PART III: LINEAR EQUATIONS (ADDITIONAL TOPICS)

23. Linear equations with analytic coefficients

We'll be considering equations

$$y'' + p(x)y' + q(x)y = 0 \tag{23.1}$$

such that both coefficients p and q are *analytic* in some neighborhood of x_0.

23.1. Power series solutions.

Recall that analyticity of $p(x)$ at x_0 means that there is a number $R > 0$ such that $p(x)$ has a power series representation

$$p(x) = \sum_n c_n (x - x_0)^n \qquad \text{in} \quad |x - x_0| < R.$$

We must have $R \leq \rho$, where ρ is the *radius of convergence* of the series,

$$\rho = \liminf |c_n|^{-1/n}.$$

Note. Sometimes the easiest way to find the radius of convergence is to use the fact that ρ is the radius of the maximal disc in $\mathbb{C}$ in which the function represented by the series is analytic. In other words, ρ is the distance from x_0 to the nearest singularity. For example, $\rho = 1$ for $p(x) = (1 + x^2)^{-1}$; note that there are no singularities on $\mathbb{R}$.

Theorem. *Suppose that the coefficients of* (23.1) *have power series representations in the interval* $|x - x_0| < R$. *Then every solution of the equation has a power series representation in this interval.*

Corollary. *If p and q are polynomials, then all solution of* (23.1) *are* entire *functions,* (i.e. they have a power series representation on the whole line).

Idea of proof:

- Write y as a power series with undetermined coefficients.
- From the equation find a *recurrence relation* between the coefficients.
- This relation plus the initial conditions determine the coefficients uniquely.
- Check that the radius of convergence is a least R.
- The sum of the power series automatically satisfies the equation.

23.2. **Example: Airy's equation.**

$$y'' - xy = 0.$$

If

$$y = \sum_{n=0}^{\infty} a_n \; x^n,$$

then

$$xy = \sum_{n=1}^{\infty} a_{n-1} \; x^n, \qquad y'' = \sum_{n=0}^{\infty} (n+2)(n+1) \; a_{n+2} \; x^n.$$

Equating the coefficients of x^n, we get

$$a_2 = 0, \qquad (n+2)(n+1) \; a_{n+2} = a_{n-1} \quad (n \geq 1).$$

It follows that

$$\begin{aligned} & a_2 = a_5 = \cdots = 0; \\ a_3 = \frac{a_0}{3 \cdot 2}, \quad & a_6 = \frac{a_3}{6 \cdot 5} = \frac{a_0}{6 \cdot 5 \cdot 3 \cdot 2}, \quad \ldots \\ a_4 = \frac{a_1}{4 \cdot 3}, \quad & a_7 = \frac{a_4}{7 \cdot 6} = \frac{a_1}{7 \cdot 6 \cdot 4 \cdot 3}, \quad \ldots \end{aligned}$$

Clearly,

$$|a_{3n}|, \; |a_{3n+1}| \; < \; \frac{C}{n!}$$

so the radius of convergence is infinite and all solutions are entire functions. The coefficients are uniquely determined by the initial conditions

$$a_0 = y_0, \quad a_1 = y_0'.$$

Let $A_1(x)$ be the solution of IVP$(0; 1, 0)$,

$$A_1(x) = 1 + \frac{x^3}{3 \cdot 2} + \frac{x^6}{6 \cdot 5 \cdot 3 \cdot 2} + \ldots$$

and let $A_2(x)$ be the solution of IVP$(0; 0, 1)$,

$$A_2(x) = x + \frac{x^4}{4 \cdot 3} + \frac{x^7}{7 \cdot 6 \cdot 4 \cdot 3} + \ldots$$

Then A_1 and A_2 are independent, and so the general solution of the Airy equation is

$$y(x) = C_1 A_1(x) + C_2 A_2(x).$$

Note. The functions $A_1(x)$, $A_2(x)$, or rather their certain linear combinations Ai(x) and Bi(x) known as the *Airy functions*, are examples of *special* functions. Airy functions can not be expressed in terms of elementary functions, but since they appear in many important applications, their properties have been extensively studied.

See Fig. 20.3 in the textbook for the graph of $A_1(x)$. Note the difference in the behavior for negative and positive x. The function oscillates on $\mathbb{R}_-$ but has a monotone, super-exponential growth on $\mathbb{R}_+$. This can be explained (informally) as follows: think of the Airy equation $y'' - xy = 0$ on the negative axis as a harmonic oscillator $y'' + \omega(x)^2 y = 0$ whose frequency depends on x, $\omega^2 = -x$.

23.3. **Aside: Riccati equation.** Second order linear equations are closely related to the (first order, non-linear) Riccati equation

$$u' = a(x)u^2 + b(x)u + c(x).$$

(Riccati's equation is linear if $a(x) = 0$ and Bernoulli if $c(x) = 0$.)

Lemma. *If y is a solutions of $y'' + py' + qy = 0$, then the function*

$$u = \frac{y'}{y}$$

satisfies the first order equation

$$u' + pu + q + u^2 = 0.$$

Proof:

$$u' = \frac{y''y - y'y'}{y^2} = \frac{-py'y - qy^2 - y'y'}{y^2}.$$

□

Example. The Airy equation $y'' - xy = 0$ corresponds to Riccati's equation

$$u' = x - u^2. \tag{23.2}$$

The general solution of (23.2) is therefore

$$u = \frac{C_1 A_1' + C_2 A_2'}{C_1 A_1 + C_2 A_2};$$

it depends on just one parameter C_2/C_1. Earlier in the course we mentioned (23.2) as an example of a "non-solvable" (in elementary functions) equation.

24. Power series method for equations with meromorphic coefficients

We will now consider equations

$$P(x)y'' + Q(x)y' + R(x)y = 0$$

such that the coefficients P, Q, R are analytic at x_0. We can rewrite the equations in the normal form,

$$y'' + p(x)y' + q(x)y = 0,$$

but the coefficients p, q will be *meromorphic* functions (the ratios of two analytic functions).

If $P(x_0) \neq 0$, then the functions p and q are analytic at x_0 and we say that x_0 is an *ordinary* point. If $P(x_0) = 0$, then x_0 is a *singular* point. In this case, we should consider the equation on the intervals $\{x > x_0\}$ and $\{x < x_0\}$ separately.

24.1. **Cauchy-Euler equation.** This is a model example:

$$x^2y'' + \alpha xy' + \beta y = 0,$$

or

$$y'' + \frac{\alpha}{x}y' + \frac{\beta}{x^2}y = 0.$$

Note that in the last equation, the coefficient of y' has a pole of order (at most) 1, and the coefficient of y has a pole of order (at most) 2.

Consider the interval $\mathbb{R}_+ = \{x > 0\}$, and make the following change of the independent variable:

$$\log x = t, \quad x = e^t, \quad \mathbb{R}_+ \leftrightarrow \mathbb{R}, \quad 0 \leftrightarrow -\infty.$$

We have

$$y' = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{\dot{y}}{x},$$

$$y'' = \frac{(\dot{y}/x)^{\cdot}}{\dot{x}} = \frac{\ddot{y}x - \dot{y}\dot{x}}{x^2\dot{x}} = \frac{\ddot{y} - \dot{y}}{x^2}.$$

The new equation is linear with constant coefficients:

$$(\ddot{y} - \dot{y}) + \alpha\dot{y} + \beta y = 0,$$

or

$$\ddot{y} + (\alpha - 1)\dot{y} + \beta y = 0.$$

The eigenvalue equation is

$$\lambda^2 + (\alpha - 1)\lambda + \beta = 0.$$

Traditional notation and terminology: $\lambda_i = r_j$ are called the *exponents at the singularity*, and the equation

$$F(r) \equiv r(r-1) + \alpha r + \beta = 0$$

is called the *indicial equation.*

(Shortcut: to derive the indicial equation, substitute $y = x^r$ in the C.-E. equation; note $e^{rt} = x^r$.)

The exponents at the singularity are

$$r_{1,2} = \frac{1 - \alpha \pm \sqrt{(1-\alpha)^2 - 4\beta}}{2}.$$

Theorem. *(i) If the exponents are real and distinct, then the general solution is*

$$y(t) = C_1x^{r_1} + C_2x^{r_2},$$

(ii) If they are equal, them

$$y(x) = C_1x^r + C_2x^r\log x.$$

(iii) If the exponents are complex $a \pm i\omega$, then

$$y(x) = x^aC_1\cos(\omega\log x) + x^aC_2\sin(\omega\log x).$$

We can describe the behavior of solutions near the singular point. For example, in the first case, if both exponents are positive, then all solutions tend to 0 at 0, if both are negative, then all solutions are unbounded, and if $r_1 > 0, r_2 < 0$, then one solution tends to zero, and the others are unbounded.

The case $\{x < 0\}$ is similar: just write $|x|$ instead of x.

24.2. **Frobenius theory.** Consider the equation

$$y'' + p(x)y' + q(x)y = 0$$

with meromorphic coefficients, and let x_0 be a singular point. The singular point is called *regular* if the singularities of p and q at x_0 are not worse than those in the CE equation, i.e. p has a pole of order at most 1, and q has a pole of order at most 2:

$$p(x) = \frac{\alpha}{x - x_0} + \dots, \qquad q(x) = \frac{\beta}{(x - x_0)^2} + \dots,$$

where the dots denote analytic functions. Let r_j be the roots ("exponents") of the *indicial equation*

$$F(r) := r(r-1) + \alpha r + \beta = 0.$$

We are interested in the behavior of the solutions near the singular point, and it is natural to expect that it is similar to the behavior of the solutions of the corresponding CE equation. This turns out to be true except for the *resonance* case

$$r_1 - r_2 \in \mathbb{Z}.$$

Here is a precise statement. We will only describe the case of real exponents (the complex case is similar and even simpler since there's no resonance). Without loss of generality, we will assume $x_0 = 0$.

