Physics 106a — Classical Mechanics

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Lecture 12

Action Angle Variables & Hamilton-Jacobi Theory

Idea

Design your Hamiltonian, and find a canonical transformation to gives this

- **Action-angle variables** (for periodic motion)

$$\{q_k\}, \{p_k\} \Rightarrow \{\psi_k\}, \{I_k\} \quad \text{such that} \quad H = H(\{I_k\})$$

$\{\psi_k\}$ are ignorable, so $\dot{I}_k = 0$ and then $\dot{\psi}_k = \partial H/\partial I_k = \Omega_k$

- **Hamilton-Jacobi theory**

$$\{q_k\}, \{p_k\} \Rightarrow \{\beta_k\}, \{\alpha_k\} \quad \text{such that} \quad \bar{H} = 0$$

New coordinates and momenta are constants $\dot{\alpha}_k = \dot{\beta}_k = 0$

It is only for very special cases that this can be done!

Action-Angle Variables

For periodic motion

$$\{q_k\}, \{p_k\} \Rightarrow \{\psi_k\}, \{I_k\} \quad \text{such that} \quad H = H(\{I_k\})$$

Then $\{\psi_k\}$ are ignorable, so $\dot{I}_k = 0$ and then $\dot{\psi}_k = \partial H / \partial I_k = \Omega_k$

- frequency without calculation of full orbit $q(t)$, $p(t)$
- orbit without solving for time evolution
- action variable is adiabatic invariant
- simple description of periodic orbit for start of perturbation theory

Adiabatic Invariant

$H(q, p, \alpha)$ with α a slowly varying function of time

For *fixed* α use $F_1(q, \psi; \alpha)$ to give action-angle variables $I, \psi \Rightarrow H(I(\alpha), \alpha)$

Now include time dependence $\quad H \to \bar{H} = H(I(\alpha), \alpha) + \dot{\alpha}(\partial F_1/\partial\alpha)$

The action becomes time dependent

$$\dot{I} = -\frac{\partial \bar{H}}{\partial \psi} = -\frac{\partial^2 F_1}{\partial \psi \partial \alpha}\dot{\alpha}$$

Average over one period T of ψ approximating $\dot{\alpha}$ as constant over this time

$$\langle \dot{I} \rangle \simeq -\frac{\dot{\alpha}}{2\pi}\left[\frac{\partial F_1}{\partial \alpha}(q, \psi + 2\pi, \alpha(T)) - \frac{\partial F_1}{\partial \alpha}(q, \psi, \alpha(0))\right] \simeq -\frac{\dot{\alpha}^2 T}{2\pi}\frac{\partial^2 F_1}{\partial \alpha^2}$$

This gives

$$\langle \dot{I} \rangle \propto \dot{\alpha}^2 \qquad \text{whereas} \qquad \dot{E} \propto \dot{\alpha}$$

Hamilton-Jacobi theory

- Time dependent canonical transformation to make new Hamiltonian zero!

$$\bar{H}(\{Q_k\}, \{P_k\}, t) = 0$$

- $\dot{Q}_k = 0$, $\dot{P}_k = 0$ (so write $P_k \to \alpha_k$, $Q_k \to \beta_k$)

- Type-2 generating function $S(\{q_k\}, \{P_k\}, t)$: then $p_k = \partial S / \partial q_k$ and

$$H\left(\{q_k\}, \left\{\frac{\partial S}{\partial q_k}\right\}, t\right) + \frac{\partial S}{\partial t} = 0$$

  the *Hamilton-Jacobi* equation for *Hamilton's principal function* $S(\{q_k\}, \{\alpha_k\}, t)$

- The mechanics problem is "reduced" to solving a nonlinear PDE!

- S is the action as a function of the endpoints $\{q_k\}, t$, given by integrating the Lagrangian along the actual dynamical path

- Method works for separable problems

$$S(\{q_k\},t) = W_1(q_1) + W_2(q_2) + \cdots + W_N(q_N) + W_0(t)$$

- May choose the new (constant) momenta $\{\alpha_k\}$ to be N separation constants (or independent combinations of them) $\Rightarrow$ $S(\{q_k\},\{\alpha_k\},t)$
- The constant $\{\alpha_k\}$ are fixed by initial conditions $p_k(0) = (\partial S/\partial q_k)_{t=0}$
- New (constant) coordinates $\{\beta_k\}$ are given by $\beta_k = \partial S/\partial \alpha_k$
- Constants $\{\beta_k\}$ are fixed by initial conditions $\{q_k(0)\}$: $\beta_k = (\partial S/\partial \alpha_k)_{t=0}$
- At general time $\partial S/\partial \alpha_k = \beta_k \Rightarrow \{q_k(t)\}$

Hamilton's characteristic function

For a time independent Hamiltonian

$$S(\{q_k\}, \{\alpha_k\}, t) = W(\{q_k\}, \{\alpha_k\}) - Et$$

with E the constant value of H.

