# Physics 106a — Classical Mechanics


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## Lecture 7

## Planetary Orbits: the Kepler Problem

# Main Points

Calculating the motion of a planet in the gravitational potential of the Sun illustrates many important ideas in Lagrangian mechanics:

- Symmetries $\rightarrow$ conserved quantities – Noether's theorem
- Use conserved quantities to reduce number of variables and EOM
  - Planetary dynamics: reduce from 6 to 1!
- Constant Hamiltonian $\rightarrow$ first integral of Euler-Lagrange equation
- Solution of one-dimensional problems
  - qualitative – motion of particle in effective potential
  - method of quadratures – reduce to integral
- Particular example of 1/r potential

# Kepler's Laws

1. Planetary orbits are ellipses, with the Sun at the focus.
2. The line joining the planets to the Sun sweeps out equal areas in equal times.
3. The square of the period of the planet is proportional to the cube of its mean distance to the Sun

# Noether's Theorem

Continuous symmetry + Lagrangian dynamics $\Rightarrow$ Conservation law

What is a symmetry?

- Operation transforms coordinates:

$$q_k \to Q_k(s), \text{ with } Q_k(s=0) = q_k$$

- Symmetry operation will leave the Lagrangian unchanged
- Want a prescription for finding the conserved quantity

# Noether's Theorem

- Symmetry operation leaves Lagrangian unchanged

$$\frac{d}{ds} L(\{Q_k\}, \{\dot{Q}_k\}, t) = 0$$

- Express the total derivative in terms of the partials

$$\begin{aligned}
\frac{dL}{ds} &= \sum_k \left[ \frac{\partial L}{\partial Q_k} \frac{\partial Q_k}{\partial s} + \frac{\partial L}{\partial \dot{Q}_k} \frac{\partial \dot{Q}_k}{\partial s} \right] \\
&= \sum_k \left[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{Q}_k} \right) \frac{\partial Q_k}{\partial s} + \frac{\partial L}{\partial \dot{Q}_k} \frac{\partial \dot{Q}_k}{\partial s} \right] \\
&= \frac{d}{dt} \left[ \sum_k \left( \frac{\partial L}{\partial \dot{Q}_k} \right) \frac{\partial Q_k}{\partial s} \right] = \frac{d}{dt} \sum_k p_k \frac{\partial Q_k}{\partial s}
\end{aligned}$$

introducing the momentum $p_k = \partial L / \partial \dot{Q}_k$.

- Putting $dL/ds = 0$ for a symmetry operation gives the conserved quantity

$$I(\{q_k\}, \{\dot{q}_k\}, t) = \sum_k p_k \left. (\partial Q_k / \partial s) \right|_{s=0}$$

# Example: Translational Symmetry

For $M$ particles with coordinates $\vec{r}_i$

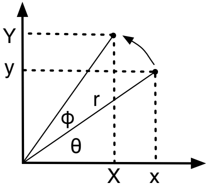
- Define the extended coordinates $\vec{R}_i(\vec{d}) = \vec{r}_i + \vec{d}$ with $\vec{d}$ a translation.
- For translational invariance in the $x$-direction the conserved quantity is

$$P_x = \sum_i \vec{p}_i \cdot (\partial \vec{R}_i / \partial d_x)_{d=0} = \sum_i p_{ix},$$

  which is the total $x$-momentum.
- For full translational invariance, the vector total momentum $\vec{P}$ is conserved.

# Example: Rotational Symmetry

- For rotations about the z-axis, define extended coordinates $(X_i(\phi), Y_i(\phi))$




$$X(\phi) = r\cos(\theta+\phi) = r\cos\theta\cos\phi - r\sin\theta\sin\phi$$
$$Y(\phi) = r\sin(\theta+\phi) = r\sin\theta\cos\phi + r\cos\theta\sin\phi$$
$$X_i(\phi) = x_i\cos\phi - y_i\sin\phi$$
$$Y_i(\phi) = x_i\sin\phi + y_i\cos\phi$$

