ACM 100c

Intro to second order ODE

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Second order ODE

- The general scalar second order ODE takes the form

$$F(x, y, y', y'') = 0.$$

- As we discussed above, for linear equations of first order, the general solution involves one undetermined constant.
- Since

$$F(x, y, y', y'') = 0.$$

involves two derivatives we expect two arbitrary constants.
- For example the trivial second order linear ODE

$$y'' = g(x)$$

can be integrated to give

$$y = \phi(x) = c_1 + c_2 x + \int^x dt \int^t g(s) ds.$$

Initial value problems (IVP)

- In order to uniquely fit the constants in

$$y = \phi(x) = c_1 + c_2 x + \int^x dt \int^t g(s) ds.$$

  we will need two conditions.
- These come typically in the form of the initial value problem (IVP) where information is given at one point (say $x = x_0$):

$$y(x_0) = y_0 \qquad y'(x_0) = y_0',$$

- The IVP might typically occur in a mechanics problem where initial position and velocity are given and the ODE then provides the acceleration.

Boundary value problems (BVP)

- Or we can obtain the constants in

$$y = \phi(x) = c_1 + c_2 x + \int^x dt \int^t g(s) ds.$$

from a boundary value problem (BVP) where information is given at two points:

$$y(x_0) = y_0 \qquad y(x_1) = y_1.$$

- A BVP arises in many settings.
- For example we might have a string clamped at two ends.
- In this case information about the string position would be provided at either end.

General second order linear ODE

- The general linear second order ODE has the form

$$y'' + p(x)y' + q(x)y = r(x).$$

- There are many important examples of such equations arising in many areas.
- For example the ODE

$$m\frac{d^2u}{dt^2} + c\frac{du}{dt} + ku = F(t)$$

is often used to describe a damped forced harmonic oscillator.

- Here $u(t)$ is deflection, m is the mass of the particle, k measures the restoring force, c is a damping coefficient and $F(t)$ is the forcing.

Second order linear ODE

- Second order ODEs are also used to define important transcendental functions which appear in many areas of physics.
- Examples include

$$\begin{aligned}(1-x^2)y'' - 2xy' + \alpha(\alpha+1)y &= 0 \qquad \text{Legendre's equation},\\ x^2y'' + xy' + (x^2-\nu^2)y &= 0 \qquad \text{Bessel's equation}.\end{aligned}$$

- We will study these in some detail later on.
- Again, we can make much more progress on the theory for linear second order ODE.
- The theory actually differs depending on whether we're solving an IVP or BVP, with the theory for IVPs being more complete so we'll focus on second order IVP problems first.

Existence and uniqueness for second order ODE

- Unlike the case of first order linear ODE, there is no explicit formula for the general solution.
- Instead we rely on specific solution techniques which we discuss later.
- There are however, very complete existence and uniqueness results (as there are for any order linear ODE).
- We'll next state such a result for real variables
- But in fact similar results hold in the complex plane.

Existence and uniqueness

- An important result (for real x) is that

Theorem

If $p(x)$, $q(x)$ and $r(x)$ are continuous in the interval $\alpha < x < \beta$ then there exists one and only one solution $y = \phi(x)$ satisfying the ODE

$$y'' + p(x)y' + q(x)y = r(x)$$

on the interval $\alpha < x < \beta$ with the initial conditions

$$y(x_0) = y_0 \qquad y'(x_0) = y_0' \qquad \alpha < x_0 < \beta.$$

- This result guarantees existence and uniqueness for solutions of the IVP in any interval in which the coefficient functions are smooth.