Physics 106b: Lecture 5 **21 January, 2014**

Nonlinear Oscillators and Perturbation Theory

We have mainly looked at solvable systems in (Newtonian) classical mechanics:

- One degree of freedom systems with a conserved Hamiltonian, that can be reduced to an integral (particle in a potential, the method of quadratures)
- Problems that reduce to this using conservation laws to eliminate other degrees of freedom (e.g. the Kepler problem)
- problems that decouple into many 1 DOF problems (the normal modes of coupled *linear* oscillators.

An important insight of Poincare in his study of the 3-body Kepler problem is that most problems are not "analytically solvable". Poincare wrote that he found

> [phase space trajectories] so tangled that I cannot begin to even draw them

and

> ... small differences in initial positions may lead to enormous differences in the final phenomena. Prediction becomes impossible.

Poincare found what we now call *chaos* and identified two of the key properties.

The three body problem is hard, so I will approach the subject instead through *nonlinear* oscillators. Trajectories in the two dimensional phase space for a time-independent, one degree of freedom problem are necessarily simple, since they cannot cross. Thus we will look at:

- 2 coupled nonlinear Hamiltonian oscillators;
- a single periodically driven nonlinear oscillator (Hamiltonian or dissipative).

In the latter case, the drive may sustain the motion in the presence of dissipation. Without drive, of course, the motion would die out without drive.

The simple pendulum is a nonlinear oscillator. The potential is

$$V(q) \propto (1 - \cos q) \simeq \tfrac{1}{2}q^2 - \tfrac{1}{24}q^4 + \cdots . \tag{1}$$

Even a system as simple as the periodically driven pendulum shows chaos for some parameter ranges! An oscillator based on a superconducting Josephson device has the same equation of motion as the simple pendulum: this is often used as a nonlinear oscillator in sophisticated experiments.

Truncating the expansion of a nonlinear potential beyond the q^4 leads to model called the Duffing oscillator. In this lecture we will study the Duffing oscillator in regions where the motion remains periodic but is no longer simple harmonic. As well as introducing the basic phenomena of nonlinear oscillators, this will serve to introduce some *perturbation methods* used to approximately solve these problems where exact solutions are no longer available or are not easy to find.

Duffing Oscillator

The general equation of motion, including damping and driving, is

$$m\frac{d^2q}{dt^2}+\Gamma\frac{dq}{dt}+Kq+aq^3=F\cos(\omega_d t), \tag{2}$$

with K the linear spring constant and α the size of the nonlinear correction. Define $\omega_0^2=K/m$, $Q=m\omega_0/\Gamma$, $\alpha=a/m\omega_0^2$, $f=F/m\omega_0^2$, and introduce the scaled time $\bar{t}=\omega_0 t$ and scaled frequency $\omega=\omega_d/\omega_0$, and let $\dot{q}$ denote $dq/d\bar{t}$. The equation of motion becomes

$$\ddot{q}+\frac{1}{Q}\dot{q}+q+\alpha q^3=f\cos(\omega\bar{t}) \tag{3}$$

I'll now drop the bar on the time - you should remember that t (standing in for $\bar{t}$) is the scaled time below.

Perturbation theory

First we consider the case with no dissipation (Hamiltonian) and no drive. The equation of motion is

$$\ddot{q}+q+\alpha q^3=0 \tag{4}$$

This is actually a problem we can reduce to a single integral in the usual way (see Hand and Finch §4.1). It can also be calculated using canonical perturbation theory as discussed in Lecture 11 of Ph106a. If the nonlinear term is absent, the solution is known, of course — the equation is a simple harmonic oscillator. Here we will try to solve for the solution approximately when the effect of the nonlinear term is small by expanding about the SHO using perturbation theory. To formalize the idea that the nonlinear term is small, we imagine that α is small, and relabel it ε to remind us of this.[1] So the equation reduces to

$$\ddot{q}+q+\varepsilon q^3=0 \qquad \varepsilon\ll 1. \tag{5}$$

We solve for some initial condition, e.g. $q(0)=A$, $\dot{q}(0)=0$.

