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Lecture 1:
Introduction to Course

Basics of Electrostatics I: Electric Force and Field,
Gauss's Law, Dirac Delta Function, $\vec{\nabla} \times \vec{E} = 0$

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Section 1

Introduction to Course

Course Material

This is a course on electrodynamics. It will review the basic material you learned in Ph1bc but will go beyond in both mathematical sophistication (boundary-value problems, multipole expansion) as well as in content (waves in matter, guided waves, radiation, electrodynamics and relativity).

The course will primarily use and follow *Introduction to Electrodynamics* by Griffiths. Supplementary material is drawn from Jackson and from Heald & Marion, both on reserve in the library, but the material presented here will be self-contained.

Course Material (cont.)

Prerequisites

Physics:

- ▶ Electricity and Magnetism: While Ph1bc is a formal prerequisite for the course, we will develop the material from scratch. However, review material will be covered quickly and a basic familiarity with the concepts will be assumed.
- ▶ Classical mechanics: Generally, mechanics at the level of Ph1a is sufficient for this course.

Course Material (cont.)

Mathematics:

- ▶ Chapter 1 of Griffiths except for Sections 1.1.5 (“How Vectors Transform”) and 1.5 (“The Dirac Delta Function”). We will review some of prerequisite material as needed.
- ▶ Solutions to second-order linear ordinary differential equations with constant coefficients (*i.e.*, simple harmonic oscillator).
- ▶ Over the course, we will develop the following more sophisticated concepts:
 - ▶ Dirac Delta function.
 - ▶ Separation of variables to reduce second-order linear partial differential equations to ordinary differential equations.
 - ▶ Series solution to solve second-order linear ordinary differential equations with polynomial coefficients
 - ▶ Tensor formalism for relativity.
- ▶ **Key point: Mathematics is the language of physics.** You must be **competent** in above basic mathematical physics in order to understand the material in this course. Intuition is important, but few can succeed in physics without learning to formalize that intuition into mathematical concepts and calculate with it.

Course Material (cont.)

Topics to be covered:

- ▶ Review of basic electrostatics — Coulomb's Law; Gauss's Law; electric field, potential, and potential energy; conductors and capacitors.
- ▶ Advanced electrostatics — boundary value problems (BVP) for determining potentials and fields; multipole expansion of potential.
- ▶ Electrostatics in Matter — polarization, susceptibility, permittivity of matter; electrostatic BVP, energy, and forces in matter
- ▶ Magnetostatics — Lorentz force; Biot-Savart Law; Ampère's Law; vector potential; boundary conditions; and multipole expansion of potential.
- ▶ Magnetostatics in Matter — magnetization, susceptibility, and permeability of matter; magnetostatic boundary conditions; ferromagnetism.
- ▶ Electrodynamics — electromotive force and electromagnetic induction; inductance and energy in magnetic fields; Maxwell's equations in vacuum and in matter; boundary conditions for Maxwell's equations.
- ▶ Conservation Laws — Continuity equation; Poynting's Theorem; electrodynamic momentum and energy.
- ▶ Electromagnetic Waves — in vacuum, in polarizable/magnetizable matter, in wave guides.
- ▶ Potentials and Radiation — potential formulation; fields and potentials of moving point charges; radiated electromagnetic waves; antennas.
- ▶ Relativity and Electrodynamics — transformation of fields and field tensor, relativistic potentials.

Notational Deviations from Griffiths

- ▶ Griffiths uses boldface notation to indicate vectors and a script \vec{r} to indicate the difference vector $\vec{r} - \vec{r}'$. In order to better match what can be written on a chalkboard, and also due to font unavailability, we use $\vec{\text{r}}$ rather than boldface and \vec{R} for the difference vector.
- ▶ Griffiths uses \vec{r} to refer to the position of the test charge Q and \vec{r}' to refer to the position of the source charge q . This seems unnecessarily confusing. We instead use q and \vec{r} for the test charge and q' and \vec{r}' for the source charge.
- ▶ Griffiths uses $\delta^3(\vec{r})$ to refer to the delta function in three spatial dimension. We use $\delta(\vec{r})$ for this for reasons that are explained after Equation 2.9.

Section 2

Review of Basics of Electrostatics

Study Guidelines

You have seen all the material in this section before in Ph1b. However, the derivations done there were not as rigorous as they could be because you were simultaneously learning vector calculus. Our goal in this section is to do more rigorous derivations to give you some practice in using the mathematical tools. We won't do any examples in lecture or the notes because they really would just be duplication of Ph1b. But you should be following the examples in Griffiths Chapter 2 and making sure you are comfortable with them.

The Assumed Conditions for Electrostatics

Electrostatics is the study of electric fields, potentials, and forces under two assumptions:

- ▶ All electric charges sourcing the electric field are stationary and have been so for a sufficiently long time that all fields are static and thus the electric field can be written in terms of the source charges' current positions.
- ▶ The source charges are held fixed and cannot react to the fields from any test charges that may be stationary or moving relative to the source charges.

We will see later that, when charges are moving, it takes time for the information about the position to propagate and thus the fields at a given point depend on the configuration of the charges at earlier times.

Coulomb's Law and the Electric Field

Coulomb's Law, Electrostatic Forces, and Superposition

We begin with two empirical facts:

- ▶ **Coulomb's Law:** the empirical fact that the force on a test charge q at position \vec{r} due to a source charge q' at \vec{r}' is given by Coulomb's Law:

$$\vec{F} = \frac{1}{4\pi\epsilon_0} \frac{q' q}{R^2} \hat{R} \quad \text{with } \vec{R} \equiv \vec{r} - \vec{r}' \quad (2.1)$$

where $\epsilon_0 = 8.85 \times 10^{-12} \text{ C}^2 \text{ N}^{-1} \text{ m}^{-2}$. The force points along the line from q' to q as indicated by the sign of the definition of \vec{R} . The electric charge is in the units of *Coulombs* (C), which is a fundamental unit that cannot be written in terms of other fundamental units.

Recall that we use $\vec{\cdot}$ rather than boldface to indicate vectors, R where Griffiths uses a script r , and a different convention from Griffiths's for the symbols for the two charges and their position vectors.

- ▶ **Superposition:** the empirical fact that Coulomb's Law obeys the principle of superposition: the force on a test charge q at \vec{r} due to N charges $\{q'_i\}$ at positions $\{\vec{r}'_i\}$ is obtained by summing the individual vector forces:

$$\vec{F} = \sum_{i=1}^N \vec{F}_i = \sum_{i=1}^N \frac{1}{4\pi\epsilon_0} \frac{q'_i q}{R_i^2} \hat{R}_i \quad \text{with } \vec{R}_i \equiv \vec{r} - \vec{r}'_i \quad (2.2)$$

Coulomb's Law and the Electric Field (cont.)

The Electric Field

Given that any test charge q placed at the position \vec{r} feels the same force, we are motivated to abstract away the test charge and define what we call the **electric field** at that position \vec{r} :

$$\vec{E}(\vec{r}) = \frac{\vec{F}}{q} = \begin{cases} \frac{1}{4\pi\epsilon_0} \frac{q'}{R^2} \hat{R} & \text{for a single source charge } q' \text{ at } \vec{r}' \\ \sum_{i=1}^N \frac{1}{4\pi\epsilon_0} \frac{q'_i}{R_i^2} \hat{R}_i & \text{for } N \text{ source charges } \{q'_i\} \text{ at positions } \{\vec{r}'_i\} \end{cases} \quad (2.3)$$

The electric field has units of N/C.

Coulomb's Law and the Electric Field (cont.)

Coulomb's Law for Continuous Charge Distributions

If a charge distribution is continuous, then the natural extension of Coulomb's Law is to *integrate* the electric field or force over the contributions from the infinitesimal charge elements dq :

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{1}{R^2} \hat{R} dq \quad (2.4)$$

where \vec{R} varies with the location of dq as the integral is performed. dq is admittedly ill-defined. However, before worrying about that, let us note that the integrand is a *vector* and so this integral requires some care. What that means is that we must break up \hat{R} into its components and individually integrate each component. For example, if we use Cartesian coordinates, then $\hat{R} = \hat{x} (\hat{R} \cdot \hat{x}) + \hat{y} (\hat{R} \cdot \hat{y}) + \hat{z} (\hat{R} \cdot \hat{z})$, and, since the Cartesian unit vectors do not depend on the location of the infinitesimal charge dq , we may write the integral out as follows:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\hat{x} \int \frac{1}{R^2} (\hat{R} \cdot \hat{x}) dq + \hat{y} \int \frac{1}{R^2} (\hat{R} \cdot \hat{y}) dq + \hat{z} \int \frac{1}{R^2} (\hat{R} \cdot \hat{z}) dq \right] \quad (2.5)$$

which is sum of three integrals with *scalar* integrands.

Coulomb's Law and the Electric Field (cont.)

Now, consider some specific charge distributions:

- ▶ **line charge distribution:**

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{C}} \frac{dl' \lambda(\vec{r}')}{R^2} \hat{R}$$

with $\lambda(\vec{r}')$ having units of $C\ m^{-1}$,
 \vec{r}' running over all points in the
line distribution \mathcal{C} , and dl' being
the differential length element
at \vec{r}' for \mathcal{C}

(2.6)

- ▶ **surface charge distribution:**

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{S}} \frac{da' \sigma(\vec{r}')}{R^2} \hat{R}$$

with $\sigma(\vec{r}')$ having units of $C\ m^{-2}$,
 \vec{r}' running over all points in the
surface distribution \mathcal{S} , and da'
being the differential area element
at \vec{r}' for \mathcal{S}

(2.7)

Coulomb's Law and the Electric Field (cont.)

► volume charge distribution:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \frac{d\tau' \rho(\vec{r}')}{R^2} \hat{R}$$

with $\rho(\vec{r}')$ having units of C m^{-3} ,
 \vec{r}' running over all points in the
volume distribution \mathcal{V} , and $d\tau'$
being the differential volume
element at \vec{r}' for \mathcal{V}

(2.8)

Using the Dirac delta function we will define below, one can write the first two as special cases of the latter, using delta functions in the dimensions in which the charge distribution has no extent.

Coulomb's Law and the Electric Field (cont.)

Aside: the Dirac Delta Function

Relating Equation 2.8 to Equation 2.2 offers us both the opportunity to rigorously connect them as well as a chance to introduce the *Dirac delta function*. The Dirac delta function at \vec{r}_0 , $\delta(\vec{r} - \vec{r}_0)$, is defined by what it does when it is multiplied against an arbitrary function $f(\vec{r})$ and integrated: For any function $f(\vec{r})$ and any volume \mathcal{V} containing the point \vec{r}_0 , it holds that

$$\boxed{\int_{\mathcal{V}} f(\vec{r}') \delta(\vec{r}' - \vec{r}_0) d\tau' = f(\vec{r}_0)} \quad (2.9)$$

and, for any volume \mathcal{V} not containing \vec{r}_0 , the integral vanishes. In particular, if $f(\vec{r})$ is unity, then the right side of the above integral is unity.

Coulomb's Law and the Electric Field (cont.)

Two notes on dimensions and notation:

- ▶ In order for the units in the above equation to work out, the delta function above must have units of m^{-3} . The general rule is that the delta function's units are the inverse of the differential that its argument says it should be integrated over. In this case, the argument is a vector in 3D space, so the differential is the differential volume element $d\tau$, and so the delta function has units of m^{-3} .
- ▶ Griffiths refers to the above delta function as $\delta^3(\vec{r} - \vec{r}_0)$. He does this because one can think of this delta function in terms of 1D delta function

$$\delta^3(\vec{r} - \vec{r}_0) = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \quad \text{where} \quad \begin{aligned}\vec{r} &= x\hat{x} + y\hat{y} + z\hat{z} \\ \vec{r}_0 &= x_0\hat{x} + y_0\hat{y} + z_0\hat{z}\end{aligned} \quad (2.10)$$

We drop the 3 because it is unnecessary: the dimension of the delta function is implied by its argument. Moreover, the 3 notation is misleading and confusing because it suggests that δ^3 is the cube of something that has $\vec{r} - \vec{r}_0$ as its argument. It is not!

Coulomb's Law and the Electric Field (cont.)

With the above, we see that if we define the charge distribution for a set of point charges $\{q'_i\}$ at positions $\{\vec{r}'_i\}$ to be

$$\rho(\vec{r}) = \sum_{i=1}^N q'_i \delta(\vec{r} - \vec{r}'_i) \quad (2.11)$$

then, when we do the integral in Equation 2.8 over any volume \mathcal{V} containing all N charges, we obtain

$$\begin{aligned}\vec{E}(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} \sum_{i=1}^N \frac{d\tau' q'_i \delta(\vec{r}' - \vec{r}'_i)}{|\vec{r} - \vec{r}'|^2} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|} \\ &= \frac{1}{4\pi\epsilon_0} \sum_{i=1}^N \int_{\mathcal{V}} d\tau' q'_i \delta(\vec{r}' - \vec{r}'_i) \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \\ &= \frac{1}{4\pi\epsilon_0} \sum_{i=1}^N q'_i \frac{\vec{r} - \vec{r}'_i}{|\vec{r} - \vec{r}'_i|^3} = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^N \frac{q'_i}{R_i^2} \hat{R}_i\end{aligned} \quad (2.12)$$

which recovers Equation 2.2.

Gauss's Law

Statement of Gauss's Law

The *flux* of the electric field through a surface is simply the integral of the component of the electric field normal to the surface over the surface:

$$\mathcal{F}_S = \int_S \vec{E} \cdot \hat{n}(\vec{r}) da \quad (2.13)$$

where \vec{r} lies in the surface S and $\hat{n}(\vec{r})$ is the surface normal at that point \vec{r} . Note that the flux has a sign based on the choice of the direction of \hat{n} .

Gauss's Law relates the flux of the electric field through any *closed* surface to the total charge enclosed by that surface:

$$\mathcal{F}_S = \oint_S \vec{E} \cdot \hat{n}(\vec{r}) da = \frac{1}{\epsilon_0} \int_{V(S)} d\tau \rho(\vec{r}) \quad (2.14)$$

where $V(S)$ is the surface enclosed by S and \oint indicates the integral over a closed surface.

Gauss's Law (cont.)

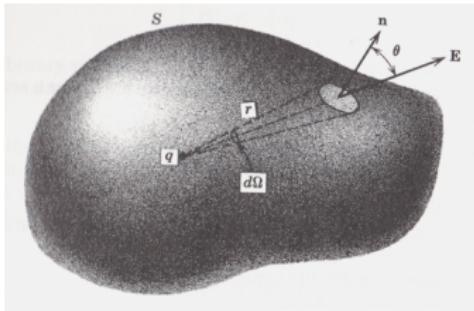
Utility of Gauss's Law

The reason we bother with Gauss's Law is that it provides a way to calculate the electric field for charge distributions having some amount of geometrical symmetry that is much easier than brute-force integration of Coulomb's Law. We will see that it will also enable us to relate the electric field's *boundary conditions* at an interface between two volumes (the conditions relating the electric field components on the two sides of the interface) to the amount of charge at that interface.

Gauss's Law (cont.)

Proof of Gauss's Law

The proof offered in Griffiths' is unnecessarily unrigorous; we follow Jackson.



First consider a charge distribution $\rho(\vec{r})$ that lies completely inside an arbitrarily shaped closed surface S . What is the infinitesimal flux through an infinitesimal portion S at a point \vec{r} due to the infinitesimal amount of charge in the infinitesimal volume $d\tau'$ at the location \vec{r}' ? It is

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$$d^2\mathcal{F}_S(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0} \frac{d\tau' \rho(\vec{r}')}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}') \cdot \hat{n}(\vec{r}) da \quad (2.15)$$

The left side is a double differential because the right side is. If one considers the geometry (see diagram above), one sees that the quantity $(\vec{r} - \vec{r}') \cdot \hat{n}(\vec{r}) da / |\vec{r} - \vec{r}'|$ is the projected area of the area element da normal to the unit vector $(\vec{r} - \vec{r}') / |\vec{r} - \vec{r}'|$ from \vec{r}' to \vec{r} . Since $|\vec{r} - \vec{r}'|^2$ is the square of the distance from \vec{r}' to \vec{r} , then the quantity $(\vec{r} - \vec{r}') \cdot \hat{n}(\vec{r}) da / |\vec{r} - \vec{r}'|^3$ is the solid angle $d\Omega(\vec{r}, \vec{r}')$ subtended by da at \vec{r} as viewed from \vec{r}' .

Gauss's Law (cont.)

The corresponding mathematical formula is

$$d^2\mathcal{F}_S(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0} d\tau' \rho(\vec{r}') d\Omega(\vec{r}, \vec{r}') \quad (2.16)$$

We know that if we integrate the solid angle over the entire closed surface S surrounding our source charge point \vec{r}' , we recover 4π , so:

$$d\mathcal{F}_S(\vec{r}') = \frac{1}{4\pi\epsilon_0} \oint_S d\tau' \rho(\vec{r}') d\Omega(\vec{r}, \vec{r}') = \frac{1}{\epsilon_0} d\tau' \rho(\vec{r}') \quad (2.17)$$

That is, for any infinitesimal volume element $d\tau'$ at \vec{r}' , the flux of the electric field due to that element through any surface S enclosing it is equal to the charge in that infinitesimal volume divided by ϵ_0 .

Gauss's Law (cont.)

It stands to reason that, if the above is true for the flux due to an infinitesimal volume of charge, then it holds for the whole distribution of charge enclosed by \mathcal{S} . We can prove this by calculating the flux through \mathcal{S} due to the entire charge distribution, using this fact that the distribution is fully contained inside \mathcal{S} :

$$\begin{aligned}\mathcal{F}_{\mathcal{S}} &= \oint_{\mathcal{S}} \vec{E}_{\mathcal{S}}(\vec{r}) \cdot \hat{n}(\vec{r}) da = \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \int_{V(\mathcal{S})} \frac{d\tau' \rho(\vec{r}')}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}') \cdot \hat{n}(\vec{r}) da \\ &= \frac{1}{4\pi\epsilon_0} \oint_{\mathcal{S}} \int_{V(\mathcal{S})} d^2 \mathcal{F}_{\mathcal{S}}(\vec{r}, \vec{r}')\end{aligned}\quad (2.18)$$

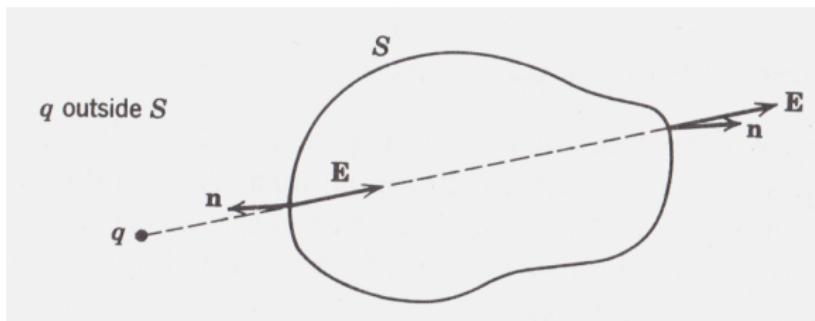
where $\vec{E}_{\mathcal{S}}(\vec{r})$ is the electric field at \vec{r} due to all the charge contained by \mathcal{S} . Note that we implicitly used superposition in the above via the formula relating $\vec{E}_{\mathcal{S}}(\vec{r})$ to the charge distribution. Therefore, exchanging the order of integration,

$$\mathcal{F}_{\mathcal{S}} = \frac{1}{4\pi\epsilon_0} \int_{V(\mathcal{S})} \oint_{\mathcal{S}} d^2 \mathcal{F}_{\mathcal{S}}(\vec{r}, \vec{r}') = \frac{1}{\epsilon_0} \int_{V(\mathcal{S})} d\mathcal{F}_{\mathcal{S}}(\vec{r}') = \frac{1}{\epsilon_0} \int_{V(\mathcal{S})} d\tau' \rho(\vec{r}') \quad (2.19)$$

which is Gauss's Law. But we are not quite done yet, as we assumed in the above that the charge distribution vanished outside of \mathcal{S} . Does the result generalize? Yes, it does.

Gauss's Law (cont.)

Returning to $d^2\mathcal{F}_S(\vec{r}, \vec{r}')$ (Equation 2.16), suppose we consider a source charge at a point \vec{r}' that lies outside of S . (See diagram below.) Then, for a given point \vec{r} on S and the solid angle it subtends $d\Omega(\vec{r}, \vec{r}')$ as viewed from the source charge point \vec{r}' , there will be second point on S that has the same unit vector to the source charge point \vec{r}' and subtends the same solid angle. But, because the direction of $\hat{n}(\vec{r})$ enters the expression for $d^2\mathcal{F}_S(\vec{r}, \vec{r}')$, and the two points subtending the same solid angle will have opposite signs of \hat{n} , their two contributions cancel. Thus, the integral over S that yields $d\mathcal{F}_S(\vec{r}')$ vanishes for \vec{r}' outside of S . Thus, the charge distribution at points outside of S do not contribute to the flux through S , and so our derivation remains valid.



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Gauss's Law (cont.)

The Divergence of \vec{E} and the Differential Version of Gauss's Law

You learned about the divergence theorem (Gauss's theorem) in Ma1abc. Applied to \vec{E} , the divergence theorem says

$$\int_{\mathcal{V}(S)} \vec{\nabla} \cdot \vec{E}(\vec{r}) d\tau = \oint_S \vec{E}(\vec{r}) \cdot \hat{n}(\vec{r}) da \quad (2.20)$$

Gauss's Law tells us

$$\frac{1}{\epsilon_0} \int_{\mathcal{V}(S)} d\tau \rho(\vec{r}) = \oint_S \vec{E}(\vec{r}) \cdot \hat{n}(\vec{r}) da \quad (2.21)$$

Combining the two, we have

$$\int_{\mathcal{V}(S)} \vec{\nabla} \cdot \vec{E}(\vec{r}) d\tau = \frac{1}{\epsilon_0} \int_{\mathcal{V}(S)} d\tau \rho(\vec{r}) \quad (2.22)$$

Since the above holds for any volume \mathcal{V} , the integrands must be equal, giving us the differential version of Gauss's Law:

$$\boxed{\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{\epsilon_0} \rho(\vec{r})} \quad (2.23)$$

Gauss's Law (cont.)

Direct Proof of Differential Version of Gauss's Law

We can prove the above differential version by simply calculating the divergence of \vec{E} using Coulomb's Law, also. This is of course not independent of Gauss's Law because Gauss's Law is proven using Coulomb's Law, but it provides some exercise in vector calculus and leads us to the Dirac delta function. We take the divergence of Coulomb's Law for \vec{E} :

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \vec{\nabla} \cdot \int_{\mathcal{V}'} \frac{1}{4\pi\epsilon_0} \frac{d\tau' \rho(\vec{r}')}{|\vec{r} - \vec{r}'|^3} (\vec{r} - \vec{r}') \quad (2.24)$$

Now, the integral is over \vec{r}' over the volume \mathcal{V}' , but the divergence is calculated relative to the \vec{r} coordinate, so we can bring the divergence inside the integral. Note that it does not act on ρ because it is a function of \vec{r}' , not \vec{r} . Thus, we have

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}'} d\tau' \rho(\vec{r}') \vec{\nabla} \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \quad (2.25)$$

One could calculate the above divergence explicitly in any particular coordinate system. But it is both more rigorous and more instructive to do it using the divergence theorem.

Gauss's Law (cont.)

The divergence theorem lets us calculate the integral of the above divergence over some arbitrary volume \mathcal{V} (with surface S , with neither necessarily related to \mathcal{V}' and S') by converting it to an easier-to-do surface integral:

$$\int_{\mathcal{V}} d\tau \vec{\nabla} \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \oint_{S(\mathcal{V})} da \hat{n}(\vec{r}) \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \quad (2.26)$$

We can apply to the right side the same argument that we used in proving Gauss's Law regarding solid angles. The integrand above is just the solid angle subtended by the area element da at \vec{r} as viewed from \vec{r}' . As before, if \vec{r}' is inside \mathcal{V} , then the above integral yields the total solid angle, 4π . If \vec{r}' is not inside of \mathcal{V} , then, for every area element da at \vec{r} , there is an area element with an equal and opposite contribution, making the integral vanish. That is,

$$\int_{\mathcal{V}} d\tau \vec{\nabla} \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \begin{cases} 4\pi & \text{if } \vec{r}' \text{ is inside } \mathcal{V} \\ 0 & \text{if } \vec{r}' \text{ is outside } \mathcal{V} \end{cases} \quad (2.27)$$

You will note that the divergence in the integrand looks a lot like a delta function, a point we will return to later.

Gauss's Law (cont.)

Let's apply the above by integrating $\vec{\nabla} \cdot \vec{E}(\vec{r})$ over the same arbitrary volume \mathcal{V} :

$$\begin{aligned}\int_{\mathcal{V}} d\tau \vec{\nabla} \cdot \vec{E}(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} d\tau \int_{\mathcal{V}'} d\tau' \rho(\vec{r}') \vec{\nabla} \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \\ &= \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}'} d\tau' \rho(\vec{r}') \int_{\mathcal{V}} d\tau \vec{\nabla} \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}\end{aligned}\quad (2.28)$$

where we have exchanged the order of integration (no prohibition on doing so because $\vec{\nabla}$ has not moved around). Now, we apply what we found out about the integral over \mathcal{V} , which says that the integral over \mathcal{V} vanishes if \vec{r}' is not inside \mathcal{V} and yields 4π if it is inside \mathcal{V} . This turns the double integral over \mathcal{V}' into a single integral over $\mathcal{V} \cap \mathcal{V}'$:

$$\int_{\mathcal{V}} d\tau \vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V} \cap \mathcal{V}'} d\tau' 4\pi \rho(\vec{r}') = \frac{1}{\epsilon_0} \int_{\mathcal{V} \cap \mathcal{V}'} d\tau' \rho(\vec{r}')\quad (2.29)$$

Gauss's Law (cont.)

Now, consider points in \mathcal{V} but outside $\mathcal{V} \cap \mathcal{V}'$. Because \mathcal{V}' is the entire volume containing charge (by Coulomb's Law), the charge density vanishes in $\mathcal{V} - \mathcal{V} \cap \mathcal{V}'$. Since the right side is the integral of charge density, we can add the region $\mathcal{V} - \mathcal{V} \cap \mathcal{V}'$ without changing the integral: the contribution we have just added vanishes. This changes the volume of integration from $\mathcal{V} \cap \mathcal{V}'$ to \mathcal{V} . Therefore,

$$\int_{\mathcal{V}} d\tau \vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{\epsilon_0} \int_{\mathcal{V}} d\tau' \rho(\vec{r}') \quad (2.30)$$

The volume \mathcal{V} is arbitrary, so the integrands must be equal:

$$\vec{\nabla} \cdot \vec{E}(\vec{r}) = \frac{1}{\epsilon_0} \rho(\vec{r}') \quad (2.31)$$

which is again the differential version of Gauss's Law.

Gauss's Law (cont.)

Aside: Relation of the Dirac Delta Function to a Divergence, Invariance under Inversion of its Argument

We can use the above manipulations to prove another property of the Dirac delta function. Let's apply Gauss's Law to the left side of Equation 2.25, yielding

$$\rho(\vec{r}) = \frac{1}{4\pi} \int_{V'} d\tau' \rho(\vec{r}') \vec{\nabla} \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \quad (2.32)$$

Now, $\rho(\vec{r})$ is an arbitrary function, so we see that the divergence we took acts like the δ function: it picks out $\rho(\vec{r}' = \vec{r})$. Thus, we have also proven

$$\boxed{\vec{\nabla} \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = 4\pi \delta(\vec{r} - \vec{r}')} \quad (2.33)$$

which we will find is a new, useful property of the delta function.

Since the delta function picks out the point where its argument vanishes, it doesn't matter what the sign of the argument is. One can prove this explicitly using change of variables: when the sign of the argument changes, the sign of the differential and of the limits of integration change also. Those two sign flips cancel each other. Thus

$$\boxed{\delta(\vec{r} - \vec{r}') = \delta(\vec{r}' - \vec{r})} \quad (2.34)$$

Gauss's Law (cont.)

It may seem that this last property is not true given the above relation between the delta function and a divergence. In particular, let's flip the sign on the function the divergence is acting on:

$$\frac{\delta(\vec{r} - \vec{r}')}{4\pi} = \vec{\nabla} \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \vec{\nabla} \cdot -\frac{\vec{r}' - \vec{r}}{|\vec{r}' - \vec{r}|^3} = -\vec{\nabla} \cdot \frac{\vec{r}' - \vec{r}}{|\vec{r}' - \vec{r}|^3} \stackrel{?}{=} -\frac{\delta(\vec{r}' - \vec{r})}{4\pi} \quad (2.35)$$

Don't we have a problem? No, because we ignored the fact that $\vec{\nabla}$ takes derivatives with respect to \vec{r} . Since $\vec{r} - \vec{r}'$ just offsets \vec{r} , then the divergence with respect to $\vec{r} - \vec{r}'$ is the same as the divergence with respect to \vec{r} . So, when we flip the sign on $\vec{r} - \vec{r}'$, we should do the same for the divergence: the divergence should be taken with respect to $\vec{r}' - \vec{r}$. But that flips the sign of the divergence operator:

$\vec{\nabla}_{\vec{r}-\vec{r}'} = -\vec{\nabla}_{\vec{r}'-\vec{r}}$. Finally, \vec{r} acts like an offset for \vec{r}' , and so the divergence with respect to $\vec{r}' - \vec{r}$ is the same as with respect to \vec{r}' . Therefore, we have

$$\vec{\nabla}_{\vec{r}} \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \vec{\nabla}_{\vec{r}-\vec{r}'} \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \left(-\vec{\nabla}_{\vec{r}'-\vec{r}} \right) \cdot \left(-\frac{\vec{r}' - \vec{r}}{|\vec{r}' - \vec{r}|^3} \right) = \vec{\nabla}_{\vec{r}'} \cdot \frac{\vec{r}' - \vec{r}}{|\vec{r}' - \vec{r}|^3}$$

or, more concisely, $\frac{\delta(\vec{r} - \vec{r}')}{4\pi} = \vec{\nabla}_{\vec{r}} \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \vec{\nabla}_{\vec{r}'} \cdot \frac{\vec{r}' - \vec{r}}{|\vec{r}' - \vec{r}|^3} = \frac{\delta(\vec{r}' - \vec{r})}{4\pi}$

(2.36)

The Electric Field has Vanishing Curl

Calculating the Curl of the Electric Field

The curl of \vec{E} can be shown to vanish simply by calculating it for an arbitrary charge distribution:

$$\begin{aligned}\vec{\nabla} \times \vec{E}(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \vec{\nabla} \times \int_V d\tau' \rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \\ &= \frac{1}{4\pi\epsilon_0} \int_V d\tau' \rho(\vec{r}') \vec{\nabla} \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}\end{aligned}\quad (2.37)$$

We could brute force calculate the curl in the integrand in Cartesian or spherical coordinates, but that would be painful because the function on which the curl is acting has no symmetry in the \vec{r} coordinate system.

The Electric Field has Vanishing Curl (cont.)

Let's take a simpler approach. As we saw above, \vec{r}' is just an offset to \vec{r} , thus

$$\vec{\nabla}_{\vec{r}} \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \vec{\nabla}_{\vec{r} - \vec{r}'} \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \quad (2.38)$$

(Note that we are considering this expression alone, not the whole integral. We need not be concerned about suddenly making $\vec{\nabla}$, which did not originally depend on \vec{r}' and thus could be brought inside the integral, now depend on \vec{r}' .) If we define $\vec{s} = \vec{r} - \vec{r}'$, then we have

$$\vec{\nabla}_{\vec{r}} \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \vec{\nabla}_{\vec{s}} \times \frac{\vec{s}}{s^3} \quad (2.39)$$

Now, the function on which the curl is acting has symmetry in the coordinate system in which the curl is acting, and hence the calculation will be simplified. You can probably see intuitively that the above curl vanishes, but let's prove it.

The Electric Field has Vanishing Curl (cont.)

With the above form, we can trivially apply the formula for the curl in spherical coordinates, which is listed in Griffiths. For the sake of being explicit, that formula is

$$\vec{\nabla} \times \vec{v} = \frac{1}{r \sin \theta} \left[\frac{\partial}{\partial \theta} (v_\phi \sin \theta) - \frac{\partial v_\theta}{\partial \phi} \right] \hat{r} + \frac{1}{r} \left[\frac{1}{\sin \theta} \frac{\partial v_r}{\partial \phi} - \frac{\partial}{\partial r} (r v_\phi) \right] \hat{\theta} + \frac{1}{r} \left[\frac{\partial}{\partial r} (r v_\theta) - \frac{\partial v_r}{\partial \theta} \right] \hat{\phi} \quad (2.40)$$

Don't get confused between \vec{s} and \vec{r} ; the r derivatives and subscripts refer to the radial coordinate of the coordinate system in which the curl is being taken. In our case, s is the radial variable and the radial component of \vec{s}/s^3 is $1/s^2$. Thus, \vec{v} has only a radial component and that radial component depends only on the radial distance from the origin. All the derivatives involving the θ and ϕ components of \vec{v} vanish because the components themselves vanish, and the derivatives involving the radial component vanish because those derivatives are with respect to θ and ϕ . (Don't be confused: \vec{v} itself depends on θ and ϕ because the direction of \vec{v} depends on them; but the curl formula takes care of that dependence.) Thus, we have $\vec{\nabla}_{\vec{s}} \times (\vec{s}/s^3) = 0$ and therefore the integrand in Equation 2.37 vanishes. So:

$$\boxed{\vec{\nabla} \times \vec{E}(\vec{r}) = 0} \quad (2.41)$$

The Electric Field has Vanishing Curl (cont.)

The Line Integral of the Electric Field

Stokes' Theorem (a mathematical theorem we will not prove here but that you saw in Malabc) then tells us that, for any surface S with boundary $\mathcal{C}(S)$,

$$\oint_{\mathcal{C}(S)} d\vec{\ell} \cdot \vec{E}(\vec{r}) = \int_S da \hat{n}(\vec{r}) \cdot \vec{\nabla} \times \vec{E}(\vec{r}) = 0 \quad (2.42)$$

Lecture 2:
Basics of Electrostatics II:
Boundary Conditions, Electric Potential and Energy,
Conductors

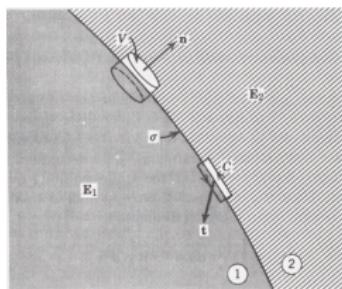
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Boundary Conditions on the Electric Field

While it makes it possible to determine the electric field for charge distributions with sufficient symmetry, the more important application of Gauss's Law and the vanishing of $\vec{\nabla} \times \vec{E}$ is to obtain generic information on the behavior of the electric field across an interface between two regions.

Relation of the normal component of the electric field to the surface charge density



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Construct a Gaussian cylinder of infinitesimal height dz whose axis is normal to the interface under question at the point of interest \vec{r} . Let \hat{n} be the surface normal at \vec{r} , with orientation from region 1 to region 2. Let's calculate the flux through the cylinder's (non-infinitesimal) faces S_1 and S_2 :

$$\begin{aligned}\mathcal{F}(\vec{r}) = & \int_{S_1} da' (-\hat{n}(\vec{r}')) \cdot \vec{E}_1(\vec{r}') \\ & + \int_{S_2} da' \hat{n}(\vec{r}') \cdot \vec{E}_2(\vec{r}') \quad (2.43)\end{aligned}$$

where \vec{E} is evaluated over the two faces.

Boundary Conditions on the Electric Field (cont.)

We neglect the flux through the cylindrical wall because we will let dz vanish in the end and so its area will vanish and it will contribute no flux. By Gauss's Law, this flux is related to the charge enclosed:

$$\mathcal{F}(\vec{r}) = \frac{1}{\epsilon_0} \int_S da' \sigma(\vec{r}') \quad (2.44)$$

where S is the area at the interface intersected by the cylinder. We neglect any charge density in the half cylinders because those contributions to the charge enclosed will vanish as we let $dz \rightarrow 0$. Equating the two expressions for $\mathcal{F}(\vec{r})$, setting $dz \rightarrow 0$ consistently, and seeing that $S_1, S_2 \rightarrow S$ yields

$$\int_S da' \hat{n}(\vec{r}') \cdot [\vec{E}_2(\vec{r}') - \vec{E}_1(\vec{r}')] = \frac{1}{\epsilon_0} \int_S da' \sigma(\vec{r}') \quad (2.45)$$

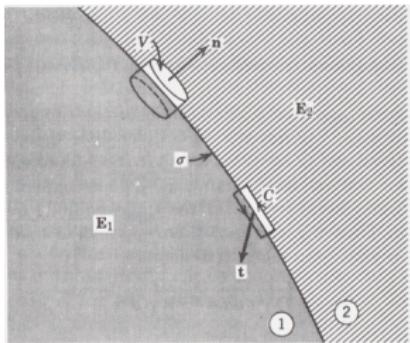
This holds for any choice of cylinder and thus any S , and so the integrands must be equal:

$$\boxed{\hat{n}(\vec{r}) \cdot [\vec{E}_2(\vec{r}) - \vec{E}_1(\vec{r})] = \frac{1}{\epsilon_0} \sigma(\vec{r})} \quad (2.46)$$

That is, the change in the normal component of the electric field across the interface is equal to the surface charge density at the interface. If there is no surface charge at the interface, this component of the electric field must be continuous.

Boundary Conditions on the Electric Field (cont.)

Continuity of the tangential component of the electric field



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Construct a rectangular loop \mathcal{C} with two legs normal to the interface of interest (i.e., along $\hat{n}(\vec{r})$) and with infinitesimal length dz and two (non-infinitesimal) legs parallel to the interface \mathcal{C}_1 and \mathcal{C}_2 . Let $\hat{t}(\vec{r})$ denote the normal to the loop area (so $\hat{n}(\vec{r}) \cdot \hat{t}(\vec{r}) = 0$). \hat{t} will set the orientation of the line integral we will do around the loop following the right-hand rule. The loop legs \mathcal{C}_1 and \mathcal{C}_2 parallel to the interface are along the vector $\hat{s}(\vec{r}) = \hat{t}(\vec{r}) \times \hat{n}(\vec{r})$. Let's calculate the line integral of \vec{E} along this loop:

$$\oint_{\mathcal{C}} d\vec{\ell} \cdot \vec{E}(\vec{r}') = \int_{\mathcal{C}_1} \vec{E}_1(\vec{r}') \cdot (-\hat{s}(\vec{r}')) ds + \int_{\mathcal{C}_2} \vec{E}_2(\vec{r}') \cdot \hat{s}(\vec{r}') ds \quad (2.47)$$

where we neglect the contributions from the infinitesimal legs because they will vanish as $dz \rightarrow 0$.

Boundary Conditions on the Electric Field (cont.)

The fact that the curl of the electric field vanishes ensures the left side is zero. Also, $\mathcal{C}_1 \rightarrow \mathcal{C}_2$ as $dz \rightarrow 0$. Thus, we have

$$\int_{\mathcal{C}_2} [\vec{E}_2(\vec{r}') - \vec{E}_1(\vec{r}')] \cdot \hat{s}(\vec{r}') ds = 0 \quad (2.48)$$

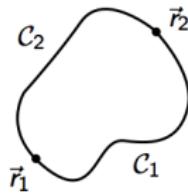
Since the contour \mathcal{C}_2 is arbitrary, the integrand must vanish, yielding

$$\boxed{\hat{s}(\vec{r}) \cdot [\vec{E}_2(\vec{r}) - \vec{E}_1(\vec{r})] = 0} \quad (2.49)$$

This expression holds for any \hat{t} and thus \hat{s} parallel to the surface, so it tells us that the tangential component of the electric field is continuous across any boundary (regardless of whether there is surface charge present).

The Electric Potential

Definition using Line Integral



We saw above that the line integral of the electric field around any closed loop \mathcal{C} vanishes. If we consider two points along the loop \vec{r}_1 and \vec{r}_2 , there are two paths along the loop from \vec{r}_1 to \vec{r}_2 , \mathcal{C}_1 and \mathcal{C}_2 . The vanishing of the full line integral tells us the line integral along these two paths are equal:

$$\int_{\mathcal{C}_1, \vec{r}_1}^{\vec{r}_2} d\vec{\ell}' \cdot \vec{E}(\vec{r}') - \int_{\mathcal{C}_2, \vec{r}_1}^{\vec{r}_2} d\vec{\ell}' \cdot \vec{E}(\vec{r}') = \int_{\mathcal{C}_1, \vec{r}_1}^{\vec{r}_2} d\vec{\ell}' \cdot \vec{E}(\vec{r}') + \int_{\mathcal{C}_2, \vec{r}_2}^{\vec{r}_1} d\vec{\ell}' \cdot \vec{E}(\vec{r}') \\ = \oint_{\mathcal{C}} d\vec{\ell}' \cdot \vec{E}(\vec{r}') = 0 \quad (2.50)$$

(Be careful with the sign flip and endpoint exchange on the second term! Think about it physically — because the line integral has a directionality, changing the order of the endpoints changes the directionality and yields a sign flip. You can prove this to yourself in a particular coordinate system by writing it out, but be careful with the multiple sign flips.) Therefore,

$$\int_{\mathcal{C}_1, \vec{r}_1}^{\vec{r}_2} d\vec{\ell}' \cdot \vec{E}(\vec{r}') = \int_{\mathcal{C}_2, \vec{r}_1}^{\vec{r}_2} d\vec{\ell}' \cdot \vec{E}(\vec{r}') \quad (2.51)$$

The Electric Potential (cont.)

The above relation tells us that the value of the above line integral depends only on the location of its endpoints, not on the path taken. Thus, we can construct a function, *the electric potential*, $V(\vec{r})$, defining it via its differences between points:

$$V(\vec{r}_2) - V(\vec{r}_1) \equiv - \int_{\vec{r}_1}^{\vec{r}_2} d\vec{\ell} \cdot \vec{E}(\vec{r}') \quad (2.52)$$

The fundamental theorem of calculus for line integrals in multiple dimensions implies

$$V(\vec{r}_2) - V(\vec{r}_1) = \int_{\vec{r}_1}^{\vec{r}_2} d\vec{\ell} \cdot \vec{\nabla}_{\vec{r}'} V(\vec{r}') \quad (2.53)$$

where $\vec{\nabla} V(\vec{r})$ is the *gradient* of the electric potential. The above two formulae hold regardless of choice of endpoints and path. Therefore, an alternate definition of the potential is

$$\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r}) \quad (2.54)$$

which makes it clear why the offset for V is not defined.

The electric potential has units of (N m/C), which we call the *volt*, V. The electric field is frequently written in units of V/m instead of N/C.

The Electric Potential (cont.)

Relation of the Electric Potential to the Charge Distribution

We know two things now:

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V d\tau' \rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \quad \text{and} \quad V(\vec{r}_2) - V(\vec{r}_1) \equiv - \int_{\vec{r}_1}^{\vec{r}_2} d\vec{l}' \cdot \vec{E}(\vec{r}')$$

We can use these to obtain an explicit expression for the potential in terms of the charge distribution. In practice, trying to explicitly do the line integral using the definition of \vec{E} is tedious and not illuminating.

Instead, let us use $\vec{E}(\vec{r}) = -\vec{\nabla}V(\vec{r})$ to make an Ansatz. If we have a point charge at the origin, then the electric field points radially outward and falls off as $1/r^2$. What function's derivative gives that dependence? $V(\vec{r}) = 1/r$. This suggests to us

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V d\tau' \rho(\vec{r}') \frac{1}{|\vec{r} - \vec{r}'|} \quad (2.55)$$

The Electric Potential (cont.)

We can prove this form by explicitly taking the gradient:

$$-\vec{\nabla}V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V d\tau' \rho(\vec{r}') \vec{\nabla} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \quad (2.56)$$

As we did earlier when calculating $\vec{\nabla} \times \vec{E}$, we make the change of variables to $\vec{s} = \vec{r} - \vec{r}'$:

$$\vec{\nabla}_{\vec{r}} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) = \vec{\nabla}_{\vec{r} - \vec{r}'} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) = \vec{\nabla}_{\vec{s}} \frac{1}{s} = -\frac{\hat{\vec{s}}}{s^2} = -\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \quad (2.57)$$

where we used the formula for the gradient in spherical coordinates from Griffiths:

$$\vec{\nabla}T(\vec{r}) = \frac{\partial T}{\partial r} \hat{r} + \frac{1}{r} \frac{\partial T}{\partial \theta} \hat{\theta} + \frac{1}{r \sin \theta} \frac{\partial T}{\partial \phi} \hat{\phi} \quad (2.58)$$

Hence, we see that our form for $V(\vec{r})$ yields the correct electric field:

$$-\vec{\nabla}V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V d\tau' \rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \vec{E}(\vec{r}) \quad (2.59)$$

The Electric Potential (cont.)

Comments on the Electric Potential

- ▶ **The electric potential obeys superposition**

This is trivial consequence of superposition for the electric field: because the electric potential is a linear function of the electric field, superposition for the electric field transfers to superposition for the electric potential. One can also see it from Equation 2.55, where the potential is a linear function of the charge density. Of critical importance is that the offset value for the electric potentials being summed be the same (implicit when using the expression in terms of the charge distribution).

- ▶ **Definition of potential offset**

There are two typical choices. When the charge distribution is confined to a finite volume, the electric field vanishes at infinity, which suggests one should define the electric potential to vanish at infinity too. When the charge distribution is not confined (e.g., a uniform electric field over all of space), it is typical to choose the origin to be the point at which the potential vanishes. Any other point would work, too, but will generally make the explicit functional form of $V(\vec{r})$ unnecessarily complicated.

- ▶ **Utility of the electric potential**

The electric potential is scalar, not a vector, function, and thus applying superposition to calculate the potential due to a charge distribution, followed by taking the gradient to find the electric field, is usually much simpler than explicitly calculating the electric field.

The Electric Potential (cont.)

Boundary Conditions on the Electric Potential

From our definition of the electric potential as the line integral of the electric field, and the corollary $\vec{E} = \vec{\nabla} V$, we can derive boundary conditions on the electric potential:

► Continuity of the electric potential

The electric potential is the line integral of the electric field. Therefore, the only way for it to be discontinuous across a boundary is if the electric field can be a delta function at a boundary. (Think about calculating the change in V by integrating $\vec{E} \cdot \hat{n} ds$ across the boundary. As the length of the path goes to zero, how do you prevent the integral from vanishing? With a delta function.) But we have established that the electric field changes by a finite amount at any boundary, and thus it cannot be a delta function on a boundary. The only place the electric field is infinite is on a point charge, and we know that even this does not result in a discontinuity in the potential (though the potential does also become infinite).

The Electric Potential (cont.)

► Change in the normal gradient

This is just a direct rewriting of the boundary condition on the normal component of the field, Equation 2.46:

$$\frac{1}{\epsilon_0} \sigma(\vec{r}) = \hat{n}(\vec{r}) \cdot [\vec{E}_2(\vec{r}) - \vec{E}_1(\vec{r})] = \hat{n}(\vec{r}) \cdot [-\vec{\nabla} V_2(\vec{r}) + \vec{\nabla} V_1(\vec{r})]$$
$$\Rightarrow \boxed{\hat{n}(\vec{r}) \cdot [\vec{\nabla} V_2(\vec{r}) - \vec{\nabla} V_1(\vec{r})] = -\frac{1}{\epsilon_0} \sigma(\vec{r})} \quad (2.60)$$

Note the sign!

► Continuity of the tangential gradient

Again, this follows directly from the continuity of the tangential component of the electric field, Equation 2.49:

$$0 = \hat{s}(\vec{r}) \cdot [\vec{E}_2(\vec{r}) - \vec{E}_1(\vec{r})] = \hat{s}(\vec{r}) \cdot [-\vec{\nabla} V_2(\vec{r}) + \vec{\nabla} V_1(\vec{r})]$$
$$\Rightarrow \boxed{\hat{s}(\vec{r}) \cdot [\vec{\nabla} V_2(\vec{r}) - \vec{\nabla} V_1(\vec{r})] = 0} \quad (2.61)$$

The Electric Potential (cont.)

The Poisson and Laplace Equations

It is natural to rewrite Gauss's Law and the curl-freeness of the electric field in terms of the electric potential. Gauss's Law:

$$\frac{1}{\epsilon_0} \rho(\vec{r}) = \vec{\nabla} \cdot \vec{E}(\vec{r}) = -\nabla^2 V(\vec{r}) \quad (2.62)$$

Rewritten more cleanly:

$$\boxed{\nabla^2 V(\vec{r}) = -\frac{1}{\epsilon_0} \rho(\vec{r})} \quad (2.63)$$

This is known as *Poisson's Equation*.

Poisson's Equation is a partial differential equation. You know from basic calculus that a differential equation alone is not sufficient to obtain a full solution $V(\vec{r})$: constants of integration are required. For partial differential equations in multiple dimensions, the constants of integration are given by specifying *boundary conditions*, conditions for how the solution or its derivatives must behave on the boundary of the volume in which we are specifying $\rho(\vec{r})$ and would like to determine $V(\vec{r})$.

The Electric Potential (cont.)

Our expression for the potential in terms of the charge distribution, Equation 2.55, is the explicit solution to this equation *for a particular boundary condition*, $V(\vec{r}) \rightarrow 0$ as $r \rightarrow \infty$. In the next major section of the notes, will develop the concept of a Green function, which is the generic tool for solving the Poisson equation for arbitrary boundary conditions.

When there is no charge and the right side vanishes, Equation 2.63 is known as *Laplace's Equation*. The importance of this equation is that it implies that, in a region where there is no charge, the second derivative vanishes everywhere, which implies there can be no local maxima or minima (they would require a positive or negative second derivative).

As for the curl, there is a theorem that the curl of a gradient always vanishes:

$$\vec{\nabla} \times (-\vec{\nabla} V) = 0 \quad (2.64)$$

This is not surprising, as the vanishing of the curl of \vec{E} is what allowed us to define the potential as a line integral, which then allowed us to write \vec{E} as the gradient of the potential.

Electrostatic Energy

Electrostatic potential energy of a point charge in an electric field

Consider moving a point charge from \vec{r}_1 to \vec{r}_2 along the contour \mathcal{C} . The work done on the charge is given by doing the line integral of the negative of the electric force along the path (because that is the mechanical force that has to be exerted to move the charge against the electric force \vec{F}_e):

$$W_{12} = - \int_{\mathcal{C}, \vec{r}_1}^{\vec{r}_2} d\vec{\ell} \cdot \vec{F}_e(\vec{r}') \quad (2.65)$$

The force is related to the electric field, and so we have

$$W_{12} = -q \int_{\mathcal{C}, \vec{r}_1}^{\vec{r}_2} d\vec{\ell} \cdot \vec{E}(\vec{r}') = q [V(\vec{r}_2) - V(\vec{r}_1)] \quad (2.66)$$

That is, the work done on the charge by the mechanical force in going from \vec{r}_1 to \vec{r}_2 is given by the charge times the change in electric potential between the two positions. Note the sign: if the potential is higher at the end point, then the work done was positive.

Electrostatic Energy (cont.)

Of course, this lets us to define the *electric potential energy* by

$$U(\vec{r}_2) - U(\vec{r}_1) = q [V(\vec{r}_2) - V(\vec{r}_1)] \quad (2.67)$$

That is, the electric potential energy of the charge and the electric potential of the field are simply related. Since it was defined in terms of work done against a force, electric potential energy obviously has units of Joules (J). That is explicit in the above form, which is $C (N \text{ m}/C) = (N \text{ m}) = J$.

Note that the electric field can also do work on the charge. In this case, the sign in the above line integral for the work is flipped and work is done as the charge loses potential energy. In this case, the work done by the electric field is converted to the kinetic energy the charge has at the end.

Electrostatic Energy (cont.)

Electric potential energy of a charge distribution

How much work must be done to assemble a distribution of charge? This is most easily understood by first considering the assembly of a set of point charges one-by-one by bringing them in from infinity. When the i th charge is brought in, work must be done against the electric field of the first $i - 1$ charges. Put another way, the i th charge starts with zero potential energy and ends with potential energy

$$U_i = \sum_{j=1}^{i-1} q_i \frac{1}{4\pi\epsilon_0} \frac{q_j}{|\vec{r}_i - \vec{r}_j|} \quad (2.68)$$

Thus, the total potential energy is

$$\boxed{U = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^N \sum_{j=1}^{i-1} \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|} = \frac{1}{8\pi\epsilon_0} \sum_{i,j=1, i \neq j}^N \frac{q_i q_j}{|\vec{r}_i - \vec{r}_j|}} \quad (2.69)$$

where the factor of $1/2$ was introduced to allow i and j to both run from 1 to N . Generalizing this to a continuous charge distribution, we have

$$\boxed{U = \frac{1}{8\pi\epsilon_0} \int_V d\tau \int_V d\tau' \frac{\rho(\vec{r}) \rho(\vec{r}')}{|\vec{r} - \vec{r}'|}} \quad (2.70)$$

Electrostatic Energy (cont.)

Electric potential energy in terms of the electric field

We can use the relations between potential, field, and charge density (Equations 2.8, 2.55, and 2.63) and the divergence theorem (Equation 2.20) to obtain an alternate expression for the electric potential energy in terms of the electric field as follows:

$$\begin{aligned} U &= \frac{1}{8\pi\epsilon_0} \int_{\mathcal{V}} d\tau \int_{\mathcal{V}} d\tau' \frac{\rho(\vec{r})\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} = \frac{1}{2} \int_{\mathcal{V}} d\tau \rho(\vec{r}) V(\vec{r}) = -\frac{\epsilon_0}{2} \int_{\mathcal{V}} d\tau [\nabla^2 V(\vec{r})] V(\vec{r}) \\ &\stackrel{i bp}{=} -\frac{\epsilon_0}{2} \int_{\mathcal{V}} d\tau \vec{\nabla} \cdot [V(\vec{r}) \vec{\nabla} V(\vec{r})] + \frac{\epsilon_0}{2} \int_{\mathcal{V}} |\vec{\nabla} V(\vec{r})|^2 \quad \text{with } \stackrel{i bp}{\equiv} \text{ integration by parts} \\ &\stackrel{\substack{\text{divergence} \\ \text{theorem}}}{=} \frac{\epsilon_0}{2} \int_{\partial\mathcal{V}} da \hat{n} \cdot [V(\vec{r}) \vec{E}(\vec{r})] + \frac{\epsilon_0}{2} \int_{\mathcal{V}} |\vec{\nabla} V(\vec{r})|^2 \end{aligned} \tag{2.71}$$

In the last line, the first term is an integral of the product of the potential and the field at the surface of the volume. In order to get the full energy of the charge distribution, \mathcal{V} must include all the charge. If we assume the charge distribution is restricted to some finite volume, then \mathcal{V} is naturally the volume containing the charge distribution. But we can add volume that does not contain charge because it contributes nothing to the initial expression for the electric potential energy.

Electrostatic Energy (cont.)

