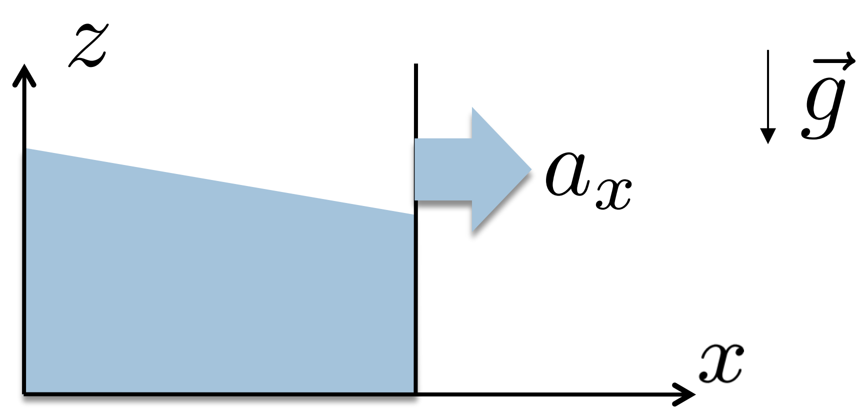
Lecture 5: Fluid statics, cont'd

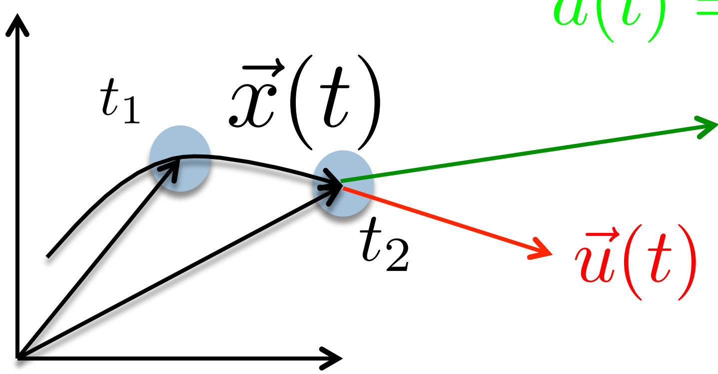
- Rigid body motion of a fluid
- Surface tension



Rigid body motion

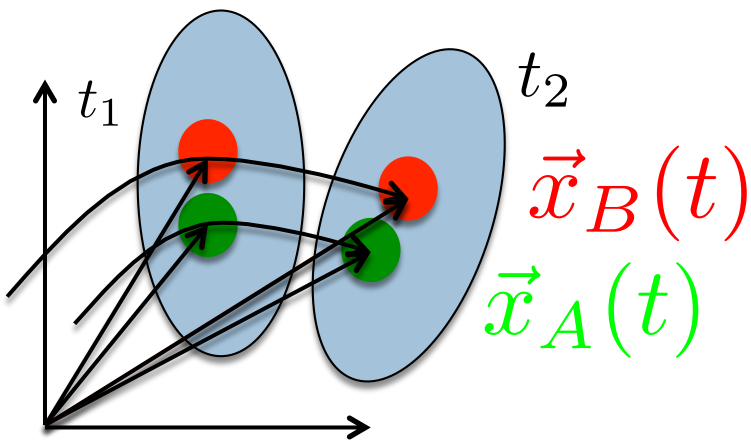
- Material is not deforming
- Distance between any two parcels of *material (as opposed to points in space)* remains constant during motion

$$\vec{a}(t) = \frac{d\vec{u}}{dt} = \frac{d^2\vec{x}}{dt^2} = \ddot{\vec{x}}$$



$$\vec{u}(t) = \frac{d\vec{x}}{dt} = \dot{\vec{x}}$$

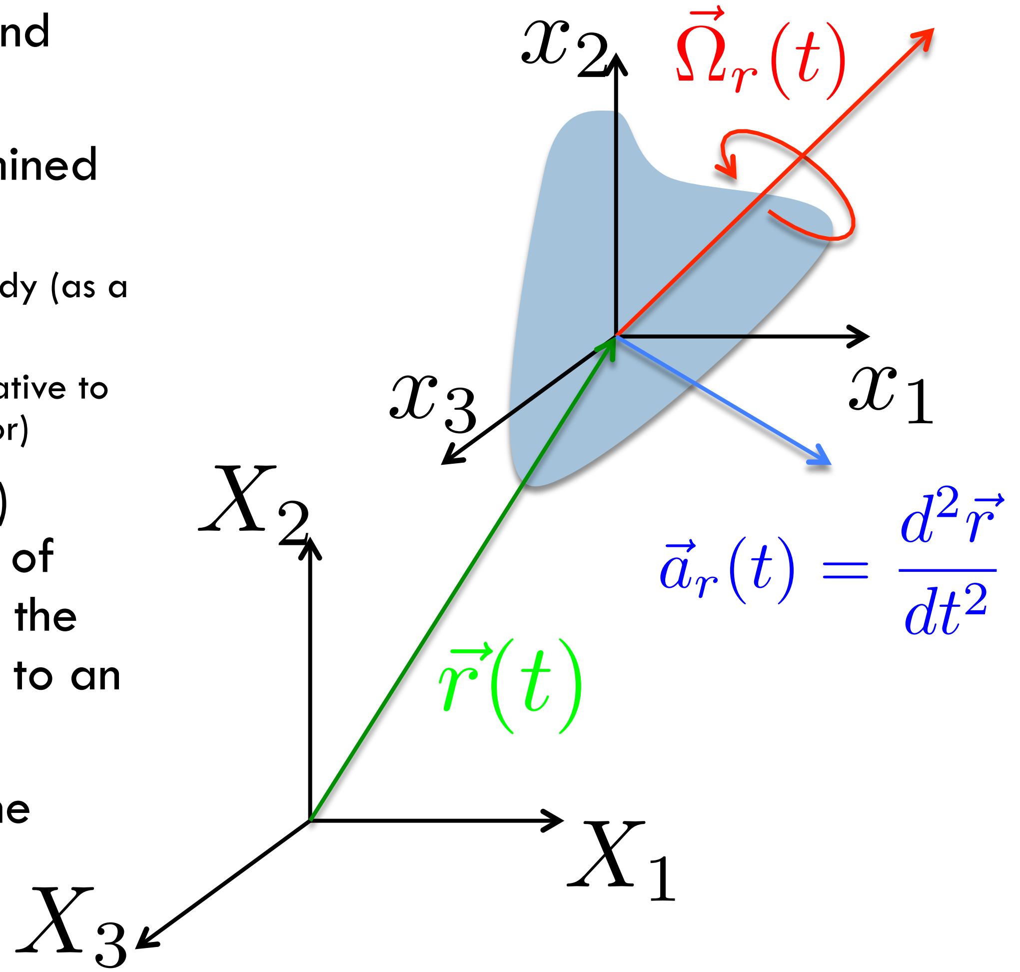
Rigid body motion



$$|\vec{x}_A(t) - \vec{x}_B(t)| = \text{const} \ \forall \ A, B \in \text{body}$$

