# Physics 106b: Lecture 4 — 16 January, 2014

## Mathematics of Relativity

### Basis vectors

A particular inertial frame is defined by the choice of basis vectors for the 4-dimensional spacetime. These are unit vectors along time and three orthogonal coordinate directions in the particular inertial frame:

$$\text{Basis:} \qquad \mathbf{e}_0, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3 \; . \tag{1}$$

The basis vector $\mathbf{e}_0$ is the 4-vector connecting the space-time origin to an event at the spatial origin at unit time in that frame of reference; the basis vector $\mathbf{e}_1$ is the coordinate basis vector along the $x$-axis at constant time, etc. The length-squared of the vectors are

$$\mathbf{e}_0{}^2 = 1, \qquad \text{(timelike)} \tag{2}$$

$$\mathbf{e}_1{}^2 = -1, \qquad \text{(spacelike)} \tag{3}$$

and different basis vectors are orthogonal in the sense

$$\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = 0 \qquad \text{for } \alpha \neq \beta \; . \tag{4}$$

These results are summarized by

$$\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = g_{\alpha\beta} \qquad \text{with} \qquad g_{\alpha\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} . \tag{5}$$

$g_{\alpha\beta}$ is called the metric tensor — strictly $g_{\alpha\beta}$ are the components of a geometric (i.e. frame independent quantity) in our chosen inertial frame — see below. $g$ is an important object: it tells us how the geometry of space-time is different from conventional Euclidean geometry, which has all $+1$ for the diagonal entries of the metric tensor. *The fact that $\mathbf{e}_\alpha \cdot \mathbf{e}_\beta$ is not the unit matrix means that there are two ways of defining the components of 4-vectors.*

### Components of a 4-vector

We can sensibly define components of a 4-vector $\mathbf{x}$ with respect to basis $\{\mathbf{e}_\alpha\}$ in two ways:

**contravariant component:** (up index) write

$$\mathbf{x} = x^\alpha \mathbf{e}_\alpha \tag{6}$$

with the *relativistic summation convention*—sum repeated up-down or down-up indices. Then $x^\alpha \equiv (t, x, y, z)$.

**covariant components:** (down index) define

$$x_\alpha = \mathbf{x} \cdot \mathbf{e}_\alpha . \tag{7}$$

Plugging Eq. (6) into Eq. (7)

$$x_\alpha = g_{\alpha\beta} x^\beta \; . \tag{8}$$

Note that $g_{\alpha\beta}$ acts to lower indices. From the form of $g_{\alpha\beta}$ we have $x_\alpha \equiv (t, -x, -y, -z)$. The inverse relation

$$x^\alpha = g^{\alpha\beta} x_\beta \tag{9}$$

defines $g^{\alpha\beta}$ (acts to raise indices): it has the same expression Eq. (5). Also from the preceding two equations or by direct calculation

$$g^{\alpha\delta} g_{\delta\beta} = g^\alpha{}_\beta = \delta^\alpha{}_\beta, \tag{10}$$

$$g_{\alpha\delta} g^{\delta\beta} = g_\alpha{}^\beta = \delta_\alpha{}^\beta, \tag{11}$$

where the first equalities define the symbols $g^\alpha{}_\beta$, $g_\alpha{}^\beta$ through the index lowering or raising property of $g$, and the second equalities evaluate these expressions to be just the Kronecker delta (unit matrix).

### Scalar product from components

The scalar product is

$$\mathbf{A} \cdot \mathbf{B} = (A^\alpha \mathbf{e}_\alpha) \cdot (B^\beta \mathbf{e}_\beta) = A^\alpha g_{\alpha\beta} A^\beta \tag{12}$$

$$= A^\alpha B_\alpha = A_\alpha B^\alpha \tag{13}$$

$$= A^0 B_0 + A^1 B_1 + A^2 B_2 + A^3 B_3 \tag{14}$$

$$= A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3. \tag{15}$$

In particular the interval is

$$\mathbf{x}^2 = \mathbf{x} \cdot \mathbf{x} = t^2 - x^2 - y^2 - z^2 \tag{16}$$

consistent with our previous definitions.