- Rotational invariance about the $z$-axis gives the conserved quantity

$$l_z = \sum_i \left[ p_{xi} \left.\frac{\partial X_i}{\partial\phi}\right|_{\phi=0} + p_{yi} \left.\frac{\partial Y_i}{\partial\phi}\right|_{\phi=0} \right]$$
$$= \sum_i \left[ p_{xi}(-y_i) + p_{yi}x_i \right] = \sum_i (\vec{r}_i \times \vec{p}_i)_z$$

  the $z$-component of the angular momentum
- Full rotational symmetry $\Rightarrow$ the conservation of the angular momentum $\vec{l}$

# Central Force Problem: Lagrangian

- Two point masses $M_1$, $M_2$ interacting with a *central force* (one directed between the points)
- 6 degrees of freedom, e.g. the position vectors $\vec{r}_1, \vec{r}_2$
- The Lagrangian is

$$L = \frac{1}{2}M_1\dot{\vec{r}}_1^2 + \frac{1}{2}M_2\dot{\vec{r}}_2^2 - V(|\vec{r}_1 - \vec{r}_2|)$$

with $V(r)$ giving the central force.

# Central Force Problem: Translational Symmetry

- Translational symmetry gives the conservation of total momentum

$$\vec{P} = \vec{p}_1 + \vec{\mathrm{p}}_2 = M_1\dot{\vec{r}}_1 + M_2\dot{\vec{r}}_2$$

- Introduce the center of mass coordinate

$$\vec{R}_{\mathrm{cm}} = \frac{M_1\vec{r}_1 + M_2\vec{r}_2}{M_1 + M_2} \quad \text{so that} \quad \vec{P} = (M_1 + M_2)\dot{\vec{R}}_{\mathrm{cm}}$$

- As the second coordinate use the separation vector $\vec{r} = \vec{r}_1 - \vec{r}_2$, and then

$$\vec{r}_1 = \vec{R}_{\mathrm{cm}} + \frac{M_2}{M_1 + M_2}\vec{r}, \quad \vec{r}_2 = \vec{R}_{\mathrm{cm}} - \frac{M_1}{M_1 + M_2}\vec{r}$$

- The Lagrangian becomes

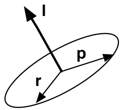
$$L = \frac{1}{2}M\dot{\vec{R}}_{\mathrm{cm}}^2 + \frac{1}{2}\mu\dot{\vec{r}}^2 - V(r), \qquad M = M_1 + M_2, \quad \mu = \frac{M_1 M_2}{M_1 + M_2}$$

- $\vec{R}_{\mathrm{cm}}$ is ignorable and $\vec{P}$ is the conserved momentum $\vec{P} = \partial L/\partial\dot{\vec{R}}_{\mathrm{cm}}$
- Only need to consider the relative motion described by the Lagrangian

$$L = \tfrac{1}{2}\mu\dot{\vec{r}}^2 - V(r)$$

# Central Force Problem: Rotational Symmetry

- The Lagrangian only involves scalars and is unchanged by any rotation of the system. By Noether's theorem the angular momentum $\vec{l} = (l_x, l_y, l_z)$ is a constant of the motion.

- 


  For constant $\vec{l} = \vec{r}(t) \times \vec{\mathrm{p}}(t)$ the vectors $\vec{r}(t)$ and $\vec{\mathrm{p}}(t)$ lie in the fixed plane perpendicular to $\vec{l}$, i.e. motion is planar

- Choosing polar coordinates with the $z$-axis along the direction of $\vec{l}$ means that the orbital plane is $\theta = \pi/2$, and the Lagrangian for motion in the plane is

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - V(r)$$

- $\dot{\phi}$ is ignorable, so $l = \partial L/\partial\dot{\phi} = \mu r^2\dot{\phi}$ is constant
- This gives Keplers's second law $\dot{A} = \frac{1}{2}r^2\dot{\phi} = l/2\mu$ = constant

# Central Force Problem: One Dimensional Motion

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) - V(r), \qquad \mu r^2\dot{\phi} = l$$

- The Euler-Lagrange equation for $r$ is

$$\mu\ddot{r} - \mu r\dot{\phi}^2 + \frac{dV}{dr} = 0$$

- Writing $\dot{\phi}$ in terms of the constant $l$

$$\mu\ddot{r} + \frac{dV_{\text{eff}}}{dr} = 0 \qquad \text{with} \qquad V_{\text{eff}}(r) = \frac{l^2}{2\mu r^2} + V(r)$$

Note: it is wrong to use expression for $\dot{\phi}$ in terms of $l$ in the Lagrangian

- This is a one dimensional problem in an effective potential with an additional term from the rotational motion.

