Physics 106a **10 October, 2013**

Lecture 4: Constraints, Virtual Work, etc.

In this lecture I go back and *deduce* the Lagrangian equations of motion from the Newtonian formulation. Along the way, we learn a number of other principles, formulate the idea of constraints carefully, and learn more about when the Lagrangian approach is valid.

One motivation for seeking a different formulation of mechanics is that many, if not all, dynamical problems we want to solve are *constrained*, and usually we do not want to investigate all the forces that maintain the constraints. For example, any macroscopic body contains of order Avogadro's number of point-like atoms (and even these may be broken down further), whereas often in mechanics we want to consider *rigid body dynamics* in which the size and shape of the macroscopic body do not change. We wish to study the constrained dynamics in terms of six coordinates (position and orientation of body) not in terms of Avogadro's number of coordinates moving under atomic constraint forces. Even if the size changes, as when the body also vibrates, we want a macroscopic description, not an atomic one.

Constraints

We imagine starting with a primitive description of the dynamics in terms of Newtonian equations of motion for M elementary objects (point particles, or small pieces of a solid body that can be treated as points, etc.) with position vectors $\vec{r}_i, i = 1 \ldots M$. The configuration of the system may be defined by the $3M$ Cartesian components of the particle positions. Newton's equations contain all the forces acting, external and interparticle.

Constraints may restrict the dynamics, so that the dynamics can be described in terms of fewer than $3M$ variables. This leads to the notion of the *number of degrees of freedom*:

> The number of degrees of freedom is the number of coordinates that can be independently varied in a small displacement.

Another way of saying this is the number of independent "directions" in which the system can move from any given initial condition. For constrained dynamics the number of degrees of freedom will usually be less than $3M$. I take this definition from Taylor's book. Hand and Finch and Goldstein et al. are less precise in their usage of this term.

For constrained dynamics we can often describe any configuration of the system allowed by the constraints using a *reduced number* N of generalized coordinates $\{q_k\}$, $k = 1 \ldots N$. A key distinction is between *holonomic* and *nonholonomic* constraints.

For a *holonomic* constraint, we can find a reduced set of N generalized coordinates such that

- the coordinates uniquely define any configuration of the system allowed by the constraints, and so we can find an expression for the positions of all the elementary components in the form
$$\vec{r}_i = \vec{r}_i(q_1, q_2 \ldots q_N, t), \qquad i = 1 \ldots M \times 3, \tag{1}$$
- the N coordinates can each be varied independently.

For holonomic constraints, the number of degrees of freedom is the same as this reduced number N of generalized coordinates.

Holonomic constraints are further classified as *time independent* or *scleronomic* if time does not appear in Eq. (1), and *time dependent* or *rheonomic* if time does appear. This distinction is most important when we consider the Hamiltonian and the conservation of energy.

A *nonholonomic* constraint is one for which the above cannot be done, i.e. "anything else". The dynamics of systems with nonholonomic constants is often hard to treat. One sort of nonholonomic constraint is a constraint in the form of an inequality: a particle must be outside a solid object for example. Rolling dynamics also often gives nonholonomic constraints. For example, consider a vertical bicycle wheel rolling on a horizontal plane (Hand and Finch, Appendix 1A). Four variables are needed to define any configuration, but there are only two degrees of freedom. This is actually an example of a *nonintegrable differential* nonholonomic constraint, since we we can specify *infinitesimal changes* in the $\vec{r}_i$ in terms of changes in a reduced set of the coordinates (e.g. θ, ϕ for the wheel)

$$\delta\vec{r}_i = \sum \vec{a}_{ik}(\{q_k\}, t)\, \delta q_k \tag{2}$$

with a_{lk} some coefficients that may depend on $\{q_k\}$ and time. Equivalently, considering the changes over an infinitesimal time δt, this is a relationship between the velocities and generalized velocities

$$\dot{\vec{r}}_i = \sum \vec{a}_{ik}(\{q_k\}, t)\, \dot{q}_k\,. \tag{3}$$

Our methods can be applied to such nonholonomic systems with a little more effort.

