PART II: LINEAR EQUATIONS

15. Basic concepts

15.1. **Linear equations.** The standard form of a second order linear equation is

$$L[x] \equiv \ddot{x} + p(t)\dot{x} + q(t)x = g(t).$$

The map $x \mapsto L[x]$ is a *differential operation.* If $g(t) \equiv 0$, then the equation

$$L[x] = 0$$

is called *homogenous.* The operation L has *constant coefficients* if $p(t)$ and $q(t)$ are constant functions.

The standard form of an n-th order equation is $L[x] = g(t)$ with

$$L[x] = x^{(n)} + p_1(t)x^{(n-1)} + \cdots + p_0(t)x.$$

15.2. **Initial value problem.** To solve the IVP$(t_0, x_0, \dot{x}_0)$ for a second order equation (not necessarily linear) is to find a solution $x(t)$ such that

$$x(t_0) = x_0, \quad \dot{x}(t_0) = \dot{x}_0.$$

In the case of n-th order equations, the initial conditions have the form

$$x(t_0) = x_0, \quad x'(t_0) = x_0', \quad \ldots, \quad x^{(n-1)}(t_0) = x_0^{(n-1)}.$$

Theorem. *Consider a linear equation $L[x] = g$. Suppose that g and the coefficients of L are continuous functions in some interval I. Then*

(i) all solutions extend to the whole interval,

(ii) every IVP with $t_0 \in I$ has a unique solution defined on I.

15.3. **Linear properties.** If the coefficients of L are in $C(I)$, then we have a linear map

$$L : C^{(n)}(I) \to C(I).$$

Denote

$$\ker L = \{x : \ L[x] = 0\}.$$

This is the set of all solutions of the homogeneous equation.

Lemma. $\ker L$ *is a linear space of dimension n.*

Proof: Fix some $t_0 \in I$ and consider the map

$$\mathbb{R}^n \to \ker L, \qquad \left(x_0, x_0', \ldots, x_0^{(n-1)}\right) \mapsto x(t),$$

where $x(t)$ is the solution of the corresponding IVP. Applying the theorem we see that this map is linear, 1-to-1, and "onto". □

Recall that n elements (or vectors) $x_1, \ldots, x_n$ of a linear space are (linearly) independent if

$$C_1x_1 + \cdots + C_nx_n = 0 \qquad \Rightarrow \qquad C_1 = \cdots = C_n = 0.$$

If the linear space has dimension n, then n independent vectors form a basis: every vector x has a unique representation as a linear combination of x_j's. Any $n+1$ vectors of an n dimensional space are dependent.

Let $x_1, \ldots, x_n$ be solutions of an n-th order homogeneous equation $L[x] = 0$. Then the collection $x_1, \ldots, x_n$ is called a *fundamental system of solutions* if the functions x_j are independent as elements of $\ker L$. It follows that every solution of $L[x] = 0$ has a unique representation as a linear combination

$$x(t) = C_1 x_1(t) + \cdots + C_n x_n(t).$$

Example. The functions $1, t, \ldots, t^{n-1}$ form a fundamental system of solutions of the equation $x^{(n)} = 0$. The *general solution* ($\equiv$ the set of all solutions) is the space of polynomials of degree $< n$.

Lemma. *If x_* is a solution of $L[x] = g$, then the general solution of $L[x] = g$ is*

$$x \in x_* + \ker L,$$

(or $x = x_* + C_1 x_1 + \cdots + C_n x_n$, "particular integral" + "complementary function".)

15.4. **Reduction of order.** The following substitution is quite useful.

Theorem. *If we know some solution x_1 of a homogeneous equation $L[x] = 0$ of order n, then the substitution $x = x_1 y$, where $y = y(t)$ is a new unknown function, transforms the equation $L[x] = g$ into a linear equation of order $n-1$ for y'.*

Proof: Consider the case $n = 3$,

$$L[x] \equiv x''' + p_1 x'' + p_2 x' + p_3 x = g,$$

(the coefficients are not necessarily constant). We have

$$\begin{aligned} L[x] = {} & x_1''' y + 3x_1'' y' + 3x_1' y'' + x_1 y''' \\ & + p_1 x_1'' y + 2p_1 x_1' y' + p_1 x_1 y'' \\ & + p_2 x_1' y + p_2 x_1 y' \\ & + p_3 x_1 y = g. \end{aligned}$$

The sum of the first terms is $yL[x_1] = 0$; the other terms do not involve y. □

16. Wronskian

16.1. **Definition.** The Wronskian of two functions $y_1(x), y_2(x) \in C^1(I)$ is the function

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \in C(I).$$

The Wronskian of n functions $y_j \in C^{(n-1)}$ is defined similarly, e.g.

$$W = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} \in C(I).$$

Let us consider a linear homogeneous equation with *continuous* coefficients,

$$L[y] \equiv y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_n(x) y = 0, \qquad p_j \in C(I).$$

Theorem. *Let $y_1, \dots, y_n$ be solutions of $L[y] = 0$. FAE:*

(i) $\{y_1, \dots, y_n\}$ is a fundamental system of solutions;

(ii) $\forall x \in I,\ W(x) \neq 0$;

(iii) $\exists x_0 \in I,\ W(x_0) \neq 0$.

Proof: As we know, the map

$$\ker L \to \mathbb{R}^n : \quad y \mapsto (y(x_0), \dots, y^{(n-1)}(x_0))$$

is a linear isomorphism for every $x_0 \in I$.

(i)⇒(ii) If the solutions y_j are independent, then the vectors of initial conditions are independent in $\mathbb{R}^n$, and therefore $W(x_0) \neq 0$ for all x_0.

(iii)⇒(i) If $W(x_0) \neq 0$, then the initial vectors are independent, and so are the solutions. □

The most interesting part of this theorem is the equivalence of (ii) and (iii).

Examples. (a) The Wronskian of $y_1(x) = x$ and $y_2(x) = \sin x$ is zero at $x = 0$ but not identically. The Wronskian of $y_1(x) = x^3$ and $y_2(x) = x^2|x|$ is identically zero but the functions are independent.

Conclusion: these pairs of functions can not be solutions of the same second order linear equation with continuous coefficients.