# Central Force Problem: Constant $H$, $E$

- The Lagrangian is time independent ($\partial L/\partial t = 0$), so the Hamiltonian $H$ is constant
- There are no time dependent constraints and so the Hamiltonian is equal to the total energy $H = E = T + V$
- Explicitly

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r), \qquad V_{\text{eff}}(r) = \frac{l^2}{2\mu r^2} + V(r)$$

# Kepler Problem

- Specialize to gravitational potential $V(r) = -k/r$ with $k = GM_1M_2 > 0$
- Problem to solve:

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r) \quad \text{with} \quad V_{\text{eff}}(r) = -\frac{k}{r} + \frac{l^2}{2\mu r^2}$$

- Qualitative solution
  - for $E < 0$ (set by initial conditions) the $r$ motion oscillates over a finite range between $r_{\text{min}}, r_{\text{max}}$, so that the motion is bound
  - for $E > 0$ the separation $r$ increases to $\infty$ (maybe after one "bounce") so that the motion is unbound

# Kepler Problem: Solution for $r(t)$

- Problem to solve:

$$E = \frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}(r) \quad \text{with} \quad V_{\text{eff}}(r) = -\frac{k}{r} + \frac{l^2}{2\mu r^2}$$

- This gives us $\dot{r}$, which we can formally integrate to get $t(r)$

$$t = \sqrt{\frac{\mu}{2}} \int_{r_0}^{r} \frac{dr'}{\sqrt{E + \frac{k}{r'} - \frac{l^2}{2\mu r'^2}}}$$

  which then implicitly gives $r(t)$
- See Hand and Finch pp148-9 for how to do this integral
- In particular, the period of the radial motion for bound orbits is

$$\text{Period} = 2\sqrt{\frac{\mu}{2}} \int_{r_{\min}}^{r_{\max}} \frac{dr'}{\sqrt{E + \frac{k}{r'} - \frac{l^2}{2\mu r'^2}}}$$

# Kepler Problem: Solution for $r(\phi)$

- Use $\dot{r} = \dot{\phi}\, dr/d\phi$ and introduce $u = 1/r$

$$\dot{r}^2 = \dot{\phi}^2 \left(\frac{dr}{d\phi}\right)^2 = \frac{l^2}{\mu^2}\left(\frac{1}{r^2}\frac{dr}{d\phi}\right)^2 = \frac{l^2}{\mu^2}\left(\frac{du}{d\phi}\right)^2$$

- Some algebra shows

$$E = \frac{l^2}{2\mu}\left[\left(\frac{du}{d\phi}\right)^2 + \left(u - \frac{1}{\mathrm{p}}\right)^2 - \frac{1}{p^2}\right] \qquad \text{with } p = \frac{l^2}{\mu k}$$

- $u(\phi)$ undergoes simple harmonic motion centered on $u = 1/p$
- Period of $u(\phi)$ is $2\pi$, so the bound orbits are *closed*
- Betrand's theorem: closed orbits occur only for $r^{-1}$ and $r^2$ potentials
- The explicit solution is

$$u = \frac{1}{r} = \frac{1}{\mathrm{p}} + \frac{\epsilon}{\mathrm{p}}\cos\phi \qquad \text{with} \qquad E = \frac{l^2}{2\mu p^2}(\epsilon^2 - 1)$$

for any amplitude $\epsilon > 0$ (additive phase constant chosen to be zero)

# Kepler Problem: Conic Sections

- The equation

$$u = \frac{1}{r} = \frac{1}{\mathrm{p}} + \frac{\epsilon}{\mathrm{p}} \cos\phi \qquad \text{or} \qquad r = p - \epsilon r \cos\phi$$

  is the equation in polar coordinates $r$, $\phi$ for a conic section with focus at the origin $r = 0$ and $\epsilon$ the *eccentricity*

