ACM 100b

Special cases for the Frobenius expansion

Dan Meiron

Caltech

January 29, 2014

Some potential difficulties

- Look at the Frobenius recursion relation:

$$P(\alpha+n)a_n = -\sum_{k=0}^{n-1}\left[(\alpha+k)p_{n-k}+q_{n-k}\right]a_k \qquad n=0,1,2,\ldots.$$

  and the associated indicial equation

$$P(\alpha)=\left[\alpha^2+(p_0-1)\alpha+q_0\right]=0.$$

- First, if we get double roots, then we seem to have only one solution.
- A less obvious problem occurs if the roots differ by an integer.
- Suppose we start the recursion relation with the lesser of the two roots.
- Then at some integer N in the recursion relation the polynomial $P(\alpha+N)$ will vanish
- In that case it looks like we will not be able to solve for a_N.

The case of repeated roots

- We know from our study of Euler-type equations that when we have double roots in the indicial equation we can expect solutions of the form
$$(x - x_0)^\alpha \text{ and } (x - x_0)^\alpha \ln(x - x_0).$$
- We'll show that something similar happens for Frobenius expansions when we get double roots.
- To see what happens we write our series solution in the following form:
$$y(x, \alpha) = (x - x_0)^\alpha \sum_{n=0}^{\infty} a_n(\alpha)(x - x_0)^n.$$
- In other words, we're letting the exponent vary freely.
- This is not going to be a solution except for special values of α.
- But it is useful to see how these special values come about.

The case of repeated roots

- Let's define

$$\mathcal{L}[\ ] \equiv \frac{d^2}{dx^2}[\ ] + \frac{p(x)}{(x-x_0)}\frac{d}{dx}[\ ] + \frac{q(x)}{(x-x_0)^2}[\ ].$$

- So in this notation our ODE is

$$\mathcal{L}[y(x)] = 0.$$

- With a little algebra you can show that

$$\mathcal{L}[y(x,\alpha)] = a_0(x-x_0)^{\alpha-2}P(\alpha),$$

where $P(\alpha)$ is the indicial polynomial.

The case of repeated roots

- Now look at the equation

$$\mathcal{L}[y(x,\alpha)] = a_0(x - x_0)^{\alpha-2} P(\alpha),$$

- We see that if we set α to be a root of the indicial equation we can get a solution.
- But this also suggests how we might find a second solution in the case of a double root.
- When there is a double root, say $\alpha = \alpha_1$, our expression above becomes

$$\mathcal{L}[y(x,\alpha)] = a_0(x - x_0)^{\alpha-2} C(\alpha - \alpha_1)^2,$$

where C is a constant.

The case of repeated roots

- This relation

$$\mathcal{L}[y(x,\alpha)] = a_0(x - x_0)^{\alpha-2}C(\alpha-\alpha_1)^2,$$

suggests that not only is $y(x,\alpha_1)$ a solution as we already know, but it would also be the case that

$$\mathcal{L}\left[\left.\frac{\partial}{\partial\alpha}y(x,\alpha)\right|_{\alpha=\alpha_1}\right] = 0,$$

- Because near a double root $P(\alpha)$ vanishes quadratically meaning the value vanishes but the derivative (with respect to α) also vanishes.

The case of repeated roots

- This means
$$\left.\frac{\partial}{\partial\alpha}y(x,\alpha)\right|_{\alpha=\alpha_1}$$
is also a solution.
- But this is
$$\left.\frac{\partial}{\partial\alpha}y(x,\alpha)\right|_{\alpha=\alpha_1} = y(x,\alpha_1)\ln(x-x_0) + \sum_{n=0}^{\infty} b_n(x-x_0)^{\alpha_1+n},$$
where
$$b_n = \left.\frac{\partial}{\partial\alpha}a_n(\alpha)\right|_{\alpha=\alpha_1}.$$
- It is tedious but not hard to show that this new solution is linearly independent
- So this is the second solution we are looking for.