Theorem. *Suppose that the functions $\tilde{p}(x) := xp(x)$ and $\tilde{q}(x) := x^2 q(x)$ have a power series representation on the interval $(-\rho, \rho)$, and suppose that the exponents r_j are real. Then the following is true in each of the two intervals $(-\rho, 0)$ and $(0, \rho)$.*

(i) If $r_1 - r_2 \notin \mathbb{Z}$, then there are two (independent) solutions of the form

$$y_j = |x|^{r_j}(1 + \dots),$$

where the dots stand for a power series which converges on $(-\rho, \rho)$ and is zero at zero.

(ii) If $0 \le r_1 - r_2 \in \mathbb{Z}$, then there is a solution y_1 of the form as above.

24.3. **Remarks.** (a) The proof (and the use of the theorem) is similar to the analytic case: we introduce undetermined coefficients and find a recurrence relation. In the equation

$$x^2 y'' + \tilde{p}(x) x y' + \tilde{q}(x) y = 0, \qquad (x > 0),$$

we set

$$y(x) = \sum_{n \ge 0} a_n x^{r+n}, \qquad \tilde{p}(x) = \sum p_n x^n, \qquad \tilde{q}(x) = \sum q_n x^n.$$

Since

$$
\begin{aligned}
xy' &= \sum_{n\ge 0}(r+n)a_n x^{r+n} \\
x^2y'' &= \sum_{n\ge 0}(r+n)(r+n-1)a_n x^{r+n} \\
\tilde{p}(x)xy' &= \sum_{n\ge 0}\left(\sum_{k+\nu=n}(r+\nu)a_\nu p_k\right)x^{r+n} \\
\tilde{q}(x)y &= \sum_{n\ge 0}\left(\sum_{k+\nu=n}a_\nu q_k\right)x^{r+n}
\end{aligned}
$$

we get the equation

$$(r+n)(r+n-1)a_n + \sum_{k+\nu=n}[(r+\nu)p_k + q_k]a_\nu = 0,$$

i.e.

$$[(r+n)(r+n-1) + (r+n)p_0 + q_0]a_n = -\sum_{\nu=0}^{n-1}[(r+\nu)p_{n-\nu} + q_{n-\nu}]a_\nu.$$

Note that

$$(r+n)(r+n-1) + (r+n)p_0 + q_0 = F(r+n).$$

In particular, for $n = 0$ the recurrence relation gives

$$F(r)a_0 = 0.$$

We see that the condition $F(r) = 0$ is necessary for the existence of a non-trivial solution of the form $y = x^r \cdot$ (analytic function). This condition is also sufficient – we only need to note that if r is the largest exponent, then $F(r+n) \ne 0$ for $n \ge 1$, and the same is true for the smallest exponent in the non-resonance case. (We also need to check the convergence of the series.)

(b) In the resonance case, we can find the second (independent) solution using the Wronskian formula:

$$y_2 = y_1 \int \frac{W}{y_1^2}, \qquad W = e^{-\int p}.$$

One can show that if $r_1 = r_2$, then y_2 has a logarithmic term (as in the CE equation), but if $0 < r_1 - r_2 \in \mathbb{Z}$, then we may or may not have a logarithmic term. In fact, it is known that in the resonance case, there is a second solution of the form

$$y_2(x) = \text{const } y_1(x)\log|x| + |x|^{r_2}(1 + \dots),$$

where the constant can be zero (or non-zero) if $r_1 \ne r_2$.

25. Hypergeometric functions

25.1. **Infinity as a singular point.** In the case of rational coefficients, we can think of infinity as a singular point. The change of variables $t = x^{-1}$ transforms $x_0 = \infty$ to $t_0 = 0$. We have

$$y' = \frac{\dot{y}}{\dot{x}} = -t^2 \dot{y},$$

and

$$y'' = -\frac{2t}{\dot{x}}\dot{y} - t^2 \frac{\ddot{y}}{\dot{x}} = 2t^3 \dot{y} + t^4 \ddot{y},$$

so the new equation is

$$\ddot{y} + P(t)\dot{y} + Q(t)y = 0$$

with

$$P(t) = \frac{2}{t} - \frac{1}{t^2} p\left(\frac{1}{t}\right), \qquad Q(t) = \frac{1}{t^4} q\left(\frac{1}{t}\right).$$

In particular, ∞ is a regular singular point iff

$$p(x) = O\left(\frac{1}{x}\right), \quad q(x) = O\left(\frac{1}{x^2}\right) \quad \text{at } \infty.$$

[Exercise: compute the exponents at infinity in the case of CE equation.]

25.2. **Hypergeometric equation.** Consider equations with rational coefficients such that all singular points (in $\hat{\mathbb{C}}$, including ∞) are regular. There are no such equations if the number of singular points is 0 or 1. If the number of singular points is 2, and the points are 0 and ∞, then the only such equation is CE. The first non-trivial case is when we have three regular singular points. Without loss of generality, the points are $0, 1, \infty$ (use linear-fractional transformations).

Theorem. *In this case, the equation is*

$$x(1-x)y'' + [\gamma - (1 + \alpha + \beta)x]y' - \alpha\beta y = 0$$

for some values of the parameters α, β, γ.

The singularities are:

$$\begin{aligned} &x = 0, \quad r_1 = 0, \quad r_2 = 1 - \gamma, \\ &x = 1, \quad r_1 = 0, \quad r_2 = \gamma - \alpha - \beta, \\ &x = \infty, \quad r_1 = \alpha, \quad r_2 = \beta. \end{aligned}$$

Consider the singular point $x_0 = 0$. If $\gamma \neq 0, -1, -2, \ldots$, then there is an analytic solution (Frobenius series corresponding to $r_1 = 0$)

$$y = \sum_{n \geq 0} a_n x^n$$

The recurrence relation is

$$a_{n+1} = \frac{(\alpha + n)(\beta + n)}{(n+1)(n+\gamma)} a_n.$$

Choosing $a_0 = 1$, we get the solution

$$F(\alpha,\beta,\gamma;x) := 1 + \frac{\alpha\cdot\beta}{1\cdot\gamma}x + \frac{\alpha\cdot(\alpha+1)\cdot\beta\cdot(\beta+1)}{1\cdot 2\cdot\gamma\cdot(\gamma+1)}x^2 + \dots$$

This is called *hypergeometric series.* The series converges in $(-1,1)$. If we denote

$$(a)_n := a(a+1)\dots(a+n-1) = \frac{\Gamma(a+n)}{\Gamma(a)},$$

then

$$F(\alpha,\beta,\gamma;x) = \sum_{n\geq 0}\frac{(\alpha)_n(\beta)_n}{(\gamma)_n n!}x^n.$$

25.3. **Special cases.** (i) If $\beta = -N \in -\mathbb{N}$, then $a_{N+1} = 0$ and the function is a polynomial of degree N. These polynomials are related to the so called *Legendre polynomials* P_N, which are defined by the equation

$$(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y = 0.$$

Here the singular points are $\pm 1, \infty$. If we change the variable $x \mapsto (1-x)/2$, then we find

$$P_N(x) = \mathrm{const}\cdot F(N+1,\ -N,\ 1,\ (1-x)/2).$$

The case $\alpha = -N$ is similar. Exercise: relate to *Chebychev's polynomials* defined by the equation

$$(1-x^2)y'' - xy' + \alpha^2 = 0.$$

(ii) There are other elementary hypergeometric functions. For example,

$$F(1,1,2;x) = 1 + \frac{x}{2} + \frac{x^2}{3} + \dots = \frac{1}{x}\log\frac{1}{1-x},$$

which follows from the relation

$$a_{n+1} = \frac{n+1}{n+2}a_n.$$

Another example is

$$F(\alpha,\beta,\alpha;x) = (1-x)^{-\beta}, \qquad a_{n+1} = \frac{\beta+n}{n+1}.$$

We also have

$$F(1/2,1/2,3/2;x^2) = \frac{\arcsin x}{x},$$
$$F(1/2,1/2,3/2;x^2) = \frac{\arctan x}{x}.$$

(Hint: differentiate the arc-functions.)

(iii) The above examples are exceptional. A typical hypergeometric function is not elementary. An important example is the elliptic integral

$$\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k(\sin\phi)^2}} = F(1/2,1/2,1;k).$$

26. Bessel's functions

The Bessel equation with parameter $\nu \geq 0$ is
$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \qquad (x > 0).$$
The point $x_0 = 0$ is regular singular with exponents $\pm\nu$:
$$r(r-1) + r - \nu^2 = r^2 - \nu^2 = 0.$$

26.1. **First solution.** We always have the Frobenius solution with $r = \nu$,
$$y_1(x) = x^\nu \sum_{n=0}^{\infty} a_n x^n.$$
The recurrence relation
$$a_n = -\frac{a_{n-2}}{n(n+2\nu)}, \quad (n \geq 2); \qquad a_1 = 0,$$
shows that
$$a_{\text{odd}} = 0,$$
and
$$a_{2m} = \frac{(-1)^m a_0}{2^{2m} m! (1+\nu)_m},$$
where
$$(1+\nu)_m := (1+\nu)\dots(m+\nu) = \frac{\Gamma(1+\nu+m)}{\Gamma(1+\nu)}.$$
Note that formula for a_n's also makes sense for non-integer negative ν's, and the corresponding series satisfies the Bessel equation.

26.2. **Definition.** For $\nu \neq -1, -2, \dots$, the *Bessel function* of order ν is
$$J_\nu(x) = x^\nu \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\ \Gamma(1+\nu+m)} \left(\frac{x}{2}\right)^{2m}.$$
This function is the solution with
$$a_0 = \frac{1}{2^\nu \Gamma(1+\nu)}.$$
The power series in the definition of J_ν has infinite radius of convergence (why?), so J_ν extends to an analytic function in $\mathbb{C} \setminus \bar{\mathbb{R}}_-$. If $\nu \in \mathbb{Z}_+$, then J_ν is an entire function:
$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{m!\ (m+\nu)!} \left(\frac{z}{2}\right)^{2m+\nu}.$$
E.g., [D. Bernoulli 1732, Bessel 1824]
$$J_0(z) = \sum_{m=0}^{\infty} \frac{(-1)^m}{[m!]^2} \left(\frac{z}{2}\right)^{2m}.$$

If $\nu \notin \mathbb{Z}$, then the functions $J_{\pm\nu}$ are independent solutions, so we have the following statement.