We attempt to solve Eq. (5) as a power series expansion in ε, also known as the *method of successive approximations*. Write

$$q(t)=q_0(t)+\varepsilon q_1(t)+\varepsilon^2 q_2(t)+\cdots. \tag{6}$$

We substitute this into the equation of motion (5) and collect terms at each order in ε

$$\epsilon^0(\ldots)+\epsilon^1(\ldots)+\epsilon^2(\ldots)+\cdots=0. \tag{7}$$

We can imagine varying ε, and powers are linearly independent, so we argue the each $(\ldots)$ must be zero. This successively gives equations of motion for the $q_k(t)$.

At order ε^0 we have

$$\ddot{q}_0+q_0=0. \tag{8}$$

This is just the linear oscillator and the solution for the given initial conditions is

$$q_0=A\cos t. \tag{9}$$

[1] There are often delicacies in setting up perturbation theories that tend to make an essentially straightforward approach seem obscure. For example, in the present case, we do not know that the effect of the nonlinearity is small, until we have set the size of q, which is the *solution* to the problem, not a parameter of the equations. In fact, we could rescale variables $\bar{q}=\varepsilon^{1/2}q$ to get an equation for $\bar{q}$ without any small parameter (try it). We develop the perturbation expansion of Eq. (5) thinking of ε as small, but the result will be valid for small εq^2. For the rescaled $\bar{q}$ problem we could put a parameter in front of the $\bar{q}^3$, expand in this parameter taking it to be small, and set it to 1 at the end. This is an often used – and correct – procedure, even though it sounds wrong!

At order ε^1 we have

$$\ddot{q}_1 + q_1 = -q_0^3 = -A^3 \cos^3 t, \tag{10}$$

and this is the structure of higher order terms as well: on the left hand side we get the linear oscillator equation for the unknown $q_k(t)$, and on the right hand side "driving" terms that are known from lower order calculations

$$\ddot{q}_k + q_k = F_k(t). \tag{11}$$

These are to be solved with initial conditions $q_k(0) = \dot{q}_k(0) = 0$ (the former because the finite initial value is already picked up by the zeroth order solution). We can use the Green function method to solve the equation

$$q_k = \int_0^t \sin(t - t') F_k(t') dt'. \tag{12}$$

In many cases this procedure will successfully generate terms at each order, and the solution Eq. (6) can be reconstructed to whatever order of accuracy is required. But in the present case there is a problem!

Consider the case $k = 1$

$$F_1(t) = -A^3 \cos t = -A^3(\tfrac{3}{4} \cos t + \tfrac{1}{4} \cos 3t), \tag{13}$$

and doing the integral gives

$$q_1(t) = A^3(-\tfrac{3}{8} t \sin t - \tfrac{1}{32} \cos t + \tfrac{1}{32} \cos 3t). \tag{14}$$

The first term proportional to $t \sin t$ is a problem. It is not periodic, and we expect a periodic solution, and furthermore, it becomes arbitrarily large at large times, so that the "small correction" $\varepsilon q_1(t)$ will eventually become as large as the zeroth order term $q_0(t)$, and the perturbation expansion breaks down. Such a perturbation is called *secular*. Physically we can understand the problem in the mathematics as arising because the driving $F_1(t)$ contains terms resonant with the linear oscillator given by the EOM for q_k. Secular terms in perturbation expansions are very important in dynamics and other situations. They can arise for technical reasons in formulating the problem (here, we will see, it arises because we have assumed the wrong frequency), or for physical reasons (e.g. in the solar system, the "small" perturbation on a planetary orbit from a distant planet may be resonant with the periodic motion of the planet, and so the effect accumulates over many periods). They can also arise because some feature of the zeroth order solution is undetermined at the zeroth order and is then fixed by removing secular terms at first order. There are many different ways of organizing the perturbation theory, but the basic idea is usually the same.

Note that I used the Green function method to solve for q_1, since this is an easy way to establish the secular term. If there are no secular terms, this is a *hard* way to solve the equation: it is much easier to assume sinusoidal solutions of unknown amplitudes and with frequencies matching those in the driving terms, and fix the amplitudes by direct substitution. Try this for the $\cos 3t$ term, which is not resonant with the linear oscillator, and so does not lead to secular terms.