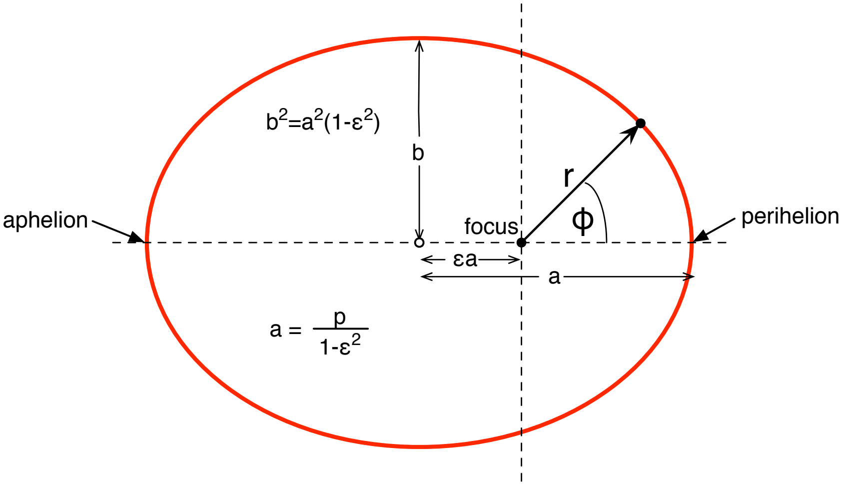
- $\epsilon < 1 \Rightarrow$ an ellipse; $\epsilon > 1 \Rightarrow$ a hyperbola; and $\epsilon = 1 \Rightarrow$ a parabola.
- Change to Cartesian coordinates $r = \sqrt{x^2 + y^2}$, $r\cos\phi = x$

$$\frac{(x - x_c)^2}{a^2} \pm \frac{y^2}{b^2} = 1$$

  with the $+$ for $\epsilon < 1$ and the $-$ for $\epsilon > 1$, and

$$a = \frac{\mathrm{p}}{|1 - \epsilon^2|} \qquad b = \frac{\mathrm{p}}{\sqrt{|1 - \epsilon^2|}} \qquad x_c = -\frac{\epsilon p}{1 - \epsilon^2}$$

# Elliptical Orbit : $\epsilon < 1$



# Keppler's Third Law

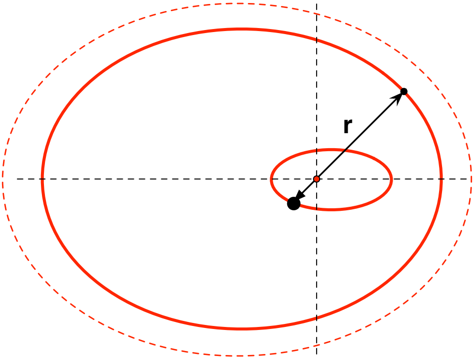
- The area of an elliptical orbit is $A = \pi ab$ and $\dot{A} = l/2\mu$
- This gives the period

$$\tau = \frac{A}{\dot{A}} = 2\pi\sqrt{\frac{\mu}{k}}a^{3/2} = \frac{2\pi}{\sqrt{G(M_1 + M_2)}}a^{3/2}$$

- If $M_{\mathrm{p}} \ll M_{\mathrm{s}}$ then $M_1 + M_2 \simeq M_{\mathrm{s}}$, so that $\tau^2 \propto a^3$

# Planetary Orbits: Geometry

$$\vec{r}_{\mathrm{p}} = \vec{R}_{\mathrm{cm}} + \frac{M_{\mathrm{s}}}{M_{\mathrm{s}} + M_{\mathrm{p}}}\vec{r}, \qquad \vec{r}_{\mathrm{s}} = \vec{R}_{\mathrm{cm}} - \frac{M_{\mathrm{p}}}{M_{\mathrm{s}} + M_{\mathrm{p}}}\vec{r}$$



# Kepler's Laws Revisited

1. Planetary orbits are ellipses, with the center of mass at the focus

2. Conservation of angular momentum: the line joining the planets to the focus sweeps out equal areas in equal times

$$\dot{A} = \frac{l}{2\mu}$$

3. The square of the period of the planet is approximately proportional to the cube of its mean distance to the Sun

$$\tau = \frac{2\pi}{\sqrt{G(M_{\mathrm{s}} + M_{\mathrm{p}})}} a^{3/2}$$

# Hyperbolic Orbit: $\epsilon > 1$