Sometimes the dynamics of a system with strictly nonholonomic constraints can be treated as holonomic over part of the dynamics. For example, consider a particle sliding without friction on the outside surface of a sphere in gravity. Starting at the top, the particle will slide down on the surface — holonomically constrained dynamics. However, the strict statement of the constraint is simply that the particle must be outside the sphere, a nonholonomic constraint. And indeed, at some time the particle will fly off the surface and can no longer be treated in terms of the surface-constrained motion.

Hand and Finch describe sliding with friction as a nonholonomic constraint. I do not agree with this. The constraint, that the body is on the surface, is holonomic. The constraint force includes the frictional force which is parallel to the surface, and this causes complications later.

Examples to illustrate these ideas:

1. The sliding block-on-ramp.
2. A particle of mass m free to slide on a frictionless elliptical wire placed vertically in a gravitational field as in Assignment 1, but now the semi-axes $a(t)$, $b(t)$ may be *time dependent*, prescribed by me.
3. A rigid body of M particles

Virtual work

I now consider the dynamics of a system with constraints. The key idea is to *project* Newton's 2nd law of motion onto directions in the $3M$ dimensional coordinate space that correspond to particle displacements that are *consistent with the constraints*. In this way most types of constraint forces are eliminated, and we may be able to derive a reduced number of equations of motion for the generalized coordinates in terms of only the external or applied forces.

It is useful to introduce the notion of a *virtual displacement* which is a coordinated displacement $\delta\vec{r}_i$ of all the particles with *time* and *velocities* held fixed that is *consistent with the constraints*.

The projection of Newton's law is then

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = \sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i \qquad \text{D'Alembert's principle} \tag{4}$$

Note that this equation is true for any choice for the $\delta\vec{r}_i$ — it is a simple identity multiplying the left and right hand sides of an equation by the same thing. We *choose* to use the *virtual displacement* rather than the *actual* displacement in the dynamics $\vec{r}_i(t+\delta t) - \vec{r}_i(\delta t) = \dot{\vec{r}}_i \delta t$ because then most types of constraint forces are eliminated. This is a very important point. Study the example of a bead on an elliptical wire with time dependent radii to convince yourself that the virtual displacement eliminates the constraint force normal to the wire from Eq. (4), but that the displacement that occurs in time δt in the dynamics does not. Also, of course, we do not know the displacements occurring in an infinitesimal time in the actual dynamics until we have solved the problem! Equation (4) can be repeated for a variety of independent virtual displacements (for example corresponding to a small change in each degree of freedom).

The quantity $\delta W = \sum_i \vec{F}_i \cdot \delta\vec{r}_i$ for a virtual displacement $\delta\vec{r}_i$ is called the *virtual work*. It is *not* necessarily the work done in the actual dynamics.

For an equilibrium (time independent) problem, the right hand side of Eq. (4) vanishes, and this gives us the *principle of virtual work* method for solving constrained equilibria

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0 \qquad \text{Principle of virtual work for equilibria} \tag{5}$$

For the case of holonomic constraints the virtual displacement coming from an infinitesimal change of a single generalized coordinate q_k is $\delta\vec{r}_i = \partial\vec{r}_i/\partial q_k$, and the corresponding virtual work is

$$\delta W = \sum_i \vec{F}_i \cdot \left(\frac{\partial\vec{r}_i}{\partial q_k}\right) \delta q_k \,. \tag{6}$$

The quantity

$$\mathcal{F}_k = \sum_i \vec{F}_i \cdot \left(\frac{\partial\vec{r}_i}{\partial q_k}\right) = \frac{\delta W}{\delta q_k} \tag{7}$$

is called the generalized force.

For the common case where the constraint forces give no contribution to the virtual work (remember the virtual displacements are consistent with the constraints) Eqs. (4-7) can be evaluated from the non-constraint ("external") forces alone $\vec{F}_i \to \vec{F}_i^{\mathrm{nc}}$, i.e. what we know.