## Lorentz Transformation

Transform between different inertial frames defined by bases $\{\mathbf{e}_\alpha\}$ and $\{\mathbf{e}'_\alpha\}$. A 4-vector $\mathbf{x}$ is represented by components $x^\alpha$ (or $x_\alpha$) with respect to $\{\mathbf{e}_\alpha\}$ and by $x'^\alpha$ (or $x'_\alpha$) with respect to $\{\mathbf{e}'_\alpha\}$. The primed frame has a velocity $\vec{v}$ with respect to the unprimed frame. The transformation is defined by the matrix $\Lambda$

$$\mathbf{e}_\alpha = \mathbf{e}'_\beta \Lambda^\beta{}_\alpha \tag{17}$$

($\beta$-summed, of course). The transformation of components is given by

$$\mathbf{x} = x^\alpha \mathbf{e}_\alpha = \mathbf{e}'_\beta \Lambda^\beta{}_\alpha x^\alpha = x'^\beta \mathbf{e}'_\beta \tag{18}$$

so that

$$x'^\beta = \Lambda^\beta{}_\alpha x^\alpha. \tag{19}$$

Also

$$x_\alpha = \mathbf{x} \cdot \mathbf{e}_\alpha = \mathbf{x} \cdot \mathbf{e}'_\beta \Lambda^\beta{}_\alpha = x'_\beta \Lambda^\beta{}_\alpha \tag{20}$$

so that

$$x_\alpha = x'_\beta \Lambda^\beta{}_\alpha. \tag{21}$$

Comparing these equations explains the notation co- (transforms as basis vectors) and contra- (transforms opposite to basis vectors) for the components.

In the "standard configuration" where the prime frame moves at speed $v$along the $x$-axis and the $x$, $x'$, $y$, $y'$, and $z$, $z'$ axes are aligned, the transformation matrix ($x^\alpha \to x'^\alpha$) is

$$\Lambda = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{22}$$

with $\gamma = (1 - v^2)^{-1/2}$, so that

$$t' = \gamma(t - vx) \tag{23}$$

$$x' = \gamma(x - vt) \tag{24}$$

$$y' = y \tag{25}$$

$$z' = z. \tag{26}$$

The $x'^\alpha \to x^\alpha$ transformation is given by $v \to -v$.

For a general orientation of the relative velocity the components of $\Lambda$ are ($i = 1, 2, 3$)

$$\Lambda^0{}_0 = \gamma, \qquad \Lambda^0{}_i = \Lambda^i{}_0 = -\gamma v_i, \qquad \Lambda^i{}_j = \delta_{ij} + (\gamma - 1)\frac{v_i v_j}{v^2} \ . \tag{27}$$

Spelling this out

$$\Lambda = \begin{bmatrix} \gamma & -\gamma v_x & -\gamma v_y & -\gamma v_z \\ -\gamma v_x & 1 + (\gamma - 1)\frac{v_x^2}{v^2} & (\gamma - 1)\frac{v_x v_y}{v^2} & (\gamma - 1)\frac{v_x v_z}{v^2} \\ -\gamma v_y & (\gamma - 1)\frac{v_x v_y}{v^2} & 1 + (\gamma - 1)\frac{v_y^2}{v^2} & (\gamma - 1)\frac{v_y v_z}{v^2} \\ -\gamma v_z & (\gamma - 1)\frac{v_x v_z}{v^2} & (\gamma - 1)\frac{v_y v_z}{v^2} & 1 + (\gamma - 1)\frac{v_z^2}{v^2} \end{bmatrix} . \tag{28}$$

You can get this by first rotating to the standard configuration, boosting, and then rotating back. For example, for a velocity in the $xy$ plane $\Lambda$ is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{29}$$

with $v_x = v\cos\theta$, $v_y = v\sin\theta$ (use e.g. Mathematica to do the matrix products).

## 4-Tensors

Going up the hierarchy of geometric objects, scalars (Lorentz invariants), 4-vectors, ... introduces the idea of 4-tensors. Just as we use tensors in 3-space when we are thinking of the consequences of rotational symmetry and need more complicated physical objects than 3-vectors, see Lecture 13 of Ph106a.