An example of repeated roots

- As an example of this approach we consider the modified Bessel equation of order 0:
$$y'' + \frac{y'}{x} - y = 0.$$
- We can see that there is a regular singular point at $x = 0$.
- In this case we have
$$p_0 = 1 \qquad p_n = 0 \quad n = 1, 2, \ldots$$
and
$$q_0 = 0 \quad q_1 = 0 \quad q_2 = -1 \qquad q_n = 0 \quad n = 3, 4, \ldots.$$
- The indicial equation is
$$\alpha^2 = 0$$
so $\alpha = 0$ is a double root.

An example of repeated roots

- The recursion relation can be shown to be
$$[(\alpha + n)^2]a_n = -\sum_{k=0}^{n-1} q_{n-k}a_k = a_{n-2},$$
- This is because only q_2 is nonzero.
- We can use the root $\alpha = 0$ to generate one Frobenius solution with a recursion relation that becomes
$$a_n = \frac{a_{n-2}}{n^2} \quad n \geq 2.$$
- We then see that we have a solution with a_0 arbitrary, $a_1 = 0$ and
$$\begin{aligned} a_2 &= \frac{a_0}{(2)(2)} \\ a_3 &= 0 \\ a_4 &= \frac{a_2}{(4)(4)}, \end{aligned}$$
and so forth.

An example of repeated roots

- We see that a general expression that captures this is

$$a_{2n} = \frac{a_0}{(n!)^2 2^{2n}} \qquad a_{2n+1} = 0 \qquad n = 1, 2, \ldots.$$

- So our first Frobenius solution is

$$y_1(x) = a_0 \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{(n!)^2}.$$

- Note that this solution is actually not singular at $x = 0$
- And in fact the series has an infinite radius of convergence.
- This illustrates that not all solutions near a regular singular point are actually singular.

An example of repeated roots

- To get the other solution we need to examine $y(x, \alpha)$.
- We can get the coefficients for this series from the recursion relation
$$[((\alpha + n)^2]a_n(\alpha) = a_{n-2}(\alpha),$$
- And we can then see that the solution is similar to what we have derived earlier:
$$a_{2n}(\alpha) = \frac{a_0}{(\alpha + 2)^2(\alpha + 4)^2 \cdots (\alpha + 2n)^2}.$$
- We next want to compute the derivative of the coefficients with respect to α and then evaluate these derivatives at $\alpha = 1$.

An example of repeated roots

- Some algebra gives us

$$b_{2n} = \left.\frac{\partial a_n(\alpha)}{\partial \alpha}\right|_{\alpha=1} = \frac{-a_0}{2^{2n}(n!)^2}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right].$$

- These are the coefficients for the second series and the total solution is

$$y = c_1 y_1(x) + c_2 y_2(x),$$

  where

$$\begin{aligned} y_1(x) &= \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^{(2n)}(n!)^2} \\ y_2(x) &= y_1(x)\ln(x) + \sum_{n=0}^{\infty} b_{2n} x^{2n}. \end{aligned}$$

When the roots differ by an integer

- For completeness, we next cover what happens when the roots differ by an integer.
- There are actually two sub-cases here.
- Recall that the recursion relation is of the form

$$P(\alpha+n)a_n = -\sum_{k=0}^{n-1}\left[(\alpha+k)p_{n-k}+q_{n-k}\right]a_k \qquad n=0,1,2,\ldots.$$

- If we choose the larger of the two roots (call it α_1), we see that there is no problem generating a series.
- This is because the expression $P(\alpha_1+n)$ will never vanish if we start with the larger root.

When the roots differ by an integer

- The problem occurs for the smaller root - call it α_2
- And remember $\alpha_1 - \alpha_2 = N$ a positive integer.
- After a finite number of steps in the recursion relation we will have

$$P(\alpha_2 + N) = 0.$$

- Recall the recursion relation is

$$P(\alpha_2 + n)a_n = -\sum_{k=0}^{n-1} \left[(\alpha + k)p_{n-k} + q_{n-k}\right] a_k \qquad n = 0, 1, 2, \ldots.$$

- So when $n = N$ the left hand side is zero but the right hand side generally is not.
- So now it looks like we have a problem because it looks like we can't solve for a_N

Sometimes a miracle happens...

