Physics 106a — Classical Mechanics

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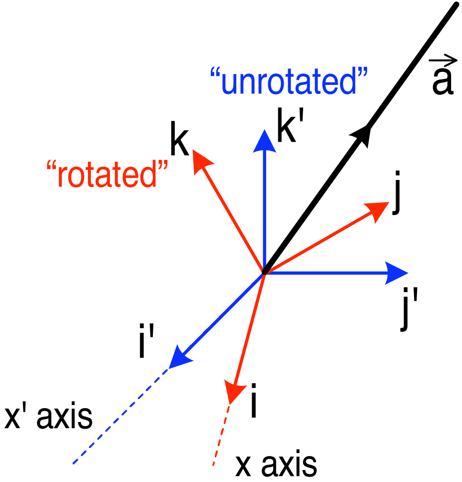
Lecture 13: Rotations

Rotations

Physics	Transformation	Ideas
Rotational symmetry	Rotated system	vectors, tensors rotation matrices (group structure)
Dynamics in rotating frame	Rotating system	fictitious forces
Dynamics of rotating bodies	Rotating axes tied to dynamics	Euler angles moment of inertia angular momentum

Rotate axes



Rotate axes

Unrotated basis vectors $\hat{\imath}'$, $\hat{\jmath}'$, $\hat{k}'$; *rotated* basis vectors $\hat{\imath}$, $\hat{\jmath}$, $\hat{k}$.
Components a_i' and a_i of $\vec{a}$ with respect to these axes

$$\vec{a} = a_1'\hat{\imath}' + a_2'\hat{\jmath}' + a_3'\hat{k}' = a_1\hat{\imath} + a_2\hat{\jmath} + a_3\hat{k}$$

Using orthonormality

$$a_1' = a_1\hat{\imath}' \cdot \hat{\imath} + a_2\hat{\imath}' \cdot \hat{\jmath} + a_3\hat{\imath}' \cdot \hat{k}$$

or in matrix notation

$$\begin{pmatrix} a_1' \\ a_2' \\ a_3' \end{pmatrix} = U \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad \text{with} \quad U = \begin{pmatrix} \hat{\imath}' \cdot \hat{\imath} & \hat{\imath}' \cdot \hat{\jmath} & \hat{\imath}' \cdot \hat{k} \\ \hat{\jmath}' \cdot \hat{\imath} & \hat{\jmath}' \cdot \hat{\jmath} & \hat{\jmath}' \cdot \hat{k} \\ \hat{k}' \cdot \hat{\imath} & \hat{k}' \cdot \hat{\jmath} & \hat{k}' \cdot \hat{k} \end{pmatrix}$$

U is called the *rotation matrix*. The matrix notation means

$$a_i' = \sum_{j=1}^{3} U_{ij} a_j \equiv U_{ij} a_j \quad \text{(repeated subscript convention)}$$

Rotation matrix

The rotation matrix U is an *orthogonal matrix*

$$U\tilde{U} = \tilde{U}U = I$$

with $\tilde{U}$ the transpose $\tilde{U}_{ij} = U_{ji}$ and I the unit matrix

$$I_{ij} = \delta_{ij} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases} \qquad \text{(Kronecker delta)}$$

Equivalently the rows (and columns) are orthonormal

$$U_{ik}U_{jk} = \delta_{ij} \text{ etc.} \quad \text{(remember } k\text{-summed)}$$

$U_{ij} \Leftarrow$ 9 numbers, orthogonality $\Rightarrow$ six constraints = 3 numbers to specify U

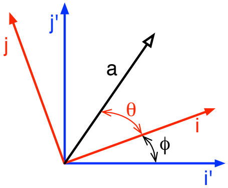
One choice of parameters is the direction of the axis of rotation $\hat{n}$ and the angle of the rotation ϕ

$$U_{ij} = (1 - \cos\phi)\hat{n}_i\hat{n}_j + \cos\phi\,\delta_{ij} - \sin\phi\,\epsilon_{ijk}\hat{n}_k$$

Example

Rotate the axes by ϕ about the z direction

- the z component of a vector is unchanged
- the x, y components transform as



$$a_1' = a\cos(\theta+\phi) = a(\cos\theta\cos\phi - \sin\theta\sin\phi) = a_1\cos\phi - a_2\sin\phi$$
$$a_2' = a\sin(\theta+\phi) = a(\cos\theta\sin\phi + \sin\theta\cos\phi) = a_1\sin\phi + a_2\cos\phi$$