In general, an $n$-th rank (or rank-$n$) tensor (3-tensor or 4-tensor) can be defined as a linear function that takes $n$ vector arguments (3-vector or 4-vector as appropriate) and produces a scalar[1]

$$\mathbf{T}(\mathbf{u}, \mathbf{v}, \ldots) = a \ , \tag{30}$$

$$\mathbf{T}(\alpha\mathbf{u}_1 + \beta\mathbf{u}_2, \mathbf{v}, \cdots) = \alpha\mathbf{T}(\mathbf{u}_1, \mathbf{v}) + \beta\mathbf{T}(\mathbf{u}_2, \mathbf{v}) + \cdots \ , \tag{31}$$

$$\vdots \tag{32}$$

[1]Or just putting in $n - 1$ vector arguments gives a vector ...

A vector is actually a rank-1 tensor and the scalar produced by acting on a second vector is just the scalar product

$$\mathbf{a}(\mathbf{b}) \equiv \mathbf{a} \cdot \mathbf{b}. \tag{33}$$

4-tensors, like 4-vectors, are geometrical frame-independent quantities, but we can represent them in terms of components with respect to a particular choice of basis vectors. The action on the basis vectors defines the (covariant) components

$$\mathbf{T}(\mathbf{e}_\alpha, \mathbf{e}_\beta) = T_{\alpha\beta} \ . \tag{34}$$

Then

$$\mathbf{T}(\mathbf{u}, \mathbf{v}) = \mathbf{T}(u^\alpha \mathbf{e}_\alpha, u^\beta \mathbf{e}_\beta) = u^\alpha T_{\alpha\beta} u^\beta \ , \tag{35}$$

giving the more familiar component form for the tensor. We can also define contravariant components, e.g. for a second rank tensor

$$\mathbf{T}(\_, \_) = T^{\alpha\beta} \mathbf{e}_\alpha \otimes \mathbf{e}_\beta \tag{36}$$

so that

$$\mathbf{T}(\mathbf{u}, \mathbf{v}) = T^{\alpha\beta} \mathbf{e}_\alpha \cdot \mathbf{u} \, \mathbf{e}_\beta \cdot \mathbf{v} = u_\alpha T^{\alpha\beta} v_\beta. \tag{37}$$

Following the same arguments as leading to Eqs. (19,21) shows that the components of a tensor transform between inertial frames as

$$T'^{\alpha\beta} = \Lambda^\alpha_{\ \mu} \Lambda^\beta_{\ \nu} T^{\mu\nu} \tag{38}$$

$$T_{\alpha\beta} = T'_{\mu\nu} \Lambda^\mu_{\ \alpha} \Lambda^\nu_{\ \beta} \tag{39}$$

with each index transforming like a the corresponding component of a vector.

The co- and contra-variant components are related by the g raising and lowering properties in the same way as vector components.

### Electromagnetic Field Tensor

I briefly introduced the *electromagnetic field tensor* **F** in the notes to Lecture 2. It can be defined in component form by

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha \tag{40}$$

with $\partial^\alpha = (\partial/\partial t, -\vec{\nabla})$ the gradient 4-vector and $\mathbf{A} = (\Phi, \vec{A})$ the potential 4-vector. We see it is a 4-tensor since it is defined in terms of the (outer) product of 4-vectors. Explicitly, in a particular frame of reference by performing the time and space derivatives and using the expressions for $\vec{E}$, $\vec{B}$ in terms of $\Phi$, $\vec{A}$ (see Ph1c or Ph106c) you should find

$$F^{\alpha\beta} \equiv \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}, \tag{41}$$

or $F^{0j} = -F^{j0} = -E_j$ , $F^{ij} = -\epsilon_{ijk} B_k$ $(i, j = 1, 2, 3)$. For example

$$F^{01} = \frac{\partial A_x}{\partial t} + \frac{\partial \Phi}{\partial x} = -E_x \ , \tag{42}$$

$$F^{12} = -\frac{\partial A_y}{\partial x} + \frac{\partial A_x}{\partial y} = -B_z \ . \tag{43}$$

Note that **F** is the geometric, frame independent object that describes electromagnetic fields at a space-time point. In different frames of reference this will be manifest as various amounts of $\vec{E}$ and $\vec{B}$.