Theorem. *If $\nu \notin \mathbb{Z}$, then the general solution of the Bessel equation of order ν is given by the formula*

$$y = C_1 J_\nu + C_2 J_{-\nu}.$$

(Note that the resonance case is $2\nu \notin \mathbb{Z}$. The second solution has a logarithmic term iff $\nu \in \mathbb{Z}$.)

EX: If $\nu \geq 0$, then all solutions excdept for the multiples of J_ν are unbounded at 0. Hint:

$$y_2 = y_1 \int \frac{dx}{x y_1^2(x)}.$$

Several usefull identities:

$$(xJ_1)' = xJ_0, \qquad J_2 = \frac{2}{x}J_1 - J_0.$$

More generally,

$$\begin{aligned} J_{\nu+1} &= \frac{2\nu}{x}J_\nu - J_{\nu-1}, \quad (\nu \geq 1). \\ J_{\nu+1} &= -2J_\nu' + J_{\nu-1}, \qquad (\nu \geq 1). \\ (x^\nu J_\nu)' &= x^\nu J_{\nu-1}, \qquad (\nu \geq 1). \\ (x^{-\nu} J_\nu)' &= -x^{-\nu} J_{\nu+1}, \qquad (\nu \geq 0). \end{aligned}$$

26.3. **Elementary Bessel's functions.**

Theorem. *If $\nu = 1/2 + n$ and $n \in \mathbb{Z}$, then J_ν is an elementary function. E.g.,*

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x, \qquad (x > 0).$$

Proof: If $\nu = 1/2$, then

$$a_n = -\frac{a_{n-2}}{n(n+1)}, \qquad a_{2m} = \frac{(-1)^n a_0}{(2m+1)!},$$

and so

$$J_{1/2}(x) = \text{const}\sqrt{x}\sum_{m\geq 0}\frac{(-1)^m}{(2m+1)!}x^{2m} = \text{const}\frac{\sin x}{\sqrt{x}}.$$

To get the right constant use the value $\Gamma(1/2) = \sqrt{\pi}$.
To get the expressions for $J_{\pm 3/2}$, $J_{\pm 5/2}$ etc use the recurrence formula. □

Fact (asymptotics at infinity):

$$J_\nu(x) = \sqrt{\frac{2}{\pi x}}\cos\left[x - (2\nu+1)\frac{\pi}{4}\right] + O(x^{-3/2}) \quad \text{as} \quad x \to \infty.$$

E.g.,

$$J_0(x) \approx \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right), \quad J_{1/2}(x) \approx \sqrt{\frac{2}{\pi x}}\sin x, \quad J_1(x) \approx \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\pi}{4}\right).$$

An informal explanation of oscillations: almost constant frequency and negligible damping if $|x| \gg 1$.

26.4. **Bessel's functions on the imaginary axis.** *Modified* Bessel's functions I_ν are defined as follows:

$$J_\nu(ix) = i^\nu I_\nu(x), \qquad I_\nu(x) := x^\nu \sum \frac{1}{m!\,\Gamma(m+\nu+1)}\left(\frac{x}{2}\right)^{2m}, \qquad (x>0).$$

Note that all coefficients are positive, so $I_\nu(x)\to\infty$ as $x\to\infty$, no oscillations. For example,

$$I_{1/2} = \sqrt{\frac{2}{\pi x}}\sinh x, \qquad I_{1/2} = \sqrt{\frac{2}{\pi x}}\cosh x.$$

Theorem. *I_ν satisfies the "modified" Bessel's equation*

$$x^2y'' + xy' - (x^2+\nu^2)y = 0.$$

Proof: The expressions $x^2(d^2y/dx^2)$ and $x(dy/dx)$ don't change if $x\mapsto ix$ but x^2y becomes $-x^2y$. □

If $\nu\notin\mathbb{Z}$, then the general solution of the modified equation is $y = C_1I_\nu + C_2I_{-\nu}$.

26.5. **Change of variables.** Many equations can be solved in terms of Bessel's functions. Just two examples:

Lemma. *Suppose $w(z)$ is a solution of the Bessel equation with parameter ν. Then the function*

$$y(x) = x^\alpha w(kx^\beta)$$

satisfies the equation

$$x^2y'' + Axy' + (B + Cx^{2\beta})y = 0$$

with

$$A = 1-2\alpha, \quad B = \alpha^2-\beta^2\nu^2, \quad C = \beta^2k^2.$$

Example. Airy's equation is $y'' = xy$, or

$$x^2y'' - x^3y = 0.$$

We have

$$A = 0, \quad B = 0, \quad C = -1, \quad \beta = \frac{3}{2},$$

so

$$\alpha = \frac{1}{2}, \quad \nu = \frac{1}{3}, \quad k = \frac{2}{3}i,$$

and the general solution is

$$y = C_1\sqrt{x}I_{1/3}\left(\frac{2}{3}x^{3/2}\right) + C_2\sqrt{x}I_{-1/3}\left(\frac{2}{3}x^{3/2}\right).$$

Example. ("Aging spring")

$$\ddot{y} + \omega^2e^{-\varepsilon t}y = 0,$$

where $0<\varepsilon\ll 1$. Change: $x = ae^{bt}$, the constants to be chosen later. We have

$$\dot{y} = y'\dot{x} = y'bx, \quad \ddot{y} = y''(bx)^2 + y'b^2x.$$

Since

$$e^{-\varepsilon t} = (x/a)^{-\varepsilon/b},$$

it follows that if we choose

$$b = -\varepsilon/2,$$

then the new equation is:

$$x^2y'' + xy' + \frac{\omega^2}{a^2b^2}x^2y = 0.$$

Finally, we set

$$a = -\omega/b = (2\omega)/\varepsilon,$$

so the equation is Bessel of order 0. Its general solution is

$$y(t) = C_1J_0(x) + C_2Y_0(x), \quad x = \frac{2\omega}{\varepsilon}e^{-(\varepsilon t)/2},$$

where Y_0 is the second independent solution.

Exercise (Riccati equations). We already mentioned that the substitution $u = y'/y$ transforms $y'' + py' + qy = 0$ into

$$u' + pu + q + u^2 = 0.$$

Similarly, the substitution $v = -y'/y$ transforms $y'' + py' + qy = 0$ into

$$v' = pv + q + v^2.$$

Solve the following equations in terms of Bessel's functions:

$$u' = x^2 - u^2, \qquad u' = x - u^2$$

and

$$v' = v^2 \pm x^2, \qquad v' = v^2 \pm x.$$

27. Aside: classical Fourier series

27.1. **Orthogonal bases in Euclidian spaces.** A Euclidean space is a finite dimensional linear space with a scalar product (a, b); in particular we can define

$$\|a\|^2 = (a, a), \qquad a \perp b.$$

Example: $\mathbb{R}^n$.

If $e_1, \dots, e_n$ are non-zero, mutually orthogonal vectors in $\mathbb{R}^n$, then they form an orthogonal basis: every vector a has a unique representation

$$a = \sum a_je_j.$$

The coefficients a_j can be found as follows:

$$(a, e_j) = a_j(e_j, e_j), \qquad a_j = \frac{(a, e_j)}{(e_j, e_j)}.$$

The Euclidean space $\mathbb{R}^n$ is the space of sequences of length n. An infinite dimensional analogue is the (Hilbert) space $l^2(\mathbb{N})$ of infinite sequences $a = \{a_j\}$ such that

$$\sum |a_j|^2 < \infty.$$

Example: the sequence $\{1/n\}$ is square summable but $\{1/\sqrt{n}\}$ is not.

The scalar product in $l^2(\mathbb{N})$ is given by the formula

$$(a, b) = \sum a_j b_j.$$

Cauchy's inequality:

$$|(a, b)| \leq \|a\| \cdot \|b\|.$$

27.2. **Square integrable functions.** Let $I \subset \mathbb{R}$ be an interval, e.g. $I = (-\pi, \pi)$ or $I = (0, \infty)$. A continuous version of $l^2(\mathbb{N})$ is the Lebesgue space $L^2(I)$ of square integrable functions $f(x)$, $x \in I$. Since Lebesgue integral is not part of the curriculum, we will be considering only the subspace $L^2_{\mathrm{pc}}(I) \subset L^2(I)$ which consists of piecewise continuous functions (the points of discontinuity are isolated and of the jump type). For such functions we define

$$(f, g)_I = \int_I f(x)g(x)dx,$$

in particular

$$\|f\|^2 = \int_I |f(x)|^2 dx,$$

where the integrals have the usual meaning.

Example: $f(x) = 1/x$ is square integrable on $(1, \infty)$ but not on $(-1, 1)$.

Two functions $f, g \in L^2_{\mathrm{pc}}(I)$ are considered equal if $\|f - g\| = 0$; e.g., changing the value of a function at finitely many points does not change the function as an element of $L^2_{\mathrm{pc}}(I)$.

