Lecture 4: Fluid statics, cont'd

- Hydrostatic pressure
- Standard atmosphere
- Manometers
- Force acting on submerged surface
- Buoyancy

$$-\nabla p + \rho \vec{g} = 0$$

Surface force due to pressure

Body force due to gravity

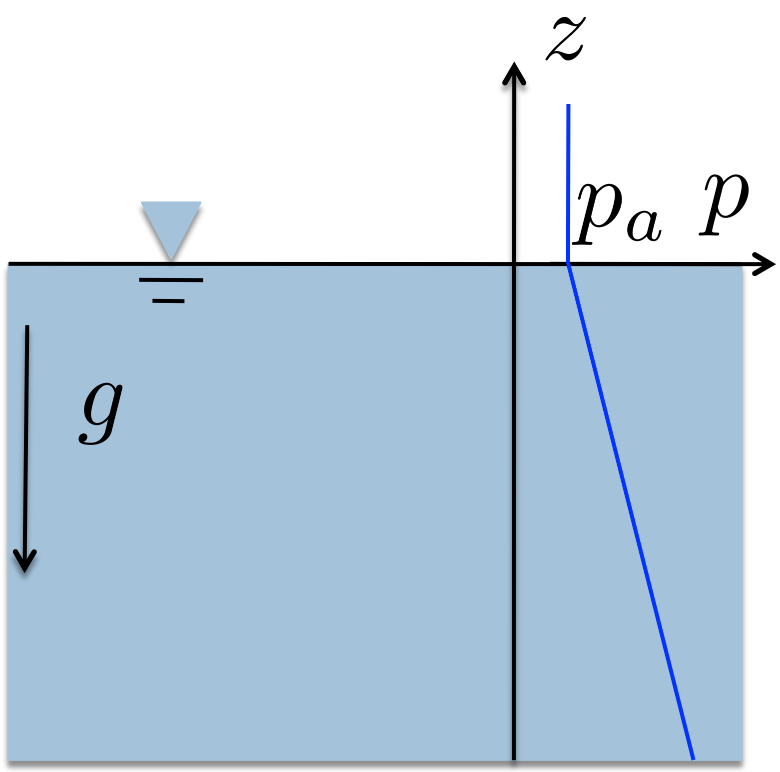
$$-\int_{\partial B} p \vec{n} dA + \int_{B} \rho \vec{g} dV = 0$$

Hydrostatic pressure

- In liquids (water in particular) the density is very nearly constant (998 kg/m^3 at STP for $H_2$0)
- Thus the pressure gradient is a constant in the direction of gravity.
- In liquids, to a very good approximation, pressure increases linearly in the direction of gravity

$$-\nabla p + \rho \vec{g} = 0$$

Hydrostatic pressure, cont'd



$$-\nabla p + \rho \vec{g} = 0$$

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0 \quad \therefore \quad p = p(z)$$

$$\begin{aligned} \frac{dp}{dz} &= -\rho g \\ p &= -\rho g z + \text{const} \\ p &= -\rho g z + p_a \end{aligned}$$

Hydrostatic pressure, cont'd

- Pressure at any elevation is simply the weight of the column of fluid above that point
- Water

$$g = 9.8 \text{ m/s}^2$$

$$\rho = 998 \text{ kg/m}^3.$$

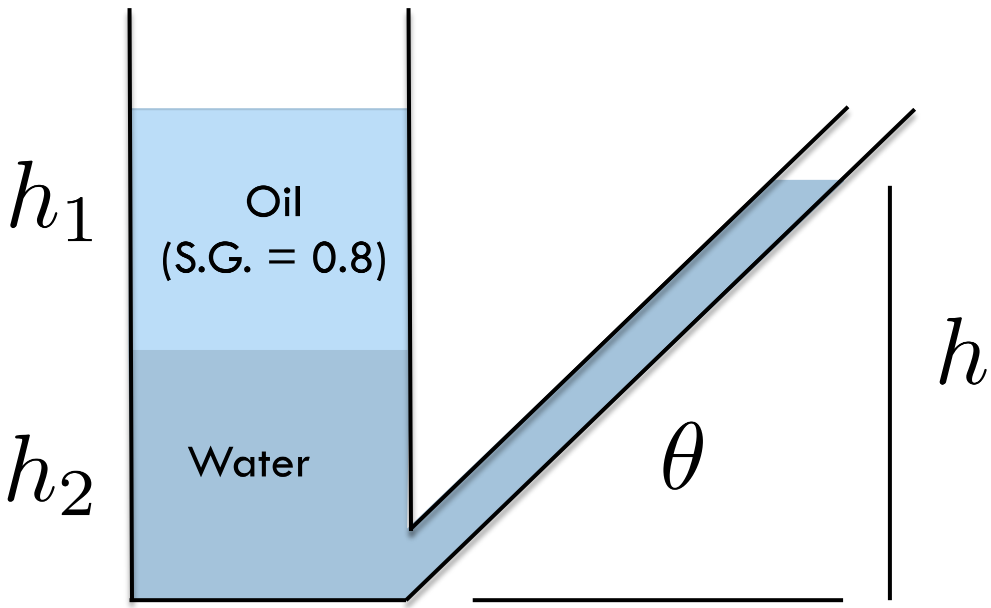
$$p_a = 1 \text{ atm} = 101{,}325 \text{ Pa} = 14.7 \text{ psi}$$

  - Each 10.33 m of depth is 1 atm of pressure

Corollaries of the hydrostatic balance

- Pascal's law: Equal pressure at equal elevation in the same (contiguous) fluid (of constant density)
- The pressure at a horizontal interface between fluids of different densities is equal (force balance as volume shrinks to a plane)
- These statements neglect surface tension which arises at the free surface between different immiscible fluids
  - Surface tension can dominate the force balance small scales O(mm) and down