Therefore, we replace \mathcal{V} with all of space and let \mathcal{S} go to infinity:

$$U = \frac{\epsilon_0}{2} \int_{r \rightarrow \infty} da \hat{n} \cdot [V(\vec{r}) \vec{E}(\vec{r})] + \frac{\epsilon_0}{2} \int_{\text{all space}} |\vec{\nabla} V(\vec{r})|^2 \quad (2.72)$$

Because the charge distribution is restricted to the finite volume \mathcal{V} and thus looks like a point charge as $r \rightarrow \infty$, the field and potential fall off like $1/r^2$ and $1/r$. The surface area of \mathcal{S} only grows as r^2 , so the integral goes like $1/r$ and thus vanishes as $r \rightarrow \infty$. (If the charge distribution is not restricted to a finite volume, the surface term may not vanish and its potential energy could indeed be infinite!)

It may seem strange that we can make this choice of \mathcal{S} , as changing \mathcal{V} and \mathcal{S} affects both integrals in the last expression. The explanation is that the choice of \mathcal{S} changes the two integrals but leaves their sum constant, and taking \mathcal{S} to infinity simple zeros out the first integral, leaving only the contribution of the second integral.

We thus find

$$U = \frac{\epsilon_0}{2} \int |\vec{E}(\vec{r})|^2 \quad (2.73)$$

where the integral is over all of space. Correspondingly, the quantity $u = \frac{\epsilon_0}{2} |\vec{E}|^2$ is an energy density. We interpret this form as indicating that the potential energy created by assembling the charge distribution is stored in the field: less charge implies a smaller field and therefore less potential energy.

Electrostatic Energy (cont.)

Superposition and Electric Potential Energy

Because the electric potential energy is a quadratic function of the electric field,

electric potential energy does not obey superposition

The energy of a sum of fields is more than just the sum of the energies of the individual fields because there is a cross term due to potential energy of the presence of the charges sourcing the second field in the first field (or vice versa, or half of both).

Electrostatic Energy (cont.)

Self-energy and Point Charges vs. Continuous Charge Distributions

We were slightly cavalier in going from Equation 2.69 to Equation 2.70 in that the “self-energy” term $i = j$ that was not included in the former did get included in the latter. In the point-charge version, this term is infinite because the denominator vanishes. In the continuous distribution version, $\rho(\vec{r})\rho(\vec{r}')d\tau \rightarrow 0$ as $|\vec{r} - \vec{r}'| \rightarrow 0$ as long as ρ remains finite over all space, and thus there is no infinite contribution. (If ρ included a delta function, as would be necessary to represent a point charge, then it would produce an infinite contribution because the integral would yield $\delta(0)/0$.) Thus, we must be careful and choose the appropriate formula depending on the situation.

The infinite self-energy of a point charge reflects the fact that we do not know how to assemble a point charge. In fundamental particle physics, the existence of point charges such as the electron is a given. In fact, there is scheme, called “renormalization,” by which the infinite self-energy one calculates for such a charge is “subtracted off” in a self-consistent fashion across all situations. While this practice is accepted and applied carefully, it is not understood. String theory, which postulates that all particles are actually vibrating string-like objects with finite extent, may offer a solution, but string theory currently is not complete — it does not offer a way to calculate the Standard Model — and there is no explicit proof it is correct.

Electric Conductors and Capacitors

Definition and Behavior of a Conductor

We now talk about *electric conductors*, both because they are interesting and because they provide a first opportunity to use boundary conditions to determine properties of the charge distribution, field, and potential.

An *electric conductor* is defined to be a material in which charge is able to flow completely freely in response to an external electric field. It is assumed, *a priori*, to contain equal and opposite amounts of positive and negative electric charge that perfectly cancel everywhere in the absence of an electric field ($\rho = 0$) but that can separate in response to an electric field. One can add charge to a conductor explicitly.

Without any calculation, we know what the response of the conductor will be to an externally applied electric field: If there is any field present in the conductor, positive and negative charge densities will separate in response to the field. That separation results in an additional field whose direction is opposite the applied field because of the direction the two polarities of charge move in response to the applied field. This movement occurs until the sum field vanishes, at which point no more charge separation occurs.

Electric Conductors and Capacitors (cont.)

Derived Properties of a Conductor

From the above behavior, we may derive the following conductor properties from the fact that $\vec{E} = 0$ inside a conductor everywhere:

- ▶ **ρ also vanishes inside a conductor**

This follows directly from Gauss's Law: because $\vec{E} = 0$ everywhere in the interior, then $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$ also vanishes.

Another way of seeing this, at least for a conductor with no net charge, is that, if there were a nonzero ρ , then there must be an equal and opposite amount of charge elsewhere in the conductor because the conductor is neutral overall.

That opposite charge will be attracted to the nonzero ρ and move to cancel it.

Electric Conductors and Capacitors (cont.)

- ▶ Any net charge or induced charge resides on the surface

The picture we described above, of charge separation being induced by the external field, does imply that there may be such *induced charge* on the surface. This does not violate Gauss's Law because \vec{E} may be nonzero outside the conductor and thus one has to be careful calculating $\nabla \cdot \vec{E}$ at the conductor boundary (we must resort to the boundary conditions we derived, Equations 2.49 and 2.49).

Also, if we intentionally add charge to a conductor, it must also move to the surface by the same Gauss's Law argument. The mechanical way of seeing this is that, if we take a neutral conductor in an external field, which has no electric field or charge density in its interior, and add charge to the interior, the added charge repels itself, pushing itself to the exterior (as far as it can go without leaving the conductor), or one can think of it as the added charge attracting charge from the surface to cancel it, leaving net charge on the boundary. Regardless, the added charge that now appears on the surface arranges itself so there is no net field in the interior.

Aside: As Griffiths notes in a footnote, this property can be interpreted to be a consequence of the fact that the electric field obeys the Coulomb's Law $1/r^2$ dependence in three dimensions (from which we derived Gauss's Law, which we used above in the proof). In a different number of dimensions, or with a different dependence on r , we would not have been able to derive Gauss's Law!

Electric Conductors and Capacitors (cont.)

- ▶ **A conductor has the same electric potential everywhere**

That is, a conductor is an *equipotential*. This occurs because \vec{E} vanishes everywhere in the conductor: any line integrals of \vec{E} between two points must therefore also vanish. The conductor may have a nonzero electric potential, but the value is the same everywhere.

One can see this using the gradient, too. If V were not constant in the conductor, there would be a nonzero $\vec{E} = -\vec{\nabla}V$, which we said above is not allowed.

- ▶ **The electric field just outside a conductor is always normal to its surface**

This arises from the boundary conditions we derived, Equations 2.46 and 2.49. Since \vec{E} vanishes inside the conductor, and the tangential component of \vec{E} is continuous across any interface, the tangential component must vanish just outside the conductor, too. There is no such condition on the normal component because there may be an induced or net surface charge density σ on the surface.

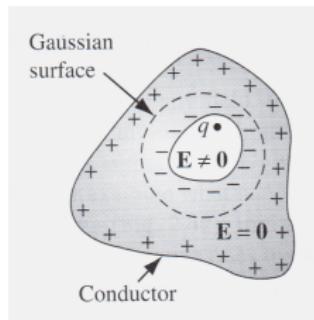
Another way of looking at this is that an electric field tangential to the surface would cause charge to move along the surface until that tangential component vanished. No such argument applies to the normal component because the charge is no longer free to move normal to the surface when it sits at the surface — it cannot leave the conductor.

Electric Conductors and Capacitors (cont.)

Conductors with Cavities

The mental image we have so far is of a conductor that has no cavities inside of it. What additional properties can we derive for a conductor with cavities?

- ▶ A charge q inside a cavity in a conductor results in an equal induced charge q on the surface of the conductor



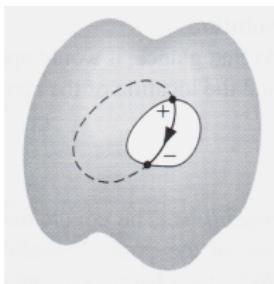
To see this, construct a surface S that lies inside the conductor but also contains the cavity. The electric field vanishes on S because it is in the conductor, so the net charge enclosed must vanish. Since a charge q is inside the cavity, there must be a canceling charge $-q$ induced at the inner surface of the cavity. If $-q$ is on the inner surface, then the canceling charge q must reside on the outer surface.

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The exact distribution of q on the surface depends on the geometry. For a spherical outer surface, it will be uniformly distributed with surface charge density $\sigma = q/4\pi r^2$. Simply on the basis of symmetry, this is clearly also the solution if one adds charge q to a spherical conductor that has no cavity. For the case with the cavity, the charge q has no way of knowing that there is a cavity in the conductor because \vec{E} vanishes inside the conductor, and thus the charge must rearrange itself in the same way as for a conductor without a cavity. A uniqueness theorem that we will prove later tells us that if we find one solution such as this one, it is the *only* solution.

Electric Conductors and Capacitors (cont.)

- If there is no net charge inside a cavity in a conductor, the electric field inside the cavity vanishes, no matter what the external field applied to or net charge added to the conductor is



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This can be seen by assuming there is a nonzero electric field in the cavity. Since there is no charge in the cavity, the field lines must start and end on charges on the surface of the cavity. Therefore, there is a path through the cavity with $\int d\vec{\ell} \cdot \vec{E} \neq 0$. Now close the path with a segment inside the conductor. This portion of the now-closed loop \mathcal{C} contributes nothing to the line integral $\oint_{\mathcal{C}} d\vec{\ell} \cdot \vec{E}$ over the entire loop because $\vec{E} = 0$ inside the conductor. To avoid violating $\oint_{\mathcal{C}} d\vec{\ell} \cdot \vec{E} = 0$, the first contribution from inside the cavity must vanish. Since this is true for any arbitrary path in the cavity, $\vec{E} = 0$ everywhere in the cavity.

Electric Conductors and Capacitors (cont.)

Aside 1: Note the technique of proof we used, proof by contradiction, similar to what is done in mathematics. We will find that such techniques are frequently applied in electrodynamics.

Aside 2: This fact is used for electric shielding of experiments from external electric fields (and also electromagnetic waves) and is called a *Faraday cage*. Note that the conductor can have some net charge on it (and correspondingly sit at some nonzero electric potential with respect to infinity) and this property still holds. As we will show later, it also holds in the presence of external electromagnetic waves, which is actually the more typical and important application.

Lecture 3:
Force on a Conductor
Capacitors and Capacitance
Laplace's Equation
Uniqueness Theorems
Method of Images

Date Revised: 2014/02/18 17:30

Date Given: 2014/02/18

pp. 69–99 assigned as reading and reviewed in class

Electric Conductors and Capacitors

Surface Charge and the Force on the Surface of a Conductor

Our boundary condition for the normal component of the electric field combined with the fact that the electric field vanishes inside a conductor tells us that the electric field infinitesimally above the surface of the conductor is

$$\vec{E} = \frac{\sigma}{\epsilon_0} \hat{n} \quad (2.74)$$

where \hat{n} points from the inside to the outside of the conductor.

There is a charge density σ at this point, and an electric field above it, so is there a force on the charge? Yes, but it is subtle to calculate. The thing to recognize is that the small element of charge σda in an infinitesimal area da cannot exert a force on itself. To find the field that this element of charge feels, we need to see what the field contribution of this charge itself is and subtract it off from the total field.

Electric Conductors and Capacitors (cont.)

We know (Griffiths Example 2.5) that the electric field of a charge sheet in the xy plane is $\vec{E} = \pm(\sigma/2\epsilon_0)\hat{z}$ where the sign applies depending on whether $z > 0$ or $z < 0$. While the small patch we are considering is not an infinite sheet, it looks like one if we are infinitesimally close to it. Thus, we may write the equations

$$\vec{E}_{\text{outside}} = \vec{E}_{\text{other}} + \frac{\sigma}{2\epsilon_0} \hat{n} \quad \vec{E}_{\text{inside}} = \vec{E}_{\text{other}} - \frac{\sigma}{2\epsilon_0} \hat{n} \quad (2.75)$$

where \vec{E}_{other} is the field due to the rest of the charge distribution excepting da . Using $\vec{E}_{\text{outside}} = (\sigma/\epsilon_0)\hat{n}$ and $\vec{E}_{\text{inside}} = 0$, we find $\vec{E}_{\text{other}} = (\sigma/2\epsilon_0)\hat{n}$. Therefore, the force per unit area is

$$\vec{f} = \frac{\vec{F}}{da} = \frac{\sigma da \vec{E}_{\text{other}}}{da} = \sigma \frac{\sigma}{2\epsilon_0} \hat{n} = \frac{\sigma^2}{2\epsilon_0} \hat{n} \quad (2.76)$$

Writing the force per unit area in terms of the field at the surface $\vec{E} = (\sigma/\epsilon_0)\hat{n}$, we have

$$\boxed{\vec{f} = \frac{\sigma^2}{2\epsilon_0} \hat{n} = \frac{\epsilon_0}{2} E^2 \hat{n}} \quad (2.77)$$

That is, the surface of a conductor always feels an outward force. Consider what would happen if you put charge on a balloon with a metallized surface.

Electric Conductors and Capacitors (cont.)

Capacitance

Consider two conductors (of arbitrary shapes) and suppose we put equal and opposite charges Q and $-Q$ on them. The potential difference ΔV between the two is of course given by the line integral of the electric field from any point on the surface of one to any point on the surface of the other. How does V scale with the charges?

The linear dependence of \vec{E} on the charge density ρ ensures that ΔV is linear in Q . Therefore, we may define the *capacitance*

$$C = \frac{Q}{\Delta V} \quad (2.78)$$

Capacitance is a purely *geometric quantity*: it does not depend on the amount of charge on the two conductors (as long as equal and opposite charges are given to each). It does depend on the shapes of the conductors and their *relative* position and orientation because those determine the shape of the electric field (while Q varies its normalization). The unit of capacitance is Coulombs/volt, which we define to be the Farad, F.

One can talk about the capacitance of a single conductor with charge Q by implicitly assuming there is another conductor at infinity that has charge $-Q$ and is defined to be at $V = 0$.

Electric Conductors and Capacitors (cont.)

We can generalize the concept of capacitance to include multiple conductors by simply assuming a generalized linear relationship between voltages and charges as we argued must be true above:

$$V_i = \sum_{j=1}^N D_{ij} Q_j \quad \text{or} \quad \underline{V} = \underline{D} \underline{Q} \quad (2.79)$$

where \underline{V} and \underline{Q} are N -element column matrices for the voltages and charges on the N conductors and \underline{D} is a $N \times N$ matrix that connects the two. It is explicit that any voltage depends linearly on all the charges. The capacitance matrix is $\underline{\underline{C}} = \underline{D}^{-1}$, with

$$Q_i = \sum_{j=1}^N C_{ij} V_j \quad \text{or} \quad \underline{Q} = \underline{\underline{C}} \underline{V} \quad (2.80)$$

This form serves to make it clear that the capacitance is not just a single quantity between two conductors, but is more general.

To calculate the capacitance or capacitance matrix, one clearly needs to find out, given a set of charges $\{Q_i\}$, what the voltages $\{V_i\}$ are. To do this analytically, there typically must be a symmetry or approximation that allows one to guess what the charge distributions on the conductors are (e.g., uniform as for an infinite parallel plate capacitor) and to calculate the field using Gauss's Law. The total charge on each electrode normalizes the voltage.

Electric Conductors and Capacitors (cont.)

For the simple case of two electrodes with equal and opposite charges $\pm Q$ and voltages $\pm V$, we can relate the elements of the capacitance matrix to what we usually call the capacitance. First, let's write the usual scalar capacitance equation in terms of the individual electrode voltages $\pm V$ instead of the voltage difference $\Delta V = 2V$:

$$Q = C \Delta V = C(V - (-V)) = 2C V \quad (2.81)$$

Therefore

$$Q_1 = Q = 2C V = C V - (-C)(-V) = C V_1 - (-C)V_2 \quad (2.82)$$

$$Q_2 = -Q = -2C V = (-C)V + C(-V) = -C V_1 + C V_2 \quad (2.83)$$

and so we may conclude

$$\underline{\underline{C}} = \begin{bmatrix} C & -C \\ -C & C \end{bmatrix} \quad (2.84)$$

where C is the scalar capacitance we usually think of. We can see the matrix is symmetric, which we will prove generically later.

Electric Conductors and Capacitors (cont.)

Electric Potential Energy of a Capacitor

In a simple two-electrode capacitor with charges $\pm q$ on the electrodes and a voltage difference $\Delta V = q/C$ between the two electrodes, the amount of work required to change the charge from q to $q + dq$ is given by the amount of work required to move a charge dq from the negative electrode (which has charge $-q$ and voltage $-\Delta V(q)/2$) to the positive electrode (which has charge $+q$ and voltage $+\Delta V(q)/2$):

$$dU = \Delta V(q) dq = \frac{q}{C} dq \quad (2.85)$$

Note that ΔV is a function of q here: the voltage is not held fixed while the charge is moved; rather, the voltage and charge increase together.

We integrate this expression from 0 to the final charge Q to find

$$U = \frac{1}{C} \int_0^Q q dq = \frac{1}{2} \frac{Q^2}{C} \quad (2.86)$$

Alternatively, using $Q = C \Delta V$,

$$U = \frac{1}{2} \frac{Q^2}{C} = \frac{1}{2} C (\Delta V)^2 \quad (2.87)$$

Electric Conductors and Capacitors (cont.)

What happens for a multi-electrode system? Rather than modeling the process using transferral of charge from one electrode to the other, we need to instead model it as bringing charge in from infinity. If we put equal and opposite charges on two electrodes, then we bring equal and opposite charges in from infinity to put on the two electrodes, which is the equivalent of transferring the charge from one electrode to another: instead of bringing negative charge in from infinity to put on a given electrode, take an equal amount of positive charge off of it and take it out to infinity, then bring that charge back in to put on the complementary positive electrode. (We implicitly define $V = 0$ at infinity.)

So, we consider bringing charge dq_i in from infinity and adding it to electrode i . The change in the electric potential energy of the system due to adding this charge is

$$dU_i = V_i dq_i = \sum_{j=1}^N C_{ij}^{-1} q_j dq_i \quad \text{Note: } C_{ij}^{-1} = D_{ij} \neq \frac{1}{C_{ij}} ! \quad (2.88)$$

There are two possible double-countings we must avoid: 1) This infinitesimal element of charge dq_i is moved from $V = 0$ at infinity to $V = V_i$ on the i th electrode, so the voltages of the other electrodes are irrelevant during this infinitesimal charge transfer and we should not bring them into the equation; 2) The charges on all the electrodes are fixed as we bring this infinitesimal element of charge dq_i in, so we do not need to somehow explicitly account for the change of the voltages on the other electrodes as this charge is brought in.

Electric Conductors and Capacitors (cont.)

Now, let's integrate over dq_i . We will later do a sum over i . The ordering of the two steps does not matter because we proved earlier that the electric potential energy does not depend on the order of assembly. But we do need to worry about what order the charges have been accumulated in that we don't want to calculate cross-terms for charges that don't yet exist. Let's assume that, if we are integrating the i th charge, then the first $i - 1$ charges have already been integrated to their full value. Thus, when we integrate, we obtain

$$U_i = \frac{1}{2} C_{ii}^{-1} Q_i^2 + \sum_{j=1, j \neq i}^{i-1} C_{ij}^{-1} Q_i Q_j \quad (2.89)$$

where the first term has a $1/2$ because we integrate $q_i dq_i$ while the second term does not because we integrate $Q_j dq_i$. Next, we need to sum over i :

$$U = \frac{1}{2} \sum_{i=1}^N C_{ii}^{-1} Q_i^2 + \sum_{i=1}^N \sum_{j=1}^{i-1} C_{ij}^{-1} Q_i Q_j = \frac{1}{2} \sum_{i=1}^N C_{ii}^{-1} Q_i^2 + \frac{1}{2} \sum_{i,j=1, i \neq j}^N C_{ij}^{-1} Q_i Q_j \quad (2.90)$$

$$= \frac{1}{2} \sum_{i,j=1}^N C_{ij}^{-1} Q_i Q_j \quad (2.91)$$

where we modified the second sum to be symmetric (assuming $\underline{\underline{C}}^{-1}$ is symmetric, which we will prove below) and included a factor of $1/2$ to correct for double-counting.

Electric Conductors and Capacitors (cont.)

We can write this more succinctly as

$$U = \frac{1}{2} \underline{Q}^T \underline{\underline{C}}^{-1} \underline{Q} \quad (2.92)$$

Using $\underline{Q} = \underline{\underline{C}} \underline{V}$, we can rewrite as

$$U = \frac{1}{2} \underline{V}^T \underline{\underline{C}} \underline{V} \quad (2.93)$$

Let's check that this gives the correct result for an elementary capacitor with the two electrodes having equal and opposite charges $\pm Q$ and voltages $\pm V$. Using the capacitance matrix we derived earlier,

$$\begin{aligned} U &= \frac{1}{2} [C_{11}(+V)^2 + C_{22}(-V)^2 + C_{12}(+V)(-V) + C_{21}(-V)(+V)] \\ &= \frac{1}{2} C V^2 [1 + (-1)^2 + (-1)^2 + (-1)^2] = 2 C V^2 = \frac{1}{2} C(\Delta V)^2 \end{aligned} \quad (2.94)$$

(recall, $C_{11} = C_{22} = C$ and $C_{12} = C_{21} = -C$) as expected.

Electric Conductors and Capacitors (cont.)

Symmetry of the Capacitance Matrix

In the above discussion, we implicitly assumed symmetry of the capacitance matrix when we converted the asymmetric cross-term sum to a symmetric cross-term sum. Why was it ok to assume this?

Let's consider two electrodes, i and j with $i \neq j$. From Equation 2.89, their contribution to the potential energy, assuming j has been charged up before i , is

$$U = \frac{1}{2} \left(C_{ii}^{-1} Q_i^2 + C_{jj}^{-1} Q_j^2 \right) + C_{ij}^{-1} Q_i Q_j \quad (2.95)$$

If we reverse the charging order, which, in our initial discussion of the electric potential energy, we argued can have no effect on the energy, then the same contribution to the electric potential energy is

$$U = \frac{1}{2} \left(C_{ii}^{-1} Q_i^2 + C_{jj}^{-1} Q_j^2 \right) + C_{ji}^{-1} Q_i Q_j \quad (2.96)$$

Equating the two and recognizing that Q_i and Q_j are arbitrary tells us

$$C_{ij}^{-1} = C_{ji}^{-1} \quad \text{and, therefore,} \quad C_{ij} = C_{ji} \quad (2.97)$$

Section 3

Advanced Electrostatics

Intuitive Approach to Laplace's Equation

As we mentioned earlier, the integral forms for the electric field or the potential

$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V d\tau' \rho(\vec{r}') \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \quad \text{and} \quad V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V d\tau' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (3.1)$$

are always correct but can be problematic to deal with in practice. Most systems will not have symmetries that make the integrals easily doable (or avoidable via Gauss's Law). Moreover, and this is the greater problem, it is rare that one completely specifies $\rho(\vec{r})$ in setting up a problem. Experimentally, what we can control are the shapes, positions, and potentials (voltages) of conductors. We don't control how the charge arranges itself on the conductors. Thus, we need to seek alternate ways to solve for the potential and field over all of space. Laplace's and Poisson's Equations are the key.

Intuitive Approach to Laplace's Equation (cont.)

Laplace's Equation in One Dimension

In one dimension, Laplace's Equation takes the simple form

$$\frac{d^2 V}{dx^2} = 0 \quad (3.2)$$

We can solve this by direct integration to obtain

$$V(x) = mx + b \quad (3.3)$$

where m and b are two constants of integration. We determine m and b by *boundary conditions*: specification of V or dV/dx at specific point(s). In the one dimensional case, there are two options for how to specify the boundary conditions:

- ▶ Specify V at two points.
- ▶ Specify V at one point and dV/dx at one point (possibly the same point).

Note that these are the only choices in one dimension. Specifying dV/dx at two points either yields a contradiction (if two different values of dV/dx are given) or insufficient information (if the same value is given). There are no other quantities to specify: all higher derivatives vanish thanks to Laplace's Equation.

Intuitive Approach to Laplace's Equation (cont.)

Let us note two important characteristics of the solutions of Laplace's Equation:

- ▶ $V(x)$ is related to the average of any pair of points $V(x + a)$ and $V(x - a)$ for any a such that $x \pm a$ belong to the region being considered:

$$\begin{aligned}\frac{1}{2} [V(x + a) + V(x - a)] &= \frac{1}{2} [(m(x + a) + b) + (m(x - a) + b)] \\ &= mx + b = V(x)\end{aligned}\tag{3.4}$$

Solutions to Laplace's Equation have this intrinsic *averaging* property.

- ▶ $V(x)$ has no *nontrivial* local maxima or minima. We already mentioned this property for the three-dimensional Laplace's Equation. The proof is straightforward in one dimension. Suppose x_0 is a local maximum or minimum. Then we have $dV/dx = 0$ at this point x_0 . Then, for any other point x_1 :

$$\left. \frac{dV}{dx} \right|_{x_1} = \left. \frac{dV}{dx} \right|_{x_0} + \int_{x_0}^{x_1} \frac{d^2 V}{dx^2} dx = 0\tag{3.5}$$

Therefore, if dV/dx vanishes anywhere, $V(x)$ is a constant. This is a trivial local maximum/minimum. If dV/dx vanishes nowhere, then the endpoints of the region give the maximum and minimum of $V(x)$ or, if there are no endpoints, there are no maxima or minima at all. Consider, for example, a uniform electric field \vec{E}_0 over all of space.

Intuitive Approach to Laplace's Equation (cont.)

Laplace's Equation in Two or Three Dimensions

We quote the analogues of the above two properties for arbitrary numbers of dimensions and prove them for three dimensions:

- ▶ The value $V(\vec{r})$ of a solution to Laplace's Equation at any point is equal to the average of its value on any sphere centered on that point in the region of interest:

$$V(\vec{r}) = \frac{\int_{\mathcal{S}_a(\vec{r})} da' V(\vec{r}')} {\int_{\mathcal{S}_a(\vec{r})} da} \quad (3.6)$$

where $\mathcal{S}_a(\vec{r})$ is the sphere of radius a centered on \vec{r} . This is straightforward to show (Griffiths Problem 3.37). Let's integrate Laplace's Equation over the volume enclosed by $\mathcal{S}_a(\vec{r})$, $\mathcal{V}_a(\vec{r})$, and use the divergence theorem:

$$\begin{aligned} 0 &= \int_{\mathcal{V}_a(\vec{r})} d\tau' \nabla_{\vec{r}'}^2 V(\vec{r}') = \int_{\mathcal{S}_a(\vec{r})} da' \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} V(\vec{r}') \\ &= \int_{\mathcal{S}_a(\vec{r})} da' \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}' - \vec{r}} V(\vec{r}') \end{aligned} \quad (3.7)$$

In the last step, we have used the fact that $\vec{\nabla}$ does not care about the location of the origin (since it is just an offset).

Intuitive Approach to Laplace's Equation (cont.)

Now, we can define $\vec{s} = \vec{r}' - \vec{r}$. In this coordinate system, where \vec{r} is at the origin, $\hat{n}(\vec{r}') = \hat{s}$, the radial unit vector in the \vec{s} coordinate system. So, we have (inserting a factor $1/4\pi a^2$):

$$0 = \frac{1}{4\pi a^2} \int_{S_a(\vec{s}=0)} a^2 d\Omega_s \left. \frac{\partial V}{\partial s} \right|_{\vec{s}} \quad (3.8)$$

where $S_a(\vec{s} = 0)$ is the sphere of radius a centered on the origin of the \vec{s} system (i.e., the same as the sphere of radius a centered on \vec{r} in the \vec{r}' coordinate system). If we pull the factor a^2 outside of the integral, the integral is now over the spherical angles in the \vec{s} coordinate system, while the derivative is in the radial direction in this coordinate system. Thus, we can pull the derivative outside the integral too, yielding

$$0 = \frac{1}{4\pi a^2} a^2 \int_{S_a(\vec{s}=0)} d\Omega_s \left. \frac{\partial V}{\partial s} \right|_{\vec{s}} = \frac{1}{4\pi} \frac{\partial}{\partial s} \int_{S_a(\vec{s}=0)} d\Omega_s V(\vec{s}) \quad (3.9)$$

(Note that, though V is a function of \vec{s} , only the angular coordinates of \vec{s} are being varied in the integration because the integral is over these angular coordinates.)

Intuitive Approach to Laplace's Equation (cont.)

Thus, the integral must be a constant

$$C = \frac{1}{4\pi} \int_{S_a(\vec{s}=0)} d\Omega_s V(\vec{s}) = \frac{1}{4\pi a^2} \int_{S_a(\vec{r})} da' V(\vec{r}') \quad (3.10)$$

where we switched the variable of integration back to \vec{r}' and we reinserted a^2 . The right side is just the average of V over the sphere of radius a centered at \vec{r} . Since this holds for any a , it must hold as $a \rightarrow 0$, which tells us $C = V(\vec{r})$. So, we have

$$V(\vec{r}) = \frac{1}{4\pi a^2} \int_{S_a(\vec{r})} da' V(\vec{r}') \quad (3.11)$$

- ▶ As a consequence of Laplace's Equation and the above property, V can have no local maxima or minima in the region of interest. The proof of this property is trivial: if there were such a candidate maximum (minimum), simply draw a sphere around it. Because the point is a maximum (minimum) there must be some radius of the sphere for which the values of all the points on the sphere are less than (greater than) the value at the candidate maximum (minimum). The average over this sphere is therefore less than (greater than) the value at the candidate maximum (minimum). This contradicts the above averaging property.

One could also prove this by calculating the gradient at any point \vec{r}' in the region by doing a line integral of Laplace's Equation from the candidate extremum \vec{r} to that point. Since the gradient vanishes at the candidate extremum, and the line integral vanishes by Laplace's Equation, the gradient vanishes at \vec{r}' .

Uniqueness Theorems

Before obtaining a solution of Laplace's and Poisson's Equations, we prove some uniqueness theorems we will need. This section draws from Jackson §1.8 and §1.9.

Green's Identities and Theorem

First, some mathematical preliminaries. Let us apply the divergence theorem to the function $\phi \vec{\nabla} \psi$ where $\phi(\vec{r})$ and $\psi(\vec{r})$ are arbitrary functions:

$$\oint_S da \hat{n} \cdot (\phi \vec{\nabla} \psi) = \int_{V(S)} d\tau \vec{\nabla} \cdot (\phi \vec{\nabla} \psi)$$

This yields *Green's First Identity*:

$$\boxed{\oint_S da \phi \hat{n} \cdot \vec{\nabla} \psi = \int_{V(S)} d\tau [\phi \nabla^2 \psi + \vec{\nabla} \phi \cdot \vec{\nabla} \psi]} \quad (3.12)$$

The function $\hat{n} \cdot \vec{\nabla} \psi$ is the *normal derivative* of ψ because it is the projection of the gradient of ψ along the direction normal to the surface. If we exchange ϕ and ψ and then difference the two versions, we have *Green's Second Identity* or *Green's Theorem*:

$$\boxed{\oint_S da [\phi \hat{n} \cdot \vec{\nabla} \psi - \psi \hat{n} \cdot \vec{\nabla} \phi] = \int_{V(S)} d\tau [\phi \nabla^2 \psi - \psi \nabla^2 \phi]} \quad (3.13)$$

Uniqueness Theorems (cont.)

Generic Uniqueness Proof

Let us use the above Green's identities to prove the uniqueness of solutions to Poisson's Equation for three types of boundary conditions:

- ▶ *Dirichlet boundary condition*

In this case, the value of the potential $V(\vec{r})$ is specified on all bounding surfaces. This is the most typical experimentally realized situation, where we attach a number of conductors to voltage sources to set their voltages.

- ▶ *Neumann boundary condition*

In this case, the value of the normal derivative of the voltage, $\hat{n} \cdot \vec{\nabla} V(\vec{r})$, is specified on the boundary. An example of such a condition is specification of the electric field (or, equivalently, the surface charge density) at the surfaces of a set of conductors; since the tangential electric field vanishes at these surfaces, the normal electric field fully defines the electric field at the conductors.

- ▶ *Mixed boundary conditions*

Dirichlet in some places, Neumann in others, is allowed as long as both are not specified at the same place.

If the volume under consideration is not bounded by a surface on which we specify the boundary condition, then we must also specify a boundary condition at infinity.

Uniqueness Theorems (cont.)

Suppose we have specified one of the above three types of boundary conditions. Assume that, for a particular given charge distribution $\rho(\vec{r})$, there are two independent solutions $V_1(\vec{r})$ and $V_2(\vec{r})$ of Poisson's Equation that satisfy this boundary condition. Let $V_3 = V_1 - V_2$. Since the charge distribution is the same, $\nabla^2 V_1 = -\rho/\epsilon_0 = \nabla^2 V_2$ and thus $\nabla^2 V_3 = 0$: V_3 satisfies Laplace's Equation. By a similar differencing argument, V_3 either satisfies the Dirichlet boundary condition $V_3(\vec{r} \in \mathcal{S}) = 0$, the Neumann boundary condition $\hat{n} \cdot \vec{\nabla} V_3(\vec{r} \in \mathcal{S}) = 0$, or a mixed boundary condition of these types. If we apply Green's first identity with $\phi = \psi = V_3$, we have

$$\oint_{\mathcal{S}} da V_3 \hat{n} \cdot \vec{\nabla} V_3 = \int_{\mathcal{V}(\mathcal{S})} d\tau \left(V_3 \nabla^2 V_3 + \vec{\nabla} V_3 \cdot \vec{\nabla} V_3 \right) \quad (3.14)$$

The left side vanishes because of the boundary condition (any type). The first term on the right side vanishes by Laplace's Equation. Thus, we have

$$\int_{\mathcal{V}(\mathcal{S})} d\tau |\vec{\nabla} V_3|^2 = 0 \quad (3.15)$$

Since the integrand is nonnegative, we may conclude

$$\vec{\nabla} V_3(\vec{r}) = 0 \iff V_3 = \text{constant} \quad (3.16)$$

which implies that our two candidate solutions $V_1(\vec{r})$ and $V_2(\vec{r})$ differ by at most a constant. Hence, uniqueness is proven.

Uniqueness Theorems (cont.)

Special Cases of Uniqueness Theorem

Given the above, we may state/prove three special cases of the uniqueness theorem, the ones given in Griffiths:

- ▶ *The solution to Laplace's Equation in some volume \mathcal{V} is uniquely specified if V is specified on the boundary surface $\mathcal{S}(\mathcal{V})$.*

This is the above uniqueness theorem with $\rho = 0$ in \mathcal{V} and a Dirichlet boundary condition on $\mathcal{S}(\mathcal{V})$.

- ▶ *The solution to Poisson's Equation in some volume \mathcal{V} is uniquely specified if $\rho(\vec{r})$ is specified throughout the region and V is specified on the boundary surface $\mathcal{S}(\mathcal{V})$.*

This is the above uniqueness theorem with arbitrary $\rho(\vec{r})$ in \mathcal{V} and a Dirichlet boundary condition on $\mathcal{S}(\mathcal{V})$.

Uniqueness Theorems (cont.)

- In a volume \mathcal{V} surrounded by conductors at the surface(s) $S(\mathcal{V})$ and containing a specified charge density $\rho(\vec{r})$, the electric field is uniquely determined if the total charge on each conductor is specified.

This one is not as obvious, but is still relatively straightforward thanks to the generic uniqueness theorem. A solution is uniquely specified if we have a Dirichlet boundary condition. Since $S(\mathcal{V})$ consists only of conductors, and conductors are equipotentials, specifying the Dirichlet boundary condition in this case consists only of specifying the N voltages $\{V_i\}$ on the N bounding surfaces. But it is the charges that are given, so how do we know the voltages? Suppose $\rho = 0$ in \mathcal{V} . Then, we know that the capacitance matrix relates the charges $\{Q_i\}$ and the voltages in the absence of ρ , which we will call $\{V_i^0\}$: $\underline{V}^0 = \underline{\mathcal{C}}^{-1} \underline{Q}$. (We don't need to know what $\underline{\mathcal{C}}$ is, we only need to know it exists, which we argued when we defined it.) Finally, when we add back $\rho(\vec{r})$, we know that, by superposition, the voltages $\{V_i\}$ are

$$V_i = V_i^0 + \int_{\mathcal{V}} d\tau \frac{\rho(\vec{r})}{|\vec{r} - \vec{r}'|} \quad (3.17)$$

Thus, we are assured that, if the $\{Q_i\}$ and $\rho(\vec{r})$ are given, they uniquely determine the $\{V_i\}$, and then we have a Dirichlet boundary condition and so the generic uniqueness theorem applies. Note how this proof relied on the boundary surfaces being conductors! Knowing the total charges on nonconducting boundary surfaces would not be sufficient.

Method of Images

The Basic Idea

The method of images uses the concept of uniqueness of solutions to Poisson's Equation. Basically, given a physical setup involving a true charge distribution $\rho(\vec{r})$ and a boundary condition for some volume \mathcal{V} , one tries to replace the region outside of \mathcal{V} with an *image charge* distribution $\rho_{\text{image}}(\vec{r})$ such that, when the image charge's potential is summed with that of $\rho(\vec{r})$, the boundary condition is met.

The technique works because of the uniqueness theorem: since the potential due to the image and original charges matches the boundary condition and satisfies Poisson's Equation with the same source term inside \mathcal{V} , it is *the* solution to Poisson's Equation for that source term and choice of boundary condition.

The imagined charge distribution is called image charge because, at least in the example of the boundary condition being imposed by the presence of a conductor, the image charges appear to be a mirror image of the original charges through the boundary. "Image charge" is also used (somewhat erroneously) to refer to the surface charge induced on a conducting boundary that sources the potential that one imagines is due to the image charge.

We will see later how the potential due to the image charge distribution (the induced surface charge) is a component of the particular problem's Green Function.

Method of Images (cont.)

A Point Charge near a Grounded Infinite Conducting Plane

For a system with the point charge q at $d\hat{z}$ and the conducting plane at $z = 0$ with $V = 0$, the appropriate image charge is $-q$ at $-d\hat{z}$. By symmetry, the (Dirichlet) boundary condition $V = 0$ at $z = 0$ is met. Thus, the solution for $V(\vec{r})$ for \vec{r} belonging to the $z > 0$ half-space is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{\sqrt{x^2 + y^2 + (z-d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z+d)^2}} \right] \quad (3.18)$$

The potential clearly satisfies $V(z=0) = 0$. Let's use this solution to do some other calculations:

► *Induced surface charge*

This we can calculate by recognizing that it is given by the change in the normal component of the electric field at the conducting boundary. Since $\vec{E} = -\vec{\nabla}V$,

$$\begin{aligned} \sigma &= -\epsilon_0 \frac{\partial V}{\partial z} \Big|_{z=0} = \frac{q}{4\pi} \left[\frac{z-d}{(x^2 + y^2 + (z-d)^2)^{3/2}} - \frac{z+d}{(x^2 + y^2 + (z+d)^2)^{3/2}} \right] \Big|_{z=0} \\ &= -\frac{q}{2\pi} \frac{d}{(x^2 + y^2 + d^2)^{3/2}} \end{aligned} \quad (3.19)$$

Method of Images (cont.)

We can calculate the total induced surface charge:

$$Q_{ind} = \int_0^\infty r dr \int_0^{2\pi} d\phi \frac{-q d}{2\pi} \frac{1}{(r^2 + d^2)^{3/2}} = q d \frac{1}{\sqrt{r^2 + d^2}} \Big|_0^\infty = -q \quad (3.20)$$

This is an example of an important general rule: *The equality of the induced surface charge and the image charge holds for any method of images situation because of Gauss's Law*: the image charge produces the same potential in \mathcal{V} as the induced surface charge, so the normal gradient of the potential at the surface is the same for the induced surface charge and the image charge, so the flux of the electric fields are the same, so the enclosed charge is the same.

► Force on the point charge

The induced charge is equal and opposite in sign to the point charge, so the two are attracted to each other. We can calculate the force by taking the gradient of the potential due to the *image charge only* (because the original point charge does not feel a force due to its own potential). Since the image charge's potential is just that of a point charge, calculating the force is straightforward:

$$\vec{F} = q \vec{E}_{image\ charge}(z = d) = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{(2d)^2} \hat{z} \quad (3.21)$$

We could have calculated this more directly by just assuming the image charge and the real charge act like two normal point charges, but it's good to see the two techniques give the same result.

Method of Images (cont.)

► Electric potential energy

Here we have to be more careful because potential energy is not linear in charge, and, moreover, because the induced charge depends on the original point charge. Let's figure this out by calculating the work one would have to do against the attractive force to bring q from $z = d$ to $z = \infty$.

$$U = - \int_d^{\infty} (-F(z)) dz = - \frac{1}{4\pi\epsilon_0} \frac{q^2}{4} \int_d^{\infty} \frac{dz}{z^2} = - \frac{1}{4\pi\epsilon_0} \frac{q^2}{4d} \quad (3.22)$$

Note that this result is half what one would get for the potential energy of two equal and opposite point charges separated by a distance $2d$:

$$U_{alt} = - \frac{1}{4\pi\epsilon_0} \frac{q^2}{2d} \quad (3.23)$$

There are two ways to understand this. The first is to recognize that, unlike in the case of two point charges, no energy is gained or lost in moving the negative charge because it is in the conductor, where $V = 0$ and thus $qV = 0$ everywhere. The second is to recognize that the above expression is the energy stored in all of space in the field of two point charges, but, in this case, the field is only real in the $z > 0$ half-space and so the integrated energy is reduced by a factor of 2.

Method of Images (cont.)

A Point Charge near a Conducting Sphere

Consider a conducting sphere of radius R centered on the origin and held at $V = 0$. Place a point charge at $a\hat{z}$.

By symmetry, the appropriate image charge must be on the z axis. Let its value be q' and its position be $b\hat{z}$, where b may be positive or negative. We can find q' and b by requiring that $V = 0$ at $\vec{r} = \pm R\hat{z}$:

$$V(z = \pm R) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{a \mp R} + \frac{q'}{R \mp b} \right] \quad (3.24)$$

Setting both equations equal to zero yields

$$q' = -q \frac{R}{a} \quad b = \frac{R^2}{a} \quad (3.25)$$

We see that both values are always physically reasonable because $R < a$. In particular, $b < R$ so the image charge remains outside the region for which the boundary condition is specified.

Method of Images (cont.)

The potential at a point $(r \geq R, \theta, \phi)$ is

$$V(r \geq R, \theta, \phi) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{|\vec{r} - a\hat{z}|} - \frac{R/a}{|\vec{r} - \frac{R^2}{a}\hat{z}|} \right] \quad (3.26)$$

$$= \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{r^2 \sin^2 \theta + (a - r \cos \theta)^2}} - \frac{R/a}{\sqrt{r^2 \sin^2 \theta + (\frac{R^2}{a} - r \cos \theta)^2}} \right] \quad (3.27)$$

Let's check that the boundary condition $V(r = R) = 0$ is satisfied:

$$\begin{aligned} V(r = R, \theta, \phi) &= \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{R^2 \sin^2 \theta + (a - R \cos \theta)^2}} - \frac{R/a}{\sqrt{R^2 \sin^2 \theta + (\frac{R^2}{a} - R \cos \theta)^2}} \right] \\ &= \frac{q}{4\pi\epsilon_0} \left[\frac{1}{\sqrt{R^2 \sin^2 \theta + (a - R \cos \theta)^2}} - \frac{1}{\sqrt{a^2 \sin^2 \theta + (R - a \cos \theta)^2}} \right] \\ &= 0 \end{aligned} \quad (3.28)$$

Method of Images (cont.)

Let's calculate the induced surface charge from $\hat{n} \cdot \vec{\nabla} V = \partial V / \partial r$:

$$\sigma = -\epsilon_0 \frac{\partial V}{\partial r} \Big|_{r=R} \quad (3.29)$$

$$\begin{aligned} &= \frac{q}{4\pi} \left[\frac{R \sin^2 \theta - (a - R \cos \theta) \cos \theta}{(R^2 \sin^2 \theta + (a - R \cos \theta)^2)^{3/2}} - \frac{R}{a} \frac{R \sin^2 \theta - (\frac{R^2}{a} - R \cos \theta) \cos \theta}{(R^2 \sin^2 \theta + (\frac{R^2}{a} - R \cos \theta)^2)^{3/2}} \right] \\ &= \frac{q}{4\pi} \left[\frac{R - a \cos \theta}{(R^2 + a^2 - 2aR \cos \theta)^{3/2}} - \frac{a^2}{R^2} \frac{R - \frac{R^2}{a} \cos \theta}{(a^2 + R^2 - 2aR \cos \theta)^{3/2}} \right] \\ &= \frac{q}{4\pi} \frac{R(1 - \frac{a^2}{R^2})}{(R^2 + a^2 - 2aR \cos \theta)^{3/2}} \\ &= -\frac{q}{4\pi R^2} \frac{R}{a} \frac{1 - \frac{R^2}{a^2}}{\left(1 + \frac{R^2}{a^2} - 2\frac{R}{a} \cos \theta\right)^{3/2}} \end{aligned}$$

One can show by integration that the total induced charge is q' , matching the image charge as one would expect from Gauss's Law. The force on the point charge and the electric potential energy can be calculated in a manner similar to that used for the conducting plane.

Method of Images (cont.)

Some Related Examples

These are drawn from Jackson Chapter 2.

- ▶ *Point charge in the presence of a charged, insulated, conducting sphere*

We can solve this problem by superposition. Suppose we want to have a charge Q on the sphere. This is the same as first bringing the point charge q in while the sphere is grounded, disconnecting the grounding wire, and then adding $Q - q'$ ($> Q$ for $q > 0$). Since the induced surface charge q' is in full equilibrium with q before $Q - q'$ is added — i.e., the net electric field is normal to the surface of the conductor, so the induced charge can move nowhere — the additional charge $Q - q'$ just spreads out over the surface evenly. This maintains the boundary condition that the surface of the sphere is an equipotential (= the field must be normal to the surface). Since we have found a configuration that respects the boundary conditions, it is the only valid configuration by the uniqueness theorem. The potential is therefore

$$V(r \geq R, \theta, \phi) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{|\vec{r} - a\hat{z}|} - \frac{R/a}{|\vec{r} - \frac{R^2}{a}\hat{z}|} \right] + \frac{Q + q\frac{R}{a}}{4\pi\epsilon_0 |\vec{r}|} \quad (3.30)$$

Method of Images (cont.)

By Gauss's Law, the field due to $Q - q'$ is the same as that of a point charge $Q - q'$ at the origin. Therefore, the electric potential at the surface of the sphere is $V(r = R) = (Q - q')/(4\pi\epsilon_0 R)$.

The surface charge density is the original induced charge for the grounded sphere with the addition of $Q - q'$ distributed uniformly, which we explained above.

The force and potential energy can be calculated in a manner similar to that for the grounded sphere.

► *Point charge in the presence of a conducting sphere at fixed potential*

We can solve this problem by applying the last problem. In the last case, the conductor acquired a voltage $(Q - q')/(4\pi\epsilon_0 R)$. So, if we want the conductor to sit at a voltage V_0 , we just have to apply the last problem with $Q = 4\pi\epsilon_0 V_0 R + q'$.

Lecture 4:
Green Functions
Obtaining Green Functions from the Method of Images

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Formal Solution to Poisson's Equation: The Green Function

Integral Equation for the Electric Potential

We apply Green's Theorem with $\phi(\vec{r}') = V(\vec{r}')$ and $\psi(\vec{r}') = |\vec{r} - \vec{r}'|^{-1}$. Note that \vec{r}' is the variable we integrate over; \vec{r} is considered a constant for the purposes of the Green's Theorem integrals.

$$\begin{aligned} & \int_{\mathcal{V}(\mathcal{S})} d\tau' \left[V(\vec{r}') \nabla_{\vec{r}'}^2 \frac{1}{|\vec{r} - \vec{r}'|} - \frac{1}{|\vec{r} - \vec{r}'|} \nabla_{\vec{r}'}^2 V(\vec{r}') \right] \\ &= \oint_S da \left[V(\vec{r}') \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} \frac{1}{|\vec{r} - \vec{r}'|} - \frac{1}{|\vec{r} - \vec{r}'|} \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} V(\vec{r}') \right] \quad (3.31) \end{aligned}$$

Next, note that

$$\nabla_{\vec{r}'}^2 \frac{1}{|\vec{r} - \vec{r}'|} = -4\pi \delta(\vec{r} - \vec{r}') \quad (3.32)$$

because (Equations 2.33 and 2.57)

$$\vec{\nabla}_{\vec{r}'} \cdot \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = 4\pi \delta(\vec{r} - \vec{r}') \quad \text{and} \quad \vec{\nabla}_{\vec{r}'} \frac{1}{|\vec{r} - \vec{r}'|} = -\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3}$$

Formal Solution to Poisson's Equation: The Green Function (cont.)

Using the above expression for the Laplacian of $|\vec{r} - \vec{r}'|^{-1}$, doing the integral over the delta function, applying Poisson's Equation, moving the second term on the right side to the left side, and multiplying everything by $-\frac{1}{4\pi}$ yields, now only for $\vec{r} \in \mathcal{V}(\mathcal{S})$:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}(\mathcal{S})} d\tau' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} + \frac{1}{4\pi} \oint_{\mathcal{S}} da \left[\frac{1}{|\vec{r} - \vec{r}'|} \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} V(\vec{r}') - V(\vec{r}') \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r} - \vec{r}'|} \right] \quad (3.33)$$

(The left side vanishes for $\vec{r} \notin \mathcal{V}(\mathcal{S})$). This is a formal equation for the electric potential. The boundary conditions are present on the right side: in the case of Dirichlet, we specify $V(\vec{r}')$ for $\vec{r}' \in \mathcal{S}$, while in the case of Neumann, we specify $\hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} V(\vec{r}')$ for $\vec{r}' \in \mathcal{S}$. Since one or the other is left unspecified, this is not a closed-form solution but rather an integral equation for $V(\vec{r}')$ for $\vec{r}' \in \mathcal{V}(\mathcal{S})$ given the unspecified boundary condition for $\vec{r}' \in \mathcal{S}$.

Note that, in the limit of $\mathcal{S} \rightarrow \infty$, the integrand of the surface integral falls off as r^{-3} and so the surface term vanishes and we recover the standard equation for $V(\vec{r})$, Equation 2.55.

Formal Solution to Poisson's Equation: The Green Function (cont.)

The Concept of Green Functions

Suppose we have the generalized linear partial differential equation

$$O_{\vec{r}} f(\vec{r}) = g(\vec{r}) \quad (3.34)$$

where O is a linear partial differential operator, f is a generalized potential, and g is a generalized source function. Poisson's Equation is an example, with $O_{\vec{r}} = -\epsilon_0 \nabla^2$, $f(\vec{r}) = V(\vec{r})$, and $g(\vec{r}) = \rho(\vec{r})$. Is there a general approach for finding f given g ?

Yes, there is, it is called the Green Function approach. The basic idea is to find the "impulse" response function for the differential equation: the generalized potential one gets if one has a point-like source. Given that impulse response function, one can convolve the impulse response function with an arbitrary source function to get the generalized potential for that source function.

Formal Solution to Poisson's Equation: The Green Function (cont.)

Mathematically, the impulse response function, or *Green Function*, is the function $G(\vec{r}, \vec{r}')$ that solves the equation

$$O_{\vec{r}} G(\vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}') \quad (3.35)$$

meaning that $G(\vec{r}, \vec{r}')$ calculates the generalized potential at the point \vec{r} for a point source of size 1 at the position \vec{r}' (i.e., the total source charge recovered by integrating over the source function is 1). If such a G exists, then, for an arbitrary source function $g(\vec{r})$, G gives us the solution $f(\vec{r})$ to the generalized linear partial differential equation:

$$f(\vec{r}) = \int d\tau' G(\vec{r}, \vec{r}') g(\vec{r}') \quad (3.36)$$

We can check that the above differential equation is satisfied by the above solution by applying the operator:

$$O_{\vec{r}} f(\vec{r}) = O_{\vec{r}} \int d\tau' G(\vec{r}, \vec{r}') g(\vec{r}') = \int d\tau' [O_{\vec{r}} G(\vec{r}, \vec{r}')] g(\vec{r}') \quad (3.37)$$

$$= \int d\tau' \delta(\vec{r} - \vec{r}') g(\vec{r}') = g(\vec{r}) \quad (3.38)$$

Assuming solutions to the generalized linear partial differential equation are unique (true for Poisson's Equation), the Green Function is the *only* solution to be found.

Formal Solution to Poisson's Equation: The Green Function (cont.)

General Discussion of Green Functions for Poisson's Equation

We have already seen that, in the absence of a bounding surface, the Green Function for Poisson's Equation is

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}'|} \quad (3.39)$$

We can see this explicitly by rewriting our usual expression for the potential for this boundary condition, Equation 2.55, in terms of G :

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} d\tau' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} = \int_{\mathcal{V}} d\tau' G(\vec{r}, \vec{r}') \rho(\vec{r}') \quad (3.40)$$

Formal Solution to Poisson's Equation: The Green Function (cont.)

More generally — i.e., for a more complex boundary condition — the Green Function for Poisson's Equation must satisfy

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{r}'|} + F(\vec{r}, \vec{r}') \quad \text{with} \quad \nabla_{\vec{r}}^2 F(\vec{r}, \vec{r}') = 0 \quad (3.41)$$

with the first term taking care of the charge placed explicitly in the volume and the second term being added to replace the integration of the first term over the charge distribution on the boundary (which we may not know) with an integral over the given boundary conditions. We may add F because it solves Laplace's Equation in the volume and thus does not affect the source side of Poisson's Equation in the volume. Finding G thus consists of finding F .

What is F ? We'll get into this a bit later when we talk about the conditions on Green Functions for specific Dirichlet or Neumann boundary conditions, as the interpretation depends on which type is imposed.

We note that both G and F are symmetric in their arguments, $G(\vec{r}', \vec{r}) = G(\vec{r}, \vec{r}')$ and $F(\vec{r}', \vec{r}) = F(\vec{r}, \vec{r}')$, for reasons we will explain later.

Formal Solution to Poisson's Equation: The Green Function (cont.)

The Green Functions for Poisson's Equation with Dirichlet or Neumann Boundary Conditions

We can apply the above concept by taking $\phi(\vec{r}') = V(\vec{r}')$ and $\psi(\vec{r}') = -\epsilon_0 G(\vec{r}, \vec{r}')$ in Green's Theorem (Equation 3.13), giving

$$V(\vec{r}) = \int_{\mathcal{V}} d\tau' \rho(\vec{r}') G(\vec{r}, \vec{r}') + \epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da' \left[G(\vec{r}, \vec{r}') \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} V(\vec{r}') - V(\vec{r}') \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} G(\vec{r}, \vec{r}') \right] \quad (3.42)$$

We see that, if we can find the appropriate G for a particular boundary condition and force the term involving the other boundary condition to vanish, our integral equation for $V(\vec{r})$ reduces to an integration over the source distribution and boundary condition. We can be more specific about this by picking a boundary condition:

Formal Solution to Poisson's Equation: The Green Function (cont.)

