Physics 106b: Lecture 9 — 4 February 2014

Dissipative Dynamical Systems

In this lecture I will introduce the language used in dynamical systems, focussing on dissipative (non-Hamiltonian) systems. I will consider equations of the form

$$\dot{x}_i = f_i(x_1, x_2, \dots x_n), \quad i = 1 \dots n. \tag{1}$$

The equations are *autonomous* (no explicit time dependence on the right hand side) and the solutions can be represented by non-crossing *trajectories* or *flows* in the n-dimensional phase space $(x_1, x_2, \dots x_n)$. I will assume the f_i are smooth and finite functions. They define a *vector field* $\mathbf{f}$ – the velocity of the flow at each point in phase space. In general the system will be non-Hamiltonian, so dynamical variables do not come in canonically conjugate pairs, and there are no symplectic constraints on the flows.

With dissipation we expect phase space volumes to contract $\nabla_{\text{ph}} \cdot \mathbf{V}_{ph} = \nabla_{\text{ph}} \cdot \mathbf{f} < 0$, at least on average.[1] Thus after transients have died out, any long time asymptotic dynamics must be confined to a lower dimensional region of phase space known as an *attractor*: examples are a point (fixed point), line (limit cycle or homo- or hetero-clinic orbit), an m-torus corresponding to oscillations at m different frequencies…, or, as we will see in Lecture 10, a *strange attractor* giving chaotic dynamics. Many (in some cases, almost all) different initial conditions will, after transients have died out, lead to trajectories on the *same* attractor, and often there is a single attractor. This makes it much easy to formulate simple questions in dissipative dynamical systems than in Hamiltonian ones, where, as we have seen, different types of solutions for different initial conditions at the same parameter values and constrains may be intricately intertwined. However, the questions may be harder to answer because we do not have the constraints of a Hamiltonian structure.

The simplest system is one dimensional $n = 1$.

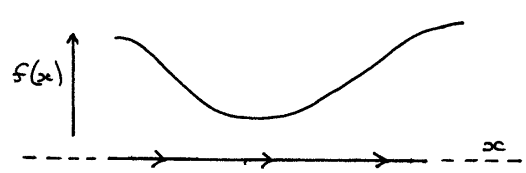
One dimensional flows

The equation of motion is

$$\dot{x} = f(x), \tag{2}$$

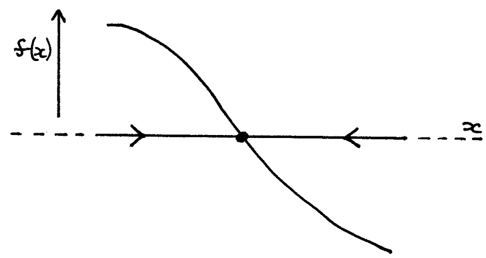
and the dynamics are flows on the line. The motion can be thought of as the damped motion in a potential, $\eta\dot{x} = -dV(x)/dx$ with η the damping constant. I now pictorially describe flows of increasing complexity to introduce some of the ideas. I will plot the flow as arrows on the x-axis, and also plot $f(x)$ which gives $\dot{x}$.

If $f(x) > 0$ everywhere, x grows continuously, and eventually diverges to infinity (and similarly for $f(x) < 0$ everywhere, x flows to minus infinity).

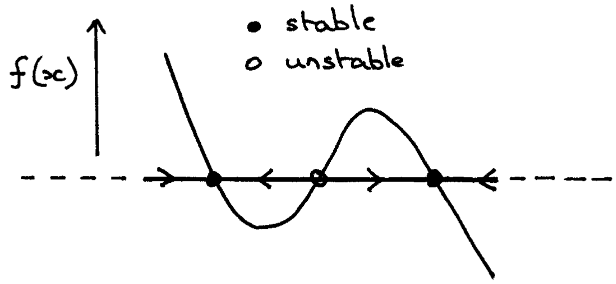


[1] I don't think there is a universally accepted definition of a *dissipative* dynamical systems. I will use $\nabla_{\text{ph}} \cdot \mathbf{V}_{ph} < 0$ in all regions of the phase space of interest as the working definition. This implies non-Hamiltonian dynamics and no time reversal symmetry.

