ACM 100c

The matrix exponential

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The matrix exponential

- Consider again the homogeneous initial value problem

  $\boldsymbol{x}' = A\boldsymbol{x} \qquad \boldsymbol{x}(z_0) = \boldsymbol{x}_0$ where A is a constant matrix.

- We know from the initial condition that

$$\boldsymbol{x}(z_0) = \boldsymbol{x}_0$$

- Let's write this as

$$\boldsymbol{x}(z_0) = I\boldsymbol{x}_0 \quad \text{where } I \text{ is the identity matrix}$$

- Then from the equation we see that

$$\boldsymbol{x}'(z_0) = A\boldsymbol{x}_0$$

- And differentiating the system with respect to z we find

$$\boldsymbol{x}''(z_0) = A\boldsymbol{x}'(z_0) = A^2\boldsymbol{x}_0$$

- Continuing this way we find

$$\boldsymbol{x}^{(n)}(\boldsymbol{x}_0) = A^n\boldsymbol{x}_0$$

The matrix exponential

- With this information we can use Taylor's theorem which allows us to expand the solution as a power series.

$$f(x) = \sum_{i=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

- Suppose we apply this to the solution of

$$\boldsymbol{x}' = A\boldsymbol{x} \qquad \boldsymbol{x}(z_0) = \boldsymbol{x}_0$$

- The Taylor series for this problem would be

$$\boldsymbol{x}(z) = I\boldsymbol{x}_0 + A\boldsymbol{x}_0(z - z_0) + \frac{A^2\boldsymbol{x}_0}{2!}(z - z_0)^2 + \ldots$$

- This looks identical to the Taylor series for the exponential

The matrix exponential

- We can then write formally

$$\boldsymbol{x}(z) = \exp(A(z - z_0))\boldsymbol{x}_0$$

  where the quantity $\exp(A(z - z_0))$ is defined by its Taylor series and is called the *matrix exponential*.
- Note that this expression has the property that the initial condition is satisfied, the ODE system is solved and

$$\exp(A(z - z_0)) = I \quad \text{at } z = z_0$$

- So in fact this must be the fundamental matrix.