Ma2a (analytical) **Fall 2012**

PART I: FIRST ORDER ODEs

1. Introduction

1.1. **Fundamental theorem of calculus.** The simplest ("trivial") d.e.:

$$y'(x) = f(x).$$

Examples:

(i) Find all (*maximal*) solutions of the equation

$$y'(x) = \frac{1}{1+x^2}.$$

Answer:

$$y(x) \in \int \frac{dx}{1+x^2} = \arctan x + C, \qquad (x \in \mathbb{R}).$$

(the set of all primitives).

(ii) Solve the IVP

$$y'(x) = \frac{1}{1+x^2}, \qquad y(1) = 0.$$

Answer:

$$y(x) = \arctan x - \frac{\pi}{4}, \qquad (x \in \mathbb{R}).$$

[Set $x = 1$ in the "general solution" $y(x) = \arctan x + C$.]

(iii) Solve the IVP

$$y'(x) = \frac{1}{x}, \qquad y(-1) = 0.$$

Answer: the (maximal) solution is

$$y(x) = \log(-x), \qquad (x < 0).$$

Theorem. *Suppose f is a continuous function on some interval $I \subset \mathbb{R}$. Let $x_0 \in I$ and $y_0 \in \mathbb{R}$. Then the IVP*

$$y' = f(x), \qquad y(x_0) = y_0,$$

has$^{(i)}$ *a unique*$^{(ii)}$ *solution on the whole interval*$^{(iii)}$ *I. The solution is given by the formula*$^{(iv)}$

$$y(x) = y_0 + \int_{x_0}^{x} f(s)ds, \qquad (x \in I).$$

(If $x < x_0$ then $\int_{x_0}^{x}$ means $-\int_{x}^{x_0}$)

Note. Several important concepts: (i) existence of a solution, (ii) uniqueness, (iii) interval/domain of a solution, (iv) solution formula.

1.2. **First order ODEs.** The most general form is

$$F(x, y, y') = 0,$$

where

- F is a given function of 3 real variables;
- x is an independent variable;
- y is the unknown function $y = y(x)$;
- $y' = dy/dx$ is the usual (*ordinary*) derivative.

Any differentiable function $y(x)$ that satisfies the equation (i.e. if we substitute $y(x)$ for y, then the equation becomes an identity) is called a *solution*.

Given two numbers x_0, y_0, the corresponding *initial value problem* ("IVP") is to find a solution $y(x)$ such that

$$y(x_0) = y_0.$$

Important: we always assume that a solution is defined on some *interval* (connected set) of the real line; the interval can be infinite, e.g. $(-\infty, +\infty)$.

Example: the function

$$y(x) = \begin{cases} \log(-x), & x < 0, \\ \log x + C, & x > 0, \end{cases}$$

is not a solution of the equation $y' = 1/x$ – it is defined on the union of two intervals $(-\infty, 0) \cup (0, +\infty)$. Otherwise, the IVP $y(-1) = 0$ would have infinitely many solutions.

A solution is *maximal* if it can not be extended to a solution on a larger interval.

1.3. **Higher order ODEs.** The general form of 2-nd order ODE is

$$F(x, y, y', y', y'') = 0,$$

and solutions are *twice* differentiable functions (defined on some intervals) satisfying the equation.

Solving for the second derivative, we obtain Newton's equation with 1 degree of freedom (the motion of a particle $m = 1$ on the real line):

$$\ddot{x} = f(t, x, \dot{x}).$$

Here we have changed the notation: now time t is an independent variable, position $x = x(t)$ is the unknown function, and dot means time derivative, so $\ddot{x}$ is acceleration. The RHS is the "force"; e.g. $f = 0$ for a free particle, $f = -kx$ for a Hooke's law spring, etc.

Example: the general solution of $\ddot{x} = 0$ is

$$x(t) = C_1 + C_2 t,$$

and the IVP

$$x(t_0) = x_0, \qquad \dot{x}(t_0) = \dot{x}_0,$$

has a unique solution on $\mathbb{R}$.

The n-th order ODE is

$$F(x, y, y', \dots, y^{(n)}) = 0.$$

Example: solve Eq. $y^{(n)} = 0$.
Answer: arbitrary polynomials of degree $< n$.

1.4. **Systems of ODEs.** A system of two first order equations (solved for the derivatives) has the form

$$\dot{x}_1 = f(t, x_1, x_2), \qquad \dot{x}_2 = g(t, x_1, x_2).$$

We can rewrite the system in the vector form:

$$\dot{\vec{x}} = \vec{v}(t, \vec{x}),$$

where

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \qquad \vec{v} = \begin{pmatrix} f \\ g \end{pmatrix}.$$

Thus $\vec{v}$ is a vector-valued function of three real variable. A solution is a vector-function $\vec{x}(t)$; it has to be defined on some time interval.

We can transform any second order d.e. into a system in $\mathbb{R}^2$. For example, Newton's equation

$$\ddot{x} = f(t, x, \dot{x}), \qquad x \in \mathbb{R},$$

is equivalent to the system

$$\dot{x} = y, \qquad \dot{y} = f(t, x, y).$$

A system for n unknown functions:

$$F(t, x, \dot{x}) = 0, \qquad t \in \mathbb{R}, \quad x(t) \in \mathbb{R}^n,$$

is the most general object of the ODE theory (or classical mechanics; e.g. the motion of N particles in $\mathbb{R}^3$ corresponds to $n = 6N$; N rigid bodies require $n = 12N$ functions). The unknown functions depend on a single variable (time) and the equations involve ordinary derivatives. We should expect that the general solution has exactly n arbitrary parameters.

Systems of the form

$$F(x, \dot{x}) = 0, \qquad x(t) \in \mathbb{R}^n,$$

are called *autonomous.* A non-autonomous system in $\mathbb{R}^n$ can be represented as an autonomous system in $\mathbb{R}^{n+1}$. For example, the scalar equation $\dot{x} = f(x, t)$, $x(t) \in \mathbb{R}$, is equivalent to the 2D autonomous system

$$\dot{x} = f(x, y), \qquad \dot{y} = 1.$$

1.5. **PDEs.** Partial differential equations have several independent variables, and involve partial derivatives. For example,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

is the (2D Laplace) equation for the unknown function $u = u(x, y)$.

Example: solve $u_x = 0$. Answer: $u(x, y) = C(y)$, an arbitrary function of y.

1.6. **Methods.**

- Formula solutions
- Qualitative analysis
- Asymptotic analysis
- Numerical methods.

Keep in mind that many (most) equations can not be solved by *quadratures*, i.e. in terms of elementary functions and their primitives, even equations as simple as $y' = y^2 - x$.

2. 1D autonomous equations (formula solution)

$$\dot{x} = v(x).$$

Translation over time property: if $x(t)$ is a solution, then for any t_0, the function $x(t - t_0)$ is also a solution. This property characterizes autonomous systems.

2.1. **Formal recipe.**

$$\frac{dx}{dt} = v(x), \qquad \frac{dt}{dx} = \frac{1}{v(x)},$$

We get a "trivial" equation for the inverse function $t(x)$, so

$$t(x) = \int \frac{dx}{v(x)}.$$

In particular, the solution to the IVP(t_0, x_0) is given by the relation

$$t - t_0 = \int_{x_0}^{x} \frac{ds}{v(s)}.$$

We obtained a solution in *implicit form*; we need to solve the last equation for $x = x(t)$ to get an explicit solution.

Note. To justify the above argument we need to know that the solution $x(t)$ is invertible, which is not always true.

2.2. **Stationary points and equilibrium solutions.** A point x_0 is a *stationary point* of the equation $\dot{x} = v(x)$, or a singular point of the *vector field* $v(x)$, if

$$v(x_0) = 0.$$

In this case, the function $x(t) \equiv x_0$ is a solution. Such solutions are called *stationary* or *equilibrium solutions.*

Clearly, a stationary solution is not an invertible function.

2.3. **Main theorem.**

Theorem. *Suppose $v(x)$ is continuously differentiable in some neighborhood of x_0.*
(i) There exists an interval I containing t_0 such that the IVP(t_0, x_0) has a unique solution on I.
(ii) If $v(x_0) = 0$, then $x(t) \equiv x_0$. If $v(x_0) \neq 0$, then the solution $x = x(t)$ satisfies the relation

$$t - t_0 = \int_{x_0}^{x} \frac{ds}{v(s)}\ , \qquad (t \in I). \tag{2.1}$$

Proof of existence. Case 1: $v(x_0) = 0$. Then there is a stationary solution.
Case 2: $v(x_0) \neq 0$. The formula (2.1) makes sense for all x's close to x_0 (we don't divide by zero) and determines a C^1-function $t = t(x)$ on some interval containing x_0. The derivative of this function is $1/v(x)$; it is non-zero at x_0, so it preserves the sign in some small interval. Thus the function $t(x)$ is monotone and therefore invertible. We can apply the inverse function theorem.

Proof of uniqueness: see homework problems.

Note. The first part of the theorem is valid in all dimensions ("fundamental theorem of ODE").

In the proof of existence we used only continuity of $v(x)$, but the assumption that $v(x)$ is smooth is crucial for uniqueness.

2.4. **Examples of non-uniqueness.** If we only assume that $v(x)$ is continuous, then it is possible that the formula (2.1) makes sense (the integral converges) and gives a non-stationary solution even if $v(x_0) = 0$. In this case, the IVP has more than one solution: the stationary solution, and the solution given by (2.1).

Example. For $0 < \alpha < 1$, consider the IVP

$$\dot{x} = |x|^\alpha, \qquad x(0) = 0.$$

The function $v(x) = |x|^\alpha$ is continuous but is not differentiable at 0. We claim that for any interval containing 0 there are at least two different solutions.

