ACM 100b

The Laplace transform - Introduction and properties

Dan Meiron

Caltech

January 28, 2014

The Laplace transform

- We will next introduce the idea of the Laplace transform
- This will be partly review
- But our approach has a different objective
- And it's a little more advanced because we will use the complex plane.
- The Laplace transform is an important application of transform methods
- We use these a lot in solving linear problems
- And there is also a very important connection with the material in the second half of the course

Definition of the Laplace transform

- Suppose we are given a real function $f(t)$ that is piece-wise continuous and satisfies

$$|f(t)| < K \exp(at)$$

  when $t \geq M$ where M is some real positive number.
- Here, a, K are also real and positive.
- The Laplace transform of $f(t)$ is given by

$$F(s) = \mathfrak{L}[f(t)] = \int_0^\infty \exp(-st) f(t) dt$$

- This integral defines a function that exists for $s > a$.
- Or more accurately $\text{Re}(s) > a$, since in what follows we will treat s as a complex variable.

The transform exists only for certain functions

- It is easy to see that for $\text{Re}(s) > a$ we have

$$\begin{aligned}
\left|\int_0^\infty \exp(-st)f(t)dt\right| &= \left|\int_0^M \exp(-st)f(t)dt + \int_M^\infty \exp(-st)f(t)dt\right| \\
&\leq \left|\int_0^M \exp(-st)f(t)dt\right| + \int_M^\infty |\exp(-st)||f(t)|dt \\
&\leq B + \int_M^\infty K\exp(-st)\exp(at)dt.
\end{aligned}$$

- The last integral is finite only if $\text{Re}(s) > a$.
- For this reason, functions for which Laplace transforms exist are called of "exponential order" as $t \to \infty$.
- So the transform exists for functions that can grow exponentially but not faster.

Example of a transform

- As an example consider $f(t) = 1$ for $t \geq 0$.
- The transform is

$$\mathfrak{L}[1] = \int_0^\infty \exp(-st)dt = \frac{1}{s}, \quad s > 0.$$

- As another example consider $f(t) = \exp(at)$ for $t \geq 0$:

$$\begin{aligned} \mathfrak{L}[f(t)] &= \int_0^\infty \exp(-st)\exp(at)dt \\ &= \int_0^\infty \exp[-(s-a)t)dt \\ &= \frac{1}{s-a}, \quad \mathrm{Re}(s) > a. \end{aligned}$$

Properties of the transform

- The transform is a linear operator since integration is linear.
- This just means

$$\mathfrak{L}[af + bg] = a\mathfrak{L}f + b\mathfrak{L}[g]$$

  where a, b are constants and f, g are functions of exponential order.
- The most important property though is the behavior under differentiation.
- Assume $f(t)$ is of exponential order and continuous and that $f'(t)$ is piece-wise continuous.
- The transform of a derivative is

$$\mathfrak{L}[f'(t)] = \int_0^\infty dt \exp(-st) f'(t) dt$$

- Integrate once by parts to get

$$\mathfrak{L}[f'(t)] = f(t)\exp(-st)\Big|_0^\infty + s\int_0^\infty f(t)\exp(-st)dt$$

The transform under differentiation

- So we have just shown that

$$\mathfrak{L}[f'(t)] = s\mathfrak{L}[f(t)] - f(0).$$

- You can use repeated integration by parts to get the transform of higher derivatives
- Suppose $f'(t), f''(t), ..., f^{(n-1)}(t)$ are continuous.
- Ans suppose $f^{(n)}$ is piece-wise continuous.
- Then repeated integration by parts of the definition of the Laplace transform gives

$$\mathfrak{L}[f^n(t)] = s^n\mathfrak{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - ... - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

The transform is very useful for IVPs

- It is this property

$$\mathfrak{L}[f^n(t)] = s^n\mathfrak{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

  that makes Laplace transforms so useful for initial value problems.
- This is because in an IVP the value of the solution and its derivatives are prescribed at one value of t.
- The idea then is to transform the ODE, insert the initial conditions, solve for the transform and then invert it to get the answer.
- The solution will already have the initial conditions satisfies.
- This is particularly valuable for initial value problems where the ODE is inhomogeneous.