ACM 100b

The Lagrange identity

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The operator form

- Denote by $L[y(x)]$ the expression

$$L[y(x)] = -\frac{d}{dx}\left(p(x)\frac{d}{dx}y(x)\right) + q(x)y(x).$$

- In terms of this expression our S-L ODE becomes

$$L[y(x)] = \lambda r(x)y(x).$$

- L is an example of a *linear operator*.
- It's certainly linear since

$$L[u(x) + v(x)] = L[u(x)] + L[v(x)]$$

- L takes as an input a function $(y(x))$ and operates on it to produce $L[u(x)]$
- Often we drop the brackets and just write

$$Ly(x) = -\frac{d}{dx}\left(p(x)\frac{d}{dx}y(x)\right) + q(x)y(x)$$

The Lagrange identity

- Now let $u(x)$ and $v(x)$ be general functions having continuous second derivatives on the interval $a \leq x \leq b$.
- We want to examine (for reasons that will hopefully become clear soon) the expression

$$\int_a^b L[u(x)]v(x)dx =$$

$$\int_a^b \left[-\frac{d}{dx}\left(p(x)\frac{d}{dx}u(x)\right)v(x) + q(x)u(x)v(x)\right]dx$$

The Lagrange identity

- We next integrate the right hand side of this expression which is

$$\int_a^b \left[-\frac{d}{dx}\left(p(x)\frac{d}{dx}u(x) \right) v(x) + q(x)u(x)v(x) \right] dx$$

by parts once to get

$$-v(x)p(x)\frac{du}{dx}\bigg|_a^b + \int_a^b p(x)\frac{dv}{dx}\frac{du}{dx}dx + \int_a^b q(x)u(x)v(x)dx$$

- Integrate by parts once more to get

$$\int_a^b L[u(x)]v(x) = -\left\{ p(x)\left[\frac{du}{dx}v - u\frac{dv}{dx} \right] \right\}\bigg|_a^b + \int_a^b u(x)L[v(x)]dx$$

The Lagrange identity

- We can rewrite this as

$$\int_a^b \{L[u(x)]v(x) - u(x)L[(v(x)]\}\, dx =$$
$$-\left\{p(x)\left[\frac{du}{dx}v - u\frac{dv}{dx}\right]\right\}\Big|_a^b .$$

- This is known as *Lagrange's identity*.
- We'll see later that it is also an application of Green's third formula.
- You can see some similarity because it relates an integral over the whole domain $a \leq x \leq b$ to an expression that just involves the boundary values of u, v at $x = a$ and $x = b$

Application to the Sturm-Liouville problem

- So far this doesn't seem too useful but we next look at what happens if we use the separable boundary conditions of the $S-L$ problem like

$$c_1 u(a) + c_2 u'(a) = 0,$$
$$d_1 u(b) + d_2 u'(b) = 0$$

- In the Lagrange identity

$$\int_a^b \{L[u(x)]v(x) - u(x)L[(v(x)]\}\, dx = -\left\{p(x)\left[\frac{du}{dx}v(x) - u\frac{dv}{dx}\right]\right\}\Bigg|_a^b$$

assume that both $u(x)$ and $v(x)$ satisfy boundary conditions of the form

$$c_1 u(a) + c_2 u'(a) = 0 \qquad c_1 v(a) + c_2 v'(a) = 0$$
$$d_1 u(b) + d_2 u'(b) = 0 \qquad d_1 v(b) + d_2 v'(b) = 0$$

Application to the Sturm-Liouville problem

- In this case a little algebra shows that

$$\left\{ p(x) \left[\frac{du}{dx} v(x) - u \frac{dv}{dx} \right] \right\} \Bigg|_a^b = 0.$$

- So in this case we have the very symmetric looking expression

$$\int_a^b \{ L[u(x)] v(x) - u(x) L[(v(x)]\} \, dx = 0$$

or

$$\int_a^b L[u(x)] v(x) dx = \int_a^b u(x) L[(v(x)] dx$$

- This identity is very special and arises because of the structure of the ODE and, equally importantly, because of the structure of the boundary conditions.