Example: similar to White P 2.36



$$SG = \frac{\rho}{\rho_{H_2O}}$$

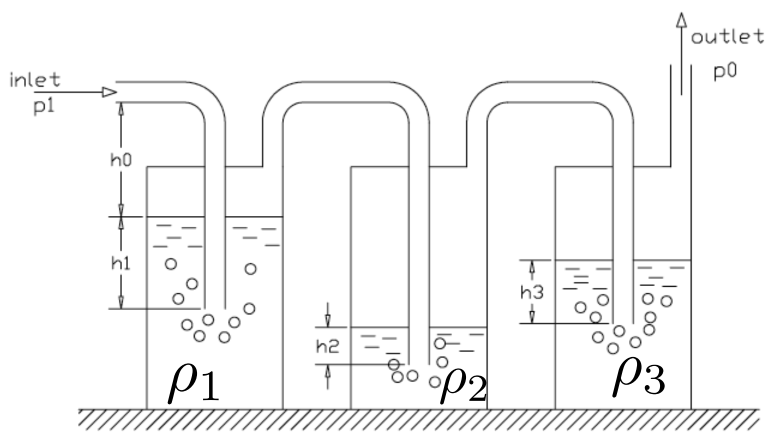
"Gravy separator"

$$h = (SG)_{oil} h_1 + h_2$$

Example: gas cleaning plant

- Gas is slowly bubbled through various liquids
- The density of the gas is negligible compared to the liquid densities

1. Find the minimum p_1 such that gas flows through the system
2. How large should h_0 be so that if p_1 is reduced to p_0, no liquid spills out of the inlet?



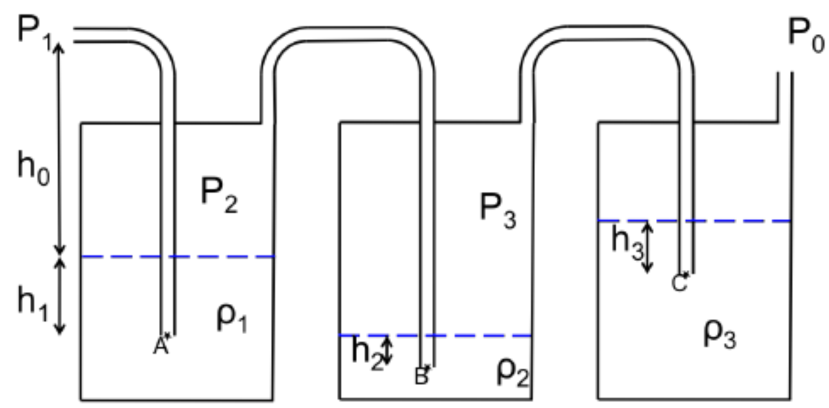
Consider static case

- Neglect all pressure variations in the gas
- The pressure at point C is $p_0 + \rho_3 g h_3 \quad \therefore \quad p_3 = p_0 + \rho_3 g h_3$
- The pressure at point B is

$$p_3 + \rho_2 g h_2 \quad \therefore \quad p_2 = p_0 + \rho_2 g h_2 + \rho_3 g h_3$$

- Similarly, then, we find

$$p_1 = p_0 + \sum_{i=1}^{3} \rho_i h_i$$



If $p_1 = p_0$

- Fluid 1 will rise to a height above the entrance at A such that

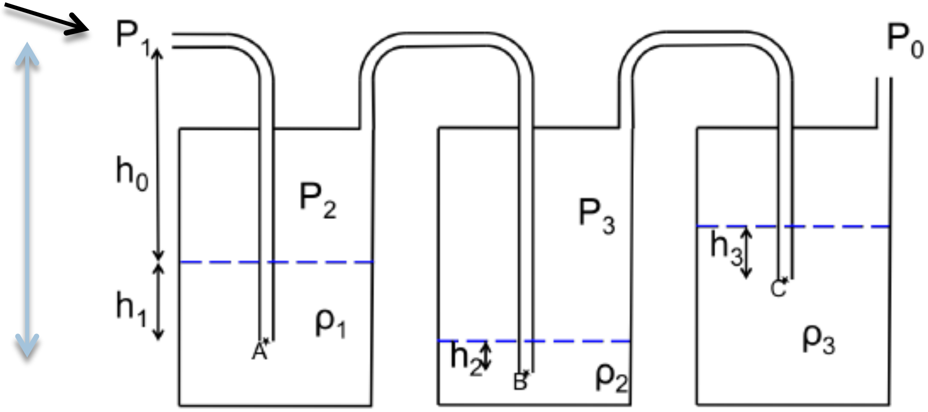
$$p_0 + \rho_1 g h = p_2 + \rho_1 g h_1 = p_0 + \rho_1 g h_1 + \rho_2 g h_2 + \rho_3 g h_3$$

$$h = h_0 + h_1$$

$$\boxed{h_0 = \frac{\rho_2 h_2 + \rho_3 h_3}{\rho_1}}$$

Change to p_0

$$h = h_0 + h_1$$



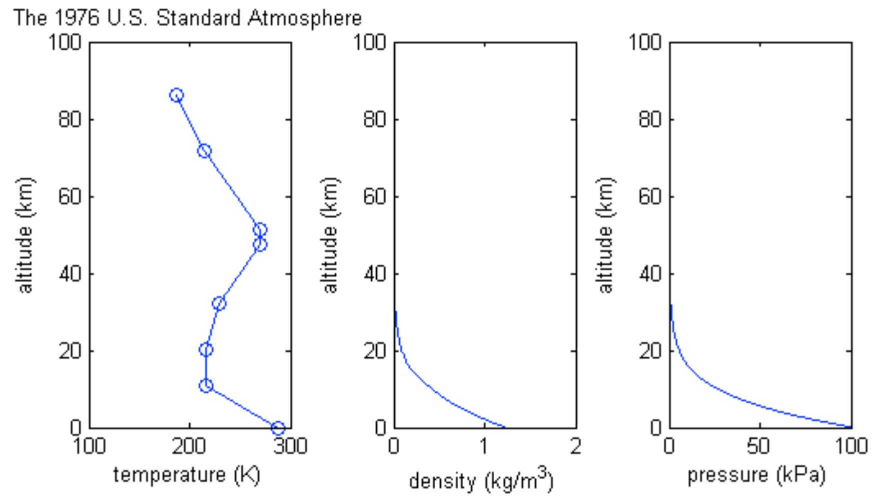
Standard atmosphere

- Over short distances we usually approximate the pressure (near the surface of earth) as a constant
- Over long distances density varies with elevation

$$\frac{dp}{dz} = -\rho(z)g$$
$$dp = -\rho(z)dz$$

Integrate from surface with known density variation

To get the density variation, we have to know something about atmospheric temperature as well, then the perfect gas law will allow us to solve these relations



