Physics 106b — Classical Mechanics

Michael Cross

California Institute of Technology

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Lecture 2: Special Relativity - Geometric Approach

Principle of Relativity

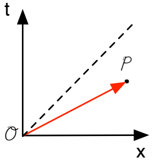
The Principle of Relativity states:

The laws of physics are the same in all inertial frames.

Alternative statement:

Every law of physics must be expressible as a geometric, frame independent relationship between geometric, frame independent objects.

Speed of light is fastest speed



Events $\mathcal{O}$, $\mathcal{P}$ separated by Δx, Δt in frame of reference S, with $\Delta x/\Delta t > 1$.

If information could travel faster than light the event $\mathcal{O}$ could influence the event $\mathcal{P}$.

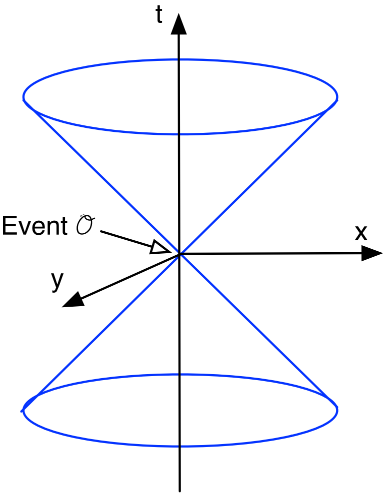
Now consider some other frame S' for which the time interval between the events is

$$\Delta t' = \gamma(\Delta t - v\Delta x) = \gamma\Delta t\left(1 - v\frac{\Delta x}{\Delta t}\right)$$

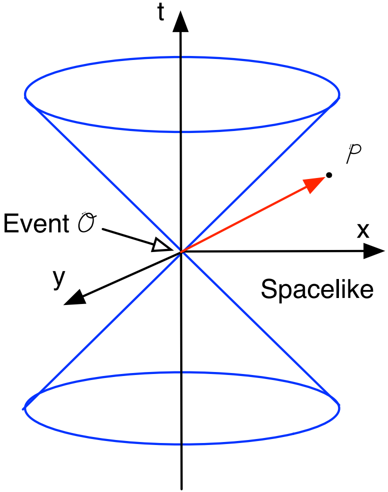
Since $\Delta x/\Delta t > 1$, we can find some frame S' given by a physical $v < 1$ for which $\Delta t' < 0$, i.e. the order of events is reversed.

This is inconsistent with our basic idea of causality.

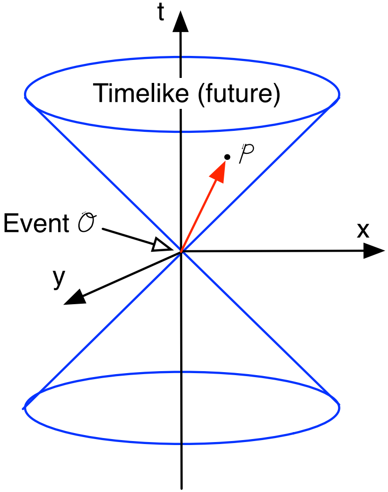
Light cone



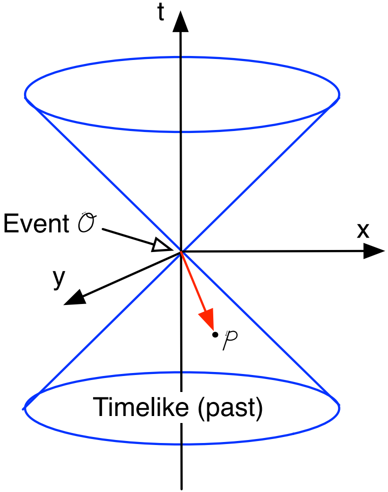
Light cone: spacelike region



Light cone: timelike (future) region



Light cone: timelike (past) region



Interval or length^2 of $\boldsymbol{\Delta x}$

Consider the 4-vector $\mathbf{\Delta x}$ joining the events $\mathcal{O}$, $\mathcal{P}$

- For a timelike 4-vector it is possible to find an inertial clock that is present at both events. The interval is defined from the time interval $\Delta\tau$ measured by this clock

$$\Delta s^2 = \Delta\tau^2$$

- For a spacelike 4-vector it is possible to find a stationary ruler joining the two events (i.e. there is an inertial frame in which the two events are simultaneous, and their separation can be measured). The interval is defined from the spatial separation Δl measured by this ruler

$$\Delta s^2 = -\Delta l^2$$

- For a lightlike 4-vector (event $\mathcal{P}$ on the lightcone of $\mathcal{O}$) the interval is zero

$$\Delta s^2 = 0$$

- In a frame S in which $\boldsymbol{\Delta x}$ has components $(\Delta t, \Delta\vec{x})$ the interval is

$$\boldsymbol{\Delta x}^2 \equiv \Delta s^2 = \Delta t^2 - \Delta\vec{x}^2 = \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

Scalar product of 4-vectors

- The scalar (dot) product of two 4-vectors is

$$\mathbf{\Delta x}_1 \cdot \mathbf{\Delta x}_2 = \frac{1}{2}\left[(\mathbf{\Delta x}_1 + \mathbf{\Delta x}_2)^2 - \mathbf{\Delta x}_1^2 - \mathbf{\Delta x}_2^2\right]$$

  where $(\mathbf{\Delta x}_1 + \mathbf{\Delta x}_2)^2$ means the interval corresponding to the sum 4-vector

- The dot product is a *scalar* – i.e. invariant under Lorentz transformations

- Evaluating in terms of the components in some inertial frame S gives

$$\mathbf{\Delta x}_1 \cdot \mathbf{\Delta x}_2 = \Delta t_1 \Delta t_2 - \Delta\vec{x}_1 \cdot \Delta\vec{x}_2 = \Delta t_1 \Delta t_2 - \Delta x_1 \Delta x_2 - \Delta y_1 \Delta y_2 - \Delta z_1 \Delta z_2$$

- Performing a Lorentz transformation on all the components to an S' frame would give the same expression in terms of the primed components

$$\mathbf{\Delta x}_1 \cdot \mathbf{\Delta x}_2 = \Delta t_1' \Delta t_2' - \Delta\vec{x}_1' \cdot \Delta\vec{x}_2' = \Delta t_1' \Delta t_2' - \Delta x_1' \Delta x_2' - \Delta y_1' \Delta y_2' - \Delta z_1' \Delta z_2'$$

Constructing 4-vectors

- Multiply a known 4-vector by a scalar (i.e. an invariant quantity, the same in all inertial frames)
- If we know that **B** is a 4-vector and an object **A** with four components gives a scalar product $\mathbf{A} \cdot \mathbf{B}$ that is a scalar (Lorentz invariant) for every value of **B**, then **A** is a 4-vector
- A quantity with four components that are related in different inertial frames by a Lorentz transformation, i.e. if A^{μ} transforms like Δx^{μ} then **A** is a 4-vector.

Electromagnetic 4-tensor

In some inertial frame S

$$F^{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

The electromagnetic field tensor can be derived from the potential 4-vector

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha .$$