Let's take for instance $\alpha = 2/3$, so the equation is

$$\dot{x}^3 = x^2.$$

In addition to the stationary solution $x(t) \equiv 0$, the function

$$x(t) = \left(\frac{t}{3}\right)^3$$

solves the initial value problem. This second solution comes from the formula (2.1): if $x > 0$, then

$$t(x) = \int_0^x s^{-2/3} ds = 3s^{1/3} \Big|_0^x = 3x^{1/3}, \qquad x(t) = (t/3)^3;$$

if $x < 0$, then

$$t(x) = \int_0^x (-s)^{-2/3} ds = \int_0^{-x} s^{-2/3} d(-s) = -3s^{1/3} \Big|_0^{-x} = 3x^{1/3}, \qquad x(t) = (t/3)^3.$$

Exercises.

(a) For what initial data (t_0, x_0) the problem $\dot{x}^3 = x^2$, $x(t_0) = x_0$ has (i) no solutions, (ii) infinitely many solutions that are defined for all x, (iii) on some nbh of x_0, only finitely many solutions?

Hint: consider the functions

$$x(t) = \begin{cases} 0, & t \le \epsilon, \\ (t-\epsilon)^3/27, & t \ge \epsilon. \end{cases}$$

(b) If v is continuously differentiable, $v(x_0) = 0$, and $v'(x_0) \neq 0$, then the integral

$$\int_{x_0}^x \frac{ds}{v(s)}$$

in the formula (2.1) diverges.

Hint:

$$v(s) = v'(x_0)(s - x_0) + o(|s - x_0|) \quad \text{as} \quad s \to x_0.$$

2.5. **Equation of normal reproduction.**

$$\dot{x} = kx.$$

(The rate of the population growth is proportional to the number of individuals.)

Claim. *All solution extend to the whole real line.*
The solution of the IVP$(0, x_0)$ is given by the formula

$$x(t) = x_0 e^{kt}.$$

Proof: if $x_0 \neq 0$ we use (2.1), and if $x_0 = 0$, then $x(t) \equiv 0$.

2.6. **"Blow up" phenomenon.**

Example. Consider the equation

$$\dot{x} = |x|^\alpha \qquad \text{with} \quad \alpha > 1.$$

Let's take $\alpha = 2$. (The rate of reproduction is proportional to the number of pairs of individuals. This has a catastrophic consequence: the population becomes infinite in finite time.)

To solve the IVP$(0, x_0)$, (we assume $x_0 > 0$), we use the formula (2.1):

$$t = \int_{x_0}^x \frac{ds}{s^2} = \frac{1}{x_0} - \frac{1}{x}, \qquad x(t) = \frac{x_0}{1 - tx_0}.$$

The last expression gives *two* maximal solutions of the equation. The solutions are defined for $t < 1/x_0$ and for $t > 1/x_0$ respectively; the first one solves our IVP. The "blow up" time is $t = 1/x_0$: the IVP solution does not extend beyond this point.

Exercise. Solve the equation

$$\dot{x} = ax^2 + bx + c, \qquad (a, b, c \text{ are constant coefficients}),$$

and describe the long term behavior of solutions.

The second part of the problem is best done using the "qualitative approach", see Section 3.4 below.

3. 1D autonomous equations (qualitative approach)

3.1. **Mechanical interpretation.**

$$\dot{x} = v(x), \qquad v \in C^1(\mathbb{R}).$$

We can think of $x(t)$ as a motion of some particle in $\mathbb{R}$, the x-axis. Then $\mathbb{R}$ is the set of all possible states of the system, the *phase space* of the equation. (The (t, x)-plane $\mathbb{R}^2$ is the *extended phase space.*)

The derivative $\dot{x}(t)$ is the velocity of the particle; it is a vector. We think of the function $v(x)$ as a vector field in the phase space. At each non-singular point the field has magnitude (length) and direction (sign). At singular points the vectors are zero.

A solution of the equation is a motion such that if the state at time t is x, i.e. $x(t) = x$, then the velocity is $v(x)$.

This interpretation extends to higher order autonomous systems.

3.2. **Sign analysis.** Suppose a *smooth* vector field $v(x)$ has three singular points,

$$x_1 < x_2 < x_3,$$

and suppose that the direction of the field is positive, $v > 0$, on $(x_1, x_2) \cup (x_3, +\infty)$ and negative, $v < 0$, on $(-\infty, x_1) \cup (x_2, x_3)$; consider for example

$$v(x) = x^3 - x.$$

The following statements are almost obvious from the mechanical interpretation, and it is not difficult to give a rigorous proof based on the existence and uniqueness theorem.

Claim. *Let $x(t)$ be the solution to the IVP(t_0, x_0).*

(i) If $x_1 < x_0 < x_2$, then the solution extends to the whole real line [time axis]*, and $x(t) \to x_1$ as $t \to -\infty$, $x(t) \to x_2$ as $t \to +\infty$.*

(ii) If $x_2 < x_0 < x_3$, then $x(t) \to x_2$ as $t \to +\infty$, $x(t) \to x_3$ as $t \to -\infty$.

(iii) If $x_0 > x_3$, then the solution extends infinitely backward in time, and $x(t) \to x_3$ as $t \to -\infty$; also, $x(t) \to +\infty$ as $t \to +\infty$ or as $t \to t_*$ (the "blow up" time).

(iv) If $x_0 < x_1$, then ... [similar statement].

Note. We can not tell from the sign analysis whether or not the solution blows up in finite time in the cases (iii) and (iv). A simpler example: the vector fields $v(x) = x$ and $v(x) = x^3$ have the same direction fields. In the first case, all solutions extend to $\mathbb{R}$. In the second case, all non-stationary solutions become infinite in finite time.

3.3. **Stability of equilibrium solutions.** In the example described above, the stationary point x_{*2} is *stable* and *attracting* (as $t \to +\infty$) but the stationary points x_{*1} and x_{*3} are *unstable*.

Stability means that a small change in the initial data does not change much the asymptotic behavior of the solution. The behavior of 1D smooth autonomous systems is very predictable: all non-stationary solutions tend to stable equilibria or infinity.

Here's a formal definition. Let x_* be a stationary point, i.e. $v(x_*) = 0$.
The equilibrium x_* is *stable* as $t \to +\infty$ if $\forall \epsilon > 0$, $\exists \delta > 0$,

$$|x_0 - x_*| < \delta \qquad \Rightarrow \qquad |x(t) - x_*| < \epsilon, \quad (\forall t \geq 0),$$

where $x(t)$ is the solution of the $\mathrm{IVP}(0, x_0)$. "Start close enough" $\Rightarrow$ "stay close".
The equilibrium x_* is *attracting* if $\exists \delta > 0$,

$$|x_0 - x_*| < \delta \qquad \Rightarrow \qquad x(t) \to x_* \quad (\text{as } t \to +\infty).$$

"Stable" + "attracting" = "asymptotically stable".

Exercises.

(a) Show "stable" $\not\Rightarrow$ "attracting".

(b) Show "attracting" $\Rightarrow$ "stable". [This is true only in 1D!]

(c) If $v'(x_*) > 0$, then x_* is unstable, and if $v'(x_*) < 0$, then x_* is asymptotically stable.

(d) If $v'(x_*) = 0$, then ???

3.4. **Logistic equation.** This is a simple model of population dynamics in 1D. Consider the equation

$$\dot{x} = kx - cx^2 - h.$$

Interpretation:

- $x(t)$ is the size of the population (say, fish) at time t;
- k is the "birth minus death" rate; we'll assume $k > 0$;
- the term $-cx^2$ accounts for overcrowding (e.g., not enough food); we'll assume $c > 0$;
- h is the harvest quota; we'll assume $h \geq 0$.

In the case $h = 0$, there are two singular points: 0, which is an unstable equilibrium, and $x_* = k/c$, which is an asymptotically stable equilibrium. The number $x_* = k/c$ is the *saturation level*: the size of the population eventually stabilizes at level x_*.

In the case $h > 0$ there are (essentially) two possibilities. If the quota is high (the parabola $y = kx - cx^2 - h$ does not intersect the x-axis), then the fish become extinct. Otherwise, there is a stable equilibrium, and so the fish and the industry are doing well.

4. Basic properties of 1st order ODEs

4.1. **Normal form of the equation.**

The general form of a 1st order equation is
$$F(x, y, y') = 0. \tag{4.1}$$

Example: $(y')^3 = y^2$. The IVP(0,0) has infinitely many solutions; we discussed this in Section 2.4.

Solving (4.1) for y', we get the *normal form* of the equation:
$$y' = f(x, y). \tag{4.2}$$

Example: the equation $(y')^2 = 1$ is the "union" of two equations $y' = \pm 1$ in the normal form. The IVP(0,0) has exactly two solutions.

Two reasons to consider equations in the normal form:

- the main result of the theory, the existence and uniqueness theorem, applies to equations in the normal form;
- the equation (4.2) has a very clear geometric meaning: $f(x, y)$ describes the direction (or slope) field of the equation, see the next section.

4.2. **Local existence and uniqueness theorem.**

Theorem. *Consider the IVP(x_0, y_0) for the equation $y' = f(x, y)$. Suppose f is continuously differentiable in some nbh of (x_0, y_0). Then*

(i) there exists a solution, at least on some small interval $(x_0 - \epsilon, x_0 + \epsilon)$, and

(ii) this solution is locally unique, i.e. for some (small) ϵ any two solutions defined on $(x_0 - \epsilon, x_0 + \epsilon)$ coincide.

Notes.

(a) Proof of uniqueness – see Problem 6.3 (homework). Proof of existence – see Problem 6.4; also see Section 6 (approximation of solutions) below.

(b) "f is continuously differentiable" (notation: $f \in C^1$) means that f and the partial derivatives f_x, f_y are continuous functions. The theorem is in fact true if only f and f_y are continuous. Continuity of f alone is sufficient for existence (but not for uniqueness).

(c) The theorem is consistent with our results concerning 1D autonomous systems. As we mentioned, the theory of 1st order equations is essentially equivalent to the theory of 2D autonomous systems.

