

D. Probability density in momentum space

The interpretation of the wave function as a probability density in real space is neat. But it turns out that the probability density picture could be extended to other spaces, for example, momentum space. Before we can deal with this, for my own peace of mind, I'd like to recount the equations of Fourier transform. Given a function $f(x)$, we can define the Fourier transform, f_k as follows:

$$f_k = \int_{-\infty}^{\infty} dx e^{-ikx} f(x) \quad (71)$$

This is useful because we can then recover the function $f(x)$ by simply:

$$f(x) = \int \frac{dk}{2\pi} f_k e^{ikx} \quad (72)$$

An interesting special case, and a number one favorite in quantum mechanics classes, is the δ -function:

$$f(x) = \delta(x - x_0) \quad (73)$$

Where $\int \delta(x) = 1$, but $\delta(x) = 0$ for all $x \neq 0$.

Plugging $f(x)$ in gives immediately:

$$f_k = e^{-ikx_0} \quad (74)$$

Every mathematical formula requires some checking. Substituting back:

$$f(x) = \int \frac{dk}{2\pi} e^{ik(x-x_0)} \quad (75)$$

wow. This is not a trivial formula. Okay - if $x \neq x_0$, then clearly this should be zero. When $x = x_0$, we get infinity. But there are many kinds of infinity, here we see that the δ -function infinity is exactly an integral with the 2π below:

$$\int dk e^{ik(x-x_0)} = 2\pi \delta(x - x_0) \quad (76)$$

This also has a dual form:

$$\int dx e^{ix(k-k_0)} = 2\pi \delta(k - k_0). \quad (77)$$

Armed with these gymnastic moves, let's look at the normalization integral:

$$1 = \int dx \psi^*(x) \psi(x) \quad (78)$$

and try to write it with the Fourier transform:

$$= \int dx \left(\int \frac{dk_1}{2\pi} \psi_{k_1}^* e^{ik_1 x} \right)^* \left(\int \frac{dk_2}{2\pi} \psi_{k_2} e^{ik_2 x} \right) \quad (79)$$

Changing the order of integration (as physicists we are allowed to do it until specified otherwise...) we get:

$$= \int \frac{dk_1}{2\pi} \psi_{k_1}^* \int \frac{dk_2}{2\pi} \psi_{k_2} \int dx e^{ix(k_2 - k_1)} \quad (80)$$

but using the integral identity we recognize a δ -function:

$$= \int \frac{dk_1}{2\pi} \psi_{k_1}^* \int \frac{dk_2}{2\pi} \psi_{k_2} 2\pi \delta(k_1 - k_2) = \int \frac{dk}{2\pi} |\psi_k|^2. \quad (81)$$

Remarkable. This ψ_k is nothing but $\psi_k = \phi(k)$ - the weight we used last time to indicate the momentum $\hbar k$ state weight in a localized wave packet solution. This integral is called the normalization integral in *the momentum representation*. We see that we can equivalently require a normalization of the momentum-representation wave function:

$$\int \frac{dk}{2\pi} |\phi(k)|^2 = 1 \quad (82)$$

indication also that $\frac{1}{2\pi} |\phi(k)|^2$ is the momentum probability density. We can now use this to do two things. First, to normalize our wave function in momentum space:

$$1 = N_k^2 \int \frac{dk}{2\pi} e^{-k^2/2\Delta k^2} = N_k^2 \frac{1}{\sqrt{2\pi}} \Delta k \quad (83)$$

and

$$N_k = \sqrt{\frac{\sqrt{2\pi}}{\Delta k}}. \quad (84)$$

By identifying $\phi(k)$ as the momentum-space probability density, we also confirm that the standard deviation of momentum is

$$\sigma_p = \hbar \sigma_k = \hbar \Delta k = \Delta p. \quad (85)$$

Let me finish this somewhat mathematical discussion with the following statement. To calculate this standard deviation, we could have written either one of two forms. Intuitively:

$$\langle p^2 \rangle \langle (\hbar k)^2 \rangle = \int \frac{dk}{2\pi} |\phi(k)|^2 (\hbar k)^2 \quad (86)$$

or:

$$\langle p^2 \rangle = \int dx \psi(x)^* \left(\frac{\hbar}{i} \frac{\partial}{\partial x} \right)^2 \psi(x) \quad (87)$$

You can see this by imagining substituting the momentum representation in $\psi(x)$, but you will prove this in the problem set. Again, this relies on the identification of momentum with the *operator*:

$$\hat{p} = \frac{\hbar}{i} \frac{\partial}{\partial x}. \quad (88)$$

IV. UNCERTAINTY AND COLLAPSE

It is high time to do some reckoning. We know that the wave function in real space gives the probability density when squared. We know that we can even convert it to the momentum-representation probability density. We know that for Gaussian wave packet, there is a width of the probability in space of the size Δx . We seemed to be able to choose Δx to make the wave function look as particle-like as we wish. But then we also saw that this imposes a width on the momentum distribution wave function, $\phi(k)$, which is Δk such that:

$$\Delta x \Delta p = \frac{\hbar}{2}. \quad (89)$$

And now this is indeed what you have anticipated. This is the uncertainty principle. In fact, this is the uncertainty principle's best case scenario. In fact:

$$\Delta x \Delta p \geq \frac{\hbar}{2}. \quad (90)$$

If you pick Δx very small, and you think that you have just managed to resolve the particle-wave duality, by writing a really localized particle-like solution which evolves according to the wave equation, this relation comes and hits you in the face: Localized in space - maybe... But in momentum - as unlocalized as could be. Vice versa - you'd like to give your particle a good well defined momentum; well, you can, but then the particle will seem to be everywhere.

This is the well known Heisenberg uncertainty principle.

This raises many problems. Classically, we can define particles with both well defined momentum and location. Quantum mechanically - doesn't look like it. Does this have physical implications? [DISCUSS]

We can measure both momentum and space. Can you think of how? location is the easiest. You let the electron go through a grid set right between a set of optical fibers, and a ccd (formerly - a photographic plate). When you want to know the location of the electron, you light up, let photons go through the optical fibers, and then look at the photographic plate. There will be a missing dot. That's where your electron is.

Imagine that the electron has a wave function of size Δx , which is bigger than the resolution of the system, how would you tell? You can repeat the experiments many times, and eventually you will get a distribution of how many times an electron was seen in each point. This distribution will be the wave function.

Here is the million dollar question - what happens after the measurement? I mean, after a measurement where an electron was seen in one of the sites of our grid. Two possibilities - (1) the electron maintains its wave function. (2) The electron reconfigures the wave function to reflect the certainty of our knowledge of its location. Maybe there is a third possibility: (3) something in between...

How do we distinguish? We do another experiment immediately afterwards. This was done, and gives result 2. The wave function collapses into a wave function with a really narrow support, which corresponds to the bin where the electron was spotted. The narrow support implies a really broad momentum distribution. The same story can be told in momentum space, by using, say - a mass spectrometer.

The situation could be yet worse. We could start with a split wave packet! Is that legal? Let me let you think and discuss this. we will revisit this question.