ACM 100b

Examples of using S-L series to solve boundary value problems

Dan Meiron

Caltech

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Recap

- In the previous lecture we explored the rules for differentiating and integrating Fourier series
- The results apply to the eigenfunctions of any Sturm-Liouville problem
- You can differentiate a Fourier series term by term as long as the series is uniformly convergent
- You can always integrate a Fourier series term by term
- The result however may not always be a Fourier series - there may be an additional term linear in x
- We then applied the Fourier series to the solution of inhomogeneous S-L problems with homogeneous boundary conditions.

An example

- Consider the boundary value problem

$$y'' + \lambda y = f(x) \qquad 0 \leq x \leq \pi$$

  with homogeneous boundary conditions $y(0) = 0$ and $y(\pi) = 0$
- For $f(x)$ let's take

$$f(x) = \begin{cases} 0 & 0 \leq x \leq \pi/2 \\ 1 & \pi/2 < x < \pi \end{cases}$$

- We took a discontinuous $f(x)$ because we want to see what happens in this case.
- Now we want to choose a set of eigenfunctions for expansion of the solution

$$y(x) = \sum_{n=1}^{\infty} A_n \phi_n(x)$$

An example

- Which eigenfunctions should we choose?
- Any set of regular S-L eigenfunctions will work but it's advantageous to pick the solutions to

$$\frac{d^2\phi_n(x)}{dx^2} + \lambda_n \phi_n(x) = 0 \qquad \phi_n(0) = 0 \quad \phi_n(\pi) = 0$$

- The solution is just the Fourier sine series

$$\phi_n(x) = \sin(nx) \qquad \lambda_n = n^2$$

- Now write the solution as

$$y(x) = \sum_{n=1}^{\infty} A_n \sin(nx)$$

An example

- Now the Fourier sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} D_n \sin(nx)$$

where

$$D_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin(nx)dx = 2\,\frac{\cos\left(\pi n/2\right) + \left(-1\right)^{1+n}}{\pi\, n}$$

- Now recall the formula we derived last lecture

$$A_n = \frac{D_n}{\lambda - \lambda_n}$$

so

$$A_n = 2\frac{\cos\left(\pi n/2\right) + \left(-1\right)^{1+n}}{\pi\, n(\lambda - n^2)}$$

An example

- You can see that the coefficients of the solution

$$A_n = 2\frac{\cos(\pi n/2) + (-1)^{1+n}}{\pi n(\lambda - n^2)}$$

decay like n^{-3} as n gets large.

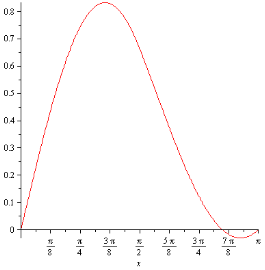
- This means the series can be differentiated twice and the resulting Fourier series would give the correct derivative.
- This is good because we did differentiate twice.
- The solution is OK even though the inhomogeneous term is only piecewise smooth.

An example

- We note our right hand side $f(x)$ was given by

$$f(x) = \begin{cases} 0 & 0 \leq x \leq \pi/2 \\ 1 & \pi/2 < x < \pi \end{cases}$$

- Below we plot the solution $y(x)$ for $\lambda = 2$ which is not an eigenvalue:



- As can be seen the solution itself is quite smooth

Solving S-L problems with inhomogeneous BC's

- Next let's turn to the solution of an inhomogeneous S-L problem but with inhomogeneous boundary conditions

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda r(x)y = r(x)f(x), \qquad a < x < b,$$

with boundary conditions $y(a) = y_a \neq 0$ and $y(b) = y_b \neq 0$

- To make things more definite let's use the previous example ODE and choose $a = 0$ and $b = \pi$

$$y'' + \lambda y = f(x) \qquad y(0) = y_0 \quad y(\pi) = y_1 \quad 0 \leq x \leq \pi$$