4.3. **Extension of solutions.** Let $y(x)$ be a solution defined on some interval (x_1, x_2). We say that the solution can be *extended forward* if there is a solution defined on a larger interval $(x_1, \tilde{x}_2)$ with $\tilde{x}_2 > x_2$ which coincides with the given solution on (x_1, x_2). In the same way we define *backward* extensions. A solution is *maximal* if it can not be extended. E.g., a solution defined on the whole real line is maximal.

Corollary. *Consider the equation $y' = f(x, y)$ with $f \in C^1(D)$, $D = \mathrm{dom}(f)$. Then every solution extends to a unique maximal solution.*

There could be only two reasons why a particular solution does not extend to infinity. A solution defined on (x_1, x_2) with $x_2 < \infty$ can not be extended forward if either

- the solution becomes unbounded as $x \to x_2$ ("blow up"), or
- the *solution curve* (= the graph of the solution in the (x, y)-plane) approaches the boundary of $D = \mathrm{dom}(f)$.

4.4. **Examples.** (a)

$$\frac{dy}{dx} = -\frac{x}{y}.$$

Then $\mathrm{dom}(f) = \mathbb{R}^2 \setminus \{y \neq 0\}$. The equation is "separable" and we can "solve" it:

$$ydy + xdx = 0, \qquad y^2 + x^2 = C.$$

Each circle contains exactly two maximal solution curves.

(b) No solution of the equation $y' = y^2 - x$ can be extended infinitely backward. Indeed, for $x < -1$ we have $y' > 1 + y^2$. Suppose we have a solution defined on $(-\infty, x_0)$ with $x_0 < -1$. Then for $x < x_0$, we have

$$\arctan y(x_0) - \arctan y(x) = \int_x^{x_0} \frac{y'(s)}{1 + y^2(s)} ds \geq \int_x^{x_0} ds = x_0 - x.$$

The LHS is bounded by π as $x \to -\infty$, but the RHS tends to infinity. A contradiction.

(There are solutions that extend infinitely forward – those "trapped" inside the parabola $x > y^2$, and there are solutions that blow up as $x \to +\infty$, see Section 5.3.)

(c) All solutions of the equation $y' = -x^2 + \sin y$ extend to $\mathbb{R}$.

Proof: the domain of the equation is the whole plane, so the only reason for non-extension could be a blow up. Suppose the solution of an IVP(x_0, y_0) is unbounded as $x \to x_* < +\infty$. We have

$$-x^2 - 1 \leq y'(x) \leq -x^2 + 1.$$

Integrating we have

$$-\frac{x^3}{3} - x - \text{const} \leq y(x) - y_0 \leq -\frac{x^3}{3} + x + \text{const},$$

and we get a contradiction: $y(x)$ is unbounded on (x_0, x_*), but the expressions to the right and to the left are bounded.

4.5. **Properties of maximal solution curves.** Consider the equation $y' = f(x, y)$ with $f \in C^1(D)$, $D = \text{dom}(f)$.

- There is one and only one maximal solution curve that passes through any given point of D. (This curve is the graph of the corresponding IVP solution.)
- Solution curves fill in D and don't intersect.
- Solution curves can not suddenly stop inside D.

5. Direction field analysis

5.1. **Geometric approach.** Consider the equation

$$y' = f(x, y).$$

Solution curves are smooth, i.e. they have a tangent at every point; there are no vertical tangents.

A smooth curve γ is a solution curve of the equation iff

$$\forall (x, y) \in \gamma, \qquad f(x, y) \;=\; \text{the slope of the tangent line at } (x, y),$$

(geometric meaning of the derivative).

The point is that we know the slopes of all solution curves without solving the equation.

5.2. **Direction fields.** A *direction field* in a domain D is a function that assigns a direction (an oriented straight line) to each point of D. The direction (or *slope*) field of the equation $y' = f(x, y)$ is the field in $D = \text{dom}(f)$ such that the direction at (x, y) is given by the line with slope $f(t, y)$.

Note. The concept of a direction field is slightly more general than that of a 1st order ODE in the normal form. Namely, there is a 1-to-1 correspondence between d.e. with $\text{dom}(f) = D$ and direction fields in D *without vertical directions.* For example, circles can be integral curves of a direction field; consider for example the field of velocities if we rotate the plane around the origin. (An *integral curve* of a direction field is a smooth curve such that the tangent has the direction of the field.)

We can visualize the direction field of an equation by drawing short line segments with slopes given by the equation and centered at some grid of points. (A computer can easily do it for you; ask you TA to show the relevant Mathematica or Matlab commands.) The field literally guides your eye along solution curves. (This is also the idea of Euler's broken line method which we discuss in the next section.) In most cases, the qualitative behavior of the solutions is already clear from the picture.

5.3. **Example.**

$$y' = y^2 - x.$$

We already mentioned that this equation can not be solved by quadratures. (This is somewhat similar to the Abel-Galois theory of algebraic equations: in general one can not solve algebraic equations of degree 5 or greater in radicals.) We can nevertheless describe qualitatively the behavior of solutions. The best bet is to use a computer but a sketch of the direction field can be easily done by hand. In fact even if we use a computer we need to zoom on essential parts of the picture. Some kind of preliminary analysis is always necessary.

- Start with *null clines*: the set of points where the direction is horizontal ($f = 0$). We get the parabola $y^2 = x$.
- Notice that the slope is negative inside the parabola, and positive outside.
- Draw some *isoclines* $f = k$ with $k = -1, 1, 2$ for example.

This gives the picture, and we can conclude the following:

- There are two kinds of solution curves – intersecting the parabola and avoiding it.
- If the curve enters the parabola, it stays there forever and tends to the lower branch.
- If the curve does not intersect the parabola, it has finite escape time: the solution becomes infinite in a finite time, it "blows up".

Try to prove these facts rigorously.

6. Approximation of solutions

6.1. **Euler's line method.** Consider the initial value problem

$$y' = f(x, y), \qquad y(x_0) = y_0.$$

Fix a small number $h > 0$ (step size) and denote

$$x_1 = x_0 + h, \quad x_2 = x_0 + 2h, \quad \ldots$$

Define recursively

$$y_1 = y_0 + f(x_0, y_0)h, \quad y_2 = y_1 + f(x_1, y_1)h, \quad \ldots$$

We can also go backward:

$$x_{-1} = x_0 - h, \quad y_{-1} = y_0 - f(x_0, y_0)h, \quad \ldots$$

We think of y_n as an approximate value of the IVP solution at x_n,

$$y_n \approx y(x_n).$$

Motivation:

$$\begin{aligned} y_{n+1} - y_n &\approx y(x_{n+1}) - y(x_n) \approx y'(x_n)(x_{n+1} - x_n) \\ &= f(x_n, y(x_n))h \approx f(x_n, y_n)h. \end{aligned}$$

Alternatively,

$$\begin{aligned} y(x_{n+1}) - y(x_n) &= \int_{x_n}^{x_{n+1}} y'(s)ds = \int_{x_n}^{x_{n+1}} f(s, y(s))ds \\ &\approx (x_{n+1} - x_n)f(x_n, y(x_n)) \approx hf(x_n, y_n). \end{aligned}$$

Connecting the points (x_n, y_n) in the plane, we get *Euler's broken line*, the graph of a piecewise linear function. We denote this function by $y^{(h)} = y^{(h)}(x)$.

Claim. *Suppose f is C^1 near the initial point. Then $\exists \epsilon$ such that for all $x \in (x_0 - \epsilon, x_0 + \epsilon)$ the following limit exists*

$$y(x) := \lim_{h \to 0} y^{(h)}(x),$$

and the limit function $y(x)$ is a solution of the IVP.

Notes. (a) Key idea: approximation of a differential equation by a difference equation.

(b) Euler's approximation can be used to prove existence of solutions.

(c) In some cases we can control the size of the interval of convergence. If f is C^1 and f_y is bounded in the strip $\{a \le x \le b,\ -\infty < y < \infty\}$, then $y^{(h)} \to y$ uniformly on $[a, b]$ (cf. no blow up for linear equation)

(d) Euler's method has the rate of convergence of order $p = 1$,

$$y^h - y = O(h^p), \qquad (h \to 0).$$

There are various refinements of the method with a better rate of convergence. E.g., the following schemes have order 2 (for sufficiently smooth f's):

- trapezoid scheme

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_{n+1})], \quad \text{solve for } y_{n+1}$$

(explain the name)

- improved Euler

$$y_{n+1} = y_n + \frac{h}{2}[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))]$$

- midpoint method (Runge of order two)

$$y_{n+1} = y_n + hf(x_n + h/2, y_n + hf(x_n, y_n)/2)$$

- Taylor of order two

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2!} f_2(x_n, y_n), \qquad f_2 := f_x + f_y f.$$

Motivation: if $y' = f(x, y)$, then $y'' = f_x + f_y y' = f_x + f_y f$

The classical Runge-Kutta method (see the end of Section 21 in textbook) has order 4. Geometrically, in all refined versions of the Euler method the solution curve is approximated not by line segments but rather by segments of higher order algebraic curves.

6.2. **Picard's method of successive approximations.** Consider the same IVP. Define $y_0(x) \equiv y_0$,

$$y_1(x) = y_0 + \int_{x_0}^{x} f(s, y_0(s))\, ds, \quad y_2(x) = y_0 + \int_{x_0}^{x} f(s, y_1(s))\, ds, \ldots$$

The functions $y_n(x)$ are defined on certain intervals which depend on n. Motivation for this method: if $y(x)$ is a solution, then it satisfies the *integral equation*

$$y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))\, ds.$$

Claim. *Suppose f is C^1 near the initial point. Then $\exists \epsilon > 0$ such that all functions $y_n(x)$ are defined on the interval $(x_0 - \epsilon, x_0 + \epsilon)$, the limit*

$$y(x) := \lim_{n\to\infty} y_n(x),$$

exists for all $x \in (x_0 - \epsilon, x_0 + \epsilon)$, and the limit function $y(x)$ is a solution of the IVP.