Standard atmosphere, cont'd

- Example: Determine the mass of Earth's atmosphere

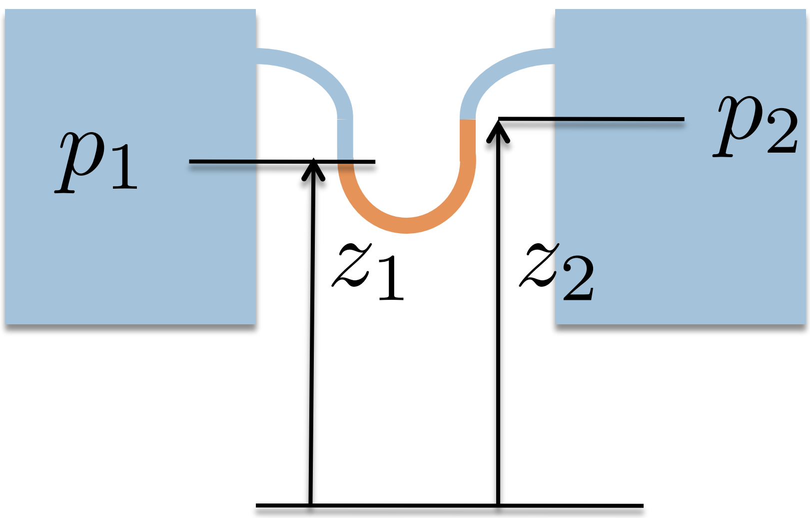
Standard atmosphere, cont'd

- Example: Determine the mass of Earth's atmosphere

$$M_a g = p_a A_{surf}$$

$$\begin{aligned} R_{surf} &= 6370 \times 10^3 \text{ m} \\ A_{surf} &= 4\pi R_{surf}^2 \\ M_a &= 5.27 \times 10^{18} \text{ kg} \end{aligned}$$

Manometry



- Length units for pressure (column of fluid)
- 1 atm =
  - 760 mm Hg
  - 10.33 m H_20

$$
\begin{aligned}
p_1 - p_2 &= -\rho_m g\left(z_1 - z_2\right) \\
\text{Let} \quad \gamma &= \rho g \qquad \text{Specific weight} \\
p_1 - p_2 &= \gamma_m h
\end{aligned}
$$

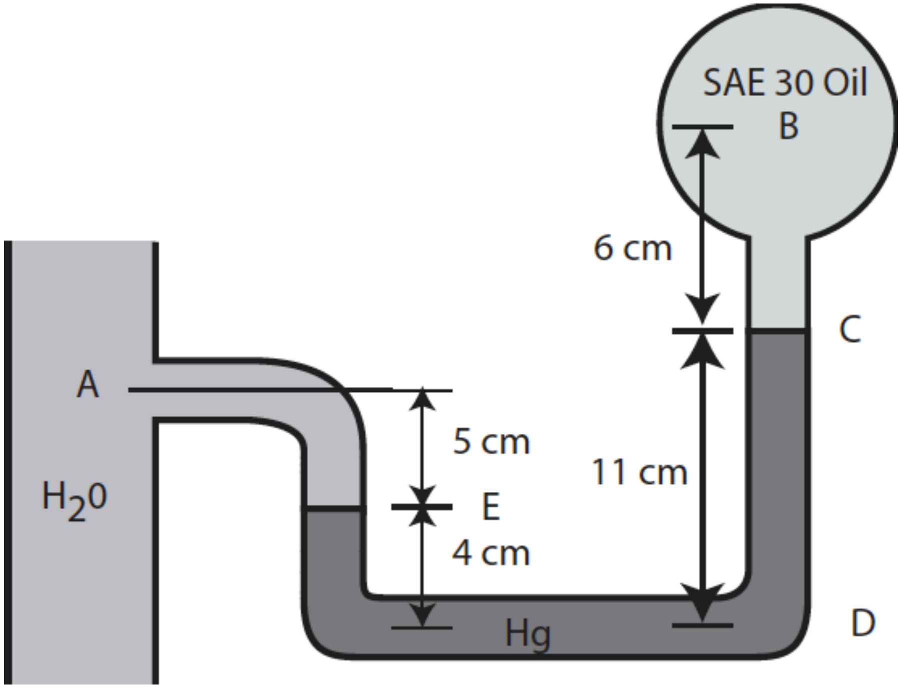
Assume density constant in manometer liquid

Manometry (2)

- Immiscible fluids (non-diffusing)
- Heavier fluids minimize the change in elevation required for a given pressure difference (hence mercury)
- Lighter fluids increase sensitivity
- Static stability: light over heavy!
- Meniscus due to surface tension will cause an error unless there are two of the same fluid combination interfaces on either side of the U

Manometry (3)

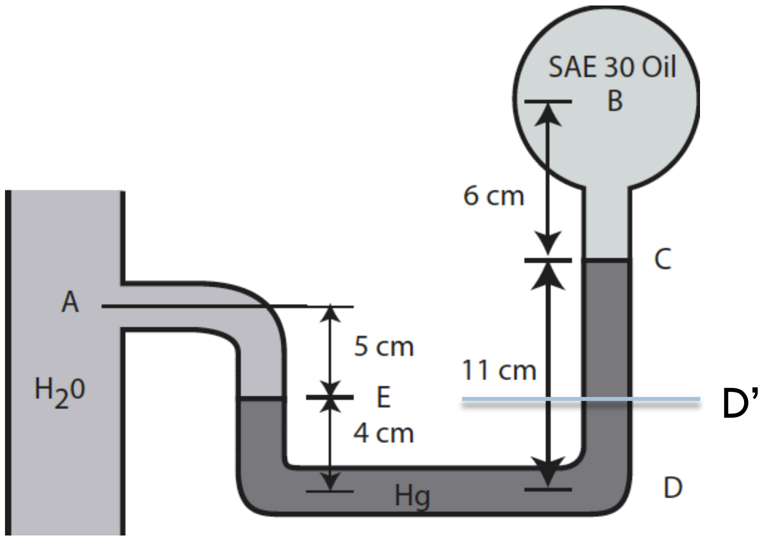
- Example.
- $p_B = 87$ kPa
- Find p_A



$$\gamma_{Hg} = 1.33 \times 10^5 \quad \text{N / m}^3$$
$$\gamma_{H_2O} = 9.79 \times 10^3 \quad \text{N / m}^3$$
$$\gamma_{SAE30} = 8.72 \times 10^3 \quad \text{N / m}^3$$

