Physics 106b: Lecture 6 — 23 January, 2014

Parametrically Driven Oscillators

Introduction: A *parameter* of an oscillator is periodically modulated. The general form for a linear oscillator is the *Hill equation*

$$\ddot{q} + a(t)\dot{q} + b(t)q = 0 \tag{1}$$

with $a(t), b(t)$ periodic functions with some period T. In the lecture I discuss systems where the spring constant b is modulated, either with a square wave or sinusoidally, and the damping a is constant (maybe zero).

Floquet's theorem: For a linear ODE with coefficients that are periodic in time with period T we can find solutions in the form

$$q(t) = e^{\sigma t} P(t) \tag{2}$$

with σ a constant, complex in general, and $P(t)$ a periodic function with the same period T: $P(t+T) = P(t)$.

Note that $\operatorname{Re}\sigma$ plays the role of a stability parameter: $\operatorname{Re}\sigma > 0$ indicates an exponentially growing solution. Hand and Finch write σ as $i\mu$.

General approach: Consider the equation

$$\ddot{\theta} + \gamma\dot{\theta} + F(t)\theta = 0 \tag{3}$$

where $F(t)$ is periodic. We solve the equation over one period of the drive, analytically if possible, numerically otherwise, and see how the values of $\theta, \dot{\theta}$ at the end are related to those at the beginning — the equation is a second order ODE so these two values determine the subsequent behavior. Then iterate to find the behavior over many periods. The solution over one period takes the form

$$\theta(t) = \theta(0)\theta_c(t) + \dot{\theta}(0)\theta_s(t) \tag{4}$$

where $\theta_c(t)$ is the solution with initial condition $\theta(0) = 1, \dot{\theta}(0) = 0$ ("cosine like"), and $\theta_s(t)$ is the solution with initial condition $\theta(0) = 0, \dot{\theta}(0) = 1$ ("sine like").

Then

$$\begin{pmatrix} \theta(T) \\ \dot{\theta}(T) \end{pmatrix} = M \begin{pmatrix} \theta(0) \\ \dot{\theta}(0) \end{pmatrix} \quad \text{with} \quad M = \begin{bmatrix} \theta_c(T) & \theta_s(T) \\ \dot{\theta}_c(T) & \dot{\theta}_s(T) \end{bmatrix}. \tag{5}$$

Since the drive is periodic, we get the same behavior each period, so that

$$\begin{pmatrix} \theta(nT) \\ \dot{\theta}(nT) \end{pmatrix} = M^n \begin{pmatrix} \theta(0) \\ \dot{\theta}(0) \end{pmatrix}. \tag{6}$$

If we find the eigenvalues λ_1, λ_2 of M, then there will be one initial condition $\theta(0), \dot{\theta}(0)$ (determined by the corresponding eigenvector of M) giving a solution that is multiplied by λ_1 at each iteration, and one (the other eigenvector) giving a solution that is multiplied by λ_2 at each iteration. These λ_i are the $e^{\sigma_i T}$ of the Floquet theorem.

Square wave modulation, no damping: Consider $\gamma = 0$ and the driving function

$$F(t) = \begin{cases} 1 + r & \text{for} \quad 0 \le t < \pi x \\ 1 - r & \text{for} \quad \pi x \le t < 2\pi x \end{cases} . \tag{7}$$

For $r = 0$ this is simple harmonic oscillator with frequency unity. The equation is easy to solve because in each half period we just have the simple harmonic oscillator equation with spring constant $1 \pm r$.

