ACM 100b

Application of the Laplace transform to ODEs

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Examples of the application to ODE's

- We give two examples of the application of Laplace transforms below
- The first is just the solution of a homogeneous constant coefficient ODE
- We can of course do that many ways
- The second has an inhomogeneous term
- Typically Laplace transforms are most useful for constant coefficient IVP's
- Because they handle the initial conditions and any inhomogeneous terms they are very useful

Homogeneous example

- Consider the initial value problem

$$y'' - y' - 2y = 0 \qquad y(0) = 1, \quad y'(0) = 0.$$

- Now apply the Laplace transform to both sides of this equation:

$$\mathcal{L}[y'' - y' - 2y] = \mathcal{L}[0].$$

- Call the transform of $y(t)$ Y(s).
- We then have

$$\mathcal{L}[y''] - \mathcal{L}[y'] - 2\mathcal{L}[y] = 0,$$

Homogeneous example

- Now recall

$$\mathfrak{L}[f^n(t)] = s^n\mathfrak{L}[f(t)] - s^{n-1}f(0) - s^{n-2}f'(0) - \ldots - sf^{(n-2)}(0) - f^{(n-1)}(0).$$

- Using the properties of derivatives of Laplace transforms we have

$$(s^2 - s - 2)Y(s) + (1 - s)y(0) - y'(0) = 0.$$

- Plug in the known initial values to get

$$Y(s) = \frac{s-1}{(s-2)(s+1)}.$$

- Note we have factored the denominator which shows two poles at $s = 2$ and $s = -1$.
- These are just the roots of the polynomial in α we solve when we substitute $\exp(\alpha x)$ into a constant coefficient ODE.

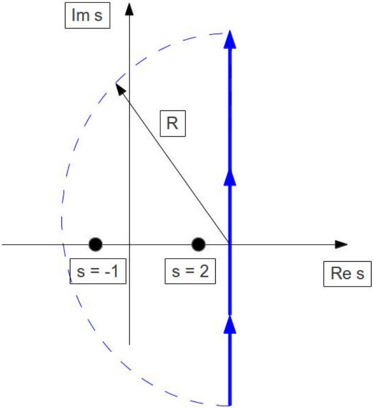
Homogeneous example

- Now comes the problem of inverting the Laplace transform.
- The Inverse Laplace transform is given by

$$f(t) = \frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{s-1}{(s-2)(s+1)} \exp(st) dt,$$

- Again the real number C is to be chosen so the vertical contour lies to the right of all singularities of the transform.
- In this case the singularities are at $s = -1$ and $s = 2$ so if we set say $C > 2$ that will work.

- To finish the inversion we use the techniques of contour integrals learned in ACM 95/100a
- The Bromwich contour looks like this



Doing the integral

- We close the vertical contour in the left half plane.
- It has to be closed there if we want to compute the solution for $t > 0$
- In that case the real part of the complex variable s will become negative and in that case we can show that the completion of the contour vanishes
- Using the residue theorem we get

$$y(t) = \frac{1}{3}\exp(2t) + \frac{2}{3}\exp(-t).$$

- Note the result automatically satisfies the IVP.

Closing the contour the other way

- Note you can close the contour for $t < 0$ in the right half plane.
- In that case we get nothing.
- This is because the integrand is analytic
- And so by Cauchy's theorem the integral is zero.
- This reflects the fact that there is a built-in definition in the Laplace transform that our function $f(t)$ vanishes for $t < 0$.
- The initial value problem is defined only for $t > 0$
- And so as a result we get no information about $t < 0$.
- Finally note the fact that our solution satisfies the initial conditions.
- This is an immediate benefit of the Laplace transform. In the next section we explore the benefits for inhomogeneous problems.

Inhomogeneous example

- In the previous lecture we showed how the Laplace transform can be used for homogeneous ODE's
- Next we'll do an inhomogeneous example
- Consider the IVP

$$y'' + 4y = h(t) \qquad y(0) = 1, \quad y'(0) = 0,$$

where

$$h(t) = \begin{cases} 0 & 0 \leq t < \pi \\ 1 & \pi \leq t < 2\pi \\ 0 & t \geq 2\pi. \end{cases}$$

- We recall that

$$h(t) = u_\pi(t) - u_{2\pi}(t).$$

in terms of the unit step function.