Manometry (4)

- Solution

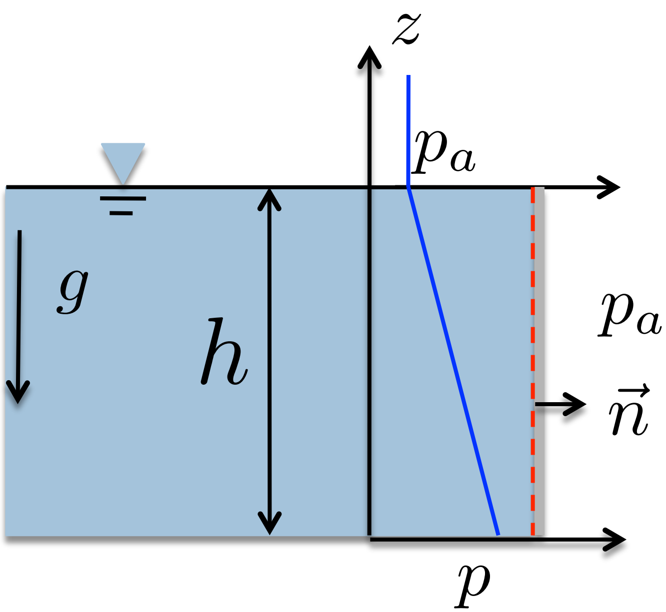


$$
\begin{aligned}
p_A &= p_E - \gamma_{H_2O}\overline{AE} \\
&= p_{D'} - \gamma_{H_2O}\overline{AE} \\
&= p_C + \gamma_{Hg}\overline{CD'} - \gamma_{H_2O}\overline{AE} \\
&= p_B + \gamma_{SAE30}\overline{BC} + \gamma_{Hg}\overline{CD'} - \gamma_{H_2O}\overline{AE} \\
&= 870000 + 8720(0.06) + \\
&+ 133100(0.07) - 9790(0.05) = 96400 \quad \text{KPa}
\end{aligned}
$$

Forces on immersed plane surfaces

- Simplest case

$$\vec{F}_{p,fluid} = -\int p\vec{n}dA$$
$$= -\int_{-h}^{0} (p_a - \rho g z)\,\vec{n}dz$$

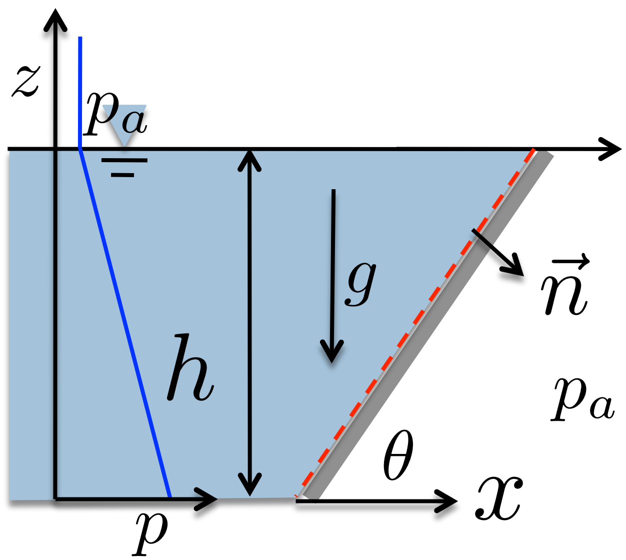


$$\vec{F}_{p,solid} = -\vec{F}_{p,fluid} - p_a A\vec{n}$$
$$= \int_{-h}^{0} (-\rho g z)\,\vec{n}dz$$
$$= \frac{1}{2}\rho g h^2 \ \hat{i}$$

(This is force per unit depth into the page)

Forces on immersed plane surfaces (2)

- More general case



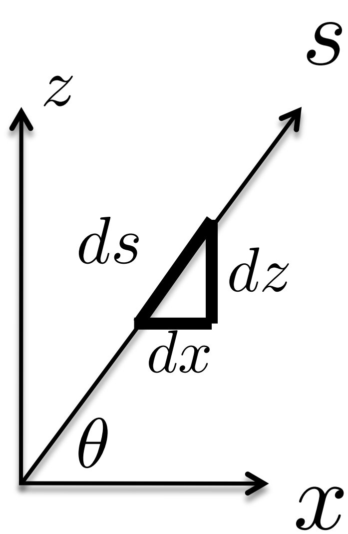
$$p = p_a - \rho g (z - h)$$

- Force on solid

$$\vec{F}_{p,solid} = \int (p - p_a)\, \vec{n} dA$$

$$\vec{n} = \begin{bmatrix} \sin\theta \\ 0 \\ -\cos\theta \end{bmatrix}$$

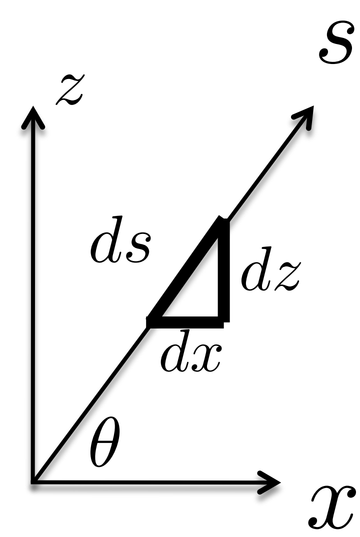
$$\begin{aligned} dz &= \sin\theta ds \\ dx &= \cos\theta ds \end{aligned}$$



Forces on immersed plane surfaces (3)

- Do x component

$$
\begin{aligned}
F_x &= \int_0^{h/\sin\theta} (p - p_a)\sin\theta ds \\
&= \int_0^{h/\sin\theta} (p - p_a)\frac{dz}{ds} ds \\
&= \int_0^h (p - p_a) dz \\
&= -\int_0^h \rho g (z - h) dz \\
&= \frac{1}{2}\rho g h^2
\end{aligned}
$$

