ACM 100b

Convergence of Fourier series

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Convergence of the Fourier series

- So if we substitute the values of α_k and β_k into I to see where the minimum lies we find

$$I = \int_{-L}^{L} f(x)^2 dx - L \sum_{k=1}^{N} \left(A_k^2 + B_k^2 \right) - 2LB_0^2$$

- Now note too that the integral I has to be ≥ 0 by definition so

$$\int_{-L}^{L} f(x)^2 dx - L \sum_{k=1}^{N} \left(A_k^2 + B_k^2 \right) - 2LB_0^2 \geq 0$$

or

$$2LB_0^2 + L \sum_{k=1}^{N} \left(A_k^2 + B_k^2 \right) \leq \int_{-L}^{L} f(x)^2 dx$$

Convergence of Fourier series

- But now note that when we let $N \to \infty$ we recover Parseval's theorem which says

$$\int_{-L}^{L} f(x)^2 dx = \left[2LB_0^2 + L \sum_{n=1}^{\infty} \left(A_n^2 + B_n^2 \right) \right]$$

- That is as we get closer and closer to the minimum we see that

$$\lim_{N\to\infty} \int_{-L}^{L} \Bigg[f(x) - \left[B_0 + \sum_{n=1}^{N} \left(A_n \sin(n\pi x/L) + B_n \cos(n\pi x/L) \right) \right] \Bigg]^2 dx = 0$$

Convergence of Fourier series

- We see then that Fourier series converge in the sense that the mean square error goes to zero as the number of terms increases.
- We see too that any function can be approximated this way as long as

$$\int_0^{2\pi} f(x)^2 dx \quad \text{is finite.}$$

- We say, then that the set of functions $\cos(n\pi x/L)$ and $\sin(n\pi x/L)$ are *complete* over the interval $-L < x < L$ in the sense of mean square.
- Again, this statement too is not unique to Fourier series.
- All solutions of regular Sturm-Liouville eigenvalue problems are complete over their respective interval.
- We have not been particularly rigorous in our derivations here but the derivations can be made rigorous.

Convergence of Fourier series

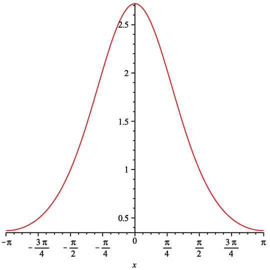
- We have shown that Fourier series converge to the function to be approximated in the sense of mean square.
- But we have not really given a picture of what this means.
- For example, does this mean that at each point x the Fourier series converges to the value of $f(x)$?
- And as a practical matter, how fast would that convergence be?
- So we will next examine what kind of convergence this is.
- We will start with some simple examples to gain some experience and then show some general results.

Some examples of convergence

- To start consider the function

$$f(x) = \exp(cos(x)) \qquad 0 \leq x \leq \pi$$

- This function is in fact very smooth for all values of x in the interval $0 < x < \pi$
- In fact it can be differentiated an infinite number of times and all the derivatives exist.
- Below is a plot of the function over the interval $-\pi \leq x \leq \pi$



Some examples of convergence

- Because it is a function of $\cos x$ it makes sense to consider expanding it as a Fourier cosine series over the interval $0 < x < \pi$ as follows:

$$f(x) = \sum_{n=1}^{\infty} B_n \cos(nx)$$

- So we need to calculate

$$B_0 = \frac{1}{\pi}\int_0^{\pi} \exp(\cos(x))dx$$
$$B_n = \frac{2}{\pi}\int_0^{\pi} \exp(\cos(x))\cos(nx)dx$$

- These integrals are not simple but they can be evaluated it turns out in terms of Bessel functions
- Since Bessel functions are well tabulated we can compute the coefficients

Some examples of convergence

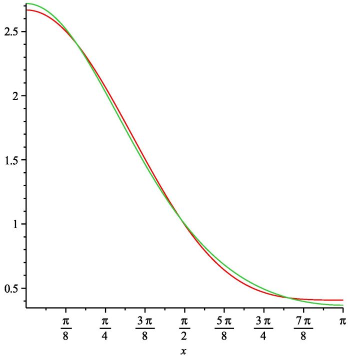
- We get
$$B_0 = I_0(1) \quad B_n = 2I_n(1)$$
where $I_n(x)$ is a well known Bessel function.
- If we evaluate the coefficients numerically we get

