ACM 100b

Analysis of the point at infinity

Dan Meiron

Caltech

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The point at infinity

- In complex analysis the point at infinity is considered a point as well.
- We can also classify the point at ∞ in the same way we have classified finite points.
- To do this we make the transformation

$$x \to 1/t,$$

- We then examine the singularity if there is one as $t \to 0$.
- Recall in doing this one also has to transform the derivatives according to the rules

$$\begin{aligned} \frac{d}{dx} &= -t^2 \frac{d}{dt} \\ \frac{d^2}{dx^2} &= t^4 \frac{d^2}{dt^2} + 2t^3 \frac{d}{dt} \\ &\vdots \end{aligned}$$

Examples of analyzing the point at infinity

- Consider the very innocent looking ODE
$$y' = y/2$$
- The coefficient function $p = 1/2$ is analytic everywhere in the complex plane.
- But now consider the point at ∞.
$$x \to 1/t,$$
- The ODE becomes
$$\frac{dy}{dt} = -\frac{y}{2t^2}$$
- As $t \to 0$ we see that since this is a first order ODE it has an irregular singular point as $t \to 0$
- Note the solution
$$y = c\exp(x/2)$$
is analytic everywhere except at ∞ where is has an essential singularity.

Examples of analyzing the point at ∞

- The ODE
$$\frac{dy}{dx} = \frac{y}{2x}$$
has a regular singular point at $x = 0$.
- If you examine the point at infinity you will find it also has a regular singular point as $x \to \infty$.
- Indeed, the solution $y = c\sqrt{x}$ is analytic except for branch points at $x = 0$ and $x \to \infty$.

Examples of analyzing the point at ∞

- The ODE
$$\frac{dy}{dx} = \frac{y}{2x^2}$$
has an irregular singular point at $x = 0$.
- But if you look at $x \to \infty$ you will find this is an ordinary point.
- And indeed the solution is
$$y = c\exp(-1/2x)$$
- This has an essential singularity at $x = 0$ but is analytic as $x \to \infty$.