ACM 100b

Laplace transform and systems

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Laplace transform for systems

- The Laplace transform can be easily extended to systems of ODE's with constant coefficients
- Consider the homogeneous system

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x} \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

- We now let

$$\boldsymbol{X}(s) = \begin{bmatrix} X_1(s) \\ \vdots \\ X_n(s) \end{bmatrix} = \int_0^\infty \exp(-st)\boldsymbol{x}(t)dt = \begin{bmatrix} \int_0^\infty \exp(-st)x_1(t)dt \\ \vdots \\ \int_0^\infty \exp(-st)x_n(t)dt \end{bmatrix}$$

Laplace transform for systems

- As before the derivative transforms as

$$\int_0^\infty \exp(-st)\frac{d\boldsymbol{x}}{dt}dt = s\boldsymbol{X}(s) - \boldsymbol{x}_0$$

- Now transform both sides of the system

$$\frac{d\boldsymbol{x}}{dt} = A\boldsymbol{x} \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0$$

  to get

$$s\boldsymbol{X}(s) - \boldsymbol{x}_0 = A\boldsymbol{X}(s)$$

- Or we can write this as

$$(sI - A)\boldsymbol{X}(s) = \boldsymbol{x}_0$$

- So the Laplace transform turns the ODE system into a linear system of equations for the transform

Laplace transform for systems

- Now define the inverse matrix

$$B(s) = (sI - A)^{-1}$$

- We then find for the solution

$$\boldsymbol{X}(s) = B(s)\boldsymbol{x}_0$$

- Once this vector is found we can then perform the inverse Laplace transform on $\boldsymbol{X}(s)$ to get the solution $\boldsymbol{x}(t)$.
- To do this we investigate the structure of the matrix $B(s)$ in more detail.

Laplace transform for systems

- Define $\Delta(s)$ to be the characteristic polynomial of A:

$$\Delta(s) = \det(sI - A)$$

- As we know $\Delta(s)$ is a polynomial of order n and its roots are the eigenvalues of A.
- Recall

$$B(s) = (sI - A)^{-1}$$

so

$$\det[B(s)(sI - A)] = \det[B(s)]\Delta(s) = 1$$

- So we know the determinant of $B(s)$:

$$\det[B(s)] = \frac{1}{\Delta(s)}$$

Laplace transform for systems

- Now from the operations used to construct an inverse matrix you can see that every component of $B(s)$ must be a rational function of s.
- A rational function is simply a fraction of two polynomials
- In other words, the i, j'th component of B is given by

$$B_{ij}(s) = \frac{p_{ij}(s)}{\Delta(s)}$$

  where $p_{ij}(s)$ is a polynomial in s.
- We must have $\Delta(s)$ in the denominator because the elements of the inverse only fail to exist if s is an eigenvalue of A.

Laplace transform for systems

- Now from

$$B(s) = (sI - A)^{-1} \text{ we have } (sI - A)B(s) = I$$

- We can write this as

$$\left(I - \frac{1}{s}A\right) sB(s) = I$$

- Now take the limit as $s \to \infty$ in the expression above to get

$$\lim_{s\to\infty} sB(s) = I$$

- So that means

$$\lim_{s\to\infty} B(s) = 0$$

- So the degree of the polynomial $p_{ij}(s)$ must be less than n where again n is the number of rows and columns of A.

Laplace transform for systems

- Now to understand the structure of the solution we first consider the case where the roots of $\Delta(s)$ are all distinct.
- Call these roots (which are the eigenvalues of A) $\lambda_1, \lambda_2 \ldots, \lambda_n$
- Now recall each entry of B is a ratio of polynomials
- And the numerator polynomial is of lower degree than the denominator polynomial.
- So each element of B has a partial fraction expansion.
- This means B can be written as

$$B(s) = \sum_{i=1}^{n} \frac{B_i}{s - \lambda_i}$$

- The numerators B_i are constant matrices

Laplace transform for systems

- Now recall the solution to our transformed system is
$$\boldsymbol{X}(s) = B(s)\boldsymbol{x}_0$$
- So it must be that
$$\boldsymbol{X}(s) = \sum_{i=1}^{N} \frac{B_i \boldsymbol{x}_0}{s - \lambda_i}$$
- And now transforming back we get
$$\boldsymbol{x}(t) = \sum_{i=1}^{n} \exp(\lambda_i t) B_i \boldsymbol{x}_0$$
- This is a result we have already derived when we looked at fundamental matrices.
- We also have an expression for the matrix exponential:
$$\exp(At) = \sum_{i=1}^{n} \exp(\lambda_i t) B_i$$

The case of multiple roots

- In general the characteristic polynomial will have the form

$$\Delta(s) = (s - \lambda_1)^{m_1}(s - \lambda_2)^{m_2} \cdots (s - \lambda_k)^{m_k}$$

- This is because there may be multiple roots.
- We always have $m_1 + m_2 + \ldots m_k = n$
- In this case the partial fraction decomposition of the elements of B takes the form

$$B(s) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{B_{ij}}{(s - \lambda_i)^j}$$

- Again the B_{ij} are constant matrices

The case of multiple roots

- So now the transform of the solution is

$$\boldsymbol{X}(s) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{B_{ij}\boldsymbol{x}_0}{(s-\lambda_i)^j}$$

- And this means the solution is of the form

$$\boldsymbol{x}(t) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{t^{j-1}\exp(\lambda_i t) B_{ij}\boldsymbol{x}_0}{(j-1)!}$$

- This approach is much easier to use than reduction of order
- This approach also allows one to see how the Jordan normal form comes about.