The function $W(\{q_k\}, \{\alpha_k\})$ is called *Hamilton's characteristic function*

Can alternatively perform a *time independent* canonical transformation with a generating function $W(\{q_k\}, \{\alpha_k\})$ to make the Hamiltonian constant rather than zero

$$\bar{H} = H\left(\{q_k\}, \left\{\frac{\partial W}{\partial q_k}\right\}\right) = E$$

Can choose E as one of the new constant momenta (other choices possible too)

Mass in gravitational potential

Hamiltonian $H = \frac{1}{2}(p_x^2 + p_z^2) + z$

Hamilton-Jacobi equation for Hamilton's principal function $S(x, z, t)]$

$$\frac{1}{2}\left(\frac{\partial S}{\partial x}\right)^2 + \left[\frac{1}{2}\left(\frac{\partial S}{\partial z}\right)^2 + z\right] + \left[\frac{\partial S}{\partial t}\right] = 0$$

Separability: $S(x, z, t) = W_1(x) + W_3(z) - Et$

$$\frac{1}{2}\left(\frac{dW_1}{dx}\right)^2 = \alpha_1 \quad \Rightarrow \quad W_1 = \pm\sqrt{2\alpha_1}x$$

$$\frac{1}{2}\left(\frac{dW_3}{dz}\right)^2 + z = \alpha_3 \quad \Rightarrow \quad W_3 = \pm\sqrt{\tfrac{8}{9}}(\alpha_3 - z)^{3/2}$$

$$\alpha_1 + \alpha_3 = E$$

Mass in gravitational potential

$$S = \pm\sqrt{2\alpha_1}x \pm \sqrt{\tfrac{8}{9}}(\alpha_3 - z)^{3/2} - (\alpha_1 + \alpha_3)t$$

Choose α_1, α_3 as the new constant momenta $\rightarrow$ new coordinates:

$$\beta_1 = \frac{\partial S}{\partial \alpha_1} = \pm\frac{1}{\sqrt{2\alpha_1}}x - t$$
$$\beta_3 = \frac{\partial S}{\partial \alpha_3} = \pm\sqrt{2(\alpha_3 - z)} - t$$

Original momenta:

$$p_x = \frac{\partial S}{\partial x} = \frac{dW_1}{dx} = \pm\sqrt{2\alpha_1}$$
$$p_z = \frac{\partial S}{\partial z} = \frac{dW_3}{dz} = \mp\sqrt{2(\alpha_3 - z)}$$

Mass in gravitational potential

Fix constants from the initial conditions: e.g. shoot from $x = z = 0$ at $t = 0$ at 45° to the horizontal with a speed 2, so that $p_x(0) = p_z(0) = \sqrt{2}$

$$
\begin{aligned}
p_x &= \pm\sqrt{2\alpha_1} && \Rightarrow && \alpha_1 = 1,\ \text{use top sign} \\
p_z &= \mp\sqrt{2(\alpha_3 - z)} && \Rightarrow && \alpha_3 = 1,\ \text{use bottom sign} \\
\beta_1 &= \pm\frac{1}{\sqrt{2\alpha_1}}x - t && \Rightarrow && \beta_1 = 0 \\
\beta_3 &= \pm\sqrt{2(\alpha_3 - z)} - t && \Rightarrow && \beta_3 = -\sqrt{2}
\end{aligned}
$$

Read off the solutions

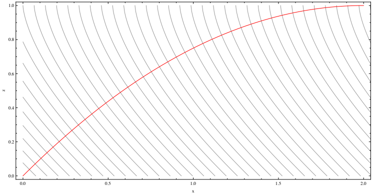
$$p_x(t) = \sqrt{2}, \quad p_z(t) = \sqrt{2(1 - z)}$$

and from the β equations

$$x(t) = \sqrt{2}t, \quad z(t) = \sqrt{2}t - \tfrac{1}{2}t^2$$

Wave description of particle motion

$$S = W(x,z) - Et = \sqrt{2}x - \tfrac{8}{9}(1-z)^{3/2} - 2t$$



- Particle trajectory is along normal to lines of constant S
- Lines of constant S propagate with speed $E/|\nabla W|$

If we interpret S as the phase of a wave:

- Frequency of wave is $\Omega = E$
- Wave vector of wave is $\vec{k} = \vec{\nabla} S = \vec{\nabla} W = \vec{p}$
- Phase speed of wave is $\Omega/k = E/p$
- Group speed of wave is $d\Omega/dk = dE/dp =$ speed of particle

Mass in gravitational potential

Alternative way of manipulating the constants

Instead use α_1 and $\alpha_0 = -E$ as the new constant momenta and write

$$S = W(x, z, \alpha_1, E) - Et$$

with

$$W = \pm\sqrt{2\alpha_1}x \pm \sqrt{\tfrac{8}{9}}(E - \alpha_1 - z)^{3/2}$$

so that

$$\begin{aligned}
\beta_1 &= \frac{\partial W}{\partial \alpha_1} = \pm\frac{1}{\sqrt{2\alpha_1}}x \mp \sqrt{2}(E - \alpha_1 - z)^{1/2} \\
p_x &= \frac{\partial W}{\partial x} = \pm\sqrt{2\alpha_1} \\
p_z &= \frac{\partial W}{\partial z} = \mp\sqrt{2(E - \alpha_1 - z)}
\end{aligned}$$

The initial conditions give $\alpha_1 = 1$, $E = 2$, $\beta_1 = \sqrt{2}$ and sign choices such that

$$W = \sqrt{2}x - \tfrac{8}{9}(1 - z)^{3/2} \qquad \text{and} \qquad z = x - \tfrac{1}{4}x^2$$

Less trivial problems I $*$

Starred items are for interest only

Particle in gravity-like potential

$$V(r, z) = -\frac{k}{r} + gz$$

Hamilton-Jacobi equation separable in parabolic coordinates
(Hand and Finch pp. 226-228)

Less trivial problems II $*$

Kepler problem in spherical polar coordiantes (GPS §10.5,§10.8)

$$H = \frac{1}{2m}\left(p_r^2 + \frac{1}{r^2}p_\theta^2 + \frac{1}{r^2\sin^2\theta}p_\phi^2\right) - \frac{k}{r}$$

$$\frac{1}{2m}\left[\left(\frac{\partial S}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial S}{\partial \theta}\right)^2 + \frac{1}{r^2\sin^2\theta}\left(\frac{\partial S}{\partial \phi}\right)^2\right] - \frac{k}{r} + \frac{\partial S}{\partial t} = 0$$

$$S = W_r(r) + W_\theta(\theta) + W_\phi(\phi) - Et$$

$$\left(\frac{dW_\phi}{d\phi}\right)^2 = \alpha_\phi^2 \quad \Rightarrow \quad I_\phi = \frac{1}{2\pi}\int_0^{2\pi}\frac{dW_\phi}{d\phi}d\phi = \alpha_\phi$$

$$\left(\frac{dW_\theta}{d\theta}\right)^2 + \frac{\alpha_\phi^2}{\sin^2\theta} = \alpha_\theta^2 \quad \Rightarrow \quad I_\theta = \frac{1}{2\pi}\int_0^{2\pi}\frac{dW_\theta}{d\theta}d\theta = \alpha_\theta - \alpha_\phi$$

$$\frac{1}{2m}\left[\left(\frac{dW_r}{dr}\right)^2 + \frac{\alpha_\theta^2}{r^2}\right] - \frac{k}{r} = E \quad \Rightarrow \quad E = -\frac{\frac{1}{2}mk^2}{(I_r + I_\theta + I_\phi)^2}$$

Connection with quantum mechanics $*$

Schrödinger's equation for a particle with Hamiltonian $H = p^2/2m + V(\vec{r}, t)$

$$i\hbar \frac{\partial \Psi}{\partial t} = H\Psi = -\frac{\hbar^2}{2m}\nabla^2\Psi + V(\vec{r}, t)\Psi \qquad (*)$$

The semiclassical limit is given by considering $\hbar$ small and looking for a solution in the WKB form

$$\Psi = \sqrt{\rho(\vec{r}, t)}e^{iS(\vec{r},t)/\hbar} \quad \text{cf. plane wave } \Psi = \sqrt{\rho}e^{i\vec{p}\cdot\vec{r}/\hbar}$$

where we assume gradients and time dependence of ρ, S are $O(1)$.

Substitute into $(*)$ and collect the leading order terms, those in $\hbar^0$

$$\frac{1}{2m}(\vec{\nabla}S)^2 + V(\vec{r}, t) + \frac{\partial S}{\partial t} = 0$$

This is exactly the Hamilton-Jacobi equation for the Hamiltonian H with Hamilton's principal function $S(\vec{r}, t)$ equal to $\hbar$ times the quantum phase.