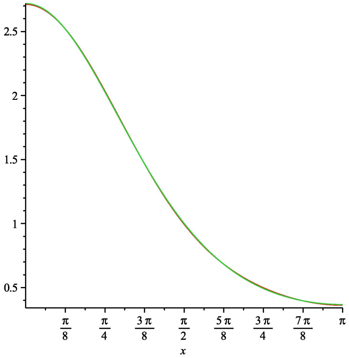
n	B_n
0	1.266065878
1	1.130318208
2	0.2714953396
3	0.04433684984
4	0.005474240442
5	0.0004497732296
6	0.00003198436462
7	$1.992124807 \times 10^{-7}$
8	$1.103677173 \times 10^{-8}$

- We can see the coefficients decay very rapidly after a while.

Some examples of convergence

- The Fourier series converges very quickly to exp(cos(x))
- Below are plots of the exact answer in green and the first 3 and 4 terms of the Fourier series





- Clearly this result is very encouraging

Some examples of convergence

- So it seems like a Fourier cosine series deals well with smooth functions
- Flush with confidence we look at the Fourier cosine series of

$$f(x) = x \qquad 0 \leq x \leq \pi$$

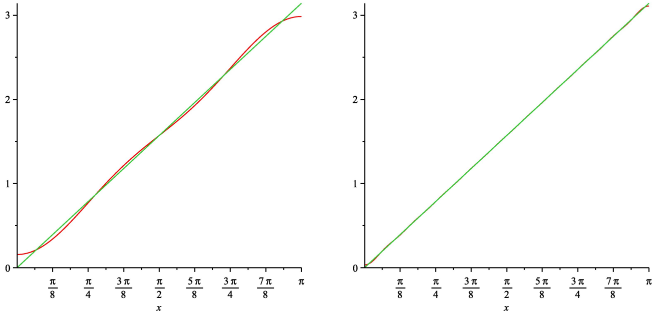
- This function is also very very smooth but it's not periodic.
- We can calculate the Fourier cosine coefficients in closed form in this case:

$$B_0 = \frac{1}{\pi}\int_0^{\pi} x dx = \pi/2$$
$$B_n = \frac{2}{\pi}\int_0^{\pi} x \cos(nx) dx = \frac{2}{\pi n^2}(-1 + (-1)^n)$$

- In this case the coefficients decrease quadratically so we expect the series to converge using standard tests of convergence.
- The series will converge but not as quickly as the previous example.

Some examples of convergence

- Let's see how well we do
- Below we plot the sum of the first 4 terms and the first 20 terms



- The results seem satisfactory but note the slope at the end points is not right
- The slope should be 1 but the cosines must have zero slope at $x = 0$ and $x = \pi$
- Still this does not contaminate things too badly.

Some examples of convergence

- Finally we try the Fourier sine series of the function

$$f(x) = 1$$

- This function is very smooth and is periodic.
- Indeed if you expand this in a Fourier cosine series you get back a one term cosine expansion because $\cos(0 \times x) = 1$
- Instead however we're trying a Fourier sine series
- We note the function is smooth but does not vanish at $x = 0$ or $x = \pi$
- The Fourier sine series coefficients are given by

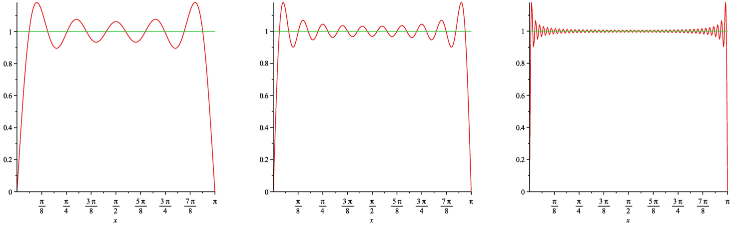
$$A_n = \frac{2}{\pi} \int_0^{\pi} \sin(nx) dx = -\frac{2}{\pi n} \left[-1 + (-1)^n\right]$$