For holonomic constraints the equation of motion Eq. (4) for the virtual displacement given by changing the generalized coordinate q_k can be written

$$\sum_i \dot{\vec{p}}_i \cdot \frac{\partial\vec{r}_i}{\partial q_k} = \mathcal{F}_k. \tag{8}$$

Note that this is the *equation of motion* and *not* the definition of the generalized force, which is given by Eq. (7).

After some uninformative algebra (see Appendix 1 or the textbooks) the left hand side of these equations can be transformed, again assuming holonomic constraints, to a form that can be calculated from the kinetic energy T, leading to the *generalized equations of motion* (or as Hand and Finch call it, Golden Rule #1)

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \mathcal{F}_k. \tag{9}$$

These are equations of motion, basically giving the "accelerations" $\ddot{q}_k$ in terms of the "forces" $\mathcal{F}_k$ just as Newton's equations do. As in Eq. (8), they are *not* the definitions of $\mathcal{F}_k$: in general you know the forces and want to calculate the dynamics. The generalized forces are defined and calculated by Eq. (7). The equations also do *not* in general express the conservation of energy (since they were derived considering virtual displacements not real ones). As we have seen, for time dependent constraints the energy need not in fact be conserved.

The scheme for using Eq. (9) is:

- calculate the kinetic energy $T = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i$ in terms of the q_k and $\dot{q}_k$: usually this is easy since it is easy to see what velocities a time derivative of each q_k gives;
- calculate the generalized force $\mathcal{F}_k$: you can use the formal expression given by the first equality in Eq. (7), but usually it is easier to make a small change in one q_k, calculate δW for this change and use the second equality in Eq. (7).

The concept of virtual work is rather tricky. I present some more details and examples in Appendix 2.

Equation (9) is true in general for holonomic constraints (or, of course, for no constraints at all). If the constraint forces give no contribution to the virtual work they do not contribute to $\mathcal{F}_k$ and so are eliminated from the equations of motion. If they *do* contribute to the virtual work (e.g. friction) they must be included in $\mathcal{F}_k$. Note that Eq. (9) is *more general* than the Lagrangian equations I will now derive, since the generalized forces may contain both constraint forces and nonconservative forces such as friction.

Conservative forces

For conservative forces

$$\mathcal{F}_k = \sum_i \vec{F}_i \cdot \left(\frac{\partial \vec{r}_i}{\partial q_k}\right) = -\sum_i \left(\frac{\partial V}{\partial \vec{r}_i}\right) \cdot \left(\frac{\partial \vec{r}_i}{\partial q_k}\right) = -\frac{\partial V}{\partial q_k} \tag{10}$$

For such forces the generalized equation of motion can be written as the *Euler-Lagrange* equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \tag{11}$$

with $L(\{q_k\}, \{\dot{q}_k\}, t)$ the Lagrangian $L = T - V$. Thus we have derived the Lagrangian formulation of mechanics starting from Newton's 2nd law.

Appendix 1: Derivation of the Generalized Equation of Motion

We start from Eq. (8)

$$\sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \mathcal{F}_k \equiv \frac{\delta W}{\delta q_k}. \tag{12}$$

The task is to evaluate the left hand side in terms of the kinetic energy

$$T = \sum_i \tfrac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i \to T(\{q_k\}, \{\dot{q}_k\}, t) \tag{13}$$

for the case of holonomic constraints.

Lemma: Dot Cancellation: First we prove the lemma *for holonomic constraints*

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} = \frac{\partial \vec{r}_i}{\partial q_k}. \tag{14}$$

The proof of the lemmas is as follows. For holonomic constraints we have $\vec{r}_i = \vec{r}_i(\{q_k\}, t)$ so that

$$\dot{\vec{r}}_i \equiv \frac{d\vec{r}_i}{dt} = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}. \tag{15}$$

This implies $\dot{\vec{r}}_i = \dot{\vec{r}}_i(\{q_k\}, \{\dot{q}_k\}, t)$ and we can imagine changing q_k and $\dot{q}_k$ independently. But $\partial\vec{r}_i/\partial q_k$ and $\partial\vec{r}_i/\partial t$ do not depend on $\{\dot{q}_l\}$ so

$$\left(\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k}\right)_{\{\dot{q}_{l\neq k}\},\{q_l\},t} = \left(\frac{\partial \vec{r}_i}{\partial q_k}\right)_{\{q_{l\neq k}\},t}. \tag{16}$$

Note that it is important to be careful about what is held constant in the partial derivatives!

