Physics 106a **17 November, 2013**

Lecture 16: Rigid Body Rotation with Torques

I discuss equations of motion for the rotation of a rigid body under torques, and illustrate the application to the motion of a spinning top under a torque from gravity and the precession of the equinoxes (the slow precession of the spin angular momentum of the earth about an axis normal to the plane of the orbit).

In the previous lecture I introduced one general strategy for approaching rigid body dynamics of asymmetric objects, where we want to take advantage of the simplicity of the moment of inertia tensor relative to the principal axes: use the Euler equations for the angular velocity components in the body frame. This still leaves the geometric problem of understanding the motion in the space frame, given by the rotation of the body with angular velocity known in terms of components with respect to axes that are themselves rotating with that angular velocity. . .

An alternative general strategy is to do the geometry first, and write the angular velocity components along the principal axes in terms of time dependent variables that define the rotation matrix giving the rotation of the body axes relative to the space frame axes. The $\hat{n}, \phi$ parameterization of the rotation matrix turns out not to be convenient for this; instead we use the *Euler angles*.

Euler angles

See the Slides or Hand and Finch for the definition of the three Euler angles ϕ, θ, ψ. Note that to cover all rotations ϕ, ψ run from 0 to 2π, but θ runs from 0 to π. The important results are:

1. Angular velocity components with respect to the (body fixed) principal axes

$$\omega_1 = \dot{\phi} \sin\psi \sin\theta + \dot{\theta} \cos\psi \tag{1}$$
$$\omega_2 = \dot{\phi} \cos\psi \sin\theta - \dot{\theta} \sin\psi \tag{2}$$
$$\omega_3 = \dot{\psi} + \dot{\phi} \cos\theta \tag{3}$$

   See Hand and Finch Eqs. (8.69) for the components along the (inertial) space axes.

2. Kinetic energy in inertial space frame (and then $L = T - V$)

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2) \tag{4}$$

3. Simplified expression for the kinetic energy for an axially-symmetric body $I_1 = I_2 = I_\perp$

$$T = \frac{1}{2}I_\perp(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 \tag{5}$$

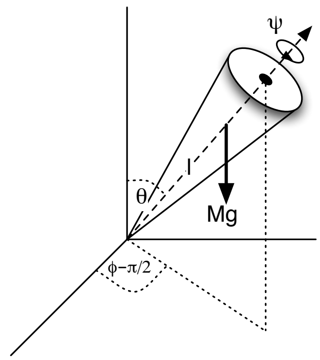
   From this the Lagrangian can be calculated, and so the equations of motion for the three Euler angles.

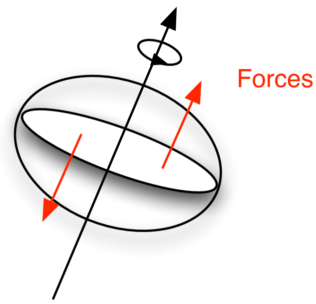
4. Alternatively the angular momentum expressed in terms of components along the principal axes is $\vec{L} \equiv (I_1\omega_1, I_2\omega_2, I_3\omega_3)$ and we can then use

$$\left.\frac{d\vec{L}}{dt}\right|_s = \left.\frac{d\vec{L}}{dt}\right|_b + \vec{\omega} \times \vec{L} = \vec{N}, \tag{6}$$

   with $d\vec{L}/dt|_b = (I_\perp\dot{\omega}_1, I_2\dot{\omega}_2, I_3\dot{\omega}_3)$.