Convergence in L^2 or *mean square convergence*:

$$f = \lim f_n \qquad \text{means} \qquad \int_I |f - f_n|^2 \to 0, \qquad n \to \infty.$$

27.3. **Complete orthogonal families.** A sequence of functions $\{g_n\} \subset L^2_{\mathrm{pc}}(I)$ is *orthogonal* on I if

$$j \neq k \quad \Rightarrow \quad (g_j, g_k) = 0.$$

Example. Let $l > 0$. The trigonometric family

$$1,\ \sin \frac{\pi}{l} x,\ \cos \frac{\pi}{l} x,\ \sin \frac{2\pi}{l} x,\ \cos \frac{2\pi}{l} x,\ \ldots \tag{27.1}$$

is orthogonal on $(-l, l)$. We also have

$$\|1\|^2 = 2l, \quad \left\|\sin \frac{\pi}{l} x\right\|^2 = \left\|\cos \frac{\pi}{l} x\right\|^2 = \cdots = l.$$

Lemma. *Let $\{g_n\}$ be an orthogonal family on I. FAE*

(i) Every $f \in L^2_{\mathrm{pc}}(I)$ can be approximated by finite linear combinations of g_n's:

$$\|f - (c_1 g_{n_1} + \ldots c_k g_{n_k})\| < \epsilon.$$

(ii) $\{g_n\}$ is a basis: every $f \in L^2_{\mathrm{pc}}(I)$ has a unique representation

$$f = \sum a_j g_j \qquad \text{(mean square convergence)} \tag{27.2}$$

Families with property (i) are called *complete* on I.

We can find the coefficients in (27.2) exactly as in the finite dimensional case:

$$a_j = \frac{(f, g_j)}{(g_j, g_j)}.$$

Theorem. *The trigonometric family is complete on* $(-l, l)$.

Corollary. *If* $f \in L^2_{\mathrm{pc}}(-l, l)$*, then*

$$f = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \Big) \tag{27.3}$$

in the sense of mean square convergence on $(-l, l)$*, and*

$$a_n = \frac{1}{l} \int_{-l}^{l} f(x) \cos \frac{n\pi x}{l} dx, \qquad b_n = \frac{1}{l} \int_{-l}^{l} f(x) \sin \frac{n\pi x}{l} dx.$$

Terminology: Fourier series, Fourier coefficients.

Exercises.

(a) Show that the trigonometric family (27.1) is an orthogonal basis on every interval of length $2l$.

(b) Show that the family $\left\{\sin \frac{n\pi x}{l}\right\}$ is an orthogonal basis on $(0, l)$. The same is true for the cosine family.

Hint: given a function on $(0, l)$, extend it to an odd function on $(-l, l)$. The Fourier series contains only the sine terms.

(c) Is the family $\{\sin kx\}$ complete on $(-\pi/2, \pi/2)$?

27.4. **Pointwise convergence.** There is a one-to-one correspondence between functions on $[-l, l)$ and $2l$-periodic functions. Note that the trigonometric functions in (27.1) are $2l$-periodic. It follows that the sum of the Fourier series (27.3) represents the $2l$-periodic extension of f.

Fact. *If* f *is a piecewise smooth function on* $[-l, l]$*, then at any point* $x \in \mathbb{R}$*, the Fourier series converges to*

$$\frac{F(x-0) + F(x+0)}{2}, \tag{27.4}$$

where F *is the* $2l$*-periodic extention of* f *from* $[-l, l)$ *to* $\mathbb{R}$.

Examples:

(a) f is periodic with period $2l = 2$, and $f(x) = x^2$ on $(0, 2)$. We have

$$a_0 = \int_0^2 x^2 dx = \frac{8}{3}, \quad a_n = \int_0^2 x^2 \cos(n\pi x) dx = \frac{4}{n^2\pi^2}, \quad b_n = -\frac{4}{n\pi},$$

and

$$f(x) \sim \frac{4}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos\, n\pi x}{n^2} - \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin\, n\pi x}{n}.$$

The value of (27.4) at $x = 0$ is 2, so we have

$$2 = \frac{4}{3} + \frac{4}{\pi^2} \sum_1^\infty \frac{1}{n^2},$$

or

$$\sum_1^\infty \frac{1}{n^2} = \frac{\pi^2}{6} \qquad \text{(Euler)}.$$

Similarly, setting $x = 1$ we find

$$\sum_1^\infty \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$

(b) f is odd, $2l$-periodic, and $f(x) = x$ on $[0, l]$. The Fourier series has only the sine terms:

$$f(x) \sim \frac{2l}{\pi} \left(\sin \frac{\pi x}{l} - \frac{1}{2} \sin \frac{2\pi x}{l} + \frac{1}{3} \sin \frac{3\pi x}{l} - \dots \right).$$

Since $f(l/2) = l/2$, we derive

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots \qquad \text{(Leibnitz)}.$$

27.5. **Fourier series solutions.** (a) Consider the equation of forced oscillations with damping, see Section 20,

$$\ddot{x} + 2c\dot{x} + \omega_0^2 x = F(t),$$

where $c > 0$ and F is a $2l$-periodic function. As we mentioned, there is a unique periodic solution $x_*(t)$, and its period is $2l$. Let's find $x_*(t)$. For simplicity assume that F is odd, so

$$F(t) = \sum B_n \sin \frac{n\pi t}{l},$$

the coefficients B_n are considered known. Then

$$x_*(t) = \sum \left(a_n \cos \frac{n\pi t}{l} + b_n \sin \frac{n\pi t}{l} \right)$$

with *undetermined* coefficients a_n, b_n, which are easy to find. Either use formal computations, or the principle of superposition (find the steady states for each n and sum them up).

(b) Undamped case:

$$\ddot{x} + \omega_0^2 x = F(t),$$

where F is as above. We have a *pure resonance* (all solutions are unbounded) if there is an n such that

$$B_n \neq 0, \qquad \omega_0 = \frac{n\pi}{l}.$$

28. Boundary value problems

28.1. **Trigonometric functions as eigenfunctions of boundary value problems.** Consider the equation

$$-u'' = \lambda u$$

with *Dirichlet* boundary conditions

$$u(0) = 0, \qquad u(\pi) = 0,$$

or with *Neumann* boundary conditions

$$u'(0) = 0, \qquad u'(\pi) = 0,$$

or with *periodic* boundary conditions

$$u(-\pi) = u(\pi), \qquad u'(-\pi) = u'(\pi).$$

The space of solutions of the equation $-u'' = \lambda u$ is a linear space of dimension 2. Since the boundary conditions (BC) are homogeneous, the space of solutions of the BVP [equation + BC] is also linear, and the dimension is ≤ 2. The values of λ for which the dimension is $\neq 0$ (i.e. there are non-trivial solutions) are *eigenvalues* of the BVP, and non-trivial solutions are the corresponding *eigenfunctions.* Let us find the eigenvalues and eigenfunctions of the three BVPs stated above.

(a) *Dirichlet* BC. If $\lambda = \omega^2 > 0$, then the general solution of the d.e. is

$$u(x) = A\cos\omega x + B\sin\omega x.$$

We get $A = 0$ from $u(0) = 0$. From $u(\pi) = 0$, we then find

$$\sin\omega\pi = 0, \qquad \text{so} \qquad \omega = \mathrm{n} \in \mathbb{Z}.$$

Similarly, we show that if $\lambda \leq 0$, then the equation has no non-trivial solutions.

Conclusion: the eigenvalues are simple (i.e. the space of BVP solutions is one-dimensional),

$$\lambda_n = n^2, \qquad (n \in \mathbb{N}),$$

and the eigenfunctions are

$$u_n(x) = \text{const } \sin nx.$$

As we mentioned earlier, these functions are orthogonal and the family is complete on $(0, \pi)$.

(b) *Neumann* BC. Clearly, $u(x) \equiv 1$ is an eigenfunction for $\lambda = 0$. If $\lambda = \omega^2 > 0$, then

$$u(x) = A\cos\omega x + B\sin\omega x, \qquad u' = -A\omega\sin\omega x + B\omega\cos\omega x.$$

From $u'(0) = 0$ we get $B = 0$, and then we use $u'(\pi) = 0$.

Conclusion: the eigenvalues $\lambda_n = n^2$, $n \in \mathbb{Z}_+$, are simple, the corresponding eigenfunctions are $u_n(x) = \text{const } \cos nx$.

(c) *Periodic* BC. It is clear that $\lambda = 0$ is a simple eigenvalue, and that if $\lambda_n = n^2$, $n \in \mathbb{N}$, then all solutions of the differential equation satisfy the BC.

Conclusion: the eigenvalues $\lambda_n = n^2$, $n \in \mathbb{Z}_+$, are double except for $\lambda = 0$, and the corresponding eigenfunctions are

$$u_n(x) = A\cos nx + B\sin nx.$$

We recover the full trigonometric family, which is complete on $(-\pi, \pi)$.

Exercises.

(d) Find the eigenvalues and eigenfunctions of the BVP

$$u'' + \lambda u = 0, \qquad u(0) = u'(\pi) = 0.$$

Answer: $\lambda_n = \omega_n^2$, where $\omega_n = n - \frac{1}{2}$, $n \in \mathbb{N}$; $u_n = \sin \omega_n x$.

(e) Consider the BVP

$$u'' + \lambda u = 0, \qquad u'(0) = u(\pi) + u'(\pi) = 0.$$

Estimate λ_n for $n \gg 1$.

28.2. **Regular Schrödinger operators.** Consider the eigenvalue problem for the equation

$$L[u] \equiv -u'' + q(x)u = \lambda u \tag{28.1}$$

on a *finite* interval $[a, b]$ assuming that $q(x)$ is a continuous *bounded* function. We will only consider the case of *separated* BC

$$h_a u(a) + u'(a) = 0, \qquad h_b u(b) + u'(b) = 0, \tag{28.2}$$

where $h_a, h_b \in \mathbb{R} \cup \{\infty\}$ are given constants; the case $h_a = h_b = \infty$ corresponds to Dirichlet boundary conditions. (There is also a theory for periodic conditions.)