[See Exercise 6.4 in the textbook for the idea of the proof.]

6.3. **Example.** Let us consider the IVP $y' = y,\ y(0) = 1$ and compute the value $y(1) = e$ applying two methods of approximation.

Picard: $y_1(x) = 1 + x$,

$$y_2(x) = 1 + \int_0^x (1 + s)\, ds = 1 + x + x^2/2,$$

$$y_3(x) = 1 + \int_0^x (1 + s + s^2/2)\, ds = 1 + x + x^2/2 + x^3/3!,$$

so

$$y_n(1) = 1 + \frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}.$$

Euler: choosing the step size $h = 1/n$, we have

$$y_{k+1} = y_k + y_k h = y_k (1 + h) = y_0 (1 + h)^k,$$

and

$$y^{(h)}(1) = y_n = \left(1 + \frac{1}{n}\right)^n.$$

Conclusion:

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n.$$

Rates of convergence:

$$e - \sum_{k=0}^{n} \frac{1}{k!} = \frac{1}{(n+1)!} + \frac{1}{(n+2)!} + \cdots = O\left(\frac{1}{(n+1)!}\right)$$

(super-exponentially fast), but

$$\left| e - \left(1 + \frac{1}{n}\right)^n \right| \asymp \frac{1}{n};$$

use

$$\left(1 + \frac{1}{n}\right)^n = e^{n \log\left(1 + \frac{1}{n}\right)} = e^{1 - \frac{2}{n} + \frac{3}{n^2} + \dots}$$

7. Difference equations

7.1. **Iterations.** A discrete analogue of a 1D autonomous system is the equation

$$x_{n+1} = f(x_n), \tag{7.1}$$

where $f(x)$ is a given function. For example, we arrive to (7.1) with $f(x) = x + hv(x)$ if we apply Euler's method to the differential equation $\dot{x} = v(x)$:

$$\frac{x_{n+1} - x_n}{h} \approx v(x_n).$$

Also, discrete models may be more appropriate in certain applications; e.g. the natural time unit in population dynamics is one generation.

Solutions of (7.1) are sequences ("orbits")

$$\{x_n\}, \quad n = 0, 1, 2, \dots.$$

The IVP for a difference equation is to find a solution $\{x_n\}$ with a given value of x_0. Clearly, this IVP has a unique solution,

$$x_1 = f(x_0), \quad x_2 = f(x_1) = f(f(x_0)), \quad x_3 = (f \circ f \circ f)(x_0), \dots$$

and in general

$$x_n = f^{\circ n}(x_0),$$

where $f^{\circ n}$ denotes the n-th iterate of f, i.e. the composition of n copies of f. The theory of difference equations is *iteration theory.*

7.2. **Cobweb diagrams.** To compute the orbit of x under the iterations of f consider the graph of $y = f(x)$ and the diagonal $y = x$ in the xy-plane. Repeat the following procedure over and over. Draw a vertical line from (x, x) to the graph, followed by a horizontal line back to the diagonal. When we reach the point (fx, fx), we repeat this to get $(f^{\circ 2}x, f^{\circ 2}x)$, etc.

Examples.

(a) $x_{n+1} = x_n^2 + c$ with $c > 1/4$. The graph does not intersect the diagonal; all orbits escape to infinity.

(b) $x_{n+1} = x_n^2 + c$ with $c = 0$. We have two *fixed* (stationary) points: $a = 1$ and $b = 0$, the solutions of the equation $f(x) = x$. The graphical analysis shows:

$$|x| > a \Rightarrow f^{\circ n}x \to \infty, \qquad |x| < a \Rightarrow f^{\circ n}x \to b.$$

This result holds if $-3/4 < c < 1/4$; the fixed points are

$$a = 1/2 + \sqrt{1/4 - c}, \quad b = 1/2 - \sqrt{1/4 - c}.$$

(c) $x_{n+1} = x_n^2 - 1$. The orbit of 0 is an attracting 2-cycle (periodic solution) $0 \mapsto -1 \mapsto 0$.

Exercises.

(a) Let x_* be a fixed point of the difference equation $x_{n+1} = f(x_n)$. Show that if $|f'(x_*)| < 1$ then x_* is stable and attracting, but if $|f'(x_*)| > 1$ then x_* is unstable. (Give accurate definitions.)

Hint:
$$\frac{|x_{n+1} - x_*|}{|x_n - x_*|} = \frac{|f(x_n) - f(x_*)|}{|x_n - x_*|} \approx |f'(x_*)|.$$

(b) Show that a 2-cycle $\{x_1, x_2\}$ is attracting if $|f'(x_1)f'(x_2)| < 1$, (again, give an accurate definition).

(c) Compare the stationary points and discuss their stability for the equations $\dot{x} = v(x)$ and $x_{n+1} = x_n + hv(x_n)$.

7.3. **Remark.** The long term behavior of difference equation (1D discrete dynamical systems) can be (and typically is) dramatically different (and infinitely more complicated) than the behavior of the corresponding differential equations (1D smooth dynamical systems). Roughly speaking the "complexity" of discrete dynamics in n dimension is the same as that of smooth dynamics in $n+1$ dimensions. Because of certain constraints of the plane topology, the 1D discrete case is, in some sense, more interesting than the case of 2D smooth systems in the plane. We will return to this topic later the course.

Further examples.

(a) **Heron of Alexandria** (250 A.D.): find a square of area A, e.g. $A = 5$.

Method: approximate the square by rectangles R_n with sides x_n, y_n. Start with any rectangle R_1 such that
$$x_1, y_1 > 0, \qquad x_1 y_1 = A,$$
then set
$$x_{n+1} = \frac{x_n + y_n}{2}, \quad y_{n+1} = \frac{2x_n y_n}{x_n + y_n}, \qquad (n = 1, 2, \dots).$$
Note that for all n, the area of R_n is $x_n y_n = A$, and
$$\min(x_n, y_n) \le \min(x_{n+1}, y_{n+1}) \le \sqrt{A} \qquad \text{(why?)}$$
Claim:
$$x_n,\ y_n \ \to \sqrt{A}, \qquad (n \to \infty).$$
Pf: we have $x_{n+1} = f(x_n)$, where
$$f(x) = \frac{1}{2}\left(x + \frac{A}{x}\right), \qquad (x > 0).$$
The point $x_* = \sqrt{A}$ is a *super-attractive* fixed point of f:
$$f(x_*) = x_*, \qquad f'(x_*) = 0.$$
Draw a cobweb diagram. For $A = 5$ and $x_1 = 1$ we have
$$x_2 = 3, \quad x_3 = \frac{7}{3} = 2.33\dots, \quad x_4 = \frac{47}{21} = 2.238\dots, \quad \dots \quad \to \sqrt{5} = 2.236\dots$$

(b) **Newton's method.** We want to find roots of the equation

$$g(x) = 0,$$

where g is a given smooth function, e.g. $g(x) = x^2 - A$.

Start with an *educated* guess x_0, then linearize g at x_0:

$$g(x) \sim g(x_0) + g'(x_0)(x - x_0),$$

and set

$$x_1 = x_0 - \frac{g(x_0)}{g'(x_0)}, \qquad \text{(the root of the linearized function)}.$$

Repeat:

$$x_{n+1} = f(x_n), \qquad f(x) = x - \frac{g(x)}{g'(x)}.$$

Show that if

$$g(x_*) = 0, \qquad g'(x_*) \neq 0,$$

then x_* is a super-attractive fixed point of f, so $x_n \to x_*$ provided that x_0 has been chosen sufficiently close to the root x_*.

8. Separable equations

In the next several sections we discuss integration techniques for certain types of 1st order ODEs. We start with separable equations:

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}. \tag{8.1}$$

We can rewrite (8.1) as follows

$$g(x)dx - h(y)dy = 0;$$

the expression to the left is an example of a *differential form.* Integrating, we have

$$G(x) - H(y) = \text{const},$$

where G and H are some primitives of g and h respectively. We get (formal) solutions of (8.1) in implicit form.

Here is a more accurate treatment of equation (8.1). Denote

$$E(x, y) = G(x) - H(y).$$

Lemma. *$E(x, y)$ is a first integral of the equation, i.e. E is constant on every solution curve.*

Proof: Let $y = y(x)$ be a solution. Then

$$\frac{d}{dx}E(x, y(x)) = \frac{d}{dx}[G(x) - H(y(x))] = g(x) - h(y(x))y'(x) = 0.$$

If a function defined on an interval has derivative zero then the function is constant. □

The concept of first integrals is familiar from mechanics; e.g. energy is a first integral of a conservative system (”conservation of energy”). Also, note that if E is a first integral, then any function of E is a first integral, e.g. $2E$, E^2, etc.

Theorem. *Suppose $g, h \in C^1(\mathbb{R})$ and consider the IVP $y(x_0) = y_0$ with $h(y_0) \neq 0$ for the equation* (8.1). *Define E as above and set*

$$C = E(x_0, y_0).$$

Then the intersection of the level set

$$L_C = \{(x, y) : \; E(x, y) = C\}$$

with the domain of the equation

$$D = \{(x, y) : \; h(y) \neq 0\}$$

is a collection of disjoint smooth simple curves (components of $L_C \cap D$). If γ denotes the component which contains the initial point (x_0, y_0), then γ is the maximal solution curve of the IVP.

Proof: The fact that the components of $L_C \cap D$ are smooth simple curves follows from implicit function theorem as $\nabla E \neq 0$ in D. We have the existence and uniqueness theorem in D by the assumption $g, h \in C^1(\mathbb{R})$, and therefore maximal solution curves have the properties mentioned in Section 4. By Lemma, the maximal solution curve of the IVP belongs to the curve γ. It coincides with γ because a maximal solution curve can not suddenly stop inside D. □

Example.

$$y' = \frac{x^2}{1 - y^2}, \qquad y(0) = 0. \tag{8.2}$$

Determine the interval of the maximal solution.