(b) The Wronskian of exponential functions $e^{\lambda_j x}$ evaluated at 0 is the so called *Vandermonde* determinant

$$W(0) = \begin{vmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_n \\ \dots & \dots & \dots & \dots \\ \lambda_1^{n-1} & \lambda_2^{n-1} & \dots & \lambda_n^{n-1} \end{vmatrix} = \prod_{j<k} (\lambda_j - \lambda_k).$$

16.2. **Abel's formula.** We don't need to solve the differential equation to compute the Wronskian of its fundamental system of solutions.

Theorem. *Let $W(x)$ be the Wronskian of some fundamental system of solutions of $L[y] = 0$. Then*

$$W(x) = W(x_0) \exp\left\{ -\int_{x_0}^{x} p_1(s) ds \right\}.$$

Proof: We will prove the statement for second order equations

$$y'' + p(x)y' + q(x)y = 0. \tag{16.1}$$

It is sufficient to show that the Wronskian satisfies the equation

$$W' = -p(x)W.$$

We have

$$\begin{aligned} W' &= \begin{vmatrix} y_1' & y_2' \\ y_1' & y_2' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ y_1'' & y_2'' \end{vmatrix} \\ &= 0 + \begin{vmatrix} y_1 & y_2 \\ -py_1' & -py_2' \end{vmatrix} + \begin{vmatrix} y_1 & y_2 \\ -qy_1 & -qy_2 \end{vmatrix} \\ &= -p \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2'. \end{vmatrix} = -pW. \end{aligned}$$

□

Corollary. *Let* y_1 *be a (known) solution of the 2nd order equation* (16.1)*. Define*

$$W(x) = \exp\left\{-\int_{x_0}^{x} p(s)ds\right\}.$$

Then

$$y_2 = y_1 \int \frac{W}{y_1^2} \tag{16.2}$$

is also a solution, and y_1*,* y_2 *are independent.*

Proof: According to the above theorem, there is an independent solution y_2 such that W is the Wronskian of y_1, y_2. To find y_2 consider the formula

$$y_1 y_2' - y_1' y_2 = W$$

as a first order linear equation for $y_2(x)$; the coefficients y_1, y_1', and W are known functions. We have

$$\left(\frac{y_2}{y_1}\right)' = \frac{y_1 y_2' - y_1' y_2}{y_1^2} = \frac{W}{y_1^2},$$

which gives us (16.2) □

Exercise: Strictly speaking, the formula (16.2) makes sense only on the intervals where $y_1 \neq 0$. Interpret the formula in the case where y_1 has zeros. E.g. $y''+y=0$, $y_1(t) = \sin t$.

Example. Consider the equation $x^2(1+x)y'' - 2y = 0$. One can verify that $y_1(x) = 1 + x^{-1}$ is a solution. Find the general solution and determine the domain of the maximal solution of the IVP$(-2; 0, 1)$.

According to the general theory of linear equations, the IVP solution extends at least to $(-\infty, -1)$ (but in principle it may exist on a larger interval.) Since the Wronskian is constant, we find

$$y_2 = y_1 \int \frac{1}{y_1^2} = y_1(x) \int \frac{x^2 dx}{(1+x)^2} = 1 + x - \frac{1}{x} - \left(1 + \frac{1}{x}\right)\log(1+x)^2.$$

The general solution is $y = C_1 y_1 + C_2 y_2$, and we have $C_2 \neq 0$ for the IVP solution. The IVP solution is not differentiable at $x = -1$, so its domain is $(-\infty, -1)$.

17. Homogeneous equations with constant coefficients

$$L[x] \equiv a_0 x^{(n)} + \cdots + a_n x = 0, \qquad a_j \in \mathbb{R}.$$

17.1. **Characteristic polynomial.** Denote

$$P(\lambda) = a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_n,$$

so (symbolically)

$$L = P(D), \qquad D = \frac{d}{dt}.$$

The polynomial P has exactly n roots ("eigenvalues")

$$\lambda_1, \ \ldots, \ \lambda_n, \qquad P(\lambda_j) = 0,$$

which may be complex and/or multiple, and we have

$$P(\lambda) = a_0(\lambda - \lambda_1)\ldots(\lambda - \lambda_n).$$

Since the characteristic polynomial P has real coefficients, complex roots appear in *conjugate pairs*, i.e. if $\lambda = \alpha + i\omega$ is an eigenvalue, then $\lambda = \alpha - i\omega$ is also an eigenvalue.

Lemma. $L[e^{\lambda t}] = P(\lambda)e^{\lambda t}$.

Proof: If $x(t) = e^{\lambda t}$, then $x^{(k)}(t) = \lambda^k e^{\lambda t}$. □

Corollary. *A real number λ is a root of the characteristic polynomial if and only if $x(t) = e^{\lambda t}$ is a solution of the homogeneous equation.*

17.2. **The case of distinct real roots.**

Theorem. *If the roots of the characteristic polynomial are all real and distinct, then the general solution of the homogeneous equation is*

$$x(t) = C_1 e^{\lambda_1 t} + \cdots + C_n e^{\lambda_n t}.$$

Proof: The exponential functions with distinct exponents are linearly independent. Indeed, suppose

$$C_1 e^{\lambda_1 t} + \cdots + C_n e^{\lambda_n t} \equiv 0$$

and $\lambda_1 < \cdots < \lambda_n$. If $C_n \neq 0$, then the LHS divided by $C_n e^{\lambda_n t}$ converges to 1 as $t \to \infty$, a contradiction. If $C_n = 0$, we repeat this argument with C_{n-1} instead of C_n, etc. □

17.3. **Second order equations.** There are 3 possible cases:

(i) the roots λ_1, λ_2 are real and distinct;

(ii) $\lambda_1 = \lambda_2 \in \mathbb{R}$;

(iii) $\lambda_{1,2} = \alpha \pm i\omega$ with $\omega \neq 0$.

