) : \ t_1 < t < t_2\},$$

is the *orbit* of the solution (or the *trajectory* of the motion). Orbits are also called *phase curves* of the system. Note that orbits don't show dependence on time: we can't tell how fast the particle moves along the trajectory.

Theorem. *Suppose $v \in C^1(D)$. Then every IVP(t_0, x_0) with $x_0 \in D$ has a unique maximal solution.*

32.2. Properties of (maximal) orbits.

We always assume $v \in C^1(D)$. The following properties are immediate consequences of the existence and uniqueness theorem.

(a) *Translation over time.*

If $x(t)$ is a solution, then $x(t + t_0)$ is a solution. These two solutions have the same orbit. In the opposite direction, if two solutions $x(t)$ and $\tilde{x}(t)$ have the same orbits, then $\tilde{x}(t) = x(t + t_0)$ for some t_0.

(b) *Two orbits either coincide or don't intersect.*

Proof: Let $x(\cdot)$ and $\tilde{x}(\cdot)$ be two solutions such that their orbits intersect. By translation, we can assume $\tilde{x}(0) = x(t_0) := x_0$. Then both $\tilde{x}(t)$ and $x(t + t_0)$ are solutions of the IVP$(0, x_0)$. □

(c) There is one and only one orbit that passes through a given point.

(d) *Three possible types of orbits:*

- a single point $\{x_*\}$, the orbit of a *stationary* solution $x(t) \equiv x_*$, a *singular* point of the vector field: $v(x_*) = 0$;
- a simple closed curve (a "*cycle*"), the orbit of a non-stationary *periodic* solution;
- a non-closed curve without self-intersections.

This property can be restated as follows. Let $x(t)$ be a maximal non-stationary solution. The following statements are equivalent:

(i) $x(t_1) = x(t_2)$ for some $t_1 \neq t_2$;

(ii) the orbit of $x(t)$ is a closed simple curve;

(iii) $x(t)$ is a periodic function [defined for all $t \in \mathbb{R}$].

The proof is similar to the proof of (b).

32.3. **Phase portraits.** The phase portrait of a system is the collection of all orbits. Usually we draw a diagram that shows typical orbits.

Examples.

(a) 1D autonomous systems, see Section 3. If the phase space is $\mathbb{R}$, then the non-stationary orbits are exactly the complementary intervals of the set of stationary points.

(b) $\dot{x} = -y^2,\ \dot{y} = x^2$, see Section 13.3.

The phase space is $\mathbb{R}^2$. There is one stationary orbit, $\{(0,0)\}$. The line $x + y = 0$ is the union of three orbits, including the stationary one. All other orbits are non-closed simple curves that extend (in finite time) to infinity both in the forward and backward directions.

(c) Lotka-Volterra model, see Section 14.6. There is one stationary point inside the population quadrant $Q = \{x > 0, y > 0\}$. All other orbits in Q are cycles.

33. Phase portraits of 2D linear systems

33.1. **Change of variables.** Let A be a 2×2 matrix. Consider the system

$$\dot{x} = Ax, \qquad x \in \mathbb{R}^2.$$

Let λ_1, λ_2 be the eigenvalues of A. Three cases: the eigenvalues are

(i) real and distinct;

(ii) real and equal ($= \lambda$);

(iii) complex ($= \alpha \pm i\omega$).

Linear algebra. Given A, there is an invertible matrix S such that the matrix

$$B = SAS^{-1}$$

has the following form (in the respective three cases):

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}; \qquad \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}; \qquad \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix}.$$

Note that A and B have the same eigenvalues, and

$$S \text{ takes eigenvectors of } A \text{ to eigenvectors of } B.$$

Change the variables:

$$x \mapsto y = Sx.$$

The new system is

$$\dot{y} = By.$$

Indeed, $\dot{y} = S\dot{x} = SAx = SAS^{-1}y = By$. The inverse transformation

$$y \mapsto x = S^{-1}y$$

maps the phase portrait of the y-system onto the portrait of the x-system.

We will only consider the case where A is *non-degenerate*, i.e

$$\det A \neq 0;$$

equivalently, $\lambda_1 \neq 0$ and $\lambda_2 \neq 0$. In this case, $x_* = 0$ is the only singular point of the system.

33.2. **Distinct real eigenvalues: nodes and saddles.** The y-system

$$\dot{y}_1 = \lambda_1 y_1, \qquad \dot{y}_2 = \lambda_2 y_2$$

has separated variables, so we find

$$y_1(t) = c_1 e^{\lambda_1 t}, \qquad y_2(t) = c_2 e^{\lambda_2 t}.$$

Excluding t, we obtain the following equation for the orbits:

$$y_2 = c|y_1|^k, \qquad k = \lambda_2/\lambda_1.$$

(a) **Stable nodes.** Suppose

$$\lambda_1 < \lambda_2 < 0, \qquad \text{so} \quad 0 < k < 1.$$

All but three orbits approach the origin tangentially to the y_2-axis as $t \to +\infty$. The exceptional orbits are the stationary point, the positive y_1-axis, and the negative y_1-axis, see Fig. 28.2 in the text.

In the x-plane the picture is similar: all but three phase curves tend to 0 tangentially to the eigenspace of λ_2. The exceptional orbits make up the eigenspace of λ_1, see Fig 31.1.

(b) **Unstable nodes.** Suppose

$$0 < \lambda_1 < \lambda_2, \qquad \text{so} \quad k > 1.$$

All but three orbits approach the origin tangentially to the eigenspace of the smallest eigenvalue as $t \to -\infty$, see Fig. 28.4 and Fig 31.1.

(c) **Saddles.** Suppose

$$\lambda_1 < 0 < \lambda_2, \qquad \text{so} \quad k < 0.$$

The reason for the name "saddle" is the shape of the graph of the first integral $E(y_1, y_2) = y_1 y_2$ in the case $k = -1$.

There are precisely two non-stationary orbits (they lie in the eigenspace of the negative eigenvalue) that tend to the stationary point as $t \to +\infty$, and there are precisely two non-stationary orbits that tend to 0 as $t \to -\infty$, see Fig. 28.6. These orbits are called stable and unstable *separatrices.*

Closely related concepts are those of the stable and unstable manifolds of a critical point, see the textbook. In the case of a saddle, the eigenspace of the negative eigenvalue is the stable manifold and the eigenspace of the positive eigenvalue is the unstable manifold.

33.3. **Equal eigenvalues: stars and improper nodes.** The phase portraits of linear systems with

$$B = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, \qquad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$$

are called (stable or unstable) star nodes and improper (or Jordan) nodes respectively; see Section 30 of the textbook.

In the first case, the eigenspace is two dimensional (all directions are eigendirections). In the second case, there is only one eigendirection, the y_1-axis, and all non-stationary orbits are tangential to it:

$$y_1(t) = [y_1(0) + t y_2(0)]\ e^{\lambda t}, \quad y_2(t) = y_2(0)\ e^{\lambda t},$$

which follows from the formula

$$e^{tB} = e^{\lambda t} \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

Thus we have

$$y_1(t) = (c + t) y_2(t),$$

and so $|y_2| \ll |y_1|$ as $t \to \pm\infty$.

33.4. **Complex eigenvalues: foci and centers.** Let us rewrite the system

$$\dot{x} = \alpha x - \omega y, \qquad \dot{y} = \omega x + \alpha y$$

in the complex form. [We denote the coordinates (x, y) instead of (y_1, y_2).] If $z(t) = x(t) + iy(t)$, then

$$\dot{z} = \lambda z, \qquad \lambda := \alpha + i\omega,$$

because

$$\dot{z} = (\alpha + i\omega)x + (-\omega + i\alpha)y = (\alpha + i\omega)(x + iy).$$

It is now easy to change the coordinates into polar coordinates:

$$z = re^{i\phi}, \qquad \dot{z} = \dot{r}e^{i\phi} + ire^{i\phi}\dot{\phi} = (\dot{r}/r + i\dot{\phi})z;$$

$$\frac{\dot{z}}{z} = \frac{\dot{r}}{r} + i\dot{\phi} = \alpha + i\omega.$$

Taking the real and imaginary parts, we separate the variables:

$$\dot{r}/r = \alpha, \qquad \dot{\phi} = \omega. \tag{33.1}$$

The second equation in (33.1) shows that we have rotation with a constant angular speed; the direction of rotation is clockwise if $\omega < 0$ and counter-clockwise if $\omega > 0$.