We are normally interested in systems with dissipation such that we know the solutions do not diverge for any initial condition. In this case there must be at least one stable *fixed point* $x = x_f$, $f(x_f) = 0$. Linear stability is given by $f'(x_f) < 0$ since a small perturbation $x = x_f + \delta x$ will then decay.



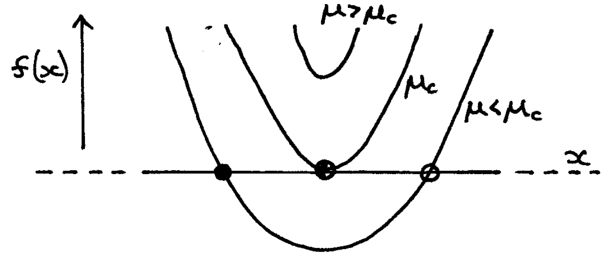
As well as this one stable fixed point, there may be any number of alternating pairs of unstable and stable fixed points. This is the only type of flow geometry possible. Thus, almost all initial conditions will end up at one of the stable fixed points (or infinity if there are diverging solutions). For an initial condition *exactly* at one of the unstable fixed points, the solution will remain at the fixed point. More complex dynamics such as oscillating solutions are not possible.



Qualitative changes of the flow can occur as parameters are changed: these are called *bifurcations*. In particular, the number of fixed points may change, or the stability of a fixed point may change. The parameter values at which they occur are called *bifurcation points*.

Saddle-node bifurcation

As a parameter μ is changed, an unstable and stable fixed point move together, collide and disappear, or are created and move apart. The critical value of μ at which the pair appear or disappear is labelled μ_c. The bifurcation can be understood in terms of a minimum or maximum of the the flow function f passing through zero. In the figure, f for three values of μ are shown straddling μ_c.

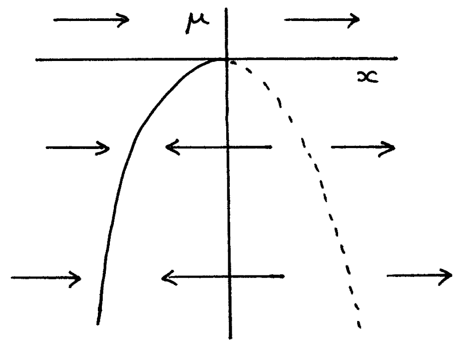


For the usual or *generic* case of a parabolic minimum (or I could use a maximum), and μ close to μ_c, shifting the x-coordinate and the parameter μ so the fixed points appear at $x = 0$, $\mu = 0$, and rescaling variables to eliminate uninteresting constants, the flow near the bifurcation point is described by

$$\dot{x} = \mu + x^2. \tag{3}$$

This type of equation, which captures the essence of the bifurcation, whilst eliminating unnecessary details, is known as the *normal form* for the bifurcation. For the positive sign of the x^2 term I have chosen, the fixed points exist for $\mu < 0$.

Stacking a series of flow diagrams as μ is varied gives the *bifurcation diagram*. The stable fixed point $x_s(\mu) = -\sqrt{-\mu}$ (check the linear stability!) is shown as a solid line, the unstable fixed point $x_u(\mu) = +\sqrt{-\mu}$ by the dashed line. Conventionally, this diagram would be rotated so that μ (the parameter we change) is the abscissa, and x the dependent variable, the ordinate.