The transformation rules for $\vec{E}$ and $\vec{B}$ between different inertial frames can be deduced from the Lorentz transformation of the electromagnetic field tensor

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta} \ . \tag{44}$$

In matrix notation this reads

$$\begin{bmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -B'_z & B'_y \\ E'_y & B'_z & 0 & -B'_x \\ E'_z & -B'_y & B'_x & 0 \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{45}$$

which gives

$$\begin{array}{ll} E'_x = E_x & B'_x = B_x \\ E'_y = \gamma(E_y - vB_z) & B'_y = \gamma(B_y + vE_z) \\ E'_z = \gamma(E_z + vB_y) & B'_z = \gamma(B_z - vE_y) \end{array} \tag{46}$$

or

$$\begin{array}{ll} E'_\parallel = E_\parallel & B'_\parallel = B_\parallel \\ \vec{E}'_\perp = \gamma(\vec{E}_\perp + \vec{v} \times \vec{B}_\perp) & \vec{B}'_\perp = \gamma(\vec{B}_\perp - \vec{v} \times \vec{E}_\perp) \end{array} \tag{47}$$

The quantity $F^{\alpha\beta}F_{\alpha\beta} = F^{\alpha\beta}g_{\alpha\gamma}g_{\beta\delta}F^{\gamma\delta}$ is a Lorentz invariant. You can show this is proportional to $E^2 - B^2$ in some inertial frame, so that the question of whether $E$ or $B$ is larger has the same answer in all inertial frames. Also, a more complicated expression shows that $\vec{E} \cdot \vec{B}$ is also the same in all inertial frames. For example if $\vec{E}$ and $\vec{B}$ are perpendicular, or one or other is zero, in one inertial frame, they are perpendicular or one is zero in all inertial frames.

The dynamics of a charged particle is elegantly expressed in covariant form in terms of 4-vectors and the electromagnetic field tensor

$$\mathbf{f} = q\mathbf{F} \cdot \mathbf{u} \quad \text{and then} \quad \frac{d\mathbf{p}}{d\tau} = \mathbf{f} = q\mathbf{F} \cdot \mathbf{u} \tag{48}$$

($\mathbf{f}$ is the Minkowski force, $\mathbf{u}$ the 4-velocity of the particle of charge $q$, etc.).

**Metric Tensor**

The metric tensor gives the scalar product of its two vector arguments

$$\mathbf{g}(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v} \ , \tag{49}$$

and so

$$g_{\alpha\beta} \equiv g(\mathbf{e}_\alpha, \mathbf{e}_\beta) = \ \mathbf{e}_\alpha \cdot \mathbf{e}_b. \tag{50}$$

This gives Eq. (5) as before. In Special Relativity this is true globally over all space time and in any inertial frame. In General Relativity the expressions is locally valid in a free falling frame, but not globally – spacetime is curved in the presence of matter.

## Group property of Lorentz transformations

So far we have defined Lorenz transformations as pure boosts keeping the axes aligned, leading to the expression (28). More generally, a Lorenz transformation is any transformation between basis vectors that leaves the metric tensor unchanged. Since

$$\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta \mathbf{e}'_\mu \cdot \mathbf{e}'_\nu \;, \tag{51}$$

if the metric tensor is the same in the two frames we must have

$$g_{\alpha\beta} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta g_{\mu\nu} \;. \tag{52}$$

Any $\Lambda$ satisfying this is a general Lorentz transformation. This is like defining a rotation matrix as any matrix leaving the length of a vector unchanged. This larger class of transformations includes pure spatial rotations, pure boosts (no axis rotations, the ones we introduced before), and combinations of boosts and rotations. The combination of two such Lorenz transformations is also a Lorenz transformation. There is an identity transform, and an inverse to any transform (e.g. $\vec{v} \to -\vec{v}$ for a boost), and so the set of all Lorenz transformations forms a group. This is a Lie group, as for rotations, since it is parameterized by continuous parameters, the transformation velocity and rotation axis direction and angle.