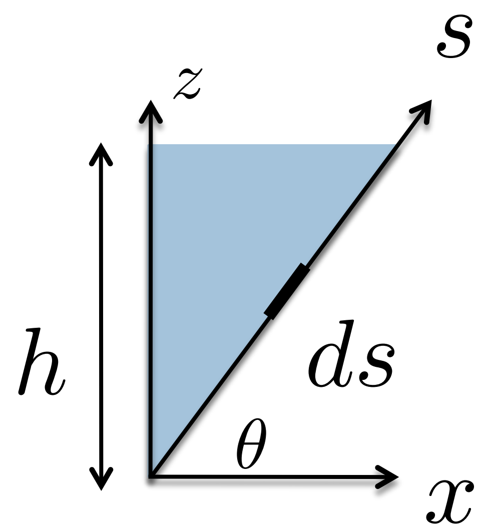


Same answer as vertical wall.

Forces on immersed plane surfaces (4)

- Do z component

$$
\begin{aligned}
F_z &= -\int_0^{h/\sin\theta} (p - p_a)\cos\theta \, ds \\
&= -\int_0^{h/\sin\theta} (p - p_a)\frac{1}{\tan\theta}\frac{dz}{ds} ds \\
&= -\frac{1}{\tan\theta}\int_0^{h} (p - p_a) dz \\
&= \frac{1}{\tan\theta}\int_0^{h} \rho g(z - h) dz \\
&= -\frac{1}{2}\frac{1}{\tan\theta}\rho g h^2
\end{aligned}
$$



(minus) Weight of fluid above surface.
Note that if we do not have atmospheric pressure below surface, then we should also include the weight of atmosphere above fluid

Forces on immersed plane surfaces (5)

- Moments

Force per unit area

$$\sum \vec{M}_0 = \int \vec{r} \times \vec{f} dA$$

$$\vec{f} = (p - p_a)\, \vec{n}$$

Force on body

$$\vec{r} \times \vec{f} = (p - p_a)\,(\vec{r} \times \vec{n})$$

$$= (p - p_a) \left(-s\hat{j} \right)$$

Right-hand-rule

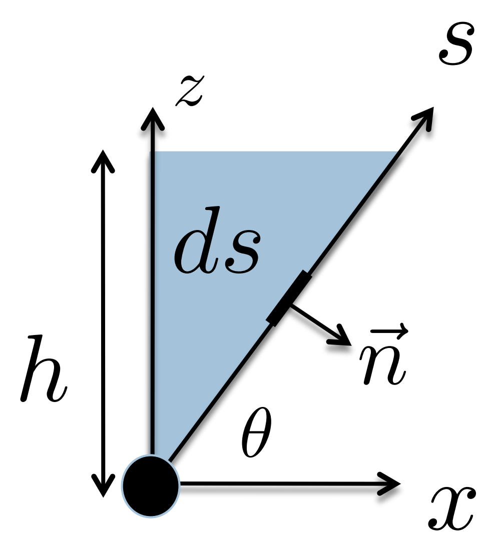
$$M_{x0} = M_{z0} = 0$$

$$M_{y0} = -\int_0^{\frac{h}{\sin\theta}} (p - p_a)\, s ds$$

Plug in hydrostatic pressure and evaluate integral

$$= \rho g \frac{1}{\sin^2\theta} \frac{h^3}{6}$$

Moment per unit depth into page



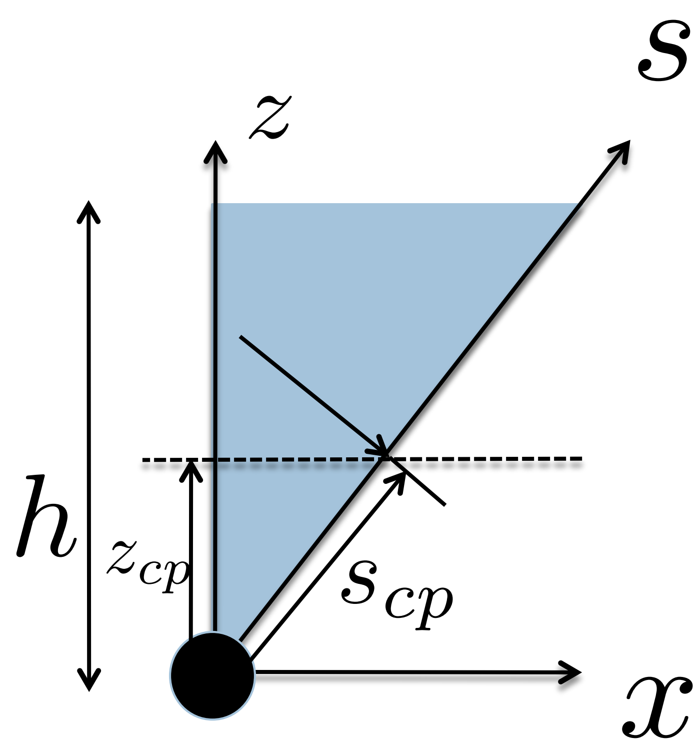
Forces on immersed plane surfaces (6)

- Center of pressure

$$s_{cp}|\vec{F}| = M_{y0}$$

$$s_{cp} = \frac{\frac{\rho g h^3}{6\sin^2\theta}}{\frac{\rho g h^2}{2\sin\theta}} = \frac{h}{3}\frac{1}{\sin\theta}$$