Fact. *(i) The BVP* (28.1)-(28.2) *has infinitely many eigenvalues*

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n < \ldots, \qquad \lambda_n \to \infty.$$

(ii) All eigenvalues are simple and the family of eigenfunctions u_n is complete on (a, b). In fact, $\{u_n\}$ is an orthogonal basis:

$$\forall f \in L^2_{\mathrm{pc}}(a, b), \qquad f \;=\; \sum \frac{(f, u_n)}{(u_n, u_n)}\, u_n \qquad \text{(mean square convergence).} \tag{28.3}$$

Analogy with linear algebra. Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be a *symmetric* matrix,

$$(Ax, y) = (x, Ay).$$

Consider the eigenvalue problem

$$Ax = \lambda x.$$

We know that all eigenvalue are real, and if simple, then the eigenvectors form an orthogonal basis in $\mathbb{R}^n$.

In the case of the BVP (28.1)-(28.2) the role of A is played by the *operator* (or rather a certain extension of this operator)

$$L = (L[\,\cdot\,],\ \mathrm{BC}) : \mathrm{dom}(L) \to C[a, b],$$

where $L[\,\cdot\,]$ is the differential operation $u \mapsto -u'' + q(x)$, BC are the boundary conditions (28.2), and $\mathrm{dom}(L)$ is the space of C^2-functions on $[a, b]$ satisfying BC. The BVP is exactly the equation

$$Lu = \lambda u.$$

The operator L is *symmetric*:

$$(Lu, v) = (u, Lv), \qquad u, v \in \operatorname{dom}(L).$$

This can be verified as follows:

$$(Lu, v) = \int_a^b (-u'')v + \int_a^b quv = -u'v\Big|_a^b + \int_a^b v'u' + \int_a^b quv,$$

$$(u, Lv) = \int_a^b u(-v'') + \int_a^b quv = -uv'\Big|_a^b + \int_a^b v'u' + \int_a^b quv,$$

and $u'v = uv'$ at $x = a$ and $x = b$ because of the boundary conditions.

Using the symmetry of L we can prove some (easier) statements of the above theorem.

Lemma. *The eigenvalues of the BVP* (28.1)-(28.2) *are simple and the eigenfunctions are orthogonal.*

Proof: (i) If u and v are independent eigenfunctions corresponding to λ, then they satisfy the same d.e. and therefore the Wronskian $W = uv' - u'v$ does not vanish. But as we just noted, the Wronskian is zero at a and b because of the BC.

(ii) If $Lu = \lambda u$ and $Lv = \mu v$, then

$$\lambda(u, v) = (Lu, v) = (u, Lv) = \mu(u, v).$$

□

Remark. We can also show that the eigenvalues of L are real. We need to consider complex-valued functions; the scalar product of two complex-valued functions is defined as follows:

$$(u, v) = \int u\bar{v}.$$

We have

$$\lambda\|u\|^2 = (\lambda u, u) = (Lu, u) = (u, Lu) = \mu(u, \lambda u) = \bar{\lambda}\|u\|^2.$$

28.3. **Regular Sturm-Liouville BVPs.**

Example. Given $\lambda > 0$, find the minimal number $L > 0$ such that λ is an eigenvalue of the BVP

$$u'' + \lambda x u = 0, \qquad u'(0) = 0, \quad u(L) = 0. \tag{28.4}$$

Hint: if $A_1(x)$ is the Airy function defined in Section 23.2, then

$$u(x) = A_1\left(-\lambda^{1/3}x\right)$$

is a solution of the equation in (28.4) satisfying the first boundary condition.

Interpretation: when a uniform vertical column will buckle under its own weight? The unknown function $u(x)$ is the angle of deflection from the vertical direction; $x = 0$ is the *free* top end and $x = L$ is the *fixed* bottom end of the column; the column parameter λ depends on the density and the Young's modulus of the material, and on the shape of the cross section of the column. The column can buckle iff there is a non-trivial solution to the BVP (28.4).

The BVP (28.4) is an example of a *regular* Sturm-Liouville BVP

$$L[u] \equiv -(pu')' + qu = \lambda \rho u,$$

$$h_a u(a) + u'(a) = 0, \qquad h_b u(b) + u'(b) = 0.$$

which generalizes (28.1)-(28.2). Here we assume

$$p,\ p',\ q,\ \rho\ \in\ C[a,b]; \qquad p > 0,\ \rho > 0 \quad \text{on} \quad [a,b].$$

We can consider the Sturm-Liouville BVP as an eigenvalue problem for the operator

$$Lu = \rho^{-1} L[u], \qquad \mathrm{dom}(L) = \{u \in C^2[a,b],\ u \in (BC)\}.$$

Lemma. *The Sturm-Liouville operator L is symmetric with respect to the scalar product*

$$(u,v) = \int_a^b u(x)v(x)\ \rho(x)dx.$$

Proof: We need to show

$$(Lu, v) = (u, Lv) \qquad u, v \in \mathrm{dom}(L).$$

Since $(Lu, v) = \int L[u]v dx$, this follows from the *Lagrange identity* (integration by parts)

$$\int_a^b L[u]vdx - \int_a^b uL[v]dx = [pW]_a^b\,,$$

where W is the Wronskian of u and v. □

[Remark. The class of Sturm-Liouville operators is exactly the class of all symmetric 2nd order differential operators with real coefficients.]

All results mentioned in the previous subsection are valid for regular Sturm-Liouville operators. In particular, there is an orthogonal basis of eigenfunctions in $L^2(\rho dx)$, the space of square integrable functions with weight $\rho(x)$.

29. Fourier method in PDE

29.1. **1D heat equation.** Heat conduction in a rod:

$$u_t = \alpha^2 u_{xx}.$$

The unknown function

$$u = u(x,t), \qquad 0 \le x \le l, \quad t \ge 0,$$

represents temperature, and α is a constant parameter, "thermal diffusitivity", α is large for metals and small for insulators.

We impose the *initial* (with respect to t) condition

$$u(x, 0) = f(x),$$

and the (Dirichlet) *boundary* (with respect to x) conditions

$$u(0, \cdot) = 0, \qquad u(l, \cdot) = 0,$$

(we maintain zero temperature at the ends of the rod).

Main idea: *separation of variables.* We first find all PDE solutions of the form

$$\phi(x,t) = X(x)T(t) \tag{29.1}$$

satisfying the *boundary* conditions. It is clear that all linear combinations of such ϕ's (including correctly defined infinite combinations) will satisfy the PDE and the boundary conditions. It will only remain to find the right linear combination to satisfy the *initial* condition.

The function (29.1) is a PDE solution if $\alpha^2 X''T = X\dot{T}$, or

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{\dot{T}}{T} := -\lambda,$$

where λ has to be constant. Indeed, the function X''/X, which depends only on x, has to be equal to a function of t.

It follows that $\phi(x,t) = X(x)T(t)$ is a solution satisfying the boundary conditions if and only if X and T satisfy the following system for some $\lambda \in \mathbb{R}$:

$$X'' + \lambda X = 0, \qquad X(0) = X(l) = 0, \tag{29.2}$$

and

$$\dot{T} + \alpha^2 \lambda T = 0. \tag{29.3}$$

The eigenvalues and eigenfunctions of the BVP (29.2) are

$$\lambda_n = \left(\frac{\pi}{l}\right)^2 n^2, \qquad X_n(x) = \sin\frac{\pi n x}{l}, \qquad (n \in \mathbb{N}). \tag{29.4}$$

Solving (29.3) with $\lambda = \lambda_n$, we find a solution

$$T_n(t) = \exp\{-n^2\pi^2\alpha^2 l^{-2}\ t\}.$$

Returning to the PDE problem, we look for a linear combination

$$u(x,t) = \sum_{n=1}^{\infty} C_n X_n(x) T_n(t) \tag{29.5}$$

that satisfies the initial condition:

$$\sum_{n=1}^{\infty} C_n X_n(x) = f(x).$$

We use the orthogonality of the eigenfunctions of BVP (29.2) to find C_n's:

$$C_n = \frac{(f, X_n)}{(X_n, X_n)} = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx.$$

Done!

Comments. (a) What we have found is a *formal* solution but it is not difficult to show that if f is "nice", then (29.5) is indeed a solution. All we need to do is to justify the convergence and differentiation of the series. Another problem is the uniqueness of the solution. We may appeal to physical intuition and actually model a mathematical proof on it.

(b) As $t \to +\infty$, the solution tends to zero exponentially fast. The exponent of convergence is $\pi^2\alpha^2 l^{-2}$; it is given by the lowest eigenvalue λ_1. The main term of the solution is the term with $n = 1$ in the series (29.5).

Further examples.

(a) *Non-homogeneous boundary conditions.* Same problem with boundary conditions

$$u(0,\cdot) = u_0, \qquad u(l,\cdot) = u_l,$$

(we maintain given constant temperatures at the ends of the rod). We expect the solution to converge, as $t \to \infty$, to the steady state $v(x)$, a stationary PDE solution satisfying the boundary conditions. Solving the BVP

$$v''(x) = 0, \qquad v(0) = u_0, \quad v(l) = u_l,$$

we find

$$v(x) = (u_l - u_0)\frac{x}{l} + u_0.$$

If we set

$$u(x,t) = v(x) + w(x,t),$$

then w satisfies the heat equation with homogeneous boundary conditions and

$$w|_{t=0} = f(x) - v(x).$$

(b) *Insulated endpoints.* Same problem with Neumann boundary conditions

$$u_x(0,t) = u_x(l,t) = 0.$$

The BVP for $X(x)$ is now

$$X'' + \lambda X = 0, \qquad X'(0) = X'(l) = 0.$$

The eigenvalues and eigenfunctions are

$$\lambda_n = n^2\left(\frac{\pi}{l}\right)^2, \qquad X_n = \cos\frac{n\pi x}{l}, \qquad (n \in Z_+).$$

As $t \to \infty$, the solution converges to the steady state

$$u_* = \text{const} = \frac{1}{l}\int_0^l f(x)dx,$$

the mean value of the initial temperature.