Rigid body motion

- The body can (only) translate and rotate
- Entire motion (all points) determined in terms of
  - The position of one particle on the body (as a function of time)
  - The angular position of a particle relative to coordinate system (rotation of a vector)
- Typically, attach a (non-inertial) coordinate system to the center of mass of the body, and measure the position and angle with respect to an inertial coordinate system.
- The velocity of particles w.r.t. the non-inertial frame is now zero



You may recall this formula from ME35

- We must write Newton's law in an inertial reference frame
- It is easier to describe the motion of the body (i.e. it is zero) in the non-inertial (unsteadily translating/rotating) reference frame
- Trick: decompose motion as part relative to reference frame + reference frame motion → plug into Newton's law and solve
- The result is the original equations plus a 'fictitious' body force (force/unit volume)

$$\vec{f}_{fict} = -\rho \left(\vec{a}_r + \vec{\Omega}_r \times \left(\vec{\Omega}_r \times \vec{x} \right) + \frac{d\vec{\Omega}_r}{dt} \times \vec{x} \right)$$

New force balance

- Force balance for a fluid that is moving as a rigid body in a coordinate system fixed to (and rotating with) the body

Translational acceleration

Centrifugal acceleration

@$#@!

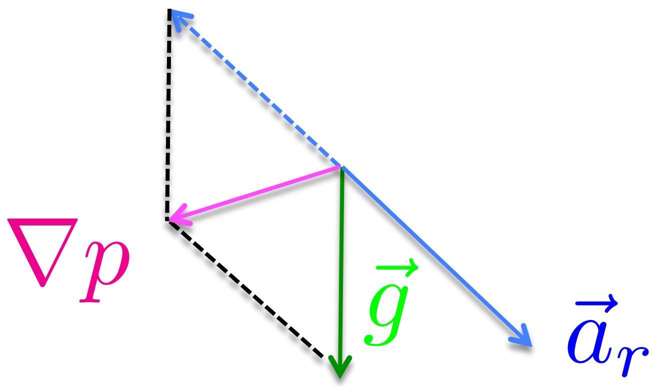
$$\vec{f}_{fict} = -\rho\left(\vec{a}_r + \vec{\Omega}_r \times \left(\vec{\Omega}_r \times \vec{x}\right) + \frac{d\vec{\Omega}_r}{dt} \times \vec{x}\right)$$

$$\boxed{\sum \vec{f} = -\nabla p + \rho\vec{g} + \vec{f}_{fict} = 0}$$

Note that the fluid parcel velocities are all zero when measured w.r.t. the accelerating coordinates → Hydrostatic problem

Translational (only) acceleration

- Consider a constant (in time) translational acceleration:
- Simple force balance



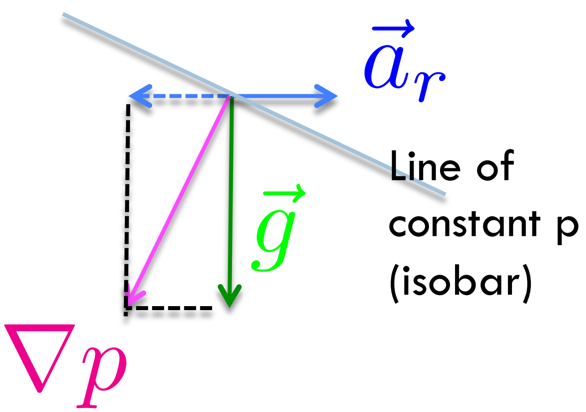
$$\rho\left(\vec{g} - \vec{a}_r\right) = \nabla p$$

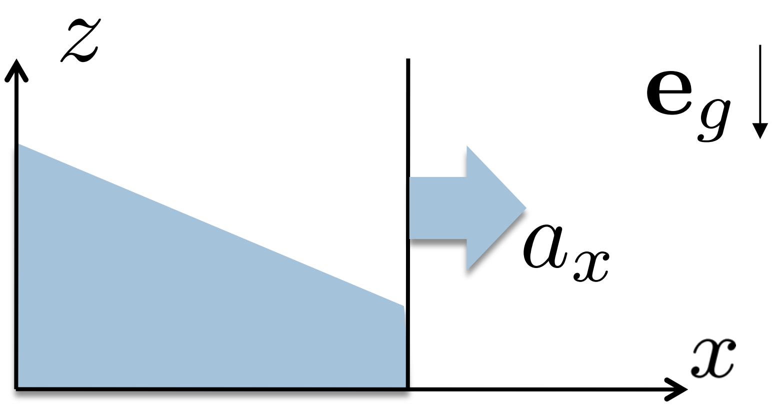
- Simplest example: Free fall (waterbomb)

$$\vec{a}_r \equiv \vec{g} \qquad\qquad p = \text{const} = p_a$$

Exercise

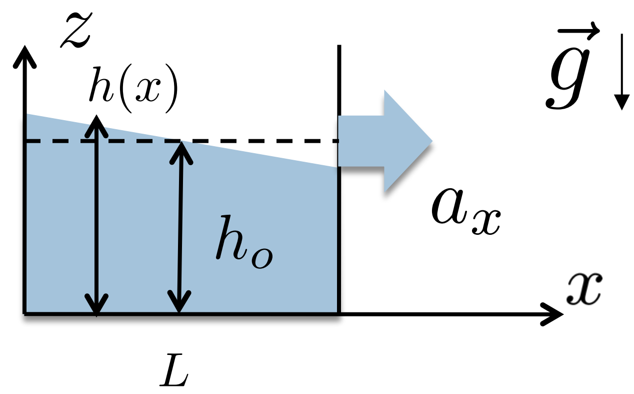
- A tank of water is accelerated at constant rate a_x
- Find an equation for the free surface and maximum pressure in the tank





Write out components

$$\frac{\partial p}{\partial x} = -\rho a_x$$
$$\frac{\partial p}{\partial y} = 0$$
$$\frac{\partial p}{\partial z} = -\rho g$$



Integrate

$$\frac{\partial p}{\partial x} = -\rho a_x \qquad \rightarrow \qquad p = -\rho a_x x + f(y, z)$$

$$\frac{\partial p}{\partial y} = 0$$

$$\frac{\partial p}{\partial z} = -\rho g$$

But…

$$\frac{\partial p}{\partial x} = -\rho a_x \qquad \rightarrow \qquad p = -\rho a_x x + f(y,z)$$

$$\frac{\partial p}{\partial y} = 0 \qquad \Leftrightarrow \qquad \frac{\partial p}{\partial y} = \frac{\partial f(y,z)}{\partial y}$$

$$\frac{\partial p}{\partial z} = -\rho g$$

$$f = f(z)$$

$$p = -\rho a_x x + f(z)$$

Now

$$p = -\rho a_x x + f(z)$$

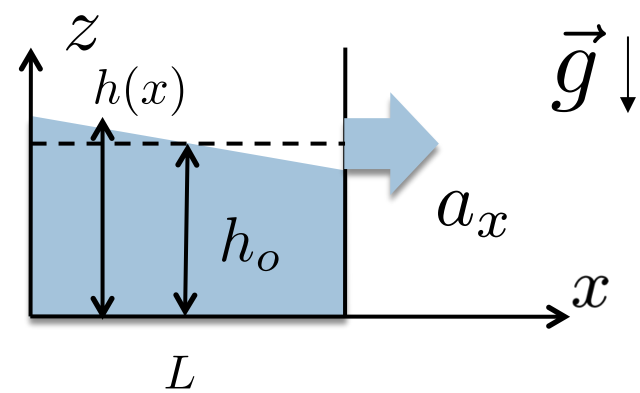
$$\frac{\partial p}{\partial z} = -\rho g = f'(z)$$

$$f(z) = -\rho g z + \text{const}$$

$$p = -\rho a_x x - \rho g z + \text{const}$$

Exercise, cont'd

$$p = -\rho a_x x - \rho g z + \text{const}$$



More explicitly:

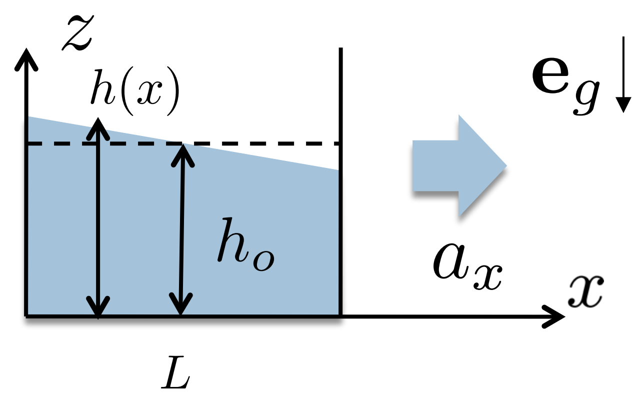
$$\begin{aligned} p(x,z) &= -\rho\left(gz + a_x x\right) + \text{const} \\ p(x,h(x)) &= -\rho\left(gh(x) + a_x x\right) + \text{const} = p_a \end{aligned}$$

$$h(x) = \frac{\text{const} - p_a}{\rho g} - x\frac{a_x}{g}$$

$$\int_0^L h(x)dx = h_o L$$

Integrate above exp and solve for const.

Exercise, cont'd



$$h(x) = h_o + \frac{a_x}{g}\left(\frac{L}{2} - x\right)$$

$$p(x,z) = p_a + \rho\left[a_x\left(\frac{L}{2} - x\right) + g\left(h_o - z\right)\right]$$

(Can check that p = p_a when z = h(x))

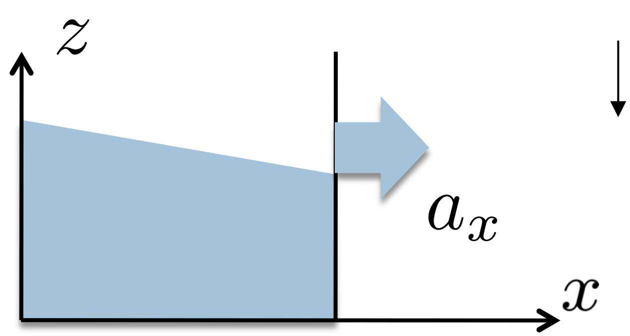
$$p_{max} = p(0,0) = p_a + \rho\left(\frac{a_x L}{2} + g h_o\right)$$

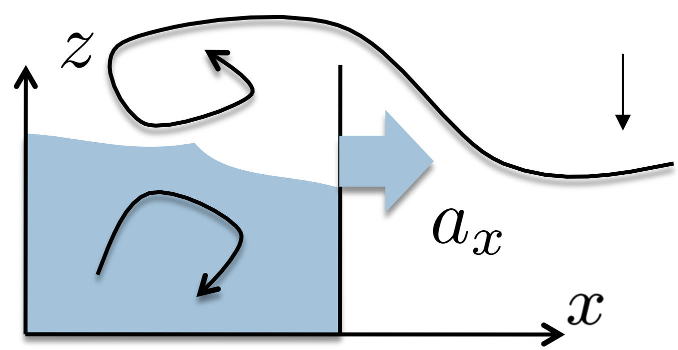
Achtung!

- It can be difficult to establish/maintain a fluid in rigid body motion without imposing some shear, and thus deforming the fluid. The results are often a reasonable first approximation, but a more detailed analysis awaits…
- Example

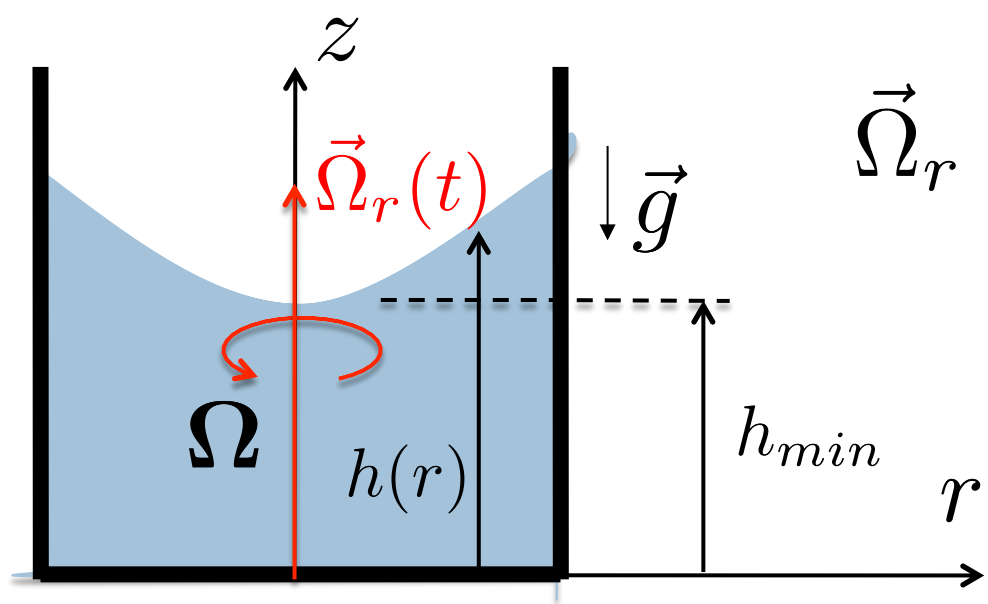
Idealized case, all fluid velocity (relative to cart) are zero

There is a flow of air around the cart and this may disturb the interface and induce motion in the liquid





Spinning cup: find h(r)



$$\vec{\Omega}_r = \Omega \hat{k}$$

Constant rotation in z dir.

$$\vec{f}_{fict} = -\rho \left(\vec{a}_r + \vec{\Omega}_r \times \left(\vec{\Omega}_r \times \vec{x} \right) + \frac{d\vec{\Omega}_r}{dt} \times \vec{x} \right)$$