- We then try a series solution using sine functions:

$$y(x) = \sum_{n=1}^{\infty} A_n \sin(nx)$$

Inhomogeneous boundary conditions

- We assume again that we can expand $f(x)$:

$$f(x) = \sum_{n=1}^{\infty} F_n sin(nx)$$

- Now substitute our trial solution into the ODE again to get

$$(\lambda - n^2)A_n = F_n \quad \text{or} \quad A_n = \frac{F_n}{\lambda - n^2}$$

- But this is the same solution as before!
- And it does not satisfy the boundary conditions - it's not supposed to go to zero at the boundaries.
- In fact we never got the opportunity to apply the boundary conditions.
- What went wrong?

Inhomogeneous boundary conditions

- What went wrong is that a Fourier sine series that is supposed to represent a function that is nonzero at the boundaries *must* exhibit the Gibbs phenomenon
- It is certainly possible to have sine series for functions with nonzero boundary values but from our experience such a series cannot be uniformly convergent
- We then blithely substituted such a series and differentiated it twice.
- There are two approaches to fixing this problem
  1. Turn the inhomogeneous boundary conditions into homogeneous boundary conditions
  2. Use integration to manipulate the Fourier series because that's always allowed - this is called the *method of finite transforms.*

Convert to homogeneous boundary conditions

- Consider the problem we tried to solve:

$$y'' + \lambda y = f(x) \qquad y(0) = y_0 \quad y(\pi) = y_1 \quad 0 \leq x \leq \pi$$

- Suppose we make the substitution

$$u(x) = y(x) + \alpha x + \beta(\pi - x)$$

  where α and β are constants.
- We can always pick these constants so that where $y(x)$ has nonzero values at the boundary, $u(x)$ vanishes.
- In fact we see easily that

$$u(0) = 0 = y_0 + \beta\pi \qquad u(\pi) = 0 = y_1 + \alpha\pi$$

- So if we choose $\beta = -y_0/\pi$ and $\alpha = -y_1/\pi$ the function $u(x)$ vanishes at the boundaries

Convert to homogeneous boundary conditions

- So we have
$$u(x) = y(x) - \frac{1}{\pi}\left[y_1 x + y_0(\pi - x)\right]$$
or
$$y(x) = u(x) + \frac{1}{\pi}\left[y_1 x + y_0(\pi - x)\right]$$
- Plug this in to our ODE and we get
$$u'' + \lambda u = g(x) \qquad 0 \le x \le \pi$$
where
$$g(x) = -\frac{\lambda}{\pi}\left[y_1 x + y_0(\pi - x)\right] + f(x)$$
but with homogeneous boundary conditions:
$$u(0) = 0 \qquad u(\pi) = 0$$
- You can always do this type of transformation for any inhomogeneous problem

Convert to homogeneous boundary conditions

- Why is this any better?
- Because we can directly substitute a Fourier series into this ODE

$$u(x) = \sum_{n=1}^{\infty} U_n \sin(nx)$$

and we know we get a sensible answer as long as the right hand side function is integrable and has a Fourier series.
- To do this we expand the right hand side of the ODE

$$-\frac{\lambda}{\pi}\left[y_1 x + y_0(\pi - x)\right] + f(x)$$

in a Fourier sine series:

$$g(x) = \sum_{n=1}^{\infty} G_n \sin(nx) \text{ where } G_n = -\frac{2\lambda}{\pi n}\left[y_0 + (-1)^{n+1} y_1\right] + F_n$$

and where F_n are the Fourier sine series coefficients for $f(x)$

Convert to homogeneous boundary conditions

- Now we can proceed as we did before to get

$$u(x) = \sum_{n=1}^{\infty} \frac{G_n}{\lambda - n^2} \sin(nx)$$

- And finally we recall the relation of $u(x)$ to $y(x)$:

$$y(x) = u(x) + \frac{1}{\pi} \left[y_1 x + y_0(\pi - x) \right]$$

- We then get

$$y(x) = \frac{1}{\pi} \left[y_1 x + y_0(\pi - x) \right] + \sum_{n=1}^{\infty} \frac{G_n}{\lambda - n^2} \sin(nx)$$