The first equation in (33.1) shows that if $\alpha < 0$ then $r(t) \to 0$ as $t \to +\infty$, and if $\alpha > 0$ then $r(t) \to 0$ as $t \to -\infty$. The corresponding phase portraits are called *stable focus* (or spiral) and *unstable focus* respectively, see Fig. 29.2-3.

If $a = 0$, then we have the case of a *center*. All non-stationary orbits are circles. The solutions are periodic with the same period $2\pi/\omega$. See Fig. 29.4.

These pictures persist if we make a linear change of variables: we get families of spirals and, in the case of a center, families of ellipses, see Fig. 31.1. The only information we need to describe the qualitative behavior of orbits is the sign of the real part of the eigenvalues and the direction of rotation.

Example:

$$\dot{x} = 2x + 5y, \quad \dot{y} = -2x.$$

The eigenvalues are $1 \pm 3i$. To determine the direction of rotation we note that $\dot{x} > 0$ on the half line $\{x = 0, y > 0\}$, so the rotation is clockwise.

Also, see Example 30.1 (textbook) for a similar argument in the case of an improper node.

Exercise. Classify all possible phase portraits of the equation

$$\ddot{x} + 2a\dot{x} + \omega_0^2 x = 0, \qquad \text{(see Section 20.2)}.$$

Answer: center (no damping), stable spiral (small damping), stable improper node (critical damping), stable node (over-damping). In the last two cases, the spring returns to the equilibrium without oscillations: the velocity changes sign at most ones.

33.5. **Stability of the equilibrium solution.**

Theorem. *The equilibrium solution of a 2D linear system with non-degenerate matrix is stable iff*

$$\forall j, \quad \Re\lambda_j \le 0,$$

and is (globally) attracting iff

$$\forall j, \quad \Re\lambda_j < 0.$$

The statement follows from the description of the phase portraits. We have asymptotically stable nodes and foci, stable centers, unstable nodes and foci. Saddles are always unstable.

Exercise. Extend the theorem to higher dimensions. (The first part of the theorem has to be modified.)

33.6. **Aside: use of complex variables.**

Rewriting the system

$$\dot{x} = g(x, y), \qquad \dot{y} = h(x, y)$$

in the complex form

$$\dot{z} = v(z), \qquad z := x + iy, \quad v := g + ih,$$

can be helpful in two ways:

(a) it is sometimes easier to express the system in polar coordinates,

(b) if $v(z)$ is *analytic*, i.e. differentiable in the complex sense, then the equation $dz/dt = v(z)$ with *complex* time t is separable.

Examples.

(a) Four bugs are at the corners of a square table. The bugs begin to move at the same instant, each crawling at the same constant speed toward the bug on its right. Describe their paths.

Solution. The paths are orbits of the system

$$\dot{z} = ce^{3\pi/4i}\frac{z}{|z|} = (-1+i)\frac{z}{|z|} \qquad (\text{if } c = \sqrt{2}).$$

In polar coordinates, we have

$$\frac{\dot{r}}{r} + i\dot{\phi} = \frac{-1+i}{r}, \quad \text{so} \quad \dot{r} = -1, \quad \dot{\phi} = r^{-1},$$

and the orbits are spirals. (Alternatively, the change of time $s = t/r$ gives us a linear system.)

(b) Describe the phase portrait of the system

$$\dot{x} = x^2 - y^2, \qquad \dot{y} = 2xy.$$

Solution. The complex form of the system is $\dot{z} = z^2$. Solving the equation

$$\frac{dz}{z^2} = dt, \qquad (z, t \in \mathbb{C}),$$

we have

$$-\frac{1}{z(t)} = t + \text{const} \equiv t + a + ib \qquad (a, b \in \mathbb{R}).$$

Restricting $z(t)$ to the real line, $t \in \mathbb{R}$, we get solutions of our system. We see that the imaginary part of $-1/z(t)$ has to be constant on each orbit of the system, so the function

$$E(x,y) = -\Im\left[\frac{1}{z}\right] = \frac{y}{x^2+y^2}$$

is a first integral in $\mathbb{R}^2 \setminus \{(0,0)\}$. The level set $\{E = 0\}$ consists of 2 orbits: $\mathbb{R}_+$ and $\mathbb{R}_-$. Each level set $\{E = b\}$ with $b \neq 0$ corresponds to a single orbit – the circle

$$x^2 + (y-c)^2 = c^2, \qquad c = 1/(2b),$$

without the point $(0,0)$. Adding the stationary orbit $\{(0,0)\}$ to the orbits described above, we obtain the phase portrait of the system.

(c) Sketch a phase diagram of the system

$$\dot{x} = x^2 - y^2, \qquad \dot{y} = -2xy.$$

Hint. The complex form of the system is

$$\dot{z} = \bar{z}^2 = \frac{|z|^4}{z^2}.$$

Changing the time variable, $ds = |z|^4 dt$, we get the equation

$$\frac{dz}{ds} = \frac{1}{z^2}.$$

Clearly, the change of time does not affect the phase portrait. The function $1/z^2$ is complex analytic, and we can proceed as in (b).

34. Linearization near a stationary point

34.1. **Linearization.** Let x_* be a stationary point of the (non-linear) system

$$\dot{x} = v(x), \qquad (x \in \mathbb{R}^2). \tag{34.1}$$

As always, we assume $v \in C^1$, so we have

$$v(x) = A(x - x_*) + o(|x - x_*|), \qquad x \to x_*,$$

where A is the *Jacobi matrix* of the map $v : \mathbb{R}^2 \to \mathbb{R}^2$,

$$A = \begin{pmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} \end{pmatrix}_{x=x_*}$$

If $\det A \neq 0$, then the linear system

$$\dot{x} = A(x - x_*), \qquad [\text{or} \quad \dot{y} = Ay \quad \text{for} \quad y = x - x_*], \tag{34.2}$$

is called the linearization of the system (34.1) at x_*.

Example. The linearization of the pendulum equation

$$\ddot{\theta} = -\sin\theta$$

at $(\theta_0, \dot{\theta}_0) = (0, 0)$ is harmonic oscillator $\ddot{y} = -y$.

The linearization at $(\theta_0, \dot{\theta}_0) = (\pi, 0)$ (the pendulum is at rest in the vertical position) is $\ddot{y} = y$.

34.2. **Hyperbolic singular points.** We can expect similar behavior (similar portraits) of (34.1) and (34.2) near x_*. This is in fact true if the singular point is *hyperbolic*.

By definition, x_* is *hyperbolic* if all eigenvalues of A have non-zero real part. Nodes, saddles, and foci are hyperbolic but centers are not hyperbolic, and of course critical points with $\det A = 0$ are not hyperbolic.

The *Hartman-Grobman* theorem states that if x_* is hyperbolic, then the phase portraits of the systems (34.1) and (34.2) are topologically equivalent near x_*. This theorem is true in all dimensions. In particular, the equilibrium solution is stable, asymptotically stable, or unstable simultaneously for both systems ("*stability by linearization*" criterion).

Corollary. *If all eigenvalues of A have negative real parts, then the equilibrium solution of* (34.1) *is asymptotically stable.*

Later we will give a proof that does not use Hartman-Grobman.