We have

$$x^2 dx + (y^2 - 1) dy = 0,$$

and so

$$E(x, y) = x^3 + y^3 - 3y$$

is a first integral. Since

$$C = E(0, 0) = 0,$$

the level set is the curve

$$x = (3y - y^3)^{1/3}.$$

We need to exclude the points with $y = \pm 1$. The corresponding values of x are $\pm 2^{1/3}$. We see that the set $L_C \cap D$ (in the notation of the theorem) has three components, and the interval of the maximal solution of the IVP is $(-2^{1/3}, 2^{1/3})$. (The reason for maximality is that the solution curve reaches the boundary of $\mathrm{dom}(f)$.)

We can also find the maximal interval using the fact that the solution curve has vertical tangents at the endpoints of the maximal interval (why?)

Note. It is not always natural to exclude the points with $h(y) = 0$ when we consider the symmetric form of the equation (8.1). Let us compare the equation (8.2) in the last example with the 2D autonomous system

$$\dot{x} = 1 - y^2, \qquad \dot{y} = x^2. \tag{8.3}$$

The system has a unique solution $(x(t), y(t))$ for all initial data $x(0) = x_0$, $y(0) = y_0$. The sets (trajectories of the motion)

$$\{(x(t), y(t)) \in \mathbb{R}^2 : \ t \in \mathbb{R}\}$$

are the *orbits* of the system. The function $E(x, y) = x^3 + y^3 - 3y$ is a first integral of the system (i.e., E is constant on each orbit, same proof as in Lemma). The system has two stationary orbits (or points) $(0, \pm 1)$. Non-stationary orbits are exactly the components of the level sets of E with excluded stationary points.

Roughly speaking, solutions $(x(t), y(t))$ of the system (8.3) represent solutions of the equation (8.2) in the *parametric form*; we need to exclude t from the equations of motion $x = x(t), y = y(t)$ to get an explicit solution $y = y(x)$. The equations of the level sets represent solutions of (8.2) in *implicit form*; we need to solve $E(x, y) = c$ for $y = y(x)$.

9. Exact equations

9.1. **Exact differential forms.** Consider the equation

$$M(x, y)dx + N(x, y)dy = 0, \tag{9.1}$$

which we can view as the symmetric form of the ODE

$$\frac{dy}{dx} + \frac{M(x, y)}{N(x, y)} = 0, \qquad y = y(x).$$

Also, we arrive to (9.1) if we exclude the time variable from the autonomous system

$$\dot{x} = N(x, y), \qquad \dot{y} = -M(x, y).$$

The differential form $\omega = M dx + N dy$ and the equation (9.1) are called *exact* in a domain D if there is a function $E = E(x, y)$, $E \in C^1(D)$, such that

$$E_x = M, \quad E_y = N \qquad \text{in } D.$$

Terminology: E is a *primitive* of ω, and $\omega = E_x dx + E_y dy$ is the *differential* of E, $dE = \omega$.

Example: separable equations are exact.

Lemma. *$E(x, y)$ is a first integral of the equation.*

Proof: Let $y(x)$ be a solution. Then

$$\frac{d}{dx}E(x, y(x)) = [E_x + y' E_y] = M + y' N = 0.$$

□

The description of (maximal) solutions in terms of E is similar to the analysis in the previous section (separable equations).

9.2. **Finding first integrals.** It is easy to find E (locally) once we know that the equation is exact.

From $E_x = M$ we find (by the fundamental theorem of calculus)

$$E(x,y) = \int_{x_0}^{x} M(s,y)ds + h(y),$$

where h is an unknown function of one variable. From the second equation $E_y = N$ we then get a "trivial" equation for h,

$$N(x,y) = \int_{x_0}^{x} M_y(s,y)ds + h'(y).$$

9.3. **Closed differential forms.** The last equation makes sense only if

$$N(x,y) - \int_{x_0}^{x} M_y(s,y)ds$$

is a function of just y. If $M, N \in C^1$, then this is equivalent to the statement

$$N_x - M_y = \frac{d}{dx}\left[N(x,y) - \int_{x_0}^{x} M_y(s,y)ds\right] = 0.$$

Alternative argument (assuming $E \in C^2$):

$$M_y = E_{xy} = E_{yx} = N_x.$$

Differential forms $Mdx + Ndy$ satisfying the equation $M_y = N_x$ are called *closed*. We have shown that exact forms are closed. This necessary condition for exactness is also sufficient if the domain D is *simply connected*, which means that there are no "holes" in D. In particular, we have the following

Theorem. *Suppose the functions M and N are defined in a rectangle (e.g. the whole plane), and suppose they are C^1 and satisfy*

$$M_y = N_x.$$

Then the equation is exact in the rectangle.

Proof: Define $E(x,y)$ using the formulae in the previous subsection and then check that $E_x = M, E_y = N$. $\square$

[Q: where did we use the rectangle assumption?]

Example. Let

$$M = \frac{-y}{x^2+y^2} = \frac{\partial}{\partial x}\arctan\frac{y}{x}, \quad N = \frac{x}{x^2+y^2} = \frac{\partial}{\partial y}\arctan\frac{y}{x}.$$

Then $M_y = N_x$ in $\mathbb{R}^2 \setminus (0,0)$ but there is no function $E(x,y)$ in $\mathbb{R}^2 \setminus (0,0)$ satisfying the system $E_x = M, E_y = N$.

Pf: E has to be equal to $\arctan\frac{y}{x} + \text{const}$ in each of the halfplanes $\{x > 0\}$ and $\{x < 0\}$, so it can not be continuous in $\mathbb{R}^2 \setminus (0,0)$. (Note that $\arctan\frac{y}{x}$ is the angle in polar coordinates.)

Relation to algebraic topology: de Rham cohomology

$$H^1_{\mathrm{dR}}(\mathbb{R}^2 \setminus (0,0)) := \{\text{closed forms}\}/\{\text{exact forms}\} \approxeq \mathbb{R}.$$

10. Integrating factors

If the equation
$$M(x,y)dx + N(x,y)dy = 0$$
is not exact, then we can *try* to find some (non-zero) function $\mu = \mu(x,y)$ such that the equation becomes exact after multiplication by μ,
$$\mu M dx + \mu N dy = 0.$$
Applying the criterion of the last section, we see that $\mu(x,y)$ must satisfy the equation $(\mu M)_y = (\mu N)_x$, i.e.
$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0. \tag{10.1}$$
(It is a first order linear PDE for $\mu(x,y)$.) We call nontrivial solutions of (10.1) *integrating factors* of the original equation $Mdx + Ndy = 0$.

In general, to find a non-trivial solution of (10.1) explicitly is no easier than to solve the original ODE. Nevertheless, in certain cases this approach works.

Example: linear equations
$$\frac{dy}{dx} + p(x)y = g(x).$$
Let us check that if $P(x)$ is a primitive of $p(x)$, then
$$\mu = e^{P(x)}$$
is an integrating factor. Rewriting the linear equation in the form
$$[p(x)y - g(x)]dx + dy = 0$$
we have
$$M\mu_y - N\mu_x + (M_y - N_x)\mu = 0 - 1 \cdot e^{P(x)}p(x) + (p(x) - 0)e^{P(x)} = 0.$$

This is an example of the special situation when there exists an integrating factor which depends only on one variable (in our case $\mu = \mu(x)$).

Lemma. *The equation $Mdx + Ndy = 0$ has an integrating factor which depends only on x if and only if the function*
$$\frac{M_y - N_x}{N}$$
does not depend on y.

Proof: An integrating factor $\mu = \mu(x)$ satisfies the equation
$$-N\mu' + (M_y - N_x)\mu = 0,$$
so the function
$$\frac{M_y - N_x}{N} = \frac{\mu'(x)}{\mu(x)}$$
does not depend on y. In the opposite direction, if $(M_y - N_x)/N$ is a function of just x, then we easily find an integrating factor $\mu(x)$ by solving the "trivial" equation
$$(\log \mu)' = \frac{M_y - N_x}{N}.$$

$\square$

Example: $(2x^2 + y)dx + (x^2y - x)dy = 0$.
We have

$$M_y = 1, \quad N_x = 2xy - 1, \quad \frac{M_y - N_x}{N} = \frac{2(1 - xy)}{x(xy - 1)} = -\frac{2}{x},$$

and the equation for $\mu(x)$ is

$$\mu' = -\frac{2}{x}\mu,$$

so

$$\mu(x) = \frac{1}{x^2}$$

is an integrating factor. Multiplying the original equation by μ, we get an exact equation

$$\left(2 + \frac{y}{x^2}\right) dx + \left(y - \frac{1}{x}\right) dy = 0.$$

To find a first integral $E(x, y)$ of the exact equation, we start with $E_x = 2 + \frac{y}{x^2}$, so

$$E(x, y) = 2x - \frac{y}{x} + h(y),$$

and then we find $h(y)$ from the equation $E_y = y - \frac{1}{x}$, i.e.

$$h'(y) - \frac{1}{x} = y - \frac{1}{x},$$

so we can take $h(y) = y^2/2$. Thus

$$E(x, y) = 2x - \frac{y}{x} + \frac{y^2}{2},$$

and it remains to do the usual analysis of the level sets $E(x, y) = C$.

Note: if we consider our equation $(2x^2 + y)dx + (x^2y - x)dy = 0$ as an equation for $x = x(y)$, then $x(y) \equiv 0$ is an obvious solution which has been lost in multiplying by $\mu = 1/x^2$.

Exercises. (a) The equation $Mdx + Ndy = 0$ has an integrating factor that depends only on y iff $(N_x - M_y)/M$ is a function of just y.

