Physics 106b — Classical Mechanics

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Mathematics of Relativity

Lorenz transformation: standard setup

Contravariant components

$$x'^{\beta} = \Lambda^{\beta}{}_{\alpha} x^{\alpha}$$

Written in matrix notation for the standard setup:

$$\begin{bmatrix} t' \\ x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix}$$

Lorenz transformation: general boost direction

For a general orientation of the boost velocity ($i = 1, 2, 3$)

$$\Lambda^0{}_0 = \gamma, \qquad \Lambda^0{}_i = \Lambda^i{}_0 = -\gamma v_i, \qquad \Lambda^i{}_j = \delta_{ij} + (\gamma - 1)\frac{v_i v_j}{v^2}$$

Spelling this out

$$\Lambda = \begin{bmatrix} \gamma & -\gamma v_x & -\gamma v_y & -\gamma v_z \\ -\gamma v_x & 1 + (\gamma - 1)\frac{v_x^2}{v^2} & (\gamma - 1)\frac{v_x v_y}{v^2} & (\gamma - 1)\frac{v_x v_z}{v^2} \\ -\gamma v_y & (\gamma - 1)\frac{v_x v_y}{v^2} & 1 + (\gamma - 1)\frac{v_y^2}{v^2} & (\gamma - 1)\frac{v_y v_z}{v^2} \\ -\gamma v_z & (\gamma - 1)\frac{v_x v_z}{v^2} & (\gamma - 1)\frac{v_y v_z}{v^2} & 1 + (\gamma - 1)\frac{v_z^2}{v^2} \end{bmatrix}$$

Lorenz transformation: general boost direction

You can get this by first rotating to the standard configuration, boosting, and then rotating back.

For example, for a velocity in the xy plane Λ is given by

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

with $v_x = v\cos\theta$, $v_y = v\sin\theta$

Electromagnetic field tensor

Defined (in component form) by

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

with $\partial^\alpha = (\partial/\partial t, -\vec{\nabla})$ the gradient 4-vector and $\mathbf{A} = (\Phi, \vec{A})$ the potential 4-vector.

Working out the derivatives

$$F^{\alpha\beta} \equiv \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix}$$

or $F^{0j} = -F^{j0} = -E_j$, $F^{ij} = -\epsilon_{ijk} B_k$ $(i, j = 1, 2, 3)$

Transforming $\vec{E}$, $\vec{B}$ between inertial frames

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\Lambda^{\nu}{}_{\beta}F^{\alpha\beta}$$

In matrix notation this reads

$$\begin{bmatrix} 0 & -E'_x & -E'_y & -E'_z \\ E'_x & 0 & -B'_z & B'_y \\ E'_y & B'_z & 0 & -B'_x \\ E'_z & -B'_y & B'_x & 0 \end{bmatrix} =$$

$$\begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{bmatrix} \begin{bmatrix} \gamma & -\gamma v & 0 & 0 \\ -\gamma v & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Transforming $\vec{E}$, $\vec{B}$ between inertial frames

This gives

$$
\begin{aligned}
E'_x &= E_x & \qquad B'_x &= B_x \\
E'_y &= \gamma(E_y - vB_z) & \qquad B'_y &= \gamma(B_y + vE_z) \\
E'_z &= \gamma(E_z + vB_y) & \qquad B'_z &= \gamma(B_z - vE_y)
\end{aligned}
$$

or

$$
\begin{aligned}
E'_\parallel &= E_\parallel & \qquad B'_\parallel &= B_\parallel \\
\vec{E}'_\perp &= \gamma(\vec{E}_\perp + \vec{v} \times \vec{B}_\perp) & \qquad \vec{B}'_\perp &= \gamma(\vec{B}_\perp - \vec{v} \times \vec{E}_\perp)
\end{aligned}
$$

Invariants

- $$F^{\alpha\beta}F_{\alpha\beta} = F^{\alpha\beta}g_{\alpha\gamma}g_{\beta\delta}F^{\gamma\delta} = 2(B^2 - E^2)$$

  Consequence: the answer to the question of whether $E > B$ or $B > E$ is frame invariant

- $$\epsilon_{\alpha\beta\gamma\delta}F^{\alpha\beta}F^{\gamma\delta} = -8\vec{E}\cdot\vec{B}$$

  Consequence: if $\vec{E}\cdot\vec{B} = 0$ in one frame ($\vec{E} \perp \vec{B}$ or $\vec{E} = 0$ or $\vec{B} = 0$) then this is true in all frames

Lorentz force

The electromagnetic field tensor gives an elegant covariant description motion of a charged particle

- Minkowski force

$$\mathbf{f} = q\mathbf{F} \cdot \mathbf{u}$$

  with **u** the particle 4-velocity

- Equation of motion

$$\frac{d\mathbf{p}}{d\tau} = q\mathbf{F} \cdot \mathbf{u}$$

$$\text{cf.} \quad \frac{d\vec{p}}{dt} = q(\vec{E} + \vec{v} \times \vec{B})$$

Composition of two parallel boosts

- Frame S'' moving with speed v' along the x' axis of S', which in turn is moving with speed v along the x axis of S, all axes aligned.
- The combined transformation from S to S'' is

$$\begin{aligned}\Lambda(S \to S'') &= \begin{bmatrix} \gamma' & -\gamma' v' \\ -\gamma' v' & \gamma' \end{bmatrix} \begin{bmatrix} \gamma & -\gamma v \\ -\gamma v & \gamma \end{bmatrix} \\ &= \gamma\gamma'(1+vv') \begin{bmatrix} 1 & -\frac{v+v'}{1+vv'} \\ -\frac{v+v'}{1+vv'} & 1 \end{bmatrix}\end{aligned}$$

- The speed of S'' relative to S is given by the addition of velocities

$$v'' = \frac{v+v'}{1+vv'}$$

- If we write $\Gamma = \gamma\gamma'(1+vv')$ some algebra gives

$$1 - \Gamma^{-2} = v''^2, \qquad \text{so that} \qquad \Gamma = \gamma_{v''} \qquad \text{and then}$$

$$\Lambda(S \to S'') = \begin{bmatrix} \gamma_{v''} & -\gamma_{v''} v'' \\ -\gamma_{v''} v'' & \gamma_{v''} \end{bmatrix}$$

Composition of nonparallel boosts

- A frame S'' is moving with small speed Δv in the y' direction relative to S', which in turn is moving with speed v in the x direction relative to frame S.
- To first order in Δv the combined transformation can be written as

$$\Lambda = \mathbf{R}\mathbf{B}$$

- $\mathbf{R}$ is a small rotation about the z axis through angle $\delta\theta$, given by a rotation matrix

$$\mathbf{R} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \delta\theta & 0 \\ 0 & -\delta\theta & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- $\mathbf{B}$ is a pure Lorentz boost with velocity $(v\hat{x} + \delta v\hat{y})$ with $\delta v = \gamma^{-1}\Delta v$, and

$$\delta\theta = -\frac{\gamma - 1}{\gamma}\frac{\Delta v}{v} \to -\frac{1}{2}v\Delta v,$$

with $\gamma = 1/\sqrt{1 - v^2}$, and the last result is for $v \ll 1$.