(c) *Non-homogeneous heat equation.* Consider the PDE

$$u_t = u_{xx} + F(x,t)$$

with Dirichlet boundary conditions and with the trivial initial condition $u(x,0) \equiv 0$. Then

$$F(x,t) = \sum a_n(t)X_n(x), \qquad u(x,t) = \sum c_n(t)X_n(x),$$

where $a_n(t)$ are known and $c_n(t)$ unknown functions. We have the following IVP for c_n's:

$$\dot{c}_n = -\lambda_n c_n + a_n(t), \qquad c_n(0) = 0,$$

where λ_n's (and X_n's) are given by (29.4).

29.2. **1D wave equation.** Vibration of elastic string:

$$u_{tt} = c^2 u_{xx}.$$

Here $u(x,t)$ is the displacement of the string, and c is the (constant) velocity of propagation of waves.

Boundary conditions: Dirichlet at fixed and Neumann at free ends of the string.

Initial conditions:

$$u|_{t=0} = f(x), \qquad u_t|_{t=0} = g(x),$$

the initial shape and velocity of the string. Note that the PDE is 2nd order in t so we need two initial conditions.

Separation of variables:

$$\frac{X''}{X} = \frac{1}{c^2}\frac{\ddot{T}}{T} = -\lambda,$$

so $\phi(x,t) = X(x)T(t)$ is a PDE solution satisfying the boundary conditions if and only if X and T satisfy the following system for some $\lambda \in \mathbb{R}$:

$$X'' + \lambda X = 0, \qquad X(0) = X(l) = 0,$$

(in the case of Dirichlet BC), and

$$\ddot{T} + c^2 \lambda T = 0.$$

The eigenvalues and the eigenfunctions of the BVP are

$$\lambda_n = \omega_n^2, \quad X_n(x) = \sin \omega_n x, \quad \omega_n = \frac{\pi n}{l}, \qquad (n \in \mathbb{N}).$$

Also,

$$T_n(t) = A_n \cos c\omega_n t + B_n \sin c\omega_n t,$$

and therefore

$$u(x,t) = \sum_{n=1}^{\infty} T_n(t) X_n(x) = \sum_{n=1}^{\infty} A_n \ \cos c\omega_n t \ \sin \omega_n x + \sum_{n=1}^{\infty} B_n \ \sin c\omega_n t \ \sin \omega_n x.$$

Since

$$T_n(0) = A_n, \qquad \dot{T}_n(0) = c\omega_n B_n,$$

we can find A_n and B_n from the initial conditions. Namely,

$$\sum A_n X_n = f, \qquad c \sum \omega_n B_n X_n = g,$$

so

$$A_n = \frac{(f, X_n)}{(X_n, X_n)} = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx,$$

and

$$B_n = \frac{1}{c\omega_n} \frac{(g, X_n)}{(X_n, X_n)} = \frac{2}{c\pi n} \int_0^l g(x) \sin \frac{n\pi x}{l} dx.$$

Done!

Remark. If $g = 0$, then $B_n = 0$ and

$$\begin{aligned} u(x,t) &= \sum A_n \cos c\omega_n t \sin \omega_n x \\ &= \frac{1}{2} \sum A_n \sin c\omega_n (x + ct) + \frac{1}{2} \sum A_n \sin c\omega_n (x - ct) \\ &= h(x + ct) + h(x - ct), \end{aligned}$$

where

$$h(x) = \frac{1}{2}\sum A_n \sin \omega_n x.$$

For any $h(x)$ the functions $h(x \pm ct)$ represent waves moving along the x-axis in the negative/positive directions with speed c, so our solution is a superposition of two waves.

In fact, it is easy to see that

$$u(x,t) = h_1(x+ct) + h_2(x-ct),$$

where h_1 and h_2 are arbitrary functions, is the general solution of the 1D wave equation (*d'Alambert solution*).

30. Laplace and wave equations in a disc

30.1. **Laplace operator.** For $u = u(x,y)$ we denote

$$\Delta u = u_{xx} + u_{yy}.$$

The equations

$$u_t = \alpha^2 \Delta u, \qquad u_{tt} = c^2 \Delta u$$

are 2D heat equation and 2D wave equation respectively. Stationary, i.e. time independent, solutions of the heat and wave equations are described by the Laplace equation

$$\Delta u = 0.$$

Lemma. *If $u = u(r,\theta)$ in the polar coordinates, then*

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}.$$

Proof: We have

$$r = \sqrt{x^2+y^2}, \qquad \theta = \arctan\frac{y}{x},$$

so

$$r_x = \frac{x}{r}, \quad r_y = \frac{y}{r}, \quad r_{xx} = \frac{1}{r} - \frac{x^2}{r^3}, \quad r_{yy} = \frac{1}{r} - \frac{y^2}{r^3}, \quad \Delta r = \frac{1}{r};$$

$$\theta_x = -\frac{y}{r^2}, \quad \theta_y = \frac{x}{r^2}, \quad \Delta\theta = 0.$$

Therefore,

$$u_x = u_r r_x + u_\theta \theta_x,$$

$$u_{xx} = u_{rr} r_x^2 + 2u_{r\theta} r_x \theta_x + u_r r_{xx} + u_{\theta\theta}\theta_x^2 + u_\theta \theta_{xx},$$

and similar for u_{yy}. We have

$$\Delta u = u_{rr}(r_x^2 + r_y^2) + u_r \Delta r + u_{\theta\theta}(\theta_x^2 + \theta_y^2).$$

□

30.2. **Laplace equation.** Consider the *Dirichlet problem* for the Laplace equation in the disc of radius r_0,

$$\Delta u = 0, \qquad u(r_0, \theta) = f(\theta),$$

where we use polar coordinates and $f(\theta)$ is a given function.

[For example, u is the steady state temperature or the electrostatic potential.]

Separating the variables we have

$$u(r, \theta) = R(r)\Theta(\theta),$$

$$\Delta u = R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0,$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} := \lambda.$$

The BVP for $\Theta(\theta)$ has periodic BC, see Section 35.1,

$$\Theta'' + \lambda\Theta = 0, \qquad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi),$$

so we have $\lambda = n^2$, $n \in \mathbb{Z}_+$, with eigenfunctions being the usual trigonometric functions. The equation for $R = R_n(r)$ is then

$$r^2 R'' + rR' - n^2 R = 0 \tag{30.1}$$

with an additional condition $|R(0)| < \infty$. The general solution of the *Cauchy-Euler* equation (30.1) is

$$R(r) = C_1 r^n + C_2 r^{-n}.$$

Because of the requirement $|R_n(0)| < \infty$ we have (up to a constant)

$$R_n(r) = r^n.$$

Conclusion: the solution of the Dirichlet problem is

$$u(r, \theta) = \frac{a_0}{2} + \sum r^n(a_n \cos n\theta + b_n \sin n\theta),$$

where $r_0^n a_n$ and $r_0^n b_n$ are the Fourier coefficients of the given boundary function $f(\theta)$.

30.3. **Vibration of a circular membrane.** Consider the wave equation

$$u_{tt} = c^2\Delta u$$

in the unit disc with Dirichlet boundary conditions. We will assume the circular symmetry of the initial conditions, so u does not depend on θ in polar coordinates, $u = u(r, t)$, and we have the PDE

$$u_{tt} = c^2\left(u_{rr} + \frac{1}{r}u_r\right)$$

with the boundary condition

$$u\Big|_{r=1} = 0$$

and the initial conditions

$$u\Big|_{t=0} = f(r), \qquad u_t\Big|_{t=0} = g(r).$$

We also require that $u(r, t)$ be bounded as $r \to 0$.

Separation of variables:

$$u(r, t) = R(r)T(t),$$

$$\frac{R'' + \frac{1}{r}R'}{R} = \frac{1}{c^2}\frac{T''}{T} := -\lambda := -\kappa^2,$$

and we have the BVP

$$R'' + \frac{1}{r}R' + \kappa^2 R = 0, \qquad R \quad \text{bounded at } 0, \qquad R(1) = 0, \tag{30.2}$$

and also the equation

$$T'' + \kappa^2 c^2 T = 0.$$

(We denoted $\lambda = \kappa^2$ assuming the physically obvious fact that all eigenvalues of the BVP are positive: in the wave equation we expect oscillations in t, but this can happen only if $\lambda > 0$ in the second equation $T'' + \lambda c^2 T = 0$.)

The BVP (30.2) is an example of a *singular* Sturm-Liouville BVP. We can rewrite the equation in the Sturm-Liouville form

$$(rR')' = \lambda r R,$$

but the weight function $\rho(r) = r$ does not meet the regularity condition $\rho > 0$ on $[0, 1]$, see Section 28.3. Nevertheless, all properties of regular BVPs happen to be true in this particular case. Namely, all eigenvalues λ_n are simple, and the eigenfunctions $R_n(r)$ form an orthogonal basis on the interval $[0, 1]$ with weight $\rho(r) = r$.

The eigenvalues and eigenfunctions can be expressed in terms of the Bessel function

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{[m!]^2}\left(\frac{x}{2}\right)^{2m}.$$

The only bounded at zero solution of the equation $(rR')' = \lambda r R$ is

$$R(r) = J_0(\sqrt{\lambda}).$$

It follows that the eigenvalues of the BVP (30.2) are

$$\lambda_n = \kappa_n^2,$$

where $\kappa_n \sim n - \frac{\pi}{4}$ are the zeros of J_0. The corresponding eigenfunctions are

$$R_n(r) = J_0(\kappa_n r).$$

The second equation (assuming for simplicity that $g = 0$) has the following solution

$$T_n(t) = \cos(c\kappa_n t).$$

Returning to the PDE problem, we have the following series representation of the solution:

$$u(r,t) = \sum_{n=1}^{\infty} a_n \cos(c\kappa_n t) J_0(\kappa_n r),$$

where the coefficients a_n are determined by the initial data

$$\sum_{n=1}^{\infty} a_n J_0(\kappa_n r) = f(r).$$

Since the eigenfunctions $R_n(r)$ are orthogonal on $[0, 1]$ with weight $\rho(r) = r$, we have

$$a_n = \frac{(f, R_n)}{(R_n, R_n)} = \frac{\int_0^1 f(r) R_n(r)\, r dr}{\int_0^1 R_n^2(r)\, r dr}.$$

31. Laplace transform

- Laplace transform $\mathcal{L}$ is a linear integral operator, a relative of Fourier transform.
- Main property: $\mathcal{L}$ transforms differentiation into multiplication by the independent variable, and so $\mathcal{L}$ transforms linear ODEs with constant coefficients into algebraic equations.
- Euler invented this integral transform, but Laplace used it later in probability theory. The application to ODEs was suggested by Heaviside.
- Laplace transform is quite useful in certain problems of ODEs (e.g., equations with periodic driving force, representation of solutions as convolutions, delay equations) as well as PDEs.