Hand and Finch discuss this case. This is a Hamiltonian system. If we work with canonical variables θ, $p_\theta = \dot{\theta}$ we know that $\theta(T)$, $p_\theta(T)$ is a canonical transformation of $\theta(0)$, $p_\theta(0)$ — see Lecture 11 of Ph106a. The Poisson bracket of this transformation is $\det M$. Thus we know

$$\det M = 1 \quad \text{which gives} \quad \lambda_1 \lambda_2 = 1 . \tag{8}$$

Call the two eigenvalues λ, $1/\lambda$. Then

$$\lambda + \frac{1}{\lambda} = \operatorname{Tr} M , \tag{9}$$

a quadratic equation for λ which can be solved to give

$$\lambda = \tfrac{1}{2} \operatorname{Tr} M \pm \sqrt{(\tfrac{1}{2} \operatorname{Tr} M)^2 - 1} . \tag{10}$$

For $|\operatorname{Tr} M| < 2$ the eigenvalues are a complex conjugate pair with magnitude 1 . This gives a solution which oscillates with an an additional frequency but remains of the same amplitude. For $|\operatorname{Tr} M| > 2$ the eigenvalues are real, with, e.g., $\lambda_1 > 1$, $\lambda_2 < 1$. This case corresponds to an exponentially growing solution, and an exponentially decaying solution, both oscillating at the drive period: the growing solution means that the rest state is *unstable* to oscillations of ever growing amplitude. In practice, nonlinear terms left out of the oscillator equation will quench the growth at a finite amplitude.

You can check that for $r = 0$ (no drive) $\operatorname{Tr} M = 2\cos T$ and the eigenvalues are $e^{\pm iT}$ — the time dependence is oscillations at the frequency 1 so that after time T the phase is also T — and the system is stable. As r increases, for T sufficiently close to one half the period 2π of the resonant motion ($x \approx 1$), $|\operatorname{Tr} M|$ passes through 2, and the oscillations begin to grow. Hand and Finch show example plots in Fig. 10.4 and 10.6. They also show plots of successive values of $(\theta(nT), \dot{\theta}(nT))$ for the two cases, in Figs. 10.5 and 10.7. Strobing the system at the drive frequency and plotting the phase space variables in this way is an example of a *Poincaré section*, a useful tool in analyzing dynamical systems. Hand and Finch show the unstable region in the r, x plane, in Fig. 10.8. The primary instability region is near $x = 1/2$ corresponding to a drive frequency at twice the resonance frequency of the oscillator. There are further instability regions at frequencies near $2/n$ times the resonant frequency, with n any integer. I believe the upper gray (unstable) region should come in to the $r = 0$ axis at $x = 1$, not $x = 0.9$ as shown. Also note that their discussion of the figure in the text confuses the white and gray regions. The instability regions are sometimes called “tongues”.

Square wave modulation with damping: Now take $\gamma \neq 0$. The solutions in each half cycle are given by solving

$$\ddot{\theta} + \gamma \dot{\theta} + (1 \pm r)\theta = 0 . \tag{11}$$

The general solutions are

$$\theta_\pm(t) = e^{-\gamma t/2}[A_\pm \cos \omega_\pm t + B_\pm \sin \omega_\pm t] , \tag{12}$$

with

$$\omega_\pm = \sqrt{1 \pm r - (\gamma/2)^2} . \tag{13}$$

From this we can calculate the "cosine" and "sine" like solutions over each half cycle

$$\theta_{c\pm}(t) = e^{-\gamma t/2}\left[\cos\omega_\pm t + \frac{\gamma}{2\omega_\pm}\sin\omega_\pm t\right], \tag{14}$$

$$\theta_{s\pm}(t) = e^{-\gamma t/2}\frac{1}{\omega_\pm}\sin\omega_\pm t\ . \tag{15}$$

We calculate the M matrix by multiplying M_+ and M_- for the two half cycles

$$M = \begin{bmatrix} \theta_{c-}(T/2) & \theta_{s-}(T/2) \\ \dot{\theta}_{c-}(T/2) & \dot{\theta}_{s-}(T/2) \end{bmatrix}\begin{bmatrix} \theta_{c+}(T/2) & \theta_{s+}(T/2) \\ \dot{\theta}_{c+}(T/2) & \dot{\theta}_{s+}(T/2) \end{bmatrix}. \tag{16}$$

It is no longer true that $\det M = 1$ and so we have to calculate the eigenvalues of M to find the instability regions. This is done in the Mathematica Notebook on the website.

