ACM 100b

Reduction of order for systems

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Reduction of order for systems

- We have presented the method of reduction of order for scalar ODE's
- The idea is that suppose you know one solution (call it $w(x)$) to a homogeneous linear ODE.
- Then you can reduce the order of the ODE by one by making the substitution
$$y(x) = v(x)w(x)$$
- Solving for $v(x)$ you find it satisfies an ODE of order reduced by 1.
- This is very useful for second order equations.
- If you know just one solution you can directly find the other.
- In that case reduction of order gives a first order linear equation which you can always solve as we did above.

Reduction of order for systems

- For systems we will show that a general $n \times n$ system can always be reduced to an $(n-1) \times (n-1)$ system if you know one solution vector.
- We start by considering again

$$\boldsymbol{x}' = A(z)\boldsymbol{x}$$

- Now assume you were able to find a solution

$$\boldsymbol{x}_1 = \begin{pmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \end{pmatrix}$$

Reduction of order for systems

- Now define the matrix

$$\Gamma = \begin{pmatrix} 1 & 0 & \cdots & 0 & x_{11} \\ 0 & 1 & \cdots & 0 & x_{12} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & x_{1n-1} \\ 0 & 0 & \cdots & 0 & x_{1n} \end{pmatrix}.$$

- Assume that $x_{1n} \neq 0$.
- This is easy to arrange since all we would need to do is relabel the variables.
- In this case Γ is clearly nonsingular.

Reduction of order for systems

- Next introduce new variables

$$\boldsymbol{x} = \Gamma \boldsymbol{y}$$

- Insert this into the system

$$\boldsymbol{x}' = A(z)\boldsymbol{x}$$

- We get

$$\Gamma' \boldsymbol{y} + \Gamma \boldsymbol{y}' = A\Gamma \boldsymbol{y}$$

- So we have

$$\boldsymbol{y}' = \Gamma^{-1}(A\Gamma - \Gamma')\boldsymbol{y} = B\boldsymbol{y}$$

- The matrix B is defined by the equation above in terms of A and Γ.

$$B = \Gamma^{-1}(A\Gamma - \Gamma')$$

Reduction of order for systems

- You can check that one solution to this new system

$$\boldsymbol{y}' = \Gamma^{-1}(A\Gamma - \Gamma')\boldsymbol{y} = B\boldsymbol{y}$$

  is just

$$\boldsymbol{y}_1 = [0, 0, \ldots, 1]^T$$

  where T denotes the transpose.
- This is just because

$$\Gamma \boldsymbol{y}_1 = \boldsymbol{x}_1$$

  and we know $\boldsymbol{x}_1$ is a solution of the original system.
- What is interesting is what happens if you substitute $\boldsymbol{y}_1$ into the system. You get

$$\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = B \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix} = \begin{pmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{nn} \end{pmatrix}$$

Reduction of order for systems

- But that means

$$\begin{pmatrix} b_{1n} \\ b_{2n} \\ \vdots \\ b_{nn} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

- So, in fact, the last equation in the system decouples from the others.
- And we have effectively an $(n-1) \times (n-1)$ system to solve.
- In component form this looks like

$$\begin{aligned} y_i' &= \sum_{j=1}^{n-1} b_{ij} y_j \quad i = 1, 2, ..., n-1 \\ y_n' &= \sum_{j=1}^{n-1} b_{nj} y_j . \end{aligned}$$

Reduction of order for systems

- You use this observation as follows:
- You first solve for the first $n-1$ entries from the $(n-1)\times(n-1)$ system (if you can)

$$y_i' = \sum_{j=1}^{n-1} b_{ij} y_j \quad i = 1, 2, \ldots, n-1$$

- Then the last entry y_n is given to you by the n'th equation.

$$y_n' = \sum_{j=1}^{n-1} b_{nj} y_j.$$

- This last equation is solved by direct integration
- This approach works in general too.
- If you know k solutions of the system this approach reduces your original $n\times n$ system to a $(n-k)\times(n-k)$ system.

Example of reduction of order for systems

- As an example of this approach consider the system

$$\boldsymbol{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -5 & -4 \end{pmatrix} \boldsymbol{x}.$$

- Using the usual approach for constant systems will give us the eigenvalues $-1, -1, -2$.
- But in this case we can only find two eigenvectors:

$$\begin{pmatrix} a \\ -a \\ a \end{pmatrix} \quad \begin{pmatrix} b \\ -2b \\ 4b \end{pmatrix}.$$

  where a and b are just constants.
- This is an example where we have a defective matrix
- We got two solutions but we know there has to be a third.
- We'll get that one by reduction of order

Example of reduction of order for systems

- To get the third solution we follow the recipe and construct the matrix Γ

$$\Gamma = \begin{pmatrix} 1 & 0 & \cdots & 0 & x_{11} \\ 0 & 1 & \cdots & 0 & x_{12} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & x_{1n-1} \\ 0 & 0 & \cdots & 0 & x_{1n} \end{pmatrix}.$$

- In our case this is

$$\Gamma = \begin{pmatrix} 1 & 0 & a\exp(-z) \\ 0 & 1 & -a\exp(-z) \\ 0 & 0 & a\exp(-z) \end{pmatrix}.$$

- We've used the eigenvector associated with the degenerate eigenvalue -1 and constructed the solution vector for that eigenvector and eigenvalue.

