Physics 106a — Classical Mechanics

Michael Cross

California Institute of Technology

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Lecture 3: Using the Lagrangian Approach

Outline

- Review
- Examples
- Conjugate momenta and ignorable coordinates
- Hamiltonian and time independent Lagrangians

Review

- Use N generalized coordinates that define the configuration at each time

$$q_1(t), q_2(t) \ldots q_N(t) \quad \rightarrow \quad \{q_k(t)\},\ k = 1 \ldots N$$

- Make the action $S = \int L\, dt$ (with $L = T - V$) stationary over paths

$$\delta S = \int \sum_k \left[\frac{\partial L}{\partial q_k} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) \right] \delta q_k(t)\, dt = 0$$

- If the N coordinates are independent

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad \text{for } k = 1 \ldots N$$

  and if a change of each coordinate is consistent with any constraints then $L = T - V$ can be evaluated without knowing any constraint forces

- For unconstrained dynamics of M particles $N = 3M$. For constrained dynamics N may be reduced.

Review

For most problems

- Use N generalized coordinates $q_1(t), q_2(t) \ldots q_N(t)$ that
  - define the configuration at each time
  - can be varied independently, consistent with any constraints

- For each coordinate use the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad \text{for } k = 1 \ldots N$$

with $L = T - V$

Justification

For the elementary description of M particles with position vectors $\vec{r}_i$ and Cartesian coordinates (x_i, y_i, z_i), for $i = 1 \ldots M$, and including all the forces, assumed conservative, in the potential $V(\{\vec{r}_i\})$

$$L = \sum_{i=1}^{M} \tfrac{1}{2} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) - V(\{\vec{r}_i\})$$

The Euler-Lagrange equation for x_i is

$$\frac{d}{dt}(m_i \dot{x}_i) + \frac{\partial V}{\partial x_i} = 0$$

which is Newton's 2nd law of motion for conservative forces

Symmetries and Conserved Quantities

The momentum p_k conjugate to the coordinate q_k is defined as

$$p_k = \frac{\partial L}{\partial \dot{q}_k}$$

If a coordinate q_m does not explicitly appear in the Lagrangian

$$\frac{\partial L}{\partial q_m} = 0$$

it is called *ignorable* or *cyclic*.

For an ignorable coordinate the corresponding momentum is a constant of the motion (conserved)

$$\dot{p}_m \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) = 0.$$

The Hamiltonian and Time Independent Lagrangians

Define the Hamiltonian as

$$H = \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \equiv \sum_k p_k \dot{q}_k - L$$

Then

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

and H is a constant of the motion if the Lagrangian does not explicitly depend on time

For velocity-independent potentials and if the kinetic energy is a quadratic form in the velocities

$$T = \tfrac{1}{2} \sum_{kl} t_{kl} \dot{q}_k \dot{q}_l$$

(with t_{kl} possibly depending on coordinates $\{q_j\}$ and time t) the Hamiltonian is the total energy

$$H = T + V$$