As a first example consider frame $S''$ moving with speed $v'$ along the $x'$ axis of $S'$, which in turn is moving with speed $v$ along the $x$ axis of $S$, and we take all axes to be aligned. Since $y, z$ components are unchanged, I will just write the matrix giving the $t, x$ transformation. The combined transformation from $S$ to $S''$ is

$$\Lambda(S \to S'') = \Lambda(S' \to S'')\Lambda(S \to S') = \begin{bmatrix} \gamma' & -\gamma' v' \\ -\gamma' v' & \gamma' \end{bmatrix} \begin{bmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{bmatrix} \tag{53}$$

$$= \gamma\gamma'(1 + v v') \begin{bmatrix} 1 & -\frac{v+v'}{1+vv'} \\ -\frac{v+v'}{1+vv'} & 1 \end{bmatrix} \tag{54}$$

The speed of $S''$ relative to $S$ is given by the addition of velocities

$$v'' = \frac{v + v'}{1 + v v'}, \tag{55}$$

the expression appearing in the off diagonal elements of the matrix in $\lambda(S'' \to S)$. If we write $\Gamma = \gamma\gamma'(1 + v v')$ some algebra gives

$$1 - \frac{1}{\Gamma^2} = v''^2, \tag{56}$$

so that $\Gamma = \gamma_{v''}$. Thus

$$\Lambda(S \to S'') = \begin{bmatrix} \gamma_{v''} & -\gamma_{v''} v'' \\ -\gamma_{v''} v'' & \gamma_{v''} \end{bmatrix}, \tag{57}$$

which is the Lorentz matrix for the transformation with velocity $v''$ along the x-axis.

It is *not* true that pure boosts form a group: performing successive pure boosts with transformation velocities that are not parallel actually leads to a transformation that is a combination of a boost and a rotation. This surprising effect is important in calculating the spin-orbit coupling of an electron in an atom. The spin of the electron precesses (rotates) because of the magnetic field from the nucleus in the rest frame of the electron acting on the magnetic moment of the electron. But there is an additional rotation in the lab frame because to relate the electron rest frame at successive times to the lab frame we need to add Lorenz boosts in the tangential direction (the circular motion) and in the radial direction (from the change in velocity due to the acceleration in the circular motion). This phenomenon, called *Thomas precession*, turns out to give a factor of 2 correction to the spin orbit coupling —rather magical that a purely kinematic effect of frame transformations has the same size as a dynamic effect involving the size of the magnetic dipole moment of the electron.[2] Here's an (algebraically rather messy) example you can work through to show this effect:

> A frame $S''$ is moving with small speed $\Delta v$ in the $y'$ direction relative to $S'$, which in turn is moving with speed $v$ in the $x$ direction relative to frame $S$. Find, to first order in $\Delta v$, the matrix $\Lambda$ that gives the combination of these transformations, i.e. from the coordinates $(t'', x'', y'', z'')$ in $S''$ to $(t, x, y, z)$ in $S$. Show that $\Lambda$ can be written as
>
> $$\Lambda = \mathbf{RB}$$
>
> where $\mathbf{R}$ is a small rotation about the $z$ axis through angle $\delta\theta$, given by a rotation matrix
>
> $$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \delta\theta & 0 \\ 0 & -\delta\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$
>
> and $\mathbf{B}$ is a pure Lorentz boost with velocity $(v\hat{x} + \delta v\hat{y})$ with $\delta v = \gamma^{-1}\Delta v$, and
>
> $$\delta\theta = -\frac{\gamma - 1}{\gamma}\frac{\Delta v}{v} \to -\frac{1}{2}v\Delta v, \tag{58}$$
>
> with $\gamma = 1/\sqrt{1 - v^2}$, and the last result is for $v \ll 1$. All calculations should be done only to first order in the small quantities $\Delta v$, $\delta v$, $\delta\theta$.

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[2] See *Classical Electrodynamics* by Jackson for more details.