Laplace transform and Jordan normal form

- Recall that we have the expression

$$B(s) = \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{B_{ij}}{(s-\lambda_i)^j}$$

- And this means

$$(sI - A) \sum_{i=1}^{k} \sum_{j=1}^{m_i} \frac{B_{ij}}{(s-\lambda_i)^j} = I.$$

- If all the λ_i are simple roots we have

$$(sI - A) \sum_{i=1}^{n} \frac{B_i}{s-\lambda_i} = I$$

Laplace transform and Jordan normal form

- Now in the expression

$$(sI - A)\sum_{i=1}^{n} \frac{B_i}{s - \lambda_i} = I$$

multiply each side by $s - \lambda$ and let $s \to \lambda_l$ for $l = 1, \ldots, n$ in turn

- We find

$$(\lambda_l I - A)B_l = 0$$

- So if $\boldsymbol{y}_l$ is a vector formed by taking one of the non-vanishing columns of the matrix B_l we must have

$$(\lambda_l I - A)\boldsymbol{y}_l = 0$$

- So the nonzero columns of the B_l are eigenvectors of A associated with the eigenvalue λ_l

Laplace transform and Jordan normal form

- We know that if we create a matrix T whose columns consist of the eigenvectors $\boldsymbol{y}_l$ then

$$T^{-1}AT = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$

- But we can also learn what happens in the more general case when the roots are repeated.

Laplace transform and Jordan normal form

- Recall the expression

$$(sI - A)\sum_{i=1}^{k}\sum_{j=1}^{m_i}\frac{B_{ij}}{(s-\lambda_i)^j} = I.$$

- Write this in the form

$$\begin{aligned}&[(s-\lambda_l)I + (\lambda_l I - A)] \times \\ &\times \left[\frac{B_{lm_l}}{(s-\lambda_l)^{m_l}} + \frac{B_{lm_l-1}}{(s-\lambda_l)^{m_l-1}} + \cdots + \frac{B_{l1}}{s-\lambda_l} + Q\right] = I\end{aligned}$$

- Here Q is all the terms where $i \neq l$

Laplace transform and Jordan normal form

- Now perform the multiplication in

$$
\begin{aligned}
&[(s-\lambda_l)I + (\lambda_l I - A)] \times \\
&\times \left[\frac{B_{lm_l}}{(s-\lambda_l)^{m_l}} + \frac{B_{lm_l-1}}{(s-\lambda_l)^{m_l-1}} + \cdots + \frac{B_{l1}}{s-\lambda_l} + Q \right] = I
\end{aligned}
$$

- We get

$$
\begin{aligned}
&\frac{(\lambda_l I - A)B_{lm_l}}{(s-\lambda_l)_l^m} + \frac{B_{lm_l} + (\lambda_l I - A)B_{lm_l-1}}{(s-\lambda_l)^{m_l-1}} + \cdots + \frac{B_{l2} + (\lambda_l I - A)B_{l1}}{s-\lambda_l} \\
&\qquad + B_{l1} + (s-\lambda_l)Q + (\lambda_l I - A)Q = I
\end{aligned}
$$

- Now the left hand side has singularities in s but the right hand side has no singularities in s
- So every numerator associated with a singular term in s must vanish.

Laplace transform and Jordan normal form

- The vanishing of these numerators means the following relations must hold:

$$
\begin{aligned}
(\lambda_l I - A) B_{lm_l} &= 0 \\
(\lambda_l I - A) B_{lm_l - 1} &= -B_{lm_l}, \\
&\vdots \\
(\lambda_l I - A) B_{l1} &= -B_{l2}
\end{aligned}
$$

- If λ_l is a simple eigenvalue then we recognize that the matrix B must have columns which are eigenvectors
- But if it is not a simple eigenvalue then these relations tell us the columns of B are a generalized type of eigenvector.

Laplace transform and Jordan normal form

- These new types of vectors for a given eigenvalue λ_l satisfy the relations

$$A\boldsymbol{y}_{lj} = \lambda_l \boldsymbol{y}_{lj} + \boldsymbol{y}_{lj+1} \qquad j = 1, 2, \ldots, m_l - 1$$
$$A\boldsymbol{y}_{lml} = \lambda_l \boldsymbol{y}_{lm_l}$$

- So no matter the multiplicity of the eigenvalues we get n vectors using this approach
- They can be shown to be linearly independent
- And if you construct a matrix T whose columns consist of these n vectors you will find that this is the Jordan normal form for the matrix A.

Laplace transform and Jordan normal form

- The Jordan normal form always exists even when you can't diagonalize a matrix
- The form says that

$$T^{-1}AT = \begin{bmatrix} M_1 & 0 & \dots & 0 \\ 0 & M_2 & \dots & 0 \\ & & \ddots & \\ 0 & 0 & \dots & M_n \end{bmatrix}$$

where the M_i are matrices with the structure

$$M_l = \begin{bmatrix} \lambda_l & 0 & 0 & \cdots & 0 \\ 1 & \lambda_l & 0 & \cdots & 0 \\ 0 & 1 & \lambda_l & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & \lambda_l \end{bmatrix}$$