Heavy symmetric top





As an application of the Euler angles consider the symmetric top. For the symmetric top θ and $\phi - \frac{\pi}{2}$ give the usual spherical polar coordinates of the symmetry axis and ψ a further rotation about the symmetry axis. In this lecture we will study the motion under a torque tending to rotate the symmetry axis so that there is a potential energy $V(\theta)$. One example is gyroscopes and toy tops where gravity acts on the center of mass a distance l from the pivot point so that

$$V(\theta) = MGl\cos\theta. \tag{7}$$

Another is the slow *precession of the equinoxes* of the Earth, due to the average gravitational effect of the moon and the sun acting on the equatorial bulge tending to rotate the symmetry axis of the Earth towards the perpendicular to the orbital plane. The potential energy takes the form (see Hand and Finch §8.11)

$$V(\theta) = \frac{1}{2}\left(\left.\frac{GM}{R^3}\right|_{\text{moon}} + \left.\frac{GM}{R^3}\right|_{\text{sun}}\right)(I_3 - I_\perp)\frac{1-3\cos^2\theta}{2}. \tag{8}$$

The gravitational forces act on the quadrupole moment of the mass distribution of the Earth, and the quantity $I_3 - I_\perp$ appears in evaluating this quadrupole moment. The quantity $GM/R^3|_{\text{sun}} = \Omega_e^2$ with $2\pi/\Omega_e$ the orbital period of the Earth (i.e. one year). The additional contribution from the moon is about twice as large. I write the sum as Ω^2 with $\Omega = 1.77 \times 2\pi/\text{year} = 3.53 \times 10^{-7}\text{s}^{-1}$. Then

$$V(\theta) = \tfrac{1}{2}I_\perp\Omega^2\epsilon \times \tfrac{1}{2}(1 - 3\cos^2\theta) \qquad \text{with} \qquad \epsilon = I_3/I_\perp - 1 \simeq 0.0034 \tag{9}$$

The Lagrangian is

$$L = \frac{1}{2}I_\perp(\dot\theta^2 + \dot\phi^2\sin^2\theta) + \frac{1}{2}I_3(\dot\psi + \dot\phi\cos\theta)^2 - V(\theta).$$

The angles ϕ, ψ are ignorable so the two conjugate (angular) momenta are constants of the motion

$$p_\psi = \frac{\partial L}{\partial\dot\psi} = I_3(\dot\psi + \dot\phi\cos\theta) = I_3\omega_3\,, \tag{10}$$

$$p_\phi = \frac{\partial L}{\partial\dot\phi} = I_\perp\dot\phi\sin^2\theta + I_3(\dot\psi + \dot\phi\cos\theta)\cos\theta\,. \tag{11}$$

The Euler-Lagrange equation for θ is (remember we must find this *before* using the constant p_ϕ, p_ψ to eliminate $\dot\psi, \dot\phi$)

$$I_\perp\ddot\theta - I_\perp\dot\phi^2\sin\theta\cos\theta + I_3(\dot\psi + \dot\phi\cos\theta)\sin\theta\,\dot\phi + \partial V/\partial\theta = 0, \tag{12}$$

and using Eq. (10) in the third term gives

$$I_\perp\ddot\theta - I_\perp\dot\phi^2\sin\theta\cos\theta + p_\psi\sin\theta\dot\phi + \partial V/\partial\theta = 0, \tag{13}$$

Now we use Eq. (11) to evaluate $\dot{\phi}$ in terms of constants and θ

$$\dot{\phi} = \frac{p_\phi - p_\psi \cos\theta}{\sin^2\theta} . \tag{14}$$

so that Eq. (13) is the single equation of motion for θ that we must solve.

Steady precession

Steady precession is given by $\dot{\theta} = 0$, $\dot{\phi} = \omega_p$ with ω_p the constant precession rate solving (from Eq. (13))

$$I_\perp \sin\theta \cos\theta\, \omega_p^2 - I_3\omega_3 \sin\theta\, \omega_p - \partial V/\partial\theta = 0, \tag{15}$$