Convert to homogeneous boundary conditions

- Note that if we now wanted to we could convert

$$\frac{1}{\pi}\left[y_1 x + y_0(\pi - x)\right]$$

  to a Fourier sine series as well.
- Such a series would be non-uniformly convergent
- This underscores that we could not just substitute such a series and differentiate it as we tried to do
- But in fact from the point of evaluating the solution it's best to leave the solution in the form

$$y(x) = \frac{1}{\pi}\left[y_1 x + y_0(\pi - x)\right] + \sum_{n=1}^{\infty} \frac{G_n}{\lambda - n^2} \sin(nx)$$

- This isolates the part of the answer that has a non-uniformly converging Fourier series.
- The series involving G_n has terms decaying like n^{-3}

The method of finite transforms

- The conversion to a homogeneous problem can always be performed
- But one might ask if there is a direct way to compute the Fourier series.
- To do this we do not substitute Fourier series and differentiate because that is problematic if the boundary conditions are non-homogeneous.
- Instead we “transform” the ODE via integration as follows.
- Recall our boundary value problem

$$y'' + \lambda y = f(x) \qquad y(0) = y_0 \quad y(\pi) = y_1 \quad 0 \leq x \leq \pi$$

- Now multiply both sides of the ODE by $(2/\pi)\sin(nx)$:

$$(2/\pi)\sin(nx)y''(x) + \lambda(2/\pi)sin(nx)y(x) = (2/\pi)f(x)sin(nx)$$

and integrate both sides of this equation from 0 to π

The method of finite transforms

- We get

$$(2/\pi)\int_0^\pi \sin(nx)y''(x)dx + \lambda(2/\pi)\int_0^\pi y(x)\sin(nx)dx =$$
$$(2/\pi)\int_0^\pi f(x)sin(nx)dx$$

- Now recall the definition of the Fourier coefficients:

$$y(x) = \sum_{n=1}^{\infty} A_n sin(nx) \qquad A_n = \frac{2}{\pi}\int_0^\pi y(x)sin(nx)dx$$

The method of finite transforms

- We recognize some of the terms in the expression

$$(2/\pi)\int_0^{\pi}\sin(nx)y''(x)dx+\lambda(2/\pi)\int_0^{\pi}y(x)\sin(nx)dx =$$
$$(2/\pi)\int_0^{\pi}f(x)sin(nx)dx$$

in terms of the Fourier series coefficients.

- We have

$$(2/\pi)\int_0^{\pi}\sin(nx)y''(x)dx+\lambda A_n=F_n \qquad n=1,2,\ldots$$

where

$$f(x)=\sum_{n=1}^{\infty}F_n sin(nx)$$

The method of finite transforms

- To relate the first term
$$(2/\pi)\int_0^\pi \sin(nx)y''(x)dx$$
to the Fourier coefficients of $y(x)$ we integrate by parts once to get
$$(2/\pi)\int_0^\pi \sin(nx)y''(x)dx =$$
$$\left(\frac{2}{\pi}\right) sin(nx)y'(x)\Big|_0^\pi - \frac{2}{\pi}n\int_0^\pi cos(nx)y'(x)dx$$
- The boundary term is 0 because the sin vanishes at $x = 0, \pi$
- We integrate by parts once more to get
$$\frac{2}{\pi}\int_0^\pi \sin(nx)y''(x)dx =$$
$$-\left(\frac{2n}{\pi}\right)\cos(nx)y(x)\Big|_0^\pi - \frac{2n^2}{\pi}\int_0^\pi y(x)\sin(nx)dx$$