Spinning cup, cont'd

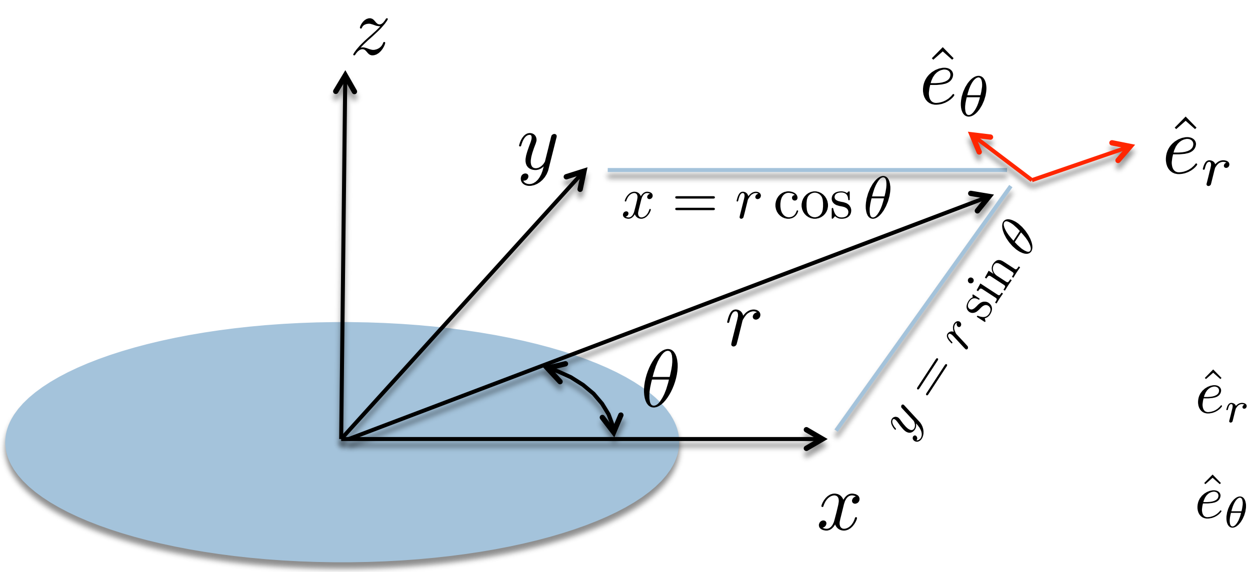
$$\vec{\Omega}_r \times \left(\vec{\Omega}_r \times \vec{x}\right) = -\Omega^2 x\hat{i} - \Omega^2 y\hat{j}$$

Because…

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \Omega \\ x & y & z \end{vmatrix} = -\Omega y\hat{i} + \Omega x\hat{j}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & \Omega \\ -\Omega y & \Omega x & 0 \end{vmatrix} = -\Omega^2 x\hat{i} - \Omega^2 y\hat{j}$$

Cylindrical coordinates



$$\hat{e}_r = \hat{i}\cos\theta + \hat{j}\sin\theta$$

$$\hat{e}_\theta = -\hat{i}\sin\theta + \hat{j}\cos\theta$$

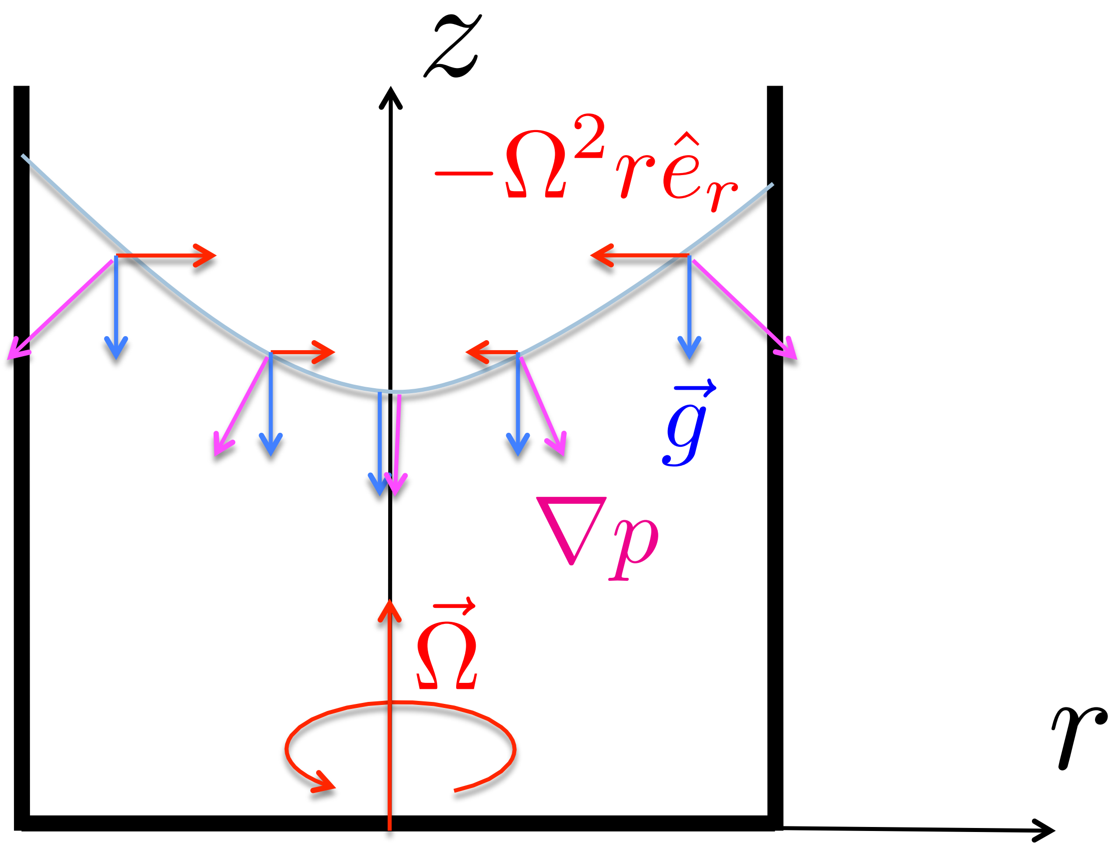
$$\hat{i} = \hat{e}_r\cos\theta - \hat{e}_\theta\sin\theta$$

$$\hat{j} = \hat{e}_r\sin\theta + \hat{e}_\theta\cos\theta$$

Spinning cup, cont'd

$$
\begin{aligned}
\vec{\Omega}_r \times \left(\vec{\Omega}_r \times \vec{x}\right) = -\Omega^2 x\hat{i} - \Omega^2 y\hat{j} \\
&= -\,\Omega^2 x \left(\hat{e}_r \cos\theta - \hat{e}_\theta \sin\theta\right) \\
&\quad -\,\Omega^2 y \left(\hat{e}_r \sin\theta + \hat{e}_\theta \cos\theta\right) \\
&= -\,\Omega^2 \left(x\cos\theta + y\sin\theta\right)\hat{e}_r \\
&\quad -\,\Omega^2 \left(-x\sin\theta + y\cos\theta\right)\hat{e}_\theta \\
&= -\,\Omega^2 \left(r\cos^2\theta + r\sin^2\theta\right)\hat{e}_r \\
&\quad -\,\Omega^2 \left(-r\cos\theta\sin\theta + r\sin\theta\cos\theta\right)\hat{e}_\theta \\
&= -\,\Omega^2 r\hat{e}_r
\end{aligned}
$$

Spinning cup, cont'd



Gradient in cylindrical coordinates?