Lindstedt-Poincaré perturbation theory

The reason for the breakdown of the perturbation expansion, signaled by the appearance of secular terms, in the present case is that our ansatz for the solution Eq. (6) did not allow for the obvious physical effect that the frequency of the oscillations might change. Including this idea is known as *Lindstedt-Poincaré perturbation theory*.

To implement the approach, we also expand the frequency in powers of ε

$$\omega = 1 + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \cdots . \tag{15}$$

We then introduce the variable $s = \omega t$, and look for a solution in the form

$$q = q(s) = \sum_{k=0}^{\infty} \varepsilon^k q_k(s). \tag{16}$$

Then

$$\dot{q} = \frac{dq}{dt} = \omega \frac{dq}{ds} \equiv \omega q'(s). \tag{17}$$

The equation of motion Eq. (5) becomes

$$\omega^2 q'' + q + \varepsilon q^3 = 0, \tag{18}$$

with

$$\omega^2 = 1 + 2\varepsilon\omega_1 + \cdots . \tag{19}$$

Again, we plug Eq. (16) into the equation of motion, and collect terms at each order in ε.

At order ε^0 we have

$$q_0'' + q_0 = 0, \tag{20}$$

giving (with the boundary conditions) $q_0 = A\cos s$.

At order ε^1 we have

$$q_1'' + q_1 = -q_0^3 + 2\omega_1 q_0 = (2A\omega_1 - \tfrac{3}{4}A^3)\cos s - \tfrac{1}{4}A^3\cos 3s. \tag{21}$$

We *remove the secular term* which sets ω_1

$$\omega_1 = \tfrac{3}{8}A^2, \tag{22}$$

and then solve for $q_1(s)$ with $q_1(0) = q_1'(0) = 0$

$$q_1 = -\tfrac{1}{32}A^3(\cos s - \cos 3s). \tag{23}$$

The important result is that the frequency of the oscillations depends on the amplitude $\omega = \omega_{\mathrm{NL}}$ with

$$\omega_{\mathrm{NL}} = 1 + \tfrac{3}{8}\varepsilon A^2 + \cdots . \tag{24}$$

For positive ε, the effective spring of the oscillator becomes stiffer as the amplitude increases, and the frequency increases with amplitude of oscillation. We could continue the approach to higher order, as needed.

Driven damped case

Now consider the equation of motion

$$\ddot{q} + \frac{1}{Q}\dot{q} + q + \alpha q^3 = f\cos(\omega t) \tag{25}$$

After transients have died out, if f is not too strong we expect a response $q(t)$ that is oscillating at frequency ω, and we want to calculate its amplitude and phase relative to the drive. This can be calculated using the same type of perturbation methods.

Introduce $\tau = \omega t$ so that $q = q(\tau)$ is periodic with period 2π and write q' for $dq/d\tau$. Then $\dot{q} = \omega q'$ and Eq. (3) becomes

$$q'' + q = \left(1 - \frac{1}{\omega^2}\right)q - \frac{1}{\omega Q}q' - \frac{\alpha}{\omega^2}q^3 + \frac{f}{\omega^2}\cos\tau \tag{26}$$

To make progress assume the terms on the right hand side are small (small driving near resonance giving a small response, small dissipation). To keep track of orders in the perturbation theory we introduce the parameter μ

$$q'' + q = \mu\left[\left(1 - \frac{1}{\omega^2}\right)q - \frac{1}{\omega Q}q' - \frac{\alpha}{\omega^2}q^3 + \frac{f}{\omega^2}\cos\tau\right] \tag{27}$$

and expand the solution in powers of μ

$$q = q_0 + q_1\mu + q_2\mu^2\cdots \tag{28}$$

At the end of the calculation set μ to unity.

We substitute Eq. (28) into Eq. (27), organize the terms at each order in μ (i.e. collect all terms multiplied by μ^p for each p) and argue that the coefficient of each μ^p must be zero since we can imagine varying μ and the μ^p are linearly independent.