Example. The function $v : \mathbb{C} \to \mathbb{C}$,

$$v(z) = -z + \frac{iz}{\log|z|} \quad (z \neq 0), \qquad v(0) = 0,$$

is continuously differentiable because $z/\log|z| = o(|z|)$ as $z \to 0$. In polar coordinates, the system $\dot{z} = v(z)$ is

$$\frac{\dot{r}}{r} = -1, \qquad \dot{\theta} = \frac{1}{\log r}.$$

We see that the orbits with $r_0 < 1$ spiral to the singular point $z_* = 0$ making infinitely many turns:

$$\theta(t) = \theta_0 + \int_0^t \frac{ds}{\log r(s)} = \theta_0 + \int_0^t \frac{ds}{\log r_0 - s} \;\to\; -\infty, \qquad (t \to +\infty),$$

so the phase portrait near the origin looks like a stable focus. On the other hand, the linearized system is $\dot{z} = -z$; its portrait is a stable star node. The two portraits are topologically (but not smoothly) equivalent.

If the vector field is sufficiently smooth, e.g. $v \in C^\infty$, one can claim more than what Hartman-Grobman theorem states. In 2D, the claim is that the systems (34.1) and (34.2) are smoothly equivalent near x_*, and so their portraits are infinitesimally the same.

In particular, if x_* is a saddle point, then the system $\dot{x} = v(x)$ has exactly two orbits that tend to x_* as $t \to +\infty$ and exactly two orbits that tend to x_* as $t \to -\infty$ tangentially to the eigenvectors. These orbits are called *stable* and *unstable separatrices* of x_*. (Note that a stable separatrix can also be an unstable separatrix, the case of a *homoclinic connection.*) The separatrices separate regions with different generic behavior of orbits and as such are the most important objects (along with singular points and limit cycles) for understanding the qualitative picture of a 2D system.

34.3. **Parabolic singular points.** If the critical point of the linearized system (34.2) is a *center*, i.e. if the eigenvalues of A are purely imaginary, then from this information only we can not tell whether the equilibrium is stable or unstable for the non-linear system (34.1). The phase portrait of (34.1) near x_* can be a stable or unstable *non-linear focus*, which means that the orbits spiral to or away from the equilibrium, or it can be a *non-linear center* - all non-stationary orbits around x_* are cycles. In fact the picture may be even more complicated.

Example. $\dot{z} = iz + az|z|^2$. The equilibrium is an unstable focus if $a > 0$, a stable focus if $a < 0$, and a center if a is purely imaginary.

It can be shown that periodic solutions near the equilibrium ["*small oscillations*"] have periods approximatively equal to the period of the linearized system, i.e. if the eigenvalues of A are $\pm i\omega$, then

$$T \approx \frac{2\pi}{\omega}.$$

(More precisely, $T \to 2\pi/\omega$ as the diameter of the cycle converges to zero.)

35. Some models of population dynamics

We'll be considering the following situation. Let $x(t)$ and $y(t)$ be the populations of two interacting species. We assume that without interaction, the dynamics of $x(t)$ is subject to the logistic equations, $\dot{x} = kx - cx^2$. We also add the term bxy, which accounts for the interaction with the y-population. We make similar assumptions about $y(t)$. Thus we arrive to the system

$$\dot{x} = k_1 x - c_1 x^2 + b_1 xy, \quad \dot{y} = k_2 y - c_2 y^2 + b_2 xy. \tag{35.1}$$

Special cases:

- $b_1, b_2 > 0$ *cooperation* (the interaction is beneficial to both species);
- $b_1, b_2 < 0$ *competition* (the interaction is detrimental for both species);
- $b_1 < 0$, $b_2 > 0$ *predator-prey* model, see Section 14.6.

We are only interested in the behavior of the orbits inside the "population quadrant"

$$Q = \{x > 0, y > 0\}.$$

The boundary of the quadrant is the union of the orbits of the corresponding 1D logistic equations.

In some cases, the qualitative picture is already clear from the position of the lines

$$L_v: \ 1 = \frac{x}{k_1/c_1} - \frac{y}{k_1/b_1}, \qquad L_h: \ 1 = -\frac{x}{k_2/b_2} + \frac{y}{k_2/c_2}.$$

The vector field is vertical on

$$N_v = L_v \cup \{x = 0\}$$

and horizontal on

$$N_h = L_h \cup \{y = 0\}.$$

(The sets N_v and N_h are the *nullclines* of the system.)

Clearly, $N_v \cap N_h$ is the set of all critical points. The Jacobi matrix at the critical point $(x_*, y_*) = L_v \cap L_h$ is

$$A = \begin{pmatrix} k_1 - 2c_1 x_* + b_1 y_* & b_1 x_* \\ b_2 y_* & k_2 - 2c_2 y_* + b_2 x_* \end{pmatrix} = \begin{pmatrix} -c_1 x_* & b_1 x_* \\ b_2 y_* & -c_2 y_* \end{pmatrix}.$$

35.1. **Cooperation.** All parameters in (35.1) are positive, so both lines L_v and L_h have positive slopes. There are two possibilities:

- The lines L_v and L_h intersect inside Q. Then the critical point (x_*, y_*) is a stable equilibrium (a node), which attracts all orbits in Q.
- $L_v \cap L_h \cap Q = \emptyset$. In this case we have an explosive growth of both species, $x(t) \to \infty$ and $y(t) \to \infty$ [in finite time?]

Example.

$$\dot{x} = x(4 - 2x + y), \qquad \dot{y} = y(4 - 2y + x).$$

The equilibrium point $(x_*, y_*) \in Q$ is $(4, 4)$. The Jacobi matrix at (x_*, y_*) is

$$A = \begin{pmatrix} -8 & 4 \\ 4 & -8 \end{pmatrix}.$$

The eigenvalues and eigenvectors of A are

$$\lambda_1 = -4,\ e_1 = (1, 1); \qquad \lambda_2 = -12,\ e_2 = (1, -1).$$

The result of cooperation is stable equilibrium. All "non-exceptional" orbits tend to the equilibrium point tangentially to the first eigenvector.

35.2. **Competition.** We have

$$k_1, k_2, c_1, c_2 > 0, \quad b_1, b_2 < 0 \qquad \text{in (35.1)}.$$

The lines L_v and L_h have negative slopes. There are three possibilities:

- *Strong competition.* The lines L_h and L_v don't intersect inside Q. Then one of the critical points on the boundary of Q is a global attractor, so one of the species will die out regardless of initial conditions.

- *Coexistence.* The lines intersect inside Q in a stable node. This happens when L_h is below L_v for small x's, i.e. when $c_1c_2 > b_1b_2$ (look at the intercepts of the lines).

- *Weak competition.* The lines L_h and L_v intersect inside Q in a saddle. This happens iff $c_1c_2 < b_1b_2$. One of the species will die out; which one depends on initial conditions. The stable separatrices of (x_*, y_*) divide Q into two regions – the basins of attraction to the equilibria $L_v \cap \{y = 0\}$ and $L_h \cap \{x = 0\}$.

It is easy to explain the last criterion: (x_*, y_*) is a saddle iff $\det A < 0$, i.e. iff $c_1c_2 < b_1b_2$.

Example.

$$\dot{x} = x(1 - x - y), \qquad \dot{y} = y(3/4 - y - x/2).$$

The critical point inside Q is $(1/2, 1/2)$ with

$$A = \begin{pmatrix} -1/2 & -1/2 \\ -1/4 & -1/2 \end{pmatrix}.$$

The eigenvalues are negative, so we have coexistence of the species. To make a better sketch of the phase portrait it's a good idea to find eigenvectors (we'll find that all non-exceptional orbits are tangential to the eigenvector $(\sqrt{2}, -1)$), and also to analyze the critical points on the boundary of Q, which are $(0, 0)$, $(0, 3/4)$, and $(1, 0)$; their types are: unstable node, saddle, and saddle respectively.

For further examples, see Section 33.1.1 in the textbook – weak competition, and Section 33.1.2 – coexistence.