$$z_{cp} = h/3$$



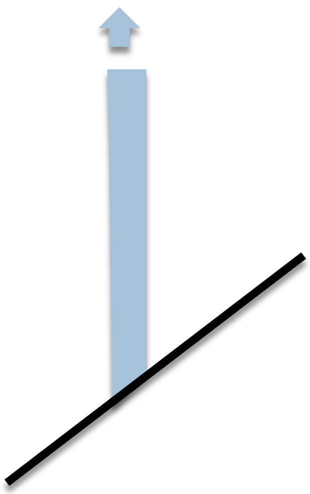
Forces on any immersed plane surfaces

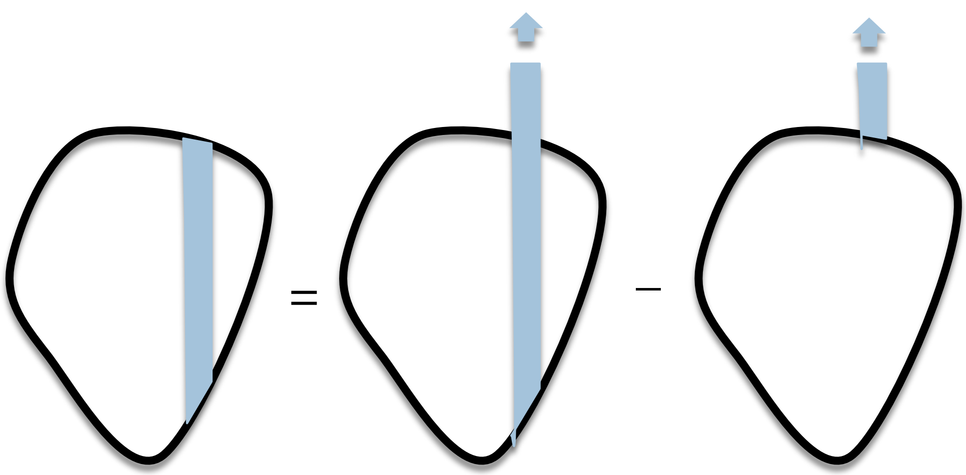
- Summary plane surface
  - Horizontal force same as that on vertical surface with same projected area
  - Vertical force: minus weight of fluid above surface
  - Force acts at an elevation of h/3

- The first two of these are general principles that apply even to curved surfaces.
- In general, the only difficult part of this problem is to figure out how to evaluate the integral for complicated surfaces

Buoyancy

- Archimedes: a immersed (totally or partially) body suffers a buoyancy force that acts *opposite* to gravity with a magnitude equal to *the weight of the displaced fluid.*
- We really already have this result…force on immersed surface equal to (minus the) weight of column of fluid above





Archimedes proof

- Consider very general case of a stratified fluid, i.e. density varies with elevation (direction opposite gravity)

$$\rho = \rho(z)$$

- Note that Pascal's law still applies (equal pressure at equal elevation), since the pressure is still *only* a function of z

$$-\nabla p + \rho \vec{g} = 0$$

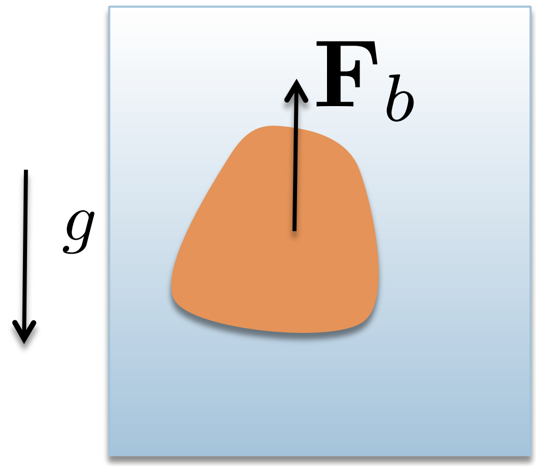
$$-\frac{\partial p}{\partial z} - \rho(z) g = 0 \qquad \frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$$

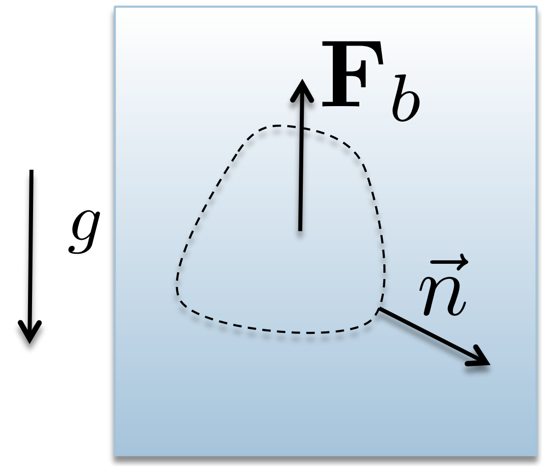
Write out components

$$-\frac{dp}{dz} - \rho(z) g = 0 \qquad p = -\int \rho(z) g dz + \text{const}$$

Integrate

Trick





Note that the pressure in the fluid outside the body is the same whether the body exists or is erased and replaced with fluid.

The proof is now simpler because it turns out to be easier to evaluate the force acting on the fluid inside the body.

$$\vec{F}_b = -\int_{\partial B} p\vec{n}dA$$

This will just be minus the force acting on the fluid outside.

Which is in turn minus the force acting on the actual body.

Now it is a two liner…

$$\vec{F}_b = -\int_{\partial B} p\vec{n}dA$$

$$= -\int_B \nabla p dA$$

Corollary of div. thm.

$$= -\int_B \rho\vec{g}dV$$

Buoyancy force is minus the weight of the displaced fluid (QED).

We never had to make density a constant or even find the pressure

How big?

- Consider a beach ball of diameter 0.5m. The mass of the vinyl is probably about ~50g = 0.05 kg.
- The volume is

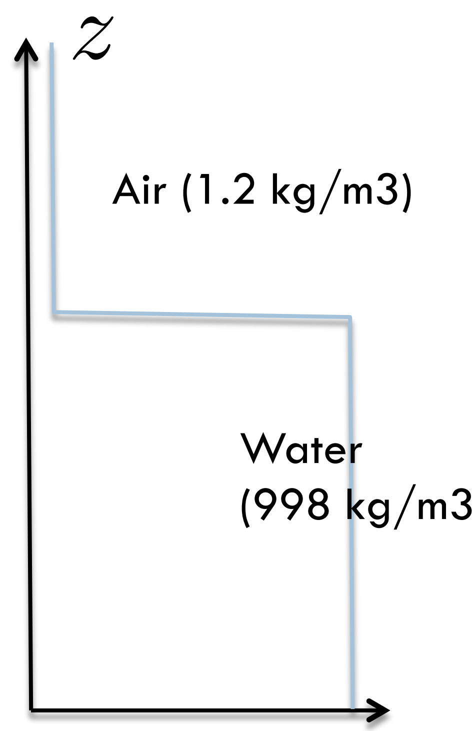
$$\frac{4}{3}\pi R^3 = 0.0654\ \mathrm{m}^3$$