At order μ^0 we get

$$q_0'' + q_0 = 0 \tag{29}$$

which has the solution

$$q_0 = \tfrac{1}{2}[A\exp(i\tau) + \text{c.c.}], \tag{30}$$

where $A = |A|e^{i\delta}$ and δ is the phase of the solution relative to the drive. Note that at zeroth order, both $|A|$, δ are free parameters. Since we do not know the phase of the solution, using complex notation reduces the algebra. Hand and Finch work the solution through using $\cos(\tau + \delta)$. They also use the notation A_1 for my A, with the subscript 1 referring to the first harmonic.[2]

The right hand side of Eq. (27) has an explicit factor of μ, so at order μ^1 we replace all the q by q_0. Note that

$$q_0^3 = \tfrac{1}{8}(A^3e^{3i\tau} + 3|A|^2Ae^{i\tau} + \text{c.c.}). \tag{31}$$

At $O(\mu^1)$

$$q_1'' + q_1 = \frac{1}{2}\left\{\left[\left(1 - \frac{1}{\omega^2}\right) - \frac{i}{\omega Q} - \frac{3\alpha}{4\omega^2}|A|^2\right]A + \frac{f}{\omega^2}\right\}e^{i\tau} + \left[-\frac{3\alpha}{8\omega^2 A^3}\right]e^{3i\tau} + \text{c.c.} \tag{32}$$

The terms in $\exp(i\tau)$ (and its complex conjugate) give secular terms in q_1 (growing linearly in time), becoming large at large times, so that the perturbation expansion would break down. We must *remove the secular terms* by setting the coefficient to zero

$$\left[\left(1 - \frac{1}{\omega^2}\right) - \frac{i}{\omega Q} - \frac{3\alpha}{4\omega^2}|A|^2\right]A = -\frac{f}{\omega^2} \tag{33}$$

Note that properties of the *zeroth* order solution are determined by the requirement of removing secular terms in the *first* order calculation. Equation (33) can be written in the form

$$A = -\frac{f}{(\omega^2 - \omega_{\rm NL}^2) - i\omega/Q}. \tag{34}$$

This is the usual resonance formula, except that the resonant frequency is now the amplitude dependent frequency $\omega_{\rm NL}$ found for the undriven case, Eq. (24), so that

$$\omega_{\rm NL}^2 \simeq 1 + \tfrac{3}{4}\alpha|A|^2 + O(|A|^4), \tag{35}$$

[2] Note that we are looking for a solution close to some simple zeroth order solution, in the present case oscillations at the drive frequency. Our solution will not include transients from some initial condition, which may be far from this solution. Correspondingly, we do not enforce an initial condition, since the transients from a particular initial condition are not included in the ansatz.

(remember we replaced $\alpha \to \varepsilon$). Equation (33) determines the magnitude of the response $|A|$ and the phase δ

$$|A|^2 = \frac{f^2}{(\omega^2 - \omega_{\mathrm{NL}}^2)^2 + (\omega/Q)^2}, \qquad \tan\delta = -\frac{\omega/Q}{\omega^2 - \omega_{\mathrm{NL}}^2}. \tag{36}$$

If Q is reasonable large, where the response is large near resonance we can approximate

$$\omega^2 - \omega_{\mathrm{NL}}^2 \simeq 2\omega_{\mathrm{NL}}(\omega - \omega_{\mathrm{NL}}) \simeq 2(\omega - \omega_{\mathrm{NL}}) \tag{37}$$

and rewrite the equation for the magnitude of the response

$$\frac{|A|^2}{f^2 Q^2} \simeq \frac{(1/2Q)^2}{(\omega - \omega_{\mathrm{NL}})^2 + (1/2Q)^2}. \tag{38}$$

Since ω_{NL} is a linear function of $|A|^2$, this is actually a cubic equation for $|A|^2$. The solution is plotted