► Dirichlet boundary condition

In this case, $V(\vec{r})$ is specified for $\vec{r} \in \mathcal{S}$. Therefore, $\hat{n}(\vec{r}) \cdot \vec{\nabla}_{\vec{r}} V(\vec{r})$ should be left unspecified — it should be determined by the solution itself — so we need for it to not appear in the integral equation. We can eliminate the term containing this normal derivative if we require the Dirichlet Green Function to satisfy

$$G_D(\vec{r}, \vec{r}') = 0 \quad \text{for } \vec{r}' \in \mathcal{S}, \vec{r} \in \mathcal{V}, \mathcal{S} \quad (3.43)$$

Note that we want the above condition to hold for not just $\vec{r} \in \mathcal{V}$ but also for $\vec{r} \in \mathcal{S}$ so the expression is usable for calculating the potential on the boundary to ensure the boundary condition remains satisfied (*i.e.*, the expression for $V(\vec{r})$ is self-consistent).

With this condition, the solution for $V(\vec{r})$ reduces to

$$V(\vec{r}) = \int_{\mathcal{V}} d\tau' \rho(\vec{r}') G_D(\vec{r}, \vec{r}') - \epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da' V(\vec{r}') \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} G_D(\vec{r}, \vec{r}') \quad (3.44)$$

where it is now clear how the Dirichlet boundary condition applies — it simply needs to be explicitly inserted into the second term, and then the integration with $\hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} G_D(\vec{r}, \vec{r}')$ calculates its contribution (or, rather, the contribution of the corresponding induced surface charge) to the overall potential.

Formal Solution to Poisson's Equation: The Green Function (cont.)

For a Dirichlet boundary condition, the symmetry of G_D in its arguments can be proven by applying Green's Theorem with $\phi = G_D(\vec{r}, \vec{x})$ and $\psi = G_D(\vec{r}', \vec{x})$ where \vec{x} is the variable that is integrated over and using the defining condition $G_D(\vec{r}, \vec{x}) = 0$ for \vec{x} on the boundary and \vec{r} in the volume and on the boundary (which also implies the same for $G_D(\vec{r}', \vec{x})$). Symmetry of G_D implies symmetry of F_D given that their difference is symmetric in \vec{r} and \vec{r}' .

The symmetry requirement makes it easier to interpret the condition $G_D(\vec{r}, \vec{r}') = 0$. As the Green Function is originally defined, where \vec{r}' is the location of the unit charge and \vec{r} is the point at which the potential is given, the condition states that the potential at some point \vec{r} in \mathcal{V} or on \mathcal{S} is not directly affected by the charge induced on the boundary at \vec{r}' . The interpretation of this is not obvious.

But when the symmetry of the Green Function is used, now one can think of the unit charge as being at \vec{r} and the potential being calculated at \vec{r}' . Thus, our condition on G_D requires that G_D yields zero contribution to the potential on the *boundary* from charges *in the volume or on the surface*. If this did not hold, then both the given charge distribution in the volume, as well as the induced charges on the boundary, would modify the Dirichlet boundary condition. (The way this is enforced physically is by the appearance of induced charge on the boundary surface due to the charges in the volume. The induced charge is drawn from the voltage source(s) that enforces the Dirichlet boundary condition.)

Formal Solution to Poisson's Equation: The Green Function (cont.)

We can also now provide an interpretation of $F(\vec{r}, \vec{r}')$ in the Dirichlet case. Because 1) $F(\vec{r}, \vec{r}')$ satisfies Laplace's Equation in the volume \mathcal{V} , and 2) when added to the potential of a unit point charge at \vec{r}' (the first term in our expression relating G and F , Equation 3.41), the sum satisfies the specified boundary condition on \mathcal{S} , $F(\vec{r}, \vec{r}')$ can be interpreted as the potential function in the volume due to the surface charge induced on the boundary by the real charges in the volume in order for the boundary condition to be satisfied.

Finally, we can now go back and reinterpret what is meant by $G_D(\vec{r}, \vec{r}') = 0$ when \vec{r}' is considered to be the location of the unit charge. It implies that the induced charge on the boundary does not modify the potential in \mathcal{V} or on \mathcal{S} calculated by the first term on our expression above for $V(\vec{r})$, the one that convolves G_D with ρ in \mathcal{V} . The first part of this statement, the fact that the induced charge on \mathcal{S} does not affect the potential in \mathcal{V} , simply avoids double-counting the effect of that surface charge — it is already accounted for by the first term. The second part of this statement, the fact that the induced charge on \mathcal{S} does not affect the potential on \mathcal{S} , is the requirement that the Dirichlet boundary condition not be modified by the induced surface charge.

Formal Solution to Poisson's Equation: The Green Function (cont.)

► Neumann boundary condition

In this case, $\hat{n} \cdot \vec{\nabla} V(\vec{r})$ is specified for $\vec{r} \in \mathcal{S}$, so we need to render irrelevant the term containing $V(\vec{r})$ because we should not have to simultaneously specify it.

While we might be inclined to require $\hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} G_N(\vec{r}, \vec{r}') = 0$ for $\vec{r}' \in \mathcal{S}$ to make this happen, this requirement is not consistent with Poisson's Equation defining G : if one integrates Poisson's Equation for G_N over $\mathcal{V}(\mathcal{S})$ and turns it into a surface integral using the divergence theorem, one obtains the requirement

$$-\epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da' \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} G_N(\vec{r}, \vec{r}') = 1 \quad (3.45)$$

Thus, the simplest condition we can impose on G_N is

$$\hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} G_N(\vec{r}, \vec{r}') = - \left[\epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da' \right]^{-1} \quad (3.46)$$

The physical interpretation of this condition is that the normal electric field at a boundary point \vec{r}' due to a unit point charge at \vec{r} is independent of \vec{r} and \vec{r}' . The physical meaning of this is not yet clear, but we will see its meaning below.

Formal Solution to Poisson's Equation: The Green Function (cont.)

Applying this condition, the solution for $V(\vec{r})$ reduces to

$$V(\vec{r}) = \int_{\mathcal{V}} d\tau' \rho(\vec{r}') G_N(\vec{r}, \vec{r}') + \epsilon_0 \oint_{S(\mathcal{V})} da' G_N(\vec{r}, \vec{r}') \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} V(\vec{r}') + \langle V(\vec{r}) \rangle_{S(\mathcal{V})}$$

$$\text{with } \langle V(\vec{r}) \rangle_{S(\mathcal{V})} \equiv \frac{\oint_{S(\mathcal{V})} da' V(\vec{r}')}{\oint_{S(\mathcal{V})} da'}$$
 (3.47)

While $V(\vec{r})$ on the boundary has not been completely eliminated, its only appearance is via its average value on the boundary. This makes sense, as the Neumann boundary condition does not specify the potential offset since it only specifies derivatives of the potential. The appearance of this term reflects the freedom we have to set the potential offset for problems with Neumann boundary conditions.

What is the interpretation of a Neumann Green Function? Since $\hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} V(\vec{r}')$ specifies the surface charge density on the boundary, $G_N(\vec{r}, \vec{r}')$ simply calculates the potential at a point \vec{r} in the volume due to the surface charge density at \vec{r}' on the boundary. This is because we see in the above equation that G_N is convolved with the volume charge density and the surface charge density in the same way. A Neumann Green Function thus has a simpler interpretation than a Dirichlet Green Function.

Formal Solution to Poisson's Equation: The Green Function (cont.)

For a Neumann boundary condition, the symmetry of G_N and F_N is not a result of the boundary condition, but it may be assumed without loss of generality; see K.-J. Kim and J. D. Jackson, *Am. J. Phys.* **61**:1144 (1993)).

To make further progress in obtaining a functional form for the Green Function, we must specify the boundary conditions in more detail. We will consider examples of this later.

Obtaining Green Functions from the Method of Images

Obtaining the Green Function from the Method of Images

We discussed earlier how the component $F(\vec{r}, \vec{r}')$ of the full Green Function $G(\vec{r}, \vec{r}')$ is determined by the method of images. Let's see how this works for the two cases we have considered:

- ▶ *Point charge near grounded conducting plane*

The full potential at a point \vec{r} for the point charge at $d\hat{z}$ is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{|\vec{r} - d\hat{z}|} - \frac{q}{|\vec{r} + d\hat{z}|} \right] \quad (3.48)$$

Thus, the Dirichlet Green Function is given by letting $\vec{r}' = d\hat{z}$ and removing q :

$$G_D(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0} \left[\frac{1}{|\vec{r} - \vec{r}'|} - \frac{1}{|\vec{r} + \vec{r}'|} \right] \quad (3.49)$$

That is, the second term accounts for the fact that induced charge appears on the grounded conducting plane and calculate the contribution to the potential due to it; it is the $F(\vec{r}, \vec{r}')$ term while the first term is the usual Coulomb's Law term. The first term solves Poisson's Equation while the second term solves Laplace's Equation. One can see how the second term's contribution depends on the position of the point charge at \vec{r}' . One can also see how $G(z=0, \vec{r}') = 0$ always, and how G_D is manifestly symmetric in \vec{r} and \vec{r}' .

Obtaining Green Functions from the Method of Images (cont.)

It is also important to notice that the surface integral term went away. In the Dirichlet case, we expect a surface term from Equation 3.44

$$-\epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da' V(\vec{r}') \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} G_D(\vec{r}, \vec{r}') \quad (3.50)$$

In this case, the Dirichlet boundary condition is $V(z = 0) = 0$. Thus, this integral vanishes and we indeed only have the volume integral term from Equation 3.44 convolving the original charge distribution with G_D .

► Point charge near conducting plane held at V_0

Suppose our boundary condition had instead been $V(z = 0) = V_0$, a constant (and also $V(r \rightarrow \infty) = V_0$ for consistency). Is the above Green Function still valid, and what is the impact of the new boundary condition? We expect it should have no impact because it just adds an offset to the potential. The charge distribution in the bulk and the type of boundary condition are unchanged, so we expect G_D is still valid. We can check that G_D provides the result we expect, an offset in the potential.

First, because $V(\vec{r}') = V_0$ for $\vec{r}' \in \mathcal{S}(\mathcal{V})$, we can pull it outside the integral, so we just have the surface integral of the normal gradient of G_D over the surface:

$$-\epsilon_0 \oint_{\mathcal{S}(\mathcal{V})} da' V(\vec{r}') \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} G_D(\vec{r}, \vec{r}') = -\epsilon_0 V_0 \oint_{\mathcal{S}(\mathcal{V})} da' \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} G_D(\vec{r}, \vec{r}')$$

Obtaining Green Functions from the Method of Images (cont.)

We recall that, by definition, $G_D(\vec{r}, \vec{r}')$ is the potential at the point \vec{r} due to a point charge of unit magnitude ($q = 1$) at \vec{r}' . By the symmetry of its arguments, it is also the potential at the point \vec{r}' due to a unit point charge at \vec{r} . Earlier, when we did the method of images solution for the grounded conducting plane, we calculated the surface charge density at \vec{r} due to the point charge at $d\hat{z}$ from $-\epsilon_0 \vec{\nabla}_{\vec{r}} V(\vec{r}, d\hat{z})$. So, in this case, $-\epsilon_0 \vec{\nabla}_{\vec{r}'} G_D(\vec{r}, \vec{r}')$ is the surface charge density at \vec{r}' due to a unit charge at \vec{r} . Since V_0 has come outside the integral, our surface integral is now just the integral of this surface charge density over the boundary, or the total induced charge on the boundary. We proved earlier the general rule (via Gauss's Law) that the total induced surface charge is equal to the image charge. In this case, the image charge is equal and opposite to the real charge so the integral gives -1. That is:

$$\begin{aligned} -\epsilon_0 \oint_{S(V)} da' V(\vec{r}') \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} G_D(\vec{r}, \vec{r}') &= -V_0 \oint_{S(V)} da' \sigma_{ind}(\vec{r}', q = 1) \\ &= -V_0 q_{image} = V_0 \end{aligned} \quad (3.51)$$

So, we see that the surface term serves to add the potential offset that the boundary condition $V(z = 0) = V_0$ requires.

Obtaining Green Functions from the Method of Images (cont.)

- ▶ Point charge near grounded conducting sphere

The full potential at a point \vec{r} for the point charge at $a\hat{z}$ was (Equation 3.27):

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\frac{q}{|\vec{r} - a\hat{z}|} - \frac{q \frac{R}{a}}{|\vec{r} - \frac{R^2}{a}\hat{z}|} \right] \quad (3.52)$$

Thus, the Dirichlet Green Function is given by letting $\vec{r}' = a\hat{z}$ and removing q :

$$G_D(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0} \left[\frac{1}{|\vec{r} - \vec{r}'|} - \frac{R/r'}{|\vec{r} - \vec{r}' \frac{R^2}{(r')^2}|} \right] \quad (3.53)$$

Again, the second term accounts for the potential due to the charge induced on the surface of the sphere and is the term that solves Laplace's Equation in this situation (the $F(\vec{r}, \vec{r}')$ term).

It is perhaps not so obvious that the second term in this Green Function is symmetric in its arguments. Let's rewrite it:

$$\frac{R/r'}{|\vec{r} - \vec{r}' \frac{R^2}{(r')^2}|} = \frac{R}{|\hat{r} r r' - R^2 \hat{r}'|} = \frac{R}{\sqrt{(r r')^2 + R^4 - 2 r r' R^2 \hat{r} \cdot \hat{r}'}} \quad (3.54)$$

Now the symmetry is manifest.

The same point about the surface integral term as for the conducting plane holds here: that term vanishes because $V(\vec{r}') = 0$ for $\vec{r}' \in \mathcal{S}$.

Obtaining Green Functions from the Method of Images (cont.)

- ▶ Point charge near conducting sphere held at fixed potential

In this case, we can see the effect of the surface integral term in Equation 3.44 because $V(\vec{r})$ on the boundary does not vanish. The integral term is, from Equation 3.44:

$$-\epsilon_0 \oint_{S(V)} da' V(\vec{r}') \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} G_D(\vec{r}, \vec{r}') \quad (3.55)$$

When we encountered this nonvanishing surface term for the prior case of a point charge near a conducting plane, we recognized that $V(\vec{r}') = V_0$ could be pulled outside the integral and that the integral of the normal gradient of the Green Function gives the total charge induced on the boundary for a unit charge at \vec{r} . That argument carries through unchanged here, and, again, the total charge induced on the boundary is equal to the image charge by Gauss's Law, so we have

$$-\epsilon_0 \oint_{S(V)} da' V(\vec{r}') \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} G_D(\vec{r}, \vec{r}') = -V_0 q_{image} = V_0 \frac{R}{r} \quad (3.56)$$

This is just the potential due to a point charge at the origin whose magnitude is such that the potential at radius R is V_0 .

Obtaining Green Functions from the Method of Images (cont.)

With this integral evaluated, we can write down the full solution for $V(\vec{r})$:

$$\begin{aligned} V(\vec{r}) &= \int_{\mathcal{V}(S)} d\tau' q \delta(\vec{r}' - a\hat{z}) G_D(\vec{r}, \vec{r}') + V_0 \frac{R}{r} \\ &= \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}(S)} d\tau' q \delta(\vec{r}' - a\hat{z}) \left[\frac{1}{|\vec{r} - \vec{r}'|} - \frac{R/r'}{|\vec{r} - \vec{r}' - \frac{R^2}{(r')^2}\hat{z}|} \right] + V_0 \frac{R}{r} \\ &= \frac{q}{4\pi\epsilon_0} \left[\frac{1}{|\vec{r} - a\hat{z}|} - \frac{R/a}{|\vec{r} - \frac{R^2}{a}\hat{z}|} \right] + V_0 \frac{R}{r} \end{aligned} \quad (3.57)$$

We thus see that, in this case, the solution consists of the superposition of the grounded-conductor solution — the potential of the point charge and its image charge — and the potential of a point charge at the origin whose magnitude would give a potential V_0 at radius R . Explicitly, that point charge has magnitude $Q = 4\pi\epsilon_0 R V_0$. This charge is what we found earlier when we discussed the same problem using the method of images.

- ▶ *Point charge in the presence of a charged, insulated, conducting sphere*

The prior situation is identical to this one: specifying the charge on a conductor is the same as specifying its potential. So the result for $V(\vec{r})$ is the same, where we must take $V_0 = (Q + (R/a)q)/(4\pi\epsilon_0 R)$.

Lecture 5:
Separation of Variables in Cartesian Coordinates
Separation of Variables in Spherical Coordinates: General Theory
Separation of Variables in Spherical Coordinates
with Azimuthal Symmetry

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pp. 122–132 assigned to be read ahead of time

Separation of Variables in Cartesian Coordinates

General Points

Griffiths makes this seem harder than it is. In separation of variables, we assume that the solution of Laplace's Equation factors into functions of single coordinates. This allows us to reduce the partial differential equation to a set of ordinary differential equations, which can be solved by standard techniques. Constants of integration appear that help to define the solutions. We apply the boundary conditions as defined by the voltages and/or the charge densities (normal derivative of voltage) at the boundaries. Once we find a set of solutions, we know from Sturm-Liouville theory that they form a *complete* set, so we are assured that we can write any solution to Laplace's Equation for the given boundary conditions in terms of these solutions.

One interesting point is that we only will seek to solve Laplace's Equation. We can do this because of the argument we made earlier: the Green Function for Poisson's Equation is of the form

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0} \frac{1}{\vec{r} - \vec{r}'} + F(\vec{r}, \vec{r}') \quad (3.58)$$

where $F(\vec{r}, \vec{r}')$ solves Laplace's Equation. So we only need to develop separation of variables for Laplace's Equation.

Separation of Variables in Cartesian Coordinates (cont.)

Cartesian Coordinates

We assume that the function $V(\vec{r})$ can be factorized as

$$V(\vec{r}) = X(x) Y(y) Z(z) \quad (3.59)$$

Plugging this into Laplace's Equation, we obtain

$$\begin{aligned} Y(y) Z(z) \frac{d^2 X}{dx^2} + X(x) Z(z) \frac{d^2 Y}{dY^2} + X(x) Y(y) \frac{d^2 Z}{dz^2} &= 0 \\ \frac{1}{X(x)} \frac{d^2 X}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y}{dY^2} + \frac{1}{Z(z)} \frac{d^2 Z}{dz^2} &= \end{aligned} \quad (3.60)$$

We have three terms, the first a function of x , the second of y , and the third of z . Given these mismatched dependences, the only way the equation can hold is if each term is a constant. That is, it must hold that

$$\frac{1}{X(x)} \frac{d^2 X}{dx^2} = C_1 \quad \frac{1}{Y(y)} \frac{d^2 Y}{dY^2} = C_2 \quad \frac{1}{Z(z)} \frac{d^2 Z}{dz^2} = C_3 \quad (3.61)$$

with $C_1 + C_2 + C_3 = 0$.

Separation of Variables in Cartesian Coordinates (cont.)

We know that the solution to these ordinary differential equations are exponentials,

$$X(x) = A \exp(x\sqrt{C_1}) + B \exp(-x\sqrt{C_1}) \quad (3.62)$$

$$Y(y) = C \exp(y\sqrt{C_2}) + D \exp(-y\sqrt{C_2}) \quad (3.63)$$

$$Z(z) = E \exp(z\sqrt{-(C_1 + C_2)}) + F \exp(-z\sqrt{-(C_1 + C_2)}) \quad (3.64)$$

We have not specified which of C_1 , C_2 , and C_3 are positive and which are negative (clearly, they cannot all be the same sign). That will be determined by the boundary conditions.

At this point, we cannot make further generic progress, we need to apply a set of boundary conditions. These will place constraints on the allowed values of the exponents and coefficients and restrict the family of solutions. There are a number of examples in Griffiths. To avoid duplication, we use a different one here from Jackson §2.9.

Separation of Variables in Cartesian Coordinates (cont.)

Box with five walls grounded and one held at a potential

Consider a box with one corner at the origin and of side lengths a , b , and c in the x , y , and z dimensions. The boundary conditions are

$$V(x = 0) = 0 \quad V(y = 0) = 0 \quad V(z = 0) = 0 \quad (3.65)$$

$$V(x = a) = 0 \quad V(y = b) = 0 \quad V(z = c) = \phi(x, y) \quad (3.66)$$

where $\phi(x, y)$ is a function that is given. Let's apply the boundary conditions in x and y :

$$A + B = 0 \quad A \exp(a\sqrt{C_1}) + B \exp(-a\sqrt{C_1}) = 0 \quad (3.67)$$

$$C + D = 0 \quad C \exp(b\sqrt{C_2}) + D \exp(-b\sqrt{C_2}) = 0 \quad (3.68)$$

Reducing,

$$A \left[\exp(a\sqrt{C_1}) - \exp(-a\sqrt{C_1}) \right] = 0 \quad (3.69)$$

$$C \left[\exp(b\sqrt{C_2}) - \exp(-b\sqrt{C_2}) \right] = 0 \quad (3.70)$$

Separation of Variables in Cartesian Coordinates (cont.)

There is no solution to these equations for $C_1 > 0$ and $C_2 > 0$: the unit-normalized decaying and rising exponentials are only equal when their arguments both vanish, which they don't. Therefore, let's take $C_1 = -\alpha^2$ and $C_2 = -\beta^2$ so these become oscillating exponentials. We thus obtain the conditions

$$\sin(\alpha a) = 0 \quad \sin(\beta b) = 0 \quad (3.71)$$

This places conditions on the allowed values of α and β :

$$\alpha_n = \frac{n\pi}{a} \quad \beta_m = \frac{m\pi}{b} \quad m, n \text{ integers} \quad (3.72)$$

Thus, we have

$$X(x) = \sum_{n=1}^{\infty} A_n \sin \alpha_n x \quad Y(y) = \sum_{m=1}^{\infty} C_m \sin \beta_m y \quad (3.73)$$

where n and m may only be positive integers (since the negative values are redundant with the positive ones and $n = 0$ and $m = 0$ contribute nothing) and the $\{A_n\}$ and $\{C_m\}$ are constants to be determined. No offset in V is allowed for these boundary conditions because $V = 0$ on some boundaries.

Separation of Variables in Cartesian Coordinates (cont.)

Now, let's apply the boundary conditions to $Z(z)$. At $z = 0$, we have

$$E + F = 0 \implies F = -E \quad (3.74)$$

Therefore, $Z(z)$ is of the form

$$Z(z) = E \left[\exp(z\sqrt{\alpha_n^2 + \beta_m^2}) - \exp(-z\sqrt{\alpha_n^2 + \beta_m^2}) \right] \quad (3.75)$$

$$= E' \sinh(\gamma_{nm} z) \quad \text{with} \quad \gamma_{nm} = \sqrt{\alpha_n^2 + \beta_m^2} \quad (3.76)$$

Thus, prior to applying the boundary condition at $z = c$, we have

$$V(\vec{r}) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z) \quad (3.77)$$

where $A_{nm} = A_n C_m E'_{nm}$ are constants to find. The boundary condition at $z = c$ yields

$$\phi(x, y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c) \quad (3.78)$$

This condition will let us determine the A_{mn} , but how, and why are we certain they exist? For this, we must briefly talk about orthonormal functions and completeness.

Separation of Variables in Cartesian Coordinates (cont.)

Digression on Orthonormal Functions

The general topic of the properties of solutions to second-order linear differential equations is beyond the scope of this course; it falls under the name *Sturm-Liouville theory*, and it is covered in ACM95/100. We will simply quote some results that are important for this course.

Sturm-Liouville theory consists of recognizing that the second-order linear ordinary differential equations we encounter in many places in this course are self-adjoint (Hermitian) operators on the Hilbert space of functions that satisfy the differential equation. You know from linear algebra that Hermitian operators are guaranteed to have a set of eigenvalues and eigenvectors (in this case, eigenfunctions), and that the eigenvectors form an orthonormal basis for the space under consideration (here, again, the space of functions that satisfy the differential equation). The same results apply here. What this means is that, for such equations, there are a set of solution functions $\{f_p(w)\}$ that are the eigenfunctions of the operator, and there are corresponding eigenvalues $\{\lambda_n\}$. These eigenfunctions form a *complete, orthonormal set*.

Separation of Variables in Cartesian Coordinates (cont.)

Orthonormality is written mathematically as

$$\int_s^t dw f_p^*(w) f_q(w) = \delta_{pq} \quad (3.79)$$

where integration over the interval of interest $[s, t]$ is the Hilbert space inner product. Completeness means

$$\sum_p f_p^*(w') f_p(w) = \delta(w' - w) \quad (3.80)$$

where the sum is over all eigenfunctions of the differential equation. Completeness implies that any solution of the differential equation is expandable in terms of this orthonormal set of functions:

$$g(w) = \sum_p A_p f_p(w) \quad (3.81)$$

Separation of Variables in Cartesian Coordinates (cont.)

Orthonormality tells us how to find the A_p : multiply the above equation by $f_q^*(w)$ and integrate:

$$\int_s^t dw f_q^*(w) g(w) = \int_s^t dw \sum_p A_p f_q^*(w) f_p(w) = \sum_p A_p \delta_{pq} = A_q \quad (3.82)$$

It is instructive to see how completeness can be used:

$$\begin{aligned} g(w) &= \int_s^t dw' g(w') \delta(w' - w) = \int_s^t dw' g(w') \sum_p f_p^*(w') f_p(w) \\ &= \sum_p f_p(w) \int_s^t dw' f_p^*(w') g(w') \end{aligned} \quad (3.83)$$

If we use Equation 3.81 and apply the orthonormality condition by multiplying by $f_q^*(w)$ and integrating over w , we explicitly recover

$$A_q = \int_s^t dw' f_q^*(w') g(w') \quad (3.84)$$

Separation of Variables in Cartesian Coordinates (cont.)

Completing the solution to the box with five walls grounded and one held at a potential

We ended with

$$\phi(x, y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c) \quad (3.85)$$

Now, it turns out that the functions $\{\sqrt{2/a} \sin(\alpha_n x)\}$ for $n \geq 1$ form an orthonormal set on the $x \in [0, a]$ interval, as do $\{\sqrt{2/b} \sin(\beta_m y)\}$ for $m \geq 1$ on $y \in [0, b]$, so we may recover the A_{nm} by multiplying by them and integrating:

$$\begin{aligned} & \int_0^a dx \int_0^b dy \phi(x, y) \sqrt{\frac{2}{a}} \sin(\alpha_p x) \sqrt{\frac{2}{b}} \sin(\beta_q y) \\ &= \int_0^a dx \int_0^b dy \sum_{n,m=1}^{\infty} A_{nm} \sinh(\gamma_{nm} c) \sin(\alpha_n x) \sqrt{\frac{2}{a}} \sin(\alpha_p x) \sin(\beta_m y) \sqrt{\frac{2}{b}} \sin(\beta_q y) \\ &= \sum_{n,m=1}^{\infty} A_{nm} \sinh(\gamma_{nm} c) \sqrt{\frac{a}{2}} \delta_{pn} \sqrt{\frac{b}{2}} \delta_{qm} = \frac{\sqrt{ab}}{2} A_{pq} \sinh(\gamma_{pq} c) \end{aligned} \quad (3.86)$$

Separation of Variables in Cartesian Coordinates (cont.)

That is,

$$A_{nm} = \frac{1}{\sinh(\gamma_{nm}c)} \int_0^a dx \int_0^b dy \phi(x, y) \frac{2}{a} \sin(\alpha_n x) \frac{2}{b} \sin(\beta_m y) \quad (3.87)$$

Thus, our full solution is

$$\begin{aligned} V(\vec{r}) &= \frac{4}{ab} \sum_{n,m=1}^{\infty} \sin(\alpha_n x) \sin(\beta_m y) \frac{\sinh(\gamma_{nm}z)}{\sinh(\gamma_{nm}c)} \\ &\quad \times \int_0^a dx' \int_0^b dy' \phi(x', y') \sin(\alpha_n x') \sin(\beta_m y') \end{aligned} \quad (3.88)$$

If we had used a more general boundary condition, specifying V to be nonzero on all six sides of the box, then we could solve the similar problem for each of the six faces independently (*i.e.*, let V be nonzero and arbitrary on that face and zero on all the other faces) and then sum the solutions since each individual solution does not affect the capability of the other solutions to satisfy their boundary conditions. (Of course, the boundary conditions themselves must be consistent with each other at the edges and corners where they meet.) *In fact, it is likely we would have to do this; it is not clear that the boundary conditions could all be satisfied by a single separated solution.*

Separation of Variables in Cartesian Coordinates (cont.)

Referring back to our discussion of Green Functions, the above solution is the surface term in Equation 3.44 for the particular boundary condition we have applied. By comparison of the two expressions, we infer (not derive!)

$$\begin{aligned} -\epsilon_0 \vec{\nabla}_{\vec{r}'} F_D(\vec{r}, \vec{r}') &= x' \hat{x} + y' \hat{y} + c \hat{z} \\ &= \frac{4 \hat{z}}{a b} \sum_{n,m=1}^{\infty} \sin(\alpha_n x) \sin(\beta_m y) \frac{\sinh(\gamma_{nm} z)}{\sinh(\gamma_{nm} c)} \sin(\alpha_n x') \sin(\beta_m y') \end{aligned} \quad (3.89)$$

Note that this expression does not fully specify F ! The above information is sufficient for the particular physical situation we have set up, which consists of no physical charge in the volume and the above boundary condition, because:

- ▶ The term consisting of the integral of the charge density in the volume convolved with G_D (and with F_D) is zero in this case because the charge density vanishes *in the volume*.
- ▶ The above surface term is the only one needed because $V = 0$ on the other boundaries.

Separation of Variables in Cartesian Coordinates (cont.)

But, for the more general problem of an arbitrary charge distribution in the volume and arbitrary Dirichlet boundary conditions on the surfaces, we need to know the full G_D by finding the full F_D term. The natural way to find this would be the method of images with the condition $V = 0$ on all the surfaces. It is left as an exercise for the reader to think about what set of image charges is appropriate.

Certainly, from the resulting G_D , we could compute the normal gradient of G_D on any surface and thus obtain the general solution for V in the volume for any Dirichlet boundary conditions. We should find that the normal gradient of F_D (not G_D , which includes the potential directly due to the charges in the volume) on the $z = c$ surface is what is given above.

It may seem like separation of variables is unsatisfactory for this reason — it does not give you the full Green Function, while the method of images does. But, as we have seen, the method of images is not a systematic procedure — one has to guess the correct image charge distribution. On the other hand, separation of variables is an entirely algorithmic procedure to give you a solution if a separable one exists for the particular boundary condition you are applying. It is less general but more reliable.

There is however, no guarantee that there will be a separable solution; this depends on the nature of the boundary conditions. The boundary conditions need to respect the separability assumed. For example, a boundary condition on a spherical boundary would not likely yield a solution via separation of variables in Cartesian coordinates!

Note also that the above method of images solution is not appropriate for a Neumann boundary condition because the method of images solution always solves the $V = 0$ on the boundaries situation, a Dirichlet boundary condition.

Separation of Variables in Spherical Coordinates: General Theory

Separation of Variables in Spherical Coordinates

We do this in a slightly more general manner than Griffiths, dropping the assumption of azimuthal symmetry until it is time to solve the separated differential equations. We then return to the azimuthally dependent case later.

Laplace's Equation in spherical coordinates is:

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left(\sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \phi^2} = 0 \quad (3.90)$$

If we assume a separable form

$$V(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi) \quad (3.91)$$

Then, after dividing through by $V(r, \theta, \phi)$ and multiplying by $r^2 \sin^2 \theta$, we have

$$\sin^2 \theta \left[\frac{1}{R(r)} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta(\theta)} \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) \right] + \frac{1}{\Phi(\phi)} \frac{d^2 \Phi}{d\phi^2} = 0 \quad (3.92)$$

Separation of Variables in Spherical Coordinates: General Theory (cont.)

We see that first term depends only on r and θ while the second term depends only on ϕ , so we can immediately assume they are each equal to a constant:

$$\frac{1}{\Phi(\phi)} \frac{d^2\Phi}{d\phi^2} = -m^2 \quad (3.93)$$

The choice of the form of the constant is motivated by what will come next, but we can see why it needs to be of this form. As we saw in Cartesian coordinates, the above differential equation is solved either by growing/decaying exponentials (right side positive) or oscillating exponentials (right side negative). Since ϕ is a coordinate that repeats on itself ($\phi = 2n\pi$ are the same physical coordinate) the solutions $\Phi(\phi)$ must also be periodic, forcing the choice of the oscillating exponential. We saw from before that the coefficient in the argument appears squared, hence m^2 instead of m .

The solutions of this equation are straightforward:

$$\Phi(\phi) = A \exp(i m \phi) + B \exp(-i m \phi) \quad (3.94)$$

Periodicity in ϕ with period 2π requires m be an integer. One can either require $m \geq 0$ and keep the $\{A_m\}$ and $\{B_m\}$ or allow m to be any integer and drop the $\{B_m\}$ (which would be redundant with the $\{A_m\}$ for $m < 0$). In either case, only A_0 or B_0 is required.

Separation of Variables in Spherical Coordinates: General Theory (cont.)

Returning to the other term, we now have

$$\sin^2 \theta \left[\frac{1}{R(r)} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta(\theta)} \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) \right] = m^2 \quad (3.95)$$

$$\frac{1}{R(r)} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) + \left[\frac{1}{\Theta(\theta)} \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} \right] = 0 \quad (3.96)$$

Now, we see that the first term depends only on r and the second only on θ , so we can separate again by setting the two terms equal to a constant. Here, we rely on prior knowledge of the result to choose the constant to be $\ell(\ell + 1)$ so that

$$\frac{1}{R(r)} \frac{d}{dr} \left(r^2 \frac{dR}{dr} \right) = \ell(\ell + 1) \quad (3.97)$$

$$\frac{1}{\Theta(\theta)} \frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) - \frac{m^2}{\sin^2 \theta} = -\ell(\ell + 1) \quad (3.98)$$

Separation of Variables in Spherical Coordinates: General Theory (cont.)

Solving the Radial Equation

Here, we pull something out of our “bag of tricks” and define $U(r)$ by $R(r) = U(r)/r$ and plug in. We find

$$\frac{d^2 U}{dr^2} - \frac{\ell(\ell + 1)}{r^2} U(r) = 0 \quad (3.99)$$

Since the two derivatives would reduce the exponent of a power-law solution by 2, and the second term does the same by dividing by r^2 , the above equation suggests $U(r)$ is a power law in r . If we plug in such a form $U(r) = r^a$, we find

$$a(a - 1)r^{a-2} - \ell(\ell + 1)r^{a-2} = 0 \implies a = \ell + 1 \quad \text{or} \quad a = -\ell \quad (3.100)$$

$$\implies R(r) = A r^\ell + \frac{B}{r^{\ell+1}} \quad (3.101)$$

There is no constraint on ℓ yet.

Separation of Variables in Spherical Coordinates: General Theory (cont.)

The Polar Equation and the Generalized Legendre Equation

We may rewrite the polar angle equation as

$$\frac{1}{\sin \theta} \frac{d}{d\theta} \left(\sin \theta \frac{d\Theta}{d\theta} \right) + \left[\ell(\ell+1) - \frac{m^2}{\sin^2 \theta} \right] \Theta(\theta) = 0 \quad (3.102)$$

If we use another one of our bag of tricks to write

$$x = \cos \theta \quad \Theta(\theta) = P(\cos \theta) = P(x) \quad 1 - x^2 = \sin^2 \theta \quad (3.103)$$

then we may rewrite the polar differential equation as

$$\boxed{\frac{d}{dx} \left[(1-x^2) \frac{dP}{dx} \right] + \left[\ell(\ell+1) - \frac{m^2}{1-x^2} \right] P(x) = 0} \quad (3.104)$$

This is called the *generalized Legendre equation*.

Separation of Variables in Spherical Coordinates: General Theory (cont.)

The generalized Legendre equation can be solved by assuming the solution is a polynomial in x . You have probably learned this technique in ACM95/100. One assumes

$$P_\ell^m(x) = \sum_{k=1}^{\infty} a_k x^k \quad (3.105)$$

and then, plugging the above form into the differential equation, one requires the series to terminate ($a_k = 0$ for some k). This condition requires ℓ to be a nonnegative integer and $-\ell \leq m \leq \ell$. These polynomials are the *associated Legendre polynomials*. We will come back to these shortly.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry

The Polar Equation Solution with Azimuthal Symmetry: the Legendre Equation and Legendre Polynomials

Consider the special case of azimuthal symmetry, for which $m = 0$ and $\Phi(\phi) = \text{constant}$. The generalized Legendre Equation reduces to the *Legendre Equation*:

$$\frac{d}{dx} \left[(1 - x^2) \frac{dP}{dx} \right] + \ell(\ell + 1) P(x) = 0 \quad (3.106)$$

One can apply the same series solution technique to this equation. Termination requires ℓ be a nonnegative integer. These solutions are the *Legendre Polynomials*. One can show they obey *Rodrigues' Formula*:

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \left(\frac{d}{dx} \right)^\ell (x^2 - 1)^\ell \quad (3.107)$$

Mathematically, there should be a second solution for a given ℓ because the equation is second order. These are the solutions one finds by not requiring termination but simply convergence for $-1 < x < 1$ (corresponding to $0 < \theta < \pi$). If one has a geometry that excludes the z-axis (where the divergence occurs), these solutions must be considered. If the z-axis is in the space, then these solutions are unphysical and can be discarded.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Properties of the Legendre Polynomials

One can see by inspection or prove the following properties:

- ▶ $P_\ell(x)$ is a ℓ th-order polynomial in x .
- ▶ $P_\ell(x)$ has only even powers of x if ℓ is even and only odd powers if ℓ is odd. Therefore, $P_\ell(x)$ is an even function of x for ℓ even and an odd function for ℓ odd.
- ▶ The Legendre polynomials are a complete orthonormal set: any function on the interval $[0, 1]$ can be written in terms of them. Their orthonormality relation is

$$\int_{-1}^1 dx \sqrt{\frac{2\ell+1}{2}} P_\ell(x) \sqrt{\frac{2\ell'+1}{2}} P_{\ell'}(x) = \delta_{\ell\ell'} \quad (3.108)$$

and their completeness relation is

$$\sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} P_\ell(x) P_\ell(x') = \delta(x - x') \quad (3.109)$$

- ▶ $P_\ell(1) = 1$ and $P_\ell(-1) = (-1)^\ell$.
- ▶ $P_\ell(0) = [(-1)^n (2n-1)!!]/2^n n!$ for even $\ell = 2n$ and vanishes for odd ℓ .

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Full Solution to Laplace's Equation with Azimuthal Symmetry

Combining our radial and polar equation solutions, we have that, for any problem with azimuthal symmetry and in which the z -axis is included, the potential must have the form

$$V(r, \theta) = \sum_{\ell=0}^{\infty} \left(A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}} \right) P_{\ell}(\cos \theta) \quad (3.110)$$

The coefficients $\{A_{\ell}\}$ and $\{B_{\ell}\}$ are set by the boundary conditions. If the volume includes the origin and the boundary conditions imply the potential must be finite there, the $\{B_{\ell}\}$ may be eliminated, and, if the volume includes infinity and requires the potential be finite (usually zero) there, the $\{A_{\ell}\}$ may be eliminated. Usually, application of the boundary conditions on V will require use of the orthonormality relations for the Legendre polynomials.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

We note that, in the process of doing separation of variables, we have proven that the components of the solution satisfy the *eigenvalue-eigenfunction equations*

$$\boxed{\nabla^2 A_\ell r^\ell = \frac{\ell(\ell+1)}{r^2} A_\ell r^\ell \quad \nabla^2 B_\ell \frac{1}{r^{\ell+1}} = \frac{\ell(\ell+1)}{r^2} B_\ell \frac{1}{r^{\ell+1}}} \quad (3.111)$$

$$\boxed{\nabla^2 P_\ell(\cos \theta) = -\frac{\ell(\ell+1)}{r^2} P_\ell(\cos \theta)} \quad (3.112)$$

The fact that there is a r^2 on the right side of the radial equations reflects the fact that it is really $U(r) = R(r)/r$ that satisfies the eigenvalue-eigenfunction equation. For the angular equation, r is like a constant and so naturally appears in the eigenvalue.

Lecture 6:
Separation of Variables in Spherical Coordinates
with Azimuthal Symmetry (cont.)
Separation of Variables in Spherical Coordinates
without Azimuthal Symmetry

Date Revised: 2014/03/13 6:00

(correction to Equation 3.181 and page that follows)

Date Given: 2014/02/27

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry

Dirichlet Boundary Condition on a Spherical Boundary with Azimuthal Symmetry

Suppose $V(R, \theta)$, the potential as a function of θ on a sphere of radius R , is specified, where the sphere is either the outer boundary or inner boundary of the space. What is the explicit form for the resulting potential?

Let's consider the two cases together. If the space is $r < R$, then we require the $\{B_\ell\}$ to vanish to ensure a finite potential at the origin. (There is no charge in the volume, so we are assured that the potential cannot be infinite there.) If the space is $r > R$, then we require the $\{A_\ell\}$ to vanish so the potential goes to zero at infinity. That is:

$$V(r, \theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos \theta) \quad \text{or} \quad V(r, \theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} P_\ell(\cos \theta) \quad (3.113)$$

Since we know $V(R, \theta)$, we evaluate the above equations at that value:

$$V(R, \theta) = \sum_{\ell=0}^{\infty} A_\ell R^\ell P_\ell(\cos \theta) \quad \text{or} \quad V(R, \theta) = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos \theta) \quad (3.114)$$

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Then, to find the coefficients, we apply orthornormality to both sides. For the case of $r < R$, we have:

$$\frac{2\ell+1}{2} \int_0^\pi \sin \theta \, d\theta \, V(R, \theta) P_\ell(\cos \theta) \quad (3.115)$$

$$= \sum_{\ell=0}^{\infty} A'_\ell R^{\ell'} \int_0^\pi \sin \theta \, d\theta \frac{2\ell+1}{2} P_\ell(\cos \theta) P_{\ell'}(\cos \theta) \quad (3.116)$$

$$= \sum_{\ell=0}^{\infty} A'_\ell R^{\ell'} \delta_{\ell\ell'} = A_\ell R^\ell \quad (3.117)$$

Therefore,

$$A_\ell = \frac{2\ell+1}{2} \frac{1}{R^\ell} \int_0^\pi \sin \theta \, d\theta \, V(R, \theta) P_\ell(\cos \theta) \quad (3.118)$$

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Similarly, for the case $r > R$,

$$B_\ell = \frac{2\ell+1}{2} R^{\ell+1} \int_0^\pi \sin \theta \, d\theta V(R, \theta) P_\ell(\cos \theta) \quad (3.119)$$

Therefore, the solutions are

$$V(r < R, \theta) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} \frac{r^\ell}{R^\ell} \int_0^\pi \sin \theta \, d\theta V(R, \theta) P_\ell(\cos \theta) \quad (3.120)$$

$$V(r > R, \theta) = \sum_{\ell=0}^{\infty} \frac{2\ell+1}{2} \frac{R^{\ell+1}}{r^{\ell+1}} \int_0^\pi \sin \theta \, d\theta V(R, \theta) P_\ell(\cos \theta) \quad (3.121)$$

Notice how the units of the coefficients have come out to cancel the powers of r .

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Other Examples

- ▶ Griffiths does an example in which a surface charge density is specified at $r = R$ and the potential has to be found over all of space. This is almost a Neumann boundary condition, but not quite, since the surface charge density specifies the change in the normal derivative of V at r , not the normal derivative of V itself. By solving for V over all of space, one effectively turns it into a Neumann boundary condition by using the solution in one region to specify the condition on the normal derivative as one approaches the surface from the other side. One writes down different solutions for the two regions: the $\{B_\ell\}$ vanish for the $r < R$ solution to avoid a divergence at the origin, and the $\{A_\ell\}$ vanish for the $r > R$ solution to ensure the potential vanishes at infinity (as we saw above). Then, one applies the conditions that the potential must be continuous at R and that the normal derivative must change by the surface charge density (divided by $-\epsilon_0$). The first condition is effectively the specification of $\langle V \rangle_R$, which we recall from our generic discussion of Green Functions for Neumann boundary conditions. The second condition is the actual Neumann boundary condition. This first condition relates the $\{A_\ell\}$ and $\{B_\ell\}$ at each ℓ . With now just a single set of coefficients to determine, the Neumann boundary condition can be used with the orthonormality relation to find a formula for the coefficient for each ℓ .

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

- Griffiths does the example of an uncharged metal sphere in a uniform electric field in the z direction, $\vec{E} = E_0 \hat{z}$. The boundary condition is a bit mixed again. Because the sphere is metal, it is an equipotential. But that doesn't specify the value of V on the sphere. Since the field is uniform, we cannot set V to vanish at infinity. Instead, $V(z=0) = 0$ is chosen. With that choice, symmetry tells us the sphere satisfies $V = 0$. But now V at infinity is not specified, so we don't yet have a Dirichlet boundary condition. The sensible thing to do is to require the potential approach $V(\vec{r}) = -E_0 z$ at infinity: whatever induced charge the sphere picks up, its contribution to the potential must fall off at infinity, leaving only the uniform field. Now we have a Dirichlet boundary condition. Because the potential is allowed to diverge at infinity, we cannot eliminate the $\{A_\ell\}$ in this case. But it is easy to see that only A_1 is nonzero: for $\ell > 1$, the behavior goes like r^ℓ , and since the potential must go like $z = r \cos \theta$ at large r , all the $\ell > 1$ terms must vanish. $A_0 = 0$ because the potential has no offset. The large r behavior sets $A_1 = -E_0$. That leaves the $\{B_\ell\}$ to be determined. Applying the boundary condition $V = 0$ at $r = R$ gives:

$$0 = A_1 R \cos \theta + \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos \theta) \quad (3.122)$$

$$-A_1 R \cos \theta = \sum_{\ell=0}^{\infty} \frac{B_\ell}{R^{\ell+1}} P_\ell(\cos \theta) \quad (3.123)$$

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Since the left side has a $\ell = 1$ term, and the Legendre polynomials are orthonormal, there can also be only a $\ell = 1$ term on the right side, implying $B_1/R^2 = -A_1 R$ or $B_1 = E_0 R^3$. Thus, the solution is

$$V(\vec{r}) = E_0 \left(r - \frac{R^3}{r^2} \right) \cos \theta \quad (3.124)$$

Generally speaking, one can see that the boundary condition is not always very obvious. One has to use whatever information one has and turn it into a boundary condition of some type.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

A Useful Expansion in Legendre Polynomials

We can easily show

$$\boxed{\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos \gamma)} \quad (3.125)$$

with $r_< = \min(|\vec{r}|, |\vec{r}'|)$ $r_> = \max(|\vec{r}|, |\vec{r}'|)$ $\cos \gamma = \hat{r} \cdot \hat{r}'$

This will have obvious utility.

To prove this, orient the coordinate system so $\vec{r}' = r' \hat{z}$. The function is the potential of a point charge $q = 4\pi\epsilon_0$ in magnitude at r' along the z-axis. It satisfies azimuthal symmetry and thus is expandable in terms of the above solutions of Laplace's Equation in spherical coordinates with azimuthal symmetry:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \left(A_\ell r^\ell + \frac{B_\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta) \quad (3.126)$$

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Consider two cases separately:

- ▶ $r < r'$

We must eliminate the B_ℓ coefficients to keep the function finite as $r \rightarrow 0$. To find the A_ℓ , let's consider the point $\vec{r} = r\hat{z}$ (i.e., $\cos\gamma = 1$), which implies

$$\frac{1}{r' - r} = \sum_{\ell=0}^{\infty} A_\ell r^\ell \quad (3.127)$$

(Recall, $P_\ell(1) = 1$.) Thus, the A_ℓ are just the coefficients of the power series expansion of the left side, which we know (recall: $(1-x)^{-1} = 1 + x + x^2 + \dots$ for $0 < x < 1$) is

$$\frac{1}{r' - r} = \frac{1}{r'} \frac{1}{1 - \frac{r}{r'}} = \frac{1}{r'} \sum_{\ell=0}^{\infty} \left(\frac{r}{r'}\right)^\ell \quad (3.128)$$

Thus, $A_\ell = 1/(r')^{\ell+1}$. This now sets the $\{A_\ell\}$ for arbitrary \vec{r} (i.e., arbitrary $\cos\gamma$ rather than the special case $\cos\gamma = 1$ we have considered).

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

- $r > r'$

We must eliminate the A_ℓ coefficients to keep the function finite as $r \rightarrow \infty$.

Again, consider $\vec{r} = r \hat{z}$, which implies

$$\frac{1}{r - r'} = \sum_{\ell=0}^{\infty} \frac{B_\ell}{r^{\ell+1}} \quad (3.129)$$

Note that here we consider an expansion in r'/r rather than r/r' because now $0 < r'/r < 1$ while, above, $0 < r/r' < 1$. Again, the B_ℓ are just the coefficients of the power series expansion of the left side, which we know is

$$\frac{1}{r - r'} = \frac{1}{r} \frac{1}{1 - \frac{r'}{r}} = \frac{1}{r} \sum_{\ell=0}^{\infty} \left(\frac{r'}{r}\right)^\ell \quad (3.130)$$

Thus, $B_\ell = (r')^\ell$.

Combining the above two cases yields Equation 3.125.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Separation of Variables for a Point Charge near a Grounded Conducting Sphere

We can use the identity we just proved to obtain the potential of a point charge near a grounded, conducting sphere. The setup is as before, with the point charge at $a\hat{z}$ and the sphere centered on the origin with radius R and $V = 0$ on its surface.

One difficulty is that the presence of the point charge implies Laplace's equation is not satisfied in the full volume! However, it is satisfied separately in the regions $R < r < a$ and $a < r < \infty$, and we have the charge density at $r = a$, so we should somehow solve separately in the two regions and then join the solutions together.

The charge density at $r = a$ is

$$\sigma(\theta, \phi) = \frac{q}{2\pi a^2 \sin \theta} \delta(\theta) \quad (3.131)$$

where one sees that this is sensible because integration returns q :

$$\int_0^\pi \int_0^{2\pi} da' \sigma(\theta, \phi) = \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi a^2 \frac{q}{2\pi a^2 \sin \theta} \delta(\theta) \quad (3.132)$$

$$= \frac{1}{2\pi} \int_0^\pi d\theta \int_0^{2\pi} d\phi q \delta(\theta) = q \quad (3.133)$$

Notice that no $\delta(\phi)$ is required.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

We can largely apply what we did in the case of a specified charge density above, except we cannot eliminate the $\{B_\ell\}$ for $r < a$ because the boundary is at $r = R$. But let's apply the boundary condition $V(r = R) = 0$:

$$0 = \sum_{\ell=0}^{\infty} \left(A_\ell R^\ell + \frac{B_\ell}{R^{\ell+1}} \right) P_\ell(\cos \theta) \quad (3.134)$$

Since the Legendre polynomials are orthonormal, the coefficient of P_ℓ at each ℓ must vanish independently, giving

$$A_\ell R^\ell = -\frac{B_\ell}{R^{\ell+1}} \implies V(r < a, \theta) = \sum_{\ell=0}^{\infty} A_\ell \left(r^\ell - \frac{R^{2\ell+1}}{r^{\ell+1}} \right) P_\ell(\cos \theta) \quad (3.135)$$

For $r > a$, the $\{A_\ell\}$ must all vanish so the potential vanishes at infinity. So we have

$$V(r > a, \theta) = \sum_{\ell=0}^{\infty} \frac{C_\ell}{r^{\ell+1}} P_\ell(\cos \theta) \quad (3.136)$$

where we use C_ℓ to avoid confusion with B_ℓ above.

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Now, matching requires that V be continuous at $r = a$ and that $\frac{\partial V}{\partial r}$ be continuous except at $\theta = 0$. Instead, it has a discontinuity there specified by $\sigma(0)$. We apply the first condition, term-by-term because of the orthonormality of the P_ℓ : matching:

$$A_\ell \left(a^\ell - \frac{R^{2\ell+1}}{a^{\ell+1}} \right) = \frac{C_\ell}{a^{\ell+1}} \quad \Rightarrow \quad C_\ell = A_\ell \left(a^{2\ell+1} - R^{2\ell+1} \right) \quad (3.137)$$

Let's do some manipulations to put the potentials into a useful form:

$$V(r < a, \theta) = \sum_{\ell=0}^{\infty} A_\ell a^{\ell+1} \left(\frac{r^\ell}{a^{\ell+1}} - \frac{\frac{R}{a} \left(\frac{R^2}{a} \right)^\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta) \quad (3.138)$$

$$V(r > a, \theta) = \sum_{\ell=0}^{\infty} A_\ell a^{\ell+1} \left(\frac{a^\ell}{r^{\ell+1}} - \frac{\frac{R}{a} \left(\frac{R^2}{a} \right)^\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta) \quad (3.139)$$

Next, we apply the derivative matching condition, which is

$$\left. \left(\frac{\partial V_>}{\partial r} - \frac{\partial V_<}{\partial r} \right) \right|_{r=a} = -\frac{\sigma(\theta)}{\epsilon_0} \quad (3.140)$$

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

The derivatives are

$$\frac{\partial V_<}{\partial r} = \sum_{\ell=0}^{\infty} A_{\ell} a^{\ell+1} \left(\frac{\ell r^{\ell-1}}{a^{\ell+1}} + (\ell+1) \frac{\frac{R}{a} \left(\frac{R^2}{a}\right)^{\ell}}{r^{\ell+2}} \right) P_{\ell}(\cos \theta) \quad (3.141)$$

$$\frac{\partial V_>}{\partial r} = \sum_{\ell=0}^{\infty} A_{\ell} a^{\ell+1} (\ell+1) \left(-\frac{a^{\ell}}{r^{\ell+2}} + \frac{\frac{R}{a} \left(\frac{R^2}{a}\right)^{\ell}}{r^{\ell+2}} \right) P_{\ell}(\cos \theta) \quad (3.142)$$

Evaluating at $r = a$ gives

$$\frac{\partial V_<}{\partial r} \Big|_{r=a} = \sum_{\ell=0}^{\infty} A_{\ell} a^{\ell+1} \left(\frac{\ell}{a^2} + (\ell+1) \frac{\frac{R}{a} \left(\frac{R^2}{a}\right)^{\ell}}{a^{\ell+2}} \right) P_{\ell}(\cos \theta) \quad (3.143)$$

$$\frac{\partial V_>}{\partial r} \Big|_{r=a} = \sum_{\ell=0}^{\infty} A_{\ell} a^{\ell+1} (\ell+1) \left(-\frac{1}{a^2} + \frac{\frac{R}{a} \left(\frac{R^2}{a}\right)^{\ell}}{a^{\ell+2}} \right) P_{\ell}(\cos \theta) \quad (3.144)$$

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

When we difference the two, the second term in the expressions cancel, leaving

$$-\sum_{\ell=0}^{\infty} (2\ell+1) A_{\ell} a^{\ell-1} P_{\ell}(\cos \theta) = -\frac{q \delta(\theta)}{2\pi a^2 \epsilon_0 \sin \theta} \quad (3.145)$$

Now, we use orthonormality, multiplying by $P_{\ell'}(\cos \theta)$, multiplying by $\sin \theta$, and integrating over θ and ϕ . (Recall $\int_0^\pi \sin \theta d\theta P_{\ell}(\cos \theta) P_{\ell'}(\cos \theta) = 2\delta_{\ell\ell'}/(2\ell+1)$). This pulls out the $A_{\ell'}$ term:

$$-2(2\pi) A_{\ell'} a^{\ell'-1} = -\frac{q}{2\pi a^2 \epsilon_0} \int_0^\pi \sin \theta d\theta \int_0^{2\pi} d\phi \frac{\delta(\theta) P_{\ell'}(\cos \theta)}{\sin \theta} \quad (3.146)$$

$$= -\frac{q}{\epsilon_0} P_{\ell'}(\cos \theta = 1) = -\frac{q}{\epsilon_0} \quad (3.147)$$

$$A_{\ell'} = \frac{1}{a^{\ell'+1}} \frac{q}{4\pi \epsilon_0} \quad (3.148)$$

Separation of Variables in Spherical Coordinates with Azimuthal Symmetry (cont.)

