Physics 106b — Classical Mechanics

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Hamiltonian Chaos II

outline

- KAM theory (review)
- Fate of the rational tori
- Homoclinic tangles
- prediction of chaos – Smale horseshoe map
- General lessons

KAM Theory

Specific statement: $N = 2$ (Mosur)

- Tori with winding numbers over a range $\propto s^{-2.5}$ about every rational r/s are destroyed by the perturbation.

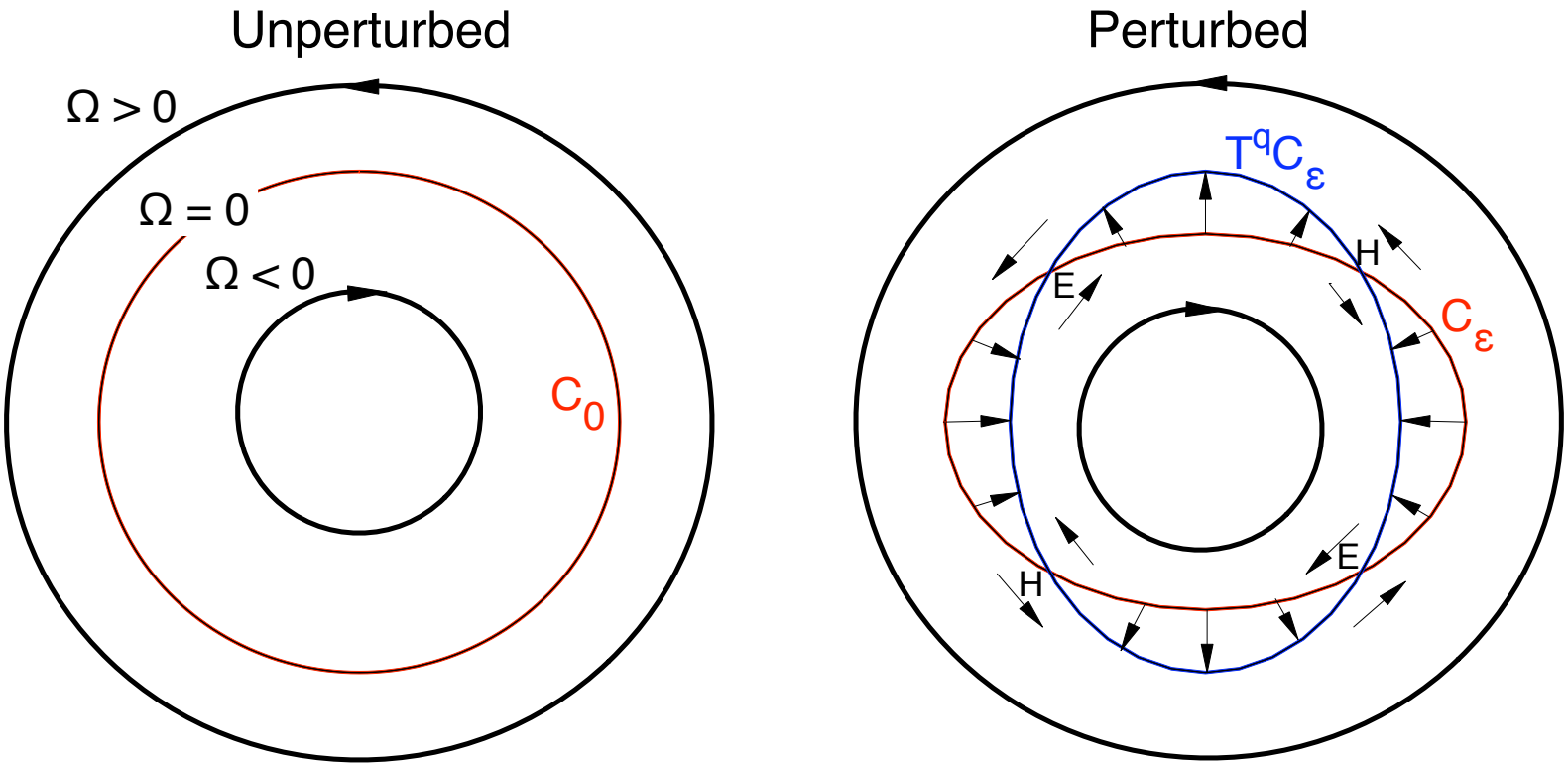
- Although there are an infinite number of rationals in the unit interval , the sum of all these ranges is finite, and goes to zero as $C(\varepsilon) \to 0$. This is because there are of order s rationals in the unit interval with denominator s (r runs from 1 to $s-1$ but some have already been counted, e.g. $2/4 \equiv 1/2$) and the following sum converges

