Physics 106a — Classical Mechanics

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Lecture 10

Applications of the Hamiltonian Formulation

Applications of the Hamiltonian Formulation

- Towards Statistical Mechanics
  - Phase space volumes are conserved
  - Liouville's theorem
  - Equal probability assumption
- Towards Quantum mechanics
  - Schrodinger's equation
  - Time dependence

Liouville's Theorem

- **Conservation of probability:**

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}_{\text{ph}}) = 0$$

Expanding the derivative

$$\frac{\partial \rho}{\partial t} + \vec{v}_{\text{ph}} \cdot \vec{\nabla}\rho + \rho \vec{\nabla} \cdot \vec{v}_{\text{ph}} = 0.$$

- **Hamiltonian dynamics:**

$$\vec{\nabla}_{\text{ph}} \cdot \vec{v}_{\text{ph}} = \sum_{k=1}^{N} \left(\frac{\partial \dot{q}_k}{\partial q_k} + \frac{\partial \dot{p}_k}{\partial p_k} \right) = \sum_{k=1}^{N} \left(\frac{\partial}{\partial q_k} \frac{\partial H}{\partial p_k} - \frac{\partial}{\partial p_k} \frac{\partial H}{\partial q_k} \right) = 0$$

- **Liouville's theorem:**

$$\frac{d\rho}{dt} \equiv \frac{\partial \rho}{\partial t} + \vec{v}_{\text{ph}} \cdot \vec{\nabla}\rho = 0$$

- **Equilibrium:** $\partial\rho/\partial t = 0$ gives

$$\vec{v}_{\text{ph}} \cdot \vec{\nabla}\rho = 0$$

Poisson Bracket

Definition

For functions $A(\{q_k\}, \{p_k\}, t)$, $B(\{q_k\}, \{p_k\}, t)$

Poisson bracket: $$[A, B]_{q,p} = \sum_{k=1}^{N} \left(\frac{\partial A}{\partial q_k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial q_k} \right)$$

Time Dependence

Physical observable $O(\{q_k\}, \{p_k\}, t)$ e.g. $\vec{L} = \sum_i \vec{r}_i \times \vec{p}_i$

Time dependence of the observable under Hamiltonian dynamics

$$\frac{dO}{dt} = \sum_{k=1}^{N} \left(\frac{\partial O}{\partial q_k} \dot{q}_k + \frac{\partial O}{\partial p_k} \dot{p}_k \right) + \frac{\partial O}{\partial t}$$

Using the Hamilton equations of motion for $\dot{q}_k$, $\dot{p}_k$ gives

$$\frac{dO}{dt} = [O, H]_{q,p} + \frac{\partial O}{\partial t}$$

If $\partial O/\partial t = 0$ and $[O, H]_{q,p} = 0$, then O is a constant of the motion

If A, B are both constants of the motion, then $[A, B]$ is a constant of the motion (although it may not be nontrivial or new).

Towards Quantum Mechanics

- In quantum mechanics a physical observable is represented by an *operator* $\hat{O}$, and the possible values of the observable that can be measured are given by the eigenvalues o of the equation $\hat{O}\Psi = o\Psi$
  - energy $\Rightarrow$ Hamiltonian
  - momentum $\Rightarrow -i\hbar\vec{\nabla}$ (in the position representation)
- Typically, operators do not commute: operating on a wave function first with $\hat{B}$ and then $\hat{A}$ is not the same as operating in the reverse order $\hat{A}\hat{B}\Psi \neq \hat{B}\hat{A}\Psi$

$$\text{Commutator:} \qquad [\hat{A}, \hat{B}] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

- Time dependence in quantum mechanics can be represented in a number of ways. In the *Heisenberg picture* the operators have a time dependence

$$\frac{d\hat{O}}{dt} = -\frac{i}{\hbar}[\hat{O}, \hat{H}] + \frac{\partial \hat{O}}{\partial t}$$

- cf. the classical result with

$$\text{Poisson bracket} \Rightarrow -i/\hbar \times \text{commutator}$$