Physics 106a — Classical Mechanics

Michael Cross

California Institute of Technology

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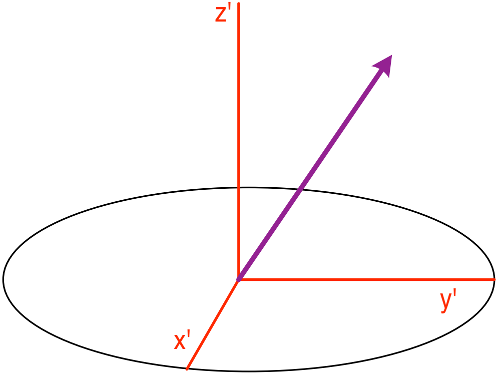
Lecture 16: Rigid Body Rotation with Torques

Outline

- Euler angles
- Angular momentum and kinetic energy in Euler angles
- Heavy symmetric top / precession of the equinoxes

Vector $\vec{a}$ with components $(a_{x'}, a_{y'}, a_{z'})$

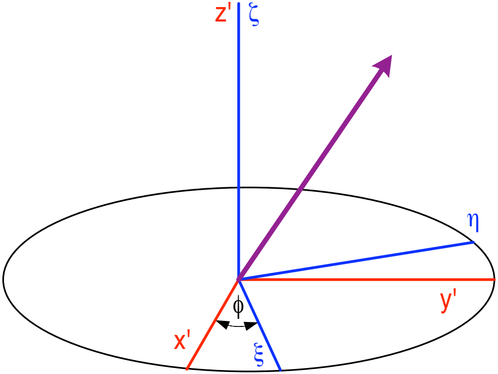


Euler Angles

First rotation

$$U_1 = \begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_\xi \\ a_\eta \\ a_\zeta \end{bmatrix} = \tilde{U}_1 \begin{bmatrix} a_{x'} \\ a_{y'} \\ a_{z'} \end{bmatrix}$$

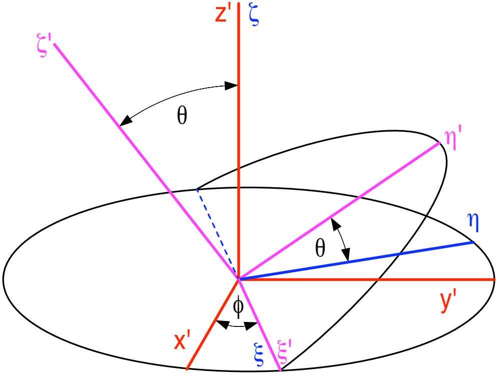


Euler Angles

Second rotation

$$U_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} a_{\xi'} \\ a_{\eta'} \\ a_{\zeta'} \end{bmatrix} = \tilde{U}_2 \begin{bmatrix} a_{\xi} \\ a_{\eta} \\ a_{\zeta} \end{bmatrix}$$

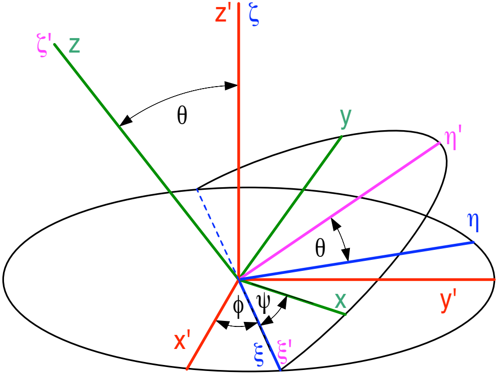


Euler Angles

Third rotation

$$U_3 = \begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \tilde{U}_3 \begin{bmatrix} a_{\xi'} \\ a_{\eta'} \\ a_{\zeta'} \end{bmatrix}$$

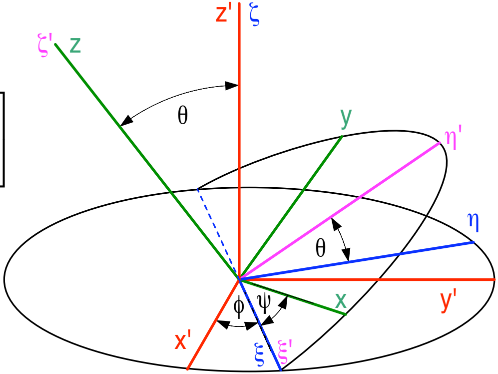


Euler Angles

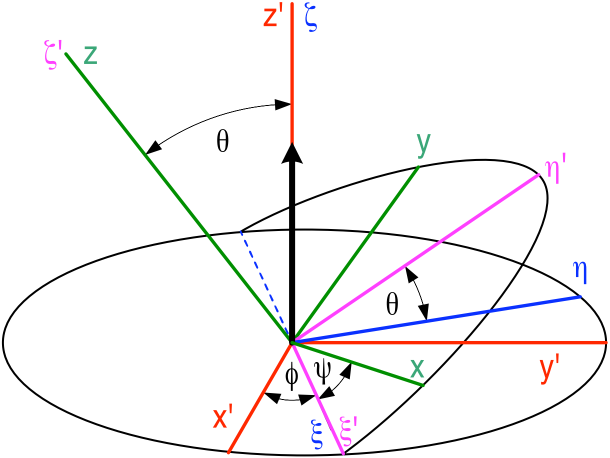
Third rotation

$$U_1U_2U_3 = \begin{bmatrix} \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

$$\begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} = \tilde{U}_3\tilde{U}_2\tilde{U}_1 \begin{bmatrix} a_{x'} \\ a_{y'} \\ a_{z'} \end{bmatrix}$$

