ACM 100b

Expansions of functions in terms of Sturm-Liouville eigenfunctions

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Recap

- In the last lecture we presented some of the important properties of the solution to Sturm-Liouville problems
- All the eigenvalues of the Sturm-Liouville ODE are real.
- If $\phi_1(x)$ and $\phi_2(x)$ are two eigenfunctions corresponding to different eigenvalues λ_1 and λ_2 then the eigenfunctions are *orthogonal* in the following sense:

$$\int_a^b r(x)\phi_1(x)\phi_2(x)dx = 0.$$

  where $r(x)$ is the function appearing in S-L eigenvalue problems as

$$L[y(x) = \lambda r(x)y(x)$$

- The eigenvalues of the Sturm-Liouville problem are all simple - that is there are no "multiple roots"

Recap

- The sequence of eigenvalues λ_1, λ_2 etc. can be ordered according to increasing magnitude.
- If this is done it is seen that $\lambda_n \to +\infty$ as $n \to \infty$.
- In other words the eigenvalues have no point of accumulation (except at ∞)
- If an eigenfunction ϕ_1 has an eigenvalue λ_1 and an eigenfunction ϕ_2 has an eigenvalue λ_2 with $\lambda_2 > \lambda_1$ then there is at least one zero of the eigenfunction ϕ_2 that lies between the zeroes of the eigenfunction of ϕ_1.
- All of these properties are proven using the Lagrange identity.

Recalling the heat equation

- At this point we can also answer one of the questions we posed when we examined the solution of the heat equation.
- Recall the problem was to solve

$$\frac{\partial \Theta}{\partial t} = D\frac{\partial^2 \Theta}{\partial x^2} \qquad 0 \leq x \leq 1$$

with boundary conditions

$$\Theta(0,t) = 0 \qquad \Theta(1,t) = 0 \qquad \Theta(x,0) = \Theta_0(x)$$

- Recall the solution was given by

$$\Theta(x,t) = \sum_{n=1}^{\infty} B_n \exp(-n^2\pi^2 t)\sin(n\pi x)$$

- To satisfy the initial condition at $t = 0$, we had to solve equation

$$\Theta_0(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

The solution involves an expansion in eigenfunctions

- We now see that the sum is actually a superposition of eigenfunctions of the equation

$$\frac{d^2X(x)}{dx^2} + \lambda^2 X = 0 \qquad X(0) = X(1) = 0,$$

- This ODE

$$\frac{d^2X(x)}{dx^2} + \lambda^2 X = 0 \qquad X(0) = X(1) = 0,$$

  is of Sturm-Liouville form with $p(x) = 1$, $q(x) = 0$, and $r(x) = 1$.
- The solutions were

$$X_n(x) = \sin(n\pi x) \qquad n = 1, 2, \ldots$$

Expansions of functions in S-L eigenfunctions

- We therefore know immediately from our S-L theory that the eigenfunctions are orthogonal meaning

$$\int_0^1 \sin(m\pi x)\sin(n\pi x)dx = \begin{cases} 1/2 & \text{if } m = n \\ 0 & \text{otherwise.} \end{cases}$$

- In fact we know this without doing any of the integrals (except for the special case $m = n$)
- Actually by normalizing the eigenfunctions we don't even have to do that case.
- Normalizing the eigenfunctions means choosing constants c_n such that $X_n = c_n \sin(n\pi x)$ and that the eigenfunctions are orthonormal:

$$c_n^2 \int_0^1 \sin(n\pi x)^2 dx = 1$$

Using orthogonality

- This orthogonality relation gives us a strategy for computing the coefficients B_n in

$$\Theta_0(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

- The idea is to think of this expression as the decomposition of the general function (or abstract vector) $\Theta_0(x)$ into a superposition of eigenfunctions $\sin(n\pi x)$
- But these eigenfunctions are mutually orthogonal
- So to get B_n all we have to do is take the scalar product of $\Theta_0(x)$ with each eigenfunction so as to project out the components of this "vector".
- In other words the eigenfunctions are acting as an infinite dimensional set of basis vectors.

Using orthogonality

- We multiply each side of

$$\Theta_0(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)$$

by $\sin m\pi x$:

$$\sin(m\pi x)\Theta_0(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x)\sin(m\pi x)$$

- Next we integrate both sides of the equation from 0 to 1.

$$\int_0^1 \sin(m\pi x)\Theta_0(x)dx = \sum_{n=1}^{\infty} B_n \int_0^1 \sin(n\pi x)\sin(m\pi x)dx$$

Computing the expansion coefficients

- But the sines are orthogonal unless $m = n$:

$$\int_0^1 \sin(n\pi x)\sin(m\pi x)dx = 0 \qquad \text{unless } n = m$$

- And we can compute that

$$\int_0^1 \sin^2(n\pi x)dx = \frac{1}{2} \qquad n = 1, 2, \ldots$$

- Because of the orthogonality of the sines we see that

$$B_n = 2\int_0^1 \Theta_0(x)\sin(n\pi x)dx.$$

- This is known as a *Fourier sine series*.
- The issue of its uniqueness and its convergence will be discussed later.

Can do this for any set of S-L eigenfunctions

- We can also see that a similar result holds for any set of eigenfunctions.
- Consider the expression

$$f(x) = \sum_{n=0}^{\infty} a_n \phi_n(x),$$

- Here the $\phi_n(x)$ are eigenfunctions for a given Sturm-Liouville problem.

Expansions of functions in S-L eigenfunctions

- Now recall we have

$$\int_a^b \phi_n(x)\phi_m(x)r(x)dx = 0 \text{ unless } n = m$$

- So we can determine the coefficients a_n by using the orthogonality of the eigenfunctions:

$$\begin{aligned}\int_a^b r(x)f(x)\phi_m(x) &= \int_a^b r(x)\sum_{n=0}^{\infty} a_n\phi_n(x)\phi_m(x)dx \\ &= \sum_{n=0}^{\infty} a_n \int_a^b r(x)\phi_n\phi_m dx\end{aligned}$$

- Using the orthogonality we get

$$a_m = \frac{\int_a^b r(x)f(x)\phi_m(x)dx}{\int_a^b r(x)\phi_m^2(x)dx}.$$

Expansions of functions in S-L eigenfunctions

- We can always normalize the eigenfunctions so that

$$\int_a^b \phi_m(x)^2 r(x) dx = 1$$

  by just multiplying each eigenfunction by an appropriate constant
- So the expression for the coefficients can be made simpler

$$a_m = \int_a^b r(x) f(x) \phi_m(x) dx$$

- There is still the issue of whether a series like this converges
- We will see later that it does and the way it does is common to all solutions of the regular S-L problem.