Theorem. *The general solution in the above three cases is*

$$\begin{aligned} &(i) \quad x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}; \\ &(ii) \quad x(t) = C_1 e^{\lambda_1 t} + C_2 t e^{\lambda_1 t}; \\ &(iii) \quad x(t) = C_1 e^{\alpha t} \cos \omega t + C_2 e^{\alpha t} \sin \omega t. \end{aligned}$$

Proof: (i) is clear.
(ii) is the limiting case of (i) as $\lambda_2 \to \lambda_1$:

$$\frac{e^{\lambda_2 t} - e^{\lambda_1 t}}{\lambda_2 - \lambda_1} \to \frac{d}{d\lambda}\bigg|_{\lambda=\lambda_1} e^{\lambda t} = t e^{\lambda_1 t}.$$

Alternatively, we can apply the reduction of order method. The equation is $x'' - 2\lambda_1 x' + \lambda_1^2 x = 0$, and $x_1 = e^{\lambda_1 t}$ is a solution. Substitution $x = x_1 y$ gives the equation $y'' = 0$, so we can take $y(t) = t$ and get the second solution $x_2 = t e^{\lambda_1 t}$.
(iii) easy to verify; also see the next section. □

17.4. **Difference equations.** *Fibonacci numbers*

$$0\ 1\ 1\ 2\ 3\ 5\ 8\ \ldots$$

satisfy the second order linear difference (or recursive) equation

$$x_n = x_{n-1} + x_{n-2}.$$

(Fibonacci's interpretation: x_n is the population of rabbits in the n-th generation.) The linear equation is homogeneous. Let us look for solutions of the form

$$x_n = a^n,$$

(discrete exponentials). Then a has to satisfy the equation

$$a^n = a^{n-1} + a^{n-2}, \qquad a^2 - a - 1 = 0.$$

We have two roots

$$a_{1,2} = \frac{1 \pm \sqrt{5}}{2},$$

(a_1 is the "*golden ratio*", see Problem 22.3). The general solution to the difference equation is

$$x_n = C_1 a_1^n + C_2 a_2^n.$$

From the initial conditions

$$x_0 = 0, \quad x_1 = 1,$$

we find the values

$$C_1 = -C_2 = \frac{1}{\sqrt{5}}.$$

This method works for all linear homogeneous recursive equations with constant coefficients. See Examples 22.1 and 22.2 for the cases of repeated and complex roots.

18. Complex eigenvalues

18.1. **Complex exponentials.** A complex-valued function of one real variable,

$$z(t) = x(t) + iy(t), \qquad t \in \mathbb{R},$$

(the functions $x(t)$ and $y(t)$ are real-valued) is differentiable if both $x(t)$ and $y(t)$ are differentiable, and in this case

$$z' = x' + iy'.$$

Example. Let $\lambda = \alpha + i\beta$. By definition,

$$e^{\lambda t} = e^{\alpha t}\cos(\omega t) + ie^{\alpha t}\sin(\omega t), \tag{18.1}$$

and it is easy to check that

$$\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}.$$

A more conceptual approach to the formula (18.1) is to introduce the *complex exponential function*

$$e^{\lambda} = \sum_{n\geq 0}\frac{\lambda^n}{n!}, \qquad \lambda \in \mathbb{C}, \tag{18.2}$$

If $\lambda = \alpha$ is real, then we get the usual (real) exponential e^{α}, and if $\lambda = i\omega$ is imaginary, then

$$e^{i\omega} = \cos\omega + i\sin\omega$$

because the real and imaginary parts of the series (18.2) are Taylor's series of cosine and sine functions respectively. The series (18.2) converges in the whole complex plane, and one can verify (by multiplication of the series) the functional equation

$$e^{\lambda_1+\lambda_2} = e^{\lambda_1}e^{\lambda_2}.$$

In particular, for $\lambda = \alpha + i\omega$ we have

$$e^{\lambda t} = e^{\alpha t}e^{i\omega t} = e^{\alpha t}\cos\omega t + ie^{\alpha t}\sin\omega t,$$

which is the same as (18.1).

18.2. **Complex solutions.** Consider the equation

$$L[x] \equiv P(D)[z] = 0$$

for complex-valued functions $z = z(t)$. Similarly to the real case, we have

$$\forall\lambda \in \mathbb{C}, \qquad L[e^{\lambda t}] = P(\lambda)e^{\lambda t},$$

which proves

Lemma. *$z(t) = e^{\lambda t}$ is a solution iff $P(\lambda) = 0$.*

We also have the following statement concerning linear equations $L[z] = 0$ with *real* coefficients.

Lemma. *$z(t) = z(t) + iy(t)$ is a (complex) solution iff both $x(t)$ and $y(t)$ are (real) solutions.*

Proof: Since the coefficients of the equation are real, we have $L[z] = L[x] + iL[y]$, so $L[z] = 0$ iff $L[x] = 0$ and $L[y] = 0$. □

Combining the two lemmas, we derive

Theorem. *Consider the equation $L[x] = 0$ with constant real coefficients, and suppose that all roots of the characteristic polynomial are distinct. Let $\lambda_1, \dots$ be the real roots, and let $\alpha_1 \pm i\beta_1, \dots$ be the complex roots. Then the collection of functions*

$$e^{\lambda_1 t},\ \dots;\quad e^{\alpha_1 t}\cos(\beta_1 t),\ e^{\alpha_1 t}\sin(\beta_1 t),\ \dots \tag{18.3}$$

is a fundamental system of (real-valued) solutions.

Note. In the case of distinct eigenvalues $\lambda_1, \dots, \lambda_n$ (real or complex), the general *complex-valued* solution is

$$z(t) = \sum_{j=1}^{n} C_j e^{\lambda_j t},$$

where C_j's are arbitrary complex numbers.

18.3. **General homogeneous equations with constant coefficients.** Consider the equation $L[x] = 0$ with real constant coefficients. If λ_j is a repeated *real* root of multiplicity k,

$$P(\lambda) = (\lambda - \lambda_j)^k \tilde{P}(\lambda), \qquad \tilde{P}(\lambda_j) \neq 0,$$

then we add the following functions to the list (18.3) of the last theorem:

$$te^{\lambda_j t},\ \dots,\ t^{k-1}e^{\lambda_j t}.$$

If $\alpha_j \pm i\beta_j$ is the conjugate pair of *complex* roots of multiplicity k, then we add the functions

$$te^{\alpha_j t}\cos(\beta_j t),\ te^{\alpha_j t}\sin(\beta_j t),\ \dots,\ t^{k-1}e^{\alpha_j t}\cos(\beta_j t),\ t^{k-1}e^{\alpha_j t}\sin(\beta_j t).$$

Using reduction of order, it is not difficult to justify the following statement.

