ACM 100c

Linear independence

Dan Meiron

Caltech

January 2, 2014

Linear independence

- There is a connection between the results above and the concept of linear independence of vectors in linear algebra.
- Only here, the results extend to functions.

Definition

Two functions $f(x)$ and $g(x)$ are said to be *linearly dependent* on an interval $\alpha < x < \beta$ if there exist two nonzero constants c_1 and c_2 such that

$$c_1 f(x) + c_2 g(x) = 0 \qquad \text{for all } x \text{ in } \alpha < x < \beta.$$

Otherwise the functions are said to be *linearly independent*.

- As an example the functions $\sin(x)$ and $\cos(x + \pi/2)$ are linearly dependent because

$$\sin(x) + \cos(x + \pi/2) = 0 \qquad \text{for all } x$$

- On the other hand, the functions $\exp(x)$ and $\exp(2x)$ are linearly independent.

Linear independence and the Wronskian

- The notion of linear independence is also connected to the solutions of ODEs and the Wronskian.
- The result is that

Theorem

If the functions $p(x)$ and $q(x)$ are continuous in $\alpha < x < \beta$ and if $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the ODE

$$y'' + p(x)y' + q(x)y = 0$$

then the Wronskian $W(y_1, y_2) \neq 0$ in the interval and so the two solutions form a fundamental set.

Linear independence and the Wronskian

- We can prove this by contradiction.
- Suppose $W(y_1, y_2) = 0$ at some point x_0 in the interval $\alpha < x < \beta$.
- But keep the assumption that y_1 and y_2 are linearly independent.
- Now we said the Wronskian vanishes at the point x_0.
- That means that the 2×2 system

$$\begin{aligned} c_1 y_1(x_0) + c_2 y_2(x_0) &= 0 \\ c_1 y_1'(x_0) + c_2 y_2'(x_0) &= 0 \end{aligned}$$

has a nontrivial solution.

Linear independence and the Wronskian

- Take such a solution for the constants c_1 and c_2.
- Use them to construct a solution

$$\phi(x) = c_1 y_1(x) + c_2 y_2(x).$$

- But that solution $\phi(x)$ satisfies the IVP

$$\phi(x_0) = 0, \qquad \phi'(x_0) = 0.$$

- But $\phi(x) = 0$ also satisfies that IVP.
- So by the uniqueness theorem it must be that

$$\phi(x) = c_1 y_1(x) + c_2 y_2(x) = 0,$$

- But that means y_1 and y_2 are linearly *dependent*.
- But that contradicts our original assumption.

Linear independence and the Wronskian

- The converse result to the theorem also holds.
- Suppose $W(y_1, y_2) \neq 0$ for any two solutions.
- Then they are linearly independent.
- As long as the coefficient functions are smooth we can think of linearly independent solutions as basis vectors.
- These basis vectors can be used to construct the solution to any IVP.
- This too is true for n'th order ODEs except in that case there are n such vectors.