$$\sum_{s=1}^{\infty} s \frac{1}{s^{2.5}}$$

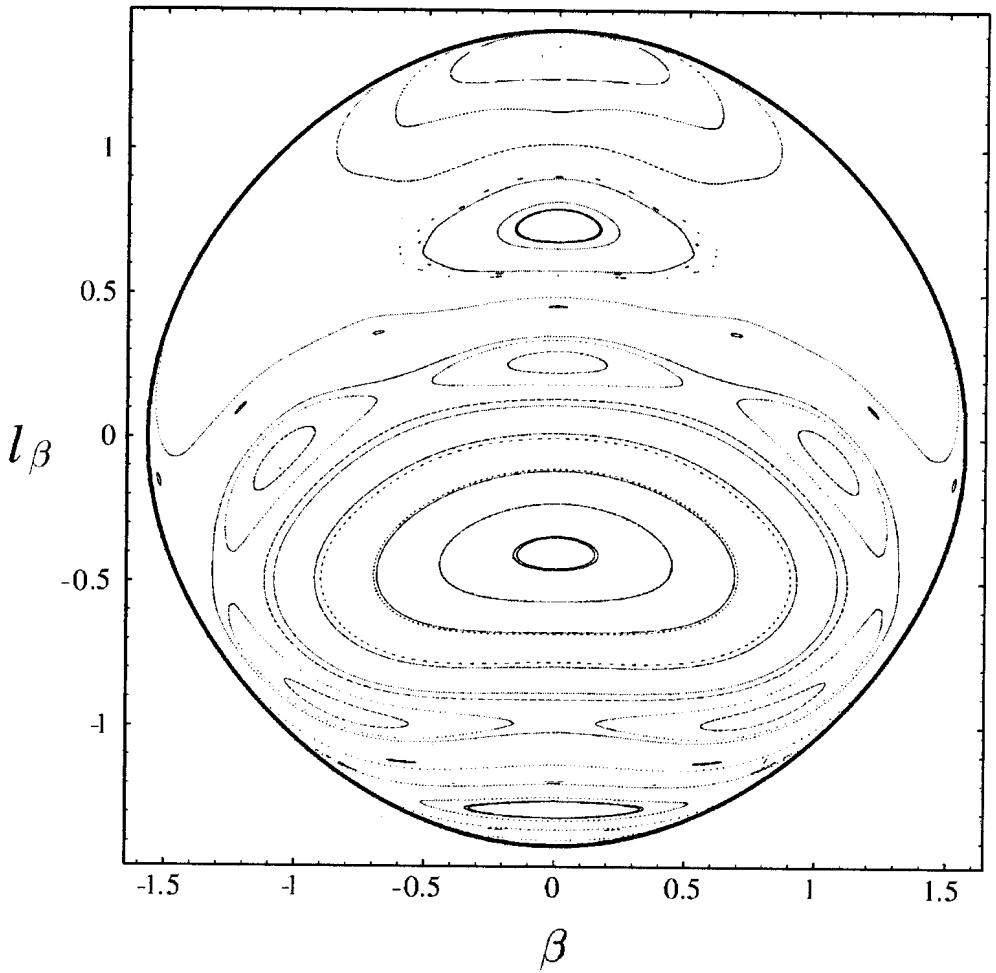
- Tori with winding numbers outside these windows survive the perturbation.

- An infinite number of tori are destroyed, but most survive!

