# ACM 100b

## Examples of Fourier series

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# Some examples of Fourier series

- We give a few examples of computing Fourier series.
- Consider first the function

$$f(x) = \sin(\pi x) + (1/3)\sin(3\pi x) \qquad 0 \leq x \leq 1$$

- Let's compute the Fourier sine series of this function.
- The Fourier sine series coefficients are

$$A_n = 2\int_0^1 f(x)\sin(n\pi x)dx$$

- We see immediately that we get

$$A_1 = 1 \qquad A_3 = 1/3 \qquad A_n = 0 \quad n \neq 1,3$$

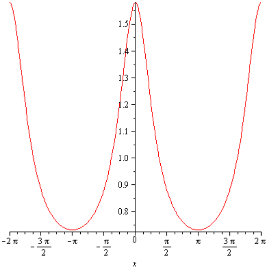
- This is just by virtue of the orthogonality.

# Some examples of Fourier series

- Let's try a less trivial example.
- Compute the Fourier cosine series of

$$f(x) = \frac{1 - \mathrm{e}^{-1}\cos(x)}{1 - 2\,\mathrm{e}^{-1}\cos(x) + \mathrm{e}^{-2}} \qquad 0 \le x \le \pi$$

- This function looks like this:




- It's clearly even and periodic and very smooth over the interval $0 \le x \le \pi$

# Some examples of Fourier series

- We now ask for the Fourier cosine series coefficients:

$$A_n = \frac{2}{\pi}\int_0^\pi f(x)\cos(nx)dx \qquad n > 0$$
$$A_0 = \frac{1}{\pi}\int_0^\pi f(x)dx$$

- Note we're doing this over the interval $0 \leq x \leq \pi$
- It is not hard to show that

$$A_n = \exp(-n)$$

- So this function has the series

$$f(x) = \sum_{n=0}^{\infty} \exp(-n)\cos(nx)$$

- Note how rapidly the coefficients decay to 0.

# Examples of Fourier series

- As another example consider

$$f(x) = x \qquad 0 \leq x \leq \pi$$

- Let's take the Fourier sine series over $0 \leq x \leq \pi$

$$A_n = \frac{2}{\pi} \int_0^{\pi} x \sin(nx) dx$$

- You get

$$A_n = \frac{2}{n}(-1)^{n+1}$$

- Note that $x$ is also a very smooth function but the coefficients don't decay very fast.
- We'll explain why this happens later on.

# Examples of Fourier series

- We can also compute the Fourier cosine series of

$$f(x) = x \qquad 0 \leq x \leq 1$$

- In this case

$$B_n = \frac{2}{\pi}\int_0^{\pi} x cos(nx) dx \quad n > 0$$
$$B_0 = \frac{1}{\pi}\int_0^{\pi} x dx$$

- You get

$$B_n = \frac{2(-1 + (-1)^n)}{\pi n^2} \quad n > 0 \qquad B_0 = \pi/2$$

- Here the coefficients decay a little faster.

# Examples of Fourier series

- Finally let's compute the full periodic Fourier series of

$$f(x) = x$$

over the interval $-\pi \leq x \leq \pi$

- Now $x$ is not periodic over this interval but let's proceed anyway.
- The sine coefficients of the periodic series over $-\pi \leq x \leq \pi$ are given by

$$A_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) sin(nx) dx$$

- The cosine coefficients are given by

$$B_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) cos(nx) dx \quad n > 0$$

$$B_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \quad n = 0$$

# Examples of Fourier series

- We can see by inspection that since $x$ is an odd function the cosine terms must vanish because the integrand is always odd.
- So

$$B_n = 0 \qquad n \geq 0$$

- On the other hand the sine terms do not vanish.
- We get

$$A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin(nx)dx = -\frac{2(-1)^n}{n}$$

- We will see in the following lecture why we get differing behaviors for the coefficients
- We'll also see how well the series approximates the function.