Physics 106a — Classical Mechanics

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Lecture 9

Hamiltonian Formulation

Hamiltonian Formulation

- Hamiltonian equations of motion
- Phase space
- Legendre transformations
- Ignorable coordinates and the Routhian
- Hamilton's principle

Ignorable Coordinates

For an ignorable coordinate q_m:

$$\frac{\partial L}{\partial q_m} = 0 \quad \Rightarrow \quad \frac{\partial H}{\partial q_m} = 0$$

The equations of motion are

$$\dot{p}_m = -\frac{\partial H}{\partial q_m} = 0 \quad \Rightarrow \quad p_m \text{ constant of motion}$$
$$\dot{q}_m = \frac{\partial H}{\partial p_m}$$

The Routhian

Coordinates $\{q_k, k = 1 \dots N\}$ with $\{q_k, k = 1 \dots s\}$ ignorable

Do a partial Legendre transformation $L \to H$ on the ignorable coordinates

Routhian: $\mathcal{R}(q_1, q_2 \dots q_s; \dot{q}_1, \dot{q}_2 \dots \dot{q}_s; p_{s+1} \dots p_N) = \sum_{k=s+1}^{N} p_k \dot{q}_k - L$

$$d\mathcal{R} = -\sum_{k=1}^{s} \left(\frac{\partial L}{\partial \dot{q}_k} d\dot{q}_k + \frac{\partial L}{\partial q_k} dq_k \right) + \sum_{k=s+1}^{N} \left(\dot{q}_k dp_k - \frac{\partial L}{\partial q_k} dq_k \right)$$

The equations of motion are

$$\frac{d}{dt}\left(\frac{\partial \mathcal{R}}{\partial \dot{q}_k}\right) - \frac{\partial \mathcal{R}}{q_k} = 0 \qquad k = 1 \dots s$$

$$\dot{q}_k = \frac{\partial \mathcal{R}}{\partial p_k} \qquad k = s+1 \dots N$$

Hamilton's Principle

Find stationary value of the action $S = \int L\,dt$ with L expressed in terms of H

$$S = \int \sum_k \left[p_k \dot{q}_k - H(\{q_k\}, \{p_k\}, t) \right] dt$$

Change in the action for a path change

$$\delta S = \int \sum_k \left(\delta p_k \dot{q}_k + p_k \delta\dot{q}_k - \frac{\partial H}{\partial q_k}\delta q_k - \frac{\partial H}{\partial p_k}\delta p_k \right) dt$$

Integrate the $\delta\dot{q}_k$ term by parts (using $\delta q_k = 0$ at endpoints)

$$\delta S = \int \sum_k \left[-\left(\frac{\partial H}{\partial q_k} + \dot{p} \right) \delta q_k - \left(\frac{\partial H}{\partial p_k} - \dot{q} \right) \delta p_k \right] dt$$

$\Rightarrow$ Hamiltonian equations of motion by requiring action to be stationary under *independent* changes of δq, δp with *no* requirement that p_k be fixed at the endpoints.

Lagrangian or Hamiltonian Approach?

Disadvantages of Hamiltonian approach

- In general, need Lagrangian $L = T - V$ anyway to form H
- Even if H is the energy, need this as a function of momenta ($p = \partial L / \partial \dot{q}$)

General strategy:

- Use Lagrangian approach for the equation of motion and solution of specific problem, initial conditions etc.
- Use Hamiltonian approach for qualitative behavior, the solution to many initial conditions, and formal arguments
- The Routhian is convenient if there are ignorable coordinates