Writing the full solution, we have

$$V(r < a, \theta) = \frac{q}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \left(\frac{r^\ell}{a^{\ell+1}} - \frac{\frac{R}{a} \left(\frac{R^2}{a}\right)^\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta) \quad (3.149)$$

$$V(r > a, \theta) = \frac{q}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} \left(\frac{a^\ell}{r^{\ell+1}} - \frac{\frac{R}{a} \left(\frac{R^2}{a}\right)^\ell}{r^{\ell+1}} \right) P_\ell(\cos\theta) \quad (3.150)$$

Comparing to Equation 3.125, we see that all four terms are of that form. The first term of the first equation has $r_< = r$ and $r_> = a$ as appropriate for $r < a$, while the first term of the second equation has $r_< = a$ and $r_> = r$ as needed for $r > a$. The second terms of both equations are of the same form with $r_< = R^2/a$, $r_> = r$, and the charge multiplied by $-R/a$. Thus, we recover

$$V(\vec{r}) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{|\vec{r} - a\hat{z}|} - \frac{R/a}{\left|\vec{r} - \frac{R^2}{a}\hat{z}\right|} \right] \quad (3.151)$$

which matches Equation 3.27.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry

The Full Polar Equation Solution: the Associated Legendre Polynomials

The associated Legendre polynomials can be derived from the Legendre polynomials for $m \geq 0$:

$$P_\ell^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x) \quad (3.152)$$

which, using Rodrigues' Formula (Equation 3.107), implies

$$P_\ell^m(x) = \frac{(-1)^m}{2^\ell \ell!} (1 - x^2)^{m/2} \frac{d^{\ell+m}}{dx^{\ell+m}} (x^2 - 1)^\ell \quad (3.153)$$

which is now valid for all m . It should be clear that $P_\ell^0 = P_\ell$. It should also be clear that parity in x (evenness/oddness) of the associated Legendre functions is given by $(-1)^{\ell+m}$ (where -1 implies oddness): the parity of P_ℓ is given by $(-1)^\ell$, and each derivative changes the parity by a factor of -1 (note that the powers of $(1 - x^2)$ have no effect on the parity because it is an even function). There are a number of other properties of these functions, but it is more useful to consider them with the ϕ solutions.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

The Full Solution to the Angular Piece of Laplace's Equation: the Spherical Harmonics

When one combines the $P_\ell^m(\cos \theta)$ and the $e^{im\phi}$ solutions of the polar and azimuthal equations, one obtains the *Spherical Harmonics*

$$Y_{\ell m}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi} \frac{(\ell - m)!}{(\ell + m)!}} P_\ell^m(\cos \theta) e^{im\phi} \quad (3.154)$$

They are an orthonormal, complete basis for functions on the sphere (θ, ϕ) . They satisfy numerous important and useful conditions:

- ▶ Conjugation:

$$Y_{\ell(-m)}(\theta, \phi) = (-1)^m Y_{\ell m}^*(\theta, \phi) \quad (3.155)$$

- ▶ Orthonormality:

$$\int_0^{2\pi} d\phi \int_0^\pi \sin \theta d\theta Y_{\ell' m'}^*(\theta, \phi) Y_{\ell m}(\theta, \phi) = \delta_{\ell\ell'} \delta_{mm'} \quad (3.156)$$

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

- Completeness:

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi) = \delta(\phi - \phi') \delta(\cos \theta - \cos \theta') \quad (3.157)$$

- $m = 0$ devolves to Legendre polynomials:

$$Y_{\ell 0}(\theta, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} P_{\ell}(\cos \theta) \quad (3.158)$$

This should be obvious from Equation 3.152, the relation between the Legendre and the associated Legendre polynomials.

- The $\theta = 0$ behavior is simple given Equation 3.152 (the $(1 - x^2)$ factor):

$$P_{\ell}^{m \neq 0}(\pm 1) = 0 \implies Y_{\ell m \neq 0}(\theta = 0, \phi) = Y_{\ell m \neq 0}(\theta = \pi, \phi) = 0 \quad (3.159)$$

Recall that we also stated $P_{\ell}(1) = 1$, $P_{\ell}(-1) = (-1)^{\ell}$, which implies

$$Y_{\ell 0}(\theta = 0, \phi) = \sqrt{\frac{2\ell + 1}{4\pi}} \quad Y_{\ell 0}(\theta = \pi, \phi) = (-1)^{\ell} \sqrt{\frac{2\ell + 1}{4\pi}} \quad (3.160)$$

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

- The above implies that any expansion in terms of $Y_{\ell m}$ simplifies at $\theta = 0, \pi$:

$$\text{given } g(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m} Y_{\ell m}(\theta, \phi) \quad (3.161)$$

$$\text{then } g(\theta = 0, \phi) = \sum_{\ell=0}^{\infty} \sqrt{\frac{2\ell+1}{4\pi}} A_{\ell 0} \quad (3.162)$$

$$\text{and } g(\theta = \pi, \phi) = \sum_{\ell=0}^{\infty} (-1)^{\ell} \sqrt{\frac{2\ell+1}{4\pi}} A_{\ell 0} \quad (3.163)$$

- The *Addition Theorem for Spherical Harmonics*: Given \hat{r} and \hat{r}' pointing in the directions (θ, ϕ) and (θ', ϕ') , respectively, then

$$P_{\ell}(\hat{r} \cdot \hat{r}') = \frac{4\pi}{2\ell+1} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi) \quad (3.164)$$

where $\hat{r} \cdot \hat{r}' = \cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$. The proof of this can be found in Jackson §3.6.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

- An important corollary of the Addition Theorem can be obtained by combining the above with Equation 3.125, the formula for the inverse of the relative distance between two points in terms of the Legendre polynomials:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos \gamma)$$

Plugging in the addition theorem gives us

$$\boxed{\frac{1}{|\vec{r} - \vec{r}'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi)} \quad (3.165)$$

The utility of this relation is even more obvious than that of Equation 3.125, especially for doing integrals over charge distributions with the relative distance function (*i.e.*, calculating the potential due to Coulomb's Law): decompose the charge distribution in terms of spherical harmonics, integrate up the charge distribution in a particular spherical harmonic $Y_{\ell m}$ over r' with weighting by $(r')^\ell$, and it gives the component of the potential at a distance r from the origin with spatial dependence $Y_{\ell m}(\theta, \phi)/r^{\ell+1}$.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

The Full Solution of Laplace's Equation in Spherical Coordinates

Putting it all together, we see that the most general solution to Laplace's Equation in spherical coordinates is

$$V(r, \theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta, \phi) \quad (3.166)$$

Again, the coefficients $\{A_{\ell m}\}$ and $\{B_{\ell m}\}$ are set by the volume under consideration and one or the other entire set may vanish. As well, application of the boundary conditions will require the orthonormality relations for the spherical harmonics.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

As with the case of azimuthal symmetry, we note that, in the process of doing separation of variables, we have proven that the components of the solution satisfy the *eigenvalue-eigenfunction equations*

$$\nabla^2 A_{\ell m} r^\ell = \frac{\ell(\ell+1)}{r^2} A_{\ell m} r^\ell \quad \nabla^2 B_{\ell m} \frac{1}{r^{\ell+1}} = \frac{\ell(\ell+1)}{r^2} B_{\ell m} \frac{1}{r^{\ell+1}} \quad (3.167)$$

$$\nabla^2 Y_{\ell m}(\theta, \phi) = -\frac{\ell(\ell+1)}{r^2} Y_{\ell m}(\theta, \phi) \quad (3.168)$$

As before, the appearance of r^2 on the right side is not surprising. Note also that m does not appear in the angular equation. This is because Laplace's Equation itself is spherically (and therefore azimuthally) symmetric. The charge distribution and boundary conditions are what may break the spherically symmetry.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

Expansion of the Green Function in Spherical Coordinates in Terms of the Spherical Harmonics

The fact that the spherical harmonics combined with the usual power laws in radius solve the Laplace Equation for problems that are separable in spherical coordinates implies that the Green Function for such problems will have a convenient expansion in terms of spherical harmonics. This just follows from the fact that the Green Function is the potential due to a point charge of unit magnitude. Let's see this explicitly for a couple geometries:

- ▶ *Free space*

The corollary of the addition theorem above is the desired expansion of the Green Function for charge in free space with no boundaries at finite radius and with the condition $V \rightarrow 0$ as $r \rightarrow \infty$.

- ▶ *Point charge near a grounded, conducting sphere*

For this geometry, we saw that the Green Function can be written as sum of the Coulomb potential of two point charges, the original one at $\vec{r}' \hat{z}$ and the image charge $q' = -q R/r'$ at $\hat{z} R^2/a$:

$$G(\vec{r}, \vec{r}') = \frac{1}{4\pi\epsilon_0} \left[\frac{1}{|\vec{r} - \vec{r}'|} - \frac{R/r'}{\left|\vec{r} - \vec{r}' \left(\frac{R}{r'}\right)^2\right|} \right] \quad (3.169)$$

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

Using the same corollary, we can then immediately write

$$G(\vec{r}, \vec{r}') = \frac{1}{\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{R}{r'} \left[r' \left(\frac{R}{r'} \right)^2 \right]^\ell \right] \frac{Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi)}{2\ell + 1} \quad (3.170)$$

$$= \frac{1}{\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[\frac{r_<^\ell}{r_>^{\ell+1}} - \frac{1}{R} \left(\frac{R^2}{r r'} \right)^{\ell+1} \right] \frac{Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi)}{2\ell + 1} \quad (3.171)$$

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

The general approach to the problem for an arbitrary (spherical) geometry is to go back to the definition of the Green Function:

$$-\epsilon_0 \nabla^2 G(\vec{r}, \vec{r}') = \delta(\vec{r} - \vec{r}') \quad (3.172)$$

and decompose both sides in terms of spherical harmonics. We don't know the Green Function yet, so its expansion is the arbitrary general form, which here we write

$$G(\vec{r}, \vec{r}') = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m}(r|\vec{r}') Y_{\ell m}(\theta, \phi) \quad (3.173)$$

where the coefficients in the expansion $A_{\ell m}$ depend on r , as usual, and they also depend parametrically on \vec{r}' because it is a parameter in the differential equation. (We don't know the solutions for the radial dependence of the $A_{\ell m}$ yet for the general case we are trying to solve, so we cannot assume they are the power laws we saw for the cases we have considered so far.)

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

The right side can be rewritten using the breakdown of the delta function into delta functions in each spherical coordinate followed by completeness of the spherical harmonics. The breakdown of the delta function is:

$$\delta(\vec{r} - \vec{r}') = \frac{\delta(r - r')}{r^2} \delta(\phi - \phi') \delta(\cos \theta - \cos \theta') \quad (3.174)$$

The $1/r^2$ on the radial component is required to cancel the r^2 in the volume element in spherical coordinates. The fact that the delta function in θ is a function of $\cos \theta$ and $\cos \theta'$ is because the volume element contains $\sin \theta d\theta = d(\cos \theta)$. One could have instead written $\delta(\theta - \theta')/\sin \theta$. Using completeness of the spherical harmonics, we have

$$\delta(\vec{r} - \vec{r}') = \frac{\delta(r - r')}{r^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi) \quad (3.175)$$

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

Thus, our differential equation for the Green Function becomes

$$-\epsilon_0 \nabla^2 \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} A_{\ell m}(r|\vec{r}') Y_{\ell m}(\theta, \phi) = \frac{\delta(r - r')}{r^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi) \quad (3.176)$$

Note that the Laplacian acts on the unprimed coordinates only. We wrote down above the eigenvalue-eigenfunction equation that the angular solutions of Laplace's Equation satisfies (Equation 3.168), which we use here to determine the action of ∇^2 on the angular coordinates:

$$\begin{aligned} & -\epsilon_0 \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[\left(\nabla^2 - \frac{\ell(\ell+1)}{r^2} \right) A_{\ell m}(r|\vec{r}') \right] Y_{\ell m}(\theta, \phi) \\ &= \frac{\delta(r - r')}{r^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi) \end{aligned} \quad (3.177)$$

Now the Laplacian on the left side is only acting with its radial derivatives on $A_{\ell m}$; its action on the spherical harmonics has yielded the $\ell(\ell+1)/r^2$ term.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

The coefficients of the individual $Y_{\ell m}(\theta, \phi)$ on the two sides must be equal because of the orthonormality relation for the spherical harmonics, implying

$$-\epsilon_0 \left[\left(\nabla^2 - \frac{\ell(\ell+1)}{r^2} \right) A_{\ell m}(r|\vec{r}') \right] = \frac{\delta(r-r')}{r^2} Y_{\ell m}^*(\theta', \phi') \quad (3.178)$$

Now, given that we have $Y_{\ell m}^*(\theta', \phi')$ on the right side, and again the spherical harmonics are orthogonal polynomials, the dependence of $A_{\ell m}(r|\vec{r}')$ on its \vec{r}' angular coordinates must be proportional to $Y_{\ell m}^*(\theta', \phi')$. Therefore, we may write

$$A_{\ell m}(r|r', \theta', \phi') = g_\ell(r, r') Y_{\ell m}^*(\theta', \phi') \quad (3.179)$$

Plugging in this form to the reduced version of Laplace's Equation, we get:

$$-\epsilon_0 \left(\nabla^2 - \frac{\ell(\ell+1)}{r^2} \right) g_\ell(r, r') = \frac{\delta(r-r')}{r^2} \quad (3.180)$$

$$\Rightarrow \frac{d}{dr} \left[r^2 \frac{d}{dr} g_\ell(r, r') \right] - \ell(\ell+1) g_\ell(r, r') = -\frac{\delta(r-r')}{\epsilon_0} \quad (3.181)$$

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

We see that $g_\ell(r, r')$ satisfies the usual radial equation in r when $r \neq r'$ (r' is a parameter, not a variable, here). Therefore, for $r < r'$ and $r > r'$,

$$g_\ell(r, r') = \begin{cases} A_\ell(r') r^\ell + B_\ell(r') r^{-(\ell+1)} & r < r' \\ C_\ell(r') r^\ell + D_\ell(r') r^{-(\ell+1)} & r > r' \end{cases} \quad (3.182)$$

Because r' is a parameter of the differential equation, the coefficients and therefore the solutions depend on it parametrically. Integration of Equation 3.181 implies $r^2 dg_\ell(r, r')/dr$ must change by $-1/\epsilon_0$ at $r = r'$. Furthermore, the function $g_\ell(r, r')$ itself must be continuous at $r = r'$. These conditions, along with the Dirichlet or Neumann boundary conditions on any boundaries in the problem, should fully specify the $A_\ell(r')$, $B_\ell(r')$, $C_\ell(r')$, and $D_\ell(r')$. In particular, we can see how our solution for the point charge near a conducting sphere has this form. Recalling Equations 3.149 and 3.150, we simply identify the terms with each other and we see they imply

$$A_\ell(r') = \frac{1}{\epsilon_0} \left(\frac{1}{r'} \right)^{\ell+1} \quad B_\ell(r') = -\frac{1}{\epsilon_0} \frac{1}{R} \left(\frac{R^2}{r'} \right)^{\ell+1} \quad (3.183)$$

$$C_\ell(r') = 0 \quad D_\ell(r') = \frac{1}{\epsilon_0} \left[(r')^\ell - \frac{1}{R} \left(\frac{R^2}{r'} \right)^{\ell+1} \right] \quad (3.184)$$

which is also consistent with Equation 3.171, where we used the addition theorem to rewrite the Green Function for this system in terms of the spherical harmonics.

Lecture 7:
Separation of Variables in Spherical Coordinates
without Azimuthal Symmetry (cont.)

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Separation of Variables in Spherical Coordinates without Azimuthal Symmetry

Examples of Using the Expansion of the Green Function in Terms of the Spherical Harmonics

We did a lot of gymnastics to get the expansion of the Green Function in terms of spherical harmonics. Let's see how it can be used. For each of the examples we will consider, it would be possible to solve for the potential by separation of variables and application of boundary conditions without explicitly using our expansion. The advantages of using the Green Function, and in particular of using its spherical harmonic expansion, will be:

- ▶ As with any Green Function, whether expanded in spherical harmonics or not, the Green Function obviates re-solving the same kind of problem many times by simply providing integrals that need to be done.
- ▶ The breakdown of the Green Function in terms of spherical harmonics makes these integrals over the charge distribution easier to do, especially when the charge distribution has azimuthal symmetry.
- ▶ More importantly, the Green function connects a particular spherical harmonic mode of the charge distribution to the corresponding spherical harmonic mode of the potential. This correspondence makes the structure of the solution much easier to understand. The effect of a spherical harmonic mode in the charge distribution at one radius r' on the potential at another radius r is just a function of the two radii, the $g(r, r')$ function.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

For our examples, we will consider charge distributions inside a conducting sphere. We quote the general result from Jackson for the Green Function expansion in spherical harmonics for a geometry consisting of the volume between two conducting spheres at $r = a$ and $r = b$

$$G_D(\vec{r}, \vec{r}') \quad (3.185)$$

$$= \frac{1}{\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[r_{<}^{\ell} - \frac{1}{a} \left(\frac{a^2}{r_{<}} \right)^{\ell+1} \right] \left[\frac{1}{r_{>}^{\ell+1}} - \frac{1}{b} \left(\frac{r_{>}}{b^2} \right)^{\ell} \right] \frac{Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi)}{\left[1 - \left(\frac{a}{b} \right)^2 \ell + 1 \right] (2\ell + 1)}$$

where, as usual, $r_{<} = \min\{r, r'\}$ and $r_{>} = \max\{r, r'\}$. Obtaining this more general result is a matter of doing the same thing as we did to obtain the result for a spherical conducting boundary at $r = R$, except now the C_ℓ term cannot be assumed to vanish. Now, taking the limit $a \rightarrow 0$, we get the result we will need for our work below:

$$G_D(\vec{r}, \vec{r}') = \frac{1}{\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} r_{<}^{\ell} \left[\frac{1}{r_{>}^{\ell+1}} - \frac{1}{b} \left(\frac{r_{>}}{b^2} \right)^{\ell} \right] \frac{Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi)}{2\ell + 1} \quad (3.186)$$

You will also be able to read off this simpler result from a method of images problem you will do in homework.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

On to our examples:

- ▶ Potential inside the conducting sphere due to an arbitrary Dirichlet boundary condition potential but no charge in the volume

With no charge in the volume, we just need to calculate the surface term in Equation 3.44, for which we need the normal gradient of G_D at the surface:

$$\begin{aligned} & \hat{n}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} G_D(\vec{r}, \vec{r}') \\ &= \frac{1}{\epsilon_0} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi)}{2\ell + 1} r^\ell \frac{d}{dr'} \left[\frac{1}{(r')^{\ell+1}} - \frac{1}{b} \left(\frac{r'}{b^2} \right)^\ell \right] \Big|_{r'=b} \\ &= -\frac{1}{\epsilon_0} \frac{1}{b^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{r}{b} \right)^\ell Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi) \end{aligned} \quad (3.187)$$

Therefore, the potential in the volume for the Dirichlet B.C. $V(b, \theta, \phi)$ is

$$V(\vec{r}) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left(\frac{r}{b} \right)^\ell Y_{\ell m}(\theta, \phi) \int d\Omega' Y_{\ell m}^*(\theta', \phi') V(b, \theta', \phi') \quad (3.188)$$

We see that the spherical harmonic component ℓm of the potential at r is determined by the spherical harmonic component ℓm of the potential on the boundary — very simple.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

- Potential inside a grounded spherical conductor with a ring of charge of radius a in the xy plane.

This time, we do the volume integral but there is no integral over the surface. The charge density due to the ring is

$$\rho(\vec{r}') = \frac{Q}{2\pi a^2} \delta(r' - a) \delta(\cos\theta') \quad (3.189)$$

Again, one can check that the charge density is correct by integrating it: the a^{-2} cancels the $(r')^2$ factor in the volume element and the argument of the θ' delta function is $\cos\theta'$ because the volume element contains $d(\cos\theta')$.

The potential is then, as usual using Equation 3.44,

$$\begin{aligned} V(\vec{r}) &= \int_{\mathcal{V}} d\tau' \rho(\vec{r}') G_D(\vec{r}, \vec{r}') \\ &= \frac{Q}{2\pi\epsilon_0 a^2} \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\theta, \phi) \\ &\quad \times \int_{\mathcal{V}} d\tau' \delta(r' - a) \delta(\cos\theta') r'_<^{\ell} \left[\frac{1}{r'_>}^{\ell+1} - \frac{1}{b} \left(\frac{r'_>}{b^2} \right)^{\ell} \right] \frac{Y_{\ell m}^*(\theta', \phi')}{2\ell + 1} \end{aligned} \quad (3.190)$$

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

Because the charge density has no azimuthal dependence, the ϕ' integral picks out the $m = 0$ term. Recall that $Y_{\ell 0} = \sqrt{(2\ell + 1)/4\pi} P_\ell$, so we may rewrite as

$$V(\vec{r}) = \frac{Q}{4\pi\epsilon_0 a^2} \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) \int_{-1}^1 d(\cos\theta') \delta(\cos\theta') P_\ell(\cos\theta') \quad (3.191)$$

$$\times \int_0^b (r')^2 dr' \delta(r' - a) r_<^\ell \left[\frac{1}{r_>^{\ell+1}} - \frac{1}{b} \left(\frac{r_>}{b^2} \right)^\ell \right]$$

$$= \frac{Q}{4\pi\epsilon_0} \sum_{\ell=0}^{\infty} P_\ell(\cos\theta) P_\ell(0) r_<^\ell \left[\frac{1}{r_>^{\ell+1}} - \frac{1}{b} \left(\frac{r_>}{b^2} \right)^\ell \right] \quad (3.192)$$

where now $r_< = \min\{r, a\}$ and $r_> = \max\{r, a\}$. Now, recall $P_\ell(0) = 0$ for odd ℓ and $P_\ell(0) = [(-1)^n (2n - 1)!!]/2^n n!$ for even $\ell = 2n$, so we may reduce the above further to (replacing ℓ with $2n$ so n runs over all nonnegative integers rather than ℓ running over all nonnegative even integers):

$$V(\vec{r}) = \frac{Q}{4\pi\epsilon_0} \sum_{n=0}^{\infty} \frac{(-1)^n (2n - 1)!!}{2^n n!} r_<^{2n} \left[\frac{1}{r_>^{2n+1}} - \frac{1}{b} \left(\frac{r_>}{b^2} \right)^{2n} \right] P_{2n}(\cos\theta) \quad (3.193)$$

This is now a complete solution.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

The induced surface charge density at $r = b$ is obtained by the normal gradient of V . Since the normal gradient is just d/dr , it does not act at all on P_{2n} . In calculating this gradient, $r_< = a$ and $r_> = r$ since we will in the end evaluate at $r = b$. Therefore:

$$\begin{aligned}\sigma(\vec{r}) &= \epsilon_0 \frac{dV}{dr} \Big|_{r=b} \\ &= -\frac{Q}{4\pi b^2} \sum_{n=0}^{\infty} \frac{(4n+1)(-1)^n (2n-1)!!}{2^n n!} \left(\frac{a}{b}\right)^{2n} P_{2n}(\cos\theta) \\ &= -\frac{Q}{4\pi b^2} \left[1 + \sum_{n=1}^{\infty} \frac{(4n+1)(-1)^n (2n-1)!!}{2^n n!} \left(\frac{a}{b}\right)^{2n} P_{2n}(\cos\theta) \right]\end{aligned}\tag{3.194}$$

The expression is written in the above suggestive form on the last line so that it is easy to obtain the total induced surface charge. Since $P_0(\cos\theta) = 1$, the integral of the $n > 0$ terms over $\cos\theta$ can be viewed as integrating P_{2n} with P_0 ; by orthonormality of the Legendre polynomials, these terms all yield zero. The first term yields $-Q$ when integrated over the sphere.

Separation of Variables in Spherical Coordinates without Azimuthal Symmetry (cont.)

We have seen in this example how the integration of the charge density with the Green Function breaks the charge density down into its spherical harmonic components, calculates the potential due to each component individually (and fairly trivially, just multiplying by a function of the radius at which the source charge is and the radius at which the potential is desired) and then sums up those components. The same kind of correspondence clearly holds for the induced surface charge density.

Note that the additional $4n + 1$ factor implies the θ dependence of the induced surface charge density is different from that of the original ring charge; i.e., the induced surface charge is not just a ring.

- ▶ *Potential inside a grounded spherical conductor with a line charge density along the z axis*

This is done in Jackson Section 3.10, you may look it up there if you would like to see another example. We also did it somewhat interactively in class (no notes provided).

Lecture 8: Multipoles

Electrostatics in Matter:

Polarizability and Polarization, Bound Charges and their Potential

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Multipole Expansions

Dipoles: Quick Review

Recall from Ph1b the idea of an *electric dipole*: two charges of equal and opposite size $\pm q$ spaced very close together at \vec{r}_+ and \vec{r}_- . The net charge cancels almost perfectly, so, rather than the potential falling off like $1/r$ at large radius, it falls off as $1/r^2$ with functional form

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2} \quad \text{as } \frac{r}{|\vec{r}_+ - \vec{r}_-|} \rightarrow \infty \quad (3.195)$$

where $\vec{p} = q(\vec{r}_+ - \vec{r}_-)$ is the dipole moment.

This idea generalizes. When one has a charge distribution with vanishing net charge, but inside of which there is a variation in the charge density, that variation is still noticeable at large distance as a set of potentials that fall off more quickly than $1/r$. The first additional term is the *dipole*, falling as $1/r^2$, the second is the *quadrupole*, falling as $1/r^3$, the third is the *octupole*, falling as $1/r^4$, and so on. The nomenclature comes from the minimum number of different source charges one must have to obtain that moment: one for monopole, two for dipole, four for quadrupole, etc.

Multipole Expansions (cont.)

Multipoles: Full Derivation

We derive the full form by considering the potential due to a charge distribution near the origin as viewed at a point \vec{r} such that r is much larger than the extent of the charge distribution. We begin with

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} d\tau' \frac{\rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (3.196)$$

We now use Equation 3.125, taking $r_< = r'$ and $r_> = r$ because r is outside the charge distribution. Thus,

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} d\tau' \rho(\vec{r}') \sum_{\ell=0}^{\infty} \frac{(r')^\ell}{r^{\ell+1}} P_\ell(\cos \gamma) \quad (3.197)$$

where $\cos \gamma = \hat{r} \cdot \hat{r}'$ is the angle between the two vectors. There is common $1/r$ we can factor out, leaving

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{1}{r^\ell} \int_{\mathcal{V}} d\tau' \rho(\vec{r}') (r')^\ell P_\ell(\cos \gamma) \quad (3.198)$$

This is the *multipole expansion* of the potential of the charge distribution. One can see that the successive terms fall off as successively higher powers of $1/r$.

Multipole Expansions (cont.)

The Monopole, Dipole, and Quadrupole Terms

Let's write out the first three terms more explicitly to get some physical intuition:

- ▶ *the monopole term*

The first term is

$$V_1(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r} \int_{\mathcal{V}} d\tau' \rho(\vec{r}') = \frac{1}{4\pi\epsilon_0} \frac{Q}{r} \quad (3.199)$$

This is the standard Coulomb's Law term due to the total charge. Far enough away, all charge distributions look pointlike. But, if $Q = 0$, this term vanishes identically and the next-order terms must be considered. Even if $Q \neq 0$, if one is close enough to the charge distribution to see its non-pointlike nature, the next-order terms will be important corrections to the monopole term.

Multipole Expansions (cont.)

► the dipole term

The second term is

$$\begin{aligned}V_2(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int_{\mathcal{V}} d\tau' \rho(\vec{r}') r' \cos \gamma = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \int_{\mathcal{V}} d\tau' \rho(\vec{r}') r' \hat{r}' \cdot \hat{r} \\&= \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} \cdot \int_{\mathcal{V}} d\tau' \rho(\vec{r}') \vec{r}'\end{aligned}\quad (3.200)$$

or

$$V_2(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^2} \hat{r} \cdot \vec{p} \quad \text{where} \quad \vec{p} = \int_{\mathcal{V}} d\tau' \rho(\vec{r}') \vec{r}' \quad (3.201)$$

is the dipole moment vector. It is the generalization of $\vec{p} = q(\vec{r}_+ - \vec{r}_-)$.

► the quadrupole term

The third term is

$$\begin{aligned}V_3(\vec{r}) &= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \int_{\mathcal{V}} d\tau' \rho(\vec{r}') (r')^2 \frac{1}{2} (3 \cos^2 \gamma - 1) \\&= \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \hat{r} \cdot \left[\int_{\mathcal{V}} d\tau' \rho(\vec{r}') (r')^2 \frac{1}{2} (3 \hat{r}' \cdot \hat{r}' - 1) \right] \cdot \hat{r}\end{aligned}\quad (3.202)$$

Multipole Expansions (cont.)

or

$$V_3(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \frac{1}{2} \hat{r} \cdot \underline{\underline{Q}} \cdot \hat{r} \quad \text{where} \quad \underline{\underline{Q}} = \int_{\mathcal{V}} d\tau' \rho(\vec{r}') \left[3\vec{r}' \vec{r}' - (r')^2 \underline{\underline{1}} \right] \quad (3.203)$$

is the quadrupole moment, where $\underline{\underline{1}} = \text{diag}(1, 1, 1)$ is the identity tensor with ones along the diagonal. Because it is composed of $\vec{r}' \vec{r}'$ and $\underline{\underline{1}}$, $\underline{\underline{Q}}$ is a tensor, implying that one can take a dot product with a vector on each side.

Origin Dependence of the Dipole Moment

Suppose we take a charge distribution and shift the origin by a vector \vec{a} such that the charge distribution is now centered around \vec{a} . Then the new dipole moment is

$$\vec{p}' = \int d\tau' \rho'(\vec{r}') \vec{r}' = \int d\tau \rho(\vec{r}) (\vec{a} + \vec{r}) = \vec{a} Q + \vec{p} \quad (3.204)$$

where we define the charge distribution in the new coordinate system $\rho'(\vec{r}')$ in terms of the original charge distribution $\rho(\vec{r})$ to be such that $\rho'(\vec{r}') = \rho(\vec{r} = \vec{r}' - \vec{a})$ when $\vec{r}' = \vec{r} + \vec{a}$. Thus, an origin shift can induce an artificial dipole moment for a charge distribution that has a monopole moment.

Multipole Expansions (cont.)

Field of an Electric Dipole

This is simply a matter of taking the gradient. If we let $\vec{p} = p \hat{z}$, then this is easy:

$$V_2(\vec{r}) = \frac{p \cos \theta}{4 \pi \epsilon_0 r^2} \quad (3.205)$$

$$\implies E_r(\vec{r}) = -\frac{\partial V_2}{\partial r} = -\frac{2 p \cos \theta}{4 \pi \epsilon_0 r^3} \quad (3.206)$$

$$E_\theta(\vec{r}) = -\frac{1}{r} \frac{\partial V_2}{\partial \theta} = \frac{p \sin \theta}{4 \pi \epsilon_0 r^3} \quad (3.207)$$

$$E_\phi(\vec{r}) = -\frac{1}{r \sin \theta} \frac{\partial V_2}{\partial \phi} = 0 \quad (3.208)$$

$$\text{or } \vec{E}(\vec{r}) = \frac{p}{4 \pi \epsilon_0 r^3} (2 \hat{r} \cos \theta + \hat{\theta} \sin \theta) \quad (3.209)$$

Multipole Expansions (cont.)

To generalize this result for an arbitrary orientation of \vec{p} requires some vector algebra. We have Equation 3.201 for the dipole potential in generic form, which we write out as

$$V_2(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{1}{r^3} \sum_i r_i p_i \quad (3.210)$$

Now, we take the gradient, first noting

$$\frac{\partial}{\partial r_j} \frac{r_i}{r^3} = \frac{\partial}{\partial r_j} \frac{r_i}{(r^2)^{3/2}} = -\frac{3}{2} \frac{r_i}{(r^2)^{5/2}} \frac{\partial r^2}{\partial r_j} + \frac{\delta_{ij}}{r^3} = -\frac{3}{2} \frac{r_i}{r^5} (2r_j) + \frac{\delta_{ij}}{r^3} \quad (3.211)$$

Where we used $r^3 = (r^2)^{3/2}$ to more easily calculate the partial derivative. Therefore, with r_j and \hat{r}_j being the j th Cartesian coordinate and unit vector,

$$\begin{aligned} \vec{E}_2(\vec{r}) &= -\vec{\nabla} V_2(\vec{r}) = -\sum_j \hat{r}_j \frac{\partial V_2}{\partial r_j} = \frac{1}{4\pi\epsilon_0 r^5} \sum_{ij} \hat{r}_j p_i [3r_i r_j - \delta_{ij} r^2] \\ &= \frac{1}{4\pi\epsilon_0 r^5} \sum_j \left[r_j \hat{r}_j \left(3 \sum_i p_i r_i \right) - p_j \hat{r}_j \right] \\ \implies \boxed{\vec{E}_2(\vec{r})} &= \frac{1}{4\pi\epsilon_0 r^3} [3(\vec{p} \cdot \hat{r}) \hat{r} - \vec{p}] \end{aligned} \quad (3.212)$$

Multipole Expansions (cont.)

Electrostatic Potential Energy of a Multipole Distribution in an External Potential

The general expression for the potential energy of a charge distribution in an external potential is

$$U = \int_{\mathcal{V}} \rho(\vec{r}') V(\vec{r}') \quad (3.213)$$

Now, we want to expand $V(\vec{r})$ about some point in the distribution. Without loss of generality, assume the charge distribution is centered around the origin, which we will expand around. We use the multidimensional Taylor expansion of $V(\vec{r})$:

$$V(\vec{r}') = V(\vec{0}) + \sum_{j=1}^3 r'_j \left. \frac{\partial V}{\partial r_j} \right|_{\vec{0}} + \frac{1}{2} \sum_{j,k=1} r'_j r'_k \left. \frac{\partial^2 V}{\partial r_j \partial r_k} \right|_{\vec{0}} + \dots \quad (3.214)$$

Multipole Expansions (cont.)

Using $E_j = -\frac{\partial V}{\partial r_j}$, we may simplify

$$V(\vec{r}') = V(\vec{0}) - \vec{r}' \cdot \vec{E}(\vec{0}) - \frac{1}{6} \sum_{j,k=1}^3 3 r'_j r'_k \left. \frac{\partial E_j}{\partial r_k} \right|_{\vec{0}} + \dots \quad (3.215)$$

$$= V(\vec{0}) - \vec{r}' \cdot \vec{E}(\vec{0}) - \frac{1}{6} \sum_{j,k=1}^3 \left(3 r'_j r'_k - \delta_{jk}(r')^2 \right) \left. \frac{\partial E_j}{\partial r_k} \right|_{\vec{0}} + \dots \quad (3.216)$$

where we were able to add the $\delta_{jk}(r')^2$ term because

$$\sum_{j,k} r^2 \delta_{jk} \frac{\partial E_j}{\partial r_k} = r^2 \vec{\nabla} \cdot \vec{E} = 0 \quad (3.217)$$

because the charge distribution sourcing V and \vec{E} is not present near the origin.

Multipole Expansions (cont.)

With the above expansion, the electrostatic potential energy is now

$$U = V(\vec{0}) \int_{\mathcal{V}} d\tau' \rho(\vec{r}') - \vec{E}(\vec{0}) \cdot \int_{\mathcal{V}} d\tau' \rho(\vec{r}') \vec{r}' \quad (3.218)$$

$$- \frac{1}{6} \sum_{j,k=1}^3 \left. \frac{\partial E_j}{\partial r_k} \right|_{\vec{0}} \int_{\mathcal{V}} d\tau' \rho(\vec{r}') [3r'_k r'_k - \delta_{jk}(r')^2] + \dots$$

$$= Q V(\vec{0}) - \vec{p} \cdot \vec{E}(\vec{0}) - \frac{1}{6} \sum_{j,k=1}^3 Q_{jk} \left. \frac{\partial E_j}{\partial r_k} \right|_{\vec{0}} + \dots \quad (3.219)$$

or, more generally, if the charge distribution is centered around \vec{r} ,

$$U = Q V(\vec{r}) - \vec{p} \cdot \vec{E}(\vec{r}) - \frac{1}{6} \sum_{j,k=1}^3 Q_{jk} \left. \frac{\partial E_j}{\partial r_k} \right|_{\vec{r}} + \dots \quad (3.220)$$

$$= Q V(\vec{r}) - \vec{p} \cdot \vec{E}(\vec{r}) - \frac{1}{6} \vec{\nabla}_{\vec{r}} \cdot \underline{\underline{Q}} \cdot \vec{E}(\vec{r}) + \dots \quad (3.221)$$

where we have written the last term in tensor dot product form. There are now contributions to the potential energy from the relative alignment of \vec{p} and \vec{E} and from the alignment of $\underline{\underline{Q}}$'s principal axes relative to the principal axes of the potential's curvature matrix (the field's derivatives). Note that $\vec{\nabla}_{\vec{r}}$ acts on the spatial dependence of \vec{E} ; \vec{r}' has already been integrated over to obtain $\underline{\underline{Q}}$.

Multipole Expansions (cont.)

Force on a Multipole Distribution in an External Field

We can calculate the force on the charge distribution by taking the derivative of U with respect to the charge distribution's nominal position \vec{r} :

$$\begin{aligned}\vec{F}(\vec{r}) &= -\vec{\nabla} U(\vec{r}) = Q \left(-\vec{\nabla} V(\vec{r}) \right) + \vec{\nabla} \left(\vec{p} \cdot \vec{E}(\vec{r}) \right) + \frac{1}{6} \sum_{j,k,m=1}^3 Q_{jk} \frac{\partial^2 E_j}{\partial r_k \partial r_m} \hat{r}_m + \dots \\ &= Q \vec{E}(\vec{r}) + (\vec{p} \cdot \vec{\nabla}) \vec{E}(\vec{r}) + \frac{1}{6} \sum_{j,k,m=1}^3 Q_{jk} \frac{\partial^2 E_j}{\partial r_k \partial r_m} \hat{r}_m + \dots \\ &= Q \vec{E}(\vec{r}) + (\vec{p} \cdot \vec{\nabla}) \vec{E}(\vec{r}) + \frac{1}{6} \vec{\nabla} \left[\vec{\nabla} \cdot \left(\underline{\underline{Q}} \cdot \vec{E}(\vec{r}) \right) \right] + \dots \quad (3.222)\end{aligned}$$

In going from the second to the third row, we used the vector identity

$\vec{\nabla} \left(\vec{a} \cdot \vec{f}(\vec{r}) \right) = (\vec{a} \cdot \vec{\nabla}) \vec{f}(\vec{r})$ when \vec{a} is a constant vector and $\vec{f}(\vec{r})$ has no curl. Note that all $\vec{\nabla}$ are with respect to \vec{r} (since \vec{r}' has been integrated over already).

We see that, in the absence of a net charge, the dipole moment responds to gradients in the electric field and, in the absence of a dipole moment, the quadrupole moment responds to the local curvature (second derivatives) of the electric field.

Multipole Expansions (cont.)

Torque on a Multipole in an External Field

Let's also calculate the torque. To calculate a torque, we need to take the gradient of the potential energy in spherical coordinates with respect to the orientation of the charge distribution relative to the electric field.

The monopole term yields no torque because there is no orientation angle involved: Q and $V(\vec{r})$ are scalars.

Considering the dipole term, we understand that there are only two vectors involved, \vec{p} and \vec{E} , and the potential energy only depends on the angle between them. So the torque will be given by the derivative with respect to this angle, which we call θ_p to differentiate it from the θ coordinate of the system in which we consider \vec{E} . This angle will be measured from \vec{E} to \vec{p} . Then,

$$\vec{N}_{elec} = -\frac{\partial}{\partial p_\theta} \left(-\vec{p} \cdot \vec{E}(\vec{r}) \right) \quad (3.223)$$

$$\begin{aligned} &= \frac{\partial}{\partial p_\theta} p \left| \vec{E}(\vec{r}) \right| \cos \theta_p = -p \left| \vec{E}(\vec{r}) \right| \sin \theta_p \\ &= \vec{p} \times \vec{E}(\vec{r}) \end{aligned} \quad (3.224)$$

This is a result you are familiar with from Ph1b.

Multipole Expansions (cont.)

Moving on to the quadrupole term, we recognize from Ph106ab that any symmetric tensor can be diagonalized via a rotation. Let's write

$$\underline{\underline{Q}} = \mathcal{R}(\phi_Q, \theta_Q, \psi_Q) \underline{\underline{\mathcal{Q}}} [\mathcal{R}(\phi_Q, \theta_Q, \psi_Q)]^T \quad \text{with} \quad \underline{\underline{\mathcal{Q}}} = \text{diag}(Q_1, Q_2, Q_3) \quad (3.225)$$

where the Q_i are quadrupole moments along the principal axes of the quadrupole tensor and $\mathcal{R}(\phi_Q, \theta_Q, \psi_Q)$ is the rotation matrix that rotates from the frame in which the coordinate axes align with the quadrupole tensor's principal axes to the arbitrary frame we started in, with the three Euler angles $(\phi_Q, \theta_Q, \psi_Q)$ defining the orientation of the principal axes of $\underline{\underline{\mathcal{Q}}}$ relative to the this arbitrary frame. This kind of diagonalization should be familiar to you from Ph106ab, with \mathcal{R} rotating from the "body" frame (the one fixed to the charge distribution's quadrupole principal axes) to the "space" frame. The quadrupole potential energy term is then

$$U_3 = -\frac{1}{6} \vec{\nabla}_{\vec{r}} \cdot \left\{ \mathcal{R}(\phi_Q, \theta_Q, \psi_Q) \underline{\underline{\mathcal{Q}}} [\mathcal{R}(\phi_Q, \theta_Q, \psi_Q)]^T \right\} \cdot \vec{E}(\vec{r}) \quad (3.226)$$

Multipole Expansions (cont.)

To calculate the torque, we need to take the gradient of U_3 with respect to the orientation of the quadrupole. This amounts to taking gradients of \mathcal{R} and \mathcal{R}^T with respect to this orientation. As you know from the case of the symmetric top, the Euler angles are particularly useful angles with respect to which these derivatives can be taken. $\partial/\partial\phi_Q$ gives the torque about the z-axis of the space frame. $\partial/\partial\theta_Q$ gives the torque that causes motion in the polar angle direction with respect to the space frame's \hat{z} . And $\partial/\partial\psi_Q$ calculates the torque about one particular principal axis of the quadrupole, chosen at will. You are familiar with symmetric tops, with $I_1 = I_2$. Here, we can have symmetric quadrupoles, with $Q_1 = Q_2$. In this case, the ψ_Q angle is the angle about the 3 axis of the quadrupole (the principal axis that aligns with the z-axis in the body frame). We do not take this further because, as you know from the study of tops in Ph106ab, the phenomenology can be quite rich.

Section 4

Electrostatics in Matter

Polarizability and Polarization

Review of Polarizability of Materials

Griffiths §4.1 does a good job of providing physical motivation for the study of the polarizability of materials, and also reviews material you saw in Ph1b, so we only summarize the basics here.

- ▶ Atoms and molecules are *polarizable*, meaning that they can acquire a dipole moment when an external electric field is applied because of the separation of the positive and negative charge in response to the applied field. The charge distribution that results is such that its field cancels the applied field at the location of the atom or molecule.
- ▶ We assume that this polarizability is a *linear process*, so that the induced dipole moment is linear in the applied electric field, though the response may be anisotropic. The *polarizability tensor* $\underline{\underline{\alpha}}$ relates the induced dipole moment to the applied field:

$$\vec{p} = \underline{\underline{\alpha}} \cdot \vec{E} \quad (4.1)$$

Polarizability and Polarization (cont.)

- As we showed in our discussion of multipoles, dipoles can experience torques and forces in an electric field. If a dipole is placed in an electric field, it feels a torque (Equation 3.224)

$$\vec{N} = \vec{p} \times \vec{E} \quad (4.2)$$

If the electric field is nonuniform, the dipole feels a force (Equation 3.222)

$$\vec{F} = (\vec{p} \cdot \vec{\nabla}) \vec{E} \quad (4.3)$$

- If a medium consists of polarizable atoms or molecules, then that medium can become polarized under the application of an electric field. The *polarization* (or *polarization density*) of the medium is

$$\vec{P} = n \vec{p} \quad (4.4)$$

where n is the density of polarization atoms or molecules and \vec{p} is the induced dipole per atom or molecule.

Polarizability and Polarization (cont.)

Bound Charges and the Potential of a Polarizable Material

When a medium is polarized and acquires a polarization vector \vec{P} , then it can generate its own electric field. This comes from the superposition of the dipole fields of the individual polarized atoms or molecules. In Ph1b, you saw how the polarization could be interpreted as yielding *bound charge* densities: when the medium polarizes, the positive components of some dipoles are cancelled by the negative components of nearby dipoles, but there can appear a net effective charge: on the boundaries, where the cancellation fails, and in the bulk if the dipole density is not uniform, also causing the cancellation to fail. This argument was made in Purcell in Ph1b to derive the bound charge densities, and Griffiths makes it in §4.2.2. Here we derive the relationship between the polarization vector and the bound charge density in rigorous fashion.

Polarizability and Polarization (cont.)

The total electric potential generated by a polarizable medium is found by summing up the dipole potentials of the individual dipoles:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V d\tau' \frac{\vec{P}(\vec{r}') \cdot (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \quad (4.5)$$

We use the identity $(\vec{r} - \vec{r}')/|\vec{r} - \vec{r}'|^3 = \vec{\nabla}_{\vec{r}'}(1/|\vec{r} - \vec{r}'|)$ (note: no minus sign because this is $\vec{\nabla}_{\vec{r}'}$, not $\vec{\nabla}_{\vec{r}}$ and we have $\vec{r} - \vec{r}'$ in the numerator, not $\vec{r}' - \vec{r}$) to rewrite this as

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int_V d\tau' \vec{P}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \quad (4.6)$$

Polarizability and Polarization (cont.)

We can integrate by parts to obtain

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\int_{\mathcal{V}} d\tau' \vec{\nabla}_{\vec{r}'} \cdot \left(\frac{\vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) - \int_{\mathcal{V}} d\tau' \frac{1}{|\vec{r} - \vec{r}'|} (\vec{\nabla}_{\vec{r}'} \cdot \vec{P}) \right] \quad (4.7)$$

The first term can be converted to a surface integral via the divergence theorem:

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\int_{\mathcal{S}(\mathcal{V})} da' \frac{\hat{n}(\vec{r}') \cdot \vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} - \int_{\mathcal{V}} d\tau' \frac{1}{|\vec{r} - \vec{r}'|} (\vec{\nabla}_{\vec{r}'} \cdot \vec{P}(\vec{r}')) \right] \quad (4.8)$$

We thus see that the potential appears to be that of a surface charge density $\sigma_b(\vec{r}')$ on $\mathcal{S}(\mathcal{V})$ and a volume charge density $\rho_b(\vec{r}')$ in \mathcal{V} with (\hat{n} is the outward normal from the polarizable material):

$$\sigma_b(\vec{r}') = \hat{n}(\vec{r}') \cdot \vec{P}(\vec{r}') \quad \rho_b(\vec{r}') = -\vec{\nabla}_{\vec{r}'} \cdot \vec{P}(\vec{r}') \quad (4.9)$$

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\int_{\mathcal{S}(\mathcal{V})} da' \frac{\sigma(\vec{r}')}{|\vec{r} - \vec{r}'|} + \int_{\mathcal{V}} d\tau' \frac{\rho_b(\vec{r}')}{|\vec{r} - \vec{r}'|} \right] \quad (4.10)$$

These charges are called “bound charges” because they are bound to the polarizable medium.

Lecture 9:
Electrostatics in Matter:
The Displacement Field, Linear Dielectrics

Date Revised: 2014/04/01 6:00
(move Sections 4.4 and 4.5 to Lecture 10)

Date Given: 2014/03/11

Polarizability and Polarization

Potential and Field of a Uniformly Polarized Sphere

This problem from Ph1b is much easier to solve with our knowledge of solutions to Laplace's Equation. The polarization density is a constant $\vec{P} = P \hat{z}$. The bound volume charge density vanishes because it is a constant and the bound surface charge density at radius R is

$$\sigma_b = \hat{n}(\vec{r}) \cdot \vec{P} = \hat{r} \cdot P \hat{z} = P \cos \theta \quad (4.11)$$

This is a problem Griffiths solves in Example 3.9 for a generic $\sigma(\theta)$, and we talked through the solution earlier. The generic solution was

$$V(r < R, \theta) = \sum_{\ell=0}^{\infty} A_{\ell} r^{\ell} P_{\ell}(\cos \theta) \quad V(r > R, \theta) = \sum_{\ell=0}^{\infty} \frac{B_{\ell}}{r^{\ell+1}} P_{\ell}(\cos \theta) \quad (4.12)$$

$$\text{with } A_{\ell} = \frac{1}{2 \epsilon_0 R^{\ell-1}} \int_0^{\pi} d\theta' \sin \theta' \sigma(\theta') P_{\ell}(\cos \theta') \quad B_{\ell} = A_{\ell} R^{2\ell+1} \quad (4.13)$$

Since $\sigma(\theta) = P P_1(\cos \theta) = P \cos \theta$, the orthonormal polynomials do their work and we get (making sure to include the normalization factor $2/(2\ell+1) = 2/3$):

$$V(r < R, \theta) = \frac{P r \cos \theta}{3 \epsilon_0} \quad V(r > R, \theta) = \frac{P R^3 \cos \theta}{3 \epsilon_0 r^2} \quad (4.14)$$

Polarizability and Polarization (cont.)

We can write these more simply. We recognize $z = r \cos \theta$ and that the total dipole moment of the sphere is $\vec{p} = 4\pi R^3 P \hat{z}/3$, yielding

$$V(r < R, \theta) = \frac{P z}{3 \epsilon_0} \quad V(r > R, \theta) = \frac{\vec{p} \cdot \hat{r}}{4 \pi \epsilon_0 r^2} \quad (4.15)$$

Thus, the field inside the sphere is *uniform*, $\vec{E} = -\vec{P}/3\epsilon_0$, and the field outside the sphere is that of a dipole \vec{p} . Note that the field outside the sphere is a perfect dipole field all the way to $r = R$; this is not an approximation.

We remind the reader from Ph1b that we could have obtained this same result by treating the sphere as two spheres of uniform charge density $\rho = q/(4\pi R^3/3)$ with their centers displaced by $\vec{d} = \vec{p}/q$. The field inside a uniform sphere of charge is proportional to the radial vector outward from its center, so the two vectors $\vec{r} - \vec{d}/2$ and $\vec{r} + \vec{d}/2$ end up differencing (because the two spheres have opposite charge) to yield \vec{d} , yielding the uniform internal field. Outside the spheres, they look like point charges, so the system looks like a point dipole \vec{p} .

One could also use this argument to figure out that the charge density on the surface is $\sigma = P \cos \theta$ and evaluate the potential and field of that charge distribution.

The Electric Displacement Field

The Electric Displacement Field

We proved earlier that the potential due to a polarization density $\vec{P}(\vec{r})$ is

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \left[\int_{S(V)} da' \frac{\hat{n}(\vec{r}') \cdot \vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} + \int_V d\tau' \frac{-\vec{\nabla}_{\vec{r}'} \cdot \vec{P}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right] \quad (4.16)$$

These are analogues of Coulomb's law for ρ_b , so the potential and field due to the polarization density satisfy

$$\nabla^2 V_b = -\frac{1}{\epsilon_0} \rho_b \quad \vec{\nabla} \cdot \vec{E}_b = \frac{1}{\epsilon_0} \rho_b = -\frac{1}{\epsilon_0} \vec{\nabla} \cdot \vec{P} \quad (4.17)$$

If there is a free charge density ρ_f , then we see that the total potential and field satisfy

$$\nabla^2 V = -\frac{1}{\epsilon_0} (\rho_f + \rho_b) \quad \vec{\nabla} \cdot \vec{E} = \frac{1}{\epsilon_0} (\rho_f - \vec{\nabla} \cdot \vec{P}) \quad (4.18)$$

The Electric Displacement Field (cont.)

We will see later that it will be convenient to have a field that depends only on the free charge density. Thus, we define the *electric displacement field* by

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \quad (4.19)$$

We immediately see that Gauss's Law can be written as

$$\vec{\nabla} \cdot \vec{D} = \rho_f \iff \oint_S da \hat{n} \cdot \vec{D} = Q_{\text{free, encl}} \quad (4.20)$$

To fully understand \vec{D} , we also need to determine its curl:

$$\vec{\nabla} \times \vec{D} = \epsilon_0 \vec{\nabla} \times \vec{E} + \vec{\nabla} \times \vec{P} = \vec{\nabla} \times \vec{P} \quad (4.21)$$

The issue here is that the right side may not vanish, so the left side may not vanish. Thus, while \vec{D} satisfies Gauss's Law in the free charge density, \vec{D} *may not have vanishing curl, so it may not satisfy Coulomb's Law in the free charge density. Also, one may not be able to apply standard Gauss's Law arguments because of the possibly nonvanishing curl (and the corresponding more complex boundary condition on the tangential component of \vec{D}) — the symmetries we usually assume may not apply.*

The Electric Displacement Field (cont.)

However, if one knows that, due to symmetry or some other consideration, $\vec{\nabla} \times \vec{P} = 0$, then \vec{D} satisfies Gauss's Law and Coulomb's Law in ρ_f for the special case $\vec{\nabla} \times \vec{P} = 0$. ($\vec{\nabla} \times \vec{P} = 0$ should be interpreted as also requiring that any boundaries be normal to \vec{P} , as we will see below that, unlike for \vec{E} , the tangential component of \vec{D} is not continuous if \vec{P} has a tangential component.)

When the above is true, \vec{D} provides a calculational convenience: if a free charge density ρ_f and a polarization field \vec{P} are specified, then we should calculate \vec{D} from the free charge density using Gauss's Law and then obtain the electric field from $\vec{E} = (\vec{D} - \vec{P})/\epsilon_0$. This simplification is possible only because of the particular form of the bound charge density, $\rho_b = -\vec{\nabla} \cdot \vec{P}$, which parallels the mathematical form of Gauss's Law, along with the condition $\vec{\nabla} \times \vec{P} = 0$.