(b) The equation $Mdx + Ndy = 0$ has an integrating factor that depends only on the sum $x + y$ iff the function $(M_y - N_x)/(M - N)$ depends only on $x + y$. E.g., solve

$$(3 + y + xy)dx + (3 + x + xy)dy = 0.$$

(c) The equation $Mdx + Ndy = 0$ has an integrating factor that depends only on the product xy iff the function $(M_y - N_x)/(xM - yN)$ depends only on xy. E.g., solve

$$(3y + 2xy^2)dx + (x + 2x^2y)dy = 0.$$

(d) Find an integrating factor of the form $x^n y^m$ and solve the equation

$$(12 + 5xy)dx + (6xy^{-1} + 3x^2)dy = 0.$$

11. Linear equations

11.1. **Main theorem.** We consider the equation

$$y' + p(x)y = g(x). \tag{11.1}$$

Theorem. *Suppose $p, g \in C(I)$, where I is an interval. Then*

(i) every solution extends to the whole interval I (no "blow up");

(ii) every IVP(x_0, y_0) with $x_0 \in I$ has a unique (maximal) solution;

(iii) this solution is given by the formula

$$y(x) = y_0 e^{-\int_{x_0}^{x} p} + \int_{x_0}^{x} g(s) e^{-\int_s^x p(t)dt} ds.$$

Except for uniqueness, everything follows from the formula: the expression to the right defines a function on the whole interval I, and it is straightforward to verify that this function is a solution.

11.2. **Linear properties.** If $g = 0$, the equation (11.1) is called *homogeneous.*

Lemma. *(i) The set of (maximal) solutions of a homogeneous equation is a linear space of dimension one.*
(ii) If $y_*(x)$ is a solution of (11.1)*, then the general solution of* (11.1) *is*

$$y(x) \in y_*(x) + \text{ general solution of } y' + p(x)y = 0.$$

[Terminology in the textbook: "particular integral + complementary function".]

The lemma is obvious from the solution formula but there is a general argument which applies in all dimensions.

Proof: (i) We need to show

$$\varphi,\ \psi \quad \text{are solutions} \quad \Longrightarrow \quad y := a\varphi + b\psi \ \text{ is a solution.}$$

We have $\varphi' + p\varphi = 0$ and $\psi' + p\psi = 0$, so

$$y' + py = a\varphi' + b\psi' + p(\varphi + \psi) = 0.$$

The dimension result follows from the uniqueness theorem. Fix some x_0, then each solution is determined by the initial value $y_0 = y(x_0)$. The space of all possible initial values is one-dimensional. (It will be 2D for second order linear equations.)

(ii) This is a simple exercise. $\square$

11.3. **Derivation of the solution formula.** We can easily derive the formula by using the integrating factor

$$\mu(x) = e^{\int_{x_0}^{x} p(s)ds}.$$

see Example in the previous section. We will outline an alternative method, **variation of parameters**, which extends to all dimensions

The general solution of the *associated* homogeneous equation $y' + p(x)y = 0$ is

$$y(t) = Ce^{-\int_{x_0}^{x} p},$$

and by the last lemma, all we need is to find a particular solution of the inhomogeneous equation (11.1). The idea is to seek this particular solution of the form

$$y(x) = C(x)\phi(x), \qquad \phi(x) := e^{-\int_{x_0}^{x} p}.$$

Note $\phi' = -p\phi$. We have

$$y' = C'\phi + C\phi' = C'\phi - Cp\varphi.$$

This should be equal to

$$-py + g = -Cp\phi + g,$$

and we find

$$C'\phi = g, \qquad C' = \frac{g}{\phi},$$

which is a "trivial" equation for $C(x)$. One solution, which is 0 at x_0, is

$$C(x) = \int_{x_0}^{x} \frac{g(s)}{\phi(s)} ds = \int_{x_0}^{x} e^{\int_{x_0}^{s} p} g(s) ds,$$

and we have

$$y(x) = e^{-\int_{x_0}^{x} p} \int_{x_0}^{x} e^{\int_{x_0}^{s} p} g(s) ds = \int_{x_0}^{x} e^{-\int_{s}^{x} p} g(s) ds.$$

Adding the term $y_0\phi(x)$, which is equal to y_0 at t_0, we solve the IVP.

11.4. **Linearization.** (Adapted from Arnold, Section 7.3. This topic is important in applications and for understanding the role of linear equations.)

Consider the IVP

$$y' = f(x, y, \epsilon), \qquad y(x_0) = a(\epsilon),$$

where the (non-linear) equation and the initial value depend on a *small parameter* ϵ. Let $y(x, \epsilon)$ denote the solution. Suppose we can solve the IVP when $\epsilon = 0$, and let

$$\varphi(x) = y(x; 0)$$

be the known solution. We will assume that $y(x; \epsilon)$ is sufficiently smooth as a function of both variables. One can show that this is true provided that f is a (sufficiently) smooth function of all three variables, see Alnold's textbook. It follows that

$$y(x, \epsilon) = \varphi(x) + \epsilon\psi(x) + o(\epsilon), \qquad \epsilon \to 0,$$

where

$$\psi(x) = \frac{\partial y}{\partial \epsilon}(x, 0).$$

We want to find this function ψ.

Theorem. *ψ is the solution of the initial value problem*

$$\psi' + p(x)\psi = g(x), \qquad \psi(x_0) = a'(0), \tag{11.2}$$

where

$$p(x) = -\frac{\partial f}{\partial y}(x, \varphi(x), 0) \qquad g(t) = \frac{\partial f}{\partial \epsilon}(x, \varphi(x), 0). \tag{11.3}$$

Proof: By definition, we have the identity

$$\frac{\partial}{\partial x} y(x, \epsilon) = f(x, y(x, \epsilon), \epsilon).$$

Differentiate it with respect to ϵ and then set $\epsilon = 0$. In the LHS we get

$$\frac{\partial}{\partial \epsilon}\frac{\partial}{\partial x} y(x, \epsilon)\Big|_{\epsilon=0} = \frac{\partial}{\partial x}\frac{\partial}{\partial \epsilon} y(x, \epsilon)\Big|_{\epsilon=0} = \psi'(x),$$

and in the RHS,

$$\frac{\partial f}{\partial y}(x, \varphi(x), 0)\psi(x) + \frac{\partial f}{\partial \epsilon}(x, \varphi(x), 0) \equiv -p(x)\psi(x) + g(x).$$

The initial condition for ψ is $\psi(x_0) = a'(0)$ because

$$a(0) + \epsilon a'(0) + o(\epsilon) = y(x_0, \epsilon) = \varphi(x_0) + \epsilon\psi(x_0) + o(\epsilon).$$

□

Note. Here is an alternative, somewhat formal way to derive equations (11.2)-(11.3). We have

$$\frac{\partial y}{\partial x} = \varphi' + \epsilon\psi' + \dots,$$

and

$$\begin{aligned} f(x, y, \epsilon) &= f(x, \varphi + \epsilon\psi + \dots, \epsilon) \\ &= f(x, \varphi, 0) + \epsilon\psi(x)\frac{\partial f}{\partial y}(x, \varphi, 0) + \epsilon\frac{\partial f}{\partial \epsilon}(x, \varphi, 0) + \dots \\ &= f(x, \varphi, 0) + \epsilon(-p\psi + g) + \dots \end{aligned}$$

We now equate the coefficients of ϵ.

Example. $y' = y^2 + y\sin x, \ \ y(0) = \epsilon$.

If $\epsilon = 0$, then the solution is of course $\varphi(x) \equiv 0$. We have

$$-p(x) = \frac{\partial f}{\partial y}(x, 0, 0) = \sin x, \qquad g(x) = \frac{\partial f}{\partial \epsilon} = 0,$$

and $a'(0) = 1$. Solving the IVP

$$\psi' - (\sin x)\psi = 0, \qquad \psi(0) = 1,$$

we get

$$\psi(x) = e^{\int_0^x \sin s ds} = e^{1-\cos x}.$$

Conclusion: in the first approximation,

$$y(x, \epsilon) \approx \epsilon e^{1-\cos x}.$$

Exercise: find the exact solution of this IVP and compare it with the first approximation obtained above. (Hint: Bernoulli, see the next section)

The method described here (linearization of the equation near the known solution) works in all dimensions – we always get an explicit linear system in the first approximation.

12. Change of variables

General idea. Let $\Phi: \tilde{D} \to D$ be a *diffeomorphism* between two planar domains. This means that Φ is invertible, and Φ and $\Phi^{\circ -1}$ (inverse map) are continuously differentiable. If $\mathcal{F}$ is a direction field in D, then we define the direction field $\tilde{\mathcal{F}} = \Phi^*\mathcal{F}$ in $\tilde{D}$ by considering preimages of infinitesimally short line segments. Clearly, Φ maps the integral curves of $\tilde{\mathcal{F}}$ onto the integral curves of $\mathcal{F}$.

Let (x, y) be coordinates in D and $(\tilde{x}, \tilde{y})$ coordinates in $\tilde{D}$, so Φ is given by two functions

$$x = \phi_1(\tilde{x}, \tilde{y}), \qquad y = \phi_2(\tilde{x}, \tilde{y}).$$

Suppose we have a differential equation for $y = y(x)$ in D,

$$\frac{dy}{dx} = f(x, y), \qquad \mathrm{dom}(f) = D,$$

and let $\mathcal{F}$ be the direction (or slope) field of this equation. If the corresponding field in $\tilde{D}$ has no "vertical" (i.e. parallel to $\tilde{y}$ axis) directions, then $\tilde{\mathcal{F}}$ defines a differential equation for $\tilde{y} = \tilde{y}(\tilde{x})$; $\tilde{x}$ is the *new* independent variable and $\tilde{y}(\tilde{x})$ is the new unknown function. Φ maps the solution curves of the new equation onto the solution curves of the original equation. If we can solve the new equation, then we know the solution curves in $\tilde{D}$, and applying Φ we get solution curves of the original equation.