Theorem. *The resulting collection of functions is a fundamental system of solutions of the equation $L[x] = 0$.*

Example. Let $P(\lambda) = \lambda^3(\lambda^2 + 1)^2$. The roots are

$$0,\ 0,\ 0,\ i,\ -i,\ i,\ -i.$$

We have the following fundamental system of solutions:

$$1,\ t,\ t^2,\ \cos t,\ \sin t,\ t\cos t,\ t\sin t.$$

The general solution of the equation $x^{(7)} + 2x^{(5)} + x^{(3)} = 0$ is therefore

$$x(t) = C_1 + C_2 t + C_3 t^2 + C_4 \cos t + C_5 \sin t + C_6 t\cos t + C_7 t\sin t.$$

19. Inhomogeneous linear equations

We will discuss here how to solve equations

$$L[x] = g.$$

We have two general methods: reduction of order (Section 15.4) and variation of constants. These methods work for all linear equations, not necessarily with constant coefficients. The third method, the method of undetermined coefficients, is often the most efficient but it applies only to equations with constant coefficients and to functions g of special type (quasi polynomials). Also, see Section *** below for the discussion of the important Laplace method.

19.1. **Variation of constants.**

We will show that if we know the general solution of the homogeneous equation $L[x] = 0$, then we can solve any inhomogeneous equation $L[x] = g$. For simplicity, we will only discuss the case of 2d order equations:

$$L[x] = x'' + p(t)x' + q(t)x.$$

Let x_1 and x_2 be two independent solutions of $L[x] = 0$. We look for a particular solution of the inhomogeneous equation of the form

$$x_* = C_1(t)x_1 + C_2(t)x_2.$$

We have

$$x_*' = (C_1x_1' + C_2x_2') + (C_1'x_1 + C_2'x_2),$$

and if the functions C_1' and C_2' are such that

$$C_1'x_1 + C_2'x_2 \equiv 0,$$

then

$$x_*'' = (C_1x_1'' + C_2x_2'') + (C_1'x_1' + C_2'x_2'),$$

so

$$L[x_*] = C_1'x_1' + C_2'x_2'.$$

Thus x_* is a particular solution if C_1' and C_2' satisfy the system of linear algebraic equations

$$C_1'x_1 + C_2'x_2 = 0, \qquad C_1'x_1' + C_2'x_2' = g(t).$$

Solving this system:

$$C_1' = \frac{\begin{vmatrix} 0 & x_2 \\ g & x_2' \end{vmatrix}}{W} = -\frac{gx_2}{W}, \qquad C_1' = \frac{\begin{vmatrix} x_1 & 0 \\ x_1' & g \end{vmatrix}}{W} = \frac{gx_1}{W}, \qquad W := \begin{vmatrix} x_1 & x_2 \\ x_1' & x_2' \end{vmatrix},$$

we derive

$$x_* = -x_1 \int \frac{gx_2}{W} + x_2 \int \frac{gx_1}{W}.$$

19.2. **Undetermined coefficients.** We will now assume that $L[\cdot]$ has constant coefficients and g is a quasi polynomial.

By definition, a *quasi polynomial* is a linear combination of the functions of the following form:

$$t^m e^{\mu t}, \qquad t^m e^{\alpha t}\cos(\omega t), \qquad t^m e^{\alpha t}\sin(\omega t). \tag{19.1}$$

Here m is a non-negative integer, and μ, α, ω are real. We will write $\mu = \alpha + i\omega$ in the last two cases in (19.1).

We will show that the equation $L[x] = g$ has a quasi polynomial solution. Because of the following (obvious) lemma, we can consider the functions (19.1) separately.

Lemma. *If $g = g_1 + g_2$, and if x_1 and x_2 are solutions of $L[x] = g_1$ and $L[x] = g_2$ respectively, then $x = x_1 + x_2$ is a solution of $L[x] = g$.*

Let g be one of the functions (19.1), and let k denote the multiplicity of μ as a root of the characteristic polynomial; e.g. $k = 0$ if μ is not an eigenvalue.

Theorem. *(i) If μ is real (i.e. $g = t^m e^{\mu t}$), then there is a solution of the form*

$$x_*(t) = t^k \text{ (some polynomial of degree } m\text{) } e^{\mu t}.$$

(ii) If μ is complex, then there is a solution of the form

$$\begin{aligned} x_*(t) &= t^k \text{ (some polynomial of degree } m\text{) } e^{\alpha t}\cos(\omega t) \\ &+ t^k \text{ (some other polynomial of degree } m\text{) } e^{\alpha t}\sin(\omega t). \end{aligned}$$

Proof: We'll only discuss the simpler case $k = 0$, i.e. $P(\mu) \neq 0$. If μ is real, then we denote

$$E_{\mu,m} = \{\text{real polynomials of degree} \le m\} \cdot e^{\mu t}.$$

This is a linear space (over $\mathbb{R}$) of dimension $m+1$. It is enough to show that the linear map $L:\ x \mapsto L[x]$ acts and is invertible in $E_{\mu,m}$. Let us compute the matrix of

$$D \equiv \frac{d}{dt}:\ E_{\mu,m} \to E_{\mu,m}$$

in the basis

$$e_j = t^j e^{\mu t}, \qquad (0 \le j \le m).$$

Since

$$De_j = jt^{j-1}e^{\mu t} + \mu t^j e^{\mu t} = je_{j-1} + \mu e_j,$$

we have

$$D = \begin{pmatrix} \mu & 1 & 0 & \dots \\ 0 & \mu & 2 & \dots \\ 0 & 0 & \mu & \dots \\ . & . & . & \dots \end{pmatrix}.$$

It follows that

$$L \equiv P(D) = \begin{pmatrix} P(\mu) & * & * & \dots \\ 0 & P(\mu) & * & \dots \\ 0 & 0 & P(\mu) & \dots \\ . & . & . & \dots \end{pmatrix}.$$

Since $P(\mu) \neq 0$, the eigenvalues of the last matrix are non-zero, and the operator $L : E_{\mu,m} \to E_{\mu,m}$ is invertible. If μ is complex, we consider complex polynomials in the definition of $E_{\mu,m}$ and use the same argument. □

19.3. **Examples.**

(i) $x'' + x = t^2$.