35.3. **Predator-prey.** We have

$$k_1 > 0,\ k_2 < 0, \quad b_1 < 0,\ b_2 > 0 \qquad \text{in} \quad (35.1).$$

(a) "Integrable" case $c_1 = c_2 = 0$, see Section 14.6. The Jacobi matrix at the critical point $(-k_2/b_2, -k_1/b_1) \in Q$,

$$A = \begin{pmatrix} 0 & -\frac{b_1 k_2}{b_2} \\ -\frac{b_2 k_1}{b_1} & 0 \end{pmatrix},$$

has purely imaginary eigenvalues,

$$\lambda_{1,2} \pm = i\sqrt{-k_1 k_2}.$$

As we showed in Section 14.6, all non-stationary orbits in Q are cycles, so the critical point is a non-linear center. The orbits near the critical point are infinitesimal ellipses. The period of small oscillations is

$$T \approx \frac{2\pi}{\sqrt{-k_1 k_2}}.$$

(b) The case $c_1 > 0$, $c_2 \geq 0$. There are two possibilities:

- No equilibrium inside Q. The predators will die out and the food population will settle down to its natural equilibrium.

- The lines L_h and L_v intersect inside Q, i.e. $k_1/c_1 > -k_2/b_2$. Then the equilibrium (x_*, y_*) is asymptotically stable. In fact, it's a global attractor.

Asymptotic stability follows from the inequalities $\det A > 0$, trace $A < 0$. For the global picture it's important to know that there are no periodic solutions, we'll prove it later.

Examples: see Section 33.2.

36. Conservative Newton's equation with 1 degree of freedom

$$\ddot{x} = F(x), \qquad x \in \mathbb{R}.$$

The corresponding 2D autonomous system:

$$\dot{x} = y, \qquad \dot{y} = F(x). \tag{36.1}$$

We assume $F \in C^1$, so we have the usual properties of the phase curves.

36.1. **Full energy.** Let $U \in C^2$ be a primitive of $(-F)$, i.e. $U'(x) = -F(x)$. As we already observed, the full energy (kinetic + potential)

$$E(x, y) = \frac{y^2}{2} + U(x)$$

is a first integral of the system.

Corollary. *Non-stationary orbits are precisely the components of the sets*

$$\{E = \text{const}\} \setminus \{\text{singular points}\}.$$

The singular points are the points $(x_*, 0)$ with $F(x_*) = 0$. This means that x_* is a critical point of the potential energy (e.g. local max or min), and the velocity is zero. The matrix of the linearized system at $(x_*, 0)$ is

$$\begin{pmatrix} 0 & 1 \\ F'(x_*) & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -U''(x_*) & 0 \end{pmatrix}. \tag{36.2}$$

36.2. **Bead-on-wire.** The motion described by Newton's equation $\ddot{x} = F(x)$ is qualitatively the same as the motion of an ideal ball rolling on the graph of $U(x)$ without friction. For this model, the conserved total energy is

$$E(x, y) = \frac{y^2}{2}\left[1 + F^2(x)\right] + U(x), \qquad (y = \dot{x}),$$

because the velocity of $(x(t), U(x))$ is $(\dot{x}(t), \dot{x}(t)U'(x))$. It follows that the motion of the ball is given by the system

$$\dot{x} = y, \qquad E_x\dot{x} + E_y\dot{y} = 0,$$

i.e.

$$\dot{x} = y, \qquad \dot{y} = F(x)\,\frac{1 - y^2F'(x)}{1 + F(x)^2}. \tag{36.3}$$

The systems (36.1) and (36.3) have the same singular points and the same linearized systems (because y^2 and $F(x)^2$ are second order at singular points). Moreover, the systems have diffeomorphic phase portraits; the diffeomorphism is

$$(x, y) \mapsto \left(x, \frac{y}{\sqrt{1 + F(x)^2}}\right).$$

(Hint: express the first integral $y^2/2 + U(x)$ in the new coordinates.)

36.3. **Phase portraits.** Let us explain how to draw level sets of E. Fix the value E_0 of the full energy.

We first assume that E_0 is *non-critical*, i.e.

$$U(x) = E_0 \quad \Rightarrow \quad U'(x) \neq 0,$$

so there are no singular points on the level set $\{E = E_0\}$, and the components of the level set are phase curves. The open set $\{U < E_0\}$ consists of intervals (a, b), which can be finite or infinite. Each interval corresponds to some component of the level set.

If (a, b) is a *finite* interval, then the orbit is a *cycle*

$$y(x) = \pm\sqrt{2(E_0 - U(x))}, \quad a \leq x \leq b, \qquad \text{[in the case (36.1)]}$$

and

$$y(x) = \pm\sqrt{\frac{2E_0 - 2U(x)}{1 + F^2(x)}}, \quad a \leq x \leq b, \qquad \text{[in the case (36.3)].}$$

If the ball starts from $x = a$ with zero velocity, then the potential energy is $U(a) = E_0$ and the kinetic energy is zero. Since the kinetic energy is non-negative, the ball can not rise above the level E_0. We can think of the graph of U as a "potential well". The ball rolls into the well picking up speed, which it then looses when ascending.

In the case of Newton's equation, the period of the motion is

$$T = 2\int_a^b \frac{dx}{\sqrt{2(E - U(x))}}. \tag{36.4}$$

The integral converges because $E_0 - U(x) = U(a) - U(x) \sim U'(a)(a - x)$ as $x \to a$, and the singularity in the integral has order $1/2$.

If the interval (a, b) is *infinite*, the orbit escapes to infinity. The escape time can be finite or infinite. Examples:

(i) If $U(x) = -x^4/2$, then $x(t) = (t - 1)^{-1}$ is a solution of Newton's equation, so we have a "blow up" case.

(ii) If $U \geq -cx^2$, then all solutions extend infinitely in time because the integral (36.4) for the escape time diverges at infinity. (Why?) The argument is even simpler in the case $U(x) = -cx^2$: the Newton equation is linear.

Consider now the case of a *critical* energy level E_0. Suppose (a, b) is a finite interval, and suppose that both endpoints are critical points. We have 4 orbits (including 2 stationary orbits) on the set

$$y = \pm\sqrt{2(E_0 - U(x))}, \quad a \leq x \leq b.$$

It takes infinite time (from $-\infty$ to $+\infty$) for the particle to trace each non-stationary orbit.

36.4. **Linearization of conservative systems.** Let $(x_*, 0)$ be a singular point, $U'(x_*) = 0$. The eigenvalues of (36.2) satisfy the equation

$$\lambda^2 = -U''(x_*).$$

There are two possibilities in the non-degenerate case $U''(x_*) \neq 0$.

(a) $U''(x_*) < 0$, so x_* is a local maximum of potential energy. The singular point is a saddle. (The graph of the full energy looks like a saddle.)

(b) $U''(x_*) > 0$, so x_* is a local minimum of U. The eigenvalues are $\pm i\sqrt{U''(x_0)}$. We know from the phase portrait that the motion is periodic near the singular point, so we have the case of a non-linear *center*. The period of *small oscillations* is

$$T \approx \frac{2\pi}{\sqrt{U''(x_*)}}.$$

Exercise: give a rigorous proof based on (36.4).

36.5. **Examples.**

(a) *Simple pendulum.*

$$\ddot{x} = -\sin x.$$

We can choose $U(x) = -\cos x$. Equilibrium points are $(k\pi, 0)$, $k \in \mathbb{Z}$; they are centers for even k's and saddles for odd k's. The orbits fall into the following 5 classes:

(i) stable equilibria (the pendulum hangs vertically downwards);

(ii) unstable equilibria (the pendulum stands upwards);

(iii) the pendulum swings between two positions of instantaneous rest;

(iv) the pendulum swings from one upright position to another (and it takes infinite time to do so);

(v) the pendulum rotates like crazy in the same direction and is never at rest.

See Fig. 35.4 in the textbook.

