Physics 106b: Lecture 10 — 6 February 2014

Dissipative Chaos

I introduce chaos in dissipative (non-Hamiltonian) systems using the example of the Lorenz model (1963) which played a major role in the resurgence of the study of chaos. I then discuss two important ways to characterize chaotic motion: the Lyapunov exponents and the fractal dimension of the attractor.

Lorenz model

Edward Lorenz was a atmospheric scientist interested in whether there are fundamental limits to the ability to forecast the weather, i.e. to predict the dynamics of the atmosphere. To investigate this question he decided to investigate some simple equations that had the type of features encountered in modeling the atmosphere, namely coupled, nonlinear systems of equations. Largely because colleagues in the department were using such equations, he studied what we now call the Lorenz model: three first order ODEs for variables $X(t)$, $Y(t)$, $Z(t)$

$$\dot{X} = -\sigma(X - Y), \tag{1a}$$
$$\dot{Y} = rX - Y - XZ, \tag{1b}$$
$$\dot{Z} = -bZ + XY, \tag{1c}$$

with σ, r, b positive parameters. See the slides to the lecture for the connection to a fluid dynamic system with some features of the atmosphere.

The Lorenz equations are

- autonomous—time does not explicitly appear on the right hand side;
- involve only first order time derivatives so that the evolution depends only on the instantaneous value of (X, Y, Z);
- non-linear—the quadratic terms XZ and XY in the second and third equations;
- dissipative—volumes in phase space contract under the flow

$$\begin{aligned}\nabla_{\text{ph}} \cdot \mathbf{V}_{\text{ph}} &= \frac{\partial}{\partial X}[-\sigma(X - Y)] + \frac{\partial}{\partial Y}[rX - Y - XZ] + \frac{\partial}{\partial Z}[-bZ + XY], \\ &= -\sigma - 1 - b < 0;\end{aligned}$$

- and have bounded solutions — trajectories can be shown to eventually enter and stay within an ellipsoidal region.

Lorenz investigated the equations with $b = 8/3$, $\sigma = 10$ and $r = 27$ and uncovered chaos. The type of results he found are illustrated in the demonstrations on the website.

Some important lessons from Lorenz's study and subsequent investigations of other models are:

- Very simple dynamical systems may show complex dynamics with random-appearing behavior that is unpredictable over long times.

- Chaotic systems show exponential *sensitivity to initial conditions* quantified by the Lyapunov exponent. This is known popularly as the butterfly effect.

- For the classic parameters of the Lorenz model, almost all initial conditions lead to motion on a two winged structure on which the dynamics are chaotic. This strange attractor has nonintegral dimension, i.e. it is a fractal, although for the usual parameters, because of the strong phase space contraction, the dimension is close to 2 (it is 2.06), and the fractal nature is not very apparent in the numerics. Other models, such as the driven damped pendulum, show the fractal nature more clearly (see Hand and Finch Figs. 11.28-9).

- The Poincaré section is again useful, and gives a 2d map (for a 3d phase space). The return map on the Poincaré section is a faithful representation of the dynamics. For example it is invertible – there is a unique inverse for every point given by running the dynamics backwards.

- For strong contraction the points on the Poincaré section almost fall on one dimensional curves and successive iterations of the Poincaré map can be approximated as a 1d map giving the return position along the curve. This gives a simple way of capturing the chaotic dynamics, numerically and conceptually. The 1d map is an approximation, and is not a faithful representation of the full dynamics. When interesting, the 1d map is not invertible, because two or more values lead to the same point after one iteration of the map, so that the dynamics cannot be run backwards.

