ACM 100b

Analysis of convergence of Fourier Series

Dan Meiron

Caltech

February 17, 2014

Analysis of convergence of Fourier series

- We'll provide an argument that shows how this comes about.
- The argument is not terribly rigorous but shows what is happening near a discontinuity.
- Let $S_N(x)$ be the sum of the full periodic Fourier series after N terms:

$$
\begin{aligned}
S_N(x) &= \frac{B_o}{2} + \sum_{n=1}^{N} [B_n \cos(nx) + A_n \sin(nx)] \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{n=1}^{N} \cos(nt)\cos(nx) + \sin(nt)\sin(nx)\right] dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2} + \sum_{n=1}^{N} \cos(n(t-x))\right] dt
\end{aligned}
$$

Analysis of convergence of Fourier series

- Now look at the sum in the brackets of

$$\frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{1}{2}+\sum_{n=1}^{N}\cos(n(t-x))\right]dt$$

- This sum can be performed in closed form
- Consider the expression

$$\sum_{n=0}^{N}\exp(iny)=\frac{1-\exp(i(N+1)y)}{1-\exp(iy)}$$

- The sum on the left hand side is a geometric series.
- Now take the real part of both sides and do a little manipulation to show that

$$\frac{1}{2}+\sum_{n=1}^{N}\cos(ny)=\frac{\sin[(N+1/2)y]}{2\sin(y/2)}$$

- So our Fourier series is now expressed by

$$S_N(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\left[\frac{\sin[(N+1/2)(t-x)]}{2\sin((t-x)/2)}\right] dt$$

- The fact that this is a single integral will make it possible to understand various behaviors by approximating the integral.
- Shift the limits of integration to rewrite this as

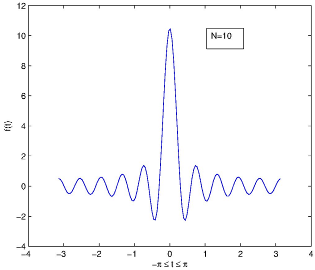
$$S_N(x) = \frac{1}{2\pi}\int_{x-2\pi}^{x} f(x-t)\left[\frac{\sin[(N+1/2)t]}{2\sin(t/2)}\right] dt$$

Analysis of convergence of Fourier series

- Now we are interested in what happens as $N \to \infty$.
- As N gets large the function

$$f(t) = \frac{\sin[(N + 1/2)t]}{2\sin(t/2)}$$

  gets more and more peaked around $t = 0$.
- A plot of $f(t)$ is shown below for $N = 10$



Analysis of convergence of Fourier series

- In fact you can see from L'Hopital's rule that the limit as $t \to 0$ of this function is $N + 1/2$
- Everywhere else within the limits of integration it's much smaller (typically oscillates around the value 1 or less).
- Now we apply a typical applied math argument (which can be made rigorous).
- As $N \to \infty$, if $f(x)$ has bounded variation - meaning it doesn't oscillate like say $\sin(1/t)$, almost all of the integral comes from the values near $t = 0$.
- So we can approximate $S_N(x)$ as follows

$$S_N \approx \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} f(x-t) \frac{\sin[(N+1/2)t]}{2\sin(t/2)} dt.$$

where ϵ is some arbitrarily small number.

Analysis of convergence of Fourier series

- We can further approximate this integral
- We notice that as long as ϵ is very small we have that

$$\sin(t/2) \approx t/2.$$

- You can't make this approximation in the numerator $\sin((N+1/2)t)$ because it oscillates. Remember N is large.
- Our interest here is typically in functions $f(x-t)$ that are smooth
- But we're particularly interested in what happens if $f(x-t)$ has a jump discontinuity near $t=x$.
- If it does, then $f(x)$ would approach two different values $f(x^+)$ and $f(x^-)$
- $f(x^+)$ is the value you get approaching from the right
- $f(x^-)$ is the value you get approaching from the left.
- Other than the jump at x, $f(x)$ is assumed to be smooth

Analysis of convergence of Fourier series

- So we can write

$$
\begin{aligned}
S_n(x) \approx & \frac{2}{2\pi} \int_{-\epsilon}^{0} f(x^+) \frac{\sin((N+1/2)t)}{t} dt + \\
& \frac{2}{2\pi} \int_{0}^{\epsilon} f(x^-) \frac{\sin((N+1/2)t)}{t} dt \\
& = \frac{1}{\pi} \left[f(x^+) + f(x^-) \right] \int_0^{\epsilon} \frac{\sin((N+1/2)t)}{t} dt
\end{aligned}
$$

- We still have to evaluate the integral

$$
S_n(x) = \frac{1}{\pi} \left[f(x^+) + f(x^-) \right] \int_0^{\epsilon} \frac{\sin((N+1/2)t)}{t} dt
$$

to get the final answer.