In technical terms, it is often convenient to work with differential forms in order to perform the change of variables. The equation

$$\omega := M(x, y)dx + N(x, y)dy = 0$$

in D corresponds to the equation $\tilde{\omega} = 0$ in $\tilde{D}$, where

$$\tilde{\omega} \equiv \Phi^*\omega = M(\phi_1, \phi_2)d\phi_1 + N(\phi_1, \phi_2)d\phi_2.$$

Example 1. It is probably not immediately clear that the equation

$$y' = \frac{1}{y^2 - x}$$

is integrable. (As we mentioned earlier, a simpler looking equation $y' = y^2 - x$ is not integrable.) Change of variables: think of x as a function of y, $x = x(y)$; in our notation this corresponds to the diffeomorphism $\mathbb{R}^2 \to \mathbb{R}^2$,

$$x = \tilde{y}, \quad y = \tilde{x}.$$

The new equation is

$$\frac{d\tilde{y}}{d\tilde{x}} = \tilde{x}^2 - \tilde{y}$$

is linear and can be easily solved.

Example 2. $y' = f(ax + by + c)$, $b \neq 0$. Consider the new unknown function

$$z = z(x), \qquad z = ax + by(x) + c.$$

We have

$$z' = a + by' = a + bf(z),$$

which is a scalar autonomous equation.

Example 3: Bernoulli equation.

$$y' + p(x)y = g(x)y^n.$$

Here n is a real number. We assume $n \neq 0, 1$; otherwise the equation is linear and we know what to do. Bernoulli equation can be transformed into a linear equation by the change of the space variable

$$z = y^\alpha, \qquad \alpha = 1 - n;$$

the independent variable x remains the same. We have $z' = \alpha y^{\alpha-1}y'$. Multiplying the equation by $\alpha y^{\alpha-1}$, we get the linear equation

$$z' + \alpha p(x)z = \alpha g(z).$$

Exercise: $y' = y^2 + y \sin x$.

Example 4: homogeneous equations. Suppose $f(x, y)$ depends only on the ratio y/x,

$$f(x, y) = F(y/x),$$

so

$$\forall \lambda, \qquad f(\lambda x, \lambda y) = f(x, y).$$

In this case the equation

$$y' = f(x, y)$$

is called *homogeneous*; e.g.

$$y' = \left(\frac{ax + by}{cx + dy}\right)^2.$$

We can separate the variables if we introduce the new unknown function

$$z(x) = \frac{y(x)}{x}.$$

This corresponds to the diffeomorphism

$$\Phi : (x, z) \mapsto (x, zx) \qquad \text{in} \quad \mathbb{R}^2 \setminus \{x = 0\}.$$

We have

$$F(z) = y' = z'x + z,$$

so the new equation is separable:

$$z'x = F(z) - z.$$

Note: use $z(y) = x(y)/y$ in the region $\{y \neq 0\}$; the corresponding Φ is $(y, z) \mapsto (zy, y)$

Exercise. Equations $y' = f(x, y)$ with f satisfying

$$\forall \lambda, \qquad f(\lambda^a x, \lambda^b y) = \lambda^c f(x, y)$$

are called *quasi-homogeneous*. Guess the right change of variables that makes such equations separable.

Example 5: polar coordinates. We can use polar coordinates (θ, r),

$$x = r\cos\theta, \quad y = r\sin\theta,$$

to separate variables in homogeneous equation. Note that we have to make a cut in the (x, y)-plane, e.g. exclude the negative real axis, to get a diffeomorphism Φ from the semi-strip $(\pi, \pi) \times (0, \infty)$ onto the slitted plane. The RHS of a homogeneous equation is a function $\Theta(\theta)$, and we have

$$\Theta(\theta) = \frac{dy}{dx} = \frac{\sin\theta dr + r\cos\theta d\theta}{\cos\theta dr - r\sin\theta d\theta} \equiv \frac{Sdr + rCd\theta}{Cdr - rSd\theta},$$

so

$$Sdr + rCd\theta = \Theta Cdr - rS\Theta d\theta,$$

and

$$\frac{dr}{r} = \frac{C + S\Theta}{-S + C\Theta} d\theta,$$

which is a separable equation.

Example 6.

$$y' = f\left(\frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}\right).$$

If $c_1 = c_2 = 0$, then we have a homogeneous equation. Assume that the lines $a_jx + b_jy + c_j = 0$ intersect at (x_0, y_0). Change of variables: translate (x_0, y_0) to the origin:

$$\tilde{x} = x - x_0, \qquad \tilde{y} = y - y_0.$$

13. 2D autonomous systems and reducible 2-d order equations

The integration techniques of 1-st order equations can be applied to certain types of 2-d order equations and 2D systems.

13.1. **Reducible equations.** This is the case of 2d order equations

$$F(x, y, y', y'') = 0$$

such that F does not depend on x or on y.

(a) $F(x, y', y'') = 0$.

Denote $p(x) = y'(x)$. Then $y'' = p'$ and we have a 1-st order equation for $p(x)$:

$$F(x, p, p') = 0.$$

(b) $F(y, y', y'') = 0$.

Again, denote $p = y'$ but now think of p as a function of y, $p = p(y)$. We have

$$y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy},$$

and the equation for $p(y)$ is 1-st order,

$$F(y, p, pp_y) = 0.$$

Once we know $p(y)$, we can find solutions $y = y(x)$ of the original equations by integrating the autonomous equation

$$\frac{dy}{dx} = p(y).$$

Example: homogeneous 2nd order linear equations with constant coefficients

$$y'' + ay' + by = 0.$$

We get a homogeneous 1st order equation for $p(y)$:

$$\frac{dp}{dy} = -a - b\frac{y}{p} = 0.$$

(Note that the term "homogeneous" has two different meanings in this example.) Later we will have a simpler method of solving linear equations with constant coefficients.

13.2. **2D autonomous systems.** As we already mentioned, to solve the system

$$\dot{x} = f(x, y), \qquad \dot{y} = g(x, y),$$

we can formally exclude t from the system and arrive to the 1-st order equation

$$gdx - fdy = 0.$$

Solving this equation for $y = y(x)$, we then find $x(t)$ from the equation

$$\frac{dx}{dt} = f(x, y(x)).$$

Exercise: $\ddot{x} = f(x, \dot{x})$. Transforming this reducible equation into a system,

$$\dot{x} = y, \qquad \dot{y} = f(x, y),$$

recover the method described in the previous subsection.

13.3. **Examples.** (a) *Conservative Newton's equation*

$$\ddot{x} = f(x).$$

The corresponding 2D system is

$$\dot{x} = y, \quad \dot{y} = f(x),$$

and we have

$$f(x)dx = ydy.$$

If U denotes a primitive of $(-f)$, i.e. $U'(x) = -f(x)$, then the full energy,

$$E(x, y) = \frac{y^2}{2} + U(x)$$

(kinetic energy + potential energy), is a first integral of the system.

(b) $\dot{x} = -y^2, \quad \dot{y} = x^2.$

The function $E(x, y) = x^3 + y^3$ is a first integral. The orbits of the system are the components of the level sets minus the stationary point $(0, 0)$. Solutions with the orbit $x^3 + y^3 = a^3$, $a \neq 0$, blow up in finite time (in forward and backward directions) because

$$T = \int_{-\infty}^{\infty} \frac{dx}{(a^3 - x^3)^{2/3}} < \infty.$$

(We used the first equation of the system and substituted $y = (a^3 - x^3)^{1/3}$.)

14. Selected applications

14.1. **Linear equations.**

(a) *Equation of normal reproduction* $y' = ky$ is encountered in a large number of problems:

- Radioactive decay ($k < 0$). Here $y(t) = y(0)e^{kt}$ is the amount of radioactive substance. Regardless of the initial amount $y(0)$, after time $T = k^{-1} \log 2$ ("half-life") the amount is reduced by one half.

- Barometric pressure, its dependence on the altitude. Let $\rho = \rho(h)$ be the density of the air at altitude h. Assume that the temperature is constant, so by Boyle-Marriott we have $\rho = kP$, ($k = \text{const}$), where $P = P(h) = \int_h^\infty \rho$ is the pressure. Thus we have $P' = -kP$.

- Newton's law of cooling. Let $x = x(t)$ be the temperature of some solid, and $x_* = \text{const}$ the surrounding temperature. The rate of change is proportional to the difference between the temperatures of the solid and the surroundings, so the equation is $\dot{x} = -k(x - x_*)$. This is a normal reproduction equation for $y(t) = x(t) - x_*$. The constant k in this model is positive, and so $x(t)$ tends to the steady state x_* exponentially fast.

- Falling objects - vertical motion with resistance proportional to velocity. Newton's equation is $m\dot{v} = mg - kv$ (in obvious notation), and we get the same equation as in the previous example. The terminal speed is a steady state $v_* = k^{-1}mg$.

(b) *Compound interest* (continuous version). The IVP

$$\dot{y} = r(t)y + g(t), \qquad y(t_0) = y_0$$

has the following interpretation.

- Investment:
  y_0 initial investment (measured in \$\$),
  $y(t)$ amount at time t (measured in years),
  $r(t)$ interest rate at time t,
  $g(t)$ rate of deposits/withdrawals.
- Loan:
  y_0 the amount you have borrowed,
  $y(t)$ debt after $t - t_0$ years,
  $r(t)$ interest rate,
  $-g(t)$ rate of payments.

Discrete compounding is described by difference equations of the type

$$y(t+1) = qy(t) + b, \qquad (q = 1 + r).$$

If r and b are constant, then (by induction)

$$y(t) = q^t y(0) + \frac{1 - q^t}{1 - q} b.$$

In particular, if $b = 0$ (i.e. no deposits/withdrawals), and the interest is compounded once a year, then after t years we have

$$y(t) = y_0(1+r)^t.$$

If the interest is compounded twice a year, then

$$y(t) = y_0(1+r/2)^{2t},$$

and if every month, then

$$y(t) = y_0(1+r/N)^{Nt}, \qquad N = 12.$$

As $N \to \infty$ (the interest is compounded continuously), we get $y(t) = y_0 e^{rt}$. (Cf. Euler's line method.)