This gives the rotation matrix

$$U(\phi,\hat{k}) = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Rotation axis and angle

- Rotation angle
  - Rotation through same the angle ϕ about a different axis $\hat{n}'$ is

$$U_{\hat{n}'} = \bar{U} R_{\hat{n}} \bar{U}^{-1} \quad \text{with} \quad \hat{n}' = \bar{U}\hat{n}$$

  - The trace is invariant under this transformation $\operatorname{Tr} U_{\hat{n}'} = \operatorname{Tr} U_{\hat{n}}$
  - The rotation angle can be found from $\operatorname{Tr} U = 1 + 2\cos\phi$

- Rotation axis
  - The rotation axis is left unchanged by U

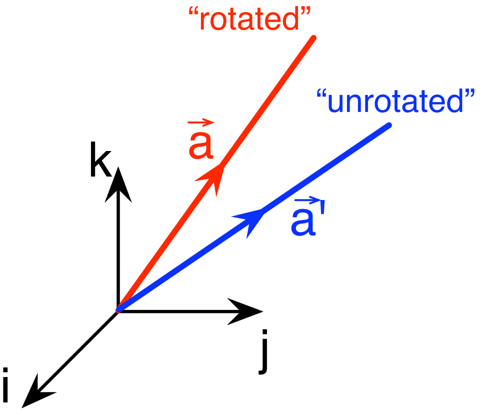
$$U\hat{n} = \hat{n}$$

  - The rotation axis is the eigenvector of U corresponding to eigenvalue 1 (for a proper rotation)

- Improper rotations

$$\det U = \begin{cases} +1 & \text{proper rotation} \\ -1 & \text{improper rotation} \end{cases}$$

Rotate vector



Rotate vector

Vector $\vec{a}'$ is rotated into a new vector $\vec{a}$ by the same physical rotation used before

Components $(\vec{a})_i$ of $\vec{a}$ and $(\vec{a}')_i$ of $\vec{a}'$ (*different* vectors, *same* axes) are

$$\vec{a} = (\vec{a})_1\hat{\imath} + (\vec{a})_2\hat{\jmath} + (\vec{a})_3\hat{k}$$
$$\vec{a}' = (\vec{a}')_1\hat{\imath} + (\vec{a}')_2\hat{\jmath} + (\vec{a}')_3\hat{k}$$

Components of the rotated vector with respect to rotated axes are the same as the components of the unrotated vector with respect to unrotated axes, so

$$(\vec{a})_i = (\vec{a}')'_i = (\vec{a}')_1\hat{\imath}' \cdot \hat{\imath} + (\vec{a}')_2\hat{\imath}' \cdot \hat{\jmath} + (\vec{a}')_3\hat{\imath}' \cdot \hat{k}$$

This gives

$$(\vec{a})_i = U_{ij}(\vec{a}')_j \qquad \text{with } U \text{ the same rotation matrix}$$

Note that the expression relates the *rotated* vector to the *unrotated* vector, whereas for the components of a vector with respect to rotated axes we had

$$a'_i = U_{ij}a_j$$

Group properties

A group is a collection of elements (here different rotations) with the properties:

- **muliplication:** the product U_2U_1 is defined as first apply U_1 and then apply U_2. Clearly
$$U_2U_1 = U_3$$
with U_3 some other rotation;
- **associative rule:** it can be shown
$$(U_1U_2)U_3 = U_1(U_2U_3)$$
- **identity:** the identity exists—do nothing!
- **inverse:** the inverse rotation is a rotation about the same axis through the negative angle.

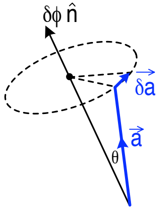
Multiplication of rotations is not commutative in general

$$U_1U_2 \neq U_2U_1$$

Rotations form a *continuous* or *Lie group*: the elements are specified by continuous parameters

Infinitesimal rotations

Infinitesimal rotation $\delta\phi$ about an axis specified by the unit vector $\hat{n}$



The change $\delta\vec{a}$ is perpendicular to $\vec{a}$ and $\hat{n}$ and is of magnitude $a\sin\theta\delta\phi$, i.e.