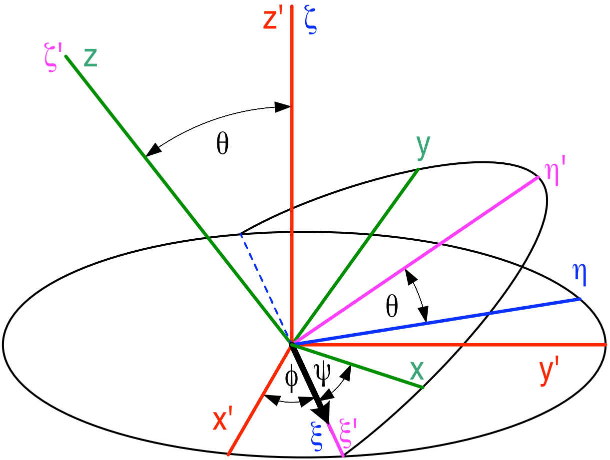


Angular velocities: $\dot{\phi}$



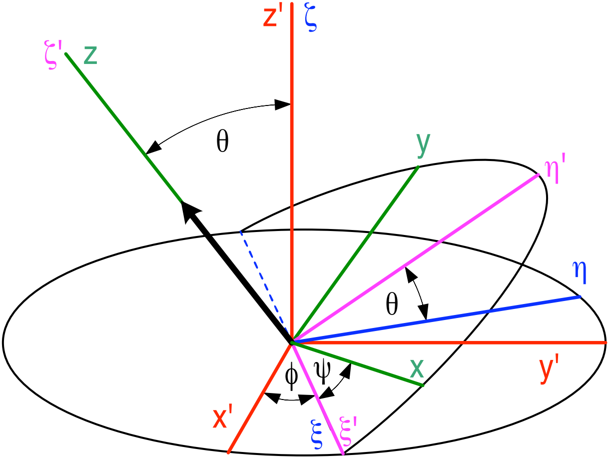
$$(\omega_x, \omega_y, \omega_z) = \dot{\phi}(\sin\theta\sin\psi, \sin\theta\cos\psi, \cos\theta)$$

Angular velocities: $\dot{\theta}$



$$(\omega_x, \omega_y, \omega_z) = \dot{\theta}(\cos\psi, -\sin\psi, 0)$$

Angular velocities: $\dot{\psi}$



$$(\omega_x, \omega_y, \omega_z) = \dot{\psi}(0, 0, 1)$$

Angular velocity and Lagrangian

Angular velocity components with respect to principal axes

$$\omega_1 \equiv \omega_x = \dot{\phi} \sin\psi \sin\theta + \dot{\theta} \cos\psi$$
$$\omega_2 \equiv \omega_y = \dot{\phi} \cos\psi \sin\theta - \dot{\theta} \sin\psi$$
$$\omega_3 \equiv \omega_z = \dot{\psi} + \dot{\phi} \cos\theta$$

Kinetic energy

$$T = \frac{1}{2}(I_1\omega_1^2 + I_2\omega_2^2 + I_3\omega_3^2)$$

Kinetic energy for axially-symmetric body $I_1 = I_2 = I_\perp$

$$T = \frac{1}{2} I_\perp(\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2} I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2$$

Calculate Lagrangian etc....

Angular velocity and angular momentum

Angular velocity components with respect to principal axes

$$\begin{aligned}\omega_1 \equiv \omega_x &= \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi \\ \omega_2 \equiv \omega_y &= \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\sin\psi \\ \omega_3 \equiv \omega_z &= \dot{\psi} + \dot{\phi}\cos\theta\end{aligned}$$

Angular momentum components with respect to principal axes

$$\vec{L} \equiv (I_1\omega_1, I_2\omega_2, I_3\omega_3)$$

Calculate

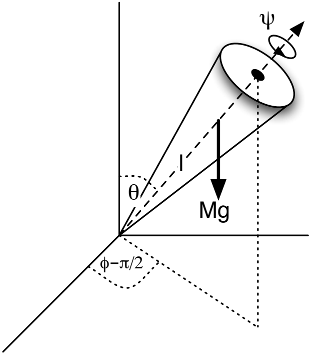
$$\left.\frac{d\vec{L}}{dt}\right|_s = \left.\frac{d\vec{L}}{dt}\right|_b + \vec{\omega}\times\vec{L} = \vec{N} \quad \text{with} \quad \left.\frac{d\vec{L}}{dt}\right|_b = (I_1\dot{\omega}_1, I_2\dot{\omega}_2, I_3\dot{\omega}_3)$$

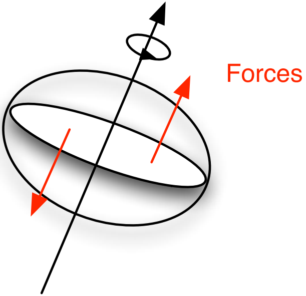
Angular velocity and behavior in space frame

Angular velocity components with respect to space (inertial) axes

$$
\begin{aligned}
\omega_{x'} &= \dot{\psi}\sin\phi\sin\theta + \dot{\theta}\cos\phi \\
\omega_{y'} &= -\dot{\psi}\cos\phi\sin\theta + \dot{\theta}\sin\phi \\
\omega_{z'} &= \dot{\psi}\cos\theta + \dot{\phi}
\end{aligned}
$$

Heavy symmetric top





Nutation of symmetric top