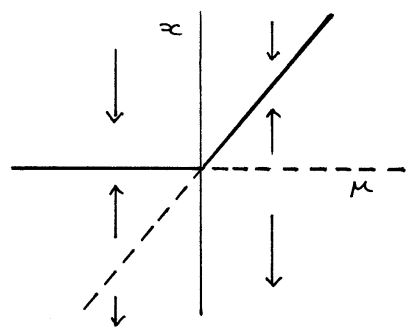


Transcritical bifurcation

Another type of bifurcation occurs when two fixed points collide and pass through each other. Necessarily one is stable and one unstable. Since the absolute position and motion of the fixed points is unimportant in the discussion of the bifurcation, let's fix one fixed point to remain at the origin.
The normal form is

$$\dot{x} = \mu x - x^2. \tag{4}$$

We see that the turning point the flow function $f_\mu(x)$ touches zero but bounces back and does not change sign. Sketch a sequence of $f_\mu(x)$ to see how this works. The bifurcation diagram (here rotated to the conventional orientation) is shown in the figure. For obvious reasons, this bifurcation is also called the *exchange of stabilities*.



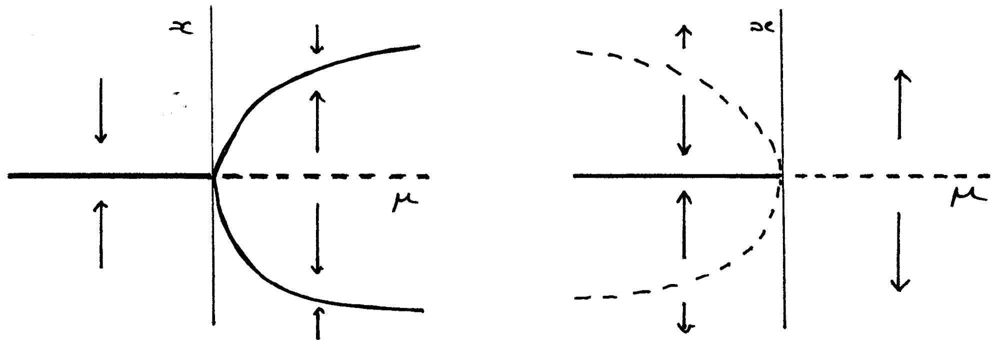
Notice that the normal form tells us that if we do a linear stability analysis of either fixed point, the stability eigenvalue goes to zero at the bifurcation point.

Pitchfork bifurcation

In the small x expansion in Eq. (4) I assumed the generic next order term, i.e. quadratic in x. If the physical system has symmetry about the fixed point ($x \to -x$ in the present case where the fixed point is at $x = 0$), such a term is not allowed, and the normal form would become

$$\dot{x} = \mu x \mp x^3. \tag{5}$$

This is called a supercritical (-) or subcritical (+) pitchfork bifurcation, depending on the sign of the x^3 term. The bifurcation diagrams for these two cases are shown in the figure.

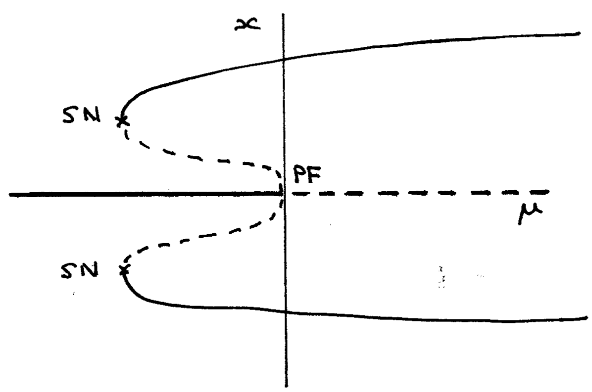


Notice the qualitative difference: if the new fixed points exist on the side of the bifurcation where the original fixed point is unstable, then they are necessarily stable (supercritical); if they exist on the stable side, then the new fixed points are unstable (subcritical). This connection between the existence of fixed points and their stability is an important result of the normal form approach. As for the saddle-node and transcritical bifurcations, the stability eigenvalue of any of the fixed points goes to zero at the bifurcation point.