$$\nabla p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k}$$

Use

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial p}{\partial \theta}\frac{\partial \theta}{\partial x} \qquad \frac{\partial r}{\partial x} = \cos\theta \qquad \frac{\partial \theta}{\partial x} = -\frac{\sin\theta}{r}$$

$$\frac{\partial p}{\partial y} = \frac{\partial p}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial p}{\partial \theta}\frac{\partial \theta}{\partial y} \qquad \frac{\partial r}{\partial y} = \sin\theta \qquad \frac{\partial \theta}{\partial y} = \frac{\cos\theta}{r}$$

Do algebra (or simply Google the result):

$$\nabla p = \frac{\partial p}{\partial r}\hat{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\hat{e}_\theta + \frac{\partial p}{\partial z}\hat{k}$$

Spinning cup, cont'd

- Force balance

$$0 = -\nabla p + \rho \vec{g} - \rho \left(-\Omega^2 r \hat{e}_r \right)$$

- Components

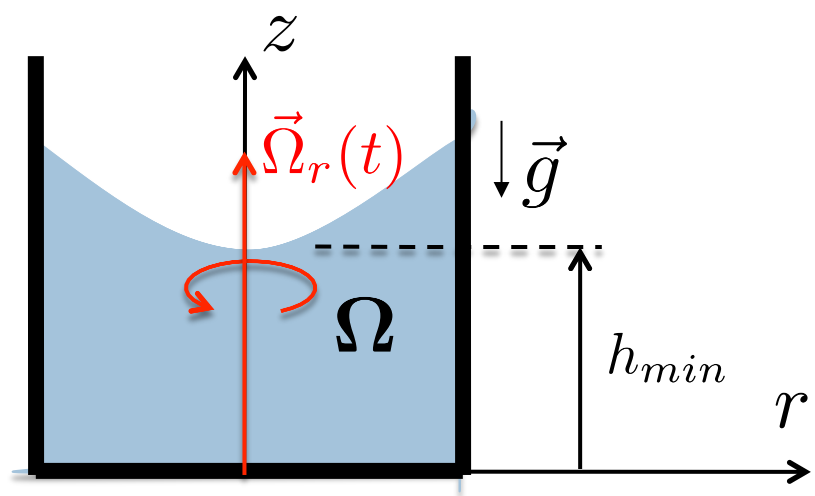
$$\frac{\partial p}{\partial r} = 0 + \rho \Omega^2 r$$

$$\frac{1}{r}\frac{\partial p}{\partial \theta} = 0 + 0 \qquad \Longrightarrow \qquad p = p(r, z)$$

$$\frac{\partial p}{\partial z} = -\rho g + 0$$

Spinning cup, cont'd

Integrate the PDE in exactly the same way as last problem (done on board)



We therefore have

$$p(r,z) = \frac{\rho\Omega^2 r^2}{2} - \rho g z + \text{const}$$

Good to double check above (by differentiating) before continuing!

$$p(0, h_{min}) = p_a \quad \Longrightarrow \quad \text{const} = p_a + \rho g h_{min}$$

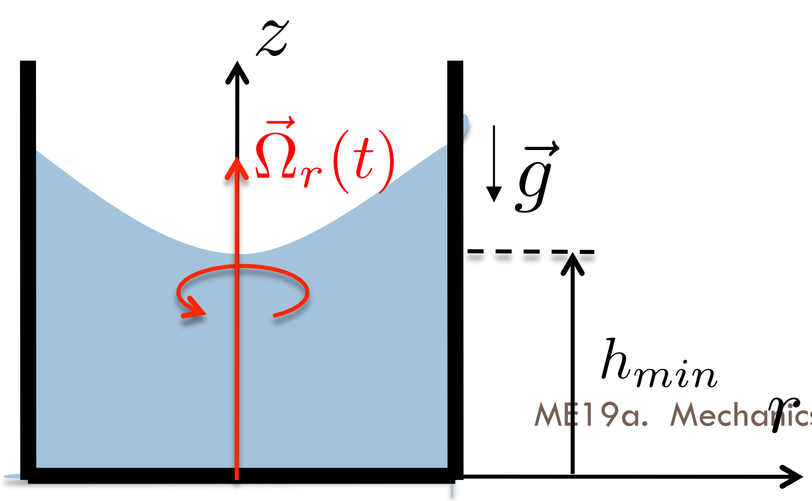
Spinning cup, cont'd

- Evaluate the constant (h_{min} is known and $p=p_a$ there)

$$p(0, h_{min}) = p_a \quad \Longrightarrow \quad \mathrm{const} = p_a + \rho g h_{min}$$

- So we obtain

$$p(r, z) = p_a + \frac{\rho \Omega^2 r^2}{2} - \rho g (z - h_{min})$$



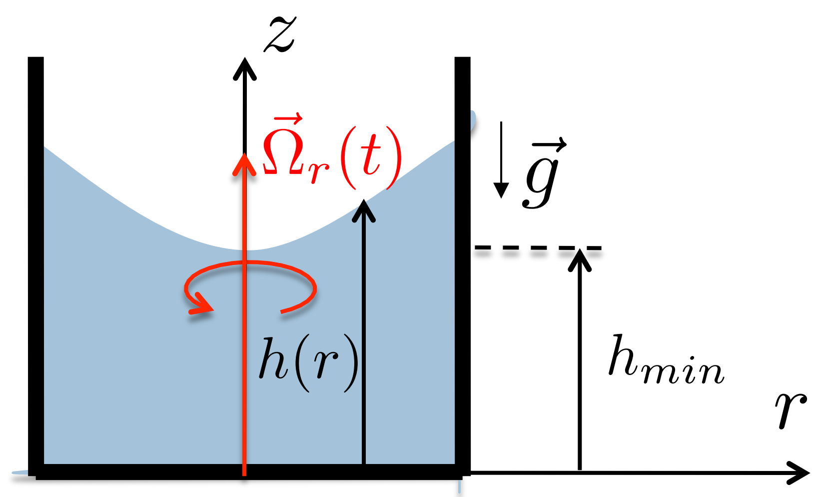
Spinning cup, cont'd

- Equation for free surface, h(r) ?

- Along free surface, z=h(r), p=p_a

$$p_a = p_a + \frac{\rho \Omega^2 r^2}{2} - \rho g (h(r) - h_{min})$$

$$h(r) = h_{min} + \frac{\Omega^2 r^2}{2g}$$



Statics with curved free surfaces

- Some situations with static (or nearly static) fluids involve curved free surfaces
  - Drops and bubbles
  - Meniscus
  - …
- Capillarity: In addition to the hydrostatic balance (pressure v. gravity), we need to include then a surface tension force at the interface in our equilibrium force balance

