ACM 100b

Fourier series

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Expansions in S-L eigenfunctions

- We will next investigate the expansion of general functions in terms of Sturm-Liouville eigenfunctions.
- We have intimated that for regular Sturm-Liouville problems the eigenfunctions bear a great deal of similarity to sine or cosine functions.
- We noted that as the value of the eigenvalue λ becomes larger the eigenfunctions do begin to look and oscillate like sines or cosines.
- This can actually be made quantitative but we will take it as an observational fact.
- Because of this, we will look at expansion of general functions in terms of sines and cosines and then extend our results to more general eigenfunctions.
- The expansion of functions in sines and/or cosines is called *Fourier analysis* and the series is called a *Fourier series*

Periodic boundary conditions

- Recall our analysis of the Sturm-Liouville problem relied on the use of local separable boundary conditions.

$$a_1 u(a) + a_2 u'(a) = 0,$$
$$b_1 u(b) + b_2 u'(b) = 0$$

- These boundary conditions lead to standard Sturm-Liouville eigenfunctions with the properties we listed earlier.
- We will look closely at the convergence of series constructed from these eigenfunctions.
- We will also look at the same time at a slight modification of the boundary conditions for S-L problems
- In some cases we deal with what are called *periodic boundary conditions*.
- These have the form

$$u(a) = u(b) \qquad u'(a) = u'(b).$$

Periodic boundary conditions

- Periodic boundary conditions come up, for example, when we deal with circular or spherical domains
- In such cases we would expect the solution at a given point has the same value when we traverse around the circle or sphere.
- These are not separable - the information at $x = a$ is connected to that at $x = b$:
- In this case, S-L eigenfunctions still exist and there is still orthogonality.
- But the guarantee that there is one eigenfunction for each eigenvalue goes away.

Periodic boundary conditions

- The simplest example of this is the ODE

$$\frac{d^2}{dx^2}y(x) + \lambda^2 y(x) = 0 \qquad 0 \le x \le 1$$

  with periodic boundary conditions: $y(0) = y(1)$ and $y'(0) = y'(1)$
- In this case we get possible solutions:

$$\sin(\lambda x) \text{ and } \cos(\lambda x).$$

- We see that the periodicity boundary condition simply means that

$$\lambda = 2n\pi \qquad n = 0, 1, 2, 3, \cdots.$$

- Now both $\sin(2n\pi x)$ and $\cos(2n\pi x)$ are eigenfunctions.
- Both eigenfunctions have the same eigenvalue: $\lambda_n = 2n\pi$

Periodic boundary conditions

- It turns out the eigenfunctions are still orthogonal:

$$\int_0^1 \sin(2n\pi x)\sin(2m\pi x)dx = 0 \qquad m \neq n$$

$$\int_0^1 \cos(2n\pi x)\cos(2m\pi x)dx = 0 \qquad m \neq n$$

$$\int_0^1 \sin(2n\pi x)\cos(2m\pi x)dx = 0 \qquad \text{always}$$

- Note too the eigenvalues are increasing ($\lambda_n \to \infty$ as $n \to \infty$)
- The oscillatory properties of the eigenfunctions are also preserved as $\lambda \to \infty$.
- The two families of sines and cosines have interlacing zeroes.

Periodic boundary conditions

- However both sets of functions (sines and cosines) must be used to expand a generic function
- Using periodic eigenfunctions for example to create an expansion of some generic function uses *all* the eigenfunctions:

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(2n\pi x) + \sum_{n=0}^{\infty} B_n \cos(2n\pi x)$$

- The coefficients A_n and B_n can be determined (uniquely) using the mutual orthogonality of the eigenfunctions.
- The only time both sets of functions are not required over a periodic interval is when the function has some sort of symmetric behavior.
- For example an even function $f(-x) = f(x)$ will only require cosine functions if you want to expand it over $0 \leq x \leq 2\pi$

An example - the heat equation again

- We return to our general discussion of expansions but consider some different boundary conditions
- We illustrate this with some examples coming from the heat equation.
- Recall the heat equation with zero temperature conditions at the ends of the rod

