Physics 106a — Classical Mechanics

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Lecture 7A
Virial Theorem

Relates the *time average* of total kinetic and potential energies for systems of bound particles interacting with a power law pair potential $U(r) \propto r^{\alpha}$

$$\langle T \rangle = \frac{\alpha}{2} \langle V \rangle$$

Time average:

- Periodic motion: average over one period
- General motion: average over very long time

Examples:

- $U(r) \propto 1/r$ (gravity, Coulomb) gives $\langle T \rangle = -\frac{1}{2} \langle V \rangle$
- $U(r) \propto r^2$ (springs) gives $\langle T \rangle = \langle V \rangle$
- Zwicky (1933) – Coma nebula cluster: argued for presence of dark matter

Virial theorem

Formulation

- Newtons law for N particles

$$\dot{\vec{p}}_i = \vec{F}_i$$

- Define the virial $G = \sum_i \vec{p}_i \cdot \vec{r}_i$

- Time derivative

$$\begin{aligned}\frac{dG}{dt} &= \sum_i \dot{\vec{p}}_i \cdot \vec{r}_i + \sum_i \vec{p}_i \cdot \dot{\vec{r}}_i \\ &= \sum_i \vec{F}_i \cdot \vec{r}_i + \sum_i m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i \\ &= \sum_i \vec{F}_i \cdot \vec{r}_i + 2T\end{aligned}$$

Virial theorem

Time average

$$\left\langle \frac{dG}{dt} \right\rangle = \frac{1}{\tau}\int_0^\tau \frac{dG}{dt} = \frac{G(\tau) - G(0)}{\tau}$$

This is zero for:

- Periodic motion: average over one period
- General bound motion (all $\vec{r}_i$, $\vec{p}_i$ finite): average over very long time

For any system for which $\langle dG/dt \rangle = 0$:

$$2\langle T \rangle = -\sum_i \langle \vec{F}_i \cdot \vec{r}_i \rangle$$

Virial theorem

Evaluate force sum

- For two-body forces, force on ith particle is sum of forces from particles j

$$\vec{F}_i = \sum_{j \neq i} \vec{F}_{ij} \qquad \text{and} \qquad \vec{F}_{ij} = -\vec{F}_{ji} \quad \text{(N3)}$$

- Sum up the force term

$$\sum_i \vec{F}_i \cdot \vec{r}_i = \sum_i \sum_{j<i} \vec{F}_{ij} \cdot (\vec{r}_i - \vec{r}_j) \equiv \sum_i \sum_{j<i} \vec{F}_{ij} \cdot \vec{r}_{ij}$$

- For central force deriving from a pair potential $U(r)$

$$\vec{F}_{ij} = -\frac{\partial}{\partial \vec{r}_i} U(r_{ij}) = -\frac{\vec{r}_{ij}}{r_{ij}} \left.\frac{dU}{dr}\right|_{r=r_{ij}}$$

- so that

$$\sum_i \vec{F}_i \cdot \vec{r}_i = -\sum_i \sum_{j<i} \left.\left(r \frac{dU(r)}{dr} \right)\right|_{r=r_{ij}}$$

- For power-law potential $U(r) \propto r^{\alpha}$

$$r\frac{dU(r)}{dr} = \alpha U(r)$$

- Then

$$\sum_i \vec{F}_i \cdot \vec{r}_i = -\alpha \sum_i \sum_{j<i} U(r_{ij}) = -\alpha V$$

- So that finally

$$\langle T \rangle = \frac{\alpha}{2}\langle V \rangle$$