Inhomogeneous example

- We transform both sides of the ODE to get

$$(s^2+4)Y(s) - sy(0) - y'(0) = \mathcal{L}[u_\pi(t)] - \mathcal{L}[u_{2\pi}(t)],$$

- This then gives us the transform of the solution

$$Y(s) = \frac{s}{s^2+4} + \frac{\exp(-\pi s)}{s(s^2+4)} - \frac{\exp(-2\pi s)}{s(s^2+4)}.$$

- We now want to invert the transform to compute

$$y(t) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \exp(st)\left[\frac{s}{s^2+4} + \frac{\exp(-\pi s)}{s(s^2+4)} - \frac{\exp(-2\pi s)}{s(s^2+4)}\right] ds.$$

Inhomogeneous example

- The best way to do this integral is to break it up into three terms:

$$y(t) = y_1(t) + y_2(t) + y_3(t).$$

- The first term is just an example of the kind of contour integral we did before.
- Close in the left half plane to get via the residue theorem that

$$\begin{aligned} y_1(t) &= \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \exp(st)\frac{s}{s^2+4}ds \\ &= \frac{1}{2\pi i}\left[2\pi i\left(\exp(2it)\left(\frac{2i}{4i}\right) + \exp(-2it)\left(\frac{-2i}{-4i}\right)\right)\right] \\ &= \cos(2t). \end{aligned}$$

Inhomogeneous example

- The second integral is a little more involved but not much harder:

$$y_2(t) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} \exp(st)\frac{\exp(-\pi s)}{s(s^2+4)}ds.$$

- If $0 < t < \pi$ we have to close the contour in the right half plane or the integral won't exist due to the argument of the exponential.
- If $t > \pi$ we close in the left half plane.
- The result is

$$y_2(t) = \begin{cases} 0 & 0 < t < \pi \\ \frac{1}{2\pi i}\left[2\pi i\left(\frac{\exp(2i(t-\pi))}{(2i)(4i)} + \frac{1}{4} + \frac{\exp(-2i(t-\pi))}{(-2i)(-4i)}\right)\right] & t > \pi, \end{cases}$$

or

$$y_2(t) = \begin{cases} 0 & 0 < t < \pi \\ 1/4 - \cos(2(t-\pi))/4 & t > \pi. \end{cases}$$

Inhomogeneous example

- A similar procedure will give you the third integral:

$$y_3(t) = \begin{cases} 0 & 0 < t < 2\pi \\ -1/4 + \cos(2(t-2\pi))/4 & t > 2\pi. \end{cases}$$

- Expanding out the cosines and putting everything together we get

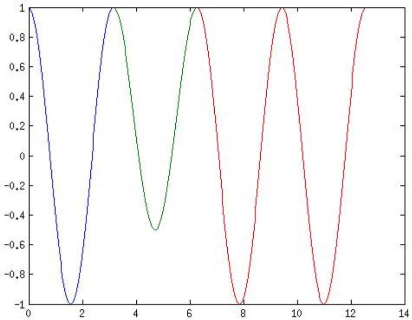
$$y(t) = \begin{cases} \cos(2t) & 0 < t < \pi \\ 3/4\cos(2t) + 1/4 & \pi \leq t \leq 2\pi \\ \cos(2t) & t > 2\pi. \end{cases}$$

Inhomogeneous example

- A few notes on the solution

$$y(t) = \begin{cases} \cos(2t) & 0 < t < \pi \\ 3/4\cos(2t) + 1/4 & \pi \leq t \leq 2\pi \\ \cos(2t) & t > 2\pi. \end{cases}$$

- A plot of the solution can be seen below



Inhomogeneous example

- Look at the values of the solution at $t = \pi$ and $t = 2\pi$ which are the locations of the discontinuities in the right hand side.
- You will see that they are continuous there.
- Their derivatives are also continuous.
- We certainly expect this from the solution because we have solved a second order ODE with a right hand side that is only piece-wise discontinuous.
- The smoothness of the solution is controlled by the highest order derivative.
- Since that is a second order derivative then we expect the solution to be continuous as well as to have continuous first derivatives.
- The Laplace transform is convenient in this regard since all initial values as well as the continuity of the solution are built into the inverse transform we computed.