Note the extra condition $\vec{\nabla} \times \vec{P} = 0$ that has to be specified; this reflects the fact that \vec{P} has more degrees of freedom than a scalar field ρ_b , so those extra degrees of freedom need to be specified (via the curl-free condition) for ρ_b to tell the whole story (and thus for \vec{D} to be derivable from ρ_f).

The situation will simplify somewhat when we consider linear, uniform dielectrics where $\vec{P} \propto \vec{E}$; then $\vec{\nabla} \times \vec{P} = 0$ is guaranteed, though the requirement that \vec{P} be normal to any boundaries may still create complications.

The Electric Displacement Field (cont.)

Boundary Conditions on the Displacement Field

We derived boundary conditions on \vec{E} earlier, Equations 2.46 and 2.49:

$$\hat{n} \cdot (\vec{E}_2 - \vec{E}_1) = \frac{1}{\epsilon_0} \sigma \quad \hat{s} \cdot (\vec{E}_2 - \vec{E}_1) = 0 \quad (4.22)$$

where \hat{n} is the normal vector pointing from region 1 into region 2 and \hat{s} is any tangential vector (i.e., $\hat{s} \cdot \hat{n} = 0$). We derived the equation for the normal component using the divergence of \vec{E} . So, here, we can use the fact that $\nabla \cdot \vec{D} = \rho_f$, which yields

$$\boxed{\hat{n} \cdot (\vec{D}_2 - \vec{D}_1) = \sigma_f} \quad (4.23)$$

Note that, by definition, we have $\sigma_b = \hat{n} \cdot \vec{P}$ where \hat{n} is the outward normal going from a region with a polarization density to vacuum. Therefore, by superposition,

$$\boxed{\hat{n} \cdot (\vec{P}_2 - \vec{P}_1) = -\sigma_b} \quad (4.24)$$

We could also have used $\rho_b = -\nabla \cdot \vec{P}$ and followed the same type of derivation as used for \vec{E} and \vec{D} . The sign on the right side of the boundary condition enters because of the sign in $\nabla \cdot \vec{P} = -\rho_b$.

The Electric Displacement Field (cont.)

In general, we know nothing about $\vec{\nabla} \times \vec{P}$, so the boundary condition on the tangential component of \vec{D} just reflects the fact that its curl is the curl of the polarization field. We obtain this condition by inserting the relation between \vec{E} , \vec{D} , and \vec{P} into the above tangential condition:

$$\hat{s} \cdot (\vec{D}_2 - \vec{D}_1) = \hat{s} \cdot (\vec{P}_2 - \vec{P}_1) \quad (4.25)$$

Note that, even in the case of linear dielectrics, the right side can be nonzero, as we will see below.

Linear Dielectrics

Susceptibility, Permittivity, and Dielectric Constant

So far, we have considered situations where \vec{P} has been specified for us. But, it is usually caused by an external field, and so what we really want to do is figure out what observed potential and field arise by summing the externally applied potential/field and that due to the polarization of the dielectric in response to that external potential/field. For most substances, at least at low fields, the relation between the two is linear:

$$\vec{P} = \epsilon_0 \chi_e \vec{E} \quad (4.26)$$

where χ_e is the *electric susceptibility*. Such materials are called *linear dielectrics*. An immediate implication of the above is:

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} = \epsilon_0 (1 + \chi_e) \vec{E} \equiv \epsilon \vec{E} \quad (4.27)$$

where $\epsilon \equiv \epsilon_0 (1 + \chi_e)$ is the *permittivity* of the material and $\epsilon_r \equiv 1 + \chi_e$ is the *relative permittivity* or *dielectric constant* of the material.

A very important point is that \vec{E} above is the *total* field, not just the externally applied field. You can think of polarization as an iterative process: an applied field \vec{E}_0 causes polarization \vec{P}_0 , which creates its own field \vec{E}_1 , which the polarization responds to by adding a contribution \vec{P}_1 , which creates its own field \vec{E}_2 , and so on. The process converges to the final total electric field \vec{E} and polarization \vec{P} .

Linear Dielectrics (cont.)

Conducting sphere with dielectric shell around it

Consider a conducting sphere of radius a with (free) charge Q on it surrounded by a shell of dielectric ϵ with inner and outer radii a and b . Because the system is spherically symmetric and contains a linear dielectric, we know that \vec{E} , \vec{D} , and \vec{P} all have the form

$$\vec{E} = E(r)\hat{r} \quad \vec{D} = D(r)\hat{r} \quad \vec{P} = P(r)\hat{r} \quad (4.28)$$

This ensures that the curl of all three vanish and that, at the boundaries, we have no tangential components of \vec{D} and \vec{P} . We have now satisfied all the conditions required for us to be able to derive \vec{D} directly from the free charge by Gauss's Law, which yields

$$\vec{D}(\vec{r}) = \frac{Q}{4\pi r^2} \hat{r} \quad r > a \quad (4.29)$$

($\vec{D} = \vec{E} = \vec{P} = 0$ for $r < a$.) Then we just apply the relation between \vec{D} and \vec{E} :

$$\vec{E}(\vec{r}) = \frac{Q}{4\pi\epsilon(r)r^2} \hat{r} = \begin{cases} (Q/4\pi\epsilon r^2)\hat{r} & a < r < b \\ (Q/4\pi\epsilon_0 r^2)\hat{r} & b < r \end{cases} \quad (4.30)$$

The electric field is *screened* (reduced) inside the dielectric and unchanged outside.

Linear Dielectrics (cont.)

Let's calculate the polarization vector and bound charge density:

$$\vec{P}(\vec{r}) = \epsilon_0 \chi_\epsilon(r) \vec{E}(\vec{r}) = (\epsilon(r) - \epsilon_0) \vec{E}(\vec{r}) = \frac{\epsilon(r) - \epsilon_0}{\epsilon(r)} \frac{Q}{4\pi r^2} \hat{r}$$
$$= \begin{cases} \frac{\epsilon - \epsilon_0}{\epsilon} \frac{Q}{4\pi r^2} \hat{r} & a < r < b \\ 0 & b < r \end{cases} \quad (4.31)$$

$$\rho_b = -\nabla \cdot \vec{P} = 0 \quad (4.32)$$

$$\sigma_b = \begin{cases} -\hat{r} \cdot \vec{P}(r = a) = -\frac{\epsilon - \epsilon_0}{\epsilon} \frac{Q}{4\pi a^2} & r = a \\ \hat{r} \cdot \vec{P}(r = b) = \frac{\epsilon - \epsilon_0}{\epsilon} \frac{Q}{4\pi b^2} & r = b \end{cases} \quad (4.33)$$

Note the ϵ in the denominator! We see that \vec{P} is radially outward and decreasing with r like $1/r^2$ as \vec{E} does. Note that, even though, \vec{P} is position-dependent, its divergence vanishes, so there is no bound charge density. There is surface charge density, negative at $r = a$ and positive at $r = b$. This is to be expected, as the dielectric polarizes so the negative ends of the dipoles are attracted to Q on the conducting sphere and the positive ends are repelled, leaving uncancelled layers of negative charge on the inner boundary and positive charge on the outer boundary.

Note that the total surface charge on the outer boundary cancels that on the inner boundary, so the net charge enclosed inside a sphere of radius $r > b$ is just Q . This explains why, outside the dielectric, no screening effect is present!

Linear Dielectrics (cont.)

Finally, let's calculate the electric potential from \vec{E} :

$$V(\vec{r}) = - \int_{\infty}^{\vec{r}} d\vec{s}' \cdot \vec{E}(\vec{r}') = - \int_{\infty}^r dr' E(r')$$
$$V(r > b) = - \frac{Q}{4\pi} \left[\int_{\infty}^r dr' \frac{1}{\epsilon_0 r'^2} \right] = \frac{Q}{4\pi} \frac{1}{\epsilon_0 r} \quad (4.34)$$

$$V(a < r < b) = - \frac{Q}{4\pi} \left[\int_{\infty}^b dr' \frac{1}{\epsilon_0 r'^2} + \int_b^r dr' \frac{1}{\epsilon r'^2} \right]$$
$$= \frac{Q}{4\pi} \left[\frac{1}{\epsilon_0 r} \Big|_{\infty}^b + \frac{1}{\epsilon r} \Big|_b^r \right] = \frac{Q}{4\pi} \left[\frac{1}{b} \left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon} \right) + \frac{1}{\epsilon r} \right] \quad (4.35)$$

$$V(r < a) = V(r = a) = \frac{Q}{4\pi} \left[\frac{1}{b} \left(\frac{1}{\epsilon_0} - \frac{1}{\epsilon} \right) + \frac{1}{\epsilon a} \right] \quad (4.36)$$

where V is constant for $r < a$ because $r < a$ is occupied by a conductor.

Linear Dielectrics (cont.)

Dielectrics and Capacitors

You all know from Ph1b that filling the volume between the plates of a parallel-plate capacitor increases the capacitance to $C = \epsilon_r C_{vac}$ where C_{vac} is the capacitance with vacuum between the plates. We remind you why this is true.

Let the capacitor plates lie parallel to the xy -plane at $z = 0$ (negative plate) and $z = a$ (positive plate) so \hat{z} is the unit vector pointing from the negative plate to the positive one. In such a geometry, we know from symmetry that \vec{E} , \vec{D} , and \vec{P} are all parallel to \hat{z} and independent of xy , assuming we ignore the capacitor edges. Thus, at the interfaces at $z = 0$ and $z = a$, all these vectors are normal to the interface and so no tangential components are present. These features of the fields imply that we can apply Gauss's Law to the free charge density to find \vec{D} .

The free charge density is $\sigma_f = \pm Q/A$ where Q is the charge on the plates ($+Q$ at $z = a$ and $-Q$ at $z = 0$) and A is the plate area. Gauss's Law for an infinite sheet of charge (Griffiths Example 2.5) tells us that the field of a single sheet is $E = \sigma/2\epsilon_0$. Therefore, we have for this case

$$\vec{D} = \begin{cases} -\frac{Q}{A}\hat{z} & 0 < z < a \\ 0 & z < 0, z > a \end{cases} \quad (4.37)$$

because the fields of the two plates cancel for $z < 0$ and $z > a$ but add for $0 < z < a$, and there is no ϵ_0 because we are calculating \vec{D} , not \vec{E} .

Linear Dielectrics (cont.)

This implies:

$$\vec{E} = \begin{cases} -\frac{1}{\epsilon} \frac{Q}{A} \hat{z} & 0 < z < a \\ 0 & z < 0, z > a \end{cases} \quad \vec{P} = \begin{cases} -\frac{\epsilon - \epsilon_0}{\epsilon} \frac{Q}{A} \hat{z} & 0 < z < a \\ 0 & z < 0, z > a \end{cases} \quad (4.38)$$

$$\rho_b = -\nabla \cdot \vec{P} = 0 \quad (4.39)$$

$$\sigma_b = \hat{n} \cdot \vec{P} = \begin{cases} \hat{z} \cdot \vec{P}(z=a) & z=a \\ -\hat{z} \cdot \vec{P}(z=0) & z=0 \end{cases} = \begin{cases} -\frac{\epsilon - \epsilon_0}{\epsilon} \frac{Q}{A} & z=a \\ \frac{\epsilon - \epsilon_0}{\epsilon} \frac{Q}{A} & z=0 \end{cases} \quad (4.40)$$

We have negative surface charge near the positive plate and positive surface charge near the negative plate. Finally, the voltage is

$$V(0 < z < a) = - \int_0^z d\vec{s}' \cdot \vec{E}(\vec{r}') = - \int_0^z dz' \left(-\frac{1}{\epsilon} \frac{Q}{A} \right) = \frac{1}{\epsilon} \frac{Q}{A} z \quad (4.41)$$

From this, we can calculate the capacitance, which comes out as expected:

$$C = \frac{Q}{\Delta V} = \frac{Q}{(1/\epsilon)(Q/A)a} = \epsilon \frac{A}{a} = \epsilon_r C_{vac} \quad (4.42)$$

C is increased because ΔV is reduced because the surface charge densities screen the electric field inside the dielectric.

Linear Dielectrics (cont.)

Capacitor with two layer dielectric

Let's repeat, but now with a capacitor that has two slabs of dielectric with different ϵ : ϵ_1 for $0 < z < a$ and ϵ_2 for $a < z < b$, where the top plate is now at $z = b$. Because the interface is normal to \vec{P} , we can apply Gauss's Law for \vec{D} as we did before, yielding no change in \vec{D} , but now the ϵ quantities in \vec{E} and \vec{P} depend on z .

The volume bound charge density vanishes again. The surface charge density at the top and bottom has the same expression, but again with ϵ being evaluated for the particular value of z . The surface bound charge density at the $z = a$ interface is

$$\sigma_b(z = a) = \hat{n}_1 \cdot \vec{P}_1 + \hat{n}_2 \cdot \vec{P}_2 = \hat{z} \cdot \vec{P}_1 - \hat{z} \cdot \vec{P}_2 = \frac{Q}{A} \left(-\frac{\epsilon_1 - \epsilon_0}{\epsilon_1} + \frac{\epsilon_2 - \epsilon_0}{\epsilon_2} \right) \quad (4.43)$$

Depending on which dielectric constant is greater, this can be positive or negative. Of course, it vanishes if $\epsilon_1 = \epsilon_2$. The potential and capacitance are

$$V(0 < z < a) = \frac{1}{\epsilon_1} \frac{Q}{A} z \quad V(a < z < b) = \frac{1}{\epsilon_1} \frac{Q}{A} a + \frac{1}{\epsilon_2} \frac{Q}{A} (z - a) \quad (4.44)$$

$$C = \frac{Q}{\Delta V} = \left(\frac{a}{\epsilon_1} + \frac{b - a}{\epsilon_2} \right)^{-1} A = \epsilon_{eff} \frac{A}{b} = \epsilon_{eff,r} C_{vac} \quad (4.45)$$

where $1/\epsilon_{eff} = [a/\epsilon_1 + (b - a)/\epsilon_2]/b$ is the thickness-weighted inverse mean of the dielectric constant and $\epsilon_{eff,r} = \epsilon_{eff}/\epsilon_0$.

Linear Dielectrics (cont.)

Capacitor with two side-by-side (parallel) dielectrics

Now, allow the capacitor to have plate spacing a but with two different dielectrics side-by-side, with ϵ_1 occupying A_1 and \mathcal{V}_1 and ϵ_2 occupying A_2 and \mathcal{V}_2 . It is a reasonable guess that one should treat this as two capacitors in parallel so that

$$C = C_1 + C_2 = \frac{1}{a} (\epsilon_1 A_1 + \epsilon_2 A_2) \quad (4.46)$$

But let's derive this from scratch, appreciating the subtlety at the interface.

Because the voltage difference between the two plates is independent of ϵ (they are equipotentials), it is reasonable to guess that \vec{E} is the same in ϵ_1 and ϵ_2 . Because the dielectrics are uniform in z , it is also reasonable to assume it is independent of z as one would have in the single-dielectric case. So, our guess for the form of the fields is:

$$\vec{E} = -E_0 \hat{z} \quad \vec{D} = \begin{cases} -\epsilon_1 E_0 \hat{z} & \text{in } \mathcal{V}_1 \\ -\epsilon_2 E_0 \hat{z} & \text{in } \mathcal{V}_2 \end{cases} \quad \vec{P} = \begin{cases} -(\epsilon_1 - \epsilon_0) E_0 \hat{z} & \text{in } \mathcal{V}_1 \\ -(\epsilon_2 - \epsilon_0) E_0 \hat{z} & \text{in } \mathcal{V}_2 \end{cases} \quad (4.47)$$

We see this form respects the tangential boundary conditions at the interface between the two dielectrics:

$$\hat{z} \cdot (\vec{E}_2 - \vec{E}_1) = 0 \quad \hat{z} \cdot (\vec{D}_2 - \vec{D}_1) = (\epsilon_1 - \epsilon_2) E_0 = \hat{z} \cdot (\vec{P}_2 - \vec{P}_1) \quad (4.48)$$

Linear Dielectrics (cont.)

Because \vec{D} is different in the two volumes, we must allow the free (and bound) charge densities to be different. This provides us a set of equations to solve for E_0 :

$$\epsilon_1 E_0 = \sigma_{f,1} \quad \epsilon_2 E_0 = \sigma_{f,2} \quad A_1 \sigma_{f,1} + A_2 \sigma_{f,2} = Q \quad (4.49)$$

$$\Rightarrow E_0 = \frac{1}{\epsilon_{eff}} \frac{Q}{A} \quad \epsilon_{eff} = \frac{\epsilon_1 A_1 + \epsilon_2 A_2}{A_1 + A_2} \quad A = A_1 + A_2 \quad (4.50)$$

$$C = \frac{Q}{\Delta V} = \frac{Q}{a E_0} = \epsilon_{eff} \frac{A}{a} = \epsilon_{eff,r} C_{vac} \quad (4.51)$$

which matches our parallel-capacitor expectation. The polarizations and free bound charge densities are

$$\vec{P} = \begin{cases} -\frac{\epsilon_1 - \epsilon_0}{\epsilon_{eff}} \frac{Q}{A} \hat{z} & \text{in } \mathcal{V}_1 \\ -\frac{\epsilon_2 - \epsilon_0}{\epsilon_{eff}} \frac{Q}{A} \hat{z} & \text{in } \mathcal{V}_2 \end{cases} \quad |\sigma_f| = \begin{cases} \frac{\epsilon_1}{\epsilon_{eff}} \frac{Q}{A} & \text{in } \mathcal{V}_1 \\ \frac{\epsilon_2}{\epsilon_{eff}} \frac{Q}{A} & \text{in } \mathcal{V}_2 \end{cases} \quad \sigma_b = \begin{cases} -|\vec{P}| & z = 0 \\ |\vec{P}| & z = a \end{cases} \quad (4.52)$$

where the sign for σ_b assumes $Q > 0$. The sign of σ_f is positive at $z = a$ and negative at $z = 0$ under this assumption.

Linear Dielectrics (cont.)

Finally, if one calculates the total charge density $\sigma_f + \sigma_b$ at $z = 0$ or $z = a$, one gets

$$\sigma_{t,1} = \sigma_{f,1} + \sigma_b = \left(\frac{\epsilon_1}{\epsilon_{\text{eff}}} - \frac{\epsilon_1 - \epsilon_0}{\epsilon_{\text{eff}}} \right) \frac{Q}{A} = \frac{\epsilon_0}{\epsilon_{\text{eff}}} \frac{Q}{A} = \sigma_{t,2} \quad (4.53)$$

This makes sense: since the electric field is the same in \mathcal{V}_1 and \mathcal{V}_2 , the total (free + bound) surface charge density sourcing it should be the same. The total charge density is a factor $\epsilon_0/\epsilon_{\text{eff}}$ smaller than would be present in the absence of dielectrics because the bound charge density screens the free charge density. The free charge density is different in the two regions because the opposite-sign bound charge density is different because of the different dielectric constants.

Lecture 10:
Electrostatics in Matter:
Boundary Value Problems with Dielectrics,
Electrostatic Energy in and Forces on Dielectrics

Date Revised: 2014/04/24 06:00

Expanded on forces in terms of capacitance matrix

Date Given: 2014/04/01

Boundary Value Problems with Dielectrics

General Conditions for Linear, Homogeneous Dielectrics

In linear, homogeneous dielectrics,

$$\rho_b = -\vec{\nabla} \cdot \vec{P} = -\vec{\nabla} \cdot \left(\frac{\epsilon - \epsilon_0}{\epsilon} \vec{D} \right) = -\left(\frac{\epsilon - \epsilon_0}{\epsilon} \right) \vec{\nabla} \cdot \vec{D} = -\left(\frac{\epsilon - \epsilon_0}{\epsilon} \right) \rho_f \quad (4.54)$$

(Homogeneity is required so the gradient does not act on ϵ .) Therefore, if there is no free charge density in a linear, homogeneous dielectric, there is no bound charge density either. Thus, the dielectric volume satisfies Laplace's Equation. All our machinery for solving Laplace's Equation applies here.

We always need boundary conditions, though, and we can use the ones we derived earlier:

$$\hat{n} \cdot [\vec{D}_2 - \vec{D}_1] = \sigma_f \quad (4.55)$$

Writing this in terms of the potential, we have

$$\hat{n} \cdot [\epsilon_2 \vec{\nabla} V_2 - \epsilon_1 \vec{\nabla} V_1] = -\sigma_f \quad (4.56)$$

And, we always require $V_1 = V_2$: the potential must be continuous.

Boundary Value Problems with Dielectrics (cont.)

Spherical cavity in a dielectric medium with uniform field applied

Let's apply the above to a spherical cavity of radius R in a medium with dielectric constant ϵ with a uniform field $\vec{E} = E_0 \hat{z}$ applied. There is no free charge anywhere. Our boundary conditions therefore are

$$V(r \rightarrow \infty) = -E_0 z = -E_0 r P_1(\cos \theta) \quad (4.57)$$

and

$$\epsilon_0 \frac{\partial V_{<}}{\partial r} \Big|_{r=R} = \epsilon \frac{\partial V_{>}}{\partial r} \Big|_{r=R} \quad \text{and} \quad V_{<}(r = R) = V_{>}(r = R) \quad (4.58)$$

where $V_{<}(r) = V(r < R)$ and $V_{>}(r) = V(r > R)$. We also choose the zero of the potential to be at $z = 0$, $V(z = 0) = 0$.

Boundary Value Problems with Dielectrics (cont.)

As usual, we begin by writing our generic solutions to Laplace's Equation in spherical coordinates:

$$V(r < R) = \sum_{\ell=0}^{\infty} C_{\ell} r^{\ell} P_{\ell}(\cos \theta) \quad V(r > R) = \sum_{\ell=0}^{\infty} \left(A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}} \right) P_{\ell}(\cos \theta) \quad (4.59)$$

where we have applied the requirement that V be finite at the origin to eliminate the $1/r^{\ell+1}$ terms for $V(r < R)$. Recall that we cannot eliminate the r^{ℓ} terms for $V(r > R)$ because the potential does not vanish at infinity.

Let's first apply the $r \rightarrow \infty$ condition. We did this before in the case of a metal sphere in a uniform field, and we found

$$A_1 = -E_0 \quad A_{\ell \neq 1} = 0 \quad (4.60)$$

Next, we apply the continuity condition at $r = R$, making use of orthonormality of the P_{ℓ} :

$$C_1 R = -E_0 R + \frac{B_1}{R^2} \quad C_{\ell \neq 1} R^{\ell} = \frac{B_{\ell \neq 1}}{R^{\ell+1}} \quad (4.61)$$

Boundary Value Problems with Dielectrics (cont.)

Finally, let's take the radial derivative and apply the matching condition on it, again using orthonormality:

$$\epsilon_0 C_1 = -\epsilon \left(E_0 + \frac{2}{R^3} B_1 \right) \quad \epsilon_0 C_{\ell \neq 1} \ell R^{\ell-1} = -\epsilon \frac{B_{\ell \neq 1}}{R^{\ell+2}} (\ell+1) \quad (4.62)$$

Doing the algebra, we find

$$C_{\ell \neq 1} = B_{\ell \neq 1} = 0 \quad B_1 = -\frac{\epsilon - \epsilon_0}{2\epsilon - \epsilon_0} E_0 R^3 \quad C_1 = -\frac{3\epsilon}{2\epsilon + \epsilon_0} E_0 \quad (4.63)$$

Thus, the potential is

$$V(r < R) = -\frac{3\epsilon}{2\epsilon + \epsilon_0} E_0 r \cos \theta = -\frac{3\epsilon}{2\epsilon + \epsilon_0} E_0 z \quad (4.64)$$

$$\begin{aligned} V(r > R) &= -E_0 r \cos \theta - \frac{\epsilon - \epsilon_0}{2\epsilon + \epsilon_0} E_0 \frac{R^3}{r^2} \cos \theta \\ &= -E_0 z + \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2} \quad \text{with} \quad \vec{p} = -\frac{4\pi}{3} R^3 E_0 \frac{3\epsilon_0}{2\epsilon + \epsilon_0} (\epsilon - \epsilon_0) \hat{z} \end{aligned} \quad (4.65)$$

Boundary Value Problems with Dielectrics (cont.)

The potential inside the cavity is that of a uniform electric field in the same direction as the applied field but multiplied by the factor $3\epsilon/(2\epsilon + \epsilon_0) > 1$, while the potential outside is that of the uniform field plus that of a dipole whose orientation is *opposite* the uniform field and whose magnitude is given above. It is as if the *cavity* acquired a polarization density in the negative z direction, though of course that cannot happen because $\chi_e(r < R) = 0$ there and thus $\vec{P}(r < R) = \epsilon_0 \chi_e(r < R) \vec{E}(r < R) = 0$. The polarization density outside the cavity is just the field times $\epsilon - \epsilon_0$ (which is not particularly illuminating).

The (bound) surface charge density is

$$\begin{aligned}\sigma_b &= \hat{n} \cdot \vec{P}(r = R) = \hat{n} \cdot (\epsilon - \epsilon_0) \vec{E}_>(r = R) \\ &= (\epsilon - \epsilon_0) \left(-\hat{r} \cdot E_0 \hat{z} - \frac{\partial}{\partial r} \frac{\epsilon - \epsilon_0}{2\epsilon + \epsilon_0} E_0 \frac{R^3}{r^2} \cos \theta \Big|_{r=R} \right) \\ &= -3\epsilon_0 \frac{\epsilon - \epsilon_0}{2\epsilon + \epsilon_0} E_0 \cos \theta\end{aligned}\tag{4.66}$$

(Notice that $\hat{n} = -\hat{r}$ because \hat{n} is taken to point *out of* the dielectric medium in the definition of σ_b .) We see the boundary of the cavity acquires a surface charge density with the same magnitude and cosine dependence as the bound charge on the surface of a uniformly polarized sphere, though with *opposite* sign (so there is negative charge at the $+z$ end and positive charge at the $-z$ end). The sign follows naturally from our arguments about cancellation of dipole charge.

Boundary Value Problems with Dielectrics (cont.)

Note that σ_b is *not* enhanced by a factor ϵ/ϵ_0 : if one calculates the field due to the free *and* bound charge (bound charge only here), it has already taken into account screening effects: there is no additional screening from the polarized medium because there is no bound charge density in the polarized medium ($\vec{\nabla} \cdot \vec{P} \propto \vec{\nabla} \cdot \vec{E} = 0$ there).

The field is enhanced in the cavity for two reasons: first, there is no polarizable material to screen the electric field, and, second there is surface charge density on the cavity's boundary that creates an additional field in the direction of the uniform field.

For reference, we note that the solution for a dielectric sphere (Griffiths Example 4.7) in a uniform field looks very similar:

$$V(r < R) = -\frac{3\epsilon_0}{2\epsilon_0 + \epsilon} E_0 z \quad V(r > R) = -E_0 z + \frac{\vec{p} \cdot \hat{r}}{4\pi\epsilon_0 r^2} \quad (4.67)$$

$$\text{with } \vec{p} = \frac{4\pi}{3} R^3 E_0 \frac{3\epsilon_0}{2\epsilon_0 + \epsilon} (\epsilon - \epsilon_0) \hat{z} \equiv \frac{4\pi}{3} R^3 \vec{P}(r < R) \quad (4.68)$$

$$\sigma_b = 3\epsilon_0 \frac{\epsilon - \epsilon_0}{2\epsilon_0 + \epsilon} E_0 \cos\theta \quad (4.69)$$

Basically, exchange ϵ_0 and ϵ everywhere to go between the two results. In this case, the sphere acquires a polarization density $3\epsilon_0(\epsilon - \epsilon_0)/(2\epsilon_0 + \epsilon)$, now in the direction of the applied field. The surface charge density is also of same form as the cavity case with the $\epsilon \leftrightarrow \epsilon_0$ exchange. That exchange flips the sign so that the $+z$ end acquires a positive charge, again as expected from the dipole charge cancellation argument.

Electrostatic Energy in and Forces on Dielectrics

Electrostatic Potential Energy due to Assembly of Free Charge

It turns out that electrostatic potential energy in dielectrics is a subtle topic because of the existence of the charges forming the dielectric. There are different kinds of electrostatic potential energy: that needed to assemble the free and bound charge distribution versus that needed to assemble the free charge distribution and polarize the preexisting dielectric. It is generally the latter we are interested in, so we consider that case.

Suppose we have a system in which an electric field $\vec{E}(\vec{r})$ and its potential $V(\vec{r})$ have already been set up and we want to bring in additional *free charge* $\delta\rho_f$ from infinity (assuming the potential vanishes at infinity). In this case, the change in potential energy is

$$\delta U = \int_V d\tau' (\delta\rho_f(\vec{r}')) V(\vec{r}') \quad (4.70)$$

The free charge density is related to the displacement field by $\vec{\nabla} \cdot \vec{D} = \rho_f$, so a change $\delta\rho_f$ corresponds to a change in the divergence of the displacement field $\delta(\vec{\nabla} \cdot \vec{D})$.

Linearity of the divergence lets us rewrite this as $\delta\rho_f = \vec{\nabla} \cdot \delta\vec{D}$.

Electrostatic Energy in and Forces on Dielectrics (cont.)

Then, we may integrate by parts and apply the divergence theorem:

$$\begin{aligned}\delta U &= \int_V d\tau' \left(\vec{\nabla} \cdot \delta \vec{D}(\vec{r}') \right) V(\vec{r}') \\ &= \int_V d\tau' \vec{\nabla} \cdot \left(V(\vec{r}') \delta \vec{D}(\vec{r}') \right) - \int_V d\tau' \left(\delta \vec{D}(\vec{r}') \right) \cdot \vec{\nabla} V(\vec{r}') \\ &= \oint_{S(V)} da' \hat{n}(\vec{r}') \cdot \left(V(\vec{r}') \delta \vec{D}(\vec{r}') \right) + \int_V d\tau' \left(\delta \vec{D}(\vec{r}') \right) \cdot \vec{E}(\vec{r}') \quad (4.71)\end{aligned}$$

Assuming the potential falls off at infinity, the surface term can be taken out to infinity to vanish. So, we are then left with

$$U = \int_0^{\vec{D}} \int_V d\tau' \vec{E}(\vec{r}') \cdot d\vec{D}(\vec{r}') \quad (4.72)$$

There are two integrals here, one over volume and one over the value of \vec{D} from zero to its final value. \vec{E} is of course tied to \vec{D} and they vary together.

Electrostatic Energy in and Forces on Dielectrics (cont.)

For the case of a linear (but perhaps not homogeneous) dielectric, we may use $\vec{D}(\vec{r}) = \epsilon(\vec{r})\vec{E}(\vec{r})$ and therefore

$$\begin{aligned} U &= \int_0^{\vec{E}} \int_{\mathcal{V}} d\tau' \epsilon(\vec{r}') \vec{E}(\vec{r}') \cdot d\vec{E}(\vec{r}') \\ &= \frac{1}{2} \int_0^{\vec{E}} \int_{\mathcal{V}} d\tau' \epsilon(\vec{r}') d(\vec{E}(\vec{r}') \cdot \vec{E}(\vec{r}')) \\ &= \frac{1}{2} \int_{\mathcal{V}} d\tau' \epsilon(\vec{r}') E^2(\vec{r}') = \frac{1}{2} \int_{\mathcal{V}} d\tau' \vec{E}(\vec{r}') \cdot \vec{D}(\vec{r}') \end{aligned} \quad (4.73)$$

If the medium is homogeneous, one could pull ϵ outside the integral at any point.

By contrast, if we wanted to know the total electrostatic potential energy stored in the assembly of the free and bound charge, we would just do the usual volume integral of E^2 with ϵ_0 instead of ϵ . That energy is smaller because $\epsilon > \epsilon_0$. The reason for this difference is that, assembling that medium in the first place, which consists of bringing positive and negative charges together, creates a system with negative potential energy, and thus the total potential energy of the system would be lower if we accounted for the energy of assembling the medium. But we will never pull the dielectric apart, so it is natural to treat that component of the potential energy as an offset that is inaccessible and neglect it in the potential energy.

Electrostatic Energy in and Forces on Dielectrics (cont.)

Energy of a Dielectric in an External Field

A topic naturally related to the above is the electrostatic energy of a polarizable material in an external field.

Suppose we start with a system with a free charge distribution ρ_f that sources a field \vec{E}_1 in a dielectric medium ϵ_1 , yielding a displacement $\vec{D}_1 = \epsilon_1 \vec{E}_1$. The initial energy is

$$U_1 = \frac{1}{2} \int d\tau \vec{E}_1 \cdot \vec{D}_1 \quad (4.74)$$

Now, *with the charge sourcing \vec{E}_1 held fixed*, let's introduce a piece of dielectric occupying the volume \mathcal{V}_2 and having dielectric constant ϵ_2 , replacing the dielectric of dielectric constant ϵ_1 there. The remainder of space outside \mathcal{V}_2 is occupied by ϵ_1 in both configurations. The electric field and displacement field everywhere change to \vec{E}_2 and \vec{D}_2 , where $\vec{D}_2(\vec{r}) = \epsilon(\vec{r}) \vec{E}_2(\vec{r})$. The energy is now

$$U_2 = \frac{1}{2} \int d\tau \vec{E}_2 \cdot \vec{D}_2 \quad (4.75)$$

Electrostatic Energy in and Forces on Dielectrics (cont.)

The difference in energy between the two configurations is

$$U_2 - U_1 = \frac{1}{2} \int d\tau \left[\vec{E}_2 \cdot \vec{D}_2 - \vec{E}_1 \cdot \vec{D}_1 \right] \quad (4.76)$$

which we rewrite as

$$U_2 - U_1 = \frac{1}{2} \int d\tau \left[\vec{E}_2 \cdot \vec{D}_1 - \vec{E}_1 \cdot \vec{D}_2 \right] + \frac{1}{2} \int d\tau \left[\vec{E}_2 + \vec{E}_1 \right] \cdot \left[\vec{D}_2 - \vec{D}_1 \right] \quad (4.77)$$

It holds that $\vec{\nabla} \times [\vec{E}_2 + \vec{E}_1] = 0$, so it can be derived from a potential V , so the second integral becomes

$$-\frac{1}{2} \int d\tau (\vec{\nabla} V) \cdot [\vec{D}_2 - \vec{D}_1] \quad (4.78)$$

We integrate by parts (the boundary term vanishes because it depends on $\vec{D}_2 - \vec{D}_1$, which should vanish as one goes far from the dielectric) to obtain

$$\frac{1}{2} \int d\tau V \vec{\nabla} \cdot [\vec{D}_2 - \vec{D}_1] \quad (4.79)$$

This divergence vanishes because the free charge has not changed between the two configurations (recall, $\vec{\nabla} \cdot \vec{D} = \rho_f$).

Electrostatic Energy in and Forces on Dielectrics (cont.)

So the second term in the energy vanishes, leaving

$$U_2 - U_1 = \frac{1}{2} \int d\tau [\vec{E}_2 \cdot \vec{D}_1 - \vec{E}_1 \cdot \vec{D}_2] \quad (4.80)$$

Now, outside \mathcal{V}_2 , it holds that $\vec{D}_2 = \epsilon_1 \vec{E}_2$ (remember, ϵ only changed inside \mathcal{V}_2), and recall also $\vec{D}_1 = \epsilon_1 \vec{E}_1$ everywhere, so the two terms cancel each other, the integrand vanishes outside \mathcal{V}_2 , and we can restrict the integral to \mathcal{V}_2 (and factoring out a sign):

$$U_2 - U_1 = -\frac{1}{2} \int_{\mathcal{V}_2} d\tau (\epsilon_2 - \epsilon_1) \vec{E}_2 \cdot \vec{E}_1 \quad (4.81)$$

If $\epsilon_1 = \epsilon_0$ (vacuum outside \mathcal{V}_2 and in \mathcal{V}_2 before the introduction of ϵ_2), then we can use $\vec{P} = (\epsilon - \epsilon_0) \vec{E}$ to rewrite as

$$W = U_2 - U_1 = -\frac{1}{2} \int_{\mathcal{V}_2} d\tau \vec{P} \cdot \vec{E} \quad \iff \quad w = -\frac{1}{2} \vec{P} \cdot \vec{E} \quad (4.82)$$

where we now define \vec{E} as the electric field in the absence of the dielectric and \vec{P} is the polarization density of the dielectric, and w refers to an energy density. This is just like the energy of a dipole in an external electric field, except that the factor of 1/2 accounts for the fact that the dipole polarizes in response to the external field (it accounts for the integration from zero field to actual field).

Electrostatic Energy in and Forces on Dielectrics (cont.)

Force and Torque on a Linear, Homogeneous Dielectric in an External Field with Free Charge Fixed

Let us first consider the force on the dielectric in the case that the free charge is held fixed. This is the case for which we calculated the above electrostatic energy, so we need only take the negative of its gradient with respect to some generalized displacement ξ to find the generalized force F_ξ :

$$F_\xi|_Q = - \left(\frac{\partial W}{\partial \xi} \right)_Q \quad (4.83)$$

ξ can be a spatial displacement coordinate like x , y , or z , or it can be an angular orientation coordinate, in which case the generalized force is actually a torque.

Any system can be reduced to a capacitance matrix, so the above can be written as

$$\begin{aligned} F_\xi|_Q &= - \left. \frac{\partial}{\partial \xi} \frac{1}{2} \sum_{i,j=1}^N Q_i Q_j C_{ij}^{-1} \right|_Q = - \frac{1}{2} \sum_{i,j=1}^N Q_i Q_j \frac{\partial C_{ij}^{-1}}{\partial \xi} \\ &= - \frac{1}{2} \underline{Q}^T \left[\frac{\partial}{\partial \xi} \underline{\underline{C}}^{-1} \right] \underline{Q} \end{aligned} \quad (4.84)$$

(Recall, $C_{ij}^{-1} = D_{ij} \neq 1/C_{ij}!$) This is not the only way, though! One can also use the derivative with respect to the energy expression given in Equation 4.82.

Electrostatic Energy in and Forces on Dielectrics (cont.)

Example: Force on a Dielectric Slab in a Parallel Plate Capacitor, Free Charge Fixed

Let's consider a parallel-plate capacitor with plate separation d , plate side dimensions ℓ and w , and with a slab of linear, homogeneous dielectric partially inserted between the plates, with vacuum from 0 to x and dielectric from x to ℓ with $0 < x < \ell$.

Neglecting the fringing fields, the electrostatic energy of the dielectric in the field is

$$W = -\frac{1}{2} d w (\ell - x) (\epsilon - \epsilon_0) E^2 \quad (4.85)$$

Recall that E here is the field *before the dielectric is present*. So it is just $E = Q/\epsilon_0 A = Q/\epsilon_0 w \ell$. Therefore,

$$F_x|_Q = - \left(\frac{\partial W}{\partial x} \right)_Q = -\frac{1}{2} d w (\epsilon - \epsilon_0) \left[\frac{Q}{\epsilon_0 w \ell} \right]^2 = -\frac{1}{2} \frac{d}{w \ell^2} \frac{Q^2}{\epsilon_0} \frac{\epsilon - \epsilon_0}{\epsilon_0} \quad (4.86)$$

We see the force is negative, pulling the slab into the capacitor. Griffiths also does this example, but by calculating the total energy of the slab in the capacitor, with E dependent on the position of the slab. If we rewrite in terms of $V = E d = Q d / \epsilon_0 w \ell$:

$$F_x|_Q = -\frac{1}{2} \frac{w}{d} (\epsilon - \epsilon_0) V^2 = -\frac{1}{2} \frac{w}{d} \epsilon_0 \chi_e V^2 \quad (4.87)$$

we see that our result matches Griffiths Equation 4.65.

Electrostatic Energy in and Forces on Dielectrics (cont.)

Force and Torque on a Linear, Homogeneous Dielectric in an External Field with Voltages Fixed

In general, we do not encounter the above situation. Rather, we hold the voltages constant on a set of electrodes while we move some dielectric by $d\xi$. Let's model this in two steps, first disconnecting the batteries and holding the charge fixed while we move the dielectric as we did above, then reconnecting the batteries so that charge flows onto or off of the electrodes and restores them to their original potentials.

Since we are now focusing on a situation with voltages on electrodes, it makes sense to think about a set of electrodes $i = 1, \dots, N$ with voltages V_i and charges Q_i . The electrodes have a capacitance matrix C_{ij} . Let's first consider the change in electrostatic energy for the first step with the charges held fixed:

$$dW_{field}|_Q = d \left[\frac{1}{2} \sum_{i,j=1}^N Q_i Q_j C_{ij}^{-1} \right]_Q = \frac{1}{2} \sum_{i,j=1}^N Q_i Q_j d[C_{ij}^{-1}] \quad (4.88)$$

(Recall, $C_{ij}^{-1} = D_{ij} \neq 1/C_{ij}!$) The change in the capacitance matrix results in a change in the voltages on the electrodes given by

$$dV_i|_Q = \sum_{j=1}^N d[C_{ij}^{-1}] Q_j \quad (4.89)$$

Electrostatic Energy in and Forces on Dielectrics (cont.)

Now, let's return the voltages to their original values by allowing charge to flow on/off the electrodes from batteries while holding the dielectrics fixed (*i.e.*, C_{ij} are constant). The charge transfer at fixed voltage required to undo the above voltage changes as fixed charge is

$$dQ_k|_V = \sum_{i=1}^N C_{ki} (-dV_i)_Q = - \sum_{i,j=1}^N C_{ki} Q_j d[C_{ij}^{-1}] \quad (4.90)$$

The change in the electrostatic energy of the configuration (energy flowing out of the battery into the field) due to this flow of charge is

$$\begin{aligned} dW_{bat}|_V &= \sum_{k=1}^N V_k dQ_k|_V = - \sum_{i,j,k=1}^N V_k C_{ki} Q_j d[C_{ij}^{-1}] = - \sum_{i,j=1}^N Q_i Q_j d[C_{ij}^{-1}] \\ &= -2 dW_{field}|_Q \end{aligned} \quad (4.91)$$

where we used $C_{ki} = C_{ik}$ and $\sum_{k=1}^N V_k C_{ik} = Q_i$.

Electrostatic Energy in and Forces on Dielectrics (cont.)

Therefore, the total infinitesimal change in energy and the generalized force are

$$dW_{tot}|_V = dW_{field}|_Q + dW_{bat}|_V = dW_{field}|_Q - 2 dW_{field}|_Q = - dW_{field}|_Q \quad (4.92)$$

$$F_\xi|_V = - \left(\frac{\partial W_{tot}}{\partial \xi} \right)_V = \left(\frac{\partial W_{field}}{\partial \xi} \right)_Q = - F_\xi|_Q \quad (4.93)$$

For the example of the dielectric slab in the capacitor, the capacitor will therefore *repel or eject* the dielectric at fixed voltage. At fixed charge, the electric field magnitude is reduced by pulling the dielectric slab into the capacitor, so the system seeks this configuration. At fixed voltage, the field configuration and thus the electrostatic energy stored in the field are not changed by the dielectric *because the voltage is held fixed*, but the battery does work to supply the charged needed to maintain fixed voltage. so it is a lower energy configuration for the dielectric to be removed so the battery does not need to do the work.

We can also write the force at fixed voltage using the energy in terms of voltages:

$$F_\xi|_V = - \frac{\partial}{\partial \xi} \left[\frac{1}{2} \sum_{i,j=1}^N V_i V_j C_{ij} \right]_V = - \frac{1}{2} \sum_{i,j=1}^N V_i V_j \frac{\partial C_{ij}}{\partial \xi} = - \frac{1}{2} \underline{V}^T \left[\frac{\partial}{\partial \xi} \underline{\underline{C}} \right] \underline{V} \quad (4.94)$$

Since $\partial \underline{\underline{C}}^{-1} / \partial \xi = - \underline{\underline{C}}^{-1} [\partial \underline{\underline{C}} / \partial \xi] \underline{\underline{C}}^{-1}$ (just evaluate $\partial [\underline{\underline{C}} \underline{\underline{C}}^{-1}] / \partial \xi = \partial \underline{1} / \partial \xi = \underline{0}$), this form combined with Equation 4.84 is consistent with $F_\xi|_V = - F_\xi|_Q$.

Section 5

Magnetostatics

Lecture 11:
Magnetostatics:
Lorentz Force, Biot-Savart Law,
Divergence and Curl of \mathbf{B} , Ampere's Law,
Magnetic Vector and Scalar Potential

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vanishing of the net current for any sphere,
and steady-state nature of currents
on pp. 235, 259, 263, 265
Date Given: 2014/04/03

Study Guidelines

As with basic electrostatics, you have seen much of the material in this section before in Ph1c. As with electrostatics, we will use more rigor here. We will also consider some more advanced topics such as the multipole expansion of the magnetic vector potential, off-axis fields for azimuthally symmetric configurations, etc. As with basic electrostatics, we won't do any examples in lecture or the notes because they would duplicate Ph1c. But you should be following the examples in Griffiths Chapter 5 and making sure you are comfortable with them.

Lorentz Force

Force on a Moving Point Charge in a Magnetic Field

The magnetic force on a point charge q moving with velocity \vec{v} in a magnetic field \vec{B} is given by the *Lorentz force Law*:

$$\vec{F}_{mag} = q(\vec{v} \times \vec{B}) \quad (5.1)$$

If an electric field is present, the total electrostatic and magnetostatic force on q is

$$\vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad (5.2)$$

Note that the electrostatic force on q is not modified by the fact that it is moving.

See the nice examples in Griffiths of cyclotron and cycloid motion (Examples 5.1 and 5.2). These are at the level of Ph1c, so we do not spend time in lecture on them.

Lorentz Force (cont.)

Magnetic Forces do No Work

Because $\vec{F}_{mag} \propto \vec{v} \times \vec{B}$, it holds that $\vec{F}_{mag} \perp \vec{v}$. Since the differential of work done by a force is $dW = \vec{F} \cdot d\vec{l} = \vec{F} \cdot \vec{v} dt$, we thus see that $dW = 0$ for magnetic forces. This may seem counterintuitive. In cases where it appears work is being done, there is usually a battery involved that is doing the work, while the magnetic force is redirecting the direction of the work (in the same way that a constraint force in mechanics does no work).

The one exception to this is the case of intrinsic magnetic moments of fundamental particles, which emerge from quantum field theory. In such cases, the magnetic moment is not identified with a current loop, it is just an intrinsic property of the particle. Since our proof requires the Lorentz force Law, and such moments are not associated with a current that experiences the Lorentz force, the proof does not apply. In cases concerning such moments, work can be done by the field of the moment or on the magnetic moment by an external magnetic field because no battery is required to maintain the magnetic moment.

Lorentz Force (cont.)

Line Currents

A current carried by a wire can be modeled as a constant line charge density λ that is moving at fixed speed v :

$$I = \lambda v \quad (5.3)$$

For the sake of the generalizations we will consider below, let us write this as a vector

$$\vec{I} = \lambda \vec{v}(\vec{r}) \quad (5.4)$$

where $\vec{v}(\vec{r})$ is a function of position and follows the wire. We have assumed λ is independent of position; as we will see below, it has to be if conservation of charge is to be satisfied and if the wire is truly one-dimensional. Thus, \vec{v} 's direction changes with position but its magnitude does not.

Lorentz Force (cont.)

Force on a Line Current

It is straightforward to calculate the force on a line current by integrating the Lorentz force Law over the wire:

$$\vec{F}_{mag} = \int dq \left[\vec{v}(\vec{r}) \times \vec{B}(\vec{r}) \right] = \int_C d\ell \lambda \left[\vec{v}(\vec{r}) \times \vec{B}(\vec{r}) \right] \quad (5.5)$$

$$\boxed{\vec{F}_{mag} = \int_C d\ell \left[\vec{I}(\vec{r}) \times \vec{B}(\vec{r}) \right]} \quad (5.6)$$

where we have used the fact that $d\vec{\ell}$, \vec{v} , and \vec{I} are all in the same direction at any point on the wire because the current flows in the wire. Now, realizing that I is independent of position along the wire (due to conservation of charge as noted above), we can pull it out in front of the integral, yielding

$$\vec{F}_{mag} = I \int_C \left[d\vec{\ell} \times \vec{B}(\vec{r}) \right] \quad (5.7)$$

Griffiths Example 5.3 is a nice example of calculating the force on a current loop and also illustrates the point of the battery supplying the energy to do the work that appears to be done by the magnetic field.

Lorentz Force (cont.)

Current Densities

Just as we generalized point charges to line, surface, and volume charge densities, we can generalize single moving point charges to line, surface, and volume current densities. We have already made the first generalization, which is straightforward to understand since one intuitively thinks of a current as an ensemble of point charges moving through a wire.

A *surface current density* is a current flowing in a sheet; think of water flowing over the surface of an object. The surface current density \vec{K} is defined by

$$d\vec{I}(\vec{r}) = \vec{K}(\vec{r}) d\ell_{\perp} = \left| \hat{\vec{K}}(\vec{r}) \times d\vec{\ell} \right| \vec{K}(\vec{r}) \quad (5.8)$$

where $d\ell_{\perp}$ is an infinitesimal length perpendicular to \vec{K} and $d\vec{\ell}$ is an arbitrary infinitesimal length. The cross-product takes the projection of $d\vec{\ell}$ perpendicular to \vec{K} .

If one thinks about the surface current density as a moving distribution of a surface charge density, then

$$\vec{K}(\vec{r}) = \sigma(\vec{r}) \vec{v}(\vec{r}) \quad (5.9)$$

where $\sigma(\vec{r})$ is the surface charge density at \vec{r} and $\vec{v}(\vec{r})$ is the velocity of the surface charge density at \vec{r} .

Lorentz Force (cont.)

A *volume current density* is a current flowing in a bulk volume; think of water flowing in a pipe or in a river. The volume current density \vec{J} is defined by

$$d\vec{I}(\vec{r}) = \vec{J}(\vec{r}) da_{\perp} = |\hat{\vec{J}}(\vec{r}) \cdot \hat{n}| da \vec{J}(\vec{r}) \quad (5.10)$$

where \hat{n} is the normal to the area element da . (If we had defined a normal \hat{n} to the line element $d\vec{l}$ in the plane of the sheet, we could have used a dot product instead of a cross product in the definition of the surface current density. But it is conventional to do it as we have done it.)

If one thinks about the volume current density as a moving distribution of a volume charge density, then

$$\vec{J}(\vec{r}) = \rho(\vec{r}) \vec{v}(\vec{r}) \quad (5.11)$$

where $\rho(\vec{r})$ is the volume charge density at \vec{r} and $\vec{v}(\vec{r})$ is the velocity of the volume charge density at \vec{r} .

Lorentz Force (cont.)

Forces on Current Densities

We can integrate the force over the current densities just as we did for the line current:

$$\vec{F}_{mag} = \int dq \left[\vec{v}(\vec{r}) \times \vec{B}(\vec{r}) \right] = \int_S da \sigma(\vec{r}) \left[\vec{v}(\vec{r}) \times \vec{B}(\vec{r}) \right] \quad (5.12)$$

$$\boxed{\vec{F}_{mag} = \int_S da \left[\vec{K}(\vec{r}) \times \vec{B}(\vec{r}) \right]} \quad (5.13)$$

$$\vec{F}_{mag} = \int dq \left[\vec{v}(\vec{r}) \times \vec{B}(\vec{r}) \right] = \int_V d\tau \rho(\vec{r}) \left[\vec{v}(\vec{r}) \times \vec{B}(\vec{r}) \right] \quad (5.14)$$

$$\boxed{\vec{F}_{mag} = \int_V d\tau \left[\vec{J}(\vec{r}) \times \vec{B}(\vec{r}) \right]} \quad (5.15)$$

It should be clear that we could have considered the last version to be the fundamental statement of the Lorentz force Law and derived the lower-dimensional versions by inclusion of appropriate delta functions in the definition of ρ or \vec{J} . Such a reduction would be cumbersome because the sheet or line carrying the current may not be easy to parameterize, but the reduction is conceptually straightforward.

Lorentz Force (cont.)

Conservation of Charge and the Continuity Equation

We defined the current densities above in terms of the infinitesimal current passing through an infinitesimal line element (for a surface current density) or through an infinitesimal area element (for a volume current density). Let's integrate the latter over a surface to obtain the total current passing through that surface:

$$I_S = \int_S da \hat{n}(\vec{r}) \cdot \vec{J}(\vec{r}) \quad (5.16)$$

If we take S to be a closed surface, we may apply the divergence theorem to the above:

$$\oint_S da \hat{n}(\vec{r}) \cdot \vec{J}(\vec{r}) = \int_{\mathcal{V}(S)} d\tau \vec{\nabla} \cdot \vec{J}(\vec{r}) \quad (5.17)$$

where $\mathcal{V}(S)$ is the volume enclosed by S . By conservation of charge, the current is just the time derivative of the charge enclosed by S , with the sign such that if a positive current is exiting S , then the charge enclosed must be decreasing, assuming that the surface itself is time-independent. With this, we have

$$I_S = -\frac{d}{dt} \int_{\mathcal{V}(S)} d\tau \rho(\vec{r}) = -\int_{\mathcal{V}(S)} d\tau \frac{\partial \rho(\vec{r})}{\partial t} \quad (5.18)$$

Lorentz Force (cont.)

Thus, we have

$$\int_{\mathcal{V}(\mathcal{S})} d\tau \vec{\nabla} \cdot \vec{J}(\vec{r}) = - \int_{\mathcal{V}(\mathcal{S})} d\tau \frac{\partial \rho(\vec{r})}{\partial t} \quad (5.19)$$

Since the surface \mathcal{S} is arbitrary, it must hold that the integrands are equal everywhere:

$$\boxed{\vec{\nabla} \cdot \vec{J}(\vec{r}) = - \frac{\partial \rho(\vec{r})}{\partial t}} \quad (5.20)$$

This is the *continuity equation* and is effectively the differential version of conservation of charge.

Biot-Savart Law

Biot-Savart Law

For a *steady current distribution* — one in which the current densities are time-independent — the magnetic field at \vec{r} due to a current distribution is given by

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d\ell' \frac{\vec{I}(\vec{r}') \times (\vec{r} - \vec{r}')}{| \vec{r} - \vec{r}' |^3} = \frac{\mu_0}{4\pi} I \int \frac{d\ell'(\vec{r}') \times (\vec{r} - \vec{r}')}{| \vec{r} - \vec{r}' |^3} \quad (5.21)$$

$\mu_0 = 4\pi \times 10^{-7} \text{ N A}^{-2}$ is the *permeability of free space*. The magnetic field carries units of *teslas*, $T = \text{N}/(\text{A m})$. The Biot-Savart Law is the analogy in magnetostatics of Coulomb's Law in electrostatics, and it has the same $1/r^2$ dependence.

You are well aware of the result that the field of a straight wire along the z -axis carrying current I at a transverse distance s from the wire is

$$\vec{B}(\vec{r}) = \frac{\mu_0}{2\pi} \frac{I}{s} \hat{\phi} \quad (5.22)$$

where $\hat{\phi}$ is the azimuthal unit vector in cylindrical coordinates. The field forms circles around the wire with orientation set by the right-hand rule. This is derived in Griffiths Example 5.5, which we will not repeat here since you saw it in Ph1c.

Biot-Savart Law (cont.)