Note. The motion of type (v) is periodic though the corresponding orbits in Fig. 35.4 are not cycles. The reason is that the correct phase space of the pendulum is the cylinder $TS^1 \cong S^1 \times \mathbb{R}$ (the tangent space of the circle). There are only two equilibrium points in TS^1.

(b) *Kepler's potential.* Consider the potential

$$V(r) = -\frac{\gamma}{r} + \frac{M^2}{2r^2}, \qquad (r > 0).$$

The unknown function $r = r(t)$ represents the distance from a comet or planet to the sun, γ is the gravitational constant, and M is the angular momentum of the comet, see next section.

If $M = 0$, then the equation $\ddot{r} = -\gamma r^{-2}$ describes the motion of a particle on the real line in the gravitational field. The full energy is $E = y^2/2 - \gamma/x$. If $E > 0$ then the orbit escapes to infinity; if $E = 0$ then the orbit also escapes but with terminal velocity zero; if $E < 0$, then the thing will crash.

Summary. Conservative systems with one degree of freedom have the following features:

- there are no asymptotically stable equilibria;
- stable motion is always periodic;
- isolated points of local minimum of potential energy are centers, and the motion near such points is similar to harmonic oscillations;
- there are no isolated (or limit) cycles.

37. Motion in a central field

Conservative Newton's equation with d degrees of freedom is

$$\ddot{x} = -\nabla U(x), \qquad x \in \mathbb{R}^d.$$

Here $U : \mathbb{R}^d \to \mathbb{R}$ is a given C^2-function (potential energy). The full energy

$$E(x, \dot{x}) = \frac{\dot{x} \cdot \dot{x}}{2} + U(x), \qquad (\cdot \quad \text{is scalar product}),$$

is a first integral of the motion. As in the case $d = 1$, we can think of an ideal ball rolling on the graph of U.

An example of a conservative system: the motion in a cental field,

$$\ddot{x} = f(r)\ \frac{x}{r}, \qquad r := \|x\|, \qquad x \in \mathbb{R}^3,$$

where $f(r)$ is a given scalar function. If $U(r)$ satisfies $U'(r) = -f(r)$, then

$$-\frac{\partial U(r)}{\partial x_j} = -U'(r)\frac{\partial r}{\partial x_j} = f(r)\frac{x_j}{r},$$

so $-\nabla U = r^{-1} f(r)\ x$ and $U(|x|)$ is the potential energy. Denote

$$M(t) = x(t) \times \dot{x}(t) \in \mathbb{R}^3, \qquad (\times \quad \text{is vector product}).$$

Lemma. *For every solution,* $M(t) = \text{const}$. *In particular, the motion is planar.*

Proof: We have

$$\dot{M} = \dot{x} \times \dot{x} + x \times \ddot{x} = \dot{x} \times \dot{x} + \frac{f(r)}{r}\ x \times x = 0.$$

The motion takes place in the plane orthogonal to M. □

Let (r, ϕ) be the polar coordinates in the plane of motion, and let $z = re^{i\phi}$. Then the equation of the motion is

$$\ddot{z} = f(r)e^{i\phi},$$

and since

$$\dot{z} = (\dot{r} + ir\dot{\phi})\ e^{i\phi}, \qquad \ddot{z} = (\ddot{r} + 2i\dot{r}\dot{\phi} + ir\ddot{\phi} - r\dot{\phi}^2)\ e^{i\phi},$$

we have the following system of 2d order equations:

$$\ddot{r} - r\dot{\phi}^2 = f(r), \qquad 2\dot{r}\dot{\phi} + r\ddot{\phi} = 0. \tag{37.1}$$

Lemma. *The system* (37.1) *has the following two first integrals:*

$$M = M(r, \dot{r}, \phi, \dot{\phi}) = r^2\dot{\phi}, \qquad E = E(r, \dot{r}, \phi, \dot{\phi}) = \frac{\dot{r}^2}{2} + \frac{M^2}{2r^2} + U(r).$$

Proof: $M = -\Im(\bar{z}\dot{z})$ is the (scalar) angular momentum, and $E = |\dot{z}|^2/2 + U(r)$ is the full energy. □

The first statement in the lemma is *Kepler's second law*: the sectorial velocity of the motion in a central field is constant.

For a fixed value of M denote

$$V(r) = \frac{M^2}{2r^2} + U(r) \qquad (\text{"effective potential"}).$$

Lemma. *Solutions of* (37.1) *with angular momentum M satisfy the equation*

$$\ddot{r} = -V'(r). \tag{37.2}$$

Proof:

$$-V'(r) = \frac{M^2}{r^3} - U'(r) = r\dot{\phi}^2 + f(r) = \ddot{r} \qquad \text{by (37.1).}$$

□

We can now apply the theory of one degree of freedom to describe the behavior of solutions $r(t)$ of (37.2). We can then use the equation $M = r^2(t)\dot{\phi}$ to understand the behavior of $\phi(t)$. For example:

(a) Critical points r_* of the effective potential, $V'(r_*) = 0$, give equilibrium solutions $r(t) \equiv r_*$ of (37.2). We then find $\phi(t) = Mr_*^{-2}t + \text{const}$, so the *trajectory* of the motion $z(t) = r(t)e^{i\phi(t)}$ is a circle (assuming $M \neq 0$).

(b) Fix the value E of the full energy, and assume that E is non-critical for the effective potential (with a fixed angular momentum M). Suppose a finite interval (r_1, r_2) is a component of the open set $\{V(r) < E\}$ so $r(t)$ periodically oscillates between r_1 and r_2, see Section 25.3. If t_1, t_2 are the times of the two successive events $r(t_1) = r_1$ and $r(t_2) = r_2$, then the angle $\Phi = \phi(t_2) - \phi(t_1)$ is given by the equation

$$\Phi = M\int_{r_1}^{r_2} \frac{dr}{r^2\sqrt{2(E - V(r))}}.$$

The trajectory of the motion $z(t)$ is a closed curve if and only if Φ/π is a rational number. Otherwise, the trajectory is dense in the annulus $r_1 < |z| < r_2$.

Exercise. Show that trajectories with full energy E and angular momentum M are described by the separable equation

$$\frac{dr}{d\phi} = \frac{r^2\sqrt{2(E - V(r))}}{M}.$$

Examples.

(a) *Gravitational field*

$$U(r) = -\frac{\gamma}{r}, \qquad V(r) = \frac{M^2}{2r^2} - \frac{\gamma}{r}.$$

Kepler's first law: trajectories are conics (ellipses, parabolas, and hyperbolas) with $z = 0$ as a focus.

Kepler's third law: the squares of the periods of revolution have the same ratio as the cubes of the major axes. [Prove it at least for circular trajectories.]

(b) *Harmonic oscillator*

$$U(r) = kr^2, \qquad V(r) = \frac{M^2}{2r^2} + kr^2.$$

Show that trajectories are ellipses, and the period does not depend on initial conditions.

38. Examples of dissipative systems

38.1. **First examples.** (a) Damped oscillations, see Section 20.2.

(b) A ball is thrown straight up from the ground level and is subject to quadratic (=Newtonian) damping: the magnitude of the damping force is proportional to the square of the velocity. Does it take longer for the ball to rise or to fall?

Hint: the full energy of the ball decreases. The speed at the same height is smaller when the ball is falling.

38.2. **Damping.** Consider a conservative Newton's equation

$$\ddot{x} = F(x), \qquad F(x) = -U'(x), \tag{38.1}$$

and add a damping term $G(\dot{x})$ (we will assume $G \in C^1$):

$$\ddot{x} = F(x) + G(\dot{x}), \tag{38.2}$$

or

$$\dot{x} = y, \qquad \dot{y} = F(x) + G(y).$$

A damping term is a force that slows down the particle:

$$y > 0 \quad \Rightarrow \quad G(y) < 0, \qquad y < 0 \quad \Rightarrow \quad G(y) > 0.$$

The simplest example is linear damping $G(y) = -ky$ with $k > 0$, see Section 19.2; quadratic damping $G(y) = -ky|y|$ with $k > 0$ is another example.