For the subcritical diagram the flows diverge diverge to infinity for any μ. In many physical systems the stable branches bend around at saddle-node bifurcations to lead to a large-x stable solution. This may be described by an equation of the form

$$\dot{x} = \mu x + x^3 - g x^5. \tag{6}$$

giving the following picture



However, this is no longer systematic, since we require the behavior at $x = O(1)$ where the truncated expansion may not be accurate, and there are other possibilities for the large x dynamics.

These are the only bifurcations typically expected in one dimensional flows on tuning one control parameter. You can imagine special cases - a quartic minimum of f passing through zero, or four fixed points colliding. However these will rely on special "lucky" features of a particular system - they are said to be not *structurally stable* - or on carefully controlling two or more parameters to tune to a special point. One "lucky" feature might be symmetries: since these are important in physical systems, it is certainly worth studying bifurcations in systems with symmetries. This is known as *equivariant bifurcation theory*, and ties together bifurcation theory and group theory. I will not discuss this topic here.

Higher-dimensional flows

The bifurcations of fixed points in two, and indeed higher, dimensional flows, are largely similar: saddle node, transcritical, and pitchfork bifurcations have the same structure along some curve (line in the vicinity of the bifurcation) with contraction of the flow towards the line. This is summarized by normal forms that describe the flows local to the bifurcation, such as

$$\dot{x} = \mu + x^2, \tag{7}$$

$$\dot{y} = -y \tag{8}$$

for a saddle-node bifurcation in a two dimensional flow (the y-contraction rate has been scaled to unity by rescaling time). However one new type of generic bifurcation may occur in two (or higher) dimensional flows: the Hopf bifurcation.

Hopf bifurcation

The easiest way to understand this new bifurcation is to consider a linear stability analysis of a fixed point, let's say at $x = y = 0$

$$\dot{\boldsymbol{\delta}\mathbf{x}} = \mathbf{J} \cdot \boldsymbol{\delta}\mathbf{x}, \tag{9}$$

with $\mathbf{J}$ the Jacobean $J_{ij} = \partial f_i/\partial x_j\big|_{\mathbf{x}=0}$. The stability is given by the eigenvalues σ_n, of $\mathbf{J}$ in the usual way. Since $\mathbf{J}$ is real the eigenvalues are real, or come in complex conjugate pairs. The bifurcations considered so for all correspond to one real eigenvalue passing through zero – and typically we do not expect two real eigenvalues to pass through zero together. The only other generic possibility is for the real part of a complex pair to pass through zero i.e. the complex pair of eigenvalues passes through the imaginary axis. This (with certain mathematical restrictions I won't go into) is the Hopf bifurcation. At the bifurcation point $(\delta x, \delta y)$ oscillate with a frequency given by the imaginary part of the eigenvalue (let's take the positive one), and beyond the bifurcation we expect a new solution oscillating with this frequency. Such an oscillation is described by a closed, attracting curve in phase space, called a *limit cycle*.

The normal form summarizing the behavior near a Hopf bifurcation is (in polar form)

$$\dot{r} = \mu r \mp r^3, \tag{10}$$

$$\dot{\theta} = \omega + br^2, \tag{11}$$

where $\mu = 0$ is the bifurcation point, and $\omega = \operatorname{Im}\sigma$ is the frequency at the bifurcation point. The term br^2 gives a dependence of the frequency on the amplitude of the oscillations r.[2] Various rescalings have been done. For example the limit cycle will be elliptical in the original coordinates: we have transformed to scaled variables along the principle axes to reduce this to a circle (cf. the treatment of the simple harmonic oscillator in Ph106a).

The bifurcation diagrams are essentially the same as the pitchfork bifurcation (compare Eq. (10) with Eq. (5)), with the θ rotation then added. The minus sign in Eq. (10) gives a *supercritical Hopf* bifurcation with the amplitude of the oscillations of th elicit cycle growing as $\mu^{1/2}$ for $\mu > 0$, and the limit cycle is *stable* (check this!). The plus sign gives a *subcritical Hopf bifurcation* with the amplitude of the oscillations growing as $|\mu|^{1/2}$ for $\mu < 0$, but the limit cycle is *unstable*. As for the subcritical pitchfork bifurcation, adding a $-gr^5, g > 0$ term to Eq. (10) would "bend around" the unstable branch to give stable, large amplitude, limit cycles.