31.1. **Definition and first examples.** Consider the map $\mathcal{L}[f] = F$,

$$\mathcal{L} : f(t) \mapsto F(s) = \int_0^\infty e^{-st} f(t) dt.$$

We assume that $f(t)$ is defined and piecewise continuous for $t > 0$, and that there are constants C and a such that

$$|f(t)| \leq Ce^{at}$$

[e.g. $f(t) = e^{t^2}$ does not satisfy the last condition.] Then $F(s)$ is defined for $s > a$, $F \in C^\infty(a, \infty)$, and $F(\infty) = 0$ [as a limit]. The correspondence

$$\mathcal{L}^{-1} : F(s) \mapsto f(t)$$

is called the *inverse Laplace transform*, see below.

Examples.

$$(1) \qquad f(t) = 1 \implies F(s) = \frac{1}{s}, \ (s > 0);$$

$$(2) \qquad f(t) = e^{at} \implies F(s) = \frac{1}{s-a}, \ (s > a);$$

$$(3) \qquad f(t) = \sin \omega t \implies F(s) = \frac{\omega}{s^2 + \omega^2}, \ (s > 0);$$

$$(4) \qquad f(t) = \cos \omega t \implies F(s) = \frac{s}{s^2 + \omega^2}, \ (s > 0);$$

$$(5) \qquad f(t) = e^{at} \sin \omega t \implies F(s) = \frac{\omega}{(s-a)^2 + \omega^2}, \ (s > a);$$

$$(6) \qquad f(t) = e^{at} \cos \omega t \implies F(s) = \frac{s-a}{(s-a)^2 + \omega^2}, \ (s > a).$$

Proof: If $\lambda = a + i\omega$, then

$$\int_0^\infty e^{-st} e^{\lambda t} dt = \int_0^\infty e^{(\lambda - s)t} dt = \frac{1}{s - \lambda}.$$

Taking the real and imaginary parts, we have

$$\frac{1}{s-\lambda} = \frac{1}{(s-a)-i\omega} = \frac{(s-a)+i\omega}{(s-a)^2+\omega^2}.$$

□

For $c \in \mathbb{R}$, let χ_c denote the step or *Heaviside* function

$$\chi_c(t) = \begin{cases} 0, & (t < c), \\ 1. & (t > c). \end{cases}$$

Claim: if $c > 0$, then

$$(7) \qquad f(t) = \chi_c(t) \quad \Longrightarrow \quad F(s) = \frac{e^{-cs}}{s}, \ (s > 0).$$

Proof:

$$\int_c^\infty e^{-st} dt = -\frac{1}{s} e^{-st}\Big|_c^\infty.$$

□

31.2. **Properties of Laplace transform.**

Lemma. *(i) $\mathcal{L}$ is a linear operator;*
(ii) $\mathcal{L}[f(kt)] = k^{-1}F(s/k)$, $(k > 0)$;
(iii) $\mathcal{L}[e^{at}f(t)] \ = \ F(s-a)$, $\qquad \mathcal{L}[\chi_c(t)f(t-c)] \ = \ e^{-cs}F(s)$

[The last property means the following: shifting the argument of a function corresponds, under the direct or inverse transform, to the multiplication by some exponential factor.]

Example. Let

$$f(t) = \begin{cases} 1, & t \in (0,1), \\ (2-t), & t \in (1,2), \\ 0, & t \in (0,\infty). \end{cases}$$

Then

$$\begin{aligned} f(t) &= 1 \cdot \chi_{(0,1)} + (2-t) \cdot \chi_{(1,2)} \\ &= (\chi_0 - \chi_1) + (2-t)(\chi_1 - \chi_2) = \chi_0 - (t-1)\chi_1 + (t-2)\chi_2, \end{aligned}$$

and

$$\begin{aligned} \mathcal{L}[f] &= \mathcal{L}[\chi_0] - e^{-s}\mathcal{L}[t] + e^{-2s}\mathcal{L}[t] \\ &= \frac{1}{s} + \frac{e^{-2s} - e^{-s}}{s^2}, \end{aligned}$$

where we used the formula $\mathcal{L}[t] = s^{-2}$, see below. The same method works for general piecewise elementary functions.

Theorem. *If f is T-periodic, then*

$$\mathcal{L}[f] = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t) dt.$$

Proof:

$$\int_0^\infty e^{-st}f(t)dt = \int_0^T e^{-st}f(t)dt + \int_0^{2T} e^{-st}f(t)dt + \dots;$$

$$\int_{nT}^{(n+1)T} e^{-st}f(t)dt = \int_0^T e^{-s(u+nT)}f(u)du = e^{-snT}\int_0^T e^{-st}f(t)dt;$$

$$\sum_{n\geq 0} e^{-nTs} = \frac{1}{1-e^{-Ts}}.$$

□

Example. Let $f = \chi_{0,1} - \chi_{1,2} + \chi_{2,3} - \dots$ ("square wave"). The period of f is $T = 2$, and we have

$$\int_0^1 e^{-st}dt - \int_1^2 e^{-st}dt = -\frac{1}{s}e^{-st}\Big|_0^1 + \frac{1}{s}e^{-st}\Big|_1^2 = s^{-1}(1-e^{-s})^2,$$

so

$$\mathcal{L}[f] = \frac{(1-e^{-s})^2}{s(1-e^{-2s})} = \frac{(1-e^{-s})}{s(1+e^{s})} = s^{-1}\tanh(s/2).$$

31.3. **Transform of the derivatives.**

Theorem.

$$\mathcal{L}[f'] = sF(s) - f(0), \qquad \mathcal{L}[tf(t)] = -F'$$

Proof: The first formula is integration by parts:

$$\int_0^\infty f'(t)e^{-st}dt = f(t)e^{-st}\Big|_{s=0}^{\infty} - \int_0^\infty f(t)(-s)e^{-st}dt.$$

We get the second formula if we differentiate the integral $\mathcal{L}[f](s)$ wrt to the parameter s:

$$F'(s) = \int_0^\infty f(t)(-t)e^{-st}dt.$$

□

Corollary.

$$\mathcal{L}[f^{(n)}] = s^nF(s) - s^{n-1}f(0) - \dots - f^{(n-1)}(0).$$

Proof:

$$\begin{aligned}\mathcal{L}[f''] &= s\mathcal{L}[f'] - f'(0) = s(sF(s) - f(0)) - f'(0)\\ &= s^2F(s) - sf(0) - f'(0),\end{aligned}$$

and so on. □

Corollary.

$$\mathcal{L}[t^nf(t)] = (-1)^nF^{(n)}.$$

Examples:

$$(8) \qquad f(t) = t^n \quad \Longrightarrow \quad F(s) = \frac{n!}{s^{n+1}},\ (s > 0);$$

$$(9) \qquad f(t) = t^n e^{at} \quad \Longrightarrow \quad F(s) = \frac{n!}{(s-a)^{n+1}},\ (s > a).$$

Proof: To get (8), we write

$$F = \mathcal{L}[1] = \frac{1}{s}, \quad \mathcal{L}[t] = -F' = \frac{1}{s^2}, \quad , \mathcal{L}[t^2] = \frac{2}{s^3}, \dots$$

By shifting, we get (9). □

Remark. Generalization of (8):

$$\mathcal{L}[t^p] = \frac{\Gamma(p+1)}{s^{p+1}}, \quad (p > -1).$$

In particular we see that the function $t^{-1/2}$ is an "eigenvector" of the Laplace transform:

$$\mathcal{L}[t^{-1/2}] = \sqrt{\pi} \cdot s^{-1/2}.$$

Gamma-function is defined by the formula

$$\Gamma(1+p) = \int_0^\infty e^{-x} x^p dx, \quad (p > -1).$$

To find the Laplace transform

$$\mathcal{L}[t^p] = \int_0^\infty e^{-st} t^p dt$$

simply change the variable $x = st$. One can easily show (integration by parts) that

$$\Gamma(1+p) = p\Gamma(p), \quad \Gamma(1) = 1, \quad \Gamma(n+1) = n!$$

Also,

$$\Gamma(1/2) = \int_0^\infty e^{-x} \frac{dx}{\sqrt{x}} = 2\int_0^\infty e^{-x}\ d\sqrt{x} = \int_{-\infty}^\infty e^{-x^2}\ dx = \sqrt{\pi}.$$

Corollary.

$$\mathcal{L}\left[\int_0^t g\right] = \frac{G(s)}{s}, \qquad \mathcal{L}[t^{-1}g(t)] = \int_s^\infty G(\sigma) d\sigma.$$

Proof: This is just another way to state the theorem. For example, denote $f(t) = \int_0^t g$. Then $f(0) = 0$, and $f' = g$. So

$$G = \mathcal{L}[f'] = sF(s) - f(0) = sF(s).$$

□

Example:

$$f(t) = \frac{\sin t}{t} \quad \Longrightarrow \quad F(s) = \arctan(1/s).$$

31.4. **Inverse transform.** We can define the inverse transform because of the following uniqueness theorem (which we don't prove):

if $F \equiv G$, then $f(t) = g(t)$ wherever both are continuous.

