ACM 100c

First order linear systems of ODE's

Dan Meiron

Caltech

January 12, 2014

First order linear ODE systems

- The most general *linear* system has the form

$$x_i' = \sum_{k=1}^{n} a_{ik} x_k + f_i(z), \quad i = 1, 2, ..., n$$

- Initial conditions for the IVP are given by

$$x_1(z_0) = x_{10}, \qquad x_2(z_0) = x_{20}, \qquad ..., \qquad x_n(z_0) = x_{n0}.$$

Existence and uniqueness for linear systems

- The system approach is actually very useful.
- As we showed with first order scalar ODE's, for given initial conditions the linear system has a unique solution provided the coefficents a_{ij} are continuous.
- Since we can convert higher order ODE's to systems we can then get results this way for any n'th order system.
- But note also that if a_{12} and a_{11} are not differentiable we won't be able to reduce the system to the second order ODE in a straight-forward way.
- So actually the system form is more general.

The matrix-vector form

- In vector form, define the vectors

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \qquad \boldsymbol{x}(z_0) = \boldsymbol{x}_0 = \begin{pmatrix} x_{10} \\ x_{20} \\ \vdots \\ x_{n0} \end{pmatrix}, \qquad \boldsymbol{F} = \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{pmatrix},$$

- And the matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

- Then the system ODE IVP is

$$\frac{d\boldsymbol{x}}{dz} = A(z)\boldsymbol{x} + \boldsymbol{F}(z), \qquad \boldsymbol{x}(z = z_0) = \boldsymbol{x}_0$$

The matrix-vector form

- Our system IVP is

$$\frac{d\boldsymbol{x}}{dz} = A(z)\boldsymbol{x} + \boldsymbol{F}(z), \qquad \boldsymbol{x}(z = z_0) = \boldsymbol{x}_0$$

- Such a system is called homogeneous if $\boldsymbol{F}(z) = \mathbf{0}$
- It's inhomogeneous otherwise.
- Suppose the coefficient matrix A has coefficients that are continuous in some interval $z_1 \leq z \leq z_2$
- And suppose that the components of $\boldsymbol{F}$ are suitably smooth (actually piecewise continuous is fine)
- Then the first order system will have a solution and it will be unique in the interval $z_1 \leq z \leq z_2$
- In this way we get existence and uniqueness results on all n' th order linear ODE IVPs

Linear independence

- Consider the system
$$\frac{d\boldsymbol{x}}{dz} = A(z)\boldsymbol{x}$$
- Assume that the matrix A is an $n \times n$ matrix and the entries $a_{ij}(z)$ are smooth in some interval
- This homogeneous system has n linearly independent solutions
- This can be seen because we can convert the system to an n'th order ODE and so there must be n constants of integration.
- Once we get these n solutions we can work backwards and use the elimination we did in reverse to get the n solution vectors $\boldsymbol{x}_i$
- The solution to the initial value problem $\dfrac{d\boldsymbol{x}}{dz} = A(z)\boldsymbol{x}$ is a linear superposition of these vectors:
$$\boldsymbol{x} = \sum_{i=1}^{n} c_i \boldsymbol{x}_i$$

Linear independence

- In order to solve the initial value problem

$$\boldsymbol{x}(z_0) = \boldsymbol{x}_0$$

we have to be able to solve the linear system

$$\sum_{i=1}^{n} c_i \boldsymbol{x}_i(z_0) = \boldsymbol{x}_0$$

- For this system to have a solution it must be the case that

$$W = \det P \neq 0 \qquad \text{where } P = \begin{bmatrix} \vdots & \vdots & & \vdots \\ \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_n \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

- W is the generalization of the Wronskian we encountered when we looked at second order linear ODE's

Abel's theorem again

- There is a generalization of Abel's theorem that holds for linear systems
- Again we can compute the Wronskian even if we don't know the n homogeneous solutions
- It is possible to show that

$$\frac{dW}{dz} = [a_{11} + a_{22} + \ldots + a_{nn}]\, W$$

where the a_{ii} are the diagonal elements of A..
- The sum of the diagonal elements of a matrix is called its *trace*.
- We write the trace of A as $\mathrm{Tr} A$
- And we can see that

$$W = C \exp\left[\int^{z} \mathrm{Tr} A(z')dz'\right]$$

- Once again we see that if the elements of A are smooth (actually just the diagonal elements) then the Wronskian can't vanish unless $C = 0$ in which case it's identically zero