The method of finite transforms

- Note now the boundary values of y don't vanish so we do get a contribution from the boundary term and this allows us to utilize the nonzero boundary conditions:

$$\frac{2}{\pi}\int_0^{\pi} \sin(nx)y''(x)dx =$$
$$\frac{2n}{\pi}(-1)^{n+1}y_1 + \frac{2n}{\pi}y_0 - \frac{2n^2}{\pi}\int_0^{\pi} y(x)\sin(nx)dx$$

- This tells us

$$\frac{2}{\pi}\int_0^{\pi} \sin(nx)y''(x)dx = \frac{2n}{\pi}(-1)^{n+1}y_1 + \frac{2n}{\pi}y_0 - n^2 A_n$$

The method of finite transforms

- Finally recalling our earlier expression

$$(2/\pi)\int_0^\pi \sin(nx)y''(x)dx + \lambda A_n = F_n$$

we get the transformed equation

$$\frac{2n}{\pi}(-1)^{n+1}y_1 + \frac{2n}{\pi}y_0 - n^2A_n + \lambda A_n = F_n \qquad n = 1, 2, \ldots$$

- We can now solve for the Fourier coefficients:

$$A_n = \frac{F_n}{\lambda - n^2} - \frac{1}{\lambda - n^2}\frac{2n}{\pi}\left[y_0 + (-1)^{n+1}y_1\right]$$

- Note that the Fourier sine series coefficients of our solution now decay only like $1/n$ because the Fourier sine series is representing a function with nonzero boundary values

The method of finite transforms

- Now let's compare the two answers we got using the two approaches.
- By making the boundary conditions homogeneous we got the solution

$$
\begin{aligned}
y(x) = & \frac{1}{\pi}\left[y_1 x + y_0(\pi - x)\right] + \\
& \sum_{n=1}^{\infty} \frac{1}{\lambda - n^2}\left[-\frac{2\lambda}{\pi n}\left[y_0 + (-1)^{n+1} y_1\right] + F_n\right]\sin(nx)
\end{aligned}
$$

- By using the method of finite transforms we got

$$
y(x) = \sum_{n=1}^{\infty}\left[\frac{F_n}{\lambda - n^2} - \frac{1}{\lambda - n^2}\frac{2n}{\pi}\left[y_0 + (-1)^{n+1} y_1\right]\right]\sin(nx)
$$

The results are the same

- To compare these two solutions let's expand

$$\frac{1}{\pi}[y_1 x + y_0(\pi - x)]$$

in a Fourier sine series - we get

$$\frac{1}{\pi}[y_1 x + y_0(\pi - x)] = \sum_{n=1}^{\infty} \frac{2}{\pi n}(y_0 + (-1)^{n+1} y_1) \sin(nx)$$

- Then let's substitute this series in the expression

$$\begin{aligned} y(x) = & \frac{1}{\pi}[y_1 x + y_0(\pi - x)] + \\ & \sum_{n=1}^{\infty} \frac{1}{\lambda - n^2} \left[-\frac{2\lambda}{\pi n} \left[y_0 + (-1)^{n+1} y_1 \right] + F_n \right] \sin(nx) \end{aligned}$$

The results are the same

- We get

$$y(x) = \sum_{n=1}^{\infty} \left[\frac{2}{\pi n} \left[y_0 + (-1)^{n+1} y_1 \right] + \frac{1}{\lambda - n^2} \left(-\frac{2\lambda}{\pi n} \right) \left[y_0 + (-1)^{n+1} y_1 \right] + \frac{F_n}{\lambda - n^2} \right] \sin(nx)$$

- But we can simplify this to

$$y(x) = \sum_{n=1}^{\infty} \left[\frac{F_n}{\lambda - n^2} - \frac{2n}{\pi(\lambda - n^2)} \left[y_0 + (-1)^{n+1} y_1 \right] \right] \sin(nx)$$

- This is identical to the series we got with the method of finite transforms so the two approaches give the same result.

