ACM 100b

An example of a boundary value problem - the heat equation

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Example of a BVP - the heat equation

- One particular class of BVP is the Sturm-Liouville ODE.
- Its associated boundary value problems occur frequently
- To motivate the importance of BVP's and also give a preview of a partial differential equation (PDE) we consider the solution of the heat equation in one space dimension.
- The heat equation is a partial differential equation that describes the evolution of heat in a uniform medium.
- The equation is given by

$$\frac{\partial \Theta(x,t)}{\partial t} = D\frac{\partial^2 \Theta(x,t)}{\partial x^2}$$

- We want to solve it in the domain

$$0 \le x \le 1, \qquad t > 0.$$

This problem has initial conditions

- We won't worry for now about where this equation came from - we'll derive it later.
- Here x describes the length along some ideal rod of material that extends from $x = 0$ to $x = 1$.
- $\Theta(x, t)$ is the temperature of the rod
- D is called the diffusivity, a measure of how well heat diffuses through the rod.
- Given some initial distribution of heat, we want to see how it evolves for $t > 0$.
- So at $t = 0$ we have

$$\Theta(x, t = 0) = \Theta_0(x).$$

- This is the initial condition for the problem.

And it has boundary conditions

- We also need to say what happens at the edges of the rod during the time we are interested in getting the evolution of the temperature.
- The PDE won't have a unique solution unless we do this.
- Intuitively this makes sense since the ends could be insulating or perhaps connected to some heat bath that maintains a constant temperature.
- To specify what happens at the ends we specify boundary conditions at $x = 0$ and $x = 1$.
- We will assume that the temperature of the rod is fixed at $\Theta = 0$ by some mechanism:

$$\Theta(x = 0, t) = 0, \qquad \Theta(x = 1, t) = 0, \qquad t > 0.$$

Solving the PDE

- One very useful approach to such problems is called the method of separation of variables.
- We will motivate this approach in great detail later on.
- But for now we will just assume that

$$\Theta(x,t) = T(t)X(x),$$

or that the solution can be written in *separable form*

- If we substitute this into the PDE (we'll omit the details right now) it turns out to be possible to get such a solution if $T(t)$ and $X(x)$ satisfy the following ODE's:

$$\frac{d^2X(x)}{dx^2} - CX = 0,$$
$$\frac{dT(t)}{dt} - DCT = 0$$

where C is a constant called the *separation constant*.

Solving the resulting ODE's

- Now let's try to solve the ODE's

$$\frac{d^2X(x)}{dx^2} - CX = 0,$$
$$\frac{dT(t)}{dt} - DCT = 0$$

- There are many types of solutions to these equations as we will discuss later.
- But we're interested in the ones that make physical sense.
- The diffusivity D is always positive ($D > 0$).
- We can solve the ODE for the time variable t to give

$$T(t) = a\exp(DCt),$$

where a is some arbitrary constant and C is again the separation constant.

Solving the spatial ODE's

- Now, physically, we don't expect that the temperature will grow exponentially without somehow externally heating the rod (which we're not doing).
- So we try to obtain solutions in which $C < 0$ meaning our temperature decays in time which seems reasonable for this type of problem.
- So we set

$$C = -\lambda^2,$$

- Here λ is hopefully real or at least λ^2 has a positive real part.
- Making this substitution in the ODE for $X(x)$ we get the following boundary value problem:

$$\frac{d^2X(x)}{dx^2} + \lambda^2 X = 0 \qquad X(0) = X(1) = 0.$$

Boundary value problems often lead to eigenvalue problems for ODE's

- We get the following boundary value problem:

$$\frac{d^2X(x)}{dx^2} + \lambda^2 X = 0 \qquad X(0) = X(1) = 0.$$

- Now it looks like we're stuck
- We have a homogeneous ODE and homogeneous boundary conditions.
- It seems the only solution is $X(x) = 0$.

Boundary value problems lead to eigenvalue problems

- This is, in fact, true for almost any value of λ
- But recall from our previous discussion that one can find nontrivial solutions to even a homogeneous boundary value problem
- This is because in the system we typically solve

$$
\begin{aligned}
c_1 y_1(z_0) + c_2 y_2(z_0) &= 0, \\
c_1 y_1(z_1) + c_2 y_2(z_1) &= 0,
\end{aligned}
$$

it might happen that the matrix has zero determinant

Getting the eigenvalues

- In that case we could get nontrivial solutions although they won't be unique.
- So we ask whether it's possible to play with λ so that solutions exist.
- Up till now, we have not said anything about λ.
- We can solve the ODE,

  $$\frac{d^2X(x)}{dx^2} + \lambda^2 X = 0 \qquad X(0) = X(1) = 0.$$

  since it's a linear constant coefficient ODE to get the general solution:

  $$X(x) = c_1 \sin(\lambda x) + c_2 \cos(\lambda x)$$

- We want $X(x)$ to satisfy the boundary conditions.
- At $x = 0$ in order to have $X(0)$ vanish, we must have $c_2 = 0$.
- Using this and trying to satisfy the other boundary condition at $x = 1$ gives us the equation $c_1 \sin(\lambda) = 0$.

Getting the eigenvalues

- Now normally, the only solution to this equation

$$c_1 \sin(\lambda) = 0$$

  is $c_1 = 0$
- We only get the trivial solution $X(x) = 0$.
- But we note that if we set λ so that the $\sin(\lambda)$ vanishes then c_1 could be arbitrary.
- We don't yet know what this means but at least we get some kind of solution.
- From the properties of the sine function we know that

$$\sin(\lambda_n) = 0 \qquad \text{where} \quad \lambda_n = n\pi, \qquad n = 1, 2, 3, \cdots .$$

- We have a set of solutions of the following type:

$$\Theta_n(x, t) = B_n \exp(-n^2\pi^2 Dt) \sin(n\pi x) \qquad n = 1, 2, 3, \cdots .$$

  where the B_n are arbitrary constants.

The general solution

- We see that we went from no solutions to a countable infinity of solutions.
- The heat equation is a linear homogeneous PDE because each term (like the time and space derivatives) appear linearly and there is no inhomogeneous term
- So it seems to be like a linear homogeneous ODE for which we know we can use the principle of superposition of solutions
- Because the whole problem is linear we can see that a more general solution of this heat equation is a superposition of the solutions we just found:

$$\Theta(x,t) = \sum_{n=1}^{\infty} B_n \exp(-n^2\pi^2 t)\sin(n\pi x).$$

The general solution

- Clearly, the sum

$$\Theta(x,t) = \sum_{n=1}^{\infty} B_n \exp(-n^2\pi^2 t)\sin(n\pi x).$$

  satisfies the boundary conditions, because the sines vanish at $x = 0, 1$
- But there is also an initial condition to satisfy.
- At $t = 0$ we have some starting distribution of heat in the rod:

$$\Theta(x,0) = \Theta_0(x).$$

- In order to satisfy this condition we substitute $t = 0$ into

$$\Theta(x,t) = \sum_{n=1}^{\infty} B_n \exp(-n^2\pi^2 t)\sin(n\pi x)$$

to get $\Theta_0(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$

Fitting the solution to the initial condition

- So we would have a solution that satisfies all the conditions if we could figure out the coefficients B_n in the expression

$$\Theta_0(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

- As promising as this looks, there are some unanswered questions:
- How does one determine B_n?
- If you can determine B_n is there only one choice that works?
- Even if there is a unique choice of B_n can you show the series converges to $\Theta_0(x)$ as $n \to \infty$?
- If it converges at $t = 0$ does it converge for $t > 0$?
- The answers to these questions will take up the next few lectures.