Now we use this lemma to manipulate Eq. (12)

$$\sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \mathcal{F}_k, \qquad T = \sum_i \tfrac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i \to T(\{q_k\}, \{\dot{q}_k\}, t). \tag{17}$$

Since we want the time derivative of $\vec{p}_i$ we look at what we get when we evaluate derivatives of T

$$\begin{aligned}
\frac{\partial T}{\partial q_k} &= \sum_i m\dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k} = \sum_i \vec{p}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k}, \\
\frac{\partial T}{\partial \dot{q}_k} &= \sum_i m\dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} = \sum_i \vec{p}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k},
\end{aligned}$$

using dot cancellation in the second expression. Now differentiate the second expression with respect to t along path of dynamics $\{q_k(t)\}$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} + \sum_i \vec{p}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_k}\right). \tag{18}$$

The first term is what we want, so we look at the second term, remembering $\vec{r}_i = \vec{r}_i(\{q_k\}, t)$

$$\begin{aligned}
\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_k}\right) &= \sum_l \frac{\partial^2 \vec{r}_i}{\partial q_l \partial q_k}\dot{q}_l + \frac{\partial^2 \vec{r}_i}{\partial t \partial q_k}, \\
&= \frac{\partial}{\partial q_k}\left(\sum_l \frac{\partial \vec{r}_i}{\partial q_l}\dot{q}_l + \frac{\partial \vec{r}_1}{\partial t}\right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_k},
\end{aligned}$$

so that

$$\sum_i \vec{p}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_k}\right) = \sum_i \vec{p}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k} = \frac{\partial T}{\partial q_k}. \tag{19}$$

This gives

$$\sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}, \tag{20}$$

and so the result we want

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \mathcal{F}_k \equiv \frac{\delta W}{\delta q_k}. \tag{21}$$

Appendix 2: More on Virtual Work

To understand virtual work better, lets consider the setting of M particles or elementary objects of the mechanical system with positions $\vec{r}_i$, $i = 1 \dots M$. For a system with holonomic constraints these positions are determined by a reduced number N of generalized coordinates q_k, $k = 1 \dots N$. Holonomic means we can write expressions for the positions $\vec{r}_i$ in terms of the generalized coordinates q_k and, maybe, time.

$$\vec{r}_i = \vec{r}_i(q_1, q_2, \dots q_N, t), \qquad i = 1 \dots M. \tag{22}$$

The virtual work corresponding to some virtual displacement in which each object i moves by the displacement $\delta\vec{r}_i$ is

$$\delta W = \sum_{i=1}^{M} \vec{F}_i \cdot \delta\vec{r}_i. \tag{23}$$

Here F_i is the total physical (Newtonian) force acting on object i.[1] A virtual displacement is a coordinated move of the objects such that the constraints remain satisfied. It is an "imaginary" set of displacements in which time and velocities are held fixed, not the actual displacements for the dynamics over a time δt (which we don't know until we solve the problem in any case). The idea is that for such virtual displacements, the constraint forces usually do not contribute to the virtual work.

To find the generalized equations of motion for the coordinates q_k we want to find the virtual work for the virtual displacements corresponding to changing each generalized coordinate q_k *separately*. (We can do this, since the q_k can be varied independently, while still maintaining the constraints.) For example, first consider just a change δq_1 in the coordinate q_1. This will correspond to a coordinated set of displacements $\delta\vec{r}_i$ of the M objects, each proportional to δq_1. The virtual work is calculated from Eq. (23): it too will be proportional to δq_1. The generalized force corresponding to q_1 is then

$$\mathcal{F}_1 = \frac{\delta W}{\delta q_1}. \tag{24}$$

Then repeat for generalized coordinate q_2, etc.

