ACM 100b

Review of ODE's - existence and uniqueness for first order linear ODE

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Existence and uniqueness for linear first order ODE

Theorem

The differential equation

$$A_1(z)y' + A_0(z)y = f(z) \qquad y(z_0) = y_0$$

will have a unique solution in every interval including the initial point z_0 *provided* $A_1(z)$, $A_0(z)$, *and* $f(z)$ *are continuous and* $A_1(z)$ *doesn't vanish within the interval.*

- This is what we call a sufficient condition
- In other words if the condition holds the solution is unique.
- It is not a necessary condition
- That is - it is not the case that if the solution is unique that the condition holds

Existence and uniqueness

- The requirements of continuity can actually be relaxed.
- For example, piecewise continuity of the coefficients would be acceptable.
- It is easily seen from the solution we got

$$y(z) = y_0 \exp\left[-\int_{z_0}^{z} \frac{A_0(t)}{A_1(t)} dt\right] + \exp\left[-\int_{z_0}^{z} \frac{A_0(t)}{A_1(t)} dt\right] \int_{z_0}^{z} \exp\left[\int_{z_0}^{t} \frac{A_0(t')}{A_1(t')} dt'\right] \frac{f(t)}{A_1(t)} dt.$$

that as long as A_1 does not vanish, we will have no problem with the integral.

Can losr existence and uniqueness if conditions are violated

- If $A_1(z)$ vanishes then problems can arise.
- For example, consider the ODE

$$zy' - 2y = 0. \quad A_0 = -2, A_1 = z$$

- Here, $A_1(z) = 0$ at $z = 0$.
- Using our solution

$$y(z) = y_0 \exp\left[-\int_{z_0}^{z} \frac{A_0(t)}{A_1(t)} dt\right]$$

we see that

$$y(z) = y_0 z^2.$$

- So we have a solution but in fact there is a problem.
- The solution vanishes at $z = 0$ no matter what y_0 is.

Example of loss of uniqueness

- Suppose we ask for the solution with initial value $y(0) = 3$?
- We can't get such a solution because the general solution is

$$y(z) = y_0 z^2.$$

- It vanishes at $z = 0$ no matter what we do.
- So we can only solve the initial value problem at $z = 0$ for special initial values (i.e. $y = 0$ there)
- And even though we get a solution to that problem it's not unique.

Loss of existence

- We might think we can avoid the problem by asking instead to solve the IVP with initial value $y(1) = 1$.
- Then we do get a solution ($y = z^2$)
- But if the interval of interest for the problem is say the whole real line then we have a different problem.
- The following solution

$$y(z) = \begin{cases} 0 & z \leq 0 \\ z^2 & z \geq 0. \end{cases}$$

  works just as well.
- So the solution exists but it's not unique.
- So we see that if $A_1(z)$ vanishes in the interval of interest we can have problems with existence and uniqueness of the solution.