Driven pendulum

We are often interested in periodically driven systems, e.g. the driven pendulum which has the equation of motion

$$\ddot{\theta} + \frac{1}{Q}\dot{\theta} + \sin\theta = g\ \cos(\omega_D t). \tag{2}$$

We convert this to autonomous phase-space form introducing the phase $\Phi(t)$ of the drive to give the three first order autonomous equations

$$\dot{\theta} = v, \tag{3}$$

$$\dot{v} = -\frac{1}{Q}v - \sin\theta + g\cos\Phi, \tag{4}$$

$$\dot{\Phi} = \omega_D. \tag{5}$$

This gives the phase space (θ, v, Φ). The phase-space velocity is $\mathbf{V}_{\mathrm{ph}} = (\dot{\theta}, \dot{v}, \dot{\Phi})$ and then $\boldsymbol{\nabla}\cdot\mathbf{V}_{\mathrm{ph}} = -Q^{-1}$, again giving a uniform rate of contraction of phase space volumes. A Poincaré section is easily constructed as successive values of $(\theta, v = \dot{\theta})$ for a particular drive phase $\Phi \mod 2\pi$, e.g. $\Phi = 2n\pi$. For certain combinations of drive strengths and frequencies the driven pendulum shows a strange attractor and chaotic dynamics, as I demonstrate with plots of the Poincaré section in class.

Lyapunov exponents

The sensitivity to initial conditions is quantified by the largest Lyapunov exponent λ. This can be defined directly from the average rate of exponential separation $|\delta\mathbf{x}|$ of two very close trajectories in the phase space $\mathbf{x}$

$$\lambda = \lim_{t_f\to\infty}\lim_{|\delta\mathbf{x}_0|\to 0}\left[\frac{1}{t_f - t_0}\ln\left|\frac{\delta\mathbf{x}_f}{\delta\mathbf{x}_0}\right|\right], \tag{6}$$

with $\delta\mathbf{x}_f$ the separation of two trajectories at time t_f with initial conditions separated by $\delta\mathbf{x}_0$ at time t_0. Note that the order of limits means that even though the separation is growing exponentially, $\delta\mathbf{x}_0$ is chosen small enough that $\delta \mathbf{x}_f$ remains small and can be calculated from the equations linearized about the trajectory $\mathbf{x}(t)$. For the iterations of a discrete map, to find the growth rate per iteration replace $|\delta \mathbf{x}_f|$ by the separation at the nth iteration $|\delta \mathbf{x}_n|$ and $t_f - t_0$ by n. More generally for an n-dimensional phase space there are n Lyapunov exponents. Let's order them $\lambda_1 \geq \lambda_2 \ldots \lambda_n$ and then

- $\lambda_1 = \lambda$ gives the rate of growth of a line of initial conditions,
- $\lambda_1 + \lambda_2$ gives the rate of growth of an area of initial conditions,
- $\ldots$
- $\sum_{i=1}^{n} \lambda_i = \langle \nabla_{\mathrm{ph}} \cdot \mathbf{V}_{\mathrm{ph}} \rangle \Rightarrow$ contraction of phase space volume.

For a mathematical model

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) \tag{7}$$

the separation can be calculated by linearizing about the solution $\mathbf{x}_s(t)$, i.e. look at $\mathbf{x}(t) = \mathbf{x}_s(t) + \delta\mathbf{x}(t)$ and then $\delta \mathbf{x}$ satisfies the equation

$$\dot{\delta\mathbf{x}} = \mathbf{J}(\mathbf{x}_s) \cdot \delta\mathbf{x}, \quad \text{with} \quad J_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\mathbf{x}=\mathbf{x}_s(t)}. \tag{8}$$

Here $\mathbf{J}(\mathbf{x_s})$ is the Jacobean calculated at the point $\mathbf{x}_s(t)$ in phase space. For example, for the Lorenz model

$$\mathbf{J} = \begin{bmatrix} -\sigma & \sigma & 0 \\ r - X & -1 & -X \\ Y & X & -b \end{bmatrix} \tag{9}$$

where we substitute in the numerically evolved $(X(t), Y(t), Z(t))$. We can imagine integrating Eq. (8) (numerically, of course) [1] to find the linear relationship between $\delta\mathbf{x}(t)$ and some chosen initial separation $\delta\mathbf{x}(t_0)$

$$\delta\mathbf{x}(t) = \mathbf{M}(t, t_0) \cdot \delta\mathbf{x}(t_0). \tag{10}$$