Analysis of convergence of Fourier series

- To do this recall we have $N \to \infty$.
- If we let $s = (N + 1/2)t$ in the integral above we can rewrite it as follows:

$$\frac{1}{\pi}\int_0^{\epsilon} \frac{\sin((N+1/2)t)}{t}dt = \frac{1}{\pi}\int_0^{(N+1/2)\epsilon} \frac{\sin(s)}{s}ds$$

- If we keep ϵ fixed but let $N \to \infty$ then the integral becomes

$$\frac{1}{\pi}\int_0^{\infty} \frac{\sin(s)}{s}ds = \frac{1}{2}$$

- And so we have that

$$S_N(x) = \frac{1}{2}\left[f(x^+) + f(x^-)\right] + \text{a remainder of size } 1/N$$

meaning that as $N \to \infty$, the Fourier series approaches the average of the two values on either side of the discontinuity.

Analysis of convergence of Fourier series

- This is indeed what we got when we looked at the Fourier series of the function
$$f(x) = \begin{cases} 0 & 0 < x < 1/2 \\ 1 & 1/2 < x < 1 \end{cases}$$
over the interval $0 \leq x \leq \pi$.
- The Fourier series gives us a value of $1/2$ when we evaluate at $x = \pi/2$
- This is exactly the average of the values you get as you approach $x = \pi$ from the left and right.
- Note that if $f(x)$ is continuous then this approach just gives us back the value of $f(x)$ at the point of continuity.

Nonuniform convergence

- We usually don't want the value that a Fourier series converges to to depend on how we take the limits

$$x \to x_0 \qquad N \to \infty$$

- If this happens it's called *nonuniform convergence*
- We stated in the theorem that if $f(x)$ has a jump discontinuity at a point (say $x = x_0$) then the convergence of the series was not uniform in any neighborhood that contains the point $x = x_0$.
- This means that the value we get will depend on how we approach the point x_0.
- For example if you tie the approach to x_0 to the number of terms you keep in the series then you get one result
- The result could be different if you approach x_0 a different way (like going directly to x_0 and then letting the number terms $N \to \infty$.
- If we had uniform convergence then it doesn't matter how you take the limits $x \to x_0$ and $N \to \infty$.

Nonuniform convergence

- To see that this really is an issue for the Fourier series near the discontinuity consider the following limit:

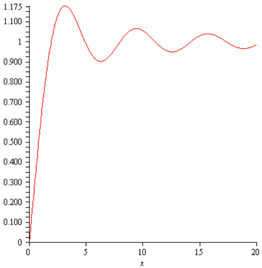
$$\lim_{N\to\infty} S_N\left(x_0 + \frac{z}{N+1/2}\right)$$

- This means you are approaching x_0 and letting the number of terms $N \to \infty$
- But you are also tying the approach to x_0 to the number of terms you take in each partial sum.
- An analysis similar to the one we did above for $S_N(z)$ will reveal that

$$\lim_{N\to\infty} S_N\left(x_0 + \frac{z}{N+1/2}\right) \approx \frac{1}{2}\left[f(x_0^+) + f(x_0^-)\right] + \frac{2}{\pi}\left[f(x_0^+) - f(x_0^-)\right] \mathrm{Si}(z)$$

Nonuniform convergence

- The function $\mathrm{Si}(z)$ is called the *sine integral function* and is defined by
$$\mathrm{Si}(z) = \int_0^z \frac{\sin s}{s} ds$$
- The sine integral function has the following shape



- Note the first max at roughly 1.18
- This is the 18% overshoot you always see when you have Gibbs phenomenon

Nonuniform convergence

- So returning to the limit we took:

$$\lim_{N\to\infty} S_N\left(x_0+\frac{z}{N+1/2}\right) \approx \frac{1}{2}\left[f(x_0^+)+f(x_0^-)\right] + \frac{2}{\pi}\left[f(x_0^+)-f(x_0^-)\right]\mathrm{Si}(z)$$

- You can see you will get a different result than the simple average

$$\frac{1}{2}\left[f(x_0^+)+f(x_0^-)\right]$$

  depending on how you take the limit
- Note that as $N\to\infty$, the overshoots become more fine-grained but they are always there.
- So the mean square error still vanishes as $N\to\infty$
- Note also that if there is no discontinuity, there is no overshoot.

