# Physics 106b — Classical Mechanics

Michael Cross

California Institute of Technology

Winter Term, 2014

Nonlinear Oscillators

# Solved problems

We have mainly looked at solvable systems in (Newtonian) classical mechanics:

- One degree of freedom systems with a conserved Hamiltonian, that can be reduced to an integral (particle in a potential, the method of quadratures)
- Problems that reduce to this using conservation laws to eliminate other degrees of freedom (e.g. the Kepler problem)
- Problems that decouple into many 1 DOF problems (the normal modes of coupled *linear* oscillators)

# Poincaré's discovery

An important insight of Poincare in his study of the 3-body Kepler problem is that most problems are not "analytically solvable". Poincare wrote that he found

> *[phase space trajectories] so tangled that I cannot begin to even draw them*

and that

> *…small differences in initial positions may lead to enormous differences in the final phenomena. Prediction becomes impossible.*

Poincare found what we now call *chaos* and identified two of the key properties.

# What we will look at

The three body gravity problem is hard, so I will approach the subject instead through *nonlinear* oscillators.

Trajectories in the two dimensional phase space for a time-independent, one degree of freedom problem are necessarily simple, since they cannot cross.

Thus we will look at:

- a single periodically driven nonlinear oscillator (Hamiltonian or dissipative)
- 2 coupled nonlinear Hamiltonian oscillators

In the former case, the drive may sustain the motion in the presence of dissipation. Without drive, of course, the motion would die out.

# Examples of nonlinear oscillators

- Simple pendulum

$$V(q) \propto (1 - \cos q) \simeq \frac{1}{2}q^2 - \frac{1}{24}q^4$$

- Duffing oscillator

$$V(q) = \frac{1}{2}q^2 + \frac{1}{4}aq^4$$

# Driven damped Duffing oscillator

The general equation of motion including damping and driving is

$$m\frac{d^2q}{dt^2} + \Gamma\frac{dq}{dt} + Kq + aq^3 = F\cos(\omega_d t)$$

Define $\omega_0^2 = K/m$, $Q = m\omega_0/\Gamma$, $\alpha = a/m\omega_0^2$, $f = F/m\omega_0^2$, and introduce the scaled time $\bar{t} = \omega_0 t$ and scaled frequency $\omega = \omega_d/\omega_0$, and let $\dot{q}$ denote $dq/d\bar{t}$.

The equation of motion becomes

$$\ddot{q} + \frac{1}{Q}\dot{q} + q + \alpha q^3 = f\cos(\omega\bar{t})$$

I'll now drop the bar on the time so that $t$ (standing in for $\bar{t}$) is the scaled time

# Driven, damped Duffing oscillator: perturbation theory

$$\ddot{q} + \frac{1}{Q}\dot{q} + q + \alpha q^3 = f\cos(\omega t)$$

We look for a periodic solution oscillating at frequency $\omega$ after transients have died out, and we want to calculate its amplitude and phase relative to the drive.

Introduce $\tau = \omega t$ so that $q = q(\tau)$ is periodic with period $2\pi$ and write $dq/d\tau = q'$

$$q'' + q = \left(1 - \frac{1}{\omega^2}\right) q - \frac{1}{\omega Q} q' - \frac{\alpha}{\omega^2} q^3 + \frac{f}{\omega^2}\cos\tau$$

Assume terms on the right hand side are small and introduce the parameter $\mu$

$$q'' + q = \mu\left[\left(1 - \frac{1}{\omega^2}\right) q - \frac{1}{\omega Q} q' - \frac{\alpha}{\omega^2} q^3 + \frac{f}{\omega^2}\cos\tau\right]$$

At the end of the calculation set $\mu$ to unity.

