Physics 106a — Classical Mechanics

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Lecture 4: Virtual Work and the Lagrangian Approach

Today's Topics

$$\text{Newton} \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad \text{Lagrange}$$

- Constraints
- Virtual displacement, virtual work, d'Alembert's principle
- Generalized equation of motion
- Lagrangian and Euler-Lagrange equation

Constraints

Starting point: primitive description in terms of Newtonian equations of motion for M elementary objects with position vectors $\vec{r}_i$, $i = 1 \ldots M$.
Newton's equations contain all the forces acting.

Constraints may restrict the dynamics, so that the dynamics can be described in terms of fewer than $3M$ variables:

> *The number of degrees of freedom is the number of coordinates that can be independently varied in a small displacement.*

For constrained dynamics

- the number of degrees of freedom will usually be less than $3M$
- any configuration allowed by the constraints can usually be specified using a *reduced number* N of generalized coordinates $\{q_k\}$, $k = 1 \ldots N$

A key distinction is between *holonomic* and *nonholonomic* constraints.

Holonomic Constraints

For a *holonomic* constraint, we can find a reduced set of N generalized coordinates such that

- the coordinates uniquely define any configuration of the system allowed by the constraints, and so we can find an expression for the positions of all the elementary components in the form

$$\vec{r}_i = \vec{r}_i(q_1, q_2 \ldots q_N, t), \qquad i = 1 \ldots M \tag{1}$$

- the N coordinates can each be varied independently

For holonomic constraints, the number of degrees of freedom is the same as this reduced number N of generalized coordinates.

Further classification

- *time independent* or *scleronomic* if time does not appear in Eq. (1)
- *time dependent* or *rheonomic* if time does appear

This distinction is most important when we consider the Hamiltonian and the conservation of energy.

Nonholonomic Constraints

Anything else! Examples:

- Inequalities
- Rolling constraints (often)

Hard to say anything general about dynamics with nonholonomic constraints, so we will establish the formalism for the case of holonomic constraints.

Special class: *nonintegrable differential* nonholonomic constraints:

Can specify *infinitesimal changes* in the $\vec{r}_i$ in terms of changes in a reduced set of the coordinates

$$\delta\vec{r}_i = \sum_k \vec{a}_{ik}(\{q_l\}, t)\,\delta q_k$$

Equivalently, this is is a relationship between the velocities

$$\dot{\vec{r}}_i = \sum_k \vec{a}_{ik}(\{q_l\}, t)\,\dot{q}_k\,.$$

Virtual Work

- Define a *virtual displacement*: a set of displacements $\delta\vec{r}_i$ of the particles with *time* and *velocities* held fixed and *consistent with the constraints*
- Project Newton's 2nd law of motion onto this virtual displacement

$$\vec{F}_i = \dot{\vec{p}}_i \quad \Rightarrow \quad \sum_i \vec{F}_i \cdot \delta\vec{r}_i = \sum_i \dot{\vec{p}}_i \cdot \delta\vec{r}_i \qquad \text{d'Alembert's principle}$$

- The quantity $\delta W = \sum_i \vec{F}_i \cdot \delta\vec{r}_i$ for a virtual displacement $\delta\vec{r}_i$ is called the *virtual work*.
- Condition for an equilibrium

$$\sum_i \vec{F}_i \cdot \delta\vec{r}_i = 0 \qquad \text{Principle of virtual work}$$

- If the constraint forces give no contribution to the virtual work only external forces contribute to the virtual work $\vec{F}_i \to \vec{F}_i^{\text{nc}}$.

Generalized Equations of Motion

- For holonomic constraints vary single coordinate q_k so $\delta\vec{r}_i = (\partial\vec{r}_i/\partial q_k)\delta q_k$

$$\delta W = \sum_i \vec{F}_i \cdot \left(\frac{\partial \vec{r}_i}{\partial q_k}\right) \delta q_k$$

- Define the generalized force

$$\mathcal{F}_k \equiv \frac{\delta W}{\delta q_k} = \sum_i \vec{F}_i \cdot \left(\frac{\partial \vec{r}_i}{\partial q_k}\right)$$

- For holonomic constraints d'Alembert's principle gives

$$\sum_i \left(\vec{F}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} - \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k}\right) \delta q_k \quad \Rightarrow \quad \sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \mathcal{F}_k$$

- The left hand side can be transformed (again for holonomic constraints) to give the *generalized equation of motion*

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \mathcal{F}_k$$

Conservative Forces

Conservative forces can be derived from a potential $V(\{\vec{r}_i\}, t) \to V(\{q_k\}, t)$

$$\mathcal{F}_k = \sum_i \vec{F}_i \cdot \left(\frac{\partial \vec{r}_i}{\partial q_k}\right) = -\sum_i \left(\frac{\partial V}{\partial \vec{r}_i}\right) \cdot \left(\frac{\partial \vec{r}_i}{\partial q_k}\right) = -\frac{\partial V}{\partial q_k}$$

For such forces the generalized equation of motion can be written as the *Euler-Lagrange* equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0$$

with $L(\{q_k\}, \{\dot{q}_k\}, t)$ the Lagrangian $L = T - V$.

Thus we have *derived* the Lagrangian formulation of mechanics starting from Newton's 2nd law.