Comparing the two approaches

- Note the method of finite transforms gets the right answer because all the operations involve *integration*.
- This operation is always formally justified regardless of the convergence of the series.
- However the result we got is a series which will not converge uniformly because of the Gibbs phenomenon
- The method of making the boundary conditions homogeneous gives us a series with no Gibbs phenomenon and so is better for evaluating the answer
- As we see both approaches lead to the same answer - just different forms.
- The idea of the method of finite transforms is quite general and we will see it again many times.

S-L expansions for general linear ODE boundary value problems

- So far we showed how we can use $S - L$ eigenfunctions to solve ODE problems where the left hand side was the Sturm-Liouville operator and the right hand side was some function
- We did this in two cases - homogeneous boundary conditions and inhomogeneous boundary conditions
- One thing we learned was that while we can always expand the solution in S-L eigenfunctions we need to understand how the series behave to get useful results
- This is one disadvantage of using expansions in regular S-L eigenfunctions - they are sensitive to what is happening at the boundary.
- In other words even if the solution is totally smooth, if it doesn't satisfy compatibility conditions at the boundary related to the eigenfunctions, convergence can be nonuniform.
- We will get around this problem shortly

S-L expansions for general linear ODE boundary value problems

- But first we ask - can we use regular S-L eigenfunctions to solve general linear boundary value problems?
- The answer is yes but most of the time you need to solve the resulting equations for the coefficients numerically.
- Consider the following example.

$$y'' + f(x)y = g(x) \qquad 0 \leq x \leq \pi \quad y(0) = 0 \quad y(\pi) = 0$$

- You can't solve this in closed form except for special cases but one could try a Fourier series:

$$y(x) = \sum_{n=1}^{\infty} A_n \sin(nx)$$

- Because the boundary conditions are homogeneous we can directly substitute the series.

S-L expansion as for general linear ODE boundary value problems

- We get

$$\sum_{n=1}^{\infty}(-n^2)A_n\sin(nx) + f(x)\sum_{n=1}^{\infty}A_n\sin(nx) = g(x)$$

- We then have to expand $f(x)$ and $g(x)$ to get

$$\sum_{n=1}^{\infty}(-n^2)A_n\sin(nx)+\sum_{n=1}^{\infty}F_n\sin(nx)\sum_{n=1}^{\infty}A_n\sin(nx) = \sum_{n=1}^{\infty}G_n\sin(nx)$$

- This is where things get complicated
- You can write the result as

$$\sum_{n=1}^{\infty}(-n^2)A_n\sin(nx)+\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}F_lA_m\sin(lx)\sin(mx) = \sum_{n=1}^{\infty}G_n\sin(nx)$$

S-L expansions for general linear boundary value problems

- But now we have to write things so we can isolate functions of $\sin(nx)$
- To do this we would write

$$\sum_{n=1}^{\infty}(-n^2)A_n\sin(nx)+$$
$$\sum_{n=1}^{\infty}\left[\frac{2}{\pi}\int_0^{\pi}\sum_{m=1}^{\infty}\sum_{l=1}^{\infty}F_lA_m\sin(lx')\sin(mx')sin(nx')dx'\right]\sin(nx)=$$
$$\sum_{n=1}^{\infty}G_n\sin(nx)$$

S-L expansions for general linear boundary value problems

- Matching terms in $\sin(nx)$ we get

$$-n^2 A_n + \sum_{m=1}^{\infty} \sum_{l=1}^{\infty} C_{lmn} F_l A_m = G_n \qquad n = 1, 2, \ldots$$

where the coefficient

$$C_{lmn} = 4 \frac{lnm\left(-1 + (-1)^{l+m+n}\right)}{\pi\,(l+m-n)(l-m+n)(l+m+n)(l-m-n)}$$

- Notice that this couples *all* the coefficients so this is a discrete linear system of equations but of infinite size.
- In practice what we would do is truncate the Fourier expansion at N terms
- This would make the linear system an $N \times N$ system for the coefficients which we could solve numerically
- This is called a *spectral method* in numerical analysis.