# ACM 100b

## The delta function

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# The delta function

- The Greens function satisfies its own inhomogeneous ODE problem
- Let's write this as

$$\frac{d^2 G(x, x'; \lambda)}{dx^2} + \lambda G(x, x'; \lambda) = Q(x, x')$$

- From our previous analysis

$$Q(x, x') = \begin{cases} 0 & x < x' \\ 0 & x > x' \end{cases}$$

- What happens at $x = x'$?
- To understand this let's integrate both sides of the equation about a symmetric neighborhood of $x = x'$:

$$\int_{x'-\epsilon}^{x'+\epsilon} \frac{d^2 G}{dx^2} dx + \lambda \int_{x'-\epsilon}^{x'+\epsilon} G dx = \int_{x'-\epsilon}^{x'+\epsilon} Q(x, x') dx$$

# The delta function

- Now as $\epsilon \to 0$ let's consider the integral

$$\int_{x'-\epsilon}^{x'+\epsilon} G dx$$

- As we saw $G$ is continuous at $x = x'$ so it must be that

$$\lim_{\epsilon \to 0} \int_{x'-\epsilon}^{x'+\epsilon} G dx = 0$$

- On the other hand the integral

$$\int_{x'-\epsilon}^{x'+\epsilon} \frac{d^2 G}{dx^2} dx = \left. \frac{dG}{dx} \right|_{x'-\epsilon}^{x'+\epsilon}$$

- In this case

$$\lim_{\epsilon \to 0} \int_{x'-\epsilon}^{x'+\epsilon} \frac{d^2 G}{dx^2} dx = 1$$

# The delta function

- But recall that

$$\int_{x'-\epsilon}^{x'+\epsilon} \frac{d^2G}{dx^2} dx + \lambda \int_{x'-\epsilon}^{x'+\epsilon} G dx = \int_{x'-\epsilon}^{x'+\epsilon} Q(x, x') dx$$

- So as $\epsilon \to 0$ we must get

$$\lim_{\epsilon \to 0} \int_{x'-\epsilon}^{x'+\epsilon} Q(x, x') dx = 1$$

- But we also just saw that $Q(x, x')$ is zero except at $x = x'$
- What kind of function is this anyway?

# The delta function

- This is not a function per se - it is a very singular object called the $\delta$ function
- The delta function $\delta(t)$ is a type of distribution (as opposed to an actual function) that is defined as follows

$$\delta(t) = \begin{cases} 0 & t < 0 \\ \infty & t = 0 \\ 0 & t > 0 \end{cases}.$$

- But $\delta(t)$ has a finite integral:

$$\int_{-\infty}^{\infty} \delta(t) dt = 1$$

- In fact the integral is the same as long as the interval of integration includes $t = 0$.

# The delta function as a kind of limit

- Yet another way to think of the delta function is as a kind of limit of a sequence of functions
- This connects to the idea of the delta function as a distribution
- For example, consider the sequence of functions

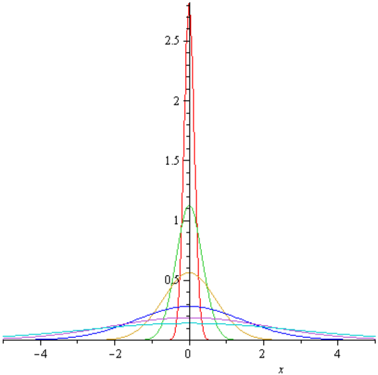
$$f_a(x) = \frac{1}{\sqrt{\pi}a} \exp\left(-\frac{x^2}{a^2}\right)$$

- Now look at this sequence of functions for various values of $a$
- As $a \to 0$ the width of the function goes to zero but the amplitude goes to $\infty$.
- You can check though that for any value of $a$ the area under the curve is 1:

$$\int_{-\infty}^{\infty} f_a(x)dx = 1$$

# The delta function as a kind of limit

- Graphically the sequence for $a \to 0$ looks like this



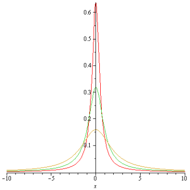

- The limit $a \to 0$ is the $\delta$ function
- It is zero everywhere except $x = 0$ but the area is always 1

# There are many ways to define a delta function

- Note that there are other ways to define a $\delta$ function.
- For example consider the sequence

$$g_a(x) = \frac{a}{\pi} \frac{1}{x^2 + a^2} \qquad a > 0$$

- This is again a sequence of functions which gets more peaked as $a \to 0$
- Graphically it looks similar to the case before:




- And again the area for each curve is 1
- The $\delta$ function is the limit $a \to 0$

# The delta function

- Yet another way to think of the $\delta$ function is that it is the derivative of the unit step function $u(t)$ defined by

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}.$$

- Using the definitions above we have

$$\delta(z-t) = \begin{cases} 0 & t < z \\ \infty & t = z \\ 0 & t > z \end{cases},$$

and

$$\int_a^b \delta(z-t)dt = 1 \qquad a < z < b.$$

# The sifting property

- A very important property of the $\delta$ function is called the *sifting property*.
- We will show below that

$$\int_a^b f(t)\delta(z-t)dt = f(z) \qquad a < z < b$$

- To show this, write

$$f(t) = f(z) + [f(t) - f(z)].$$

- Now note that

$$\int_a^b f(z)\delta(z-t)dt = f(z)\int_a^b \delta(z-t)dt = f(z)$$

# The sifting property

- But we also have
$$\int_a^b [f(t) - f(z)]\delta(t-z)dt = \int_{z-\epsilon}^{z+\epsilon} [f(t) - f(z)]\delta(z-t)dt$$
because the $\delta$ function is zero for all values except when you evaluate at 0.
- But for any continuous function $f(z)$, it must be that
$$|f(z) - f(t)| < \tau \qquad |t-z| < \epsilon.$$
- So we must have
$$\left| \int_{z-\epsilon}^{z+\epsilon} [f(t) - f(z)]\delta(z-t)dt \right| < \tau \int_{z-\epsilon}^{z+\epsilon} \delta(z-t)dt = \tau.$$
- So as $z \to t$ the remainder $\tau$ will go to zero and that shows the result.
- A more rigorous development uses the theory of distributions.