We have $\mu = 0$, which is not a root of the characteristic polynomial $\lambda^2 + 1$, so $k = 0$. There is a solution of the form

$$x(t) = At^2 + Bt + C.$$

Let's find the (undetermined) coefficients. We need to satisfy

$$x'' + x = At^2 + Bt + (2A + C) = t^2,$$

so

$$A = 1, \quad B = 0, \quad C = -2A = -2.$$

Thus we get a particular solution $x(t) = t^2 - 2$. The general solution is then

$$x(t) = C_1 \cos t + C_2 \sin t + t^2 - 2.$$

(ii) $x'' + x = e^{2t}$.

We have $\mu = 2$, which is not a root of the characteristic polynomial, so $k = 0$. There is a solution of the form $x(t) = Ae^{2t}$. We find $A = 1/5$.

(iii) $x'' + x = te^{-t}$.

There is a solution of the form $x(t) = (At + B)e^{-t}$.

(iv) $x'' + x = t^3 \sin t$.

Now $\alpha = 0$, $\omega = 1$, and $\mu = i$, which is a root of multiplicity $k = 1$. There is a solution of the form

$$x(t) = (At^4 + Bt^3 + Ct^2 + Dt) \cdot \sin t + (\tilde{A}t^4 + \tilde{B}t^3 + \tilde{C}t^2 + \tilde{D}t) \cdot \cos t.$$

(v) $x^{(4)} + 4x'' = \sin 2t + te^t + 4$.

The characteristic polynomial is

$$P(\lambda) = \lambda^4 + 4\lambda^2 = \lambda^2(\lambda^2 + 4), \qquad \lambda_{1,2} = 0,\ \lambda_3 = 2i,\ \lambda_4 = -2i.$$

We have

$$\begin{aligned} g_1 = \sin 2t \quad &\Rightarrow \quad x_1 = t(A \sin 2t + B \cos 2t); \\ g_2 = te^t \quad &\Rightarrow \quad x_2 = e^t(Ct + D); \\ g_1 = 4 \quad &\Rightarrow \quad x_3 = Et^2, \end{aligned}$$

and so there is a solution of the form

$$x(t) = At \sin 2t + Bt \cos 2t + Cte^t + De^t + Et^2.$$

19.4. **Inhomogeneous linear difference equations.** Example: the sequence

$$1,\ 1,\ 6,\ 12,\ 29,\ 59,\ \ldots$$

is described by the IVP

$$x_n = x_{n-1} + 2x_{n-2} + n, \qquad x_1 = x_2 = 1.$$

The associated homogeneous equation is $x_n = x_{n-1} + 2x_{n-2}$, and if $x_n = a^n$ is its solution, then

$$a^2 = a + 2; \qquad a_1 = -1,\ a_2 = 2.$$

Since $n = n1^n$ and 1 is not a root of $a^2 = a+2$, we will look for a particular solution of the inhomogeneous equation of the form

$$x_n = An + B.$$

We have

$$An + B = An - A + B + 2An - 4A + 2B + n,$$

and therefore

$$A = A + 2A + 1, \qquad B = -A + B - 4A + 2B.$$

Thus $A = -1/2, \quad B = -5/4$. Conclusion: the general solution is

$$x_n = -(n+5)/2 + (-1)^n C_1 + 2^n C_2,$$

and it remains to determine C_1 and C_2 from the initial conditions.

20. Oscillations

In this section we discuss applications to mechanical and electrical vibrations. We will consider 2nd order equations

$$\ddot{x} + 2a\dot{x} + \omega_0^2 y = g(t) \tag{20.1}$$

with constant coefficients and *periodic* functions $g(t)$. It is remarkable that such simple equations have many meaningful applications.

20.1. **Interpretations.**

(a) *Harmonic oscillator*: the motion of a mass m on a Hooke's law spring with spring constant $k > 0$. Newton's equation for the elongation of the spring $x = x(t)$ is

$$m\ddot{x} + \gamma\dot{x} + kx = g(t),$$

where γ is the damping coefficient, and $g(t)$ is the external force.

(b) *Electrical circuits.* An RLC circuit consists of a *resistor* with resistance R (ohms), an *inductor* with inductance L (henrys), and a *capacitor* with capacitance C (farads).

The resistor converts electrical energy into heat or light, e.g. a light bulb element. The inductor has a special geometry such as coils which creates a magnetic field that induces a voltage drop. A typical capacitor consists of two plates separated by an insulator; charges of opposite signs will build up on the two plates.

The impressed *voltage* $E(t)$ is a given function of time (seconds). We consider the following characteristics of the circuit: the *charge* $Q(t)$ (coulombs) on the capacitor, and the *current* $I(t) = \dot{Q}(t)$ (amperes). By Kirchhoff,

$$E(t) = L\dot{I} + RI + C^{-1}Q.$$

($E(t)$ is equal to the sum of voltage drops across R, L, and C; the respective drops are RI by Ohm's law, $C^{-1}Q$ by Coulomb's law, and $L\dot{I}$ by Faraday's law.) Thus we get equations similar to the mass-spring model:

$$E(t) = L\ddot{Q} + R\dot{Q} + C^{-1}Q,$$

or

$$\dot{E}(t) = L\ddot{I} + R\dot{I} + C^{-1}I,$$

where R is responsible for damping.

20.2. **Free vibrations: $g = 0$ in (20.1).**

(a) *Undamped case:* $a = 0$. The general solution is

$$x(t) = C_1 \cos\omega_0 t + C_2 \sin\omega_0 t = M\cos(\omega_0 t - \phi).$$

We have a periodic motion with *period* equal to $2\pi/\omega_0$; $\omega_0 >$ is the *natural frequency*, $M > 0$ is the *amplitude* of the motion (maximal displacement), and ϕ is the *phase* angle.

