Physics 106b — Classical Mechanics

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Lecture 3: Special Relativity - Mechanics

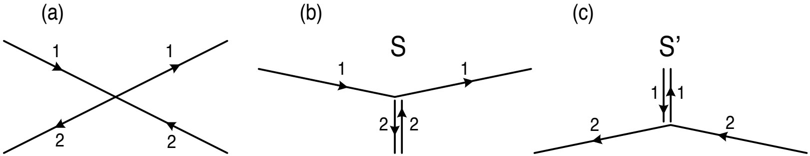
Energy - Momentum

- 4-momentum: $\mathbf{p} = m\mathbf{u} \Rightarrow (m\gamma_u, m\gamma_u\vec{u}) = \left(\frac{m}{\sqrt{1-u^2}}, \frac{m\vec{u}}{\sqrt{1-u^2}}\right)$
  - Relativistic momentum: $\vec{p} = m\gamma_u\vec{u} = \frac{m\vec{u}}{\sqrt{1-u^2}} \xrightarrow[u\to 0]{} m\vec{u}$
  - Relativistic energy: $E = m\gamma_u = \frac{m}{\sqrt{1-u^2}} \xrightarrow[u\to 0]{} m + \frac{1}{2}mu^2$
- Thus components are $\mathbf{p} \to (E, \vec{p})$
- Conservation of 4-momentum implies conservation of 3-momentum and energy in any frame of reference

- Useful relationships
  - Magnitude squared: $\mathbf{p}^2 = m^2 \quad \Rightarrow \quad E^2 = p^2 + m^2$
  - Velocity from momentum: $\vec{u} = \frac{\vec{p}}{E} = \frac{\vec{p}}{\sqrt{p^2+m^2}}$
- Photon: $E = |\vec{p}|$ so that $\mathbf{p} \Rightarrow E(1, \hat{n})$ with $\hat{n}$ the direction of propagation

Other approaches

- Gedanken collision experiment



- Hand and Finch: investigate behavior of electromagnetic energy and momentum

Lorentz transformation

In our standard configuration

$$
\begin{aligned}
p'_x &= \gamma(p_x - vE) \\
p'_y &= p_y \\
p'_z &= p_z \\
E' &= \gamma(E - vp_x)
\end{aligned}
$$

and the inverse

$$
\begin{aligned}
p_x &= \gamma(p'_x + vE') \\
p_y &= p'_y \\
p_z &= p'_z \\
E &= \gamma(E' + vp'_x)
\end{aligned}
$$

Relativistic mechanics

Topics

- Collisions
  - Particle experiments
  - Rocket thrust
- Forces and acceleration
  - 4-acceleration
  - (Minkowski) 4-force, relativistic 3-force
- Lagrangian approach
  - Free particle
  - Charged particle in electromagnetic field
  - Relativistic correction to planetary motion: precession of the perihelion

Collisions

An important topic in relativistic mechanics is collisions, since colliding elementary particles at high energies is the main tool of particle experiments.

Collisions can be classified as:

- **Elastic:** masses of the particles are unchanged
- **Inelastic:** masses change
  - some kinetic energy and mass are exchanged
  - new particles may even be formed

Fundamental physics: conservation of 4-momentum

Center of momentum frame (CM)

General approach (not necessary in simple cases)

- Transform to the frame in which the total 3-momentum $\vec{P}$ is zero.
- Solve collision in this frame
  - Collision is particularly simple in this frame
  - In a binary collision, the incoming particles must have equal and opposite 3-momenta, and the same applies for the outgoing particles if there remain only two.
- Transform back to original frame

Center of momentum frame (CM)

Method 1

- Choose the x-direction along the total 3-momentum so that $P_y = P_z = 0$, and use the standard configuration for Lorentz transforming from the lab frame S to the center of momentum frame S';
- Since $P'_x = 0$, the speed of S' relative to S is $v = P_x/E$;
- Transform the energies and 3-momenta of all the particles to S' using the Lorentz transformation with the speed v;
- Solve the collision in S' (outgoing energies and momenta etc.);
- Transform back to S.

Method 2

- Use the invariance of scalar products such as

$$\mathbf{p}_a \cdot \mathbf{p}_b = \mathbf{p}'_a \cdot \mathbf{p}'_b$$

  with $\mathbf{p}_a$, $\mathbf{p}_b$ and $\mathbf{p}'_a$, $\mathbf{p}'_b$ any two of the particle 4-momenta in the lab frame and in the CM frame, respectively

4-force or Minkowski force

The 4-force is defined as

$$\mathbf{f} = \frac{d\mathbf{p}}{d\tau}$$

Components in inertial frame S where velocity is $\vec{u}$, momentum $\vec{p}$, energy E:

$$\mathbf{f} \rightarrow \left(\gamma_u \frac{dE}{dt}, \gamma_u \frac{d\vec{p}}{dt} \right)$$