- In some cases it turns out the right hand side of the recursion relation also vanishes:

$$P(\alpha_2+N)a_N = 0 = -\sum_{k=0}^{N-1}\left[(\alpha_2+k)p_{n-k}+q_{n-k}\right]a_k = 0.$$

- In this case, the coefficient a_N is arbitrary.
- And in fact it is the arbitrary constant of the second series.
- The recursion relation can then be continued with a_N arbitrary and this creates the second series.
- This seems unlikely, but there are important examples where it happens.
- One important example is the Bessel equation where ν is a half integer meaning $\nu = 1/2, 3/2, \ldots$.

But often a miracle does not happen...

- In most cases, however, this magical cancellation does not happen
- We then need another approach to find the missing series solution.
- Recall when we had double roots, an effective technique was to try the series

$$\left.\frac{\partial}{\partial\alpha}y(x,\alpha)\right|_{\alpha=\alpha_1} = y(x,\alpha_1)\ln(x-x_0) + \sum_{n=0}^{\infty} b_n(x-x_0)^{\alpha+n},$$

where

$$b_n = \left.\frac{\partial}{\partial\alpha}a_n(\alpha)\right|_{\alpha=\alpha_1}.$$

When the roots differ by an integer

- If we try this we find this does not quite work.
- Instead, we get

$$\mathcal{L}\left[\left.\frac{\partial y(x,\alpha)}{\partial \alpha}\right|_{\alpha=\alpha_1}\right] = a_0 P'(\alpha_1)(x-x_0)^{\alpha_2+N-2} \quad \text{where} \quad \alpha_1 = \alpha_2 + N,$$

- The problem here is that $P'(\alpha_1) \neq 0$
- So this is not a solution in this case.
- However, if we could perhaps find another solution $y_P(x)$ such that

$$\mathcal{L}\left[y_P(x)\right] = a_0 P'(\alpha_1)(x - x_0)^{\alpha_2+N-2},$$

- Then we could solve our problem by forming

$$y_2(x) = \left.\frac{\partial y(x,\alpha)}{\partial \alpha}\right|_{\alpha=\alpha_1} - y_P(x),$$

- And this would solve

$$\mathcal{L}\left[y_2(x)\right] = 0.$$

When the roots differ by an integer

- It turns out this is possible.
- And further, that this solution is itself in the form of a Frobenius expansion:

$$y_P(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+\alpha_2}.$$

- If we substitute this series and equate coefficients we get at order $(x - x_0)^{\alpha_2 - 2}$

$$P(\alpha_2)c_0 = 0$$

- So this says c_0 is arbitrary
- Then for the next terms from $n = 1, \ldots, N - 1$

$$P(\alpha_2 + n)c_n + \sum_{k=0}^{n-1} [(\alpha_2 + k)p_{n-k} + q_{n-k}]c_k = 0 \quad n \neq 0, N$$

- This gives us $c_1, c_2, \ldots c_{N-1}$

When the roots differ by an integer

- And finally at order N

$$\sum_{k=0}^{N-1}[(\alpha_2+k)p_{N-k}+q_{N-k}]c_k = a_0 P'(\alpha_1) \neq 0.$$

- The last equation is a relationship between the c_n and a_0.
- It basically defines a_0 in terms of those c coefficients.
- After that, one can continue developing the c coefficients starting with c_N.
- You can see that there are still only two arbitrary coefficients: c_0 and c_N.
- And everything else is defined via the recursion relations.
- Note the final result for the second solution is that it still has a similar form to the logarithmic solution we found when we had double roots.
- But it is more complicated in terms of the relationships among the coefficients.