We might calculate δW for the coordinate change δq_1in two ways:

- **Geometrically:** Draw pictures showing the vector displacements $\delta\vec{r}_i$ for the change δq_1. Form *force dot displacement* $\vec{F}_i \cdot \vec{\delta} r_i$ for each object i, and sum over the objects.

- **Algebraically:** Calculate $\delta\vec{r}_i$ for the coordinate change δq_1 using the explicit equations Eq. (22)

$$\delta\vec{r}_i = \frac{\partial\vec{r}_i}{\partial q_1}\delta q_1. \tag{25}$$

  Now calculate the virtual work, etc.

Some other thoughts:

- Varying a single coordinate δq_1 leads, in general, to displacements of all the objects. The displacements will be consistent with the constraints.

- There is a generalized force $\mathcal{F}_k$ is associated with each generalized coordinate q_k

- The generalized force does not act on one object: it is compiled from the forces acting on all the objects.

[1] An important point is that $\vec{F}_i$ only includes the real physical forces, due to gravity, the contact between the objects (i.e. the constraint forces) etc., and not fictitious forces such as the "centripetal force".

- The Golden Rule No.1

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \mathcal{F}_k \tag{26}$$

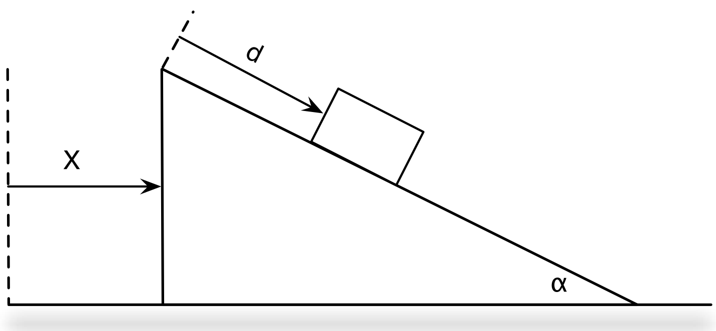
is the equation of motion, giving $\ddot{q}_k$ in terms of the generalized force $\mathcal{F}_k$ calculated as above. It is equivalent to Newtons $m\vec{a} = \vec{F}$. It is *not* the definition of $\mathcal{F}_k$.

- Since Eq. (26) is more complicated than Newton's second law, we need to work out the consequences of the equation. For example, $\mathcal{F}_k = 0$ does not alway mean $\ddot{q}_k = 0$ (an example is equation for the X generalized coordinate for the block-on-wedge problem).

- The virtual work can be defined for any set of object displacements that satisfy the constraints, including general nonholonomic constraints. In these more general cases, the virtual displacement cannot be associated with a single generalized coordinate, and so the notion of a virtual force cannot be extracted from the virtual work. Also, in these cases it is usually not possible to write the "other side of the equation" in a simple form such as the left hand side of Eq. (26). Thus, although the virtual work can be calculated in these cases, it may not be useful to solve for the dynamics.

Example: Frictionless wedge and block

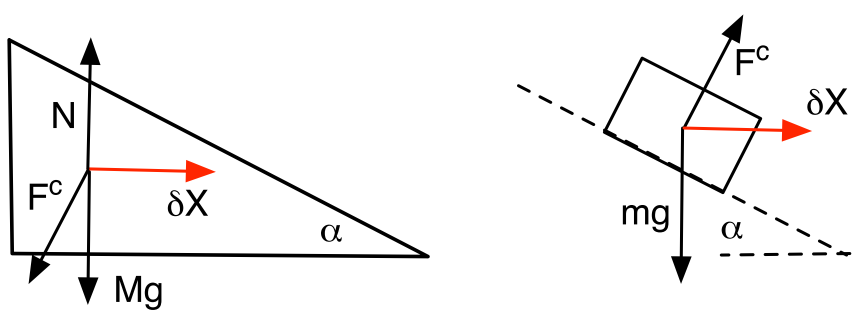
As an example, consider the frictionless sliding block on wedge problem we discussed in class.



