Physics 106a — Classical Mechanics

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Lecture 6

Equilibria and Small Oscillations

Outline

- Lagrangian Approach
  - Equilibria
  - Dynamics near an equilibrium
  - Simple harmonic motion
    - Undriven
    - Driven
- Add Dissipation

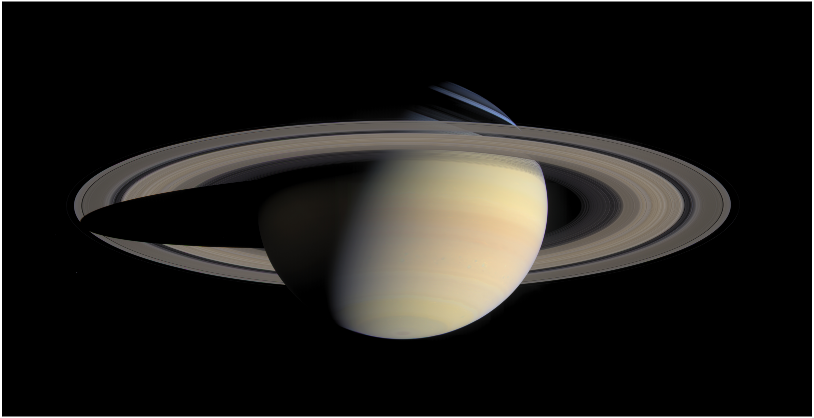
Solution strategy

Look for simple solutions

- Time independent solution in some coordinate system (equilibrium, fixed point)
- Behavior near equilibrium (linearize equations about equilibrium point)
  - Unstable equilibrium: deviation from equilibrium grows exponentially
  - Stable equilibrium: deviation from equilibrium remains small (may decay)
    - Study small oscillations near stable equilibrium (simple harmonic motion)
- Study change in behavior of equilibria as parameters change (bifurcation theory)

Resonance

- Tacoma Narrows bridge [Movie, Article: Am. J. Phys. 59, 118 (1991)]
- Rings of Saturn



Damped simple harmonic oscillator

Mechanical systems usually have some dissipation or damping. A common form (*not* valid for solid on solid friction) is a force proportional to the negative of the velocity. This leads to the equation of motion

$$\ddot{q} + \frac{1}{Q}\dot{q} + q = 0$$

with Q^{-1} proportional to the dissipation coefficient.

Q is the *quality factor* of the oscillator

- large Q — high quality, small dissipation
- small Q — low quality, high dissipation

Damped simple harmonic oscillator

$$\ddot{q} + \frac{1}{Q}\dot{q} + q = 0$$

The general solution is

$$q(t) = A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t}$$

with $\lambda_\pm$ the roots of the polynomial $\lambda^2 + Q^{-1}\lambda + 1 = 0$

$$\lambda_\pm = -\frac{1}{2Q} \pm \sqrt{\frac{1}{4Q^2} - 1}.$$

The behavior depends on the sign inside the $\surd$

$Q > \frac{1}{2}$	Underdamped	$q(t) = Ae^{-\frac{t}{2Q}} \cos(\omega' t + \phi)$ with $\omega' = \sqrt{1 - \frac{1}{4Q^2}}$
$Q < \frac{1}{2}$	Overdamped	$q(t) = A_+ e^{\lambda_+ t} + A_- e^{\lambda_- t}$ with $\lambda_+, \lambda_- < 0$
$Q = \frac{1}{2}$	Critically damped	$q(t) = (A + Bt)e^{-t}$ (special case)

Damped simple harmonic oscillator

Energy $E = \frac{1}{2}(q^2 + \dot{q}^2)$ decays as

$$\dot{E} = \dot{q}(\ddot{q} + q) = -\frac{1}{Q}\dot{q}^2 < 0$$

For large Q oscillatory motion, the energy decays little in one cycle, and averaging over one cycle gives

$$\langle V \rangle = \langle \tfrac{1}{2}q^2 \rangle \simeq \langle T \rangle = \langle \tfrac{1}{2}\dot{q}^2 \rangle \simeq \tfrac{1}{2}\langle E \rangle$$

and so the average energy decays as

$$\langle \dot{E} \rangle \simeq -\frac{\langle E \rangle}{Q}$$

- exponential decay with a $1/e$ time of Q (in units of ω_0^{-1})
- $\simeq Q/2\pi$ oscillations in this decay time
- fractional energy loss per period (cycle) is $\simeq 2\pi/Q$.

