ACM 100b

Interpretation in terms of linear algebra

Dan Meiron

Caltech

February 9, 2014

Interpretation in terms of linear algebra

- We can interpret this identity

$$\int_a^b L[u(x)]v(x)dx = \int_a^b u(x)L[(v(x)]dx$$

  in terms of linear algebra but the "vectors" and "matrices" are functions and operators not finite dimensional objects as in regular linear algebra.
- Instead the idea here is one of *abstract vector spaces*.
- That is we can use an analogy with vectors and matrices but the spaces are infinite dimensional (because we're working with functions and derivatives)

Generalized scalar product

- We can think of the integral

$$\int_a^b u(x)v(x)dx$$

as a type of scalar product like the dot product $\boldsymbol{x} \cdot \boldsymbol{y}$ in vector calculus.

- In fact, the integral satisfies all the requirements of being a scalar product in an abstract vector space where the vectors are the functions $u(x)$ and $v(x)$.
- If we define the scalar product (u, v) by

$$(u(x), v(x)) = \int_a^b u(x)v(x)dx,$$

then Lagrange's identity becomes the expression

$$(Lu, v) - (u, Lv) = 0.$$

Generalized scalar product

- We see also that Lu can be thought of as a linear operator on the vector $u(x)$.
- Indeed suppose we had the expression

$$\int_a^b vMudx,$$

where M is *any* linear expression involving derivatives (of which our L is a special case)

- Then we can think of this as similar to a scalar product of the form

$$(\boldsymbol{y}, A\boldsymbol{x}) = \boldsymbol{y} \cdot (A\boldsymbol{x})$$

where $\boldsymbol{x}$ and $\boldsymbol{y}$ are n-dimensional vectors and A is an $n \times n$ matrix.

Adjoint operators

- By picking special boundary conditions on u and v, we can also always use integration by parts to rewrite

$$\int_a^b vMudx$$

as

$$\int_a^b u(x)M^*[v(x)]dx$$

- Recall that when we have a scalar product of the form

$$(\boldsymbol{x}, A\boldsymbol{y})$$

we can also write this as

$$(A^*\boldsymbol{x}, \boldsymbol{y})$$

where A^* is the matrix adjoint of A.

- For real matrices A^* is the same as the transpose of A

Self adjoint operators

- Here M^* now formally plays the role of an *adjoint matrix* except it's now an *adjoint operator*.
- Note that for a general linear differential operator M, the adjoint M^* is not necessarily identical to M.
- In fact, in general it isn't - just like the transpose of a matrix is not identical to the original matrix unless the matrix is symmetric.
- But in the special case of the S-L operator given by L we can see from Lagrange's identity that

$$L^* = L$$

- We see that L is its own adjoint matrix
- We say L is a *self-adjoint operator*

Interpretation in terms of linear algebra

- Recall from courses on linear algebra that self-adjoint matrices have special properties.
- A real self-adjoint matrix is a symmetric matrix
- Recall from linear algebra that a symmetric matrix has real eigenvalues
- A symmetric matrix also has mutually orthogonal eigenvectors
- A similar property holds for the Sturm-Liouville ODE
- The S-L ODE also has some additional properties associated with the existence of positive eigenvalues.
- We will show that these special properties carry over to the continuous case.
- However, it's important to recall that if the S-L problem is not a regular one then some of these special properties get violated.