The generalized coordinates d, X are shown in the figure. Note that a change in X moves both the wedge and the block, since the constraint "block is on wedge" must be maintained.

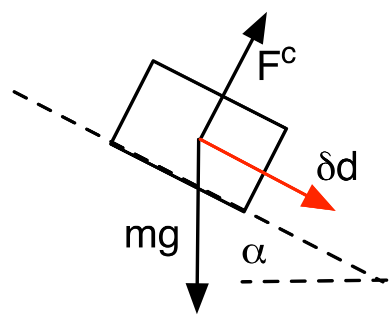
Geometrical method

First consider a change δX in the generalized coordinate X.



Draw the "free body diagrams" as you were taught in freshman physics. The forces acting are gravity, and action and reaction forces normal to the contact surfaces. In the figures, the black arrows denote the directions of the forces, and the symbols the magnitudes. The displacements $\delta\vec{r}_i$ of the two objects due to the change δX are the red arrows. Now calculate the virtual work for this δX. For object 1 (the wedge) the contribution to the virtual work is $\delta W_1 = -F^c \sin\alpha\, \delta X$ (gravity and the normal surface force N don't contribute since $\vec{F} \cdot \vec{\delta r}$ is zero for these forces and the horizontal displacement.). For object 2 (the block) the contribution to the virtual work is $\delta W_2 = F_c \sin\alpha\, \delta X$. The total virtual work for the change δX is $\delta W = \delta W_1 + \delta W_2 = 0$: the contributions from the constraint forces vanish as promised! This is of course because the forces are equal and opposite, whereas the displacements of the two bodies are the same to maintain the constraint. Since the virtual work is zero, this means the generalized force $\mathcal{F}_X$ corresponding to X is zero.[2]

Now consider a change δd in the generalized coordinate d.



There is not displacement of the wedge and so I haven't shown the wedge in the figure. The virtual work is $\delta W = mg \sin\alpha \delta d$. The constraint force does not contribute, since it is perpendicular to the displacement. The generalized force for the generalized coordinate d is $\mathcal{F}_d = \delta W / \delta d = mg \sin\alpha$.

Algebraic method

The relationship between the positions of the objects and the generalized coordinates is, using the notation (u, v) for a vector with x-component (horizontal, to the right) u and y-component (vertical, up) v,

$$\vec{r}_1 = (X, 0), \tag{27}$$
$$\vec{r}_2 = (X + d\cos\alpha, h - d\sin\alpha), \tag{28}$$

(with h the height of the wedge). The forces on the objects are

$$\vec{F}_1 = -F^c(\sin\alpha, \cos\alpha) + (0, N - Mg), \tag{29}$$
$$\vec{F}_2 = F^c(\sin\alpha, \cos\alpha) + (0, -mg). \tag{30}$$

Now use $\mathcal{F}_k = \sum_i \vec{F}_i \cdot \partial\vec{r}_i / \partial q_k$.

For the generalized coordinate X we have $\partial\vec{r}_1/\partial X = (1, 0)$ and $\partial\vec{r}_2/\partial X = (1, 0)$. Summing over wedge and block

$$\sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial X} = 0, \tag{31}$$

[2]This does *not* mean that $\ddot{X} = 0$ – check the generalized equation of motion.

and so $\mathcal{F}_X = 0$.

For the generalized coordinate d we have $\partial\vec{r}_1/\partial d = (0,0)$ and $\partial\vec{r}_2/\partial d = (\cos\alpha, -\sin\alpha)$. Summing gives

$$\sum_i \vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial d} = mg\sin\alpha, \tag{32}$$

so that $\mathcal{F}_d = mg\sin\alpha$.

The two methods agree, as they should. The geometric method may seem longer, since I went through it in some detail. But I think it is easier once you get familiar with it.

Michael Cross, October 8, 2013