Surface tension

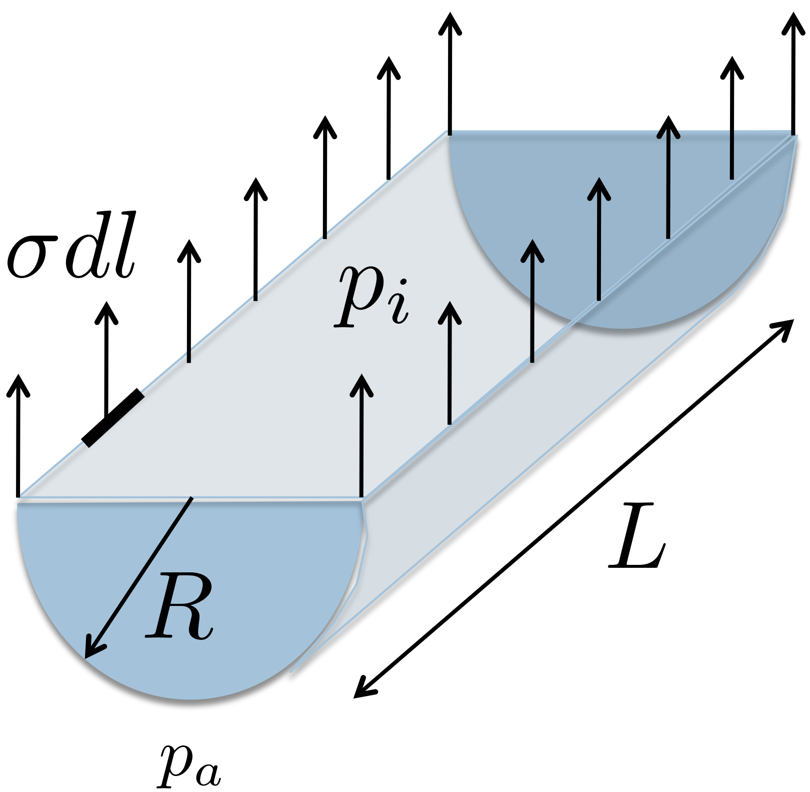
- Stronger intermolecular forces on liquid side of interface causes interface to contract (at least in the absence of gravity)
- At the continuum level, we can represent this as if there is a infinitely thin membrane stretched across the interface, with a tension that resists increasing the radius of curvature
- The surface tension, σ, is a property of the **pair of materials** (e.g. water/air), and has units of force per unit length

$$\sigma_{air,H_2O} = 0.074\mathrm{N/m}$$

At STP

Simple geometries – ignore gravity for now

- Cylinder



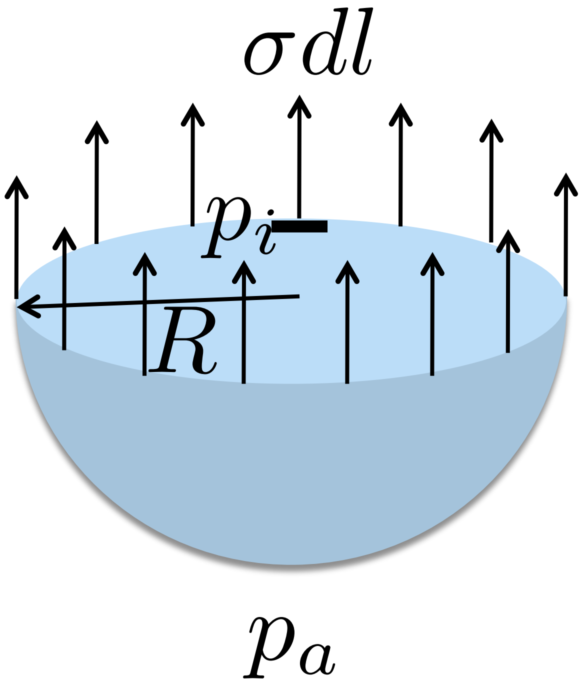
- Ignoring gravity, a force balance gives

$$2\sigma L + p_a(2RL) - p_i(2RL) = 0$$

$$p_i - p_a = \frac{\sigma}{R}$$

Simple geometries – ignore gravity for now

- Sphere



- Ignoring gravity, a force balance gives

$$2\pi R\sigma + p_a(\pi R^2) - p_i(\pi R^2) = 0$$

$$p_i - p_a = \frac{2\sigma}{R}$$

Simple geometries

- Exercise: A soap bubble is a thin layer of liquid with air inside and out
- If the surface tension is σ, what is the pressure difference across the bubble?

Simple geometries

- There are two interfaces with (starting from inside) a pressure decrease of σ / 2 R across each, so

$$p_i - p_a = \frac{4\sigma}{R}$$

Simple geometries

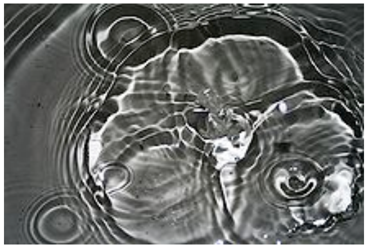
- For spherical case with air/water (droplet or bubble) @ 20 C where $\sigma = 0.074$ N/m

R	Δ p
1 μm	0.74 atm
1 mm	74 Pa
1 m	0.074 Pa

- For tiny drops and bubbles, the pressure jump is very large compared to the hydrostatic pressure – approximately spherical with ~constant pressure inside
- For long waves on the ocean, the pressure jump due to surface tension is miniscule, pressure variation with depth significant on scale of wavelength

Aside: Surface waves

- Small wavelength free surface waves (ripples) wave is balance between fluid inertia and surface tension