There are systematic ways to compute the inverse transform but we'll just read the list of formulae (1)-(9) right to left, and also use the properties of $\mathcal{L}$ mentioned above. In particular, we can find the inverse transform of any rational function

$$R(s) = \frac{P(s)}{Q(s)}, \qquad \deg P < \deg Q.$$

To do this, use the *partial fraction* decomposition:

- Factorize Q into linear functions and irreducible quadratic factors corresponding to real and complex zeros of Q.
- The part of the decomposition corresponding to a linear factor $(s-a)$ of multiplicity ν is the sum (with undetermined coefficients which you have to find)
$$\frac{A_1}{(s-a)} + \cdots + \frac{A_\nu}{(s-a)^\nu}.$$
- The part of the decomposition corresponding to a quadratic factor $(s-a)^2 + \omega^2$ of multiplicity ν is the sum
$$\frac{A_1 s + B_1}{(s-a)^2+\omega^2} + \cdots + \frac{A_\nu s + B_\nu}{[(s-a)^2+\omega^2]^\nu}.$$
- Use the formulae
$$\mathcal{L}^{-1}: \quad \frac{1}{(s-a)^n} \quad \mapsto \quad \frac{1}{(n-1)!} t^{n-1} e^{at},$$
$$\mathcal{L}^{-1}: \quad \frac{1}{(s-a)^2+\omega^2} \quad \mapsto \quad \frac{1}{\omega} e^{at} \sin \omega t,$$
$$\mathcal{L}^{-1}: \quad \frac{s-a}{(s-a)^2+\omega^2} \quad \mapsto \quad e^{at} \cos \omega t.$$
- The inverse transform of other elementary fractions can be derived by differentiating the Laplace transforms of sin and cos. For example,
$$\mathcal{L}[t \sin \omega t] = \frac{2\omega s}{(s^2+\omega^2)^2},$$
$$\mathcal{L}[\sin \omega t - \omega t \cos \omega t] = \frac{2\omega^3}{(s^2+\omega^2)^2}.$$
[One should expect such terms in the resonance situation.]

31.5. **Application to differential equations.** Consider the IVP$(0, y_0, y_0')$ for the eq.

$$my'' + cy' + ky = g(t).$$

Denote $Y = \mathcal{L}[y]$. Then

$$m[s^2Y - sy_0 - y_0'] + c[sY - y_0] + kY = G,$$

or

$$P(s)Y - I(s) = G,$$

where $P(s)$ is the characteristic polynomial, and

$$I(s) = msy_0 + my_0' + cy_0$$

depends on the initial conditions only. We get

$$Y = \frac{G}{P} + \frac{I}{P}.$$

The miracle of the Laplace transform is that the DE has been transformed into an algebraic equation.

Special cases:

(a) In the case of forced vibrations with damping, the inverse transforms

$$\mathcal{L}^{-1}[G/P], \qquad \mathcal{L}^{-1}[I/P]$$

are the steady state and the transient solution respectively.

(b) For the equation

$$y'' + \omega_0^2 y = F_0 \sin \omega t$$

with $I = 0$ we have

$$Y(s) = \frac{F_0\omega}{(s^2 + \omega^2)(s^2 + \omega_0^2)},$$

and we need to use different formulae in the resonance and non-resonance cases.

EX:

The Laplace transform method can be used to solve some systems of DEs. Analyze the following example (*"coupled springs"*):

$$m_1 y_1'' + k_1 y_1 = k_2(y_2 - y_1),$$
$$m_2 y_2'' + k_2(y_2 - y_1) = g(t).$$

31.6. **Application to difference and delay-differential equations.**

Example 1. Consider the following difference equation:

$$x(t) = ax(t-b) + f(t) \quad \text{for} \quad t > 0, \qquad x(t) = 0 \quad \text{for} \quad t \le 0.$$

Denote $F = \mathcal{L}[f]$ and $X = \mathcal{L}[x]$. Then

$$X = ae^{-bs}X + F,$$

so

$$X(s) = \frac{F(s)}{1 - ae^{-bs}},$$

and we are done if we can find the inverse transform. For example, consider the problem

$$3x(t) - 4x(t-1) = 1, \qquad x(t) = 0 \quad \text{for} \quad t < 0.$$

We find

$$3X - 4e^{-s}X = s^{-1},$$

$$X = \frac{1}{s(3 - 4e^{-s})} = \frac{1}{3s(1 - \frac{4}{3}e^{-s})} = \sum_{n \ge 0} \frac{4^n}{3^{n+1}} s^{-1} e^{-ns},$$

and

$$x = \sum_{n \ge 0} \frac{4^n}{3^{n+1}} \chi_n = \frac{1}{3}\chi_0 + \frac{4}{9}\chi_1 + \dots$$

so $x = 1/3$ on $(0,1)$, $x = 7/9$ on $(1,2)$, etc.

Example 2. For positive parameters τ and λ consider the problem

$$\dot{v}(t + \tau) = \lambda[t - v(t)], \qquad v(t) = 0 \quad \text{for} \quad t \le \tau.$$

If $V = \mathcal{L}[v]$, then

$$\mathcal{L}[v(t+\tau)] = e^{\tau s}V(s), \qquad \mathcal{L}[\dot{v}(t+\tau)] = se^{\tau s}V(s),$$

the latter is because $v(0 + \tau) = 0$. Thus

$$se^{\tau s}V = \lambda(s^{-2} - V),$$

and

$$\begin{aligned} V &= \frac{\lambda}{s^2(\lambda + se^{s\tau})} = \frac{\lambda}{s^3 e^{s\tau}}(1 + \lambda s^{-1}e^{-\tau s})^{-1} \\ &= \frac{\lambda}{s^3}e^{-s\tau} - \frac{\lambda^2}{s^4}e^{-2s\tau} + \frac{\lambda^3}{s^5}e^{-3s\tau} - \dots \end{aligned}$$

applying the inverse transform, we have

$$v = \chi_\tau \frac{\lambda}{2!}(t - \tau)^2 - \chi_{2\tau}\frac{\lambda^2}{3!}(t - 2\tau)^3 + \dots$$

Note. The equation in the last example has an interesting interpretation. Suppose two cars are stopped at a red light. The light turns green, and the first car accelerates with a constant acceleration 1, so its velocity and position are

$$\tilde{v}(t) = t, \qquad \tilde{x}(t) = t^2/2.$$

The driver of the second car, which starts from the position $x_0 < 0$, responds with an acceleration proportional to the magnitude of the relative velocity. The response is of course not instantaneous; let τ be the response time (i.e. the delay). The equation is then

$$\dot{v}(t+\tau) = \lambda[\tilde{v}(t) - v(t)].$$

The position of the second car is given by the function

$$x(t) = x_0 + \int_0^t v(s)ds,$$

so

$$f(t) = x_0 + \frac{\lambda}{6}(t-\tau)^3, \qquad , t \in (2\tau, 2\tau),$$

$$f(t) = x_0 + \frac{\lambda}{6}(t-\tau)^3 - \frac{\lambda^2}{24}(t-2\tau)^4, \qquad t \in (2\tau, 3\tau),$$

etc. Clearly, for certain combination of parameters there will be a collision $x(t) = \tilde{x}(t)$ for some t.

31.7. **Convolutions.** For two (piecewise continuous) functions $f(t)$ and $g(t)$ defined for $t > 0$, we denote

$$(f * g)(t) = \int_0^t f(t-u)g(u)du.$$

Lemma. *Convolution is commutative, associative, and distributive:*

$$f * g = f * g,$$
$$(f * g) * h = f * (g * h),$$
$$(f + g) * h = f * h + g * h.$$

This can be proven directly, but it also follows from the following fundamental fact.

Theorem.

$$\mathcal{L}[f * g] = FG.$$

Proof:

$$\int_0^\infty e^{-st}dt \int_0^t g(u)f(t-u)\ du = \int_0^\infty e^{-su}g(u)\ du \int_u^\infty e^{-s(t-u)}f(t-u)dt.$$

□

Corollary. *Consider the IVP for $L[y] = g$ with trivial initial conditions. Then*

$$y(t) = \int_0^t h(t-u)g(u)du, \qquad h = \mathcal{L}^{-1}[1/p],$$

where p is the characteristic polynomial.

Proof:

$$p\mathcal{L}[y] = \mathcal{L}[g], \qquad \mathcal{L}[y] = \mathcal{L}[g]\mathcal{L}[h] = \mathcal{L}[g * h]$$

□

Example: tautochrone. Huygens showed in the 17-th century that (an arc of) the cycloid has the following property. It's a curve down which a particle sliding freely in the gravitational field will reach the bottom in the same time regardless of the starting point.

Suppose a curve $x = x(u)$ is in the first quadrant of the xu-plane, where u is height. The starting height is u_0, and let $v = v(u)$ denote the velocity as a function of height. By conservation of energy,

$$v^2(y) = \mathrm{const}(u_0 - u).$$

On the other hand,

$$|v| = \sqrt{\dot{x}^2 + \dot{u}^2} = \sqrt{1 + (dx/du)^2}|\dot{u}| := f(u)|\dot{u}|.$$

All we need is to find the function f. We'll show that $f = \mathrm{const} \cdot k$, where k is the eigenfunction of the Laplace transform $k(u) = u^{-1/2}$; then it's elementary to get the equation of the cycloid. Equating the two expressions of the velocity, we have

$$\frac{dt}{du} = \mathrm{const}\frac{f(u)}{\sqrt{u_0 - u}},$$

and the time to reach the bottom is const times

$$\int_0^{u_0} \frac{f(u)}{\sqrt{u_0 - u}} du = (f * k)(u_0).$$

This has to be independent of u_0, and so we have $f * k \equiv \mathrm{const}$. Applying $\mathcal{L}$, we get

$$\mathcal{L}[f] = \mathrm{const}\frac{\mathcal{L}[1]}{\mathcal{L}[k]} = \mathrm{const} \cdot k.$$