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \qquad 0 < x < 1,$$
$$T(x,0) = T_0(x), \qquad T(0,t) = 0 \quad T(1,t) = 0$$

- We noted the solution could be expressed as a series in terms of sine functions

$$T(x,t) = \sum_{n=1}^{\infty} A_n \exp(-n^2\pi^2 t)\sin(n\pi x) \qquad T_0(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

An example - the heat equation again

- The sine functions were eigenfunctions from a regular Sturm-Liouville problem and they are orthogonal
- If $m = n$ we have

$$\int_0^1 \sin^2(n\pi x)dx = \int_0^1 \left[\frac{1-\cos(2n\pi x)}{2}\right] dx = \frac{1}{2}$$

- Applying these relations to $T_0(x)$ we get

$$\int_0^1 T_0(x)\sin m\pi x dx = \sum_{n=1}^{\infty} A_n \int_0^1 \sin(n\pi x)\sin(m\pi x)dx = A_m/2$$

- So our coefficients are given by

$$A_m = 2\int_0^1 T_0(x)\sin(m\pi x)dx$$

- These coefficients A_m define the *Fourier sine series* for $T_0(x)$ over the interval $0 < x < 1$

Different boundary conditions lead to a different series

- We can get a different expansion by considering the heat equation with different boundary conditions.
- Consider

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} \qquad 0 < x < 1, \qquad T(x,0) = T_0(x),$$

- But this time with *insulating* boundary conditions:

$$\frac{\partial T}{\partial x}(0,t) = 0 \quad \frac{\partial T}{\partial x}(1,t) = 0$$

- In this case, when we perform separation of variables we get the same Sturm-Liouville ODE
- But the boundary conditions are different and involve derivatives
- Note the boundary conditions are still separable

Different boundary conditions lead to a different series

- In this case the solution of the Sturm-Liouville problem gives

$$T_0(x) = \sum_{n=0}^{\infty} B_n \cos(n\pi x).$$

- The cosines are again mutually orthogonal.
- They have to be, because this is a regular Sturm-Liouville problem.
- So we have

$$\begin{aligned} B_n &= 2\int_0^1 T_0(x)\cos(m\pi x)dx & \qquad m \neq 0 \\ B_0 &= \int_0^1 T_0(x)dx & \qquad m = 0 \end{aligned}$$

- The coefficients B_m form what is called the *Fourier cosine series* for $T_0(x)$

Different boundary conditions lead to a different series

- Finally, we should point out a third kind of Fourier series.
- Suppose we ask that the solution be *periodic* at $x = 0, 1$.
- In that case we don't get a regular Sturm-Liouville problem
- We get an S-L problem with nonlocal boundary conditions
- But we can get the answer here too.
- We recall both sines and cosines are solutions.

Heat equation with periodic boundary conditions

- So we try to write

$$T_0(x) = \sum_{n=0}^{\infty} B_n \cos(2n\pi x) + \sum_{n=1}^{\infty} A_n \sin(2n\pi x).$$

- Again using orthogonality of the eigenfunctions we get

$$A_n = 2\int_0^1 T_0(x)\sin(2n\pi x)dx$$

$$B_n = 2\int_0^1 T_0(x)\cos(2n\pi x)dx \qquad n \neq 0$$

$$B_0 = \int_0^1 T_0(x)dx$$

- This is called the *full periodic Fourier series* of $T_0(x)$ over the interval $0 < x < 1$.
- We will next look at how the three types of Fourier series are related.

Relations among the 3 Fourier series

- We have so far introduced three versions of Fourier transforms:
  1. Sine series
  2. Cosine series
  3. Full periodic Fourier series
- We will now relate these series to one another.
- Rather than use a single interval like $0 < x < 1$ we'll examine a function $f(x)$ defined on $-L < x < L$.
- The full periodic Fourier series of some function $f(x)$ is given by

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/L) + \sum_{n=0}^{\infty} B_n \cos(n\pi x/L) \qquad -L < x < L$$

