ACM 100b

Intro to series solutions for ODE's

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Series solution of ODE's

- In treating systems of constant coefficient ODE's we have already introduced the idea of series expansions
- Here we will focus on second order scalar ODE's and discuss the use of series to get solutions.
- Everything we say can be applied to systems as well.
- In most cases we cannot provide analytical solutions to second order initial value problems for linear ODE's of the form

$$y'' + p(x)y' + q(x)y = 0 \qquad y(x_0) = y_0 \quad y'(x_0) = y_1 .$$

- One way to make progress is to write the solution as a power series expansion about the point $x = x_0$.

Series solutions of ODE's

- We assume that for the ODE

$$y'' + p(x)y' + q(x)y = 0$$

  the coefficients $p(x)$ and $q(x)$ are sufficiently smooth in some neighborhood of x_0 so that we can take as many derivatives of $p(x)$ and $q(x)$ as necessary.
- We then assume a solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

- To get a solution we have to determine the coefficients a_n so that the ODE is satisfied
- It turns out that to get the coefficients uniquely we need to make use of the initial conditions

Series solutions for ODE's

- Given this series form of the solution

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n.$$

  it is straightforward to compute as many derivatives as we need:
- For example the first two derivatives of $y(x)$ are

$$y'(x) = \sum_{n=0}^{\infty} n a_n (x - x_0)^{n-1}$$
$$y''(x) = \sum_{n=0}^{\infty} n(n-1) a_n (x - x_0)^{n-2}$$

- The best way to see how this works is to first do an example.

An example - the Airy ODE

- Consider the Airy equation

$$y'' = xy \qquad y(0) = y_0 \quad y'(0) = y_1 .$$

- The Airy function is important in microscopy and astronomy.
- It describes the pattern, due to diffraction and interference, produced by a point source of light that is smaller than the resolution limit of a microscope or telescope.
- Actually the original ODE due to Airy is

$$y'' = -xy$$

  but this is immaterial to the discussion.
- For the initial point we take $x_0 = 0$.

An example - the Airy equation

- Before we start in we note that the series expression for the first derivative

$$y'(x) = \sum_{n=0}^{\infty} na_n(x - x_0)^{n-1}$$

is equivalent to the expression

$$y'(x) = \sum_{n=0}^{\infty} (n+1)a_{n+1}(x - x_0)^n$$

- And the expression for the second derivative

$$y''(x) = \sum_{n=0}^{\infty} n(n-1)a_n(x - x_0)^{n-2}$$

is equivalent to the expression

$$y''(x) = \sum_{n=0}^{\infty} (n+1)(n+2)a_{n+2}(x - x_0)^n.$$

An example - the Airy equation

- These expressions

$$y'(x) = \sum_{n=0}^{\infty}(n+1)a_{n+1}(x-x_0)^n$$
$$y''(x) = \sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}(x-x_0)^n.$$

are more convenient to work with.

- What we want to do is substitute these expressions into the ODE and develop series solutions by matching like powers of $(x-x_0)^n$
- When we do this we will get relations for the coefficients a_n that we can then solve.

- Using the formulas for the derivatives above and substituting into our ODE

$$y'' = xy$$

we get

$$\begin{aligned}
\sum_{n=0}^{\infty}(n+1)(n+2)a_{n+2}x^n &= x\sum_{n=0}^{\infty} a_n x^n \\
&= \sum_{n=0}^{\infty} a_n x^{n+1} \\
&= \sum_{n=1}^{\infty} a_{n-1} x^n.
\end{aligned}$$

An example - the Airy equation

- Now we can equate the coefficients of like powers of x. We get the following equations:

$$
\begin{aligned}
x^0 &: & 2a_2 &= 0 \\
x^1 &: & 6a_3 &= a_0 \\
x^2 &: & 12a_4 &= a_1 \\
x^3 &: & 20a_5 &= a_2 \\
&\vdots & & \\
x^n &: & (n+1)(n+2)a_{n+2} &= a_{n-1}.
\end{aligned}
$$

- Note these equations give no information about a_0 or a_1.
- a_0 and a_1 come from the initial value problem.
- Once these are known the relations above can be used to get all the rest of the coefficients.

An example - the Airy equation

- In a series expansion about $x = 0$ we have

$$y(x) = \sum_{n=0}^{\infty} a_n x^n,$$

- So a_0 and a_1 are respectively $y(0)$ and $y'(0)$
- These are given as part of the initial value problem.
- Once these values are given to us we can then get all the coefficients from the *recursion relation*

$$a_{n+3} = \frac{a_n}{(n+2)(n+3)} \qquad n > 0.$$

An example - the Airy equation

- Now look at the values of a_2, a_3 etc. gotten from the recursion relation
$$a_{n+3} = \frac{a_n}{(n+2)(n+3)} \qquad n > 0.$$
- We assume a_0 and a_1 are known.
- We can eventually spot a pattern and write down the coefficients a_n:

$$\begin{aligned} a_{3n} &= \frac{a_0}{(3n)(3n-1)(3n-3)(3n-4)\cdots 3\cdot 2} \\ a_{3n+1} &= \frac{a_1}{(3n+1)(3n)(3n-2)(3n-3)\cdots 4\cdot 3} \\ a_{3n+2} &= 0. \end{aligned}$$

where $n = 1, 2, \ldots$ and $a_2 = 0$.

An example - the Airy equation

- We can then see there is a general solution of the form

$$y(x) = a_0 \left[1 + \sum_{n=1}^{\infty} a_{3n} x^{3n} \right] + a_1 \left[x + \sum_{n=1}^{\infty} a_{3n+1} x^{3n+1} \right],$$

- And the coefficients a_0 and a_1 are given by

$$a_0 = y_0 \qquad a_1 = y_1 .$$

- The solution is in the form of a linear superposition of two series.
- Both series satisfy the ODE and are linearly independent.
- This can easily be checked by computing their Wronskian.
- A series solution, however is only useful if it converges for some values of x.
- An application of the ratio test for series reveals that in fact both series converge of all values of x.
- In fact they converge for all complex values of x and each define entire functions.