ACM 100b

The Sturm-Liouville ODE

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Recap - boundary value problems

- Last lecture we introduced the concept of boundary value problems
- We illustrated some of the issues by solving the heat equation by separation of variables

$$\frac{\partial \Theta}{\partial t} = D \frac{\partial^2 \Theta}{\partial x^2} \qquad 0 \leq x \leq 1$$

  with homogenous boundary conditions

$$\Theta(0, t) = 0 \quad \Theta(1, t) = 0 \qquad t \geq 0$$

  and an initial condition $\Theta(x, 0) = \Theta_0(x, 0)$
- This led us to a homogeneous boundary value problem of the form

$$\frac{d^2 X}{dx^2} + \lambda^2 X = 0 \qquad 0 \leq x \leq 1$$

  which had a countably infinite number of solutions for special values of λ: $X_n(x) = A \sin(n\pi x) \quad \lambda_n = n\pi \qquad n = 1, 2, \ldots$

Recap - boundary value problems

- The general solution to the heat problem was

$$\Theta(x,t) = \sum_{n=1}^{\infty} B_n \exp(-n^2\pi^2 t)\sin(n\pi x).$$

which satisfies the boundary conditions, because the sines vanish at $x = 0, 1$

- But there is also an initial condition to satisfy.
- At $t = 0$ we have some starting distribution of heat in the rod:

$$\Theta(x,0) = \Theta_0(x).$$

- In order to satisfy this condition we substitute $t = 0$ into

$$\Theta(x,t) = \sum_{n=1}^{\infty} B_n \exp(-n^2\pi^2 t)\sin(n\pi x)$$

to get $\Theta_0(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$

Recap - boundary value problems

- So we would have a solution that satisfies all the conditions if we could figure out the coefficients B_n in the expression

$$\Theta_0(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

- As promising as this looks, there are some unanswered questions:
- How does one determine B_n?
- If you can determine B_n is there only one choice that works?
- Even if there is a unique choice of B_n can you show the series converges to $\Theta_0(x)$ as $n \to \infty$?
- If it converges at $t = 0$ does it converge for $t > 0$?

The Sturm-Liouville ODE

- We will answer all these questions shortly.
- But what we want to emphasize right now is that the type of ODE problem we just solved is actually quite common.
- It turns out that the ODE we solved above in the x-direction is a special case of a second order ODE boundary value problem called the *Sturm-Liouville problem*.
- The Sturm-Liouville ODE is given by

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda r(x)y(x) = 0, \qquad a < x < b,$$

- You will also see this ODE written as

$$-\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y(x) = \lambda r(x)y(x), \qquad a < x < b,$$

- This is called the *positive definite form.*

Boundary conditions for the Sturm-Liouville ODE

- The boundary conditions are homogeneous

$$\begin{aligned} c_1 y(a) + c_2 y'(a) &= 0, \\ d_1 y(b) + d_2 y'(b) &= 0 \end{aligned}$$

- For the boundary conditions

$$\begin{aligned} c_1 y(a) + c_2 y'(a) &= 0, \\ d_1 y(b) + d_2 y'(b) &= 0 \end{aligned}$$

  the coefficients c_1, c_2, d_1, d_2, are assumed to be real constants.
- The boundary conditions above are said to be *separated* because they provide conditions on only one end point at a time.
- Later on we will relax this a bit.

The heat equation problem led to a Sturm-Liouville ODE

- If we recall the boundary value problem we solved earlier for the heat equation

$$\frac{d^2 X(x)}{dx^2} + \beta^2 X = 0 \qquad X(0) = 0 \quad X(1) = 0.$$

we see this problem is indeed an example of the Sturm-Liouville ODE:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda r(x)y = 0, \qquad a < x < b,$$

- We have $p = 1$, $q = 0$ $r(x) = 1$ and $\lambda = \beta^2$
- We'll see some other examples shortly where the coefficients p, q, r are not just constants.

Requirements for the Sturm-Liouville ODE

- We will assume in the ODE

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda r(x)y = 0, \qquad a < x < b,$$

  that the coefficient functions $p(x)$, $q(x)$ and $r(x)$ are all continuous in the interval $a \le x \le b$.
- We also assume that $p'(x)$ is also continuous in this interval.
- Most importantly, we will assume that $p(x)$ and $r(x)$ are *strictly positive* over the interval $a \le x \le b$.
- And, as usual, there is no loss of generality if we restrict our attention to a specific interval so we will assume in what follows that $a = 0$ and $b = 1$.

Regular vs. singular Sturm-Liouville ODE

- Note we have already insisted in the S-L ODE
$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda r(x)y = 0, \qquad a < x < b,$$
that $p(x)$ and $r(x)$ be strictly positive
- There are a few more restrictions that we will impose and then relax later.
- First we'll insist the boundary conditions be of the separable form
$$c_1 y(a) + c_2 y'(a) = 0,$$
$$d_1 y(b) + d_2 y'(b) = 0$$
- Second, we insist that the domain $a \le x \le b$ be *finite*
- You can see that if $p(x)$ vanishes our ODE becomes singular
- But the ODE will also be singular if the domain is made infinite
- A S-L problem on a finite domain with separable boundary conditions and $p(x) > 0$ and $r(x) > 0$ is called a *regular Sturm-Liouville problem*