ACM 100b

Solvable second order ODE

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Second order constant coefficient ODE

- Consider the homogeneous 2nd order constant coefficient ODE:
$$ay'' + by' + cy = 0.$$
where a, b, c are all constants and $a \neq 0$.
- Because of this the solution is a function whose derivatives keep the same form as the solution itself.
- An elementary function that does this is $\exp(\alpha x)$.
- Substituting this exponential function into the differential equation yields
$$a\alpha^2 + b\alpha + c = 0$$
- This is called the *characteristic equation.*
- This is a quadratic equation with two roots
$$\alpha_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$
which can either be real and equal, real and unequal, or complex.

Second order constant coefficient ODE

- The roots again are

$$\alpha_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

- If $b^2 - 4ac > 0$ then $\alpha_1 \neq \alpha_2$ and the solution is given by

$$y(x) = c_1 \exp(\alpha_1 x) + c_2 \exp(\alpha_2 x).$$

- If $b^2 - 4ac = 0$ then $\alpha_1 = \alpha_2 = -b/2a$ and one solution is given by $\exp[(-b/2a)x]$.
- The other can be found by reduction of order:

$$y(x) = c_1 \exp(\alpha x) + c_2 x \exp(\alpha x).$$

- If $b^2 - 4ac < 0$ then $\alpha = u \pm iv$ and the solution is given by

$$\begin{aligned} y(x) &= c_1 \exp\{(u + iv)x\} + c_2 \exp\{(u - iv)x\} \\ &= d_1 \exp(ux)\cos(vx) + d_2 \exp(ux)\sin(vx). \end{aligned}$$

The Euler ODE

- Another solvable second order ODE is called the *Euler ODE*.
- It is given by
$$y'' + \frac{\alpha}{x} y' + \frac{\beta}{x^2} y = 0$$
where α and β are constants.
- This can be solved exactly by substituting
$$y(x) = x^r \quad \text{where } r \text{ is a constant}$$
- This yields a quadratic equation for r and one can then get both solutions.
- If the roots are double the solution is of the form
$$y(x) = c_1 x^r + c_2 x^r \ln x$$
- If the roots are complex it is still possible to write the solution in real form by recalling that
$$x^{\alpha + i\beta} = x^{\alpha} \left[\cos(\beta \ln x) + i \sin(\beta \ln x)\right]$$