Here $\mathbf{M}$ is analogous to the mondromy matrix of Lecture 6 but for a long integration time $t - t_0$ rather than over one period of a periodic drive. The most important result is that the ratio of $|\delta\mathbf{x}(t)|/|\delta\mathbf{x}(t_0)|$ gives the largest exponent λ via Eq. (6). The other exponents can be obtained from eigenvalues of $\mathbf{M}$ by one of the following (which I will state without proving):

- It is possible to find a particular orthonormal set of initial displacements $\delta\mathbf{x}(t_0) = \mathbf{V}_i$ such that

$$\lambda_i = \lim_{t\to\infty} \frac{1}{t - t_0} \ln |\mathbf{M}(t, t_0) \cdot \mathbf{V}_i|; \tag{11}$$

- The λ_i are given via the eigenvalues $\mathrm{Ev}[\tilde{\mathbf{M}}\mathbf{M}]$ of $\tilde{\mathbf{M}}\mathbf{M}$, which is a symmetric matrix (here $\tilde{\mathbf{M}}$ is the transpose)

$$\lambda_i = \lim_{t\to\infty} \frac{1}{2(t - t_0)} \mathrm{Ev}_i[\tilde{\mathbf{M}}(t, t_0)\mathbf{M}(\mathbf{t}, \mathbf{t_0})]. \tag{12}$$

[1] In practice we would simultaneously integrate Eqs. (7) and (8) since we need $\mathbf{x}(t)$ to calculate the Jacobean at each t.

Notice that almost any initial choice of $\delta\mathbf{x}_0$ will have some component along the "most stretching" direction $\mathbf{V}_1$, so I do not have to be careful in choosing $\delta\mathbf{x}_0$ to use Eq. (6).

For the Lorenz model I simulated, the exponents are: $\lambda_1 = 0.906$, $\lambda_2 = 0$, $\lambda_3 = -14.572$. For a system defined by ODEs (i.e. continuous time) one of the exponents is always zero, unless the attractor is a fixed point. This corresponds to an initial separation $\delta\mathbf{x}_0$ *along* the trajectory, which is equivalent to a time shift and so does not grow exponentially over long times.

We could use a similar procedure looking at the exponential growth of separation of points on the Poincaré section (finding the Jacobean of the map, iterating its effect on an initial separation vector $\delta\mathbf{x}_0 \ldots$), and use the time between intersections to convert this to the exponents for the original flow. We would "lose" the zero exponent, which corresponds to the continuous trajectory between successive intersections.

Dimensions

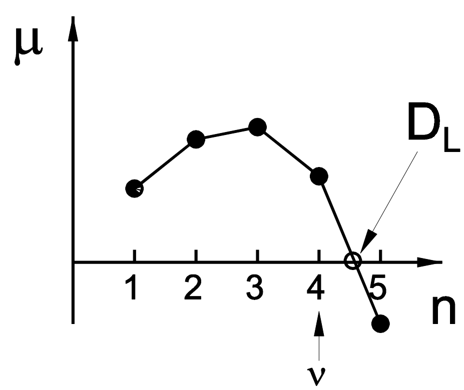
A second important way of characterizing chaos is the *fractal dimension* of the attractor. This characterizes the finer and finer structure found on increasing the resolution of the view of the attractor. There are many ways of defining a "dimension". Here are a couple:

Capacity: I illustrated the *capacity* or *box counting* dimension D_C, defined from the number of boxes $N(\epsilon)$ of side ϵ (of dimension equal to the phase space dimension) needed to cover the attractor (i.e. contain all points on the attractor)

$$D_C = \lim_{\epsilon \to 0} \frac{\ln N(\epsilon)}{\ln \epsilon^{-1}}. \tag{13}$$

It is often easier to find the capacity on the Poincaré section and add 1 to get the dimension of the flow.

Lyapunov dimension:



This is given by the dimension D_L of a ball of initial conditions that neither grows nor decays under evolution. It is estimated from the sum of the first n Lyapunov exponents $\mu(n) = \sum_1^n \lambda_n$, and interpolating to a noninteger value D_L where $\mu(D_L) = 0$.

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