In higher dimensional flows, there will be more eigenvalues, but again we do not expect multiple complex pairs of eigenvalues to pass through the imaginary axis at the same time, except coincidentally or

[2] The nonlinear terms are of relative order r^2 not r to be smooth at the origin: e.g. do the Taylor expansion in Cartesian coordinates, and recognize that the θ independence implies symmetry under $x \to -x$, etc., and then transform to polar coordinates.

because of some additional symmetry. So the Hopf bifurcation is the only new generic bifurcation of a fixed point in higher dimensional flows.

Bifurcations *of* limit-cycles:

As you might imagine, since a limit cycle in an extended object in phase space, the types of qualitative changes that can occur to the flow structure flow structure of a limit cycle are harder to classify, and a simple study becomes more *ad hoc*. Here are some of the ideas that arise.

Bifurcations on the Poincaré section: Since the analysis about a point is easier than about a curve, we can use Poincaré's trick of taking the intersections of the trajectory with a surface in phase space. Then a limit cycle becomes a fixed point of the Poincaré map $\mathcal{T}$ for successive intersections with this Poincaré section. A linear stability analysis of this fixed point is done in the usual way, by finding the eigenvalues λ of the Jacobean of the map at the fixed point. The magnitude $|\lambda|$ gives the magnification of a perturbation under each iteration of the map, i.e. after one period of the limit cycle. This is nothing but Floquet analysis as in Lecture 6, and the Floquet exponent is $\sigma = T^{-1} \log \lambda$, with T the period of the limit cycle. The bifurcation corresponds to $|\lambda|$ passing through the unit circle ($|\lambda| > 1$ signals instability).

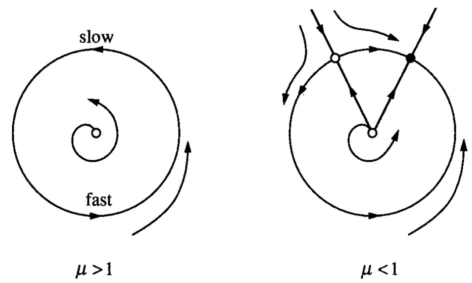
$\lambda = 1$ corresponds to a change in the stability of the limit cycle, but no change in the frequency. There are bifurcations analogous to the saddle-node, transcritical, and pitchfork bifurcations of fixed points, but now for limit cycles.

$\lambda = -1$ corresponds to a period doubling bifurcation – it takes two of the original periods of the original limit cycle for the motion to repeat. One *route to chaos* is through repeated period doubling bifurcations - an infinite number of them - leading to the repeat time of the motion getting larger and larger, until the motion no longer repeats, i.e. chaos!

A complex pair of λ passing through the unit circle gives oscillations at a new frequency, and so, at least naïvely, to motion on a two torus in phase space. This seems quite analogous to the original Hopf bifurcation, and at one time it was expected that this type of analysis could continue i.e. a bifurcation of the two frequency motion would lead to three frequency motion (motion on a 3-torus), and eventually after many bifurcations to chaotic-like motion of m-frequencies with $m \to \infty$. However the m-frequency motion for $m > 1$ but finite is subject to the same issues of resonances as we saw in the Hamiltonian system, and the nonlinear terms in the dynamics may cause locking between some of the m frequencies (usually those with winding numbers close to some low-order rational), leading to lower m-frequency periodic motion, or the smooth m-torus may break down to chaos in ways analogous to the Hamiltonian phenomena. This can happen even for $m = 2$.

Other bifurcations of limit cycles There are also *global* bifurcations of limit cycles, in which the period of the limit cycle diverges to infinity at the bifurcation point (the cycle continuously slows down). These are not given by the Poincaré section analysis.