- The air inside (assume it is just barely inflated) has mass

$$1.2\mathrm{kg/m}^3 \times 0.0654\mathrm{m}^3 = 0.0785\mathrm{kg}$$

- The total mass is thus about 0.13 kg
- The mass of displaced water is 65 kg!
- To hold the beach ball underwater I need to (stably) push down with a force of 9.8*(65-0.13) N.
- The same beach ball above the surface experiences a buoyancy force of only 9.8*0.0785 N (duh…)

Iceberg, revsited

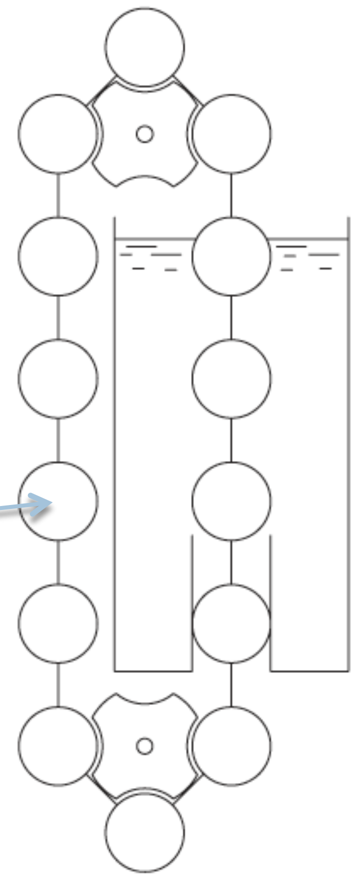


The buoyancy force from the part above the water is negligibly small compared to the part below, we usually neglect it

$$\rho = \rho(z)$$

Example

- Perpetual motion machine (?)

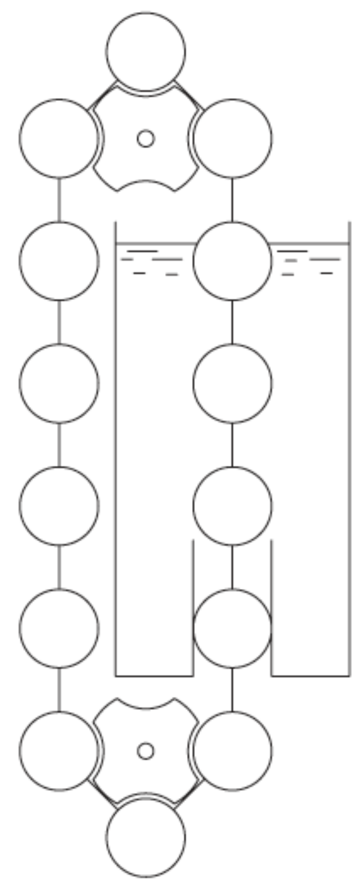


Hollow spheres.
Buoyancy force on submerged sphere is much greater than their weight

Example, cont'd

- The weight of the spheres balances on either side, so the condition for the machine to rotate clockwise is that the net force from the fluid on the spheres is positive.

$$\mathbf{F}_{net} > 0$$



Example, cont'd

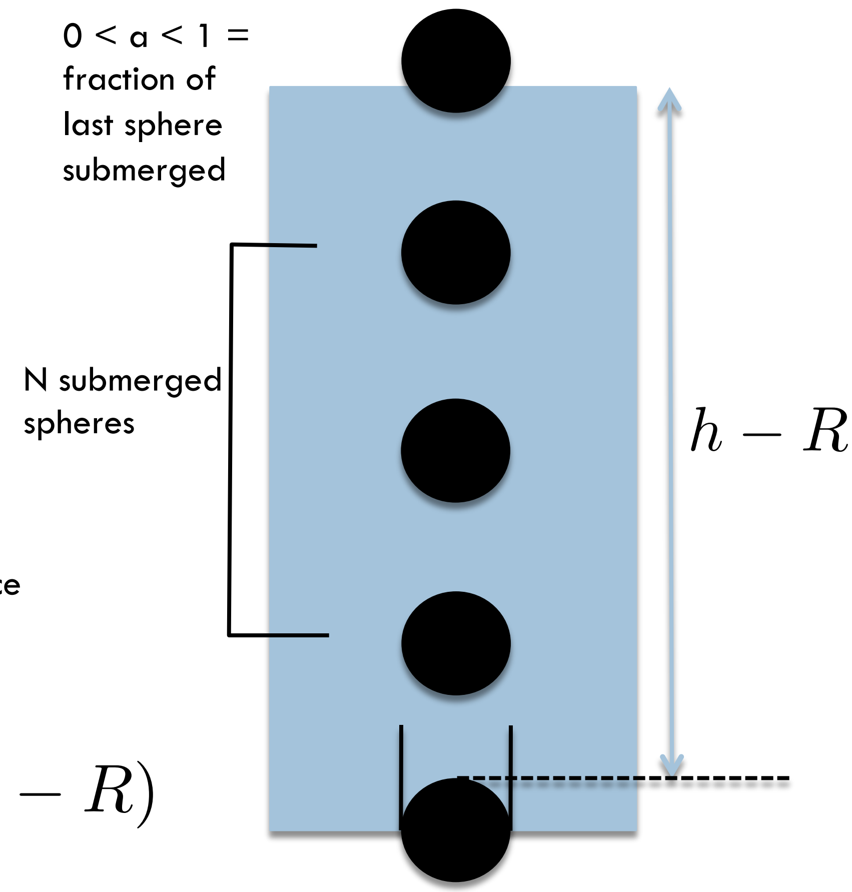
- Consider the case when one sphere is just entering the gate

$$
\begin{aligned}
F_{net} &> 0 \\
\rho g V(N+a) &> \rho g A(h-R)
\end{aligned}
$$