14.2. **Geometric meaning of 1st order ODEs.**

(a) Suppose the lines $x = 0$ and $x = a$ in the xy-plane are the banks of the river flowing in the positive direction of the y-axis. Let $v(x, y)$ be the velocity of the water flow at the point (x, y). A swimmer starts at $(0, 0)$ and always heads orthogonally to the right bank. Her speed is constant, say equal to 1. Describe the trajectory $y = y(x)$ of his motion.

Answer: solution curve of $y' = v(x, y)$, $y(0) = 0$.

Pf: the resulting velocity, and therefore the tangent line to the trajectory at the point (x, y), $y = y(x)$, has slope $\tan\theta = v(x, y)$.

(b) *Orthogonal families of curves.* Find all smooth curves that intersect the curves $y = cx^2$ at the right angle.

Solution. Suppose the graph $y = \phi(x)$ has the stated property. At the point $(x, \phi(x))$ the graph intersects the curve $y = cx^2$ with

$$c = \frac{\phi(x)}{x^2}.$$

The slope of this curve is

$$\tan\theta = 2cx = \frac{2\phi(x)}{x},$$

and therefore the slope of the graph is

$$\phi'(x) = \tan\left(\theta \pm \frac{\pi}{2}\right) = -\cot\theta = -\frac{x}{2\phi(x)}.$$

We have

$$2\phi d\phi + x dx = 0, \qquad 2\phi^2 + x^2 = C.$$

Answer: the orthogonal trajectories are ellipses $x^2 + 2y^2 = C$.

Alternative argument. Let $F(x, y) = C$ be the equation for the family of curves, e.g. $F(x, y) = y/x^2$ as in our example. The curves satisfy the differential equation

$$F_x dx + F_y dy = 0,$$

so the slope is $dy/dx = -F_x/F_y$. The slope of an orthogonal curve is just the negative reciprocal. It follows that the equation for the orthogonal family is

$$F_y dx - F_x dy = 0.$$

Exercises: (i) Find orthogonal trajectories of the families

$$y^2 = x + c, \qquad y = cx^4, \qquad x^2 = y + cx.$$

(ii) Find d.e. for the orthogonal trajectories of $y = cx + c^3$.

(iii) Find all smooth curves that intersect the lines $y = cx$ at angle $60°$.

14.3. **Functional equations.** Find all differentiable functions f satisfying the following equation:

$$f(x+y) = f(x)f(y).$$

Answer: $f \equiv 0$, or $f(x) = e^{cx}$.

Solution. We'll try to find a differential equation for f. Differentiate the functional equation with respect to x:

$$(*) \qquad f'(x+y) = f'(x)f(y).$$

First we set $y = 0$ in (*), so $f'(x) = f'(x)f(0)$. Either $f' \equiv 0$, and then $f \equiv C$ with $C = C^2$, or $f(0) = 1$. In the latter case we set $x = 0$ in (*) and get the equation $f'(y) = f'(0)f(y)$. Thus we have the IVP $f' = cf$, $f(0) = 1$.

Exercise: solve the functional equation

$$f(x+y) = \frac{f(x) + f(y)}{1 - f(x)f(y)}.$$

14.4. **Conservation laws.**

(a) *Escape velocity.* A body of mass m is projected away from the earth, which is a ball of radius R, in the direction perpendicular to the surface with an initial velocity v_0. The Newton equation (assuming the gravity law) is

$$m\ddot{x} = -\frac{mgR^2}{(R+x)^2}, \qquad \ddot{x} = -\frac{gR^2}{(R+x)^2},$$

where $x = x(t)$ is the distance to the earth surface, and the initial conditions are $x(0) = 0$, $\dot{x}(0) = v_0$. Since the equation is conservative, the full energy

$$E(x,v) = \frac{v^2}{2} + U(x) \equiv \frac{v^2}{2} - \frac{gR^2}{R+x}$$

is a first integral of the motion, where $v = \dot{x}$, of course. Note that $U(x)$ is negative and zero at the infinite altitude. The energy of the IVP solution is

$$E_0 = v_0^2/2 - gR.$$

The body escapes to infinity iff $E_0 \geq 0$, (i.e. $v_0 \geq \sqrt{2gR}$) with terminal velocity $v(\infty) = \sqrt{v_0^2 - 2gR}$. Indeed, if it escapes then taking $x \to \infty$ in the expression for $E(x,y) = E_0$ we see that $E_0 \geq 0$; if it does not escape, then there should be some time t_* at which the altitude is maximal and $v(t_*) = 0$, so $E_0 = U(t_*) < 0$. [Find the maximal altitude $x(t_*)$ – exercise.]

(b) *Torricelli's law.* Suppose we have a water tank with a hole of area a at the bottom. Let $A(y)$ be the area of the tank cross-section at height y above the hole. Claim: if $y(t)$ is the water level in the tank, then

$$\dot{y} = -c\frac{\sqrt{y}}{A(y)}, \qquad c = a\sqrt{2g}.$$

Proof. The velocity of the water exiting the tank is

$$v(y) = \sqrt{2gy}.$$

To see this balance the energy: let m be a small amount of water exiting at speed v when the water level is y; its kinetic energy is $mv^2/2$; at the same time the potential energy of the water in the tank has decreased by mgy. It follows that if $V(t)$ is the volume, then

$$\dot{V} = -av(y(t)) = -a\sqrt{2gy},$$

and we also have

$$\dot{V} = \frac{dV}{dy}\frac{dy}{dt} = A(y)\dot{y}.$$

14.5. **Pursuit curves.** A bird wants to fly from $A(a, 0)$ to the nest $B(0, 0)$ but a steady wind is blowing perpendicular to AB. Bird's speed is 1, the speed of wind is $w < 1$. The bird always points toward the nest, but the wind blows it off course so the heading angle changes all the time. Find the flight path and the time needed to reach the nest. Is the bird's strategy optimal? Note that if the heading angle is constant so that the path is a straight line segment AB, then the flight time is $a(1 - w^2)^{-1/2}$.

For simplicity, let $a = 1$. Denote by $\theta = \theta(t)$ the heading angle. From the resultant velocity, we find

$$\dot{x} = -\cos\theta = -\frac{x}{\sqrt{x^2 + y^2}}, \qquad \dot{y} = w - \sin\theta = w - \frac{y}{\sqrt{x^2 + y^2}},$$

and for the parts of the path of the form $y = y(x)$, we have

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \tan\theta - \frac{w}{\cos\theta} = \frac{y}{x} - w\frac{\sqrt{x^2 + y^2}}{x}.$$

This equation is homogeneous. For $z = z(x) = y/x$, we get the IVP

$$\frac{dz}{\sqrt{1 + z^2}} = -w\frac{dx}{x}, \qquad z(1) = 0.$$

Integrating we have

$$\log(z + \sqrt{1 + z^2}) = -w \log x,$$

and

$$\sqrt{1 + z^2} = x^{-w} - z.$$

Solving for z, after some algebra we have

$$y(x) = \frac{x}{2}\left[x^{-w} - x^{w}\right].$$

To find the flight time, we use the first equation of the system

$$-\dot{x} = \frac{x}{\sqrt{x^2 + y^2}} = \frac{1}{\sqrt{1 + (y/x)^2}} = \frac{2}{x^{-w} + x^{w}}.$$

Integrating we get

$$2T = \int_0^1 (x^{-w} + x^{w})dx = \frac{2}{1 - w^2},$$

which is greater than $(1 - w^2)^{-1/2}$.

For $a \neq 1$, the solution is

$$y(t) = \frac{a}{2}\left[\left(\frac{x}{a}\right)^{1-w} - \left(\frac{x}{a}\right)^{1+w}\right].$$

Exercise. Suppose that a truck starts at a point $(a, 0)$ on a desert and moves north (the direction of the y-axis). Let a police car start at a point $(0, 0)$ and pursue the truck across the desert so that the velocity vector of the police car always points towards the truck. Assume that the police car travels twice as fast as the pursued. Find the point at which the police captures the truck.

14.6. **Lotka-Volterra model.** Consider the system

$$\dot{x} = kx - axy, \qquad \dot{y} = -ly + bxy,$$

where k, l, a, b are positive constants. This is a special case of the predator-prey model in 2D population dynamics: $y(t)$ is the population of predators (sharks) and $x(t)$ is the population of food fish (sardines). Without interaction, $x(t)$ increases (no predators) and $y(t)$ decreases (no food) according to the equation of normal reproduction. [A more general model would include logistic terms.] The xy-terms account for interaction, which is beneficial for the sharks and detrimental for the food; the interaction terms are proportional to the number of possible encounters, i.e. the number of pairs. Note that the system has an equilibrium at

$$(x_*, y_*) = (l/b, k/a).$$

Claim. *All non-stationary solutions with $x_0 > 0, y_0 > 0$ are periodic.*

Proof: We have

$$\frac{dy}{dx} = \frac{b - l/x}{k/y - a}, \qquad (b - l/x)dx + (a - k/y)dy = 0,$$

so

$$E(x, y) = F(x) + G(y)$$

with

$$F(x) = bx - l\log x, \quad G(y) = ay - k\log y,$$

is a first integral. Both functions F and G are strictly convex, and so $E(x, y)$ is strictly convex, and tends to infinity at the boundary of the quadrant $\{x > 0, y > 0\}$. The singular point is the minimum of E. It follows that all other level sets are simple closed curves. $\square$

Interpretation of the result. Consider the following cycle. First we have few predators and plenty of food, both populations increase. At some point we have enough predators to stop the growth of food, from now on $x(t)$ decreases but $y(t)$ still increases. Then predators do not have enough food anymore and their population starts to decline, etc.

If we start fishing sardines (with constant effort), it will be bad for sharks, but not for sardines. The equations will be

$$\dot{x} = (k - h)x - axy, \qquad \dot{y} = -ly + bxy,$$

with $h > 0$, and the new equilibrium (which is actually the mean value of the populations over the period - exercise) will be

$$(\tilde{x}_0, \tilde{y}_0) = (l/b, (k-h)/a).$$