# Driven, damped Duffing oscillator: perturbation theory

$$q'' + q = \mu \left[ \left( 1 - \frac{1}{\omega^2} \right) q - \frac{1}{\omega Q} q' - \frac{\alpha}{\omega^2} q^3 + \frac{f}{\omega^2} \cos \tau \right]$$

Expand the solution in powers of $\mu$

$$q(\tau) = q_0 + q_1(\tau)\mu + q_2(\tau)\mu^2 \cdots$$

Substitute into the EOM, organize the terms at each order in $\mu$, and argue that the coefficient of each $\mu^p$ must be zero

# Driven, damped Duffing oscillator: perturbation theory

At order $\mu^0$ we get

$$q_0'' + q_0 = 0$$

which has the solution

$$q_0 = \tfrac{1}{2}[Ae^{i\tau} + \text{c.c.}] = \tfrac{1}{2}[Ae^{i\tau} + A^* e^{-i\tau}],$$

where $A = |A|e^{i\delta}$ and $\delta$ is the phase of the solution relative to the drive.
At $O(\mu^1)$

$$q_1'' + q_1 = \frac{1}{2}\left\{\left[\left(1 - \frac{1}{\omega^2}\right) - \frac{i}{\omega Q} - \frac{3\alpha}{4\omega^2}|A|^2\right]A + \frac{f}{\omega^2}\right\}e^{i\tau} + \left[-\frac{3\alpha A^3}{8\omega^2}\right]e^{3i\tau} + \text{c.c.}$$

using

$$q_0^3 = \tfrac{1}{8}(A^3 e^{3i\tau} + 3|A|^2 A e^{i\tau} + \text{c.c.})$$

Remove the secular terms by setting the coefficient of $e^{i\tau}$ to zero

$$\left[\left(1 - \frac{1}{\omega^2}\right) - \frac{i}{\omega Q} - \frac{3\alpha}{4\omega^2}|A|^2\right]A = -\frac{f}{\omega^2}$$

# Driven, damped Duffing oscillator: perturbation theory

The (complex) amplitude of the periodic motion is

$$A = -\frac{f}{(\omega^2 - \omega_{\mathrm{NL}}^2) - i\omega/Q}$$

This is the usual resonance formula, except that the resonant frequency is now the amplitude dependent frequency $\omega_{\mathrm{NL}}$ found for the undriven case

$$\omega_{\mathrm{NL}}^2 = 1 + \frac{3}{4}\alpha|A|^2 + O(|A|^4)$$

This determines the magnitude of the response $|A|$ and the phase $\delta$

$$|A|^2 = \frac{f^2}{(\omega^2 - \omega_{\mathrm{NL}}^2)^2 + (\omega/Q)^2}, \qquad \tan\delta = -\frac{\omega/Q}{\omega^2 - \omega_{\mathrm{NL}}^2}.$$

# Driven, damped Duffing oscillator: perturbation theory

If $Q$ is reasonable large, where the response is large near resonance we can approximate

$$\omega^2 - \omega_{\mathrm{NL}}^2 = (\omega + \omega_{\mathrm{NL}})(\omega - \omega_{\mathrm{NL}}) \simeq 2(\omega - \omega_{\mathrm{NL}})$$

and rewrite the equation for the magnitude of the "gain" $|A|/f$

$$\frac{|A|^2}{f^2 Q^2} \simeq \frac{(1/2Q)^2}{(\omega - \omega_{\mathrm{NL}})^2 + (1/2Q)^2}.$$

with

$$\omega_{\mathrm{NL}} \simeq 1 + \tfrac{3}{8}\alpha |A|^2 = 1 + \left(\tfrac{3}{8}\alpha f^2 Q^2\right) \frac{|A|^2}{f^2 Q^2}$$

a *cubic* equation for $|A|^2$ or $|A|^2/f^2 Q^2$ [but easy to solve for $\omega(|A|^2)$]

# Driven, damped Duffing oscillator: perturbation theory

We can now solve the $O(\mu^1)$ equation.

Since the remaining driving is at frequency $3\tau$ and "off resonance", the solution is $q_1 \propto e^{3i\tau}$ with a finite amplitude

$$q_1 = \frac{3\alpha}{64\omega^2} A^3 e^{3i\tau} + \text{c.c.} \simeq \frac{3}{32}\,\alpha|A|^3 \,\cos[3(\tau+\delta)].$$

The nonlinearity induces *harmonics* (here the third) of the drive frequency.

Going to higher order in the expansion, would lead to further harmonics with frequencies 5,7,..., all odd multiples of the drive frequency, with the amplitude of each higher harmonic reduced in size by the factor $\alpha|A|^2$.