writing the more intuitive $I_3\omega_3$ for p_ψ. This is a quadratic equation giving two possible steady precession rates for a given θ. (Of course, the top will need to be started with carefully chosen initial conditions to gives this motion, or some dissipation mechanism must damp out other motion —see below.) Often we are interested in the case where the top is "rapidly spinning" compared with the applied torque. The appropriate comparison is the range of the potential energy which we can estimate as $|\partial V/\partial\theta|$ to eliminate constant parts, and the kinetic energy of the spin $\frac{1}{2}I_3\omega_3^2$. For the rapidly spinning top the quadratic equation gives the solutions

$$\omega_p \simeq \frac{I_3\omega_3}{I_\perp \cos\theta}, \quad \omega_p \simeq -\frac{1}{I_3\omega_3 \sin\theta}\frac{\partial V}{\partial\theta}. \tag{16}$$

The first value is $\omega_p \sim \omega_3$ and is independent of the applied torque in the approximation used. This is just a special case of the Chandler wobble (from the space frame perspective) where the amplitude of the motion is the same as the tilt angle. The second expression is the slow precession $\omega_p \ll \omega_3$ that is the familiar gyroscopic motion. For the spinning top $\omega_p \simeq Mgl/I_3\omega_3$, as found from simple arguments. The precession rate of the equinoxes is given using Eq. (9) and is

$$\omega_p = -\frac{3}{2}\epsilon\frac{\Omega^2\cos\theta}{\omega_3}, \tag{17}$$

(using $I_\perp \simeq I_3$). Putting in $\epsilon = 0.0034$, $\Omega/\omega_3 \simeq 1.77/365$, and $\theta = 23.4°$ gives $\omega_p/\Omega_e = (24891)^{-1}$ so that the predicted precession period is 24891 years compared with the measured value of 25730 years (a 3% difference).

Motion from given initial condition

More generally we look for the solution from given initial conditions $\dot{\psi}, \dot{\phi}, \theta, \dot{\theta}$. Use Eqs. (10,11) to fix the constant angular momenta. The Lagrangian is a quadratic form in the angular velocities $\dot{\phi}, \dot{\theta}, \dot{\psi}$ and has no explicit time dependence so $H = T + V = E$ with E the constant energy. It is convenient to use the constant E'

$$E' = E - \frac{1}{2I_3}p_\psi^2 = \frac{I_\perp}{2}\dot{\theta}^2 + V_{\text{eff}}(\theta) \tag{18}$$

and

$$V_{\text{eff}}(\theta) = \frac{1}{2I_\perp}\frac{(p_\phi - p_\psi\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta . \tag{19}$$

Remember p_ϕ, p_ψ and E (and so E') are constants of the motion set by the initial conditions. Therefore we have reduced the problem to a one dimensional problem of the dynamics of θ in the effective potential $V_{\text{eff}}(\theta)$. We can try to integrate the equation for $\dot{\theta}$ or can proceed qualitatively as in Lecture 8.[1]

[1] For a fuller understanding, it is useful to introduce $u = \cos\theta$ and perform the analysis in terms of u. The effective potential for u is a cubic polynomial, which simplifies the work. For this approach see Hand and Finch §8.10. For simple results on precession and nutation, a direct analysis in terms of θ is sufficient.

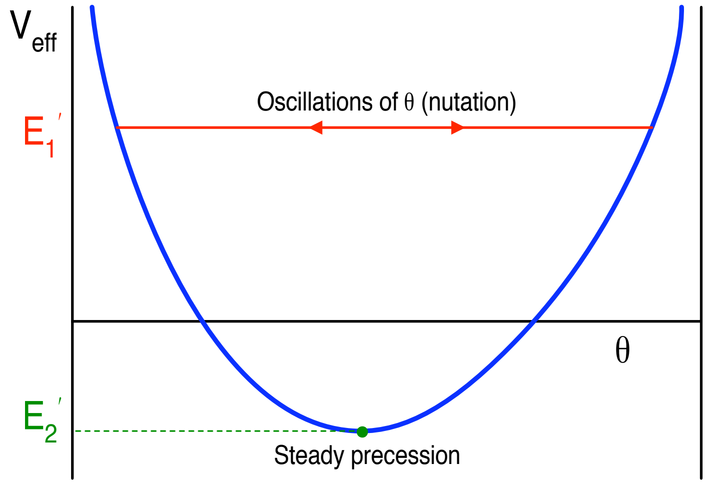


Figure 1: Effective potential for heavy top.