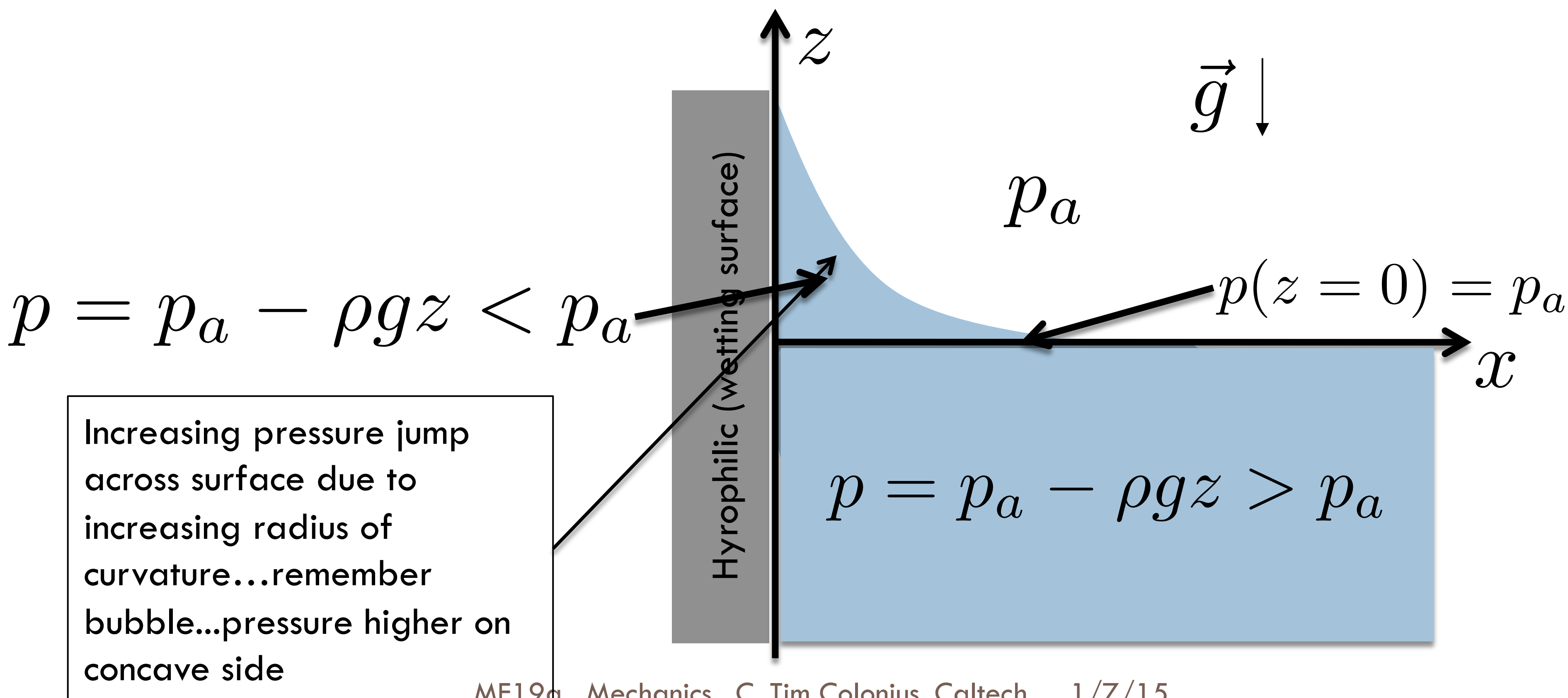


- Large wavelength free surface waves are a balance between gravity and fluid inertia

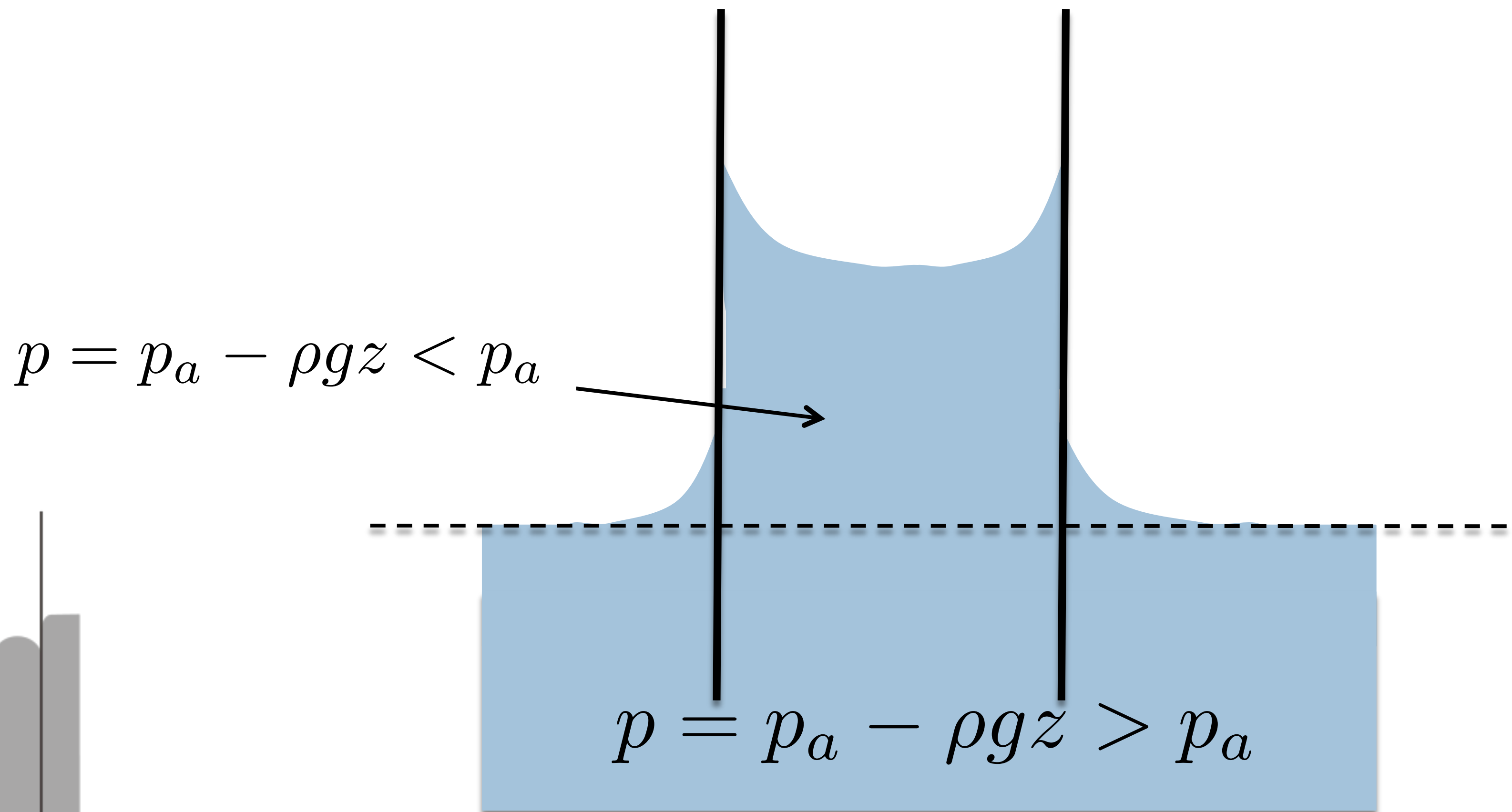
- In between waves involve all three effects

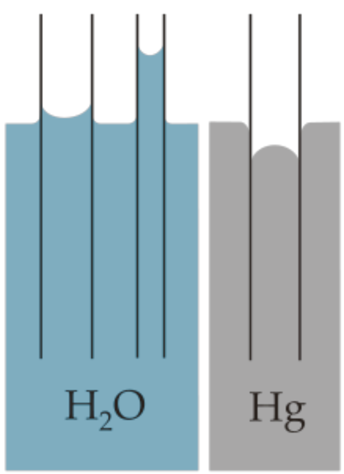
Meniscus

- Inside the fluid, the force balance is unaltered: pressure increases linearly with depth

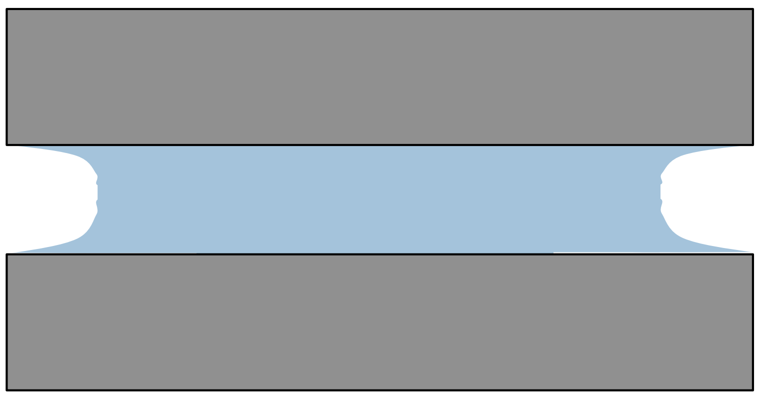


Capillary rise in a small tube





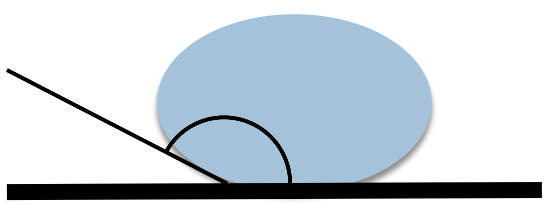
Why do wetted solids 'stick' together?