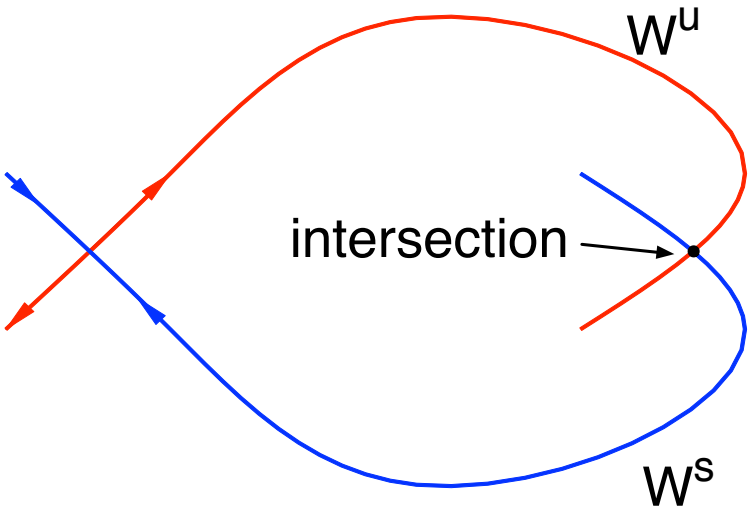
Fate of the Rational Tori

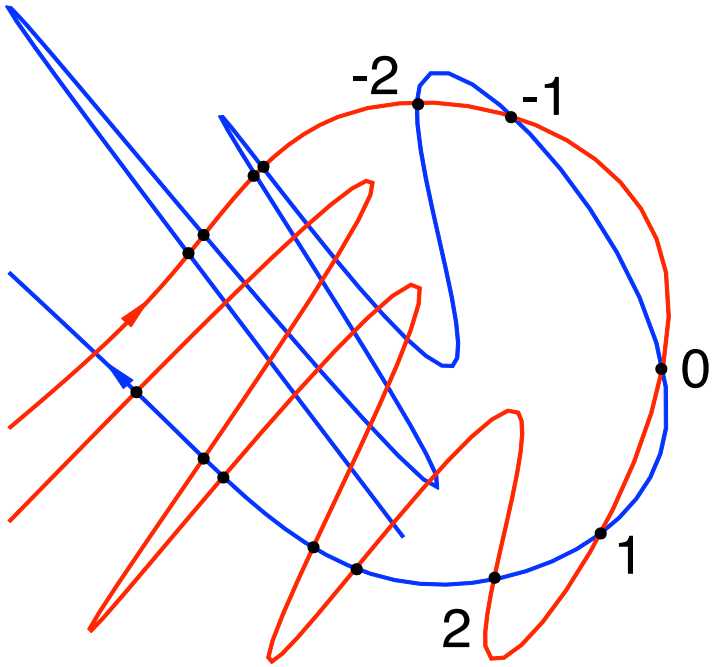


Double Pendulum

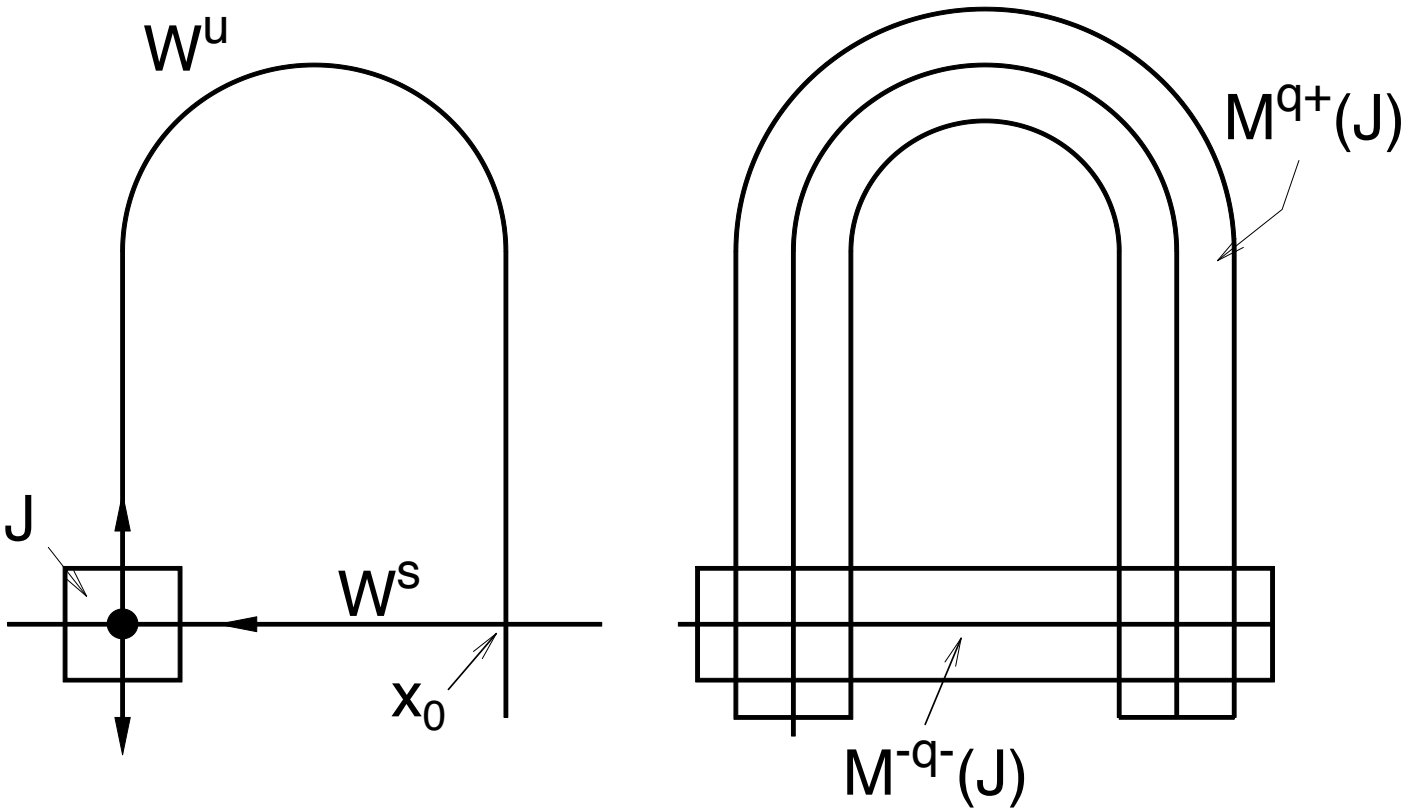


Homoclinic Tangle





Smale Horseshoe Map



Lessons

- Resonances and the problem of small divisors
- Relative simplicity of maps compared with flows
- Think geometrically about families of solutions
  - invariant torus
  - unstable and stable manifolds
  - homoclinic tangles
- Qualitative analysis — the sorts of things that can happen
  - Mosur twist map
  - Smale horseshoe map