Force between Two Current-Carrying Wires

We can combine the Lorentz force Law and the Biot-Savart Law to calculate the force between two current-carrying wires; this force is the empirical basis for magnetostatics, as it is much easier to measure the force between two wires than it is to create ideal test charges and measure their motion in the magnetic field of a wire. We just plug the Biot-Savart Law into the Lorentz force Law for a line current distribution, Equation 5.6:

$$\vec{F}_{mag} = I_1 \int_{C_1} d\vec{\ell} \times \vec{B}(\vec{r}) \quad (5.23)$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 \int_{C_1} \int_{C_2} \frac{d\vec{\ell}(\vec{r}) \times [d\vec{\ell}'(\vec{r}') \times (\vec{r} - \vec{r}')] }{|\vec{r} - \vec{r}'|^3} \quad (5.24)$$

Consider the special case of both wires running along the z axis separated by $s\hat{s}$ in the xy -plane, with the first wire on the z -axis itself. Then $d\vec{\ell} = \hat{z} dz$, $d\vec{\ell}' = \hat{z} dz'$, $\vec{r} = z\hat{z}$, $\vec{r}' = s\hat{s} + z'\hat{z}$.

Biot-Savart Law (cont.)

Therefore,

$$d\vec{\ell}(\vec{r}) \times [d\vec{\ell}(\vec{r}') \times (\vec{r} - \vec{r}')] = dz \, dz' \, \hat{z} \times [\hat{z} \times ((z - z') \hat{z} - s \hat{s})] \quad (5.25)$$

$$= dz \, dz' \, s \hat{s} \quad (5.26)$$

$$\text{and} \quad |\vec{r} - \vec{r}'|^3 = [(z - z')^2 + s^2]^{3/2} \quad (5.27)$$

Thus, the integral becomes

$$\vec{F}_{mag} = \frac{\mu_0}{4\pi} I_1 I_2 s \hat{s} \int_{-\infty}^{\infty} dz \int_{-\infty}^{\infty} dz' [(z - z')^2 + s^2]^{-3/2} \quad (5.28)$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 s \hat{s} \int_{-\infty}^{\infty} dz \left[\frac{z' - z}{s^2 [(z - z')^2 + s^2]^{1/2}} \right] \Big|_{-\infty}^{\infty} \quad (5.29)$$

$$= \frac{\mu_0}{4\pi} I_1 I_2 s \hat{s} \int_{-\infty}^{\infty} dz \frac{2}{s^2} = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{s} \hat{s} \int_{-\infty}^{\infty} dz \quad (5.30)$$

where we did the integral using a trigonometric substitution. The total force is infinite, but we can abstract out of the above expression the force per unit length on the first wire:

$$\vec{f}_{mag} = \frac{\mu_0}{2\pi} \frac{I_1 I_2}{s} \hat{s} \quad (5.31)$$

Biot-Savart Law (cont.)

General Expressions for Fields due to Current Densities

The obvious generalizations of the Biot-Savart Law are

$$\boxed{\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int da' \frac{\vec{K}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \quad \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d\tau' \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3}} \quad (5.32)$$

Griffiths notes that a line current distribution is the lowest-dimensional current distribution one can have because the zero-dimensional version — a point charge moving with velocity \vec{v} — does not constitute a steady current: the charge passing a given point in space is time-dependent.

As with the Lorentz force Law, it should also be clear that one could consider the volume version to be the fundamental statement of the Biot-Savart Law and one can derive the lower-dimensional versions by including delta functions in the definition of \vec{J} . This does not apply to a reduction to zero dimensionality, as noted above.

There are good examples of the use of the Biot-Savart Law in Griffiths. Again, these are at the level of Ph1c, so we do not spend time in lecture on them.

Biot-Savart Law (cont.)

Another Form for the Biot-Savart Law

We begin by using Equation 2.57 to rewrite the Biot-Savart Law expression for the magnetic field:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V d\tau' \frac{\vec{J}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = -\frac{\mu_0}{4\pi} \int_V d\tau' \vec{J}(\vec{r}') \times \vec{\nabla} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \quad (5.33)$$

We use one of the product rules for the curl, $\vec{\nabla} \times (f\vec{a}) = f(\vec{\nabla} \times \vec{a}) - \vec{a} \times (\vec{\nabla} f)$, and notice that $\vec{\nabla} \times \vec{J}(\vec{r}') = 0$ because $\vec{J}(\vec{r}')$ is a function of \vec{r}' while $\vec{\nabla}$ is with respect to \vec{r} , to obtain

$$\boxed{\vec{B}(\vec{r}) = \vec{\nabla} \times \frac{\mu_0}{4\pi} \int_V d\tau' \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|}} \quad (5.34)$$

where we have brought $\vec{\nabla}$ outside the integral over \vec{r}' because it acts with respect to \vec{r} . This is a useful form for \vec{B} , so we highlight it with a box.

We note that, while our derivation of this equation did not appear to require any assumptions about the way the current behaves at infinity, we will see later that the assumption of steady-state currents does imply the net current through any sphere must vanish.

Curl and Divergence of the Magnetic Field; Ampere's Law

Curl of the Magnetic Field

From the field of a current-carrying wire, Equation 5.22, we get the clear impression that \vec{B} has curl and that the curl is related to the current sourcing the field. Here, we explicitly calculate this curl from the Biot-Savart Law. Griffiths Section 5.3.2 provides one technique for this; we use Jackson's technique instead to avoid duplication.

We take the curl of Equation 5.34 and apply the *BAC – CAB* rule for the triple vector product, $\vec{\nabla} \times (\vec{\nabla} \times \vec{a}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \nabla^2 \vec{a}$:

$$\vec{\nabla} \times \vec{B}(\vec{r}) = \vec{\nabla} \times \left[\vec{\nabla} \times \frac{\mu_0}{4\pi} \int_V d\tau' \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right] \quad (5.35)$$

$$= \frac{\mu_0}{4\pi} \left[\vec{\nabla} \int_V d\tau' \vec{\nabla} \cdot \left(\frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) - \int_V d\tau' \nabla^2 \left(\frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) \right] \quad (5.36)$$

Because $\vec{\nabla}$ is with respect to \vec{r} and \vec{J} is a function of \vec{r}' , \vec{J} passes through the divergence in the first term and the Laplacian in the second one, preserving the necessary dot product in the first term and the vectorial nature of the second term:

$$\vec{\nabla} \times \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \left[\vec{\nabla} \int_V d\tau' \vec{J}(\vec{r}') \cdot \vec{\nabla} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) - \int_V d\tau' \vec{J}(\vec{r}') \nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \right] \quad (5.37)$$

Curl and Divergence of the Magnetic Field; Ampere's Law (cont.)

We know from electrostatics that

$$\vec{\nabla}_{\vec{r}} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) = -\vec{\nabla}_{\vec{r}'} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right)$$
$$\nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) = -4\pi \delta(\vec{r} - \vec{r}')$$

The first equation can be derived by changing variables in the derivative from \vec{r} to $\vec{r} - \vec{r}'$ to $\vec{r}' - \vec{r}$ to \vec{r}' . The second equation combines Equations 2.57 and 2.33. Applying them, we obtain

$$\vec{\nabla} \times \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \left[-\vec{\nabla}_{\vec{r}} \int_V d\tau' \vec{J}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) + 4\pi \int_V d\tau' \vec{J}(\vec{r}') \delta(\vec{r} - \vec{r}') \right] \quad (5.38)$$

The second term just becomes $4\pi \vec{J}(\vec{r})$.

Curl and Divergence of the Magnetic Field; Ampere's Law (cont.)

We can apply the product rule $\vec{\nabla} \cdot (\vec{f}\vec{a}) = \vec{a} \cdot \vec{\nabla}f + f\vec{\nabla} \cdot \vec{a}$ to rewrite the first term:

$$\int_{\mathcal{V}} d\tau' \vec{J}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) = \int_{\mathcal{V}} d\tau' \vec{\nabla}_{\vec{r}'} \cdot \left(\frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) - \int_{\mathcal{V}} d\tau' \frac{\vec{\nabla}_{\vec{r}'} \cdot \vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (5.39)$$

$$= \oint_{S(\mathcal{V})} da' \hat{n}(\vec{r}') \cdot \left(\frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) = 0 \quad (5.40)$$

We used the divergence theorem to transform the first term into a surface integral, and then we take the surface to infinity. Assuming the currents are localized, the integrand vanishes on that surface, causing the first term to vanish. The second term vanishes because $\vec{\nabla}_{\vec{r}'} \cdot \vec{J}(\vec{r}') = 0$ for steady-state currents by the continuity equation with $\partial\rho/\partial t = 0$. Thus, we obtain

$$\boxed{\vec{\nabla} \times \vec{B}(\vec{r}) = \mu_0 \vec{J}(\vec{r})} \quad (5.41)$$

This equation is the differential version of *Ampere's Law*, which we will return to shortly.

Curl and Divergence of the Magnetic Field; Ampere's Law (cont.)

Notes added after lecture:

Let's discuss some subtleties in the above derivation connected to the vanishing of the $\vec{\nabla}(\vec{\nabla} \cdot \vec{a})$ term. There are two points to make:

- ▶ When we get to the definition of the vector potential \vec{A} , we will be able to interpret the vanishing of that term as implying $\vec{\nabla} \cdot \vec{A} = 0$ for the form of the vector potential implied by Equation 5.34. $\vec{\nabla} \cdot \vec{A}$ will not vanish for any other form of the vector potential that yields the same field. Just keep this point in mind, we'll provide more explanation later.
- ▶ We assumed that the currents are localized (confined to a finite volume) to make the surface term vanish. This is not the minimal condition required. We only need the integral to vanish. If we let the surface go off to infinity while keeping the point \vec{r} at which we want to know the field at finite distance from the origin, then $1/|\vec{r} - \vec{r}'| \rightarrow 1/r'$. We can also make the integral vanish by simply requiring that the net flux of \vec{J} through a surface of radius r' vanishes. Griffiths notes this subtlety in Footnote 14 in §5.3.2. It explains how Ampere's Law works for an infinitely long wire: for any sphere at large radius, as much current flows in as out of that sphere, so the integral vanishes.

Do we have to make this requirement? It may seem that we do not; we would just get a nonstandard Ampere's Law if we did not. But we do have to make it to be self-consistent with our assumption of steady-state currents. If there were a net current through some sphere, then the charge contained in that volume would be changing with time, violating our steady-state assumption. This is the point we made in connection to Equation 5.34.

Curl and Divergence of the Magnetic Field; Ampere's Law (cont.)

Divergence of the Magnetic Field

The vector identity $\vec{\nabla} \cdot (\vec{\nabla} \times \vec{a}) = 0$ combined with Equation 5.34 immediately implies

$$\boxed{\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0} \quad (5.42)$$

The magnetic field has no divergence. This immediately implies *there are no magnetic point charges*: magnetic fields are sourced by currents only. It should be realized that this apparent fact is really an assumption inherent in the Biot-Savart Law. If we had added to the Biot-Savart Law a second term that looks like Coulomb's Law, due to magnetic monopoles, then the above divergence would have yielded that density of magnetic charge on the right side. It is *an empirical observation* that there are no magnetic monopoles, and hence we *assume* that magnetic fields are only sourced by currents via the Biot-Savart Law. That magnetic fields are sourced by currents at all is also *an empirical fact*; the Biot-Savart Law codifies that fact.

Curl and Divergence of the Magnetic Field; Ampere's Law (cont.)

General Thoughts on the Curl and Divergence of the Electric and Magnetic Field

Considering the corresponding expressions for electrostatics, we recognize that the electric field has divergence equal to the charge density because of the empirical observation of Coulomb's Law describing the electric field. It has a vanishing curl because of the empirical absence of a current that sources electric fields in the way that electric currents source magnetic fields; if there were a Biot-Savart-like term that added to Coulomb's Law, then the electric field would have curl. We can in fact guess that, if magnetic monopoles existed, moving magnetic monopoles would generate an electric field in the same way that moving electric monopoles generate a magnetic field.

The key point in all of the above is that the nature of the divergence and the curl of the electric and magnetic fields reflect *empirical observations* about the way these fields are generated. These are not *derivable* facts: they are inherent in the formulae we wrote down for the electric and magnetic fields, which themselves are based on *observations*.

We will see later that we can replace the assumption of Coulomb's Law and the Biot-Savart Law with an assumption about a potential from which the electric and magnetic fields can be derived. But, again, we can only make that assumption because it yields the correct empirical relations, Coulomb's Law and the Biot-Savart Law.

Curl and Divergence of the Magnetic Field; Ampere's Law (cont.)

Integral form of Ampere's Law

We obtained the differential version of Ampere's Law above by taking the curl of the Biot-Savart Law for the magnetic field. We may obtain the integral form of Ampere's Law from it. We begin by integrating over a surface \mathcal{S} with normal $\hat{n}(\vec{r})$:

$$\int_{\mathcal{S}} da \hat{n}(\vec{r}) \cdot [\vec{\nabla} \times \vec{B}(\vec{r})] = \mu_0 \int_{\mathcal{S}} da \hat{n} \cdot \vec{J}(\vec{r}) \quad (5.43)$$

The left side can be transformed using Stokes' Theorem into a line integral around the edge of \mathcal{S} , which we denote by the closed contour $\mathcal{C}(\mathcal{S})$, while the right side is just total current passing through $\mathcal{C}(\mathcal{S})$, I_{encl} :

$$\oint_{\mathcal{C}(\mathcal{S})} d\vec{\ell} \cdot \vec{B}(\vec{r}) = \mu_0 I_{encl} \quad (5.44)$$

yielding the integral version of Ampere's Law.

As before, there are a number of examples in Griffiths that are at the level of Ph1c, so we do not spend time on them here.

Magnetic Vector Potential

Form for the Magnetic Vector Potential

We saw (Equations 5.41 and 5.42) that the magnetic field has no divergence and has curl. You know from vector calculus (Griffiths §1.6) that this implies the magnetic field can be written purely as the curl of a *vector potential*. Equation 5.34 gave us its form:

$$\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A}(\vec{r}) \quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V d\tau' \frac{\vec{j}(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (5.45)$$

But this form, implied by the Biot-Savart Law, is not the only form. We had an analogous freedom with the electrostatic potential to add an offset. Here, we can add any curl-less function to \vec{A} without affecting \vec{B} . The form above corresponds to the additional condition

$$\vec{\nabla} \cdot \vec{A}(\vec{r}) = 0 \quad (5.46)$$

In deriving Equation 5.41, we showed, *for this form of \vec{A} only*, $\vec{\nabla} \cdot \vec{A} = 0$ is equivalent to requiring steady-state currents and that the net current through a surface of any radius vanishes (and also how the latter implies the former). For a different choice of \vec{A} (and thus of $\vec{\nabla} \cdot \vec{A}$), the mathematical manifestation of this physical requirement will be different. In fact, it must be, because $\vec{\nabla} \cdot \vec{A} = 0$ is unique to this form.

Magnetic Vector Potential (cont.)

Explicit Proof that $\vec{\nabla} \cdot \vec{A} = 0$ Can Always Be Obtained

It is interesting to prove “mechanically” that the choice $\vec{\nabla} \cdot \vec{A}$ is possible even if one, for some reason, started out with a form that did not satisfy this condition. Suppose one has a vector potential \vec{A}_0 that is not divergenceless. We need to add to it a function that makes the result divergenceless. For reasons we will see below, let’s add a function $\vec{\nabla}\lambda(\vec{r})$:

$$\vec{A} = \vec{A}_0 + \vec{\nabla}\lambda \quad (5.47)$$

Then

$$\vec{\nabla} \cdot \vec{A} = \vec{\nabla} \cdot \vec{A}_0 + \nabla^2\lambda \quad (5.48)$$

If we require the left side to vanish, then we have a version of Poisson’s Equation:

$$\nabla^2\lambda = -\vec{\nabla} \cdot \vec{A}_0 \quad (5.49)$$

One thus sees the motivation for the assumed form $\vec{\nabla}\lambda$.

Magnetic Vector Potential (cont.)

Let's choose boundary conditions that place the boundary at infinity. For these boundary conditions, we know the Green Function for the Poisson Equation and thus

$$\lambda = \frac{1}{4\pi} \int_V d\tau' \frac{\vec{\nabla} \cdot \vec{A}_0(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (5.50)$$

The vector calculus identity $\vec{\nabla} \times \vec{\nabla} a = 0$ implies that $\vec{\nabla} \times \vec{A} = \vec{\nabla} \times \vec{A}_0$ and thus the magnetic field is the same for the two vector potentials. We thus have an explicit formula for the term that has to be added to \vec{A}_0 so that the resulting form \vec{A} is divergenceless while leaving the magnetic field unchanged.

The above explicit formula may not be valid if we assume different boundary conditions, but we know the Poisson Equation always has a solution, so we are guaranteed that the desired function $\lambda(\vec{r})$ exists.

Let us make a final point about how the above relates to the physical meaning of $\vec{\nabla} \cdot \vec{A} = 0$. It is *not true* that starting with $\vec{\nabla} \cdot \vec{A}_0 \neq 0$ corresponds to a different physical assumption about the currents at infinity. Our standard formula for \vec{A} is only valid under the assumption $\vec{\nabla} \cdot \vec{A} = 0$, and so the relation between $\vec{\nabla} \cdot \vec{A}$ and the assumption about how the currents behave is only valid for that form. If one assumes a different form for \vec{A} , one that has $\vec{\nabla} \cdot \vec{A} \neq 0$, then taking its divergence will not necessarily result in the particular expressions that we encountered before, so the interpretation of $\vec{\nabla} \cdot \vec{A}$ will be different.

Magnetic Vector Potential (cont.)

Alternate Proof of the Form for the Magnetic Vector Potential

We can arrive at Equation 5.45 via a slightly different path, which makes uses of Ampere's Law and the same triple vector identity we used to prove Ampere's Law, $\vec{\nabla} \times (\vec{\nabla} \times \vec{a}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \nabla^2 \vec{a}$:

$$\text{Ampere's Law: } \vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad (5.51)$$

$$\text{use vector identity: } \vec{\nabla}(\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J} \quad (5.52)$$

$$\text{set } \vec{\nabla} \cdot \vec{A} = 0: \quad \nabla^2 \vec{A} = -\mu_0 \vec{J} \quad (5.53)$$

Note that the vector components of \vec{A} and \vec{J} line up. Thus, the last equation is a component-by-component Poisson Equation. Again, under the assumption that the currents are localized (as we assumed in providing the alternate version of the Biot-Savart Law that we previously used to define \vec{A}), we know the solution:

$$\boxed{\nabla^2 \vec{A}(\vec{r}) = -\mu_0 \vec{J}(\vec{r}) \quad \xrightleftharpoons{\text{localized currents}} \quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V d\tau' \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (5.54)}$$

This is just Equation 5.45 again. Essentially, we can think of the three components of the current density as sourcing the three components of the vector potential in the same way that the electric charge density sources the electric potential.

Magnetic Vector Potential (cont.)

The Vector Potential for Line and Surface Currents

We can consider the specific cases of line and surface current densities as volume current densities that include a delta functions specifying the localization to a line or sheet. When one does the volume integral, the delta function reduces the three-dimensional integral over the volume to one- or two-dimensional integrals over a line or sheet, yielding:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_C d\ell \frac{\vec{I}(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_S da' \frac{\vec{K}(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (5.55)$$

Magnetic Vector Potential (cont.)

Examples: Spinning Sphere of Charge and Solenoid

You can find the details for these in Griffiths Examples 5.11 and 5.12. We note the techniques here:

- ▶ The calculation of the vector potential for a spinning spherical shell of charge is a straightforward application of the definition of the vector potential. The only complication is the vector arithmetic. So please take a look at the technique to get some familiarity with handling the vectorial nature of the integrand.

Magnetic Vector Potential (cont.)

- ▶ The calculation of the vector potential for a solenoid, which is the equivalent of a spinning cylinder of charge if one ignores the small axial current contribution, is more interesting because one cannot do it by brute force application of the definition of \vec{A} . Instead, one must use some intuition along with the combination of Stokes' Theorem and the relation between \vec{B} and \vec{A} :

$$\oint_{C(S)} d\vec{\ell} \cdot \vec{A} = \int_S da \hat{n} \cdot \vec{\nabla} \times \vec{A} = \int_S da \hat{n} \cdot \vec{B} \quad (5.56)$$

The intuition part is to recognize that, because \vec{B} is along the z -axis inside the solenoid and vanishing outside and because \vec{A} "wraps around" \vec{B} , it is natural to assume \vec{A} is along $\hat{\phi}$. Then one can do the calculation in the same way as one applies Ampere's Law, except that instead of current through a surface ("enclosed current"), we have enclosed magnetic flux, and, instead of a line integral of magnetic field around the edge of the surface, we have a line integral of vector potential. Please study the technique, as a variant on this problem will be given in homework.

Lecture 12: Magnetostatics:

Uniqueness Theorem, Magnetostatic Scalar Potential,
Boundary Conditions, Multipole Expansion, Magnetic Dipoles,
Force, Torque, and Energy of Magnetic Dipoles

Most vector algebra was not covered in class;
please review on your own.

Date Revised: 2014/04/08 14:00

Reorganized how material was split among lectures,
no changes to content aside from uniqueness theorem

Date Given: 2014/04/08

Magnetic Vector Potential

Uniqueness Theorem for Magnetic Fields (Griffiths Problem 5.56)

Just as we did for electric fields, we can show that, given a current distribution and a well-defined set of boundary conditions, the magnetic field obtained is unique. We assume that a current distribution $\vec{J}(\vec{r})$ in a volume \mathcal{V} is specified. We will see later how specific we must be about the boundary conditions.

First, we need something analogous to the Green's Identities we used in the case of electrostatics. Using the vector identity $\vec{\nabla} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot \vec{\nabla} \times \vec{a} - \vec{a} \cdot \vec{\nabla} \times \vec{b}$, letting \vec{u} and \vec{v} be two arbitrary vector fields, and applying the identity with $\vec{a} = \vec{u}$ and $\vec{b} = \vec{\nabla} \times \vec{v}$, we may write

$$\int_{\mathcal{V}} d\tau \vec{\nabla} \cdot (\vec{u} \times (\vec{\nabla} \times \vec{v})) = \int_{\mathcal{V}} d\tau [(\vec{\nabla} \times \vec{v}) \cdot (\vec{\nabla} \times \vec{u}) - \vec{u} \cdot (\vec{\nabla} \times (\vec{\nabla} \times \vec{v}))] \quad (5.57)$$

Since the expression on the left-hand side is a divergence, we may turn it into a surface integral using the divergence theorem:

$$\oint_{S(\mathcal{V})} da \hat{n} \cdot (\vec{u} \times (\vec{\nabla} \times \vec{v})) = \int_{\mathcal{V}} d\tau [(\vec{\nabla} \times \vec{u}) \cdot (\vec{\nabla} \times \vec{v}) - \vec{u} \cdot (\vec{\nabla} \times (\vec{\nabla} \times \vec{v}))] \quad (5.58)$$

We will use this below.

Magnetic Vector Potential (cont.)

Now, suppose that we have two different magnetic field configurations $\vec{B}_1 \neq \vec{B}_2$, derived from two different magnetic vector potentials $\vec{A}_1 \neq \vec{A}_2$, that both satisfy Ampere's Law for the same current distribution. Let $\vec{A}_3 = \vec{A}_2 - \vec{A}_1$ and $\vec{B}_3 = \vec{B}_2 - \vec{B}_1$. We apply the above vector identity with $\vec{u} = \vec{v} = \vec{A}_3$:

$$\oint_{S(\mathcal{V})} da \hat{n} \cdot (\vec{A}_3 \times (\vec{\nabla} \times \vec{A}_3)) = \int_{\mathcal{V}} d\tau \left[(\vec{\nabla} \times \vec{A}_3) \cdot (\vec{\nabla} \times \vec{A}_3) - \vec{A}_3 \cdot (\vec{\nabla} \times (\vec{\nabla} \times \vec{A}_3)) \right] \quad (5.59)$$

We have that $\vec{\nabla} \times (\vec{\nabla} \times \vec{A}_3) = \vec{\nabla} \times \vec{B}_3 = \vec{\nabla} \times \vec{B}_2 - \vec{\nabla} \times \vec{B}_1 = \mu_0(\vec{J} - \vec{J}) = 0$ by Ampere's Law and the assumption that both field configurations are sourced by the same current distribution. Exchanging the two sides, plugging in $\vec{B}_3 = \vec{\nabla} \times \vec{A}_3$, and using the cyclic property of the triple scalar product, $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a})$, we have

$$\int_{\mathcal{V}} d\tau \left| \vec{B}_3 \right|^2 = \oint_{S(\mathcal{V})} da \hat{n} \cdot (\vec{A}_3 \times \vec{B}_3) = \oint_{S(\mathcal{V})} da \vec{B}_3 \cdot (\hat{n} \times \vec{A}_3) \quad (5.60)$$

$$= - \oint_{S(\mathcal{V})} da \vec{A}_3 \cdot (\hat{n} \times \vec{B}_3) \quad (5.61)$$

Magnetic Vector Potential (cont.)

From the above equation, we can see what (minimal) boundary condition information we must have to obtain uniqueness of \vec{B} : we must have that, at any given point on the surface, $\hat{n} \times \vec{A}$ or $\hat{n} \times \vec{B}$ is specified. If this is true, then

$\hat{n} \times \vec{A}_3 = \hat{n} \times (\vec{A}_1 - \vec{A}_2) = 0$ where $\hat{n} \times \vec{A}$ is specified and $\hat{n} \times \vec{B}_3 = \hat{n} \times (\vec{B}_1 - \vec{B}_2) = 0$ where $\hat{n} \times \vec{B}$ is specified. This ensures the integrand on the right side vanishes at every point on $S(\mathcal{V})$ and thus the right side vanishes. Since the integrand on the left side is nonnegative, it must therefore vanish everywhere: $\vec{B}_3 = 0$. Hence, $\vec{B}_1 = \vec{B}_2$ and the fields are identical and the field solution is unique.

This is a lot like our Dirichlet and Neumann boundary conditions in electrostatics: specifying $\hat{n} \times \vec{A}$ is like a Dirichlet boundary condition where we specify the electrostatic potential on the boundary, and specifying $\hat{n} \times \vec{B} = \hat{n} \times (\vec{\nabla} \times \vec{A})$ is a lot like a Neumann boundary condition where we specify the normal gradient of the electrostatic potential $\hat{n} \cdot \vec{\nabla} V$ (which is proportional to the normal component of the electric field, $\hat{n} \cdot \vec{E}$).

Magnetic Vector Potential (cont.)

We have already discussed how the \vec{A} that generates a particular \vec{B} is unique up to the gradient of an additional function if its divergence is left unspecified, and how it is fully unique if its divergence is specified. The above theorem therefore now tells us that specification of \vec{J} in the volume and of $\hat{n} \times \vec{A}$ or $\hat{n} \times \vec{B}$ on the boundary gives a vector potential that is unique up to the gradient of an additional function if its divergence is unspecified and one that is completely unique if its divergence is specified as well. This is also a lot like what we found for electrostatics in connection to the offset of the potential.

Why does specification of $\hat{n} \times \vec{A}$ on the boundary not make \vec{A} unique through the volume? Recall our “mechanical” proof of the ability to make \vec{A} divergenceless. Specifying $\hat{n} \times \vec{A}$ on the boundary implies that $\hat{n} \times \vec{\nabla} \lambda = 0$ on the boundary. But if $\vec{\nabla} \times \vec{A}$ is not given, then $\nabla^2 \lambda$ is unknown. This situation is analogous to trying to solve Poisson’s equation for a potential with knowledge only of the boundary condition but not of the source charge distribution (the right-hand side of Poisson’s Equation). This is insufficient information and thus λ is not known. To specify λ , one needs to know $\vec{\nabla} \cdot \vec{A}$.

Specifying $\hat{n} \times \vec{B}$ on the boundary provides even less information because it specifies $\hat{n} \times (\vec{\nabla} \times \vec{\nabla} \lambda)$, which always vanishes, regardless of the choice of λ .

Magnetic Vector Potential (cont.)

The Magnetostatic Scalar Potential

If one considers current-free regions, then we have $\vec{\nabla} \times \vec{B} = 0$ and the magnetic field should be derivable from a scalar potential:

$$\vec{B}(\vec{r}) = -\vec{\nabla} U(\vec{r}) \quad (5.62)$$

One must take some care, though: in addition to being current-free, the region under consideration must be *simply connected*. Griffiths Problem 5.29 shows a situation where the current in a region may vanish but $\vec{\nabla} \times \vec{B} \neq 0$ because the region is not simply connected and the enclosed volume outside the region contains current.

With the above assumptions, and noting $\vec{\nabla} \cdot \vec{B} = 0$, we can infer that U satisfies Laplace's Equation:

$$\nabla^2 U(\vec{r}) = -\vec{\nabla} \cdot \vec{B}(\vec{r}) = 0 \quad (5.63)$$

Our usual assumption of simple boundary conditions — everything falls off to zero at infinity — yields a trivial result here, $U(\vec{r}) = 0$, so we must assume less trivial boundary conditions to obtain a nonzero U . We will return to the use of the magnetostatic scalar potential in connection with magnetically polarizable materials.

Boundary Conditions on Magnetic Field and Vector Potential

We will use techniques similar to those we used in determining the boundary conditions on the electric field. We will not immediately apply these conditions to boundary value problems for currents in vacuum because there are no nontrivial boundary-value problems of this type. That is because there is no way to directly set the vector potential, unlike for the electrostatic potential. One only has Neumann boundary conditions, with current densities on surfaces, from which one can calculate the field directly via the Biot-Savart Law rather than solving Laplace's or Poisson's Equation. We will find the boundary conditions more useful in the context of magnetic materials.

Boundary Conditions on the Magnetic Field

Recall that Gauss's Law, $\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0$, implied that the normal component of the electric field satisfied Equation 2.46

$$\hat{n}(\vec{r}) \cdot [\vec{E}_2(\vec{r}) - \vec{E}_1(\vec{r})] = \frac{1}{\epsilon_0} \sigma(\vec{r}) \quad (5.64)$$

Since $\vec{\nabla} \cdot \vec{B} = 0$, we can conclude by analogy that

$$\boxed{\hat{n}(\vec{r}) \cdot [\vec{B}_2(\vec{r}) - \vec{B}_1(\vec{r})] = 0} \quad (5.65)$$

That is, the normal component of the magnetic field is continuous at any boundary.

Boundary Conditions on Magnetic Field and Vector Potential (cont.)

For the tangential component, we return to the derivation leading to Equation 2.49. In that case, we considered a contour \mathcal{C} that consisted of two contours \mathcal{C}_1 and \mathcal{C}_2 parallel to the interface and to each other and two legs normal to the interface whose length would be shrunk to zero. We saw

$$\oint_{\mathcal{C}} d\vec{l}' \cdot \vec{E}(\vec{r}') = \int_{\mathcal{C}_1} \vec{E}_1(\vec{r}') \cdot (-\hat{s}(\vec{r}')) ds + \int_{\mathcal{C}_2} \vec{E}_2(\vec{r}') \cdot \hat{s}(\vec{r}')) ds \quad (5.66)$$

where \hat{t} is the normal to the loop area, $\hat{s} = \hat{t} \times \hat{n}$, and ds is a line element along \hat{s} . In that case, the left side of the above expression vanished. In the case of the magnetic field, Ampere's Law tells us that it is the current enclosed flowing in the direction \hat{t} . That is:

$$\mu_0 \int_{\mathcal{C}'} ds \hat{t}(\vec{r}') \cdot \vec{K}(\vec{r}') = \int_{\mathcal{C}_1} \vec{B}_1(\vec{r}') \cdot (-\hat{s}(\vec{r}')) ds + \int_{\mathcal{C}_2} \vec{B}_2(\vec{r}') \cdot \hat{s}(\vec{r}')) ds \quad (5.67)$$

where \mathcal{C}' is the contour parallel to \mathcal{C}_1 and \mathcal{C}_2 but now in the plane of the interface. We neglect any volume current density passing through the area enclosed by the contour \mathcal{C} because we will shrink the area of \mathcal{C} to zero.

Boundary Conditions on Magnetic Field and Vector Potential (cont.)

When we shrink the area of \mathcal{C} to zero, $\mathcal{C}_1 \rightarrow \mathcal{C}_2$ and $\mathcal{C}' \rightarrow \mathcal{C}_2$, so we have

$$\int_{\mathcal{C}_2} ds \left[\vec{B}_2(\vec{r}') - \vec{B}_1(\vec{r}') \right] \cdot \hat{s}(\vec{r}') = \mu_0 \int_{\mathcal{C}_2} ds \hat{t}(\vec{r}') \cdot \vec{K}(\vec{r}') \quad (5.68)$$

Since the contour \mathcal{C}_2 is arbitrary, the integrands must be equal

$$\left[\vec{B}_2(\vec{r}) - \vec{B}_1(\vec{r}) \right] \cdot \hat{s}(\vec{r}) = \mu_0 \hat{t}(\vec{r}) \cdot \vec{K}(\vec{r}) \quad (5.69)$$

Next, we use $\hat{t} = \hat{n} \times \hat{s}$ and the cyclic nature of triple vector products ,
 $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{c} \times \vec{a})$:

$$\left[\vec{B}_2(\vec{r}) - \vec{B}_1(\vec{r}) \right] \cdot \hat{s}(\vec{r}) = \mu_0 \left[\hat{n}(\vec{r}) \times \hat{s}(\vec{r}) \right] \cdot \vec{K}(\vec{r}) \quad (5.70)$$

$$\boxed{\left[\vec{B}_2(\vec{r}) - \vec{B}_1(\vec{r}) \right] \cdot \hat{s}(\vec{r}) = \mu_0 \left[\vec{K}(\vec{r}) \times \hat{n}(\vec{r}) \right] \cdot \hat{s}(\vec{r})} \quad (5.71)$$

Note that this condition holds for any \hat{s} tangential to the interface.

Boundary Conditions on Magnetic Field and Vector Potential (cont.)

Finally, we can combine these two conditions to obtain one compact expression for the boundary condition on the magnetic field. By the definition of the cross product, $\vec{K} \times \hat{n}$ is always perpendicular to \hat{n} and thus has no component along \hat{n} . Therefore, the expression

$$\vec{B}_2(\vec{r}) - \vec{B}_1(\vec{r}) = \mu_0 \vec{K}(\vec{r}) \times \hat{n}(\vec{r}) \quad (5.72)$$

captures both boundary conditions: the projection of \vec{B} normal to the interface (along \hat{n}) is continuous because the projection of the right side along that direction vanishes, and the projection of \vec{B} along any \hat{s} parallel to the interface can be discontinuous by the projection of $\mu_0 \vec{K} \times \hat{n}$ along that direction.

Boundary Conditions on Magnetic Field and Vector Potential (cont.)

We can rewrite the above in another way. Take the cross product of both sides with $\hat{n}(\vec{r})$ from the left. The right side becomes a triple vector product, which we can rewrite using the *BAC – CAB* rule, $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$. The second term has $\hat{n} \cdot \vec{K}$, which vanishes, while the first term has $\hat{n} \cdot \hat{n} = 1$. Thus, we have

$$\hat{n}(\vec{r}) \times [\vec{B}_2(\vec{r}) - \vec{B}_1(\vec{r})] = \mu_0 \vec{K}(\vec{r}) \quad (5.73)$$

The earlier form is more useful when \vec{K} is specified, and the second form would more easily yield \vec{K} if the fields are specified. Note, however, that this form does not preserve the information about the normal component of \vec{B} because the contribution of that component to the left side vanishes.

Boundary Conditions on Magnetic Field and Vector Potential (cont.)

Boundary Conditions on the Vector Potential

As one might expect by analogy to the electrostatic case, the vector potential itself has to be continuous across a boundary:

$$\vec{A}_2(\vec{r}) - \vec{A}_1(\vec{r}) = 0 \quad (5.74)$$

This is seen easily:

- ▶ We have chosen the divergence of \vec{A} to vanish, so the normal component of \vec{A} must be continuous, just as we found the normal component of \vec{B} is continuous for the same reason.
- ▶ The curl of \vec{A} does not vanish, $\vec{\nabla} \times \vec{A} = \vec{B}$. This implies the line integral of \vec{A} around the contour \mathcal{C} used above is nonzero and equals $\Phi_{S(\mathcal{C})} = \int_{S(\mathcal{C})} da \hat{n} \cdot \vec{B}$, the *magnetic flux* of \vec{B} through the surface $S(\mathcal{C})$ defined by \mathcal{C} . But, as the area of the contour is shrunk to zero, the magnetic flux vanishes because the magnetic field cannot have a delta function singularity in the same way that the current can (though the field can go to infinity as a power law in $1/r$).

Boundary Conditions on Magnetic Field and Vector Potential (cont.)

While the vector potential itself is continuous, its derivatives are not necessarily continuous because its derivatives are related to \vec{B} , which is not necessarily continuous. Evaluating these discontinuities is a bit harder than in the case of the electric potential because the derivatives are not related in a trivial component-by-component way to the field. We need an expression involving second derivatives of \vec{A} if we want to obtain boundary conditions on the first derivatives of \vec{A} . Let's use Equation 5.54:

$$\nabla^2 \vec{A}(\vec{r}) = \mu_0 \vec{J}(\vec{r}) \quad (5.75)$$

Consider a projection of this equation in Cartesian coordinates, rewritten so the divergence is clear:

$$\vec{\nabla} \cdot \vec{\nabla} (\hat{x} \cdot \vec{A}(\vec{r})) = \mu_0 \hat{x} \cdot \vec{J}(\vec{r}) \quad (5.76)$$

We have used Cartesian coordinates rather than a coordinate system using \hat{n} , \hat{t} , and \hat{s} because the latter vary in direction depending on where one is on the surface; their derivatives do not vanish, so we would not have been able to pull them inside the gradient as we did.

Boundary Conditions on Magnetic Field and Vector Potential (cont.)

Given the above, we now apply the same kind of geometry we used to derive the boundary condition on the normal component of \vec{E} . That yields

$$\hat{n} \cdot [\vec{\nabla} (\hat{x} \cdot \vec{A}_2(\vec{r})) - \vec{\nabla} (\hat{x} \cdot \vec{A}_1(\vec{r}))] = \mu_0 \hat{x} \cdot \vec{K}(\vec{r}) \quad (5.77)$$

$$\hat{n} \cdot \vec{\nabla} [\hat{x} \cdot \vec{A}_2(\vec{r}) - \hat{x} \cdot \vec{A}_1(\vec{r})] = \quad (5.78)$$

where $\hat{x} \cdot \vec{K}$ is what is left of $\hat{x} \cdot \vec{J}$ as the Gaussian volume used in that proof shrinks to zero thickness in the direction normal to the interface, just as ρ reduced to σ in the case of the electric field.

The above argument holds for the \hat{y} and \hat{z} projections of \vec{A} and \vec{K} also, so we may combine them to obtain

$$\boxed{\hat{n} \cdot \vec{\nabla} [\vec{A}_2(\vec{r}) - \vec{A}_1(\vec{r})] = \mu_0 \vec{K}(\vec{r})} \quad (5.79)$$

Thus, we see that the normal derivative of each component of the vector potential has a discontinuity set by the surface current density in the direction of that component of the vector potential. This implies that the normal derivative of the normal component of \vec{A} has no discontinuity since there can be no surface current in that direction!

Boundary Conditions on Magnetic Field and Vector Potential (cont.)

Let's consider the tangential derivatives of the vector potential. Here, we use the vector identity

$$\vec{\nabla} \times \vec{\nabla} \vec{A}(\vec{r}) = 0 \quad (5.80)$$

where again we consider each component of \vec{A} as a scalar function and the above equation holds for all three components. If we again project by Cartesian components; e.g.

$$\vec{\nabla} \times \vec{\nabla} (\hat{x} \cdot \vec{A}(\vec{r})) = 0 \quad (5.81)$$

then we can apply the same type of argument as we applied for calculating the boundary condition on the tangential components of \vec{E} , which in this case yields

$$\hat{s} \cdot [\vec{\nabla} (\hat{x} \cdot \vec{A}_2(\vec{r})) - \vec{\nabla} (\hat{x} \cdot \vec{A}_1(\vec{r}))] = 0 \quad (5.82)$$

$$\hat{s} \cdot \vec{\nabla} [\hat{x} \cdot \vec{A}_2(\vec{r}) - \hat{x} \cdot \vec{A}_1(\vec{r})] = \quad (5.83)$$

Boundary Conditions on Magnetic Field and Vector Potential (cont.)

Since the argument again generalizes to any Cartesian component, we may combine the three expressions to obtain

$$\hat{s} \cdot \vec{\nabla} [\vec{A}_2(\vec{r}) - \vec{A}_1(\vec{r})] = 0 \quad (5.84)$$

for any \hat{s} parallel to the interface: the tangential derivatives of \vec{A} are continuous.

Magnetic Multipoles

Derivation of Magnetic Multipole Expansion

Since the vector potential is sourced by the current distribution in a manner similar to the way the charge distributions sources the electric potential, it is natural to apply the same multipole expansion. We follow Jackson for the sake of generality and variety; you can of course read the derivation in Griffiths, too. We start with the equation for the vector potential in terms of the current distribution:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V d\tau' \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (5.85)$$

We recall Equation 3.125:

$$\frac{1}{|\vec{r} - \vec{r}'|} = \sum_{\ell=0}^{\infty} \frac{r'_<^\ell}{r'_>^{\ell+1}} P_\ell(\cos \gamma) \quad (5.86)$$

where $r_<$ and $r_>$ and the smaller and larger of r and r' .

Magnetic Multipoles (cont.)

For the sake of specificity, and because it is the case we will generally consider, we assume \vec{r} is outside the current distribution so $r > r'$ and we can take $r_< = r'$ and $r_> = r$:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V d\tau' \vec{J}(\vec{r}') \sum_{\ell=0}^{\infty} \frac{(r')^\ell}{r^{\ell+1}} P_\ell(\cos \gamma) \quad (5.87)$$

where $\cos \gamma = \hat{r} \cdot \hat{r}'$ is the angle between the two vectors.

There is common $1/r$ we can factor out, leaving

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r} \sum_{\ell=0}^{\infty} \frac{1}{r^\ell} \int_V d\tau' \vec{J}(\vec{r}) (r')^\ell P_\ell(\cos \gamma) \quad (5.88)$$

Magnetic Multipoles (cont.)

Now, consider the first term, which is just the volume integral of the current density. We can prove that it vanishes for steady-state currents. First, we use the vector identity $\vec{\nabla} \cdot (f\vec{a}) = f\vec{\nabla} \cdot \vec{a} + \vec{a}\vec{\nabla}f$ with $\vec{a} = \vec{J}$ and $f = r_i$ any of the Cartesian coordinates:

$$\vec{\nabla} \cdot (r_i \vec{J}) = r_i \vec{\nabla} \cdot \vec{J} + \vec{J} \cdot \vec{\nabla} r_i = 0 + \sum_{j=1}^3 J_j \frac{\partial}{\partial r_j} r_i = \sum_{j=1}^3 J_j \delta_{ij} = J_i \quad (5.89)$$

where the first term vanishes because the currents are steady-state and so continuity implies $\vec{\nabla} \cdot \vec{J} = 0$. With this, we can compute the integral using the divergence theorem:

$$\int_V d\tau' J_i(\vec{r}') = \int_V d\tau' \vec{\nabla}' \cdot [r'_i \vec{J}(\vec{r}')] = \oint_{S(V)} da' \hat{n}(\vec{r}') \cdot [r'_i \vec{J}(\vec{r}')] = 0 \quad (5.90)$$

where the surface integral in the last term vanishes because the current distribution is localized.

Magnetic Multipoles (cont.)

So, we are left with

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r} \sum_{\ell=1}^{\infty} \frac{1}{r^\ell} \int_V d\tau' \vec{J}(\vec{r}') (r')^\ell P_\ell(\cos \gamma) \quad (5.91)$$

This is the *multipole expansion* of the vector potential of the current distribution. As with the multipole expansion of the electric potential, one can see that the successive terms fall off as successively higher powers of $1/r$.

Magnetic Multipoles (cont.)

The Magnetic Dipole Term

Let's consider the first nonzero term in more detail, which we subscript with a 2 because it will look like the electric dipole potential:

$$\vec{A}_2(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^2} \int_V d\tau' \vec{J}(\vec{r}') r' P_\ell(\cos\gamma) \quad (5.92)$$

$$= \frac{\mu_0}{4\pi} \frac{1}{r^3} \int_V d\tau' \vec{J}(\vec{r}') \vec{r} \cdot \vec{r}' \quad (5.93)$$

We must first prove an identity. We start with the same vector identity as before, now with $f = r_i r_j$ and $\vec{a} = \vec{J}$:

$$\vec{\nabla} \cdot (r_i r_j \vec{J}) = r_i r_j \vec{\nabla} \cdot \vec{J} + \vec{J} \cdot \vec{\nabla}(r_i r_j) = 0 + r_j \vec{J} \cdot \vec{\nabla} r_i + r_i \vec{J} \cdot \vec{\nabla} r_j = r_j J_i + r_i J_j \quad (5.94)$$

where we have again used $\vec{\nabla} \cdot \vec{J} = 0$. We apply the same technique of integrating over volume and turning the left side into a surface term that vanishes, so we are left with

$$\int_V d\tau' \left[r'_i J_j(\vec{r}') + r'_j J_i(\vec{r}') \right] = 0 \quad (5.95)$$

Magnetic Multipoles (cont.)

We can use this identity to rewrite the \vec{A}_2 term as (starting first by expanding \vec{J} in terms of its components):

$$\vec{A}_2(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3} \sum_{i,j=1}^3 \hat{r}_i \int_V d\tau' J_i(\vec{r}') r_j r'_j \quad (5.96)$$

$$= \frac{\mu_0}{4\pi} \frac{1}{r^3} \sum_{i,j=1}^3 \hat{r}_i r_j \int_V d\tau' \frac{1}{2} [J_i(\vec{r}') r'_j - J_j(\vec{r}') r'_i] \quad (5.97)$$

where we split out half of the $J_i r'_j$ factor and used the identity to exchange the indices. You have learned in Ph106a and hopefully elsewhere that the cross-product can be written

$$(\vec{a} \times \vec{b})_k = \sum_{m,n=1}^3 \epsilon_{kmn} a_m b_n \quad \text{with} \quad \epsilon_{kmn} = \begin{cases} 1 & \text{for cyclic index permutations} \\ -1 & \text{for anticyclic index permutations} \\ 0 & \text{when any two indices are identical} \end{cases} \quad (5.98)$$

where ϵ_{kmn} is the *Levi-Civita* symbol.

Magnetic Multipoles (cont.)

Multiplying this definition by ϵ_{ijk} and summing over k gives

$$\sum_{k=1}^3 \epsilon_{ijk} (\vec{a} \times \vec{b})_k = \sum_{k,m,n=1}^3 \epsilon_{ijk} \epsilon_{kmn} a_m b_n = \sum_{k,m,n=1}^3 \epsilon_{kij} \epsilon_{kmn} a_m b_n \quad (5.99)$$

There is an identity for the Levi-Civita symbol

$$\sum_{k=1}^3 \epsilon_{kij} \epsilon_{kmn} = \delta_{im} \delta_{jn} - \delta_{in} \delta_{jm} \quad (5.100)$$

(this is the identity that produces the $BAC - CAB$ rule,
 $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$) which lets us rewrite the above as

$$\sum_{k=1}^3 \epsilon_{ijk} (\vec{a} \times \vec{b})_k = \sum_{m,n=1}^3 a_m b_n (\delta_{im} \delta_{jn} - \delta_{in} \delta_{jm}) = a_i b_j - a_j b_i \quad (5.101)$$

This is exactly the expression we have inside the integral above.

Magnetic Multipoles (cont.)

Using the above identity, we may rewrite the \vec{A}_2 term as

$$\vec{A}_2(\vec{r}) = \frac{\mu_0}{4\pi} \frac{1}{r^3} \sum_{i,j,k=1}^3 \hat{r}_i r_j \int_V d\tau' \frac{1}{2} \epsilon_{ijk} [J(\vec{r}') \times \vec{r}']_k \quad (5.102)$$

$$= -\frac{\mu_0}{4\pi} \frac{1}{r^3} \frac{1}{2} \sum_i^3 \hat{r}_i \left\{ \vec{r} \times \int_V d\tau' [\vec{r}' \times J(\vec{r}')] \right\}_i \quad (5.103)$$

$$= -\frac{\mu_0}{4\pi} \frac{1}{r^3} \frac{1}{2} \vec{r} \times \int_V d\tau' [\vec{r}' \times J(\vec{r}')] \quad (5.104)$$

Magnetic Multipoles (cont.)

If we define the *magnetization density* $\vec{M}(\vec{r})$ and the *magnetic dipole moment* \vec{m} by

$$\boxed{\vec{M}(\vec{r}) = \frac{1}{2} \vec{r} \times \vec{J}(\vec{r}) \quad \text{and} \quad \vec{m} = \int_V d\tau' \vec{M}(\vec{r}')} \quad (5.105)$$

then the *magnetic dipole vector potential* is

$$\boxed{\vec{A}_2(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}} \quad (5.106)$$

Interestingly, this form has the same radial dependence as that of the electrostatic potential of a dipole, but the cross-product in the numerator differs from the dot product in the numerator of the electric dipole potential. However, because the magnetic field is obtained from the curl of the vector potential, while the electric field is obtained from the gradient of the electric potential, we will see that the two forms result in the same field configuration!

Magnetic Multipoles (cont.)

Specialization to a Current Loop

Now, let us consider a current loop. The only assumption we make is that the current throughout the loop is the same so that we can extract it from the integral. The volume integral reduces to a line integral over the loop contour:

$$\vec{A}_2(\vec{r}) = -\frac{\mu_0}{4\pi} \frac{1}{r^3} \frac{1}{2} \vec{r} \times \oint_C \vec{r}' \times I d\vec{l}'(\vec{r}') = -\frac{\mu_0}{4\pi} \frac{1}{r^3} \vec{r} \times I \oint_C \frac{\vec{r}' \times d\vec{l}'(\vec{r}')}{2} \quad (5.107)$$

The integral is now just a geometric quantity that has units of area. Separating out the magnetic moment, we have

$$\boxed{\vec{A}_{2,loop}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m}_{loop} \times \vec{r}}{r^3} \quad \vec{m}_{loop} = I \oint_C \frac{\vec{r}' \times d\vec{l}'(\vec{r}')}{2}} \quad (5.108)$$

Magnetic Multipoles (cont.)

For the case of a loop confined to a plane that contains the origin, the quantity $\vec{r}' \times d\vec{\ell}/2$ is the differential area element for the loop: it is the area of the triangle formed by \vec{r}' , the vector from the origin to a point on the loop, and $d\vec{\ell}$, the line element tangent to the loop at \vec{r}' and in the direction of the current, and this cross product has the standard right-hand-rule orientation. Thus, for a planar loop, the above reduces to

$$\vec{A}_2(\vec{r}) = -\frac{\mu_0}{4\pi} \frac{1}{r^3} \vec{r} \times I \hat{n} a \quad (5.109)$$

where a is the loop area and \hat{n} is the normal to the loop with orientation defined by the current via the right-hand rule. For this case, we have

$$\boxed{\vec{A}_2(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3} \quad \vec{m}_{flat \ loop} = I \hat{n} a} \quad (5.110)$$

Magnetic Multipoles (cont.)

Field of a Magnetic Dipole

If we let $\vec{m} = m\hat{z}$, then the dipole vector potential is

$$\vec{A}_2(\vec{r}) = \frac{\mu_0}{4\pi} \frac{m \sin \theta}{r^2} \hat{\phi} \equiv A_{2,\phi} \hat{\phi} \quad (5.111)$$

This form offers some intuition about how $\vec{A}_2(\vec{r})$ behaves. In general, \vec{A}_2 “circulates” around \vec{m} using the right-hand rule in the same way that \vec{A} “circulates” around \vec{B} or \vec{B} “circulates” around \vec{J} using the right-hand rule. Since we are considering the distribution from far enough away that it is indistinguishable from a simple circular current loop in the xy -plane, the direction of \vec{A}_2 just results from the fact that \vec{A}_2 is the convolution of \vec{J} with a scalar function.

If we take the curl of this in spherical coordinates, we obtain

$$B_{2,r}(\vec{r}) = \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\sin \theta A_{2,\phi}) = 2 \frac{\mu_0}{4\pi} \frac{m \cos \theta}{r^3} \quad (5.112)$$

$$B_{2,\theta}(\vec{r}) = -\frac{1}{r} \frac{\partial}{\partial r} (r A_{2,\phi}) = \frac{\mu_0}{4\pi} \frac{m \sin \theta}{r^3} \quad (5.113)$$

$$B_{2,\phi}(\vec{r}) = 0 \quad (5.114)$$

$$\text{or } \vec{B}_2(\vec{r}) = \frac{\mu_0}{4\pi} \frac{m}{r^3} (2\hat{r} \cos \theta + \hat{\theta} \sin \theta) \quad (5.115)$$

Magnetic Multipoles (cont.)

Let's derive the more generic result by releasing the condition $\vec{m} = m\hat{z}$:

$$\vec{B}(\vec{r}) = \vec{\nabla} \times \vec{A} = \sum_{i,j,k=1}^3 \epsilon_{ijk} \hat{r}_i \frac{\partial A_k}{\partial r_j} = \frac{\mu_0}{4\pi} \sum_{i,j,k,\ell,m=1}^3 \epsilon_{ijk} \hat{r}_i \frac{\partial}{\partial r_j} \epsilon_{k\ell m} \left(\frac{m_\ell r_m}{r^3} \right) \quad (5.116)$$

$$= \frac{\mu_0}{4\pi} \sum_{i,j,k,\ell,m=1}^3 \epsilon_{ijk} \epsilon_{k\ell m} \hat{r}_i \left[\frac{m_\ell \delta_{jm}}{r^3} - \frac{3}{2} \frac{m_\ell r_m}{r^5} (2r_j) \right] \quad (5.117)$$

We use the cyclicity of the Levi-Civita symbol in its indices and the identities $\sum_{k=1}^3 \epsilon_{ijk} \epsilon_{k\ell m} = \delta_{i\ell} \delta_{jm} - \delta_{im} \delta_{j\ell}$ and $\sum_{j,k=1}^3 \epsilon_{jki} \epsilon_{jk\ell} = 2\delta_{i\ell}$ to rewrite the above in a form identical to that of the electric dipole, Equation 3.212:

$$\begin{aligned} \vec{B}(\vec{r}) &= \frac{\mu_0}{4\pi} \sum_{i=1}^3 \hat{r}_i \left[\frac{2m_i}{r^3} - \frac{3}{r^5} \left(m_i \sum_{j=1}^3 r_j r_j - r_i \sum_{j=1}^3 m_j r_j \right) \right] \\ &= \frac{\mu_0}{4\pi} \sum_{i=1}^3 \hat{r}_i \frac{3r_i (\vec{m} \cdot \vec{r}) - m_i (\vec{r} \cdot \vec{r})}{r^5} \end{aligned}$$

$$\implies \boxed{\vec{B}_2(\vec{r}) = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{r}) \hat{r} - \vec{m}}{r^3}} \quad (5.118)$$

Magnetic Multipoles

Force on a Magnetic Dipole (à la Jackson)

As we did for electric multipoles, let's consider the problem of the force and torque on a magnetic dipole. However, because there is no magnetic potential energy function, we must begin from the Lorentz force on the current distribution, which is given by

$$\vec{F}_{mag} = \int_{\mathcal{V}} d\tau \vec{J}(\vec{r}) \times \vec{B}(\vec{r}) \quad (5.119)$$

As we Taylor expanded the $1/|\vec{r} - \vec{r}'|$ expression to calculate the vector potential's multipole expansion, here we can Taylor expand $\vec{B}(\vec{r})$. For simplicity, let's place the multipole at the origin; we can generalize the result later. The expansion is

$$B_k(\vec{r}) = B_k(\vec{0}) + \sum_{m=1}^3 r_m \left. \frac{\partial B_k}{\partial r_m} \right|_{\vec{0}} + \dots \quad (5.120)$$

Magnetic Multipoles (cont.)

