ACM 100b

The Fourier transform

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February 23, 2014

The Fourier transform

- We saw in the previous lecture some of the things that can happen in the case of a singular Sturm-Liouville problem.
- In some cases the results are not that different from a non-singular problem.
- In the previous lecture we saw examples of discrete eigenvalues and unique eigenfunctions.
- But as we mentioned in previous lectures, this is not guaranteed.
- Another possibility is that the eigenvalues become dense in what is called continuous spectrum.
- Remarkably these results can still be used to represent functions.
- A very important example of this is the *Fourier transform* which we describe below.

The Fourier transform

- Suppose $f(x)$ is a bounded integrable function defined on $-\infty < x < \infty$. That is

$$\int_{-\infty}^{+\infty} |f(x)| dx \text{ exists}$$

- Suppose we only sample $f(x)$ on the finite interval $-L \le x \le L$
- And we assume periodicity so that $2L$ is the period.
- Then we know that $f(x)$ can be represented by a Fourier series written below in its complex periodic form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp(in\pi x/L),$$

- Here c_n are the Fourier coefficients defined by

$$c_n = \frac{1}{2L} \int_{-L}^{L} f(\zeta) \exp(-in\pi\zeta/L) d\zeta.$$

The Fourier transform

- Now let's substitute the expression for the c_n in the series for $f(x)$:

$$f(x) = \frac{1}{2L} \sum_{n=-\infty}^{\infty} \int_{-L}^{L} f(\zeta) \exp(-in\pi(\zeta - x)/L) d\zeta.$$

- Make the assignments

$$k = n\pi/L \qquad \Delta k = \pi/L.$$

- Then in terms of these variables we have

$$f(x) = \frac{1}{2\pi} \sum_{k=n\pi/L} \Delta k \int_{-L}^{L} f(\zeta) \exp(-ik(\zeta - x)) d\zeta \qquad -\infty < n < \infty.$$

The Fourier transform

- Now suppose we have that

$$\int_{-\infty}^{\infty} |f(x)| dx \quad \text{exists.}$$

- In that case we can see that

$$\int_{-L}^{L} f(\zeta) \exp(-ik(\zeta - x)) d\zeta \to g(k, x) \quad \text{as} \quad L \to \infty.$$

- We're just pointing out the integral inside the series is itself a function
- This basically follows from the comparison test for integrals.

The Fourier transform

- Now recall that

$$f(x) = \frac{1}{2\pi} \sum_{k=n\pi/L} \Delta k \int_{-L}^{L} f(\zeta) \exp(-ik(\zeta - x)) d\zeta \qquad -\infty < n < \infty,$$

- And let $L \to \infty$
- So we must then have that

$$\frac{1}{2\pi} \sum_{k=n\pi/L} g(k,x) \Delta k \to f(x) \quad \text{as} \quad L \to \infty.$$

- But note in the expression

$$\frac{1}{2\pi} \sum_{k=n\pi/L} g(k,x) \Delta k \to f(x)$$

that as $L \to \infty$, we have that $\Delta k \to 0$ because $\Delta k = \pi/L$

The Fourier transform

- So this sum
$$\frac{1}{2\pi}\sum_{k=-n\pi/L} g(k,x)\Delta k \to f(x)$$
is becoming an integral:
$$\frac{1}{2\pi}\int_{-\infty}^{\infty} g(k,x)dk.$$
- Putting this all together we have that
$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} g(k,x)dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(\zeta)\exp(-ik(\zeta-x))d\zeta\right] dk \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(ikx)dk\int_{-\infty}^{\infty} f(\zeta)\exp(-ik\zeta)d\zeta. \end{aligned}$$

The Fourier transform

- Now look at

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx) \left[\int_{-\infty}^{\infty} f(\zeta) \exp(-ik\zeta) d\zeta \right] dk$$

- Note that if we define

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\zeta) \exp(-ik\zeta) d\zeta,$$

- Then our result above says that

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \exp(ikx) dk$$

- This is called the *Fourier transform pair*.
- The first integral defines $F(k)$ which is called the *Fourier transform* of $f(x)$.
- The second recaptures $f(x)$ in terms of its transform.

The Fourier transform and singular S-L problems

- But what does all of this have to do with singular Sturm-Liouville problems?
- What we have effectively done here is tried to extend the results we know about Fourier series over the finite interval $-L < x < L$ to the domain $-\infty < x < \infty$
- The attempt to solve a Sturm-Liouville problem which is regular over a finite interval results in a singular Sturm-Liouville problem, when the domain is extended to become infinitely long.
- What has happened is that the eigenvalues over the finite domain which were given by

$$\lambda_n = n^2\pi^2/L^2 \qquad 0 < n < \infty$$

  have now become a dense set as $L \to \infty$.
- This is an example of what we mean by continuous spectrum.

The Fourier transform and singular S-L problems

- But even so, we end up with a result that now involves integrals that looks quite similar to the Fourier series.
- If we get the Fourier transform we can reconstruct the function.
- This is typical of what happens in singular Sturm-Liouville problems.
- If you get a discrete spectrum then you have results somewhat similar to what we have in the theory of Fourier series
- But even if you get dense spectrum, you get a *transform pair* which can still be used to represent functions.
- We shall see that this idea can be used to solve problems over fully infinite domains as well as semi-infinite domains.

Forward and inverse Fourier transforms

- The expression

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ikx) dx,$$

  is often called the *inverse Fourier transform*

- And the expression

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \exp(ikx) dk$$

  is often called the *forward Fourier transform*

- But the names are not all that useful.