An obvious feature of $V_{\text{eff}}(\theta)$ is that it diverges positively at the ends of the range $\theta = 0, \pi$ and is bounded below by a finite value $-Mgl$. It therefore has a minimum somewhere between (and maybe more than one). Figure 1 shows a sketch of V_{eff} for some choice of parameters, and examples of two energies E' giving oscillations of θ called nutation as well as precession $\dot{\phi} \neq 0$ (red line), or just steady precession $\dot{\phi} \neq 0, \dot{\theta} = 0$ (green dot). Since $\dot{\phi}$ is given by Eq. (14), it may or may not change sign during the variation of θ depending on p_ϕ / p_ψ and the range of θ, and so the precession may be monotonic, or involve backwards and forwards motion (see Fig. 2). For the common initial condition of $\dot{\theta} = \dot{\phi} = 0$ the motion involves cusps (third panel).

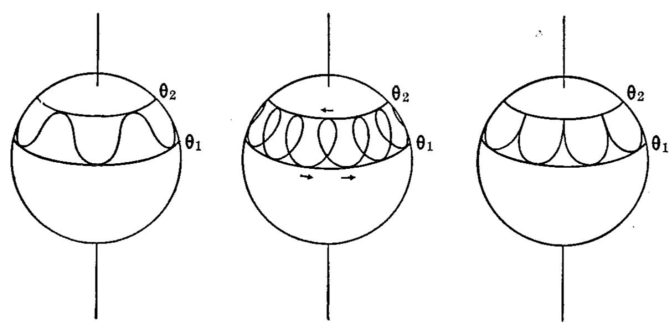


Figure 2: Nutation motion of heavy top. The third panel corresponds to a top released from rest $\dot{\phi} = \dot{\theta} = 0$.

We can investigate the small periodic nutational motion of θ near the uniformly precessing solution by expanding V_{eff} about the minimum to quadratic order. For the large spin case the leading order term is $V''_{\text{eff}} \simeq p_\psi^2 / I_\perp$. Together with the $\dot{\theta}^2$ term in the energy E', this gives simple harmonic motion of θ at the frequency $p_\psi / I_\perp = I_3 \omega_3 / I_\perp$. For the Earth example this is the Chandler wobble in the space frame at a frequency close to 2π/day. There are very small $O(\varepsilon)$ corrections from the gravitational torque.

For given p_ϕ, p_ψ and E Eq. (18) for $\dot{\theta}$ can in principal be integrated, although only numerically in general. Some special cases and limits can be investigated analytically. A simple example is an initial condition of $\theta = \dot{\theta} = 0$, i.e. the top spinning about a vertical axis. For this case $p_\phi = p_\psi$ and the effective potential reduces to

$$V_{\text{eff}}(\theta) = \frac{p_\psi^2}{2I_\perp}\frac{(1-\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta\ . \tag{20}$$

Expanding for small θ

$$V_{\text{eff}}(\theta) = Mgl + \frac{\theta^2}{2}\left(\frac{p_\psi^2}{4I_\perp} - Mgl\right). \tag{21}$$

The $\theta = 0$ equilibrium is *stable* for $p_\psi^2 > 4I_\perp Mgl$ and *unstable* for $p_\psi^2 > 4I_\perp Mgl$. A fast spinning top will rotate smoothly about the vertical (called a "sleeping top"). Friction will gradually decrease p_ψ, and when it reaches the critical value $2\sqrt{I_\perp Mgl}$ the top will begin to wobble and precess.

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