Damped simple harmonic oscillator

Sometimes, this type of dissipation force proportional to velocity is shoe-horned into a Lagrangian type approach by introducing a dissipation function $\mathcal{D}(\{\dot{q}_k\})$ such that the equations of motion are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} + \frac{\partial \mathcal{D}}{\partial \dot{q}_k} = 0$$

For the damped simple harmonic oscillator the dissipation function (using the scaled time) would be

$$\mathcal{D} = \tfrac{1}{2} Q^{-1} \dot{q}^2$$

The rate of energy dissipation is $2\mathcal{D}$

Driven, damped simple harmonic oscillator

Oscillatory forcing

$$\ddot{q} + \frac{1}{Q}\dot{q} + q = F(t) \quad \text{with} \quad F(t) = \cos\omega t = \mathrm{Re}[e^{i\omega t}]$$

- Calculate the response $q_c(t)$ to the forcing $F_c(t) = e^{i\omega t}$, and take the real part at the end.
- For $F_c(t)$ the steady state solution is

$$q_c(t) = A_c e^{i\omega t} = A e^{i(\omega t + \phi)}$$

with

$$A_c = \frac{1}{(-\omega^2 + 1) + \frac{i\omega}{Q}}$$

- This gives the solution for $F(t) = \cos\omega t$ as $q(t) = A\cos(\omega t + \phi)$.

Driven, damped simple harmonic oscillator

The amplitude is given by

$$A^2 = \frac{1}{(\omega^2 - 1)^2 + \omega^2/Q^2} \simeq \frac{1}{4}\frac{1}{(\omega - 1)^2 + 1/(4Q^2)}$$

For large Q the response is a sharp *resonance peak* centered at $\omega_r \simeq 1$, with full width at half intensity (half height in A^2) $1/Q$ and amplitude at resonance Q larger than the static response

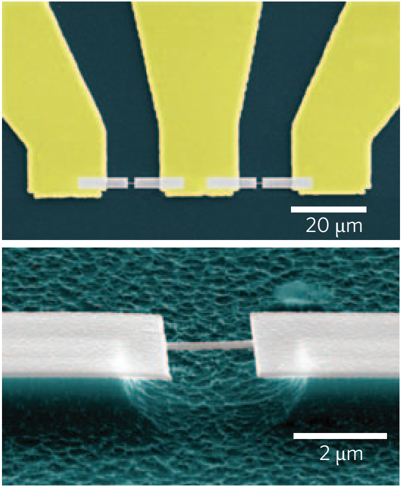
The phase is

$$\phi = -\tan^{-1}\left(\frac{\omega/Q}{1 - \omega^2}\right)$$

and varies from slightly negative for low frequencies, passing rapidly through $-\pi/2$ on resonance, and decreasing to $-\pi$ for large frequencies.

Single molecule mass sensor

Roukes group, Caltech, Nature Nanotechnology 4, 445, (2009)



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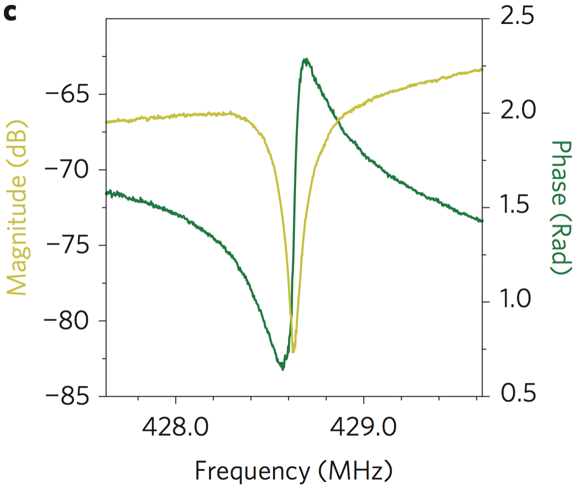

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