$$\delta\vec{a} = \delta\phi\hat{n}\times\vec{a} = \delta\vec{\phi}\times\vec{a} \qquad \text{with } \delta\vec{\phi} = \delta\phi\,\hat{n}$$

rotated vector $\quad \vec{a} = \vec{a}' + \delta\vec{\phi}\times\vec{a}'$

For successive infinitesimal rotations $\vec{a}'' \to \vec{a}' \to \vec{a}$ vectors:

$$\vec{a} = \vec{a}' + \delta\vec{\phi}_2\times\vec{a}' = (\vec{a}'' + \delta\vec{\phi}_1\times\vec{a}'') + \delta\vec{\phi}_2\times(\vec{a}'' + \delta\vec{\phi}_1\times\vec{a}'')$$

$$\Rightarrow \quad a = \vec{a}'' + (\delta\vec{\phi}_1 + \delta\vec{\phi}_2)\times\vec{a}'' + O(\delta\vec{\phi}^2)$$

i.e. just add the $\delta\vec{\phi}$

Infinitesimal rotations commute; finite rotations about different axes do not.

Angular velocity: small rotation divided by a small time increment $\vec{\omega} = d\vec{\phi}/dt$

Infinitesimal rotation matrices

Rotated vector is $\vec{a}$ and the unrotated one $\vec{a}'$

$$\vec{a} = \vec{a}' + \delta\vec{\phi} \times \vec{a}'$$

or in component notation with respect to a fixed basis

$$(\vec{a})_i = (\delta_{ij} - \epsilon_{ijk}\delta\phi_k)(\vec{a}')_j$$

$$\text{Levi-Civita symbol} \quad \epsilon_{ijk} = \begin{cases} 0 & \text{if any repeated indices} \\ 1 & \text{if } ijk \text{ is an even permutation of } 123 \\ -1 & \text{if } ijk \text{ is an odd permutation of } 123 \end{cases}$$

The corresponding rotation matrix is

$$\delta U = I + \delta\vec{\phi} \cdot \vec{M}$$

with $\vec{M}$ a vector of matrices with components $(M_k)_{ij} = -\epsilon_{ijk}$

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; M_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \; M_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Lie group properties

Build up a finite rotation ϕ about $\hat{n}$ as N rotations $\delta\phi = \phi/N$ for $N \to \infty$

$$U = \lim_{N\to\infty}\left(I + \frac{\phi}{N}\hat{n}\cdot\vec{M}\right)^N = e^{\phi\hat{n}\cdot\vec{M}}$$

where the exponential of a matrix is a compact way of writing the power series

$$e^A = 1 + A + \tfrac{1}{2!}A^2 + \cdots$$

- $\vec{M}$ are called *generators* of the Lie group
- Lie algebra

$$M_i M_j - M_j M_i = c_{ij}^k M_k$$

with the *structure constants* c_{ij}^k given by $c_{ij}^k = \epsilon_{ijk}$

Example: Noether's theorem for rotations

- Define the rotation transformed paths in coordinate notation

$$r_i(\delta\vec{\phi}, t) = \delta U_{ij} r_j$$

with $\delta U = I + \delta\phi_k M_k$ and $(M_k)_{ij} = -\epsilon_{ijk}$

- For a system with rotational symmetry Noether's theorem gives the conserved quantities

$$\begin{aligned} I_k &= p_i \frac{\partial r_i}{\partial \delta\phi_k} = p_i (M_k)_{ij} r_j \\ &= -\epsilon_{ijk} p_i r_j = (\vec{r} \times \vec{p})_k \end{aligned}$$

so that $\vec{I}$ is the angular momentum vector $\vec{r} \times \vec{p}$.

Rigid body motion

- For a rigid body the velocity of the ith point or mass element is

$$\vec{v}_i = \vec{V} + \vec{\omega} \times \vec{r}_i$$

  with $\vec{r}_i$ the displacement of the ith point relative to reference point K at $\vec{R}$ moving with velocity $\vec{V} = \dot{\vec{R}}$.
- Change the reference point to K' at $\vec{R}' = \vec{R} + \vec{a}$ so that the vector from the ith point to K' is $\vec{r}_i'$ with $\vec{r}_i = \vec{r}_i' + \vec{a}$ then

$$\vec{v}_i = \vec{V}' + \vec{\omega} \times \vec{r}_i' \quad \text{with} \quad \vec{V}' = \vec{V} + \vec{\omega} \times \vec{a}$$

- The angular velocity is independent of the reference point.
- May choose reference point to make $\vec{V}' \parallel \vec{\omega}$: the motion is instantaneously rotation about a stationary axis of rotation plus translation along this axis
- For rolling motion on a stationary surface the contact point or line is instantaneously stationary, and the motion is instantaneously pure rotation about an axis passing through the contact