Lemma. *The full energy*

$$E(x, \dot{x}) = \frac{\dot{x}^2}{2} + U(x)$$

dissipates along the orbits of system (38.2)*, i.e. if $x(t)$ is a non-stationary solution, then*

$$\frac{d}{dt}E(x(t), \dot{x}(t)) < 0.$$

Proof:

$$\dot{E} = \dot{x}\ddot{x} - \dot{x}F(x) = \dot{x}[\ddot{x} - F(x)] = \dot{x}G(\dot{x}) < 0.$$

□

Corollary. *Dissipative systems have no periodic non-stationary solutions.*

In particular, singular points can not be centers.

38.3. **Singular points.** Systems (38.1) and (38.2) have the same singular points, namely the points $(x_*, 0)$ with $U'(x_*) = 0$.

It is physically "obvious" that the equilibrium $(x_*, 0)$ is stable for both (38.1) and (38.2) if x_* is a strict local minimum of U; in the dissipative case the equilibrium is also attracting. See the next section for rigorous proofs.

If the local minimum is non-degenerate, i.e. if $U''(x_*) > 0$, then the Jacobi matrix of the dissipative system is

$$A = \begin{pmatrix} 0 & 1 \\ -U''(x_*) & -k \end{pmatrix}$$

in the case of linear damping, and we find that the critical point is a stable node, improper node, or focus according as the damping coefficient is large, critical, or small.

Exercise: saddle points of (38.1) are saddle points of (38.2).

38.4. **Examples of phase portraits.**

(a) $\qquad \ddot{x} = -4(x^3 - x) - k\dot{x}, \qquad 0 < k \ll 1.$

This is the case of a small linear damping in the conservative system with potential

$$U(x) = x^4 - 2x^2$$

("double well"). The potential has 3 critical points: a local maximum at $x_* = 0$, and local minima at ± 1. The equilibrium $(x_*, 0) = (0, 0)$ is a saddle, and $(\pm 1, 0)$ are stable foci. The stable separatrices of the saddle spiral out to infinity in the backward direction of time, and unstable separatrices approach the foci as $t \to +\infty$. The stable manifold (the union of the stable separatrices and the saddle point) is a simple curve that divides the phase plane into two snake-like regions. One the regions in the basin of attraction to the focus $(1, 0)$, and the other one is the basin of $(-1, 0)$.

(b) Damped pendulum, see Section 35.2 in the textbook.

39. Lyapunov functions and stability

39.1. **Lyapunov functions.** Let x_* be a singular point of the system

$$\dot{x} = v(x), \qquad x(t) \in \mathbb{R}^n. \tag{39.1}$$

By definition, a C^1-function $\Phi(x)$, which is defined in some neighborhood of x_*, is a *Lyapunov function* of (39.1) at x_* if

(i) x_* is a *strict* minimum of Φ, i.e. $\Phi(x_*) < \Phi(x)$ for all $x \neq x_*$, and

(ii) $D_v\Phi(x) < 0$ for all $x \neq x_*$.

Also, $\Phi(x)$ is called a *weak Lyapunov function* if instead of (ii) it satisfies

(ii') $D_v\Phi(x) \leq 0$ for all x.

We use the notation $D_v\Phi$ for the *directional derivative* of the function Φ. If $v \in \mathbb{R}^n$ is a constant vector, then

$$\begin{aligned} D_v\Phi(x) &= \frac{d}{d\tau}\Big|_{\tau=0} \Phi(x+\tau v) \\ &= \nabla\Phi(x)\cdot v \qquad \text{(scalar product)} \\ &= \frac{\partial \Phi}{\partial x_1} v_1 + \cdots + \frac{\partial \Phi}{\partial x_n} v_n. \end{aligned}$$

For a vector field $v(x)$, $D_v\Phi$ is the short notation for the function

$$x \mapsto [D_{v(x)}\Phi](x).$$

Clearly, if $x(t)$ is a solution of $\dot{x} = v(x)$, then

$$\frac{d}{dt}\Phi(x(t)) = D_v\Phi(x(t)),$$

and therefore condition (ii) has the following meaning:

$$\frac{d}{dt}\Phi(x(t)) < 0, \qquad \text{and} \qquad \Phi(x(t)) \quad \text{is descreasing}$$

for all non-stationary solutions near x_*; in other words, we have the dissipation of Φ along the orbits.

Example. If we have a dissipative system, and x_* is a strict local minimum of the potential energy, and if there are no other critical points near x_*, then the full energy is a Lyapunov function at x_*. Similarly, the full energy is a weak Lyapunov function for conservative systems at the points of strict local minimum of the potential.

39.2. **Three theorems.**

Theorem. *If the system* (39.1) *has a weak Lyapunov function at x_*, then the equilibrium x_* is stable.*

Proof: Wlog, $x_* = 0$, and $\Phi(x_*) = 0$. We want to show

$$\forall \varepsilon > 0\ \exists \delta > 0, \qquad |x(0)| < \delta \ \Rightarrow\ |x(t)| < \varepsilon \quad \text{for all } t > 0.$$

[More accurately, $x(t)$ exists for all $t > 0$ and $|x(t)| < \varepsilon$.] By condition (ii'), we have

$$\Phi(x(t)) \le \Phi(x(0)) \tag{39.2}$$

as long as $x(t)$ stays in N, a neighborhood where Φ is defined and satisfies (i), (ii'). Let us assume that ε is so small that N contains the ball $|x| \le 2\varepsilon$, and let $2c$ denote the minimal value of Φ in the ring $\varepsilon \le |x| \le 2\varepsilon$. Clearly, $c > 0$. The open set $\{\Phi < c\}$ contains the point $x_* = 0$ together with its δ-neighborhood (for some $\delta > 0$). If $x(0)$ is inside this δ-neighborhood, then $\Phi(x(0)) < c$, and by (39.2) we have $\Phi(x(t)) < c$ and therefore $|x(t)| < \varepsilon$ as long as $|x(t)| \le 2\varepsilon$. This shows that the orbit can not leave the ε–neighborhood, in particular, the solution extends infinitely forward. $\square$

Corollary. *Let x_* be a strict local minimum of the potential energy of the conservative system $\ddot{x} = -U'(x)$. Then the stationary solution $x(t) \equiv x_*$ is stable.*

[Q: is the statement true for points of non-strict local minimum?]

Theorem. *If the system* (39.1) *has a Lyapunov function at* x_**, then the equilibrium* x_* *is asymptotically stable.*

The proof is similar to the proof of the previous theorem. Note that (ii) implies that x_* is an isolated singular point of the system.

Theorem. *Suppose* x_* *is a singular point of* (39.1) *and suppose that there is a function* Φ *in some neighborhood of* x_* *such that* x_* *is* not *a point of local minimum of* Φ *but* $D_v\Phi(x) < 0$ *for all* $x \neq x_*$ *in some neighborhood of* x_**. Then the equilibrium* x_* *is unstable.*

Proof: The function Φ strictly decreases along the orbits of the system in some ϵ-neighborhood of x_*. Taking a solution $x(t)$ with $x(0)$ arbitrarily close to x_* and with $\Phi(x(0)) < \Phi(x_*)$, we get smaller and smaller values of $\Phi(x(t))$ along the orbit, so the orbit has to leave the ϵ-neighborhood of x_*. □

Exercise. Prove the following global version of the second theorem.

Consider the system $\dot{x} = v(x)$ in some domain $D \subset \mathbb{R}^n$, $v \in C^1(D)$. Suppose there is a *bounded* function $\Phi \in C^1(D)$ such that x_* is a minimum point and $D_v\Phi < 0$ in $D \setminus \{x_*\}$. Then x_* is a global attractor.

39.3. **Stability by linearization.** We'll show how the technique of Lyapunov's functions can be used to prove the "stability by linearization" criterion mentioned in Section 34.2. Consider a non-linear system

$$\dot{x} = v(x), \qquad v \in C^1(\mathbb{R}^n), \tag{39.3}$$

and let $\dot{x} = A(x - x_*)$ be its linearization at a critical point x_*.