(b) *Damped case.* The eigenvalues are

$$\lambda_{1,2} = -a \pm \sqrt{a^2 - \omega_0^2}.$$

There are no oscillations in the *over-damped case* $a^2 \geq \omega_0^2$. Otherwise (*"under-damping"*), we have complex eigenvalues

$$\lambda_{1,2} = -a \pm i\mu,$$

and the motion is a *damped vibration*:

$$x(t) = Re^{-at}\cos(\mu t - \delta);$$

μ is called *quasi frequency* and $2\pi/\mu$ *quasi period.*

20.3. **Forced vibrations: undamped case.** Suppose the exterior force g in (20.1) is T-periodic,

$$g(t + T) \equiv g(t),$$

and denote $\omega = 2\pi/T > 0$, the *forcing frequency.* We will study the question of existence and uniqueness of periodic solutions and also the question of their stability. We only discuss the case

$$g(t) = A\cos(\omega t - \delta),$$

and refer to *Arnol'd* for general theory. Assume first that there is no damping, so the equation is

$$\ddot{x} + \omega_0^2 x = A\cos(\omega t - \delta).$$

Theorem. *(i) If the ratio ω_0/ω is not a rational number, then there exists a unique periodic solution. The period of the periodic solution is T.*

(ii) If ω_0/ω is a rational number but $\omega \neq \omega_0$, then all solutions are periodic.

(iii) If $\omega = \omega_0$, then every solution is unbounded and non-periodic

The case (iii) is called (pure) *resonance*: "bounded input, unbounded output"

Proof: For simplicity assume $\delta = 0$. If $\omega \neq \omega_0$, then the general solution is

$$x(t) = C_1 \cos \omega_0 t + C_1 \sin \omega_0 t + \frac{A}{\omega_0^2 - \omega^2} \cos \omega t,$$

(use the method of undetermined coefficients). The choice $C_1 = C_2 = 0$ gives a periodic solution. All other solutions are not periodic unless the frequencies are commensurable. In the resonance case, we have

$$x(t) = C_1 \cos \omega_0 t + C_1 \sin \omega_0 t + \frac{A}{2\omega_0} t \sin \omega_0 t,$$

(again, use undetermined coefficients). □

Comments.

(a) The results in the case of a general periodic exterior force are similar. In particular, if the frequency of the exterior force is equal to the natural frequency, then all solutions are unbounded. Soldiers used to break step when crossing a bridge to eliminate the periodic force of their marching that could resonate with a natural frequency of the bridge.

(b) *Beats.* In the non-resonance case, every solution is the sum of two periodic functions,

$$x(t) = A \cos(\omega_0 t + \alpha) - B \cos(\omega t + \beta).$$

The graphs of such functions could be quite intriguing. For example, consider the case where the amplitudes are equal, $A = B$, (or almost equal), and the frequencies are not very different in the sense that

$$|\omega - \omega_0| \ll |\omega + \omega_0|.$$

(Think of two flutes playing slightly out of tune with each other.) We have (for $\alpha = \beta = 0$, otherwise up to a phase shift)

$$x(t) = 2A \sin \frac{\omega - \omega_0}{2} t \;\; \sin \frac{\omega + \omega_0}{2} t.$$

The first term oscillates slowly compared to the second one. We can think of the function

$$\left| 2A \sin \frac{\omega - \omega_0}{2} t \right|$$

as a slowly changing amplitude of the vibrations with frequency $|\omega + \omega_0|/2$. We say that the solution exhibits the phenomenon of beats.

(c) Example of case (ii):

$$\ddot{x} + 36x = 3 \sin 4t.$$

Every solution is the sum of a $2\pi/6$-periodic function and a $2\pi/4$-periodic function, so every solution is π-periodic.

Relation to *Lissajous' curves*: consider the orbit of the motion $\{x(t), \dot{x}(t)\}$ in the "phase space" $\mathbb{R}^2 = \{(x, \dot{x})\}$. For example, draw the curve

$$x(t) = -\frac{1}{10}\sin 6t + \frac{3}{20}\sin 4t, \qquad \dot{x}(t) = -\frac{3}{5}\cos 6t + \frac{3}{5}\cos 4t,$$

which is the orbit of the IVP(0;0,0).

20.4. **Forced vibrations: damped case.**

$$\ddot{x} + 2a\dot{x} + \omega_0^2 x = A\cos(\omega t - \delta).$$

Theorem. *If $a > 0$, then there is a unique periodic solution $x_*(t)$. This solution has period $2\pi/\omega$ and is a steady state.*

The last sentence means that $x_*(t)$ is *globally asymptotically stable*: for any solution $x(t)$, we have

$$|x(t) - x_*(t)| \to 0 \quad \text{as} \quad t \to \infty$$

(convergence is exponentially fast). In other words, no matter what initial conditions are, we asymptotically get the same function $x_*(t)$. Note that there are no steady states in the undamped case – the effect of the initial conditions would persist at all times.

Proof: The eigenvalues are $\lambda_{1,2} = -a \pm \sqrt{a^2 - \omega_0^2}$. Clearly, $\Re\lambda_1 < 0$ and $\Re\lambda_2 < 0$, in particular $i\omega$ is not an eigenvalue. By the method of undetermined coefficients, there is a particular solution $x_*(t)$ of the form $B\cos(\omega t - \phi)$. On the other hand, every "complementary function" (solution to the associated homogeneous equation) is exponentially small as $t \to +\infty$, so the general solution is

$$x(t) = B\cos(\omega t - \phi) + o(1), \qquad t \to +\infty.$$

□

This theorem generalizes to arbitrary linear systems with constant coefficients such that *all eigenvalues have negative real parts.* If the non-homogeneous term is periodic, then there is a unique periodic solution, which is a steady state.

Terminology: the periodic force $g(t)$ is the *input*, and the steady state is the *output* or *forced response* of the system. The $o(1)$ part of the solution is the *transient solution.* The ratio of the amplitudes $G = B/A$ is the *gain.*

Exercise: show

$$G = \frac{1}{\sqrt{(\omega^2 - \omega_0^2)^2 + 4a^2\omega^2}}. \tag{20.2}$$

Proof: If $x = \tilde{B}e^{i\omega t}$ with $|\tilde{B}| = B$, then $L[x] = \tilde{A}e^{i\omega t}$ with $|\tilde{A}| = A$ and $p(i\omega)\tilde{B} = \tilde{A}$. We have

$$G = \frac{|\tilde{B}|}{|\tilde{A}|} = \frac{1}{|p(i\omega)|} = \frac{1}{|(\omega_0^2 - \omega^2) + 2a\omega i|}.$$

□

The gain can be large if the frequencies are almost equal and the damping is small, the case of "practical resonance".

