Physics 106b — Classical Mechanics

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Dissipative Dynamical Systems

Outline

- Dissipative systems
- One dimensional flows
  - Fixed points
  - Bifurcations of fixed points
- Higher dimensional flows
  - Hopf bifurcation of fixed point
  - Bifurcations of periodic orbits
  - Chaos

Equations of Motion

Consider equations of the form

$$\dot{x}_i = f_i(x_1, x_2, \ldots x_n), \quad i = 1 \ldots n.$$

with f_i smooth and finite functions

- Equations are **autonomous**: no explicit time dependence on RHS
- Solutions can be represented by non-crossing *trajectories* or *flows* in the n-dimensional phase space $(x_1, x_2, \ldots x_n)$
- f_i define a *vector field* $\mathbf{f}$ – the velocity of the flow at each point in phase space
- In general the system will be non-Hamiltonian, so dynamical variables do not come in canonically conjugate pairs, and there are no symplectic constraints on the flows

Example of a Dissipative System

Lorenz Model

$$\begin{aligned}\dot{X} &= -\sigma(X - Y) \\ \dot{Y} &= rX - Y - XZ \\ \dot{Z} &= -bZ + XY\end{aligned}$$

Crudely, the diagonal terms such as $\dot{X} = -\sigma X$ correspond to decaying motion

Contraction of phase space volumes:

$$\begin{aligned}\boldsymbol{\nabla}_{\text{ph}} \cdot \mathbf{V}_{\text{ph}} &= \frac{\partial}{\partial X}[-\sigma(X - Y)] + \frac{\partial}{\partial Y}[rX - Y - XZ] + \frac{\partial}{\partial Z}[-bZ + XY] \\ &= -\sigma - 1 - b < 0\end{aligned}$$

Phase space volumes contract uniformly and exponentially

Dissipation

With dissipation we expect phase space volumes to contract

$$\nabla_{\mathrm{ph}} \cdot \mathbf{V}_{ph} = \nabla_{\mathrm{ph}} \cdot \mathbf{f} < 0$$

at least on average

After transients have died out, the long time asymptotic dynamics must be confined to a lower dimensional region of phase space known as an *attractor*

- point: fixed point, equilibrium
- curve:
  - limit cycle, periodic orbit
  - homo- or hetero-clinic orbit
- surface: m-torus corresponding to oscillations at m different frequencies
- fractal: *strange attractor* giving chaotic dynamics (Lecture 10)

Many (in some cases, almost all) different initial conditions will lead to trajectories on the *same* attractor after transients have died out, and often there is a single attractor. This makes it easier to formulate simple questions than in Hamiltonian systems, but harder to answer.

One Dimensional Flows

The simplest system is one dimensional $n = 1$

Equation of motion is

$$\dot{x} = f(x)$$

giving flows on the line.

The motion can be thought of as the damped motion in a potential, $\eta\dot{x} = -dV(x)/dx$ with η the damping constant.

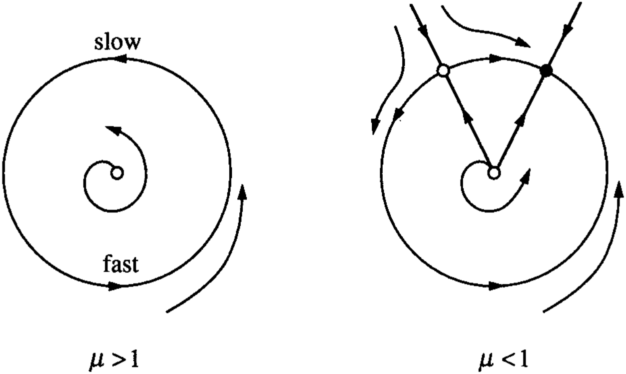
Pictures of the flow: plot the flow as arrows on the x-axis, and also plot $f(x)$ which gives $\dot{x}$.

Bifurcations of Limit Cycles

Bifurcations on the Poincaré section

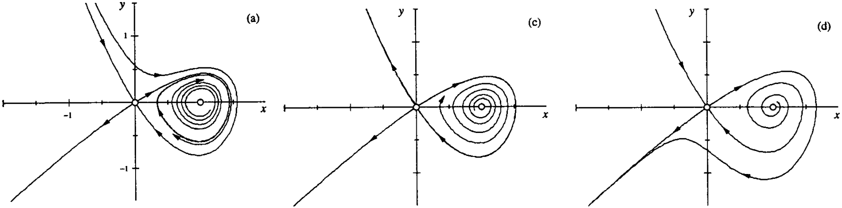
- Limit cycle $\Rightarrow$ fixed point of Poincaré map $\mathcal{T}$: $\mathbf{x}_{n+1} = \mathcal{T}(\mathbf{x}_n)$
- Linear stability given by eigenvalues λ of the Jacobean of the map at the fixed point
- $|\lambda|$ gives the magnification of a perturbation under each iteration of the map $\Rightarrow$ Floquet exponent $\sigma = T^{-1} \log \lambda$ (with T the limit cycle period)
- Bifurcation: $|\lambda|$ passes through the unit circle
  - $\lambda = 1$: change in the stability of the limit cycle, but no change in the frequency, giving bifurcations analogous to the saddle-node, transcritical, and pitchfork bifurcations of fixed points
  - $\lambda = -1$: period doubling bifurcation – it takes two of the original periods of the original limit cycle for the motion to repeat
  - Complex pair λ, λ^* passing through the unit circle: oscillations at a new frequency, and, naïvely, motion on a two torus in phase space

Infinite period bifurcation: saddle-node bifurcation of fixed points where the one-dimensional line of the the analysis is the θ variable



$$\dot{\theta} = \mu - \sin\theta$$

Homoclinic or **saddle-loop** bifurcation: limit cycle grows towards a saddle fixed point, becomes a homoclinic orbit at the bifurcation point, and then disappears



Chaos

- For flows in a phase space of dimension $n > 2$, the attractors are not limited to fixed points and limit cycles
- For dissipative systems where the phase space volume contracts, the dimension of the attractor must be less than n
- Simplest case: three dimensional system $n = 3$
  - Does the phase space volume of initial conditions contract to a planar (and so zero volume) attractor, ruling out chaos by the usual argument that trajectories cannot intersect?
  - No: chaotic motion corresponds to a *strange attractor* that has non-integral dimension, i.e. is a *fractal* (here $2 < D < 3$) – next lecture