Sinusoidal drive: Sinusoidal drive gives the *Mathieu equation*

$$\ddot{\theta} + \gamma\dot{\theta} + [1 + h\cos\omega_d t]\theta = 0\ . \tag{17}$$

Again, in the high Q limit (γ small) there is instability towards the growth of oscillations at a frequency near the resonant frequency (1 for our equation) for drive frequencies ω_d near $2/n$ and for h above a critical value. I will look at the $n = 1$ tongue, using the type of perturbation scheme introduced in the last chapter, assuming small dissipation γ and drive h.

Floquet's theorem tells us to look for a solution which is $e^{\sigma t}$ times a function with period $2\pi/\omega_d$. Since we expect oscillations near the resonant frequency I use the method of harmonic analysis and look for a solution in the form

$$\theta(t) = e^{\sigma t}\,[Ae^{i\omega_d t/2} + \text{c.c.}] + \text{harmonics}, \tag{18}$$

where, since we do not know the phase of the growing solution relative to the drive, I use complex notation. The "harmonics" include terms involving $e^{in\omega_d t/2}$ (as well as the exponential growth or decay): they will be small relative to the first term for small h, and I will neglect them. Substitute in to the equation of motion and collect the various harmonic terms. The coefficient of each must be zero. It is sufficient just to look at the $e^{i\omega_d t/2}$ term. Setting the coefficient to zero gives

$$\left[1 - \left(\frac{i\omega_d}{2} + \sigma\right)^2 + \gamma\left(\frac{i\omega_d}{2} + \sigma\right)\right]A + \frac{h}{2}A^* = 0\ . \tag{19}$$

For $\omega_d \approx 2$ and leaving out terms proportional to σ^2 and $\sigma\gamma$ this can be approximated

$$[(2 - \omega_d) + i\,(2\sigma + \gamma))]\,A = -\frac{h}{2}A^*\ . \tag{20}$$

Taking the magnitude-squared of this equation gives the growth rate

$$\sigma = -\frac{\gamma}{2} \pm \frac{1}{4}\sqrt{h^2 - (\omega_d - 2)^2}\ . \tag{21}$$

For $\omega_d = 2$ instability occurs for $h > 2\gamma$. It is instructive to plot the path of complex σ as h increases from zero for a fixed nonzero (but small) value of $\omega_d - 2$, and the region of instability $\operatorname{Re}\sigma > 0$ in the $\omega_d - 2$, h plane giving the instability tongue.

Note that below the instability, the decay rate of the oscillations is less than $\gamma/2$. This corresponds to an *enhanced* Q of the oscillator, and gives a larger response of the oscillator to an additional periodic direct drive on resonance — the parametric drive acts as an *amplifier*. The Q and linear gain of the amplifier go to infinity at the value of h where the instability occurs, and the system turns into an $oscillator$ with the energy of oscillations coming from the parametric drive.

From Eq. (20) we can also find the *phase* for amplification below the instability and of the growing solution above the instability. For drive at $\omega_d = 2$, and writing $A = |A|e^{i\delta}$, Eq. (20) gives $\delta = \pi/4$ for the larger eigenvalue and $\delta = -\pi/4$ for the smaller eigenvalue. Thus the amplified signal or the growing oscillation has a phase $\pi/4$ relative to the drive. This phase relationship is plotted in the slides posted on the website. The slides also show the experimental implementation using nanomechanical oscillators, described in the papers by Karabalin et al. Nano Lett. **9**, 116 (2009) and App. Phys. Lett. **97**, 183101 (2010). Parametric amplifiers are often used these days as amplifiers adding very little noise to the very delicate measurements of tiny quantum systems such as being developed for quantum computers.

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