Results for Fourier sine and cosine series

- We just saw that for a fully periodic Fourier series the presence of a discontinuity led to nonuniform convergence.
- That's not too surprising since the function is not smooth.
- For Fourier sine and cosine series you can still get Gibbs phenomenon even if the function is completely smooth in its interval of definition
- This is because a sine series is really a periodic series for the *odd extension of* $f(x)$
- That is you are getting the full Fourier series for

$$F(x) = \begin{cases} f(x) & 0 < x < \pi \\ -f(-x) & -\pi < x < 0 \end{cases}$$

- So even if $f(x)$ is totally smooth the odd extension can have a discontinuity for example if $f(x) \neq 0$ at $x = 0$ or $x = \pi$
- This is what happened when we computed the sine transform of

$$f(x) = 1 \quad 0 \leq x \leq \pi$$

Rate of convergence

- We have seen cases where the Fourier series converges incredibly fast
- This occurred for the cosine expansion of

$$f(x) = \exp(cos(x)) \quad 0 \leq x \leq \pi$$

- We also saw a case where it converged uniformly but not very fast
- This occurred for the cosine expansion of

$$f(x) = x \qquad 0 \leq x \leq \pi$$

- And finally we saw a case where we got nonuniform convergence for the sine series of

$$f(x) = 1 \qquad 0 \leq x \leq \pi$$

Rate of convergence of Fourier series

- Integration by parts can be used to find out how fast the Fourier series coefficients will decay.
- This is important in applications because we sometimes want to use Fourier series to approximate a function.
- The faster the coefficients decrease with increasing n, the fewer terms we would need to represent $f(x)$ accurately.
- Suppose $f(x)$ is periodic and has continuous derivatives of order $p = 0, 1, 2, \ldots, k-1$,
- Suppose $f^{(k)}(x)$ is integrable meaning the integral is finite.

Rate of convergence of Fourier series

- Now consider the coefficients of the full Fourier series as written in complex form:
$$C_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \exp(inx) dx$$
- Then repeated integration by parts can be used to show that
$$C_n = \frac{1}{2\pi (in)^k} \int_{-\pi}^{\pi} f^{(k)}(x) \exp(inx) dx$$
- In other words, the number of continuous derivatives of f within the period determine the rate of decay of the Fourier coefficients
- If you have $k-1$ continuous derivatives the coefficients decay at least as fast as n^{-k}
- In many cases the coefficients decay like $n^{-(k+1)}$ but this is not guaranteed to always happen.

Rate of convergence of Fourier series

- This is consistent with the examples we presented
- For the cosine expansion of the function

$$f(x) = x \qquad 0 \leq x \leq \pi$$

  the even extension of $f(x)$ is continuous but its first derivative is not
- We saw the coefficients decay like n^{-2}
- For the sine expansion of

$$f(x) = 1 \qquad 0 \leq x \leq \pi$$

  the odd extension of $f(x)$ is discontinuous and the coefficients decay like n^{-1}
- Finally for the cosine expansion $f(x) = \exp(cos(x))$ any derivative of the even extension exists so the coefficients decay faster than any power of n (that is exponentially or faster)

The Riemann-Lebesgue Lemma

- The *Riemann-Lebesgue lemma* says that if $g(x)$ is any integrable function then

$$\int_0^{2\pi} g(x)\exp(ikx)dx \to 0 \text{ as } k \to \infty$$

- We won't prove this result here.
- The upshot is that it tells you the integral goes to zero.
- It does not tell you how fast it goes to zero.

Rate of decay of Fourier coefficients

- So consider again

$$C_n = \frac{1}{2\pi (in)^k} \int_{-\pi}^{\pi} f^{(k)}(x) \exp(inx) dx$$

- From the Riemann-Lebesgue lemma we can conclude that

$$C_k \to 0 \text{ faster than } \frac{1}{k^n}$$

- So suppose you have a Fourier series where the coefficients C_k go to zero like k^{-n} as $k \to \infty$.
- And you determine that they decay no faster than this.
- Then you can conclude that $f^{(n-1)}(x)$ is discontinuous.
- This means that if $f(x)$ is infinitely differentiable and periodic, the Fourier series terms decay faster than any power of k.
- This is also consistent with the examples above.

Rate of convergence of Fourier series

- We have gotten estimates on how the coefficients of a Fourier series go to zero
- We can also obtain from those results the error in a Fourier series after you sum say K terms.
- From the discussion above, suppose x is a fixed distance away from a point of discontinuity x_0 of $f^{(n-1)}(x)$.
- Now we want to consider

$$G_K(x) = \sum_{n=-K}^{K} C_n exp(inx)$$

- Then the difference between $G_K(x)$ and the function $f(x)$ goes to zero like K^{-n}.

Rate of convergence of Fourier series

- However, if you approach the point of discontinuity x_0 in the following way
$$x - x_0 = \frac{D}{K}$$
then $|G_K(x) - f(x)|$ goes to zero like $1/K^{n-1}$.
- This is in keeping with what we found out about the Gibbs phenomenon.
- Suppose $f(x)$ has a discontinuity (either because it is discontinuous or because you use a Fourier series with a certain symmetry)
- Then we see that the Fourier series coefficients decrease like $1/n$.
- And the error between the Fourier series and the function decreases like $1/N$ where N is the number of terms you keep as long as you are a fixed distance from the point of discontinuity.
- If you approach the point of discontinuity as we did in the case of Gibbs phenomenon the error does not decrease as we saw.