Thus, the Lorentz force is

$$\vec{F}_{mag} = \sum_{i,j,k=1}^3 \epsilon_{ijk} \hat{r}_i \int_V d\tau J_j(\vec{r}) B_k(\vec{r}) \quad (5.121)$$

$$= \sum_{i,j,k=1}^3 \epsilon_{ijk} \hat{r}_i \left[B_k(\vec{0}) \int_V d\tau J_j(\vec{r}) + \sum_{m=1}^3 \left(\frac{\partial B_k}{\partial r_m} \Big|_{\vec{0}} \right) \int_V d\tau J_j r_m + \dots \right] \quad (5.122)$$

We have done both these integrals before. The first one contains the monopole of the current distribution which vanishes as in Equation 5.90. Since we will see that the second term is in general nonzero and is proportional to the magnetic dipole moment, let's call it \vec{F}_{dip} and focus on it, dropping the higher-order terms. It is very similar in structure to what we encountered in calculating the dipole term in Equation 5.96.

Applying the same tricks we used there to obtain Equation 5.102, we may rewrite it as

$$\vec{F}_{dip} = \sum_{i,j,k,m,n=1}^3 \epsilon_{ijk} \hat{r}_i \left(\frac{\partial B_k}{\partial r_m} \Big|_{\vec{0}} \right) \int_V d\tau \frac{1}{2} \epsilon_{jmn} [\vec{J}(\vec{r}) \times \vec{r}]_n \quad (5.123)$$

$$= - \sum_{i,j,k,m,n=1}^3 \epsilon_{ijk} \epsilon_{jmn} \hat{r}_i \left(\frac{\partial B_k}{\partial r_m} \Big|_{\vec{0}} \right) m_n \quad \text{with} \quad \vec{m} = \frac{1}{2} \int_V d\tau [\vec{r} \times \vec{J}(\vec{r})] \quad (5.124)$$

Magnetic Multipoles (cont.)

We use the vector identity Equation 5.100, $\sum_{j=1}^3 \epsilon_{ijk} \epsilon_{jmn} = \delta_{im} \delta_{kn} - \delta_{in} \delta_{km}$, and also use $\epsilon_{ijk} = -\epsilon_{jik}$ to adjust the indices to match this expression, yielding

$$\vec{F}_{dip} = \sum_{i,k,m,n=1}^3 (\delta_{im} \delta_{kn} - \delta_{in} \delta_{km}) \hat{r}_i \left(\frac{\partial B_k}{\partial r_m} \Big|_{\vec{0}} \right) m_n \quad (5.125)$$

$$= \sum_{i,k=1}^3 \hat{r}_i \left[\left(\frac{\partial B_k}{\partial r_i} \Big|_{\vec{0}} \right) m_k - \left(\frac{\partial B_k}{\partial r_k} \Big|_{\vec{0}} \right) m_i \right] \quad (5.126)$$

$$= \vec{\nabla} (\vec{m} \cdot \vec{B}) \Big|_{\vec{0}} - \vec{m} (\vec{\nabla} \cdot \vec{B}) \Big|_{\vec{0}} \quad (5.127)$$

The second term vanishes. Generalizing the first term to a dipole at an arbitrary position, we have

$$\boxed{\vec{F}_{dip} = \vec{\nabla} [\vec{m} \cdot \vec{B}(\vec{r})] \quad \text{with} \quad \vec{m} = \frac{1}{2} \int_V d\tau [\vec{r} \times \vec{J}(\vec{r})]} \quad (5.128)$$

The force causes the magnetic dipole to move to a local extremum of $\vec{m} \cdot \vec{B}$. Note how it is identical to the force on an electric dipole in an electric field.

Magnetic Multipoles (cont.)

Torque on a Magnetic Dipole (à la Jackson)

We may obtain from the Lorentz force Law on a current distribution the corresponding torque:

$$\vec{N}_{mag} = \int_V d\tau \vec{r} \times [\vec{J}(\vec{r}) \times \vec{B}(\vec{r})] \quad (5.129)$$

where we have just added up the torque volume element by volume element in the same way we summed the force. When we Taylor expand the magnetic field, we have

$$\vec{N}_{mag} = \int_V d\tau \vec{r} \times [\vec{J}(\vec{r}) \times \vec{B}(\vec{0})] + \dots \quad (5.130)$$

Because of the $\vec{r} \times$ inside the integrand, the zeroth-order term no longer vanishes and so we do not need to consider the next order term in the Taylor expansion. We will write the zeroth-order term as \vec{N}_{dip} below for reasons that will become clear.

Magnetic Multipoles (cont.)

To get the above expression into a familiar form, we need to repeat the same kinds of vector arithmetic tricks we have used before. First, we apply the *BAC – CAB* rule, $\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a} \cdot \vec{c}) - \vec{c}(\vec{a} \cdot \vec{b})$, which we can do without having to write things in terms of indices because there are no derivatives floating around:

$$\vec{N}_{dip} = \int_V d\tau \vec{r} \times [\vec{J}(\vec{r}) \times \vec{B}(0)] = \int_V d\tau \vec{J}(\vec{r}) [\vec{r} \cdot \vec{B}(0)] - \int_V d\tau \vec{B}(0) [\vec{r} \cdot \vec{J}(\vec{r})] \quad (5.131)$$

We can make the second term vanish by the same kinds of tricks we used earlier during the vector potential multipole expansion:

$$\vec{r} \cdot \vec{J}(\vec{r}) = [r \vec{\nabla} r] \cdot \vec{J}(\vec{r}) = \frac{1}{2} [\vec{\nabla} r^2] \cdot \vec{J}(\vec{r}) = \frac{1}{2} \left\{ \vec{\nabla} \cdot [r^2 \vec{J}(\vec{r})] - r^2 \vec{\nabla} \cdot \vec{J}(\vec{r}) \right\} \quad (5.132)$$

The second term vanishes for steady-state currents, and the first term can be turned into a surface integral with integrand $r^2 \vec{J}(\vec{r})$. Since we are considering a localized current distribution, the surface can be taken far enough out that the $\vec{J}(\vec{r})$ vanishes on the surface.

Magnetic Multipoles (cont.)

The first term looks again like the expression we have encountered at Equation 5.96, which becomes apparent when we write it out in component form:

$$\vec{N}_{dip} = \sum_{i,j=1}^3 \hat{r}_i B_j(\vec{0}) \int_V d\tau J_i(\vec{r}) r_j \quad (5.133)$$

We again apply the same tricks used to arrive at Equation 5.102:

$$\vec{N}_{dip} = \sum_{i,j=1}^3 \hat{r}_i B_j(\vec{0}) \int_V d\tau \frac{1}{2} \epsilon_{ijk} [\vec{J}(\vec{r}) \times \vec{r}]_k = -\frac{1}{2} \vec{B}(\vec{0}) \times \int_V d\tau \vec{r} \times \vec{J}(\vec{r}) \quad (5.134)$$

$$= -\vec{B}(\vec{0}) \times \vec{m} \quad \text{with} \quad \vec{m} = \frac{1}{2} \int_V d\tau [\vec{r} \times \vec{J}(\vec{r})] \quad (5.135)$$

Generalizing to a multipole distribution centered on an arbitrary point, the zeroth-order term in the torque is (and hence the dip subscript)

$$\boxed{\vec{N}_{dip} = \vec{m} \times \vec{B}(\vec{r}) \quad \text{with} \quad \vec{m} = \frac{1}{2} \int_V d\tau [\vec{r} \times \vec{J}(\vec{r})]} \quad (5.136)$$

The magnetic dipole feels a torque that tends to align it with the magnetic field (the torque vanishes when \vec{m} is aligned with \vec{B}), again like the situation for an electric dipole in an electric field.

Magnetic Multipoles (cont.)

Potential Energy of a Magnetic Dipole

We can do the line integral of the force or the angular integral of the torque to determine that we can write a potential energy

$$U(\vec{r}) = -\vec{m} \cdot \vec{B}(\vec{r}) \quad (5.137)$$

This form for the potential energy expresses two features of magnetic dipoles: they like to be aligned with the local magnetic field, and they seek the region of largest $\vec{m} \cdot \vec{B}$.

The thing that should be concerning about this expression is that we argued earlier that magnetic fields can do no work, yet here we have the possibility of such work. The resolution will come when we study electrodynamics — *i.e.*, the effect of changing electric and magnetic fields. We will see that something must be doing work to keep the current flowing to maintain \vec{m} , and that is the object providing the energy in this situation.

Section 6

Magnetostatics in Matter

Lecture 13:
Magnetostatics in Matter:
Field of a Magnetized Object,
Auxiliary Field \vec{H} ,
Magnetic Permeability in Linear Systems

Date Revised: 2014/04/10 08:00

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no other changes)

Date Given: 2014/04/10

Paramagnetism and Diamagnetism

See Griffiths Sections 6.1.1 and 6.1.3 and Purcell Sections 11.1 and 11.5 for discussions of paramagnetism and diamagnetism. This will be discussed in class briefly, but there is little to add to their discussions.

The Field of a Magnetized Object

Bound Currents

Suppose we have an object with a position dependent macroscopic density of magnetic moments, or *macroscopic magnetization density* $\vec{M}(\vec{r})$, where the magnetic moment of an infinitesimal volume $d\tau$ is

$$d\vec{m} = \vec{M}(\vec{r}) d\tau \quad (6.1)$$

\vec{M} is not to be confused with the magnetization density $M(\vec{r})$; the latter can be for some arbitrary current distribution, while the former is specifically to be considered to be a density of magnetic dipole moments. The latter should give the former for this special case of pure dipoles. The contribution to the vector potential at \vec{r} due to the above infinitesimal volume at \vec{r}' is therefore

$$d\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{d\vec{m}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} = \frac{\mu_0}{4\pi} \frac{d\tau' \vec{M}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \quad (6.2)$$

Integrating over the volume containing the magnetization density, we have

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V d\tau' \frac{\vec{M}(\vec{r}') \times (\vec{r} - \vec{r}')}{|\vec{r} - \vec{r}'|^3} \quad (6.3)$$

The Field of a Magnetized Object (cont.)

Now, we use $(\vec{r} - \vec{r}')/|\vec{r} - \vec{r}'|^3 = \vec{\nabla}_{\vec{r}'} |\vec{r} - \vec{r}'|^{-1}$ (note that the gradient is with respect to \vec{r}' , not \vec{r}), which allows us to apply the product rule for curl,
 $\vec{\nabla} \times (f \vec{a}) = f \vec{\nabla} \times \vec{a} - \vec{a} \times \vec{\nabla} f$:

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} d\tau' \vec{M}(\vec{r}') \times \vec{\nabla}_{\vec{r}'} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \quad (6.4)$$

$$= \frac{\mu_0}{4\pi} \int_{\mathcal{V}} d\tau' \frac{\vec{\nabla}_{\vec{r}'} \times \vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} - \frac{\mu_0}{4\pi} \int_{\mathcal{V}} d\tau' \vec{\nabla}_{\vec{r}'} \times \left(\frac{\vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} \right) \quad (6.5)$$

$$= \frac{\mu_0}{4\pi} \int_{\mathcal{V}} d\tau' \frac{\vec{\nabla}_{\vec{r}'} \times \vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} + \frac{\mu_0}{4\pi} \int_{S(\mathcal{V})} da' \frac{\vec{M}(\vec{r}') \times \hat{n}(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (6.6)$$

where, in the last step, we have used a vector identity that we will prove on the following slide.

The Field of a Magnetized Object (cont.)

Let's prove the vector identity we just used. Let $\vec{a}(\vec{r})$ be an arbitrary vector field and let \vec{c} be an arbitrary constant vector. Then the divergence theorem tells us

$$\int_{\mathcal{V}} d\tau \vec{\nabla} \cdot [\vec{a}(\vec{r}) \times \vec{c}] = \oint_{S(\mathcal{V})} da \hat{n} \cdot [\vec{a}(\vec{r}) \times \vec{c}] \quad (6.7)$$

Now, apply the cyclicity of triple vector products (using the fact that \vec{c} is constant and can thus be exchanged with $\vec{\nabla}$) and bring \vec{c} outside the integrals (since it is a constant vector):

$$\vec{c} \cdot \int_{\mathcal{V}} d\tau [\vec{\nabla} \times \vec{a}(\vec{r})] = \vec{c} \cdot \oint_{S(\mathcal{V})} da [\hat{n} \times \vec{a}(\vec{r})] \quad (6.8)$$

Since \vec{c} is arbitrary, the expression must hold with \vec{c} :

$$\int_{\mathcal{V}} d\tau [\vec{\nabla} \times \vec{a}(\vec{r})] = \oint_{S(\mathcal{V})} da [\hat{n} \times \vec{a}(\vec{r})] \quad (6.9)$$

which is what we wanted to prove.

The Field of a Magnetized Object (cont.)

Making some definitions, we recognize that the vector potential can be considered to be sourced by a *bound volume current density* $\vec{J}_b(\vec{r})$ and a *bound surface current density* $\vec{K}_b(\vec{r})$:

$$\boxed{\vec{J}_b(\vec{r}) = \vec{\nabla} \times \vec{M}(\vec{r}) \quad \vec{K}_b(\vec{r}) = \vec{M}(\vec{r}) \times \hat{n}(\vec{r})} \quad (6.10)$$

$$\boxed{\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_V d\tau' \frac{\vec{J}_b(\vec{r}')}{|\vec{r} - \vec{r}'|} + \frac{\mu_0}{4\pi} \oint_{S(V)} da' \frac{\vec{K}_b(\vec{r}')}{|\vec{r} - \vec{r}'|}} \quad (6.11)$$

The way in which these current densities source \vec{A} is identical to the way in which free current densities do so. Moreover, we can see the clear analogy to bound volume and surface charges in the case of polarized materials.

The Field of a Magnetized Object (cont.)

Physical Interpretation of Bound Currents

Griffiths Section 6.2.2 gives a nice discussion of this that will be presented in class, but there is not much to add here.

The Field of a Magnetized Object (cont.)

Example: Uniformly Magnetized Sphere

Center the sphere of radius R at the origin. Let $\vec{M} = M\hat{z}$. Then

$$\vec{J}_b(\vec{r}) = \vec{\nabla} \times M\hat{z} = 0 \quad \vec{K}_b(\vec{r}) = M\hat{z} \times \hat{n} = M\hat{z} \times \hat{r} = M \sin \theta \hat{\phi} \quad (6.12)$$

We need to calculate

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} R^2 \int_0^{2\pi} d\phi' \int_0^\pi d\theta' \sin \theta' \frac{M \sin \theta' \hat{\phi}}{|\vec{r} - \vec{r}'|} \quad (6.13)$$

$$= \frac{\mu_0}{4\pi} R^2 \int_0^{2\pi} d\phi' \int_0^\pi d\theta' \sin \theta' \frac{M \sin \theta' (-\hat{x} \sin \phi' + \hat{y} \cos \phi')}{|\vec{r} - \vec{r}'|} \quad (6.14)$$

(The R^2 out front is because an area integral, not just a solid angle integral, needs to be done.) This is done in Griffiths Example 5.11 via explicit integration. For the sake of variety, let's use a different technique. We use Equation 3.165, which expands $|\vec{r} - \vec{r}'|^{-1}$ in terms of spherical harmonics:

$$\frac{1}{|\vec{r} - \vec{r}'|} = 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_{<}^\ell}{r_{>}^{\ell+1}} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi) \quad (6.15)$$

The Field of a Magnetized Object (cont.)

Let's consider one piece of the above angular integral; the other term will be similar in spirit:

$$\int_0^{2\pi} d\phi' \int_0^\pi d\theta' \sin \theta' \frac{\sin \theta' (-\sin \phi')}{|\vec{r} - \vec{r}'|} \quad (6.16)$$

Let's write the numerator in terms of spherical harmonics and use the expansion; we abbreviate $\int_0^{2\pi} d\phi' \int_0^\pi d\theta' \sin \theta' = \int d\Omega'$ and recall $Y_{\ell,-m} = (-1)^m Y_{\ell,m}^*$

$$\int d\Omega' \sqrt{\frac{8\pi}{3}} \frac{Y_{1,1}(\theta', \phi') + Y_{1,-1}(\theta', \phi')}{2i} 4\pi \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{1}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y_{\ell m}^*(\theta', \phi') Y_{\ell m}(\theta, \phi) \quad (6.17)$$

The integral over Ω' gives $\delta_{\ell,1}\delta_{m,1}$ and $\delta_{\ell,1}\delta_{m,-1}$, eliminating the sum and yielding

$$\frac{4\pi}{2i} \sqrt{\frac{8\pi}{3}} \frac{1}{3} \frac{r_<^\ell}{r_>^{\ell+1}} [Y_{1,1}(\theta, \phi) + Y_{1,-1}(\theta, \phi)] = -\frac{4\pi}{3} \frac{r_<}{r_>^2} \sin \theta \sin \phi \quad (6.18)$$

where the $1/3$ came from $1/(2\ell+1)$.

The Field of a Magnetized Object (cont.)

We can repeat the same kind of manipulation for the \hat{y} term, yielding

$$\int_0^{2\pi} d\phi' \int_0^\pi d\theta' \sin\theta' \frac{\sin\theta' (\cos\phi')}{|\vec{r} - \vec{r}'|} = \frac{4\pi}{3} \frac{r_{<}}{r_{>}^2} \sin\theta \cos\phi \quad (6.19)$$

Therefore,

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} R^2 M \frac{4\pi}{3} \frac{r_{<}}{r_{>}^2} \sin\theta \hat{\phi} = \begin{cases} \frac{\mu_0}{3} M r \sin\theta \hat{\phi} & r \leq R \\ \frac{\mu_0}{4\pi} \left(\frac{4\pi}{3} R^3 M\right) \frac{\sin\theta}{r^2} \hat{\phi} & r \geq R \end{cases} \quad (6.20)$$

Note that $\vec{A}(\vec{r})$ is continuous at $r = R$. Evaluating the curl of the first term to obtain the magnetic field, we have inside the sphere

$$\vec{B}(r \leq R) = \vec{\nabla} \times \vec{A}(r \leq R) = \frac{2}{3} \mu_0 M \left[\hat{r} \cos\theta + \hat{\theta} \sin\theta \right] = \frac{2}{3} \mu_0 \vec{M} \quad (6.21)$$

which is a uniform field pointing in the *same* direction as the magnetization.

The Field of a Magnetized Object (cont.)

For $r \geq R$, we have

$$\vec{A}(r \geq R) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} \quad \vec{m} = \frac{4\pi}{3} R^3 \vec{M} \quad (6.22)$$

which is the vector potential (thus yielding the field of) a pure dipole with magnetic moment given by integrating the uniform magnetization density over the sphere.

Let's compare to the case of a uniformly polarized dielectric medium:

$$r \leq R \quad \vec{E}(\vec{r}) = -\frac{1}{3\epsilon_0} \vec{P} \quad \vec{B}(\vec{r}) = \frac{2}{3} \mu_0 \vec{M} \quad (6.23)$$

$$r \geq R \quad V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \frac{\vec{p} \cdot \hat{r}}{r^2} \quad \vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2} \quad (6.24)$$

$$\vec{p} = \frac{4\pi}{3} R^3 \vec{P} \quad \vec{m} = \frac{4\pi}{3} R^3 \vec{M} \quad (6.25)$$

Inside the sphere, the difference is a factor of -2 and the exchange of $1/\epsilon_0$ for μ_0 .

Outside the sphere, the two potentials result in fields identical up to the replacement of \vec{P} by \vec{M} and again $1/\epsilon_0$ by μ_0 . The difference in the $r \leq R$ expressions reflects the fact that the magnetic field of the surface current is aligned with \vec{M} while the electric field of the surface charge density is opposite to \vec{P} .

The Auxiliary Field \vec{H} and Magnetic Permeability

Definition of the Auxiliary Field

We saw that \vec{A} is sourced by the bound current density $\vec{J}_b = \vec{\nabla} \times \vec{M}$ in the same way it would be sourced by a free current density \vec{J}_f . Therefore, Ampere's Law is satisfied with the sum of the two currents:

$$\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} = \vec{J}_f + \vec{J}_b = \vec{J}_f + \vec{\nabla} \times \vec{M} \quad (6.26)$$

If we want to write an Ampere's Law in terms of the free currents only, in the same way that we wanted to write Gauss's Law in terms of the free charges only, then we can define the *auxiliary field*

$$\boxed{\vec{H} \equiv \frac{\vec{B}}{\mu_0} - \vec{M}} \quad (6.27)$$

which then satisfies

$$\vec{\nabla} \times \vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \vec{\nabla} \times \vec{M} = \vec{J}_f + \vec{J}_b - \vec{J}_b = \vec{J}_f \quad (6.28)$$

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

Therefore, we have a modified Ampere's Law

$$\vec{\nabla} \times \vec{H} = \vec{J}_f \iff \oint_C d\vec{\ell} \cdot \vec{H}(\vec{r}) = \int_{S(C)} da \hat{n}(\vec{r}) \cdot \vec{J}_f(\vec{r}) = I_{f,enc} \quad (6.29)$$

Thus, as intended, we have an Ampere's Law in terms of the free currents only, which (partially) source \vec{H} . The fact that \vec{H} satisfies Ampere's Law in the free current leads some to use the name *applied field* for it. That may be misleading, though, because the free current does not tell one everything one must know to determine \vec{H} .

To fully specify \vec{H} , we need to know its divergence, which is given by applying $\vec{\nabla} \cdot \vec{B} = 0$:

$$\vec{\nabla} \cdot \vec{H} = -\vec{\nabla} \cdot \vec{M} \quad (6.30)$$

This nonvanishing of $\vec{\nabla} \cdot \vec{H}$ is analogous to the nonvanishing of $\vec{\nabla} \times \vec{D}$ in electrostatics.

There is a not-so-useful example of how to calculate \vec{H} using the above Ampere's Law in Griffiths Example 6.2.

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

What Sources \vec{H} ? When Does It Vanish?

Considering the uniformly magnetized sphere example we just looked at, we see

$$\vec{H}(r \leq R) = \frac{\vec{B}(r \leq R)}{\mu_0} - \vec{M} = \frac{2}{3}\vec{M} - \vec{M} = -\frac{1}{3}\vec{M} \quad (6.31)$$

$$\vec{H}(r \geq R) = \frac{\vec{B}(r \geq R)}{\mu_0} = \frac{\text{field of the magnetic dipole } \vec{m} = \frac{4\pi}{3}R^3\vec{M}}{\mu_0} \quad (6.32)$$

This example highlights the importance of the nonvanishing of $\vec{\nabla} \cdot \vec{H}$. There is no free current in this problem, so one might be inclined to think \vec{H} vanishes by analogy to the fact \vec{B} would vanish if there were no total current. But the nonzero nature of $\vec{\nabla} \cdot \vec{H}$ means that \vec{H} has another sourcing term that is not captured by Ampere's Law alone. This is analogous to the way that, even if there is no free charge, there may be a displacement field \vec{D} sourced by $\vec{\nabla} \times \vec{P}$. To have \vec{H} vanish identically, one needs to have $\vec{\nabla} \cdot \vec{M} = 0$ and also appropriate boundary conditions.

This all makes sense given the Helmholtz theorem: since $\vec{\nabla} \cdot \vec{H}$ does not vanish, \vec{H} is not just the curl of a vector potential, but must be the sum of the gradient of a scalar potential and the curl of a vector potential. Ampere's Law for \vec{H} tells us that the free current density sources the vector potential, while $\vec{\nabla} \cdot \vec{M}$ sources the scalar potential. We will see later that the latter point allows us to use our electrostatic boundary value problem techniques.

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

Who Cares About \vec{H} ?

Is \vec{H} any more useful than \vec{D} was?

The thing that limited the utility of \vec{D} is that, in practice, one rarely controls free charge, the thing that sources \vec{D} . In practice, one sets potentials using batteries or other voltage sources. Potentials specify \vec{E} , not \vec{D} .

On the other hand, \vec{H} is sourced by the free currents, which is the thing one explicitly controls in the lab. For that reason alone, we expect \vec{H} is of greater utility than \vec{D} . We will see this more clearly when we consider specific types of permeable materials.

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

Boundary Conditions on \vec{H}

From the boundary conditions on \vec{B} at an interface, we can derive boundary conditions on \vec{H} . The continuity of the normal component of the magnetic field (Equation 5.65) along with Equation 6.27 implies

$$\hat{n}(\vec{r}) \cdot [\vec{H}_2(\vec{r}) - \vec{H}_1(\vec{r})] = -\hat{n}(\vec{r}) \cdot [\vec{M}_2(\vec{r}) - \vec{M}_1(\vec{r})] \quad (6.33)$$

Applying the same arguments using Ampere's Law for \vec{H} as we did using Ampere's Law for \vec{B} , we can also conclude the analogy of Equation 5.71:

$$[\vec{H}_2(\vec{r}) - \vec{H}_1(\vec{r})] \cdot \hat{s}(\vec{r}) = [\vec{K}_f(\vec{r}) \times \hat{n}(\vec{r})] \cdot \hat{s}(\vec{r}) \quad (6.34)$$

where \vec{K}_f is the free surface current density at the interface.

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

We can manipulate the latter expression to obtain a different form more useful when \vec{H} is given and one wants to find \vec{K}_f . Let's use $\hat{s} = \hat{t} \times \hat{n}$ and then use the cyclicity of the triple vector product on both sides:

$$[\vec{H}_2 - \vec{H}_1] \cdot [\hat{t} \times \hat{n}] = [\hat{n} \times \hat{s}] \cdot \vec{K}_f \quad (6.35)$$

$$\hat{t} \cdot (\hat{n} \times [\vec{H}_2 - \vec{H}_1]) = \hat{t} \cdot \vec{K}_f \quad (6.36)$$

The same equation holds trivially with \hat{t} replaced by \hat{n} : the left side vanishes because \hat{n} is perpendicular to any cross product involving \hat{n} and the right side vanishes because \vec{K}_f is always perpendicular to \hat{n} . This, combined with the fact that \hat{t} in the above can be any vector in the plane of the boundary, implies the more general statement

$$\boxed{\hat{n}(\vec{r}) \times [\vec{H}_2(\vec{r}) - \vec{H}_1(\vec{r})] = \vec{K}_f(\vec{r})} \quad (6.37)$$

But note that this equation provides no information about the normal component of \vec{H} .

The Auxiliary Field \vec{H} and Magnetic Permeability

Magnetic Permeability in Linear Materials

Many magnetic materials we will consider have a linear relationship between the field and the magnetization. The *magnetic susceptibility* of a material is defined to be the constant of proportionality between \vec{M} and \vec{H} :

$$\vec{M} = \chi_m \vec{H} \quad (6.38)$$

Since $\vec{B} = \mu_0 (\vec{H} + \vec{M})$, we have

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) = \mu_0 (1 + \chi_m) \vec{H} \equiv \mu \vec{H} \quad (6.39)$$

where we have defined the *magnetic permeability* $\mu = \mu_0(1 + \chi_m)$. The definition of χ_m and μ follows a different convention than the definition of χ_e and ϵ . One natural rationale is the same as we discussed above: we experimentally control the free current and thus \vec{H} , whereas in electrostatics we control the voltages and thus \vec{E} . We define the permittivity and the permeability to be the ratio of the thing we do not control to the thing we do control.

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

Paramagnetic materials have $\chi_m > 0$ because the magnetization is in the same direction as the field and so the field due to the free currents is added to by the field from the magnetization.

Diamagnetic materials have $\chi_m < 0$ because the magnetization is in the direction opposite the field and so the field due to the free currents is reduced by the field from the magnetization.

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

Boundary Conditions for Linear Magnetic Materials

With the linear relationship between \vec{H} , \vec{M} , and \vec{B} , we can rewrite the boundary conditions we derived earlier in a somewhat simpler form.

The continuity of the normal component of \vec{B} implies

$$\hat{n}(\vec{r}) \cdot [\mu_1 \vec{H}_1(\vec{r}) - \mu_2 \vec{H}_2(\vec{r})] = 0 \quad (6.40)$$

$$\hat{n}(\vec{r}) \cdot \left[\frac{\mu_1}{\chi_{m,1}} \vec{M}_1(\vec{r}) - \frac{\mu_2}{\chi_{m,2}} \vec{M}_2(\vec{r}) \right] = 0 \quad (6.41)$$

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

We saw earlier that the tangential component of \vec{H} changes by the free surface current density (Equations 6.34 and 6.37). That implies

$$\left[\frac{\vec{B}_2(\vec{r})}{\mu_2} - \frac{\vec{B}_1(\vec{r})}{\mu_1} \right] \cdot \hat{s}(\vec{r}) = [\vec{K}_f(\vec{r}) \times \hat{n}(\vec{r})] \cdot \hat{s}(\vec{r}) \quad (6.42)$$

or

$$\hat{n}(\vec{r}) \times \left[\frac{\vec{B}_2(\vec{r})}{\mu_2} - \frac{\vec{B}_1(\vec{r})}{\mu_1} \right] = \vec{K}_f(\vec{r}) \quad (6.43)$$

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

and

$$\left[\frac{\vec{M}_2(\vec{r})}{\chi_{m,2}} - \frac{\vec{M}_1(\vec{r})}{\chi_{m,1}} \right] \cdot \hat{s}(\vec{r}) = [\vec{K}_f(\vec{r}) \times \hat{n}(\vec{r})] \cdot \hat{s}(\vec{r}) \quad (6.44)$$

or

$$\hat{n}(\vec{r}) \times \left[\frac{\vec{M}_2(\vec{r})}{\chi_{m,2}} - \frac{\vec{M}_1(\vec{r})}{\chi_{m,1}} \right] = \vec{K}_f(\vec{r}) \quad (6.45)$$

Vanishing of K_f will of course simplify these expressions, yielding the continuity of the tangential component of \vec{B}/μ and \vec{M}/χ_m .

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

Example: Magnetizable Rod with Uniform Current

Let's consider a rod of radius R whose axis is in the z direction and which carries a current I distributed uniformly across its cross section. Assume the material is linear with magnetic susceptibility χ_m . Let's find \vec{H} , \vec{M} , and \vec{B} .

Let's first see how far we can get without using χ_m . Ampere's Law for \vec{H} tells us

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_{S(C)} da \hat{n} \cdot \vec{J}_f \quad (6.46)$$

This system has azimuthal symmetry as well as translational symmetry in z , so we can guess $\vec{H} = \vec{H}(s)$ where s is the radial coordinate in cylindrical coordinates. By the right-hand rule and the z translational symmetry, we expect $\vec{H} = H(s) \hat{\phi}$. Also, we know $\vec{J}_f = \hat{z} I / \pi R^2$. Therefore, Ampere's Law tells us

$$s \leq R : \quad 2\pi s H(s) = \pi s^2 \frac{I}{\pi R^2} \iff \vec{H}(s) = \frac{I}{2\pi s} \frac{s^2}{R^2} \hat{\phi} \quad (6.47)$$

$$s \geq R : \quad 2\pi s H(s) = \pi R^2 \frac{I}{\pi R^2} \iff \vec{H}(s) = \frac{I}{2\pi s} \hat{\phi} \quad (6.48)$$

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

If we do not know χ_m , we do not know \vec{M} inside the material and so we cannot calculate \vec{B} for $s \leq R$. For $s \geq R$, we have vacuum and so $\vec{M} = 0$ and $\vec{B} = \mu_0 \vec{H}$, which implies

$$\vec{B}(s \geq R) = \mu_0 \vec{H}(s \geq R) = \frac{\mu_0 I}{2\pi s} \hat{\phi} \quad (6.49)$$

Now, if we assume the material is linear with susceptibility χ_m , we have

$$\vec{M}(s \leq R) = \chi_m \vec{H}(s \leq R) = \chi_m \frac{I}{2\pi s} \frac{s^2}{R^2} \hat{\phi} = \frac{\mu - \mu_0}{\mu_0} \frac{I}{2\pi s} \frac{s^2}{R^2} \hat{\phi} \quad (6.50)$$

and therefore

$$\vec{B}(s \leq R) = \mu \vec{H}(s \leq R) = \frac{\mu I}{2\pi s} \frac{s^2}{R^2} \hat{\phi} \quad (6.51)$$

All three fields are azimuthal inside and outside R . For paramagnetic materials, $\chi_m \geq 0$ ($\mu \geq \mu_0$), so \vec{M} is parallel to \vec{H} and $|\vec{B}| > \mu_0 |\vec{H}|$ inside R . For diamagnetic materials, $\chi_m < 0$ ($\mu \leq \mu_0$), so \vec{M} is antiparallel to \vec{H} and $|\vec{B}| \leq \mu_0 |\vec{H}|$ inside R .

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

Let's check the boundary conditions. All the fields are tangential at the boundary, so the normal conditions — continuity of the normal components of \vec{B} , $\mu \vec{H}$, and $\mu \vec{M}/\chi_m$ — are trivially satisfied. There is no free surface current density, so we expect the tangential components of \vec{H} , \vec{B}/μ , and \vec{M}/χ_m to be continuous. We see that indeed holds, with them taking on the values

$$\hat{\phi} \cdot \vec{H}(s = R) = \hat{\phi} \cdot \frac{\vec{B}(s = R)}{\mu} = \hat{\phi} \cdot \frac{\vec{M}(s = R)}{\chi_m} = \frac{I}{2\pi R} \quad (6.52)$$

The last one is a bit tricky because both the numerator \vec{M} and the denominator χ_m vanish for $s > R$, but L'Hopital's rule allows evaluation of the ratio in the limit $\chi_m \rightarrow 0$. The \hat{z} tangential components are trivially continuous since they all vanish.

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

For the sake of completeness, let's calculate the *bound* surface current and check that the boundary conditions on \vec{B} are correct. The bound surface current is $\vec{K}_b = \vec{M} \times \hat{n}$ (Equation 6.10). In this case, $\hat{n} = \hat{s}$, the radial unit vector in cylindrical coordinates, so

$$\vec{K}_b(s = R) = M(s = R) \hat{\phi} \times \hat{s} = -\chi_m \frac{I}{2\pi R} \hat{z} \quad (6.53)$$

For a paramagnetic materials ($\chi_m > 0$), the surface current points along $-\hat{z}$ while, for diamagnetic materials ($\chi_m < 0$), it points along $+\hat{z}$. One can see this physically by considering the direction of alignment of the dipoles and which direction the uncancelled current on the boundary flows. From the direction of this surface current, one can then see that the field of this surface current adds to the field of the free current for the paramagnetic case and partially cancels it for the diamagnetic case. Finally, let's check the boundary conditions on \vec{B} . It has no normal component in either region, so continuity of the normal component is trivially satisfied. The discontinuity in the tangential component matches (Equation 5.73):

$$\hat{n} \times [\vec{B}_2 - \vec{B}_1] = \hat{s} \times [\mu_0 - \mu] \frac{I}{2\pi R} \hat{\phi} = -\mu_0 \chi_m \frac{I}{2\pi R} \hat{z} = \mu_0 \vec{K}_b \quad (6.54)$$

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

Let's also calculate the bound volume current density, $\vec{J}_b = \vec{\nabla} \times \vec{M}$ from Equation 6.10. It is

$$\vec{J}_b(\vec{r}) = \vec{\nabla} \times \vec{M} = \chi_m \vec{\nabla} \times \vec{H} = \chi_m \vec{J}_f = \chi_m \frac{I}{\pi R^2} \hat{z} \quad (6.55)$$

For paramagnetic materials, \vec{J}_b is parallel to \vec{J}_f and thus its field adds to the field of the free current, while, for diamagnetic materials, it is antiparallel and it partially cancels the free current's field.

A modest extension to this problem would be to include a free surface current in the \hat{z} direction, which would then cause a discontinuity in the $\hat{\phi}$ component of \vec{H} , \vec{B}/μ and \vec{M}/χ_m .

Lecture 14:
Magnetostatics in Matter:
Nonlinear Materials and Ferromagnetism
Boundary Value Problems in Magnetostatics

Date Revised: 2014/04/17 09:00

Correct minus sign error in Equation 6.67 for definition of ρ_m

Correct algebra error in 6.115 on permeable spherical shell

Expanded on material on permeable spherical shell

Date Given: 2014/04/15

The Auxiliary Field \vec{H} and Magnetic Permeability

Nonlinear Materials and Ferromagnetism

There are materials that are *nonlinear*. In such materials, in addition to the tendency of magnetic dipoles due to unpaired electrons to align with the applied magnetic field, these dipoles interact with each other in such a way to prefer this alignment too. This extra preference for magnetization causes the magnetization to depend nonlinearly on \vec{H} .

Beyond nonlinearity, there is the phenomenon of *ferromagnetism*, in which there are additional interactions that cause the magnetization to be preserved even after the applied field is reduced.

Both phenomena are caused by unpaired electrons as paramagnetism is; one might have guessed this by the fact that all three phenomena involve the alignment of magnetic dipoles with the applied field. The additional dipole-dipole interaction that causes nonlinearity is due to the *exchange effect* in quantum mechanics.

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

The basic idea of the effect is as follows. As you likely know, electrons in atoms occupy *shells* corresponding to different energies for the electron-nucleus Coulomb interaction. Only a certain number of states are allowed for each shell (n^2 for shell n), and electrons can be put in a shell with spin “up” or spin “down” (multiplying by 2 the number of allowed states). When a shell is partially filled, the electrons prefer to be *unpaired*, meaning that they have different orbital wavefunctions (probability distributions) and *the same spin direction* (i.e., *aligned spins*) rather than the same orbital wavefunction and different spin directions. (Having both the same is prevented by the *Pauli exclusion principle*.) The reason for this preference against pairing is that the electrostatic repulsive energy of two electrons in the same orbital state is high: in quantum mechanics, that energy is determined by the integral of the product of their wavefunctions weighted by $1/|\vec{r} - \vec{r}'|$ where \vec{r} and \vec{r}' are their positions, so the less similar their wavefunctions are, the lower the (positive) electrostatic repulsive energy is. We thus have a mechanism for alignment of unpaired electrons in a single atom.

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

In addition, though, one needs a mechanism for unpaired electrons in nearby atoms to align with each other; alignment of the unpaired electrons in a single atom is not enough. A similar exchange interaction is required, of which there are many types that depend on the details of the material and how the electrons in nearby atoms interact. The key requirement for such exchange effects to occur, though, is delocalization — the electron wavefunctions must be large enough that they spread to nearby atoms — so that there can be exchange interactions between electrons in adjacent atoms. This explains why nonlinearity occurs only in atoms with d - and f -shell electrons — the electrons in these orbitals are more weakly bound than s - and p -shell electrons, allowing the delocalization necessary.

So far, what we have talked about leads to a nonlinear magnetic permeability, where the exchange interactions cause the magnetic dipoles to prefer to align with each other when they have been pushed into alignment by an applied field. This would yield a relationship of the form $\vec{B} = \vec{F}(\vec{H})$, where the relation between the two cannot be explained by a simple constant of proportionality, but the relation is at least well-defined.

Ferromagnetism requires yet one more effect, which is some sort of interaction between the magnetic dipoles and crystal defects that causes the magnetic dipole orientation to remain *pinned* in a particular direction even when the applied field is reduced to zero.

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

Ferromagnets have domains, which are regions of coaligned magnetic dipoles. By default, these domains are macroscopic in size (fractions of a mm, perhaps), but they tend to be unaligned with each other because alignment would create a large magnetic field outside the material, which is not a low-energy state (which we will see when we talk about magnetic energy). When a large field is applied, though, the domains tend to align and the pinning energy is large enough to overcome the preference for no net magnetic field.

This business of pinning leads ferromagnetic materials to be *hysteretic*, meaning that not only is \vec{B} a nonlinear function of \vec{H} , but, in addition, \vec{B} depends on the history of \vec{H} . Hysteresis curves are shown in Griffiths Figures 6.28 and 6.29. Suppose the applied field \vec{H} is very large, leading to large \vec{B} . Suppose, in fact, that \vec{H} is so large that all the dipoles are ordered and they cannot further order; \vec{B} saturates and can get no larger. ($|\vec{B}| \gg \mu_0 |\vec{H}|$ in such cases, so continuing to increase $|\vec{H}|$ has a negligible effect on $|\vec{B}|$.) Then, as \vec{H} is reduced to zero, \vec{B} stays large because of the pinning effect. In fact, to reduce \vec{B} to zero requires a significant \vec{H} in the direction opposite to \vec{B} . After \vec{B} goes through zero, it can then begin to align with \vec{H} again and one can reach saturation in the other direction. And so on.

The Auxiliary Field \vec{H} and Magnetic Permeability (cont.)

We note that ferromagnets have a *Curie or transition temperature*, T_c . This temperature corresponds to roughly the interaction energy of nearby dipoles. When the temperature is larger than T_c , the thermal energy available overcomes the exchange energy, causing magnetic ordering to go away. If a saturated ferromagnetic is raised above T_c , the ordering will dissipate. Then, when recooled in zero applied field, randomly oriented domains will appear but there will be no overall ordering of the magnetic dipoles.

There is not much more we can say about ferromagnetism without considering specific cases.

Boundary Value Problems in Magnetostatics

Griffiths does not really consider boundary value problems in magnetostatics, so we follow Jackson Sections 5.9 through 5.12.

The General Technique

In general, it always holds that

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \vec{H} = \vec{H}(\vec{B}) \quad \vec{\nabla} \times \vec{H} = \vec{J}_f \quad (6.56)$$

Therefore, one can always write the differential equation

$$\vec{\nabla} \times \vec{H}(\vec{\nabla} \times \vec{A}) = \vec{J}_f \quad (6.57)$$

If the relation between \vec{H} and \vec{B} is not simple, the above equation may be difficult to solve.

Boundary Value Problems in Magnetostatics (cont.)

For linear magnetic materials, though, the above reduces to

$$\vec{\nabla} \times \left(\frac{1}{\mu} \vec{\nabla} \times \vec{A} \right) = \vec{J}_f \quad (6.58)$$

If we further specify that μ is constant over some region, then in that region we have

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{\nabla} (\vec{\nabla} \cdot \vec{A}) - \nabla^2 \vec{A} = \mu \vec{J}_f \quad (6.59)$$

Finally, if we specify $\vec{\nabla} \cdot \vec{A} = 0$, this simplifies to a component-by-component Poisson Equation:

$$\nabla^2 \vec{A} = -\mu \vec{J}_f \quad (6.60)$$

One can apply the same techniques as we used for solving Poisson's Equation in electrostatics to solve this component by component. Boundary conditions must be specified either directly (recall that we proved that if $\hat{n} \times \vec{B}$ or $\hat{n} \times \vec{A}$ are specified at every point on the boundary, then solutions are unique) or by matching using the conditions on the normal and tangential components at boundaries.

Boundary Value Problems in Magnetostatics (cont.)

No Free Currents: Magnetic Scalar Potential

If there are no free currents, then $\vec{\nabla} \times \vec{H} = 0$ and we are assured that \vec{H} can be derived from a *magnetic scalar potential*

$$\vec{H} = -\vec{\nabla} V_M(\vec{r}) \quad (6.61)$$

Again, if we know the relationship $\vec{B} = \vec{B}(\vec{H})$, then we can use the divergence equation:

$$\vec{\nabla} \cdot \vec{B} \left(-\vec{\nabla} V_M \right) = 0 \quad (6.62)$$

Again, if the relation between \vec{H} and \vec{B} is not simple, the above equation may be difficult to solve.

Boundary Value Problems in Magnetostatics (cont.)

Again, though, for the case of linear magnetic materials, we have

$$\vec{\nabla} \cdot (\mu \vec{\nabla} V_M) = 0 \quad (6.63)$$

In a region where μ is constant, it can be passed through the divergence and we can reduce this to

$$\nabla^2 V_M = 0 \quad (6.64)$$

We now have Laplace's Equation. Again, boundary conditions and/or matching conditions will allow one to solve for V_M . In a region where μ is constant, we could equally well write $\vec{B} = -\vec{\nabla} U_m$ and solve $\nabla^2 U_m = 0$ with appropriate boundary conditions.

Boundary Value Problems in Magnetostatics (cont.)

Hard Ferromagnets (\vec{M} fixed and $\vec{J}_f = 0$) via Scalar Potential

This is a different special case of the above magnetostatic scalar potential approach.

In this case, instead of $\vec{B} = \mu \vec{H}$, we use $\vec{B} = \mu_0 (\vec{H} + \vec{M})$ with \vec{M} fixed. Then $\vec{\nabla} \cdot \vec{B} = 0$ gives

$$\vec{\nabla} \cdot \mu_0 (\vec{H} + \vec{M}) = 0 \quad (6.65)$$

$$-\nabla^2 V_M + \vec{\nabla} \cdot \vec{M} = 0 \quad (6.66)$$

$$\nabla^2 V_M = -\rho_M \quad \text{with} \quad \rho_M = -\vec{\nabla} \cdot \vec{M} \quad (6.67)$$

(note the canceling minus signs in the definitions!) where ρ_M is termed the *magnetic charge density*. Note the close similarity to the definition of the bound charge density $\rho_b = -\vec{\nabla} \cdot \vec{P}$ for dielectrics. This equation can be solved by the standard techniques for solving Poisson's Equation. In particular, if the boundary is at infinity and we require the fields to fall off to zero there, we know the Green Function for the above equation, which yields

$$V_M(\vec{r}) = -\frac{1}{4\pi} \int_{\mathcal{V}} d\tau' \frac{\vec{\nabla}_{\vec{r}'} \cdot \vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (6.68)$$

Boundary Value Problems in Magnetostatics (cont.)

Assuming \vec{M} is well behaved (has no discontinuities or infinite derivatives except at well-defined boundaries) and using similar techniques as we have used before, we use the product rule for the divergence to do an integration by parts of the above expression, which yields the integral of a divergence and the complementary expression. The integral of the divergence can be turned into a surface integral and the surface can be taken to infinity. With our assumption that \vec{M} falls off at infinity, the surface term vanishes, leaving us only the complementary term

$$V_M(\vec{r}) = \frac{1}{4\pi} \int_V d\tau' \vec{M}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}'} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \quad (6.69)$$

We change variables on the gradient from \vec{r}' to \vec{r} in the usual way, picking up a sign:

$$V_M(\vec{r}) = -\frac{1}{4\pi} \int_V d\tau' \vec{M}(\vec{r}') \cdot \vec{\nabla}_{\vec{r}} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \quad (6.70)$$

We then apply the product rule again, which allows us to bring $\vec{\nabla}_{\vec{r}}$ outside the integral since it does not act on $\vec{M}(\vec{r}')$:

$$V_M(\vec{r}) = -\frac{1}{4\pi} \vec{\nabla}_{\vec{r}} \cdot \int_V d\tau' \frac{\vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (6.71)$$

Boundary Value Problems in Magnetostatics (cont.)

If we want to know the potential and field far from the region that is magnetized, we can make the approximation $|\vec{r} - \vec{r}'|^{-1} \approx r^{-1}$ and pull this factor outside the integral, which gives

$$V_M(\vec{r}) = -\frac{1}{4\pi} \vec{\nabla}_{\vec{r}} \cdot \left[\frac{1}{r} \int_V d\tau' \vec{M}(\vec{r}') \right] \quad (6.72)$$

$$= \frac{1}{4\pi} \frac{\vec{m} \cdot \vec{r}}{r^3} \quad \text{with} \quad \vec{m} = \int_V d\tau' \vec{M}(\vec{r}') \quad (6.73)$$

That is, the scalar potential is equal to that of an electric dipole with $\vec{p} = \epsilon_0 \vec{m}$, implying the field is equal to that of a magnetic dipole \vec{m} . (The factor of μ_0 will reappear when one calculates \vec{B} instead of \vec{H}).

Boundary Value Problems in Magnetostatics (cont.)

We assumed \vec{M} dropped off to zero smoothly at infinity in obtaining Equation 6.68 from Poisson's Equation for the magnetostatic scalar potential. If we consider a case where there is a boundary — such as the boundary of the magnetized region, with $\vec{M} = 0$ outside — then we know that the solution to Poisson's Equation has a surface term due to the charge density on the boundary. By analogy to our consideration of surface charge densities at boundaries in electrostatics, we see that we need to add a surface term:

$$V_M(\vec{r}) = -\frac{1}{4\pi} \int_{\mathcal{V}} d\tau' \frac{\vec{\nabla}_{\vec{r}'} \cdot \vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} + \frac{1}{4\pi} \oint_{\partial\mathcal{V}} da' \frac{\hat{n}(\vec{r}') \cdot \vec{M}(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (6.74)$$

This second term looks like the bound surface charge density term in the corresponding expression in electrostatics, so we define a *magnetic surface charge density*

$$\sigma_M(\vec{r}) = \hat{n}(\vec{r}) \cdot \vec{M}(\vec{r}) \quad (6.75)$$

and see that it sources the magnetostatic scalar potential in the same way that ρ_M does. Together, both terms look identical to Equation 4.8. One must take some care about the sign of the surface term. \hat{n} is defined to be the *outwardly* directed normal from the magnetized region out into vacuum. This is why σ_M has the sign definition that it does. This convention is consistent with the definition of σ_b , which also used the outwardly directed normal.

Boundary Value Problems in Magnetostatics (cont.)

Example: Uniformly Magnetized Sphere, Again

Let's apply the above kind of formalism for the uniformly magnetized sphere. Again, $\vec{M} = M \hat{z}$. This implies $\rho_M = -\vec{\nabla} \cdot \vec{M} = 0$ and $\sigma_M = \hat{n} \cdot \vec{M} = M \cos \theta$. We solved this same problem before for the uniformly polarized dielectric sphere via separation of variables in spherical coordinates, which yielded Equation 4.15. Making the replacement $P \rightarrow M$ and noting the missing factor of ϵ_0 in Equation 6.74, we obtain

$$V_M(r \leq R) = \frac{M z}{3} \quad V_M(r \geq R) = \frac{\vec{m} \cdot \hat{r}}{4 \pi r^2} \quad \text{with} \quad \vec{m} = \frac{4 \pi}{3} \pi R^3 \vec{M} \quad (6.76)$$

$$\vec{H} = -\vec{\nabla} V_M = \begin{cases} -\frac{\vec{M}}{3} & r \leq R \\ \vec{H} \text{ field of a magnetic dipole } \vec{m} & r \geq R \end{cases} \quad (6.77)$$

$$\vec{B} = \mu_0 (\vec{H} + \vec{M}) \implies \vec{B}(r \leq R) = \mu_0 \left(-\frac{1}{3} \vec{M} + \vec{M} \right) = \frac{2}{3} \mu_0 \vec{M} \quad (6.78)$$

$$\vec{B}(r \geq R) = \mu_0 \vec{H} = \vec{B} \text{ field of a magnetic dipole } \vec{m} \quad (6.79)$$

This matches our previous solution for the magnetic field of this system that we obtained by calculating the vector potential of the bound surface current.

Boundary Value Problems in Magnetostatics (cont.)

Hard Ferromagnets (\vec{M} fixed and $\vec{J}_f = 0$) via Vector Potential

We have already done this analysis, yielding Equations 6.10 and 6.11:

$$\vec{J}_b(\vec{r}) = \vec{\nabla} \times \vec{M}(\vec{r}) \quad \vec{K}_b(\vec{r}) = \vec{M}(\vec{r}) \times \hat{n}(\vec{r})$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} d\tau' \frac{\vec{J}_b(\vec{r}')}{|\vec{r} - \vec{r}'|} + \frac{\mu_0}{4\pi} \oint_{S(\mathcal{V})} da' \frac{\vec{K}_b(\vec{r}')}{|\vec{r} - \vec{r}'|}$$

Example: Uniformly Magnetized Sphere, Again

We don't need to do this again: the above vector potential based on the bound current density (in this case, only a bound surface current density) is exactly how we solved this system before. We used the spherical harmonics technique to do the integral, which is different from what Griffiths did, but the starting point was the same.

Boundary Value Problems in Magnetostatics (cont.)

Magnetically Permeable Sphere in External Field

Let's first set this up to see how it is directly analogous the corresponding case of a dielectrically polarizable sphere in an external electric field, then we will show a shortcut around the calculation.

We need to make a correspondence between potentials, fields, and boundary conditions:

$$\rho_b = -\vec{\nabla} \cdot \vec{P} \quad \rho_M = -\vec{\nabla} \cdot \vec{M} \quad (6.80)$$

$$\sigma_b = \hat{n} \cdot \vec{P} \quad \rho_M = \hat{n} \cdot \vec{M} \quad (6.81)$$

$$\epsilon_0 \vec{E} = -\epsilon_0 \vec{\nabla} V \quad \vec{H} = -\vec{\nabla} V_M \quad (6.82)$$

$$\epsilon_0 \nabla^2 V = -\rho_b \quad \nabla^2 V_M = -\rho_M \quad (6.83)$$

$$\vec{D} = \epsilon_0 \vec{E} + \vec{P} \quad \vec{B}/\mu_0 = \vec{H} + \vec{M} \quad (6.84)$$

$$\epsilon_0 \vec{E} \xrightarrow{r \rightarrow \infty} \epsilon_0 \vec{E}_0 \quad \vec{H} \xrightarrow{r \rightarrow \infty} \vec{B}_0/\mu_0 \quad (6.85)$$

$$\vec{D} \xrightarrow{r \rightarrow \infty} \epsilon_0 \vec{E}_0 \quad \vec{B}/\mu_0 \xrightarrow{r \rightarrow \infty} \vec{B}_0/\mu_0 \quad (6.86)$$

$$\vec{P} \xrightarrow{r \rightarrow \infty} 0 \quad \vec{M} \xrightarrow{r \rightarrow \infty} 0 \quad (6.87)$$

Boundary Value Problems in Magnetostatics (cont.)

Let's also compare the matching conditions. For the electric field, the normal and tangential component matching conditions are

$$\hat{n} \cdot [\epsilon_0 \vec{E}_>(R) - \epsilon_0 \vec{E}_<(R)] = \sigma_b \quad (6.88)$$

$$\hat{s} \cdot [\epsilon_0 \vec{E}_>(R) - \epsilon_0 \vec{E}_<(R)] = 0 \quad (6.89)$$

and the corresponding matching conditions for the \vec{H} field are

$$\hat{n} \cdot [\vec{H}_>(R) - \vec{H}_<(R)] = -\hat{n} \cdot [\vec{M}_>(R) - \vec{M}_<(R)] = \hat{n} \cdot \vec{M}_<(R) = \sigma_M \quad (6.90)$$

$$\hat{s} \cdot [\vec{H}_>(R) - \vec{H}_<(R)] = \hat{s} \cdot [\vec{K}_f \times \hat{n}] = 0 \quad (6.91)$$

There is thus a perfect correspondence between the two problems with $\epsilon_0 \vec{E} \leftrightarrow \vec{H}$, $\vec{P} \leftrightarrow \vec{M}$, and $\vec{D} \leftrightarrow \vec{B}/\mu_0$. Thus, we can just apply the solution to the dielectric problem.