Contact angle

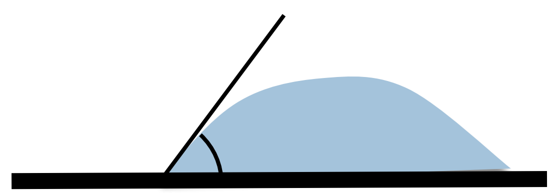
- When the free surface intersects a solid material, we form a *contact line*
- *At least for the static case, we can model this as there being a contact angle between the two surfaces (measured from wet to dry)*

Hydrophobic



$$\theta > 90^o$$

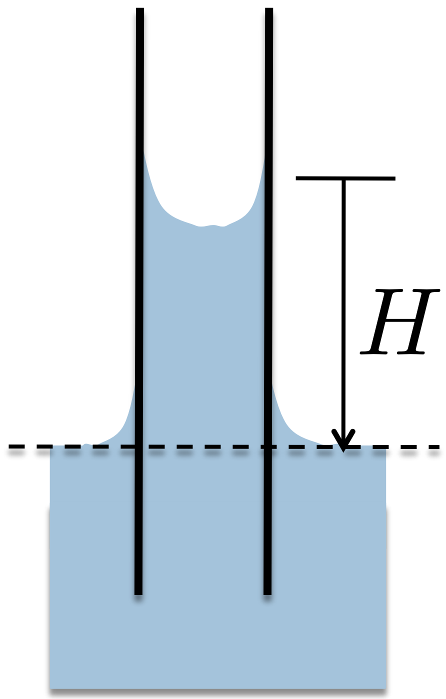
Hydrophilic

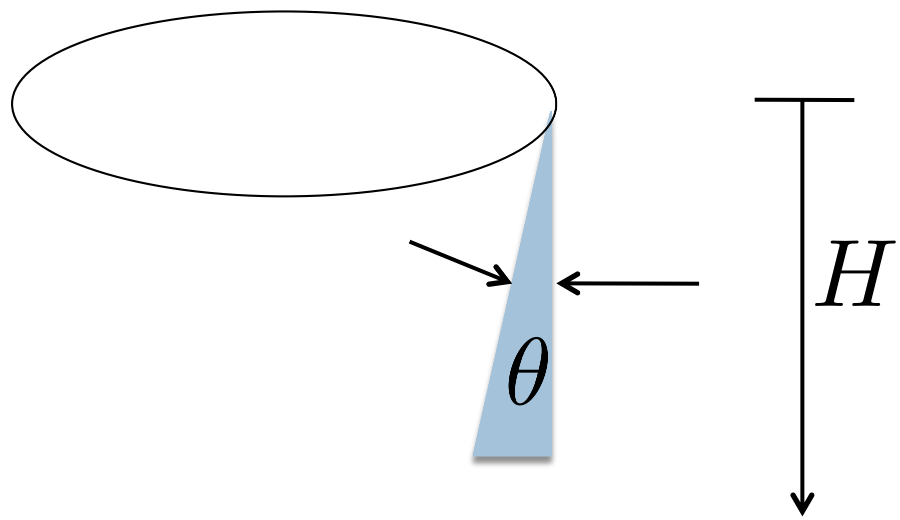


$$\theta > 90^o$$

Capillary tube

- For surface tension, σ, and contact angle θ, find the capillary rise/depression in a tube of radius R.





Capillary tube, cont'd

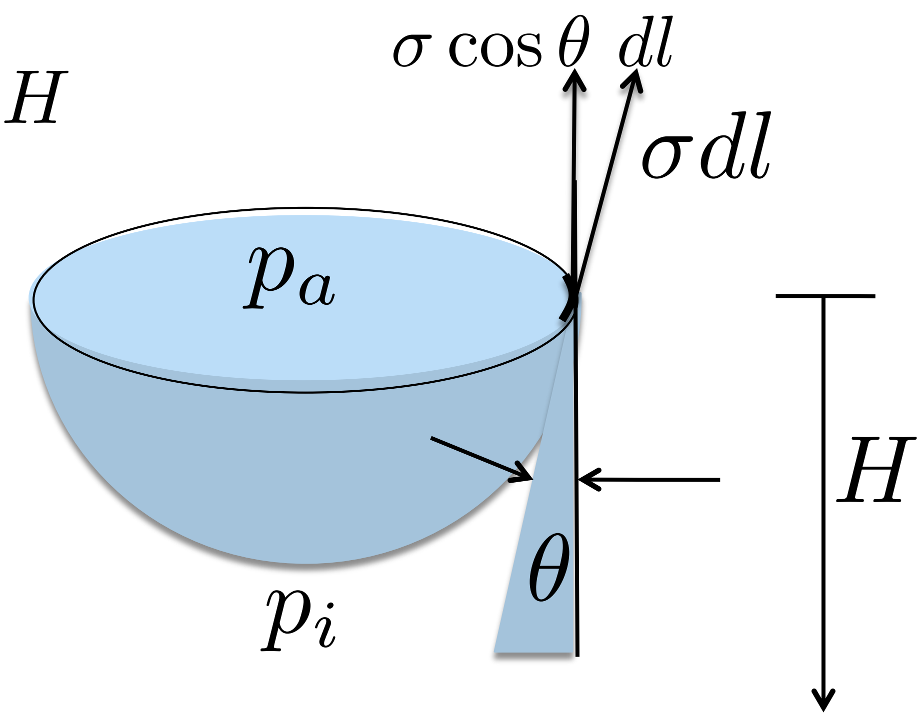
$$\pi R^2(p_a - p_i) = 2\pi R\sigma\cos\theta$$

$$p_a - p_i = \frac{2\sigma\cos\theta}{R}$$

$$p_i \approx p_a - \rho g z = p_a - \rho g H$$

$$H = \frac{2\sigma\cos\theta}{\rho g R}$$

$$H \begin{cases} > 0 & 0 < \theta < \frac{\pi}{2} \\ < 0 & \frac{\pi}{2} < \theta < \pi \end{cases}$$



Capillary tube, cont'd

$$H = \frac{2\sigma \cos\theta}{\rho g R}$$

- Air/water/clean glass, $\theta \approx 0$

$$R = \begin{cases} 1 \quad \mu\text{ m} & H = 15.1 \quad \text{m} \\ 1 \quad \text{mm} & H = 1.5 \quad \text{cm} \\ 1 \quad \text{cm} & H = 1.5 \quad \text{mm} \end{cases}$$

!!!!!

Summary

- We can easily extend fluid static force balance to case when fluid moves as a rigid body
- We attach coordinate system to translating/rotating body of fluid, and add a fictitious body force to account for the non-inertial reference frame
  - We will do this plenty more in winter quarter, and have a chance to look more carefully at the transformations
- Simple examples treated: translating tank of water, spinning cup
- Surface tension alters the hydrostatic balance at a liquid/gas interface and can lead to large (and strange) effects when the scale is small (~millimeter and less)