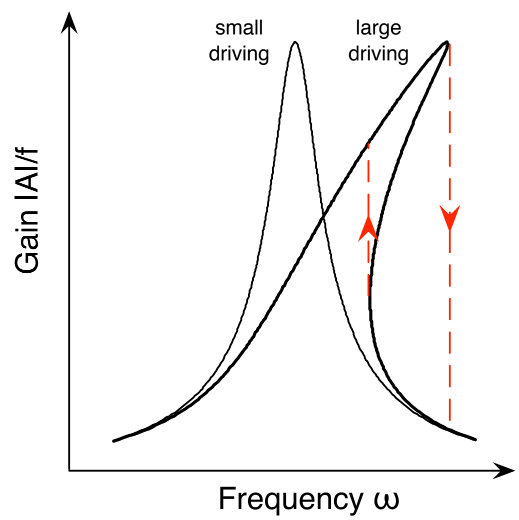

Figure 1: Ratio of oscillator amplitude to strength of forcing for weak and strong driving for the driven damped Duffing oscillator with "spring stiffening". The red arrows show the jumps in response that would be seen on up and down frequency sweeps.

in Fig. 1 for a small value of the forcing strength f, where the linear theory is sufficient, and for a larger value where the nonlinear effects are evident. (See also the course website for a Mathematica Notebook.) It is convenient to plot the ratio of the response to the forcing (the gain) since the peak value at the shifted resonance frequency takes on a value independent of the strength of forcing with the approximations we are using. Notice the *multiplicity of solutions*: in the frequency range where there are 3 solutions for the amplitude, it turns out that the upper and lower ones are stable, but the intermediate one is unstable. On sweeping the drive frequency up and down, as in a typical resonance measurement, there will be *hysteresis* for large enough f – over some range different values are measured as the trace follows the stable branch until it ends at a *saddle node bifurcation*. An interesting recent application of this phenomenon to quantum nondemolition measurements using a Josephson device as the nonlinear oscillator is *Siddiqi et al.*, Phys. Rev. Lett. **93**, 207002 (2004).

Now that we have taken care of the secular terms, we can solve Eq. (32). Since the remaining driving is at frequency 3τ and "off resonance", the solution is $q_1 \propto e^{3i\tau}$ with a finite amplitude

$$q_1 = \frac{3\alpha}{64\omega^2} A^3 e^{3i\tau} + \text{c.c.} \simeq \frac{3}{32}\alpha |A|^3 \cos[3(\tau + \delta)], \tag{39}$$

(using $\omega \simeq 1$). Note that the nonlinearity induces *harmonics* (here the third) of the drive frequency. The perturbation expansion is based on the idea that q_1 is small compared with q_0: this will be true if $\alpha |A|^2 \ll 1$. Going to higher order in the expansion, would lead to more harmonics. You should be able to convince yourself that there will be further harmonics with frequencies 5,7,..., all odd multiples of the drive frequency, and that the amplitudes of the higher order harmonics are reduced in size by the factor $\alpha |A|^2$ at each successive order.

Harmonic Analysis

The driven problem just treated is actually easier to solve than the undriven, undamped problem, since the solution (after transients) is oscillations with the same period as the drive. Once we understand that the nonlinearity induces harmonics, we could solve the problem a different way, by expanding

$$q(t) = \sum_n [A_n(\omega)\cos(n\omega t) + B_n(\omega)\sin(n\omega t)] . \tag{40}$$

We expect A_n, B_n to decrease with n if the effect of the nonlinearity is small, as in the previous section. I consider here the solution of the driven but undamped case

$$\ddot{q} + q + \alpha q^3 = f\cos(\omega t), \tag{41}$$

This is easier than the damped case because $B_n = 0$ (this follows from the symmetry of the equation $t \to -t$) and also there are only n odd terms (which follows from the symmetry $t \to t + \pi/\omega$, $q \to -q$). Thus

$$q(t) = \sum_{n \text{ odd}} A_n(\omega)\cos(n\omega t), \tag{42}$$

where for f small we expect the A_n to decrease with increasing n.