The first (known as the *infinite period bifurcation*) can be understood as a saddle-node bifurcation of fixed points where the one-dimensional line of the the analysis is the θ variable. The figure shows before-and-after pictures of the flow.

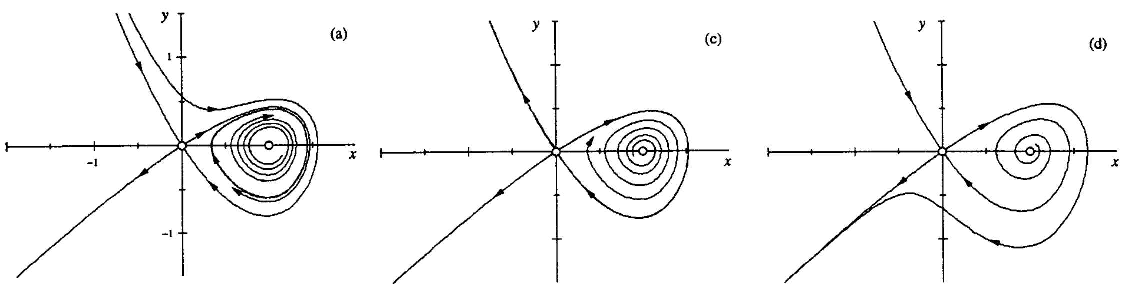


The difference from the one dimensional flow is that the outward flow reconnects to the inwards flow far from the bifurcation point. The equation

$$\dot{\theta} = \mu - \sin\theta, \tag{12}$$

captures the behavior.

In the second, a limit cycle grows towards a saddle fixed point, becomes a homoclinic orbit at the bifurcation point, and then disappears. This is consequently known as the *homoclinic* or *saddle-loop* bifurcation. The flow is sketched before, at, and after the bifurcation in the following figure.



Chaos

For flows in a phase space of dimension $n > 2$, the attractors are not limited to fixed points and limit cycles as for $n = 2$. But for dissipative systems where the phase space volume contracts, the dimension of the attractor must be less than n. For a three dimensional system, does the phase space volume of initial conditions contract to a planar (and so zero volume) attractor, ruling out chaos by the usual argument that trajectories cannot intersect? The answer is no: chaotic motion corresponds to a *strange attractor* that has non-integral dimension, i.e. is a *fractal*. Chaotic dynamics can exist for a system with a three dimensional phase space $n = 3$ and the attractor has non-integral dimension between two and three. We will study chaotic motion in such dissipative systems in Lecture 10.

Math or physics?

The discussion of this lecture has been essentially mathematics, arguing what might be expected to happen based on general smooth mathematical equations, informed by very general physical ideas such as symmetry. Using mathematics to aid the discussion of physics is of course essential. But physics, is not just arbitrary equations, and so what might be "typical" mathematically might not be typical physically, and vice versa. Consider for example the familiar equation

$$\ddot{x} + \mu\dot{x} + x = 0, \tag{13}$$

as the parameter μ is changed. For all positive μ there is a single attractor, a stable fixed point at $x = 0$. At $\mu = 0$ there is a qualitative change of behavior: suddenly on infinity of periodic orbits arises.[3] This is not in the list of typical bifurcations we arrived at from the mathematical considerations, although it is certainly of physical interest.

Further reading

If you would like to read more in this area, beyond the level of the class, the book *Nonlinear Dynamics and Chaos* by Steven Strogatz is an excellent pedagogical introduction. I have taken much of the discussion, and some of the figures, from there.

Michael Cross February 3, 2014

[3] These are not limit cycles in the language of dynamical systems: a limit cycle as the name implies has to be the infinite time limit of at least one other trajectory (positive or negative infinity is allowed, so that unstable limit cycles are OK. I have sometimes used the term limit cycle for a periodic orbit in a Hamiltonian, which is not strictly correct.