Applications. (a) For a given circuit or spring (ω_0 and a are fixed), the gain is a function of ω, $G = G(\omega)$. Sometimes we want to find the "resonant frequency" ω_r for which the gain is maximal. Assuming $a < \omega_0^2$ (small damping), we have

$$\omega_r^2 = \omega_0^2 - 2a^2, \qquad G_{\max} = \frac{1}{2a\sqrt{\omega_0^2 - a^2}},$$

(differentiate the expression under the square root in (20.2) with respect to ω^2).

(b) *Tuning.* Consider the equation.

$$L\ddot{I} + R\dot{I} + C^{-1}I = A_1 \cos\omega_1 t + A_2 \cos\omega_2 t,$$

where L and R are fixed but C is a tuning parameter. Think of two radio station broadcasting on different frequencies. We want to tune the radio to one of the stations by reducing the amplitude of the other frequency. The output (up to phase shifts) is

$$x_*(t) = G_1 A_1 \cos\omega_1 t + G_2 A_2 \cos\omega_2 t,$$

where the gains G_1 and G_2 are given by (20.2). By tuning $C^{-1} \approx L\omega_1$ we can often make $G_2 \ll G_1$.

21. Linear systems with constant coefficients

$$\dot{x} = Ax + g(t), \qquad x(t) \in \mathbb{R}^n.$$

Here A is a constant $n \times n$ matrix, $g(t)$ is a given vector-valued function, and $x(t)$ is the unknown vector-valued function,

$$x(t) = \begin{pmatrix} x_1(t) \\ \cdots \\ \cdots \\ x_n(t) \end{pmatrix}.$$

21.1. **Converting the system into a single n-th order linear equation.** This is usually the most practical method of finding an explicit solution, especially in the inhomogeneous case. Let us consider the 2D system,

$$\begin{cases} \dot{x} = ax + by + f(t) \\ \dot{y} = cx + dy + h(t) \end{cases},$$

i.e.

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad g = \begin{pmatrix} f \\ h \end{pmatrix}.$$

To solve the system, do the following:

(1) use the first equation to express y in terms of $\dot{x}$, x, t, namely

$$y = \frac{\dot{x}}{b} - \frac{ax}{b} - \frac{f(t)}{b}; \tag{21.1}$$

(2) find $\dot{y}$ in terms of $\ddot{x}$, $\dot{x}$, t by differentiating (21.1);

(3) use the second equation of the system and the expressions for y and $\dot{y}$ to derive a 2nd order equation for $x(t)$;

(4) solve this 2nd order equation and find $x(t)$;

(5) use (21.1) to find $y(t)$.

[See Examples 26.1, 26.2 in the text.]

21.2. **Vector exponentials.** This method applies to homogeneous systems

$$\dot{x} = Ax, \qquad x(t) \in \mathbb{R}^n.$$

We look for solutions of the form

$$x(t) = e^{\lambda t} w, \tag{21.2}$$

where λ is a real number and w is a constant vector in $\mathbb{R}^n$. Since

$$\frac{d}{dt}\left(e^{\lambda t} w\right) = \lambda e^{\lambda t} w, \qquad A\left(e^{\lambda t} w\right) = e^{\lambda t} Aw,$$

we see that (21.2) is a solution if and only if

$$\lambda w = Aw,$$

i.e. if and only if λ is an eigenvalue of A and w is a corresponding eigenvector.

Recall that the eigenvalues of A are the roots of the characteristic polynomial

$$P(\lambda) = \det(\lambda I - A),$$

where I is the identity matrix. We have n eigenvalues, real or complex, possibly multiple. For simple eigenvalues, the eigenvectors (in $\mathbb{C}^n$) are uniquely determined up to a scalar multiplicative constant. Since the matrix A is real, we can choose $w \in \mathbb{R}^n$ if the eigenvalue is real. Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Theorem. *Suppose the eigenvalues $\lambda_1, \dots, \lambda_n$ of A are real and distinct, and let $w_1, \dots w_n \in \mathbb{R}^n$ be corresponding eigenvectors. Then*

$$x(t) = C_1 e^{\lambda_1 t} w_1 + \cdots + C_n e^{\lambda_n t} w_n, \qquad C_j \in \mathbb{R},$$

is the general solution of the system $\dot{x} = Ax$.

Proof: The set of solutions is a linear space of dimension n (same argument as in the case of higher order linear differential equations, see Section 15 of the notes). The n solutions $e^{\lambda_j t} w_j$ are linearly independent, and therefore form a basis in the space of solutions. $\square$

Complex eigenvalues. Let

$$w = u + iv, \qquad u \equiv \Re w \in \mathbb{R}^n, \quad u \equiv \Im w \in \mathbb{R}^n,$$

be an eigenvector corresponding to $\lambda = \alpha + i\omega$. Then

$$z(t) = e^{\lambda t} w = e^{\alpha t}(\cos\ \omega t + i \sin\ \omega t)\ (u + iv)$$

is a solution of the complexified equation

$$\dot{z} = Az, \qquad z(t) \in \mathbb{C}^n.$$

Since A is real, we get two real solutions:

$$\Re z(t) = e^{\alpha t}[\cos\ \omega t\ u - \sin\ \omega t\ v],$$

$$\Im z(t) = e^{\alpha t}[\sin\ \omega t\ u + \cos\ \omega t\ v].$$

If $n = 2$ and if the eigenvalues are complex, then the general solution of $\dot{x} = Ax$ is

$$x(t) = C_1 \Re z(t) + C_2 \Im z(t).$$

Note. The characteristic polynomial has real coefficients, so complex eigenvalues appear in conjugate pairs and $\bar{w} := u - iv$ is an eigenvector corresponding to $\bar{\lambda} = \alpha - i\omega$. The complex solution $e^{\bar{\lambda}t}\bar{w}$ produces the same real solutions as $e^{\lambda t}w$.

