ACM 100b

Poles of the Laplace transform

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The poles of the Laplace transform

- We note that the types of exponentials we get in the solution come from the roots of the denominator of the Laplace transform.
- A first order pole in a location $s = \alpha$ will lead to a solution $\exp(\alpha t)$ when one uses the residue theorem to evaluate the inverse Laplace transform.
- We could also have higher order poles in the Laplace transform.
- In that case the subsequent use of the inverse Laplace transform will also gives us solutions like $\exp(\alpha a t)$
- But also solutions typically of the form

$$t^m \exp(\alpha t).$$

where the power will depend on the order of the pole.

The poles of the Laplace transform

- To see how this arises consider the ODE

$$y'' - (\alpha+\beta)y' + \alpha\beta y = 0 \qquad t > 0.$$

- A Laplace transform gives us

$$\left[s^2 - (\alpha+\beta) + \alpha\beta\right] Y(s) = sy(0) + y'(0) - (\alpha+\beta)y(0),$$

- Or we can write it as

$$Y(s) = \frac{y(0) + y'(0) - (\alpha+\beta)y(0)}{s^2 - (\alpha+\beta) + \alpha\beta}.$$

The poles of the Laplace transform

- The roots of the transform were designed to be at $s = \alpha$ and $s = \beta$.
- We can now let $\alpha \to \beta$.
- This is the case of a double root.
- In that case we get

$$Y(s) = \frac{y(0)}{s-\alpha} + \frac{y'(0) - \alpha y(0)}{(s-\alpha)^2}.$$

- We see that there is now both a first order and a second order pole at $s = \alpha$.
- Through the use of the residue theorem we recover the two types of solutions:

$$\exp(\alpha t) \qquad t\exp(\alpha t).$$