Some examples of convergence

- This result is a bit more disconcerting
- The coefficients
$$A_n = -\frac{2}{\pi n}[-1 + (-1)^n]$$
decay only like n^{-1}.
- Such series may not converge
- It turns out the situation is not that bad here because the coefficients are oscillating and the sine functions oscillate too
- But it's hard to know by looking at the series whether things will converge
- So we plot some partial sums to see what is happening

Some examples of convergence

- Plotted below are the partial sums at 10, 20, and 100 terms



- We can see that in the in the interior we do converge albeit with an error that goes to zero something like $1/n$
- As we go to the boundary the sines must vanish
- But also the error seems to grow as we tend to the boundary
- We also note there is an overshoot as $x \to 0$ and $x \to \pi$
- If you look closely the overshoot is always about 18% and never goes away

Some examples of convergence

- This behavior is known as *Gibbs phenomenon* and was first explained by J. Willard Gibbs, a physicist at Yale who also made profound contributions to statistical mechanics and thermodynamics
- We have so far seen three very different convergence behaviors for what are nominally very smooth functions.
- How do we know which one to expect?
- Before we answer this we will do one more example

Some examples of convergence

- Consider the discontinuous function

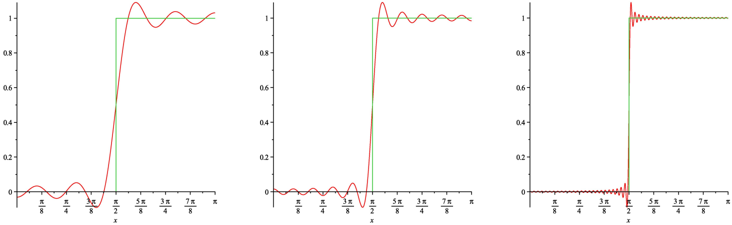
$$f(x) = \begin{cases} 0 & 0 < x < 1/2 \\ 1 & 1/2 < x < 1 \end{cases}$$

- We will try to expand this in a Fourier cosine series
- We have

$$B_0 = \frac{1}{\pi}\int_0^{\pi} f(x)dx = 1/2$$
$$B_n = \frac{2}{\pi}\int_0^{\pi} f(x)cos(nx)dx = -2\,\frac{\sin(1/2\,\pi\,n)}{\pi\,n}$$

- Again the coefficients decay like $1/n$ but this is not too surprising given our function is discontinuous.

- Below we plot the partial sums for $n = 10, 20, 100$



- Again we see the oscillatory behavior
- Strangely it's not "at" the location of the discontinuity (which is $x = \pi/2$) but on either side of it.
- There are overshoots (again 18% or so) on either side.
- Note the Fourier series gives a value at $x = \pi/2$ which is exactly the average of the right and left hand limits as you approach the discontinuity.

Convergence of Fourier series

Theorem (Uniform convergence of Fourier series)

If $f(x)$ is piece-wise differentiable and absolutely integrable on $0 \leq x \leq 2\pi$, then the (full periodic) Fourier series for $f(x)$ converges uniformly to $f(x)$ but only where $f(x)$ is differentiable. If $f(x)$ has a jump discontinuity say at $x = a$, the Fourier series converges to

$$F(a) = \frac{f(a^+) + f(a^-)}{2}$$

where $f(a^\pm)$ is short hand for the value you get when you approach a from positive (respectively negative) values. However, the convergence to $F(a)$ is not uniform in any neighborhood of $x = a$.

- Because the sine and cosine series can be converted to periodic series we'll be able to understand shortly convergence for these series as well.

Convergence of Fourier series

- One thing we should point out is that all of the functions we tried are square integrable

$$\int_0^L f(x)^2 dx \text{ is finite}$$

- So we should expect that the mean square error of the Fourier series should go to zero.
- In fact it does for all the examples we considered
- But we see that convergence in mean square doesn't always mean convergence to the function in a point-wise sense
- In some cases we do see this and in other cases we see the results depending on how limits are taken
- We'll next examine things more closely to see why we got the results we did