22. The exponential of a matrix

22.1. **Definition.** Let A be a constant n by n matrix. Consider the following IVP for the unknown matrix-valued function $X(t)$:

$$\dot{X} = AX, \qquad X(0) = I \quad \text{(identity matrix).} \tag{22.1}$$

If $x_{jk}(t)$ are the matrix elements of $X(t)$, then

$$\dot{X}(t) = \begin{pmatrix} \dot{x}_{11}(t) & \dots & \dot{x}_{1n}(t) \\ \dots & \dots & \dots \\ \dot{x}_{n1}(t) & \dots & \dot{x}_{nn}(t) \end{pmatrix},$$

so (22.1) is a *linear* system in $\mathbb{R}^{n^2}$. By definition,

$$e^{tA} = X(t), \qquad \text{in particular} \quad e^A = X(1).$$

Theorem.

$$e^{tA} = \sum_{k=0}^{\infty} \frac{A^k}{k!} t^k = \lim_{k\to\infty} \left(I + \frac{tA}{k}\right)^k.$$

Remarks.

(a) Relate to Picard's and Euler's approximations respectively, see Section 6 of the notes.

(b) In the above statement, the limit is understood in terms of the limits of all matrix elements; same for the infinite sum.

(c) It is not difficult to justify the following formal computations:

$$\frac{d}{dt}\left(\sum_{k=0}^{\infty} \frac{A^k}{k!} t^k\right) = \sum \frac{d}{dt} = A + \frac{2t}{2!}A^2 + \frac{3t^2}{3!}A^3 + \dots = A\left(\sum_{k=0}^{\infty} \frac{A^k}{k!} t^k\right),$$

and

$$\frac{d}{dt}\left[\lim \left(I + \frac{tA}{k}\right)^k\right] = \lim \frac{d}{dt} = \lim\ k\left(I + \frac{tA}{k}\right)^{k-1} \frac{A}{k} = A\left[\lim \left(I + \frac{tA}{k}\right)^k\right].$$

(d) It is not difficult to prove that if $AB = BA$ (which is not always the case!), then

$$e^{A+B} = e^A e^B,$$

in particular,

$$\left(e^A\right)^{-1} = e^{-A}.$$

22.2. **Solving homogeneous and inhomogeneous systems of equations.**

Theorem. *For every $x_0 \in \mathbb{R}^n$, the vector-valued function*

$$x(t) = e^{At}x_0$$

is the solution of the IVP

$$\dot{x} = Ax, \quad x(0) = x_0.$$

Proof:

$$\dot{x}(t) = \frac{d}{dt}\left[e^{At}x_0\right] = \left[\frac{d}{dt}e^{At}\right]x_0 = Ae^{At}x_0 = Ax(t).$$

□

Variation of constants. Consider now the inhomogeneous system

$$\dot{x} = Ax + g(t), \qquad x(t) \in \mathbb{R}^n.$$

Since $e^{tA}C$ with $C \in \mathbb{R}^n$ is the general solution of the associated homogeneous system of equations, we look for a particular solution [of the inhomogeneous system] of the form

$$x(t) = e^{tA}C(t),$$

where $C(t)$ is the unknown vector-valued function. We have

$$\dot{x} = Ae^{tA}C + e^{tA}\dot{C},$$

and this has to be equal to

$$Ax + g = Ae^{tA}C + g,$$

so $C(t)$ satisfies

$$e^{tA}\dot{C} = g, \qquad \text{i.e.} \quad \dot{C} = e^{-tA}g.$$

22.3. **Computation of e^{tA}.**

(a) *Diagonal matrices:*

$$A = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix} \qquad \Rightarrow \qquad e^{tA} = \begin{pmatrix} e^{\lambda_1 t} & 0 & \dots & 0 \\ 0 & e^{\lambda_2 t} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & e^{\lambda_n t} \end{pmatrix}.$$

Proof:

$$A^k = \begin{pmatrix} \lambda_1^k & 0 & \dots & 0 \\ 0 & \lambda_2^k & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n^k \end{pmatrix}.$$

(b) *Nilpotent matrices:*

$$N = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \Rightarrow \qquad e^{tN} = \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof:

$$N^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad N^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \qquad N^4 = N^5 = \cdots = 0.$$

(c) *Jordan cells:*

$$A = \begin{pmatrix} \lambda_0 & 1 & 0 & 0 \\ 0 & \lambda_0 & 1 & 0 \\ 0 & 0 & \lambda_0 & 1 \\ 0 & 0 & 0 & \lambda_0 \end{pmatrix} \qquad \Rightarrow \qquad e^{tA} = e^{\lambda_0 t} \begin{pmatrix} 1 & t & \frac{t^2}{2!} & \frac{t^3}{3!} \\ 0 & 1 & t & \frac{t^2}{2!} \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Proof: $A = \lambda_0 I + N$. Since $IN = NI$, we have

$$e^{tA} = e^{t\lambda_0 I} e^{tN} = e^{\lambda_0 t} e^{tN}.$$

Note. If $P(\lambda)$ is the characteristic polynomial of A, then $P(A) = 0$ (Cayley-Hamilton theorem). In particular, if $P(\lambda) = (\lambda - \lambda_0)^n$, then $(A - \lambda_0 I)^n = 0$ and the computation of

$$e^{tA} = e^{\lambda_0 t} e^{t(A - \lambda_0 I)}$$

is very simple (similar to (b)).

(d) *Complex eigenvalues:*

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad \Rightarrow \qquad e^{\beta J} = \begin{pmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{pmatrix}.$$

Proof: Since $J^2 = -I$, $J^3 = -J$, $J^4 = I$, ... , we have

$$e^{\beta J} = I + \beta J - \frac{\beta^2}{2!} I - \frac{\beta^3}{3!} J + \frac{\beta^4}{4!} I + \cdots = (\cos\beta) I + (\sin\beta) J.$$

Corollary:

$$A = \begin{pmatrix} \alpha & -\omega \\ \omega & \alpha \end{pmatrix} \qquad \Rightarrow \qquad e^{tA} = e^{\alpha t} \begin{pmatrix} \cos\omega t & -\sin\omega t \\ \sin\omega t & \cos\omega t \end{pmatrix}.$$

Pf: $A = \alpha I + \omega J$ and $IJ = JI$.

(e) *Linear change of variables.*

If $B = SAS^{-1}$, where S is an invertible $n \times n$ matrix, then

$$e^{tB} = S e^{tA} S^{-1}.$$

Pf: $B^2 = SA^2S^{-1}$, $B^3 = SA^3S^{-1}$, ...

Remark. The last observation plus Examples (a)–(d) allow us to compute e^{tA} for all matrices: use S that puts the matrix A into its Jordan canonical form.