Relations among the 3 Fourier series

- Again using orthogonality of the eigenfunctions we get

$$A_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin(n\pi x/L) dx \quad n \neq 0$$

$$B_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos(n\pi x/L) dx \quad n \neq 0$$

$$B_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

Relations among the 3 Fourier series

- Now we note that the periodic series consists of both a Fourier sine series and a Fourier cosine series.
- The Fourier sine series coefficients are

$$
\begin{aligned}
A_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\pi x/L)dx \\
&= \frac{1}{L}\int_{-L}^{0} f(x)\sin(n\pi x/L)dx + \frac{1}{L}\int_{0}^{L} f(x)\sin(n\pi x/L)dx \\
&= \frac{1}{L}\int_{0}^{L} -f(-x)\sin(n\pi x/L)dx + \frac{1}{L}\int_{0}^{L} f(x)\sin(n\pi x/L)dx \\
&= \frac{2}{L}\int_{0}^{L} \left[\frac{f(x)-f(-x)}{2}\right]\sin(n\pi x/L)dx
\end{aligned}
$$

Relations among the 3 Fourier series

- The Fourier cosine series coefficients are

$$
\begin{aligned}
B_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos(n\pi x/L)dx \\
&= \frac{1}{L}\int_{0}^{L} f(-x)\cos(n\pi x/L)dx + \frac{1}{L}\int_{0}^{L} f(x)\cos(n\pi x/L)dx \\
&= \frac{2}{L}\int_{0}^{L} \left[\frac{f(x)+f(-x)}{2}\right]\cos(n\pi x/L)dx \qquad n \neq 0, \\
B_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx \\
&= \frac{1}{2L}\int_{0}^{L} f(-x)dx + \frac{1}{2L}\int_{0}^{L} f(x)dx \\
&= \frac{1}{L}\int_{0}^{L} \left[\frac{f(x)+f(-x)}{2}\right]dx
\end{aligned}
$$

Relations among the 3 Fourier series

- We know that every function $f(x)$ can be written in terms of a sum of an even function and an odd function:

$$f(x) = \left[\frac{f(x) - f(-x)}{2}\right] + \left[\frac{f(x) + f(-x)}{2}\right]$$

- We now see that the fully periodic Fourier series is made up of
  - the Fourier sine series of the odd part of $f(x)$ and
  - the Fourier cosine series of the even part of $f(x)$
- Note the Fourier sine and cosine series are defined over the interval $0 < x < L$.
- The Fourier sine series therefore defines an odd function of x over the interval $-L \leq x \leq L$.
- The Fourier cosine series defines an even function of x over the interval $-L \leq x \leq L$

Relations among the 3 Fourier series

- And the sum of the Fourier sine series and Fourier cosine series is the full periodic Fourier series defined over $-L < x < L$.
- Conversely, a Fourier sine series can be thought of as a full Fourier series for a periodic function defined over $-L \leq x \leq L$
- For this function all the cosine terms are zero because the function $f(x)$ is odd.
- And a Fourier cosine series can be thought of as a full Fourier series for a periodic function define over $-L \leq x \leq L$
- For this function all the sine terms have vanished because $f(x)$ is even.

Relations among the 3 Fourier series

- This also means that if you form a Fourier sine series from some function $f(x)$ over the interval $0 < x < L$ then it is as if you formed a full Fourier series from the function
$$\begin{cases} f(x) & 0 < x < L \\ -f(-x) & -L < x < 0 \end{cases}$$
- And if you form a Fourier cosine series from some function $f(x)$ over the interval $0 < x < L$ then it is as if you formed a full Fourier series from the function
$$\begin{cases} f(x) & 0 < x < L \\ f(-x) & -L < x < 0 \end{cases}$$
- You may be wondering why we're belaboring all this.
- It all seems quite obvious (and it is)
- But it has has some important implications for the rate of convergence of Fourier sine and cosine series as we will see later.

A note about domains and boundary conditions

- It's important to note that the boundary conditions are what determine the type of S-L expansion one might use
- For example for a function $f(x)$ defined on any finite domain one could do an expansion in
  - Fourier sine series
  - Fourier cosine series
  - Periodic Fourier series
  - Or the eigenfunctions from any regular S-L ODE defined over the finite interval with separable boundary conditions
- Each of these expansions will be slightly different although there can be interrelations among them like the ones we just derived for Fourier series.