Theorem. *(i) If all eigenvalues of A have negative real parts, then the equilibrium solution* $x(t) \equiv x_*$ *of* (39.3) *is asymptotically stable.*

(ii) If the real part of at least one eigenvalue is positive, then the equilibrium is unstable.

Proof: [for $n = 2$] We will prove (i); the proof of (ii) is similar.

Changing variables we can assume that A has one of the following forms:

$$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}, \quad \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix}, \quad \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}.$$

In the first two cases we consider

$$\Phi(x_1, x_2) = \frac{1}{2}(x_1^2 + x_2^2), \qquad \text{so} \quad \nabla\Phi = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Then $x_* = 0$ is a strict minimum of Φ, and since

$$v(x) = Ax + o(|x|) \quad \text{as} \quad x \to 0,$$

we have

$$D_v\Phi = Ax \cdot \nabla\Phi = \lambda_1 x_1^2 + \lambda_2 x_2^2 + o(|x|^2) \qquad \text{in the first case,}$$

and

$$D_v\Phi = Ax \cdot \nabla\Phi = \alpha(x_1^2 + x_2^2) + o(|x|^2) \qquad \text{in the second case.}$$

Since $\lambda_1, \lambda_2 < 0$ or $\alpha < 0$, we have $D_v\Phi < 0$ in some neighborhood of x_*, and we can apply the second theorem.

In the third case we consider the function

$$\Phi(x_1, x_2) = \frac{1}{2}(x_1^2 + Kx_2^2), \qquad \nabla\Phi = \begin{pmatrix} x_1 \\ Kx_2 \end{pmatrix},$$

where $K > 0$ is sufficiently large. Then $x_* = 0$ is a strict minimum of Φ, and

$$D_v\Phi = Ax \cdot \nabla\Phi = \lambda x_1^2 + K\lambda x_2^2 + x_1 x_2 + o(|x|^2).$$

By assumption, $\lambda < 0$. If $K \gg 1$, namely if $K > 1/(4\lambda^2)$, then the quadratic form in the last expression is negatively definite. $\square$

[Recall the following fact: the form $ax^2 + bxy + cy^2$ is positive definite iff $a > 0$, $c > 0$, and $4ac > b^2$.]

39.4. **Further examples.** There is no universal method to construct Lyapunov's functions. In some cases the choice $\Phi(x, y) = ax^2 + bxy + cy^2$ may work. It's remarkable that we don't need to solve the differential equation to make a conclusion about stability.

(a) $\dot{x} = -x^3 + xy^2$, $\dot{y} = -2x^2y - y^3$.

The critical point $(0, 0)$ is degenerate: $A = 0$. Let us try $\Phi(x, y) = (x^2 + y^2)/2$. We have

$$D_v\Phi = x(-x^3 + xy^2) + y(-2x^2y - y^3) = -x^4 + x^2y^2 - 2x^2y^2 - y^4 < 0$$

for $(x, y) \neq (0, 0)$. Conclusion: the equilibrium is asymptotically stable.

(b) $\dot{x} = y - 3x - x^3$, $\dot{y} = 6x - 2y$.

The critical point $(0, 0)$ is degenerate:

$$A = \begin{pmatrix} -3 & 1 \\ 6 & -2 \end{pmatrix} \qquad \lambda_1 = 0,\ \lambda_2 = -5.$$

Let us try $\Phi(x, y) = (ax^2 + y^2)/2$ with an undetermined coefficient $a > 0$. We have

$$\begin{aligned} D_v\Phi &= ax(y - 3x - x^3) + y(6x - 2y) = axy - 3ax^2 - ax^4 + 6xy - 2y^2 \\ &= -(3ax^2 - (6 + a)xy + 2y^2) - ax^4 < 0 \end{aligned}$$

for $(x, y) \neq (0, 0)$, provided that

$$4 \cdot 3a \cdot 2 \geq (6 + a)^2, \qquad \text{i.e.} \qquad 24a \geq 36 + 12a + a^2,$$

i.e. $0 \geq 36 - 12a + a^2 = (a - 6)^2$. The choice of $a = 6$ will do! Conclusion: the equilibrium is asymptotically stable.

(c) $\dot{x} = x^3 - y$, $\dot{y} = x + y^3$.

The eigenvalues of the critical point $(0, 0)$ are $\pm i$, so once again the linearization criterion does not apply. Consider $\Phi(x, y) = (x^2 + y^2)/2$. We have

$$\nabla\Phi \cdot v = x^4 + y^4 > 0, \qquad (x, y) \neq (0, 0),$$

and therefore the function $(-\Phi)$ satisfies the conditions of the third theorem. The equilibrium is unstable.

(d) $\dot{x} = 2y^3 - x^5,\ \dot{y} = -x - y^3 + y^5$.

Try $\Phi(x, y) = (ax^2 + y^4)/2$. We have

$$D_v\Phi = 2axy^3 - ax^6 - y^6 - xy^3 + y^8 = -ax^6 - y^6 + y^8 \quad \text{if} \quad a = \frac{1}{2}.$$

(e) $\dot{z} = \bar{z}^2,\ z \in \mathbb{C}$. We don't need to use Lyapunov's method here. The answer (the equilibrium is unstable) is obvious if we look at the orbits which lie on the real axis.

40. Limit cycles

40.1. **Definition.** Consider a 2D autonomous system. Recall that cycles are orbits of periodic solutions.

A cycle C is a *limit* cycle if C is the limit set of some non-periodic orbit as $t \to +\infty$ or $t \to -\infty$. For example, isolated cycles are limit cycles.

[A cycle C is *isolated* if there is a neighborhood of C that does not intersect any other cycle. E. g., cycles of conservative system are not isolated.]

Example: $\dot{z} = -iz + a(1 - |z|^2)z, \quad (z \in \mathbb{C},\ a$ is a real parameter).

In polar coordinates we have

$$\dot{r} = ar(1 - r^2), \quad \dot{\theta} = -1.$$

The second equation means clockwise rotation with constant angular speed. The first equation is a 1D autonomous system. Applying the sign analysis (see Section 3.2), we see that the stationary point $r_* = 1$ of the 1D autonomous equation is attracting if $a > 0$ and repelling if $a < 0$. This gives us examples of *attracting* and *repelling* limit cycles.

[A cycle C is *attracting* or *orbitally asymptotically stable* if it attracts every orbit with initial point in some neighborhood of C.]

Obvious modifications of the equation, e.g.

$$\dot{r} = ar(1 - r^2)(4 - r^2), \quad \text{or} \quad \dot{r} = r\sin(1/r),$$

provide examples with two, infinitely many isolated cycles.

40.2. **Van der Pol systems.**

$$\dot{x} = y - \mu f(x), \qquad \dot{y} = -x. \tag{40.1}$$

We assume that the function f satisfies the following conditions:

- f is odd, $f(0) = 0$, and $f(\infty) = \infty$;
- $f < 0$ on $(0, a)$ and $f > 0$ on (a, ∞) for some $a > 0$,

and μ is a positive parameter. For example, we can take $f(x) = x - 2\arctan x$, or we can take

$$f(x) = \frac{x^3}{3} - x. \tag{40.2}$$

There are two ways to transform the system (40.1) into a single second order equation:

$$\ddot{y} + \mu f(\dot{y}) + y = 0,$$

and

$$\ddot{x} + x + \mu f'(x)\dot{x} = 0. \tag{40.3}$$

E.g., if f is given by (40.2), then we get the equations

$$\ddot{y} + y - \mu(\dot{y} - \dot{y}^3/3) = 0 \qquad \text{(Rayleigh)}$$

and

$$\ddot{x} + x + \mu(x^2 - 1)\dot{x} = 0 \qquad \text{(van der Pol)}. \tag{40.4}$$

Theorem. *Every van der Pol system* (40.1) *has a unique cycle.*

The cycle is in fact a global attractor – it attracts all non-stationary orbits.