Relativistic 3-force

Most convenient definition

$$\vec{f} = \frac{d\vec{p}}{dt}$$

- Relationship to 4-force

$$\mathbf{f} \to \left(\gamma_u \frac{dE}{dt}, \gamma_u \vec{f} \right)$$

- Force and work: using $\mathbf{p}^2 = m^2$ gives

$$\frac{d\mathbf{p}^2}{d\tau} = 0 = 2\mathbf{p} \cdot \frac{d\mathbf{p}}{d\tau} = 2m\mathbf{u} \cdot \mathbf{f}$$

  In an inertial frame where $\mathbf{u} = \gamma_u(1, \vec{u})$ and $\mathbf{f} = \gamma_u(dE/dt, \vec{f})$

$$\frac{dE}{dt} = \vec{f} \cdot \vec{u}$$

- Electromagnetism

$$\vec{f} = q(\vec{E} + \vec{u} \times \vec{B})$$

Acceleration

- In general the acceleration $\vec{a} = d\vec{u}/dt$ is not even parallel to the force:

$$\vec{f} = \frac{d}{dt}(m\gamma_u \vec{u}), \qquad \gamma_u = \frac{1}{\sqrt{1-\vec{u}^2}}$$
$$= \gamma_u^3 m(\vec{u}\cdot\vec{a})\vec{u} + \gamma_u m\vec{a}$$

- For forces parallel and perpendicular to $\vec{u}$ the result is simpler

$$f_\parallel = \gamma_u^3 m a_\parallel$$
$$\vec{f}_\perp = \gamma_u m \vec{a}_\perp$$

  but with different proportionality constants (sometimes called longitudinal and transverse masses). For other directions $\vec{f}$ and $\vec{a}$ are not parallel.

- See Assignment 2 for a discussion of the 4-acceleration $\mathbf{a} = d\mathbf{u}/d\tau$

Lagrangian Approach

Basic idea

- Basic assumption: action S is a Lorentz invariant
- Since time is *not* the same in different frames write the action as

$$S = \int_{\mathcal{P}_1}^{\mathcal{P}_2} \mathcal{L}(\tau)\, d\tau$$

  with τ the *proper time*

- Look for a Lorentz invariant $\mathcal{L}$, a function of particle velocity and position, that satisfies any other symmetries of the problem.

Lagrangian Approach

Free particle

Use arguments such as:

- for a free particle $\mathcal{L}$ must be independent of space-time;
- $\mathbf{u}^2 = 1$, and so we cannot use the 4-velocity $\mathbf{u}$ to give any interesting dependence;
- ...

to deduce that the only possible function is a particle dependent constant.

To connect with the Newtonian Lagrangian in the small velocity limit we use

$$\mathcal{L} = -m$$

with m the mass of the particle.

In a particular inertial frame where the velocity of the particle is $\vec{u}$, and time is t with $d\tau = dt/\gamma$ and $\gamma = 1/\sqrt{1-u^2}$ (time dilation).

$$S = \int_{t_1}^{t_2} L\,dt \quad \text{with} \quad L = -m\sqrt{1-u^2} \xrightarrow[u \to 0]{} -m + \tfrac{1}{2}u^2 + \cdots$$

Lagrangian Approach

Free particle

Lagrangian

$$L = -m\sqrt{1-u^2}$$

Momentum

$$\vec{p} = \frac{\partial L}{\partial \vec{u}} = \frac{m\vec{u}}{\sqrt{1-u^2}} = m\gamma\vec{u}$$

Hamiltonian

$$H = \vec{p}\cdot\vec{u} - L = m\gamma = E$$

Lagrangian Approach

Particle in electromagnetic field

The only new Lorentz invariant (with the right symmetry properties, linear in the field strength etc.) is a constant times the scalar product $\mathbf{u} \cdot \mathbf{A}$ where $\mathbf{u}$ is the *velocity 4-vector* and $\mathbf{A}$ is the *electromagnetic potential 4-vector* $\mathbf{A} \Rightarrow (\Phi, \vec{A})$.

Thus for a particle in an electromagnetic field

$$\mathcal{L} = -m - q\mathbf{u} \cdot \mathbf{A}$$

where q is the charge of the particle.

In our inertial frame, the 4-vectors are $\mathbf{u} = \gamma(1, \vec{u})$, $\mathbf{A} = (\Phi, \vec{A})$ and then

$$L = -m\sqrt{1 - u^2} - q\Phi(\vec{x}, t) + q\vec{u} \cdot \vec{A}(\vec{x}, t)$$

The Euler-Lagrange equation gives the equation of motion

$$\frac{d\vec{\pi}}{dt} = \vec{f} = q(\vec{E} + \vec{u} \times \vec{B}) \quad \text{with} \quad \vec{\pi} = m\gamma\vec{u}$$