Substituting into Eq. (41) gives

$$\left[(1-\omega^2)A_1 + \tfrac{3}{4}\alpha A_1^3 \cdots\right]\cos\omega t + \left[(1-9\omega^2)A_3 + \tfrac{1}{4}\alpha A_1^3 + \cdots\right]\cos 3\omega t + \cdots = f\cos\omega t, \tag{43}$$

where the $\cdots$ contain higher harmonics or higher order terms. Balancing the coefficients of each harmonic gives

$$\left[(1-\omega^2) + \tfrac{3}{4}\alpha A_1^2\right] A_1 = f, \tag{44}$$

$$(1-9\omega^2)A_3 = -\tfrac{1}{4}\alpha A_1^3. \tag{45}$$

The first equation gives the undamped version of Eq. (34), and the second gives the third harmonic corrections.

Subharmonics

Nonlinearity naturally introduces harmonics of the drive, growing steadily from infinitesimal values as the drive is increased. For strong enough drive, sometimes subharmonic response, at some rational fraction of the drive frequency develops. Such behavior is harder to find, since you have to guess the existence, and see if it works. Here's the example from Hand and Finch pp 409-11 for you to work through.

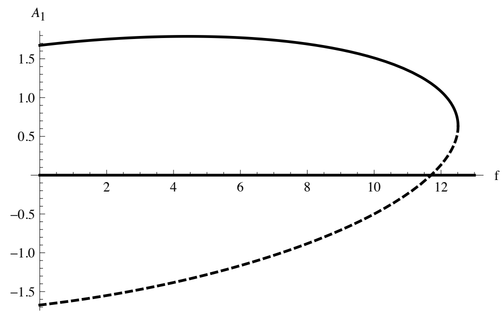

Figure 2: Amplitude A_1 of subharmonic response for Eq. (46) with $\alpha = 0.1$, $\omega = 1.1$

For example consider the undamped Duffing oscillator driven at a frequency near 3 times the linear resonant frequency. The large response of the resonator is near the resonant frequency, but there is no drive at this frequency. Can such a motion develop?

The equation of motion is

$$\ddot{q} + q + \alpha q^3 = f \cos 3\omega t, \tag{46}$$

with $\omega \simeq 1$. Now, although there is no component present in the drive, we guess a solution of the form [3]

$$q(t) = A_1 \cos \omega t + A_3 \cos 3\omega t + \cdots . \tag{47}$$

We substitute into the equation of motion, and keep terms at ωt and $3\omega t$. The messy term is $q^3 \simeq (A_1 \cos \omega t + A_3 \cos 3\omega t)^3$, which we want to write as a sum of $\cos n\omega t$ terms. Mathematica's *TrigReduce* does this for us

$$q^3 \simeq (A_1 \cos \omega t + A_3 \cos 3\omega t)^3 = (\tfrac{3}{4}A_1^3 + \tfrac{3}{4}A_1^2 A_3 + \tfrac{3}{2}A_1 A_3^2) \cos \omega t + \cdots , \tag{48}$$

where we will not need the higher order $\cdots$ terms. Balancing coefficients of the first and third harmonics in Eq. (46) to the order we need gives

$$A_1 \left[(1 - \omega^2) + \tfrac{3}{4}A_1^2 + \alpha(\tfrac{3}{4}A_1 A_3 + \tfrac{3}{2}A_3^2) \right] = 0, \tag{49}$$

$$(1 - 9\omega^2) A_3 = f. \tag{50}$$

The second equation just gives the off-resonance response at the drive frequency. The first equation *always* has the solution $A_1 = 0$ (no subharmonic response) but *may* have the second pair of solutions

$$A_1 = -\frac{f}{2(1 - 9\omega^2)} \pm \frac{1}{2\sqrt{3}} \sqrt{\frac{16(\omega^2 - 1)}{\alpha} - 21 \frac{f^2}{(1 - 9\omega^2)^2}} , \tag{51}$$

(using the value of A_3). The solutions are plotted in Fig. 2 assuming $f > 0$. It turns out the lower branch, shown dashed, is unstable. Remember $A_1 = 0$ is also a solution, and it turns out to be stable.

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[3] There are also solution with $\sin \omega t$ dependence, but these just correspond to a first harmonic phase shifted by 1 or 2 periods of the drive, and so are not different solutions.