Remarks.

(a) Informal explanation of the theorem.

Think of equation (40.4) as a damped harmonic oscillator with damping coefficient $\mu(x^2 - 1)$, which is positive only if $|x| > 1$ but is negative if $|x| < 1$. In other words, if $|x|$ is large the oscillation is slowed, but for small $|x|$ the oscillation is driven. The system must settle down to some particular oscillation.

(b) Interpretation of (40.3) in terms of electric circuits.

Take the usual CLR circuit but replace the resistor R by a semiconductor. The corresponding voltage drop will be equal to $\mu f(I)$, where $I = I(t)$ is the current, and f has the properties similar to the properties stated above – the semiconductor acts like a "negative resistor" at low current levels pumping energy into the circuit but dissipates energy at high levels. Energize the circuit, then remove the power supply. The equation for the current will be

$$L\dot{I} + \mu f(I) + C^{-1}Q = 0, \qquad I = \dot{Q},$$

or

$$L\ddot{I} + \mu f'(I)\dot{I} + C^{-1}I = 0.$$

The coefficient μ may serve as a tuning parameter.

40.3. **Poincaré-Bendixson.**

Theorem. *Let Ω be an annulus region between two smooth curves and let a smooth vector field v be defined in a neighborhood of clos Ω. Suppose that the vector field*

- *has no singular points in clos Ω;*
- *points into Ω along both boundary components.*

Then the system $\dot{x} = v(x)$ has a cycle inside Ω.

Proof: [in a special case]. Assume that the origin is inside the interior curve and that v never points in the radial direction. From the hypothesis we conclude that every orbit stays inside Ω (the vector field points inside) and never stops (no singular points), so it rotates around the hole in the same direction (by our assumption). Let us assume in addition that there is a closed radial segment I joining the two boundaries. We can define the *first return* map $F : I \to I$ as follows:

$$x_0 \in I \mapsto \phi(T),$$

where ϕ is the solution of the IVP $\phi(0) = x_0$ and $T > 0$ is the first time when the orbit of ϕ intersects (returns to) the segment I. The first return map is a continuous function, and therefore it has a fixed point. [Why?] The orbit through the fixed point of F is a cycle. $\square$

Remark. It is not difficult to construct a Poincaré-Bendixson annulus for any van der Pol system.

40.4. **Small perturbations of conservative systems.** [Adapted from Arnold, Section 12.10]

What is the amplitude of the periodic solution of the van der Pol equation (40.4) in the case $\mu = \epsilon \ll 1$? We will show that the cycle is approximately a circle of radius $R = 2$.

The system (40.1),

$$\dot{x} = y + \epsilon f(x), \qquad \dot{y} = -x,$$

is a small perturbation of the harmonic oscillator $\ddot{x} + x = 0$. Let us parametrize the set of cycles of the harmonic oscillator by the values of the (conserved) full energy

$$E(x, y) = \frac{x^2 + y^2}{2},$$

so the cycle C_E with energy E is the circle of radius $\sqrt{2E}$. The phase curves of the perturbed system are no longer closed, but they are "almost cycles", and we can define the first return map as in the proof of the PB theorem. Let

$$E \mapsto E + \Delta E$$

be the change of the quantity $E(x, y)$ at the time of the first return. If $\Delta E < 0$, then the curve spirals inward, if $\Delta E > 0$ then it spirals outward, and we have a cycle if $\Delta E = 0$. To find an expression for ΔE, we note that

$$\frac{d}{dt} E(x(t), y(t)) = x\dot{x} + y\dot{y} = -\epsilon x f(x).$$

If $T = T(E)$ is the time of the first return, then

$$\Delta E = -\epsilon \int_0^T x(t) f(x(t))\ dt.$$

Clearly, $T(E) \approx 2\pi$, the period of the unperturbed system, and the solution $\{x(t), y(t)\}$ is close to the motion $\{x_E(t), y_E(t)\}$ of the harmonic oscillator. In fact, we have

$$\Delta E = -\epsilon \int_0^{2\pi} x_E(t) f(x_E(t))\ dt + o(\epsilon^2), \qquad \epsilon \to 0.$$

Since $x_E(t)dt = \dot{y}_E(t)dt = dy_E(t)$, we can rewrite the last integral using Green's formula:

$$\int_0^{2\pi} f(x_E(t))\ x_E(t)dt = \int_{C_E} f(x)\ dy = \int\int_{G_E} f'(x)\ dxdy,$$

where G_E is the domain bounded by C_E, i.e. the disc of radius $R = \sqrt{2E}$. Thus we have

$$\Delta E \approx -\epsilon \int\int_{G_E} f'(x)\ dxdy.$$

In the case $f(x) = \frac{x^3}{3} - x$, we have $f'(x) = x^2 - 1$, and since

$$\int\int_{\Omega_E} x^2\ dxdy = \frac{1}{2}\int\int_{\Omega_E} (x^2 + y^2)\ dxdy = \pi \int_0^R r^3 dr = \frac{\pi R^4}{4},$$

we have

$$\Delta E \approx \epsilon\pi \left[R^2 - \frac{R^4}{4}\right],$$

so $\Delta E = 0$ if $R \approx 2$.

40.5. **Bendixson-Dulac.**

Theorem. *Suppose Ω is a simply connected domain (i.e. Ω has no holes), and suppose that a smooth vector field v in Ω has the following property: the function*

$$\text{div } v \equiv \frac{dv_1}{dx_1} + \frac{dv_2}{dx_2}$$

has the same sign throughout Ω. Then the system $\dot{x} = v(x)$ has NO cycles inside Ω.

Proof: Suppose there is a cycle $C \subset \Omega$. Denote by G the domain bounded by C. Since Ω has no holes, $G \subset \Omega$. Gauss theorem [or Green's formula in the divergence form] states that

$$\int_C v \cdot n\ ds = \int\int_G (\text{div } v)\ dx_1 dx_2.$$

Here n is the exterior unit normal to C, $v \cdot n$ is scalar product, and ds is the arclength. Since C is an orbit, we have $v \cdot n \equiv 0$ on C (orbits are tangent to the vector field). On the other hand, the double integral is strictly positive or negative by assumption, a contradiction. □

Corollary. *The conclusion of the theorem is true if there is a* scalar *function $h \in C^1(\Omega)$ such that* div (hv) *has the same sign throughout Ω.*

Proof: The systems $\dot{x} = v(x)$ and $\dot{x} = h(x)v(x)$ have the same orbits. $\square$

Example. *The ecological models in Section 24 have no limit cycles.*

Proof: Let us show that the system

$$\dot{x} = k_1 x - c_1 x^2 + b_1 xy, \qquad \dot{y} = k_2 y - c_2 y^2 + b_2 xy$$

with $c_1 < 0$, $c_2 < 0$ has no periodic solutions in the population quadrant $\Omega = \{x > 0, y > 0\}$. Define $h(x, y) = x^{-1}y^{-1}$. Then

$$hv = \begin{pmatrix} \frac{k_1}{y} - c_1 \frac{x}{y} + b_1 \\ \frac{k_2}{x} - c_2 \frac{y}{x} + b_2 \end{pmatrix},$$

and

$$\text{div } (hv) = -\frac{c_1}{y} - \frac{c_2}{x} > 0 \quad \text{in} \quad \Omega.$$

$\square$

40.6. **Poincaré-Hopf.**

Claim. *Suppose C is a cycle and G is a domain bounded by C. Then G contains at least one critical point.*

The statement will look "almost obvious" if you try to draw a vector field without singular points in a disc so that on the field is tangential to the boundary circle.

A related statement: if the critical point inside G is unique, then it can not be a saddle. Also, there is a vector field without critical points on a torus, but not on a sphere.

See Ch. 5 of [Arnold] for an introduction to topological methods.