Example of reduction of order for systems

- Now to use the formula we will need the inverse of Γ.
- This is not hard to calculate:

$$\Gamma^{-1} = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & \exp(z)/a \end{pmatrix}.$$

- Now we need to calculate

$$B = \Gamma^{-1}(A\Gamma - \Gamma')$$

- This is

$$B = \begin{pmatrix} 2 & 6 & 0 \\ -2 & -5 & 0 \\ -(2/a)e^z & -(5/a)e^z & 0 \end{pmatrix}.$$

- Note that as advertised, the last column is all zeros

Example of reduction of order for systems

- If we look at the resulting system of equations we have

$$
\begin{aligned}
y_1' &= 2y_1 + 6y_2 \\
y_2' &= -2y_1 - 5y_2 \\
y_3' &= -(2/a)e^z y_1 - (5/a)e^z y_2
\end{aligned}
$$

- Note that once y_1 and y_2 are obtained by solving the 2×2 system

$$
\begin{aligned}
y_1' &= 2y_1 + 6y_2 \\
y_2' &= -2y_1 - 5y_2
\end{aligned}
$$

we get y_3 by simply integrating the last equation

$$y_3' = -(2/a)e^z y_1 - (5/a)e^z y_2$$

Example of reduction of order for systems

- We solve the 2×2 system

$$\begin{aligned} y_1' &= 2y_1 + 6y_2 \\ y_2' &= -2y_1 - 5y_2 \end{aligned}$$

  and we get

$$y_1 = -2c\exp(-z) \qquad y_2 = c\exp(-z)$$

- This is the solution of interest to us.
- The other one gives us information about a solution we already got earlier with eigenvalue $\lambda = -2$.
- With this choice of y_1 and y_2 we integrate

$$y_3' = -(2/a)e^z y_1 - (5/a)e^z y_2$$

  and get

$$y_3 = (-c/a)z$$

Example of reduction of order for systems

- Now to find the last solution vector we compute

$$\boldsymbol{x}_3 = \Gamma \begin{bmatrix} -2c\exp(-z) \\ c\exp(-z) \\ (-c/a)z \end{bmatrix} = \exp(-z)\begin{bmatrix} -2c \\ c \\ 0 \end{bmatrix} + z\exp(-z)\begin{bmatrix} -c \\ c \\ -c \end{bmatrix}$$

  where c is arbitrary.

- Finally we can use the previous solutions to put together the general solution:

$$\boldsymbol{x} = \exp(-z)\begin{bmatrix} a-2c \\ -a+c \\ a \end{bmatrix} + z\exp(-z)\begin{bmatrix} -c \\ c \\ -c \end{bmatrix} + \exp(-2z)\begin{bmatrix} b \\ -2b \\ 4b \end{bmatrix}$$

- You can see that because of the double root and the defective matrix we get terms of the form of $z\exp(-z)$.

Application to linear systems with constant coefficients

- From the previous example we have seen that there are always n linearly independent solutions for an $n \times n$ linear system of ODE's with constant coefficients.
- If the eigenvalues are distinct getting all the solutions is straightforward
- If the roots are repeated it may still be the case that one can get distinct eigenvectors.
- An example where this happens is the identity matrix - all the eigenvalues have value 1 but there are n linearly independent eigenvectors.
- Finally if the matrix is defective we can use reduction of order to get the remaining solutions.
- If we have double roots and the matrix is defective we expect terms like $z \exp(-\lambda z)$
- If we have triple roots then we can get up to terms like $z^2 \exp(-\lambda z)$ etc.

Application to linear systems with constant coefficients

- In general we can make the following statement:
- The first order system with constant coefficients with

$$\boldsymbol{x}' = A\boldsymbol{x}$$

and with eigenvalues

$$|A - \lambda I| = 0$$

has roots $\lambda_1, ...\lambda_n$ with multiplicities $m_1, ..., m_n$ respectively.
- This system has a solution of the form

$$\boldsymbol{x} = \sum_{j=1}^{k} \sum_{i=1}^{m_j} C_{ij} z^{i-1} \exp(\lambda_j z)$$

where $m_1 + m_2 + ... + m_k = n$ and the C_{ij} are n x n coefficient matrices.
- The sum will contain n constants and n linearly independent terms.