Forward and inverse Fourier transforms

- For example we could define

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(+ikx) dx,$$

and with this $F(k)$ we have

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \exp(-ikx) dk.$$

- If you use the $\exp(ikx)$ to get $F(k)$ then you must undo it with $\exp(-ikx)$ to get back $f(x)$
- So as long as you're consistent about the use of the signs in the exponential the transform will work.
- Note that whatever you do you must normalize by an overall factor of $1/(2\pi)$ when transforming forward and then back.
- We did this in a symmetric way in the expressions above.

The Fourier transform for discontinuous functions

- If $f(x)$ is discontinuous the Fourier transform pair has to be modified.
- This is done in accordance with our understanding of how a Fourier series converges near a discontinuity:

$$\frac{f(x^+) + f(x^-)}{2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} d\zeta f(\zeta) \exp(-ik(\zeta - x)),$$

- Or if we let

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-ik\zeta) f(\zeta) d\zeta,$$

- Then we have

$$\frac{f(x^+) + f(x^-)}{2} = \frac{1}{\sqrt{2\pi}} \int_{\infty}^{\infty} F(k) \exp(ikx) dk.$$

- Note that the recovered $f(x)$ has no Gibbs phenomenon
- You recover $f(x)$ except at the discontinuity you get back the average of the two values on either side.

The Fourier transform is viewed as a complex function

- Note that the expression

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(+ikx) dx,$$

  defines $F(k)$ the *Fourier transform of* $f(x)$ for k real as long as $f(x)$ is absolutely integrable.
- Recall this means

$$\int_{-\infty}^{\infty} |f(x)| dx \text{ is finite.}$$

- But we can then use analytic continuation to define $F(k)$ for k complex.
- This will be very important later as complex analysis is often used to recover $f(x)$ from $F(k)$.
- In other words you will need those contour integration skills from ACM 95/100a

An example - transform of a Gaussian

- Consider the function

$$f(x) = N\exp(-ax^2) \qquad -\infty < x < \infty.$$

- This is a Gaussian function whose value is N at $x = 0$.
- It is an even function whose rate of decay is governed by the value of a.
- If a is large then the function decays rapidly and if a is small the decay is less rapid.
- More precisely, a defines a “scale” over which decay occurs.
- The Fourier transform of $f(x)$ is defined by

$$\begin{aligned} F(k) &= \frac{N}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\exp(-ax^2)\exp(-ikx)dx \\ &= \frac{N}{\sqrt{2\pi}}\exp(-k^2/4a)\int_{-\infty}^{\infty}\exp\left[-a(x+(ik/2a)^2\right]dx \\ &= \frac{N}{\sqrt{2\pi}}\sqrt{\frac{\pi}{a}}\exp(-k^2/(4a)). \end{aligned}$$

An example - transform of a Gaussian

- If we normalize our function by selecting N so that the area of the function is 1 we get

$$f(x) = \sqrt{\frac{a}{\pi}} \exp(-ax^2),$$

- And so the transform becomes

$$F(k) = \frac{1}{\sqrt{2\pi}} \exp(-k^2/(4a)).$$

- Notice there is a kind of duality between these two results.
- If a is large then the function $f(x)$ decays over a scale like $1/\sqrt{a}$.
- The larger a is, the faster the decay and the smaller the scale over which $f(x)$ decays.

An example - transform of a Gaussian

- In contrast the Fourier transform

$$F(k) = \frac{1}{\sqrt{2\pi}} \exp(-k^2/(4a)).$$

  behaves in exactly the opposite way.
- The larger a is, the less rapidly $F(k)$ decays.
- And notice the scale of F is inversely proportional to that of $f(x)$
- The way to understand this is that the Fourier transform is an attempt to synthesize $f(x)$ in terms of oscillating functions $\exp(ikx)$.
- If a function has a sharp gradient then it requires a large value of k to capture this behavior.
- This is because the function $\exp(ikx)$ oscillates with a characteristic scale of $1/k$.
- Slowly varying parts of $f(x)$ can be captured with small values of k (large scales)
- Rapidly varying regions require large values of k.

The δ function again

- Note that in the limit as $a \to \infty$ our function $f(x)$

$$f(x) = \sqrt{\frac{a}{\pi}} \exp(-ax^2),$$

  becomes very sharply peaked at $x = 0$ with an amplitude of $\sqrt{a/\pi}$.
- Note we also fixed $f(x)$ so that its integral from $x \to -\infty$ to $x \to \infty$ is always 1.
- So in the limit we see $f(x)$ goes to zero everywhere except $x = 0$ as $a \to \infty$ and in addition

$$\int_{-\infty}^{\infty} f(x)dx = 1$$

- So we must have

$$f(x) \to \delta(x),$$

  where δ is the infamous delta function.

The δ function again

- The Fourier transform $F(k)$ in this limit becomes

$$F(k) \to \frac{1}{\sqrt{2\pi}} \qquad a \to \infty.$$

- We have thus shown

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) \exp(-ikx) dx = \frac{1}{\sqrt{2\pi}},$$

- This result is also consistent with the sifting property for the δ function.
- But what is more interesting is the other part of the transform pair:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx) dk.$$

The completeness relation again

- A more general way to write this

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ik(x - x_0))dk.$$

- Now compare this to the completeness relation we got for regular Sturm-Liouville problems over finite domains.

$$\sum_{n=0}^{\infty} \phi_n(x)\phi_n(x') = \frac{\delta(x - x')}{r(x)}.$$

- You can rewrite the expression above as

$$\delta(x - x_0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \exp(ikx) \exp(-ikx_0)dk.$$

- This is the completeness relation for the Fourier integral.
- And in fact there are completeness relations for other singular Sturm-Liouville problems