ACM 100c

Brief introduction to singular Sturm-Liouville problems

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Singular Sturm-Liouville problems

- Singular Sturm-Liouville problems occur for example whenever the function $p(x)$ vanishes at one or both of the end points:

$$p(a) = 0 \quad \text{and/or} \quad p(b) = 0 \qquad a \leq x \leq b$$

- In this case either $x = a$ or $x = b$ are singular points for the ODE.
- They also occur if the domain is infinite in extent:

$$a \leq x < +\infty \qquad -\infty < x < b \qquad -\infty < x < +\infty$$

- In this case it is not possible to ask for example that the solution satisfy

$$y(a) = y(b) = 0,$$

Singular Sturm-Liouville problems

- This is because at a singular point one usually does not have the freedom to restrict the value of the solution.
- All you can really reasonably ask for is that the solutions are finite or smooth at such points
- Such a condition usually corresponds to a physically realistic solution.
- Insisting that things not blow up at the singular points is basically the boundary condition.
- This will typically determine the eigenvalues
- But now all the usual results about discrete eigenvalues, etc. may not hold.

Singular Sturm-Liouville problems

- Some singular Sturm-Liouville problems do have discrete sets of eigenvalues
- We will study some of these in various applications
- But it's also possible to have dense sets of eigenvalues
- These are known to the cognoscenti as "continuous spectrum"
- The eigenvalues are no longer discrete but now form a dense set.
- These too arise in important applications
- We will explore these as well.
- And there can be regions of discrete eigenvalues mixed with continuous.
- Whether a singular Sturm-Liouville problem has discrete or continuous sets of eigenvalues is a difficult analysis problem
- For certain classes of coefficients in the S-L ODE problem was solved by Hermann Weyl
- This is an advanced topic and we talk about this further in ACM 101.

Singular Sturm-Liouville problems

- If, however, there are discrete eigenvalues, then results on orthogonality still hold
- In addition results about interlacing of zeros as discussed above will still hold.
- We will examine several problems where such singular Sturm-Liouville ODE's play an important role.
- Typically they come up whenever we use a coordinate transform that has a singularity such as polar coordinates or spherical coordinates and the problem demands that we consider solutions at the singular points.