ACM 100b

Parseval's theorem

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Recap

- In our last lecture we demonstrated several forms of Fourier series
- We introduced the Fourier sine, Fourier cosine and periodic series
- We developed relations among the periodic series and the cosine series
- We also computed some examples of such series
- In the following lectures we will develop the theory of convergence for Fourier series
- Recall everything we do for Fourier sine and cosine series will apply to other regular Sturm-Liouville problems.

Parseval's theorem

- We showed in the last lecture that we can represent the Fourier series of $f(x)$ in a complex form:

$$f(x) = \sum_{n=-\infty}^{\infty} C_n \exp(in\pi x/L)$$

- In what we do here we will consider $f(x)$ as real even though the coefficients C_n are complex.
- Consider next the integral

$$\int_{-L}^{L} f(x)^2 dx$$

Parseval's theorem

- Using our complex form of the Fourier series we can work out this integral in terms of the coefficients:

$$\int_{-L}^{L} f(x)^2 dx =$$

$$\int_{-L}^{L} \left(\sum_{n=-\infty}^{\infty} C_n \exp(in\pi x/L) \right) \left(\sum_{m=-\infty}^{\infty} C_m \exp(im\pi x/L) \right) dx$$

- Now look at the series

$$\sum_{n=-\infty}^{\infty} C_n \exp(in\pi x/L)$$

- This is a Fourier series for a real function $f(x)$
- Because f is real, the terms with $n < 0$ must be the complex conjugates of the terms with $n > 0$

Parseval's theorem

- So we can write the series as

$$f(x) = \overline{f(x)} = \sum_{n=-\infty}^{\infty} \overline{C_n} \exp(-in\pi x/L)$$

and we have changed nothing.

- So we can use this identity to write

$$\int_{-L}^{L} f(x)^2 dx = \int_{-L}^{L} \left(\sum_{n=-\infty}^{\infty} C_n \exp(in\pi x/L) \right) \times \left(\sum_{m=-\infty}^{\infty} \overline{C_m} \exp(-im\pi x/L) \right) dx$$

- But this is

$$\int_{-L}^{L} f(x)^2 dx = 2L \sum_{n=-\infty}^{\infty} C_n \overline{C_n}$$

Parseval's theorem

- Or in terms of A_n and B_n we have

$$\int_{-L}^{L} f(x)^2 dx = L\left[2B_0^2 + \sum_{n=1}^{\infty}\left(A_n^2 + B_n^2\right)\right]$$

- This result is known as *Parseval's theorem*.
- Parseval's theorem is actually not specific only to Fourier series.
- There is a relation of this type for every family of regular Sturm-Liouville eigenfunctions.
- Note there is a big assumption in deriving this result.
- You may have noticed that we switched an infinite sum and an integral again.
- We have not justified this and so have to take it on faith right now.
- We next show another seemingly disconnected result
- When put together with the Parseval theorem gives us insight into the way Fourier series converge.