(conservative…actual force slightly larger)

$$
\begin{aligned}
V &= \frac{4}{3}\pi R^3 \\
A &= \pi R^2 \\
\frac{V}{A} &= \frac{4}{3}R
\end{aligned}
$$

$$\frac{4}{3}R(N+a) > (h-R)$$



Example, cont'd

$$\frac{4}{3}R(N+a)+R \quad > \quad h$$

$$h > (N+1)L$$

L is center to center spacing of balls, so L / R > 2

$$\frac{4}{3}(N+a)+1 > (N+1)\frac{L}{R}$$

(worst case, set a=1)

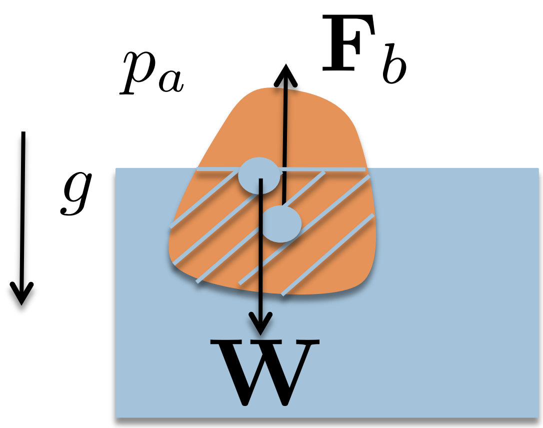
$$\frac{L}{R} < \frac{\frac{4}{3}N+\frac{7}{3}}{N+1}$$

$$\frac{L}{R} < \begin{cases} \frac{11}{6} & N=1 \\ \frac{15}{9} & N=2 \\ \frac{4}{3} & N \quad \text{large} \end{cases}$$

All are smaller than 2, so we conclude that there is no net upthrust. In fact, it will turn the other way, and the water will drain → no perpetual motion

Buoyancy (3)

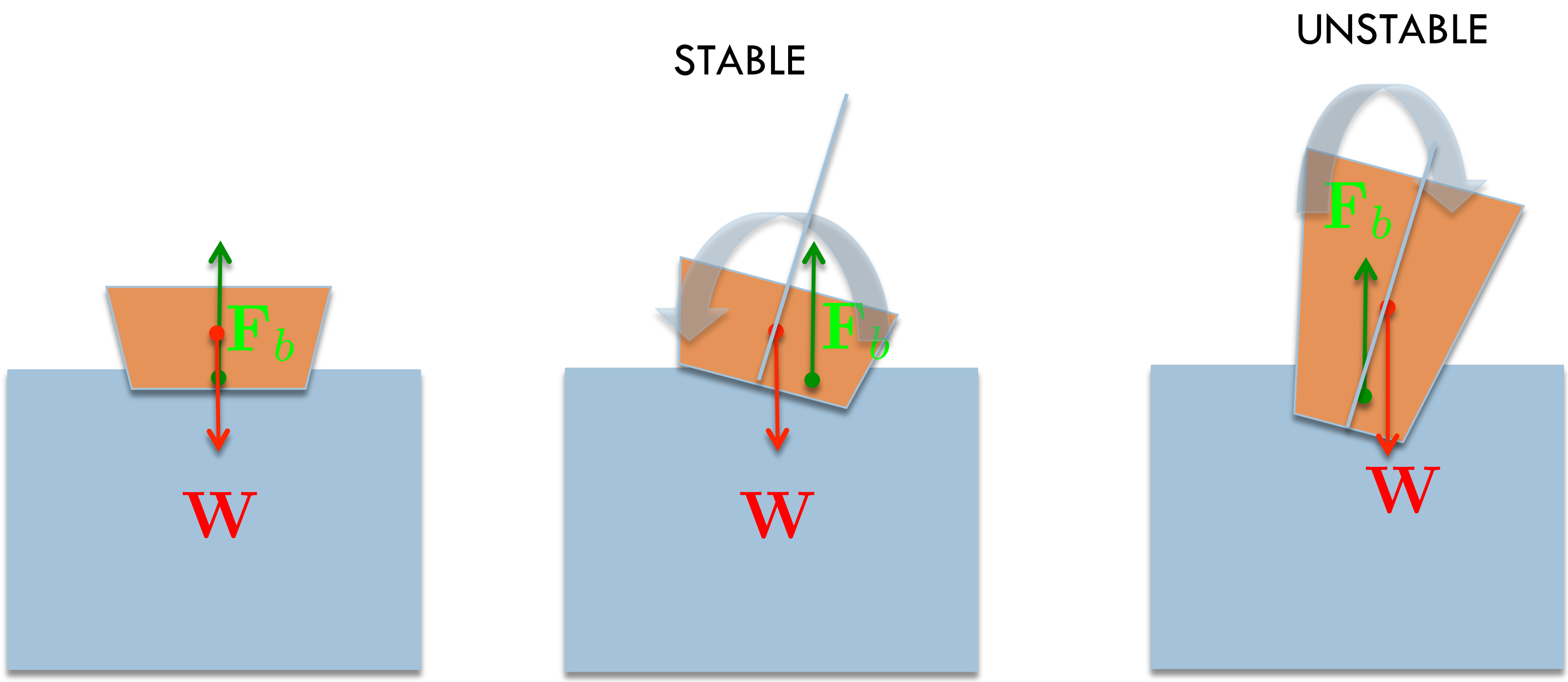
- Note: If the body is static, then the sum of forces must be zero, hence the buoyancy force must also be equal to the weight of the body (i.e. it is "neutrally buoyant" or "floats")
- Center of mass and buoyancy



$$\vec{x}_{CM} = \frac{\int_B \rho_b \vec{x} dV}{\int_B \rho_b dV}$$

$$\vec{x}_{CB} = \frac{\int_B \rho_f \vec{x} dV}{\int_B \rho_f dV}$$

Stability of floating bodies (or why not to stand up in your rowboat)



In this case the moment tends to right the vessel (counters the disturbance)

In this case the moment tends to overturn the vessel (enhances the disturbance)

Summary

- The hydrostatic pressure distribution produces forces on immersed surfaces and immersed bodies
  - Planar surfaces: easy to evaluate integrals to determine net forces, moments, and center of pressure
  - Immersed bodies (fully or partially submerged) : Archimedes principle, buoyancy force
  - The buoyancy force must balance the weight of the body (and have the same line of action) if it is static, but, depending on the relative locations of the center of gravity and center of buoyancy, it may be statically stable or unstable