Boundary Value Problems in Magnetostatics (cont.)

This gives us

$$V_M(r < R) = -\frac{3\mu_0}{2\mu_0 + \mu} \frac{B_0}{\mu_0} z \quad (6.92)$$

$$\vec{H}(r < R) = \frac{3\mu_0}{2\mu_0 + \mu} \frac{\vec{B}_0}{\mu_0} = \frac{\vec{B}_0}{\mu_0} - \frac{\vec{M}(r < R)}{3} \quad (6.93)$$

$$\vec{M}(r < R) = 3 \frac{\mu - \mu_0}{2\mu_0 + \mu} \frac{\vec{B}_0}{\mu_0} \quad \sigma_M = 3 \frac{\mu - \mu_0}{2\mu_0 + \mu} \frac{B_0}{\mu_0} \cos \theta \quad (6.94)$$

$$\begin{aligned} \vec{B}(r < R) &= \mu_0 \left[\vec{H}(r < R) + \vec{M}(r < R) \right] = \mu_0 \left[\frac{\vec{B}_0}{\mu_0} - \frac{\vec{M}(r < R)}{3} + \vec{M}(r < R) \right] \\ &= \vec{B}_0 + \frac{2}{3} \mu_0 \vec{M}(r < R) = \left(\frac{3\mu}{2\mu_0 + \mu} \right) \vec{B}_0 \end{aligned} \quad (6.95)$$

$$V_M(r > R) = -\frac{B_0}{\mu_0} z + \frac{\vec{m} \cdot \hat{r}}{4\pi r^2} \quad (6.96)$$

$$\vec{m} \equiv \frac{4\pi}{3} R^3 \vec{M}(r < R) = \frac{4\pi}{3} R^3 \frac{B_0}{\mu_0} \frac{3}{2\mu_0 + \mu} (\mu - \mu_0) \hat{z} \quad (6.97)$$

Boundary Value Problems in Magnetostatics (cont.)

The shortcut method is to first use superposition of a uniform field and a uniformly magnetized sphere without relating \vec{M} and \vec{H} , then use the relation between the two. Superposition gives

$$\vec{B}(r < R) = \vec{B}_0 + \frac{2}{3} \mu_0 \vec{M} \quad (6.98)$$

$$\vec{H}(r < R) = \frac{\vec{B}_0}{\mu_0} - \frac{1}{3} \vec{M} \quad (6.99)$$

Now, use $\vec{B}(r < R) = \mu \vec{H}(r < R)$ which relates the above two equations, and then solve for \vec{M} . One finds one gets the same result. Admittedly, this technique is somewhat backhanded; when trying to understand things for the first time, reapplying the scalar potential to the full problem is more straightforward.

Boundary Value Problems in Magnetostatics (cont.)

Magnetically Permeable Spherical Shell

Consider a spherical shell of inner radius a and outer radius b consisting of a highly permeable ($\mu/\mu_0 \gg 1$) material placed in a uniform external field \vec{B}_0 . We shall see that this shell shields its inner volume from the external field by a factor μ/μ_0 . This technique is of great importance for magnetically sensitive experiments and equipment.

There are no free currents, so may use the magnetostatic scalar potential technique. Furthermore, $\vec{\nabla} \cdot \vec{H} = 0$ in each region since μ is constant in each region. So the scalar potential V_M satisfies Laplace's Equation, allowing us to apply our techniques for the solution of Laplace's Equation from electrostatics.

Boundary Value Problems in Magnetostatics (cont.)

In particular, given the azimuthal symmetry, we may assume the solution in each of the three regions is of the form given in Equation 3.110

$$V_M(r < a, \theta) \equiv V_1(r, \theta) = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos \theta) \quad (6.100)$$

$$V_M(a < r < b, \theta) \equiv V_2(r, \theta) = \sum_{\ell=0}^{\infty} \left(C_\ell r^\ell + \frac{D_\ell}{r^{\ell+1}} \right) P_\ell(\cos \theta) \quad (6.101)$$

$$V_M(a < r < b, \theta) \equiv V_3(r, \theta) = -H_0 r \cos \theta + \sum_{\ell=0}^{\infty} \frac{E_\ell}{r^{\ell+1}} P_\ell(\cos \theta) \quad (6.102)$$

where we have already applied the requirements that V_M be finite as $r \rightarrow 0$ and that it yield the uniform field as $r \rightarrow \infty$ with $H_0 = B_0/\mu_0$. We have also assumed that V_M has no constant offset as $r \rightarrow \infty$.

Boundary Value Problems in Magnetostatics (cont.)

There are no free currents, so our matching conditions are that the normal component of \vec{B} and the tangential component of \vec{H} be continuous. Using $\vec{H} = -\vec{\nabla}V_M$, we thus have the four conditions

$$\mu_0 \frac{\partial V_1}{\partial r} \Big|_a = \mu \frac{\partial V_2}{\partial r} \Big|_a \quad \mu \frac{\partial V_2}{\partial r} \Big|_b = \mu_0 \frac{\partial V_3}{\partial r} \Big|_b \quad (6.103)$$

$$\frac{\partial V_1}{\partial \theta} \Big|_a = \frac{\partial V_2}{\partial \theta} \Big|_a \quad \frac{\partial V_2}{\partial \theta} \Big|_b = \frac{\partial V_3}{\partial \theta} \Big|_b \quad (6.104)$$

Note that we do not impose continuity on V_M . In the electrostatic case, we imposed continuity of V and the boundary condition on the normal derivative, ignoring continuity of the tangential derivative. In electrostatics, continuity of V comes from constructing it as the line integral of the electric field, which we in turn were motivated to write down in order to calculate the work done by the electric field on a point charge. Since \vec{H} does not do such work, writing down the line integral is not physically motivated, though it is mathematically reasonable to do so because $\vec{H} = -\vec{\nabla}V_M$. So, here, we instead use continuity of the radial and tangential derivatives. This is an arbitrary choice driven by our physical intuition. We will see below that continuity of V_M would yield information redundant with tangential derivative continuity.

Boundary Value Problems in Magnetostatics (cont.)

Before we dive into a lot of algebra, let's see what we can figure out without doing much work. The radial derivative equations only connect terms on the two sides of the equations with the same ℓ because they do not modify the orthonormal $P_\ell(\cos \theta)$. What about the angular derivative equations? Recall Equation 3.152:

$$P_\ell^m(x) = (-1)^m (1 - x^2)^{m/2} \frac{d^m}{dx^m} P_\ell(x) \quad (6.105)$$

Let's write $\frac{\partial P_\ell(\cos \theta)}{\partial \theta}$ using this:

$$\frac{\partial P_\ell(\cos \theta)}{\partial \theta} = \frac{dP_\ell(\cos \theta)}{d \cos \theta} \frac{d \cos \theta}{d \theta} = \frac{P_\ell^1(\cos \theta)}{(-1)^1 (1 - \cos^2 \theta)^{1/2}} (-\sin \theta) \quad (6.106)$$

$$= P_\ell^1(\cos \theta) \quad (6.107)$$

where we note that, since $0 < \theta < \pi$, there is no sign ambiguity and thus $\sin \theta = (1 - \cos^2 \theta)^{1/2}$. The $P_\ell^1(\cos \theta)$ are also orthonormal polynomials (the P_ℓ^m over all ℓ at fixed m form an orthonormal set), so the same point we made above about the equations connecting terms at the same ℓ holds for these equations also. Note however that, for $\ell = 0$, the $\partial/\partial\theta$ matching condition yields zero.

Note also that, for $\ell \geq 1$, these equations are the same as one would have obtained by requiring continuity of V_M since $\partial/\partial\theta$ doesn't modify the coefficients of each term.

Boundary Value Problems in Magnetostatics (cont.)

Taking the necessary derivatives for the radial derivative equations and then equating the two sides of all six equations term-by-term gives us:

$$\ell > 0 : \quad \mu_0 \ell A_\ell a^{\ell-1} = \mu \ell C_\ell a^{\ell-1} - \mu (\ell + 1) \frac{D_\ell}{a^{\ell+2}} \quad (6.108)$$

$$\mu \ell C_\ell b^{\ell-1} - \mu (\ell + 1) \frac{D_\ell}{b^{\ell+2}} = -\mu_0 H_0 \delta_{\ell 1} - \mu_0 (\ell + 1) \frac{E_\ell}{b^{\ell+2}} \quad (6.109)$$

$$A_\ell a^\ell = C_\ell a^\ell + \frac{D_\ell}{a^{\ell+1}} \quad (6.110)$$

$$C_\ell b^\ell + \frac{D_\ell}{b^{\ell+1}} = -H_0 b \delta_{\ell 1} + \frac{E_\ell}{b^{\ell+1}} \quad (6.111)$$

$$\ell = 0 : \quad 0 = -\mu \frac{D_0}{a^2} \quad -\mu \frac{D_0}{b^2} = -\mu_0 \frac{E_0}{b^2} \quad (6.112)$$

We explicitly write out the $\ell = 0$ equations because they yield qualitatively different conditions than the $\ell > 0$ terms.

Boundary Value Problems in Magnetostatics (cont.)

For $\ell > 1$, solving for C_ℓ and D_ℓ results in both vanishing, so then A_ℓ and E_ℓ vanish for $\ell > 1$.

For $\ell = 0$, the radial derivative matching equations imply $D_0 = E_0 = 0$. We expect $E_0 = 0$ because it would yield a magnetic monopole potential for $r > b$, which we know is physically disallowed.

There are no equations that explicitly determine A_0 and C_0 , which correspond to offsets of V_M for $r < a$ and $a < r < b$. We actually don't need to find them, since they do not affect \vec{H} when the gradient is taken. (Recall, there is no issue of this potential being related to work or a potential energy, so we do not need to worry about discontinuities due to offsets.) But we can specify them by applying a restricted version of continuity of V_M , which is that we require V_M have the same offset in all regions. The lack of an offset for $r > b$ then implies $A_0 = 0$ and $C_0 = 0$.

Boundary Value Problems in Magnetostatics (cont.)

For $\ell = 1$, we can do a lot of algebra to find explicit formulae for all the coefficients (you can find these in Jackson Section 5.12). These formulae are not particularly illuminating, but they become interesting when we take the limit $\mu/\mu_0 \gg 1$. Inserting those coefficients into the solutions, we obtain

$$V_1(r, \theta) \xrightarrow{\frac{\mu}{\mu_0} \gg 1} A_1 r \cos \theta = -\frac{9}{2 \frac{\mu}{\mu_0} \left(1 - \frac{a^3}{b^3}\right)} H_0 r \cos \theta \quad (6.113)$$

$$V_2(r, \theta) \xrightarrow{\frac{\mu}{\mu_0} \gg 1} \left(C_1 r + \frac{D_1}{r^2}\right) \cos \theta = -\frac{3}{\frac{\mu}{\mu_0} \left(1 - \frac{a^3}{b^3}\right)} H_0 \left(r + \frac{1}{2} \frac{a^3}{r^2}\right) \cos \theta \quad (6.114)$$

$$V_3(r, \theta) \xrightarrow{\frac{\mu}{\mu_0} \gg 1} \left(-H_0 r + \frac{E_1}{r^2}\right) \cos \theta = H_0 \left(-r + \frac{b^3}{r^2} \left(1 - \frac{3 \left(1 + \frac{1}{2} \frac{a^3}{b^3}\right)}{\frac{\mu}{\mu_0} \left(1 - \frac{a^3}{b^3}\right)}\right)\right) \cos \theta \quad (6.115)$$

Note that we include the term of order μ_0/μ in the $r > b$ solution so we can see that the matching condition on the tangential derivative at $r = b$ is explicitly satisfied even in this limit.

Boundary Value Problems in Magnetostatics (cont.)

Here are the resulting fields in the three regions:

$$\vec{H}_1(r, \theta) \xrightarrow{\frac{\mu}{\mu_0} \gg 1} -\frac{9}{2} \frac{H_0 \hat{z}}{\frac{\mu}{\mu_0} \left(1 - \frac{a^3}{b^3}\right)} \quad \vec{B}_1(r, \theta) = \mu_0 \vec{H}_1(r, \theta) \quad (6.116)$$

$$\vec{H}_2(r, \theta) \xrightarrow{\frac{\mu}{\mu_0} \gg 1} \frac{3 H_0 \hat{z}}{\frac{\mu}{\mu_0} \left(1 - \frac{a^3}{b^3}\right)} + \frac{3 (\vec{m}_a \cdot \hat{r}) \hat{r} - \vec{m}_a}{4 \pi r^3} \quad \vec{B}_2(r, \theta) = \mu \vec{H}_2(r, \theta) \quad (6.117)$$

$$\vec{H}_3(r, \theta) \xrightarrow{\frac{\mu}{\mu_0} \gg 1} H_0 \hat{z} + \frac{3 (\vec{m}_b \cdot \hat{r}) \hat{r} - \vec{m}_b}{4 \pi r^3} \quad \vec{B}_3(r, \theta) = \mu_0 \vec{H}_3(r, \theta) \quad (6.118)$$

$$\text{with } \vec{m}_a = -\frac{9}{2} \frac{H_0}{\frac{\mu}{\mu_0} \left(1 - \frac{a^3}{b^3}\right)} \left(\frac{4 \pi}{3} a^3\right) \hat{z} \quad (6.119)$$

$$\vec{m}_b = 3 \left[1 - \frac{3 \left(1 + \frac{1}{2} \frac{a^3}{b^3}\right)}{\frac{\mu}{\mu_0} \left(1 - \frac{a^3}{b^3}\right)}\right] H_0 \left(\frac{4 \pi}{3} b^3\right) \hat{z} \quad (6.120)$$

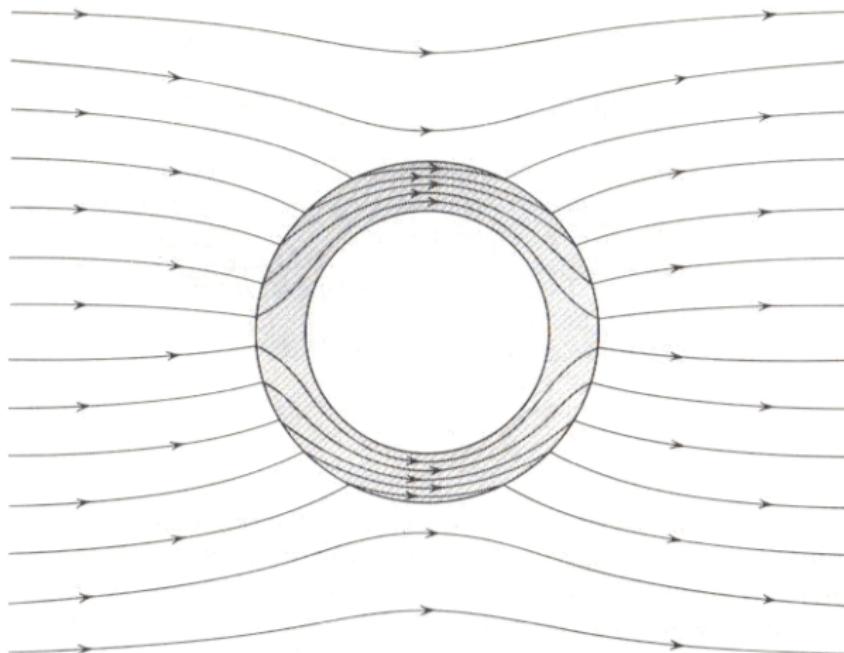
Boundary Value Problems in Magnetostatics (cont.)

The following features can be pointed out:

- ▶ Inside $r < a$, we have a uniform field weakened by a factor of μ/μ_0 (for both B and H).
- ▶ In the permeable material, we have a uniform H field as well as a dipole field, but both are of order $(\mu_0/\mu) H_0$ (*i.e.*, attenuated) with the dipole moment pointed to $-\hat{z}$. The dipole field cancels the uniform field at the poles at $r = a$ and adds to it at the equator.
- ▶ In the permeable material, the B field receives a factor of μ , though, so the B field receives uniform field and dipole contributions of order B_0 in the permeable material, though the vanishing at the poles at $r = a$ remains.
- ▶ Finally, the field outside is the uniform field (for H and B) plus that of a dipole in the $+\hat{z}$ direction. The dominant part of the dipole field cancels the uniform field at the equator at $r = b$, leaving a small residual field of order μ_0/μ smaller. At the poles, the dipole field adds to the uniform field, increasing the fields to $3H_0$ and $3B_0$ there.

Boundary Value Problems in Magnetostatics (cont.)

Here is a picture from Jackson of \vec{B} . Note the concentration of field lines in the permeable material and their absence in the empty central region.



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Boundary Value Problems in Magnetostatics (cont.)

While we have benefited from our boundary-value problem techniques to get directly to the fields without having to calculate the bound surface currents, it would be nice to see how the bound surface currents give the observed fields. Recall Equation 5.73, which gives the bound surface current density from the change in the tangential component of the magnetic field:

$$\vec{K}(\vec{r}) = \frac{1}{\mu_0} \hat{n}(\vec{r}) \times [\vec{B}_>(\vec{r}) - \vec{B}_<(\vec{r})] \quad (6.121)$$

where \hat{n} points from the < region to the > region. In our case, $\vec{K} = \vec{K}_b$ because there are no free currents. Since $\hat{n} = \hat{r}$ for our spherical surfaces and \vec{B} only has components in the \hat{r} and $\hat{\theta}$ directions, this reduces to

$$K_b(r) \hat{\phi} = \frac{1}{\mu_0} [B_{>,\theta} - B_{<,\theta}] \hat{\phi} = \frac{1}{\mu_0} \left[-\frac{\mu_>}{r} \frac{\partial V_M}{\partial \theta} \Big|_{r_>} + \frac{\mu_<}{r} \frac{\partial V_M}{\partial \theta} \Big|_{r_<} \right] \hat{\phi} \quad (6.122)$$

$$= \frac{\mu_< - \mu_>}{\mu_0 r} \frac{1}{r} \frac{\partial V_M}{\partial \theta} \Big|_r \hat{\phi} \quad (6.123)$$

where < and > indicate the two sides of the particular boundary at r and we use the fact that the tangential component of \vec{H} , which is given by $-(1/r) \partial V_M / \partial \theta$ here, is continuous and thus has the same value on both sides of the interface at r . So it is straightforward to calculate the surface currents given V_M .

Boundary Value Problems in Magnetostatics (cont.)

We know how to calculate \vec{B} given surface currents: we did it in our first calculation of the magnetic field of the permanently magnetized sphere (Equation 6.21) and saw

$$\vec{B}_{\vec{M}}(r < R) = \frac{2}{3} \mu_0 \vec{M} = \frac{2}{3} \mu_0 \frac{\vec{K}_b \cdot \hat{\phi}}{\sin \theta} \hat{z}$$

$$\vec{B}_{\vec{M}}(r > R) = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} \quad \vec{m} = \frac{4}{3}\pi R^3 \vec{M} = \frac{4}{3}\pi R^3 \frac{\vec{K}_b \cdot \hat{\phi}}{\sin \theta} \hat{z}$$

where the relation between \vec{M} and \vec{K}_b comes from the definition of the bound surface current, $\vec{K}_b = \vec{M} \times \hat{n} = M \hat{z} \times \hat{r} = \hat{\phi} M \sin \theta$. This applies here because the directions of the magnetizations and surface currents are the same as we have here. (The fact that we have permeable materials present is irrelevant for the calculation of \vec{B} : once one has all the bound currents, one can calculate \vec{B} directly from them.) So, we expect that, in this case, we can just add the field of the above form due to the bound currents to the uniform applied field to get the total field in the three regions.

Boundary Value Problems in Magnetostatics (cont.)

The total fields then are

$$\vec{B}_1(r, \theta) = \vec{B}_0 + \frac{2}{3} \mu_0 \frac{\vec{K}_b(a) \cdot \hat{\phi} + \vec{K}_b(b) \cdot \hat{\phi}}{\sin \theta} \hat{z} \quad (6.124)$$

$$\vec{B}_2(r, \theta) = \vec{B}_0 + \frac{2}{3} \mu_0 \frac{\vec{K}_b(b) \cdot \hat{\phi}}{\sin \theta} \hat{z} + \frac{\mu_0}{4\pi} \frac{3(\vec{m}_a \cdot \hat{r})\hat{r} - \vec{m}_a}{r^3} \quad (6.125)$$

$$\vec{B}_3(r, \theta) = \vec{B}_0 + \frac{\mu_0}{4\pi} \frac{3(\vec{m}_b \cdot \hat{r})\hat{r} - \vec{m}_b}{r^3} \quad (6.126)$$

$$\vec{m}_a = \frac{4}{3} \pi a^3 \frac{\vec{K}_b(a) \cdot \hat{\phi}}{\sin \theta} \hat{z} \qquad \vec{m}_b = \vec{m}_a + \frac{4}{3} \pi b^3 \frac{\vec{K}_b(b) \cdot \hat{\phi}}{\sin \theta} \hat{z} \quad (6.127)$$

and then we can obtain \vec{H} from the usual relation $\vec{H}(\vec{r}) = \vec{B}(\vec{r})/\mu(\vec{r})$.

Boundary Value Problems in Magnetostatics (cont.)

There is an important subtlety in trying to do this calculation using the approximate forms for the fields we have written down (valid for $\mu_0/\mu \ll 1$). We expect the magnetic field for $r < a$ to be of order $(\mu_0/\mu) B_0$. But \vec{B}_0 is in the expression for \vec{B}_1 , so that implies the second term in that expression due to the surface currents will carry one term of order B_0 to cancel B_0 and then a second term of order $(\mu_0/\mu) B_0$ to give the residual field. As we explained above, our expressions for the contribution of the surface current to the field are of the following form for $r \lesssim b$:

$$B_K \sim \mu_0 K \sim \pm \mu_0 \frac{\mu - \mu_0}{\mu_0} \frac{1}{r} \frac{\partial V_M}{\partial \theta} \sim \mu_0 \left(\frac{\mu}{\mu_0} - 1 \right) H_\theta \quad (6.128)$$

$$\sim \mu_0 \left(\mathcal{O} \left(\frac{\mu}{\mu_0} \right)^1 + \mathcal{O} \left(\frac{\mu_0}{\mu} \right)^0 \right) \mathcal{O} \left(\frac{\mu_0}{\mu} \right)^1 H_0 \quad (6.129)$$

$$\sim \left[\mathcal{O} \left(\frac{\mu_0}{\mu} \right)^0 + \mathcal{O} \left(\frac{\mu_0}{\mu} \right)^1 \right] B_0 \quad (6.130)$$

In the second line, we used $H \sim \mathcal{O}(\mu_0/\mu)^1 H_0$, which one can see from the expressions for \vec{H}_1 and \vec{H}_2 . It is not so obvious that this is true for \vec{H}_3 at $r \sim b$, but it must be true because H_θ is continuous. It turns out to be true because the dipole field cancels the applied field to first order in H_0 (*i.e.*, zeroth order in μ_0/μ) at the equator, leaving a residual field of order $\mathcal{O}(\mu_0/\mu)^1 H_0$. The cancellation does not happen at the poles, but $H_\theta = 0$ at the poles.

Boundary Value Problems in Magnetostatics (cont.)

We now see the problem. The term that is $\mathcal{O}(\mu_0/\mu)^0$ will cancel the \vec{B}_0 term. So then the $\mathcal{O}(\mu_0/\mu)^1 B_0$ term is all that is left and is our full field, as expected. But we have not done the approximation self-consistently. We would have obtained a term of the same order by including terms $\mathcal{O}(\mu_0/\mu)^2$ in the expression for H because they would yield $\mathcal{O}(\mu_0/\mu)^1$ terms when multiplied by the $\mathcal{O}(\mu/\mu_0)^1$ term from the $(\mu/\mu_0 - 1)$ prefactor. Without including that term, we will get the incorrect coefficient for the residual field.

We could have included that higher order term, but then we would run into the same problem at the next order: our calculation of the field using the surface currents would be correct to $\mathcal{O}(\mu_0/\mu)^1$, but our expression for the fields would have terms of order $\mathcal{O}(\mu_0/\mu)^2$ that we would not be able to fully reproduce. Given that it would be algebraically challenging to do this even to $\mathcal{O}(\mu_0/\mu)^1$ correctly, we punt on trying to calculate the residual field.

Boundary Value Problems in Magnetostatics (cont.)

However, we can self-consistently check our results (Equations 6.116-6.120) to $\mathcal{O}(\mu_0/\mu)^0$, so let's do that because it will show us that the zeroth order field does vanish at $r < a$ and it will tell us interesting things for other regions. The explicit results for the bound surface currents are

$$\vec{K}_b(a, \theta) = \frac{\mu_- - \mu_+}{\mu_0} \frac{1}{a} \left. \frac{\partial V_M}{\partial \theta} \right|_a \hat{\phi} = -\hat{\phi} \left(\frac{\mu_-}{\mu_0} - 1 \right) \frac{9}{2 \frac{\mu_-}{\mu_0} \left(1 - \frac{a^3}{b^3} \right)} \frac{B_0}{\mu_0} \sin \theta$$
$$\underset{\mathcal{O}(\mu_0/\mu)^0}{\approx} -\hat{\phi} \frac{9}{2 \left(1 - \frac{a^3}{b^3} \right)} \frac{B_0}{\mu_0} \sin \theta \quad (6.131)$$

$$\vec{K}_b(b, \theta) = \frac{\mu_- - \mu_+}{\mu_0} \frac{1}{b} \left. \frac{\partial V_M}{\partial \theta} \right|_b \hat{\phi} = \hat{\phi} \left(\frac{\mu_-}{\mu_0} - 1 \right) \frac{3 \left(1 + \frac{1}{2} \frac{a^3}{b^3} \right)}{\frac{\mu_-}{\mu_0} \left(1 - \frac{a^3}{b^3} \right)} \frac{B_0}{\mu_0} \sin \theta$$
$$\underset{\mathcal{O}(\mu_0/\mu)^0}{\approx} \hat{\phi} \frac{3 \left(1 + \frac{1}{2} \frac{a^3}{b^3} \right)}{\left(1 - \frac{a^3}{b^3} \right)} \frac{B_0}{\mu_0} \sin \theta \quad (6.132)$$

Boundary Value Problems in Magnetostatics (cont.)

First, for $r < a$, we have, to zeroth order in μ_0/μ ,

$$\hat{z} \cdot \vec{B}_1(r, \theta) \stackrel{\mathcal{O}(\mu_0/\mu)^0}{\approx} B_0 + \frac{2}{3} \frac{B_0}{1 - \frac{a^3}{b^3}} \left[-\frac{9}{2} + 3 \left(1 + \frac{1}{2} \frac{a^3}{b^3} \right) \right] = 0 \quad (6.133)$$

$$\hat{z} \cdot \vec{H}_1(r, \theta) = \frac{\vec{B}_1(r, \theta)}{\mu_0} \stackrel{\mathcal{O}(\mu_0/\mu)^0}{\approx} 0 \quad (6.134)$$

As expected, both the magnetic and auxiliary fields vanish to zeroth order in μ_0/μ inside the cavity.

Boundary Value Problems in Magnetostatics (cont.)

For $a < r < b$, we have

$$\begin{aligned}\hat{z} \cdot \vec{B}_2(r, \theta) &\stackrel{\mathcal{O}(\mu_0/\mu)^0}{\approx} B_0 + \frac{2}{3} \frac{B_0}{1 - \frac{a^3}{b^3}} (3) \left(1 + \frac{1}{2} \frac{a^3}{b^3}\right) + \frac{\mu_0}{4\pi} \frac{3(\vec{m}_a \cdot \hat{r}) \cos \theta - m_a}{r^3} \\ &= \frac{3B_0}{1 - \frac{a^3}{b^3}} + \frac{\mu_0}{4\pi} m_a \frac{3\cos^2 \theta - 1}{r^3}\end{aligned}\quad (6.135)$$

$$\hat{z} \cdot \vec{H}_2(r, \theta) = \frac{\vec{B}_2(r, \theta)}{\mu} \stackrel{\mathcal{O}(\mu_0/\mu)^0}{\approx} 0 \quad (6.136)$$

$$\text{with } \vec{m}_a \stackrel{\mathcal{O}(\mu_0/\mu)^0}{\approx} -\frac{4\pi}{3} a^3 \left(\frac{9}{2}\right) \frac{1}{1 - \frac{a^3}{b^3}} \frac{B_0}{\mu_0} \hat{z} \quad (6.137)$$

The total field in the permeable material is of order B_0 because both terms shown are of order B_0 . In the limit $a \ll b$, one recovers $3B_0$ as we expect from the case of the permeable sphere (Equation 6.95 with $\mu_0/\mu \rightarrow 0$). There is a correction factor that amplifies both terms by the ratio of the volume of the sphere of radius b to the volume of the permeable material. This intensification reflects the fact that the field lines that would have threaded through the entire sphere now are restricted to flow in the permeable material only. The auxiliary field vanishes in the permeable material to order $(\mu_0/\mu)^0$ because one must divide the entire expression by μ to get H from B , which combines with the μ_0 in the expression for B to give a prefactor of μ_0/μ , which vanishes at the level of approximation we are considering.

Boundary Value Problems in Magnetostatics (cont.)

Finally, let's look at $r > b$, for which we obtain

$$\hat{z} \cdot \vec{B}_3(r, \theta) \stackrel{\mathcal{O}(\mu_0/\mu)^0}{\approx} B_0 + \frac{\mu_0}{4\pi} \frac{3(\vec{m}_b \cdot \hat{r}) \cos \theta - m_b}{r^3} \quad (6.138)$$

$$\begin{aligned} \text{with } \vec{m}_b &\stackrel{\mathcal{O}(\mu_0/\mu)^0}{\approx} \frac{4\pi}{3} \frac{1}{1 - \frac{a^3}{b^3}} \frac{B_0}{\mu_0} \left[-\frac{9}{2} a^3 + b^3 (3) \left(1 + \frac{1}{2} \frac{a^3}{b^3} \right) \right] \hat{z} \\ &= 3 \left(\frac{4\pi}{3} b^3 \right) \frac{B_0}{\mu_0} \hat{z} \end{aligned} \quad (6.139)$$

$$\hat{z} \cdot \vec{H}_3(r, \theta) = \frac{\vec{B}_3(r, \theta)}{\mu_0} \quad (6.140)$$

One can see that the expressions for \vec{B}_0 and \vec{m}_b match to zeroth order in μ_0/μ the results we obtained via the boundary value problem technique, Equations 6.116-6.120. The expression for H has the same form with B_0 replaced by H_0 and it also matches the expressions we obtained earlier, again to zeroth order in μ_0/μ .

Boundary Value Problems in Magnetostatics (cont.)

Let's now consider the analogy to electrostatic shielding. Electrostatic shielding is easily provided by conductors, and perfect conductors ($\epsilon/\epsilon_0 \rightarrow \infty$) provide perfect electrostatic shielding. They do this by setting up surface charge that perfectly cancels the externally applied field.

Magnetostatic shielding effect is very similar, though it occurs for completely different reasons. In this case, the high magnetic permeability of the materials causes magnetic dipoles to be set up to almost perfectly cancel the externally applied magnetic field (a residual field of order μ_0/μ times the externally applied field reaches the interior). Effectively, a large bound surface current is set up and its magnetic field cancels the externally applied magnetic field.

This calculation has important practical implications: such highly permeable materials are in widespread use for magnetic shielding from, especially, Earth's field in magnetically sensitive experiments and equipment such as SQUIDs (very sensitive magnetometers) and photomultiplier tubes (where the electrons paths can be substantially bent and thus the gain modified by magnetic fields).

Section 7

Electrodynamics

Lecture 15:
Currents and Ohm's Law
Electromotive Forces, Faraday's Law
Induced Fields in the Absence of Charges

Date Revised: 2014/04/18 06:00

Clarified surface normals in generic derivation of motional emf

Modified lecture break

Date Given: 2014/04/17

Currents and Ohm's Law

Ohm's Law and Joule Heating: Differential Version

We state the very nonobvious point that the current due to an ensemble of flowing charges is proportional to the *force per unit charge* \vec{f} acting on them:

$$\vec{J} = \sigma \vec{f} \quad (7.1)$$

Since current is proportional to velocity, and force is proportional to acceleration, why is this true? In an ideal conductor, it would not be true, we would expect current to be proportional to the integral of the force over time. But in all real conductors, there are two important effects that change this picture:

- ▶ The first is the random thermal motion of the charge carriers. The forces we can apply yield velocities that are small perturbations to this random thermal motion. So the mean *speed* of the carriers is dominated by the thermal speed v_{thermal} and the added velocity from the applied force is a small perturbation.
- ▶ The second is scattering. This scattering is in fact the cause of the randomness of the thermal motion. The charge carriers scatter off of impurities and defects in the material and off of the thermal vibrations present in the material. This scattering is elastic in general, resulting in no loss of energy but in a redirection of velocity.

Currents and Ohm's Law (cont.)

In the presence of such effects, our picture should not be of a charge carrier smoothly accelerating under the influence of an external force, but rather of a carrier with a large randomly directed velocity, scattering frequently, and with acceleration by the force between scatters. The scattering randomly redirects the velocity, so the velocity due to the externally applied force is, on average, reset to zero after each collision. If the thermal speed is $v_{thermal}$ and the typical distance traveled between scatters is λ , then the time available for the externally applied force to accelerate a carrier between scatters is

$$t = \frac{\lambda}{v_{thermal}} \quad (7.2)$$

The average velocity acquired from the applied force during this time is

$$\vec{v}_{ave} = \frac{1}{2} \vec{a} t = \frac{1}{2} \frac{\vec{f}}{m} \frac{\lambda}{v_{thermal}} \quad (7.3)$$

This velocity is the average overall velocity because of the zeroing of the instantaneous velocity after each collision.

Currents and Ohm's Law (cont.)

If we then use $\vec{J} = n q \vec{v}_{ave}$ where n is the number density of charge carriers and q is the charge per carrier, and we use $\vec{f} = q \vec{E}$, we then can write

$$\vec{J} = \left(\frac{n q^2 \lambda}{2 m v_{thermal}} \right) \vec{E} \quad \Rightarrow \quad \vec{J} = \sigma \vec{E} \quad \sigma = \frac{n q^2 \lambda}{2 m v_{thermal}} \quad (7.4)$$

Thus, we see our earlier expression is justified. This is *Ohm's Law*.

There is power dissipated in this process — the work done on the charge carriers by the electric field is lost to random motion when they scatter. The amount of energy lost per unit time and per unit volume is

$$P = \text{number density} \cdot \text{velocity} \cdot \frac{\text{force}}{\text{carrier}} = n \vec{v}_{ave} \cdot \vec{f} = n \frac{\vec{J}}{n q} \cdot q \vec{E} = \vec{J} \cdot \vec{E} \quad (7.5)$$

This is known as *Joule Heating*.

We note that the possibility of $\vec{E} \neq 0$ does not contradict our earlier discussions of electrostatics; here, we have non-stationary charges, where in that case we considered the final static situation after any currents had flowed.

Currents and Ohm's Law (cont.)

Integral Version of Ohm's Law and Joule Heating

We integrate the above to obtain a more familiar version of Ohm's Law. We start with:

$$I = \int_S da \hat{n} \cdot \vec{J} = \int_S da \sigma \hat{n} \cdot \vec{E} \quad (7.6)$$

Let's assume the cross-sectional area of the conductor is constant and the conductor is uniform. This lets us do the area integral trivially:

$$I = \sigma A \hat{n} \cdot \vec{E} \quad (7.7)$$

If we then do a line integral directed along the wire, such that $d\vec{\ell} \propto \hat{n}$, we have

$$I \int d\ell = \sigma A \int d\ell \hat{n} \cdot \vec{E} = \sigma A \int d\vec{\ell} \cdot \vec{E} \implies I\ell = \sigma A V \quad (7.8)$$

$$\implies V = IR \quad \text{with} \quad R = \frac{\ell}{A} \frac{1}{\sigma} \quad (7.9)$$

which is the familiar version of Ohm's law in terms of current, voltage, and resistance. We could call this the *integral version of Ohm's Law* and $\vec{J} = \sigma \vec{E}$ the differential (or local) version.

Currents and Ohm's Law (cont.)

We can also integrate the Joule heating expression to get the integrated power dissipation in a wire:

$$P = \int_V d\tau \vec{J} \cdot \vec{E} = \int_S da \int d\ell \frac{I}{A} \hat{n} \cdot \vec{E} = I V = I^2 R = \frac{V^2}{R} \quad (7.10)$$

This is the usual integral expression for Joule heating.

Steady Currents and Uniform Conductivity \implies Zero Charge Density

Do we need to worry about charge accumulation in conductors? Let's calculate the divergence of \vec{E} to find the charge density, assuming uniform conductivity:

$$\vec{\nabla} \cdot \vec{E} = \frac{1}{\sigma} \vec{\nabla} \cdot \vec{J} = 0 \quad (7.11)$$

where the first step was possible by Ohm's Law and the assumed uniformity of the conductivity and the second step by the assumption of steady-state currents. So the answer is no, as long as the conductivity is uniform, no charge density accumulates.

Currents and Ohm's Law (cont.)

Uniformity of Electric Field in a Uniform Wire

We implicitly assumed in proving the integral version of Ohm's Law above that the uniformity of the conductor implied that the field and thus the current were uniform over the cross-sectional area. We can prove this. We did not explicitly require that the electric field also be uniform with position along the wire, but we can prove that, too.

We proved above that the charge density vanishes in a uniform conductor with steady currents. Therefore, the conductor satisfies Laplace's Equation. The boundary conditions are set by the potential at the two ends with potential difference ΔV . We take these equipotentials to be transverse to the wire axis at $z = 0$ and $z = \ell$. On the outer surface of the wire, $\vec{J} \cdot \hat{n} = 0$ because no current flows out of the wire, which implies that $\vec{E} \cdot \hat{n} = 0$, which provides a boundary condition on the normal gradient of the potential. (Equivalently, this implies the charge density vanishes at the surface.)

Currents and Ohm's Law (cont.)

We guess a solution that satisfies these boundary conditions,

$$V(\vec{r}) = \frac{\Delta V}{\ell} z \quad (7.12)$$

The resulting electric field is

$$\vec{E} = -\vec{\nabla} V = -\frac{\Delta V}{\ell} \hat{z} \quad (7.13)$$

This satisfies the boundary conditions: equipotential surfaces at $z = 0$ and $z = \ell$ and normal derivative at outer surface (which is always perpendicular to \hat{z}) vanishing. Therefore, it is valid to assume that the field is uniform over the cross-sectional area of the wire *and* along the length of the wire if the conductivity is uniform and the currents are steady state, the two conditions that told us Laplace's Equation is satisfied.

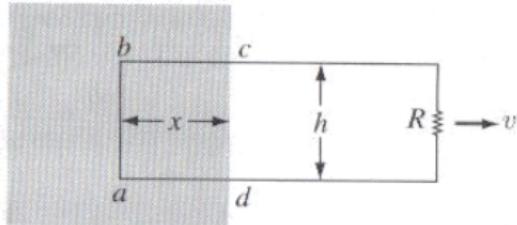
You may want to think about how this argument fails if the wire changes in some way along its length; e.g., the conductivity changes, or the wire diameter changes.

Electromotive Forces

We deviate from Griffiths somewhat in the introduction of electromotive forces; his section 7.1.2 just seems confusing.

Motional Electromotive Force

Consider a rectangular loop with a resistor in it with part of the loop's area intersecting a region of uniform magnetic field perpendicular to the loop into the page, as shown in the figure.



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Electromotive Forces (cont.)

Let's consider the force on a charge carrier in the portion of the wire that intersects the field. If $\vec{v} = v \hat{x}$ and $\vec{B} = -B \hat{z}$ (into the page), then these charges feel a Lorentz force

$$\vec{F}_{mag} = (q v \hat{x}) \times (-B \hat{z}) = q v B \hat{y} \quad (7.14)$$

Since this force is aligned with the vertical portion of the wire, the carriers in that section can move. Assuming for the moment the charge carriers are positive (the argument can be reversed if they are negative), they would start to collect at b at the top end of the vertical portion and a deficit would appear at a at the bottom end. The local electrostatic repulsion between like charges would cause the charge carriers to start flowing through the rest of the circuit and would prevent this clumping of carriers. In this way, a current is generated around the loop *without the influence of a large-scale electric field in the circuit*. If the loop is pulled at constant speed, one has steady-state current with no charge buildup.

Electromotive Forces (cont.)

There is work done on a charge carrier during its movement up the vertical portion of the wire

$$W_{ab} = \int_a^b d\vec{\ell} \cdot \vec{F}_{mag} = v B h \quad (7.15)$$

We will see below that this work is not done by the Lorentz force as suggested above (recall, the Lorentz force can do no work because $\vec{F}_{mag} \perp \vec{v}$), but it is nevertheless done. The energy gained by the charge carriers via this work is dissipated as Joule heating in the resistor.

We define the work done per unit charge on the charges as they move from a to b as the *motional electromotive force* or *motional emf*:

$$\mathcal{E} = \frac{W_{ab}}{q} = v B h \quad (7.16)$$

It has units of voltage. We thus are inclined to think of it like a voltage: it has the right units and it drives a current. But be sure to remember that it is generated by movement of the circuit in a magnetic field; it is not due to an electric field!

Electromotive Forces (cont.)

Let's also think about what force is doing the work. As we discussed some time ago, the Lorentz force does no work because $\vec{F} \perp \vec{v}$. However, a force must pull the loop. There is a force counteracting this force that the pulling force must match to keep the loop at constant speed: the Lorentz force due to the velocity the carriers have acquired in the \hat{y} direction, which we will denote by $\vec{u} = u\hat{y}$. This force is

$$-\vec{F}_{pull} = \vec{F}'_{mag} = q u \hat{y} \times -B \hat{z} = -q u B \hat{x} \quad (7.17)$$

The total velocity of the charge carriers is

$$\vec{w} = \vec{v} + \vec{u} = v \hat{x} + u \hat{y} \quad (7.18)$$

The pulling force must cancel \vec{F}'_{mag} , so the work done per unit time by the pulling force is

$$\frac{dW_{pull}}{dt} = \vec{F}_{pull} \cdot \vec{w} = q u B \hat{x} \cdot (v \hat{x} + u \hat{y}) = q u B v \quad (7.19)$$

Electromotive Forces (cont.)

Note that the charge carriers move on a diagonal line relative to the lab frame as they move from point a to point b on the wire, with this line partly in the direction of \vec{F}_{pull} . It takes the charge carriers a time $t = h/u$ to move on this trajectory since their \hat{y} direction speed is u . Therefore, the work done by \vec{F}_{pull} during the movement of a charge from a to b is:

$$W_{\text{pull}} = \frac{dW_{\text{pull}}}{dt} \frac{h}{u} = q B v h \quad (7.20)$$

$$\implies \frac{W_{\text{pull}}}{q} = B v h = \mathcal{E} \quad (7.21)$$

That is, the work done by the pulling force, per unit charge, matches the motional emf — the pulling force provides the energy that is eventually dissipated as heat as the carriers flow through the resistor. Yet all this is done without an electric field.

Electromotive Forces (cont.)

Returning to the emf itself, we can rewrite it in a useful form. We define the *magnetic flux* to be the integral of \vec{B} dotted into the normal to a surface over the surface:

$$\Phi = \int_S da \hat{n} \cdot \vec{B}(\vec{r}) \quad (7.22)$$

Using the definition of x in the figure, we have in this case

$$\Phi = B h x \quad (7.23)$$

The time derivative is

$$\frac{d\Phi}{dt} = B h \frac{dx}{dt} = -B h v \quad (7.24)$$

(x decreases with time for $v > 0$) which is just the negative of the motional emf. That is, we have

$$\mathcal{E} = -\frac{d\Phi}{dt} \quad (7.25)$$

Electromotive Forces (cont.)

Let's prove rigorously that this rule holds more generally for any shape of loop with any type of motion through an arbitrary magnetic field.

Consider the motion of the loop over a time dt . The loop is defined by a contour $\mathcal{C}(t)$ that depends on t . Each point on the loop has a velocity \vec{v} that may depend on the position on the loop. Regardless, each piece of the loop moves by the vector $\vec{v} dt$ during this time where \vec{v} is position-dependent. The charges in that piece of the loop acquire a velocity \vec{u} along the direction of the loop due to the action of the Lorentz force during that time.

The motion \vec{v} also describes the change in the area of and flux through the loop. The flux changes by

$$d\Phi = \int_{S(\mathcal{C}(t+dt))} da \hat{n}_{\mathcal{C}} \cdot \vec{B} - \int_{S(\mathcal{C}(t))} da \hat{n}_{\mathcal{C}} \cdot \vec{B} \quad (7.26)$$

We subscript \hat{n} with \mathcal{C} to distinguish it from a different \hat{n} we define below.

Electromotive Forces (cont.)

Let's rewrite the above expression in a more usable form. Consider a closed surface that consists of the surfaces defined by $\mathcal{C}(t)$ and $\mathcal{C}(t + dt)$ as well as the ribbon-like surface connecting the two contours. (If the two contours were circular loops, the ribbon-like surface would be the wall of the cylinder formed by the two contours. See Griffiths Figure 7.13 for an illustration, but note that he perversely has the loop move in the direction opposite to the obvious \hat{n}_C , while we have it move in the direction of \hat{n}_C , resulting in a flipped sign relative to what we have below.) $\vec{\nabla} \cdot \vec{B} = 0$ tells us the surface integral of $\hat{n}_S \cdot \vec{B}$ (where \hat{n}_S is the outward surface normal, different from \hat{n}_C) through this surface vanishes. That surface integral is related to the above integrals by

$$0 = \oint_{closed \ S} da \hat{n}_S \cdot \vec{B} = \int_{S(\mathcal{C}(t+dt))} da \hat{n}_C \cdot \vec{B} - \int_{S(\mathcal{C}(t))} da \hat{n}_C \cdot \vec{B} + \int_{ribbon} da \hat{n}_S \cdot \vec{B} \quad (7.27)$$

where we have used the fact that $\hat{n}_S = \hat{n}_C$ on the $S(\mathcal{C}(t + dt))$ surface but $\hat{n}_S = -\hat{n}_C$ on the $S(\mathcal{C}(t))$ surface. The negative sign is present in the latter because the orientation of \hat{n}_C that maintains the same orientation of \hat{n}_C as the contour moves has \hat{n}_C on this surface pointing into the enclosed volume rather than outward.

Electromotive Forces (cont.)

The first two terms give $d\Phi$, so

$$d\Phi = - \int_{ribbon} da \hat{n}_S \cdot \vec{B} \quad (7.28)$$

The area element on the ribbon (with outwardly directed normal as in the above integral; again, note the flipped sign relative to Griffiths Figure 7.13) is given by

$$\hat{n}_S da = d\vec{\ell} \times [\vec{r}(t + dt) - \vec{r}(t)] \quad (7.29)$$

where: $d\vec{\ell}$ is the line element along $\mathcal{C}(t)$ with orientation set by consistency with $\hat{n}_{\mathcal{C}}$ for $S(\mathcal{C}(t))$ and the right-hand rule; and $\vec{r}(t + dt)$ and $\vec{r}(t)$ are the position of that line element at t and $t + dt$. The difference between these two positions is related to \vec{v} :

$$\hat{n}_S da = d\vec{\ell} \times \vec{v} dt \quad (7.30)$$

$$\text{Therefore, } d\Phi = - \oint_{\mathcal{C}(t)} (d\vec{\ell} \times \vec{v} dt) \cdot \vec{B} \quad (7.31)$$

We turned an area integral into a line integral, but it still calculates magnetic flux.

Electromotive Forces (cont.)

Since $\vec{u} \parallel d\vec{\ell}$, we can add \vec{u} to \vec{v} to obtain \vec{w} without affecting the integral:

$$d\Phi = - \oint_{C(t)} (d\vec{\ell} \times \vec{w} dt) \cdot \vec{B} \quad (7.32)$$

Using the cyclic property of the triple vector product and moving dt to the left side, we obtain

$$\frac{d\Phi}{dt} = - \oint_{C(t)} d\vec{\ell} \cdot (\vec{w} \times \vec{B}) \quad (7.33)$$

The quantity $\vec{w} \times \vec{B}$ is just the Lorentz force per unit charge:

$$\frac{d\Phi}{dt} = - \oint_{C(t)} d\vec{\ell} \cdot \frac{\vec{F}_{mag}}{q} \quad (7.34)$$

Electromotive Forces (cont.)

The integral of the Lorentz force per unit charge integrated around the loop is the generalization for arbitrary loops of our earlier expression for the motional emf (earlier, we integrated over only the section of length h from a to b of the rectangular wire for which \vec{F}_{mag} was nonzero), so

$$\mathcal{E} = \oint_{C(t)} d\vec{\ell} \cdot \frac{\vec{F}_{mag}}{q} = -\frac{d\Phi}{dt} \quad \text{moving circuit} \quad (7.35)$$

The motional emf, as defined by the line integral of the Lorentz force per unit charge around the loop, is given by the negative of the rate of change of the magnetic flux through the loop. The signs of the line integral and the flux are set by requiring that the orientation of the line integral (via $d\vec{\ell}$) be consistent via the right-hand rule with the orientation of the surface normal \hat{n}_C used for the flux calculation.

Electromotive Forces (cont.)

Example: Alternating Current Generator

The classic and pervasive use of the above relationship is the alternating current generator. Consider a square loop placed in a uniform magnetic field and rotated about a midline at constant angular speed ω . That is, the rotation is such that, at one point of the motion, the magnetic field is normal to the loop while, one fourth of the period before or after this time, the magnetic field is in the plane of the loop. What is the motional emf around the loop generated by this motion?

The magnetic field is constant, so the flux is just given by B times the area of the loop projected onto the direction of \vec{B}_0 :

$$\Phi(t) = A \vec{B}_0 \cdot \hat{n}(t) = A B_0 \cos \omega t \quad (7.36)$$

where we have chosen $\hat{n} \parallel \vec{B}_0$ at $t = 0$. Thus, the motional emf is

$$\mathcal{E}(t) = -\frac{d\Phi}{dt} = A B_0 \omega \sin \omega t \quad (7.37)$$

This is of course how 60-Hz AC voltage is generated.

Electromotive Forces (cont.)

It is instructive to think again about the Lorentz force experienced by the charge carriers in the loop and see how it generates the motional emf. Suppose the loop is just moving past having \hat{n} pointing along \vec{B}_0 . If the loop's axis of rotation is \hat{y} , then the carriers all have a velocity parallel to \hat{x} due to the motion of the loop (this is \vec{v}).

(They also have motion in the \hat{z} direction, but this is parallel to \vec{B}_0 and thus no Lorentz force is generated.) The carriers in the sections of the loop parallel to \hat{x} cannot move in response to this force because they feel a force in the \hat{y} direction, transverse to the section of wire they are in. Those along the \hat{y} direction also feel a force along \hat{y} and they can move along \hat{y} . As the loop turns away from this orientation, the arm at $-\hat{x}\sqrt{A}/2$ has velocity in the $+\hat{x}$ direction and vice versa for the arm at $+\hat{x}\sqrt{A}/2$. Positive charge carriers in these arms feel forces in the $-\hat{y}$ and $+\hat{y}$ directions, respectively. This forces a current to flow in the counterclockwise direction ($+\hat{n}$ orientation by right-hand rule), generating a field in the $+\hat{n}$ direction.

One can see that the driven current is in the direction needed to counter the change in magnetic flux. This is an expression of *Lenz's Law*, which we will return to later.

If one taps the loop as is typical for such a generator, the tap connected to the $-\hat{x}\sqrt{A}/2$ arm will have positive voltage and the tap connected to the $+\hat{x}\sqrt{A}/2$ arm will have negative voltage because they need to drive a current in an external circuit that carries current in the right direction, from the $-\hat{x}\sqrt{A}/2$ arm to the $+\hat{x}\sqrt{A}/2$ arm.

Lecture 16:
Inductance
Magnetic Energy and Forces

Date Revised: 2014/04/24 10:00

Cleaned up and augmented material on magnetic energy,
modified lecture break

Date Given: 2014/04/22

Electromagnetic Induction

Faraday's Law

We are going to consider three different physical situations:

- ▶ *Moving loops*: As we considered above, the magnetic field is stationary but the loop is moving.
- ▶ *Moving magnetic fields*: The loop is held fixed but the magnetic field is changing because the currents sourcing the field are being translated.
- ▶ *Changing magnetic fields*: Both the loop and the sources of the field are stationary, but the currents sourcing the field are changing.

We just proved using the Lorentz force Law that the first situation results in a *motional emf*: a force that causes the flow of a current around the loop, given by Equation 7.35:

$$\mathcal{E} = \oint_{C(t)} d\vec{\ell} \cdot \frac{\vec{F}_{mag}}{q} = -\frac{d\Phi}{dt} \quad \text{moving circuit} \quad (7.38)$$

Electromagnetic Induction (cont.)

Faraday's Law consists of the *empirical observation* that the same rule applies for the second and third situations. However, and here is the subtlety, this law could not have been derived using the Lorentz force applied to the situation described above of a fixed loop and a moving and/or changing magnetic field: there is no magnetic force if the charge carriers are not moving. A natural and important corollary is that the emf that appears is not due to a magnetic force. Rather, since the charges are at rest in the second situation, the force that appears arises from a true *electric field*. Mathematically, we write Faraday's Law as

$$\mathcal{E} = \oint_{\mathcal{C}(t)} d\vec{l} \cdot \frac{\vec{F}_{elec}}{q} = -\frac{d\Phi}{dt} \quad \text{moving or changing magnetic field} \quad (7.39)$$

We see that it is identical in form to the Lorentz force law applied to a moving loop with the replacement of \vec{F}_{mag} by \vec{F}_{elec} .

Electromagnetic Induction (cont.)

Combining the two forms, and recognizing $\vec{F}_{elec}/q = \vec{E}$, we then may write a common law that applies in any situation:

$$\mathcal{E} = \oint_{C(t)} d\vec{l} \cdot \left[\vec{E} + \frac{\vec{F}_{mag}}{q} \right] = -\frac{d\Phi}{dt} = -\frac{d}{dt} \int_{S(C(t))} da \hat{n}(\vec{r}, t) \cdot \vec{B}(\vec{r}, t) \quad (7.40)$$

If there is any ambiguity in the sign, one should apply *Lenz's Law*: the emf has a sign such that the polarity of the current it would drive produces a magnetic field that counters the change in magnetic field.

Electromagnetic Induction (cont.)

Motional EMF, Faraday's Law, Galilean Relativity, and Galilean Field Transformations

When first proposed, Faraday's Law was an empirical observation. However, it could have been justified using the principle of Galilean relativity: physics is the same in all frames moving at constant speed.

Consider the problem of the moving magnetic field. One could go to the rest frame of the magnetic field and consider the loop to be moving. The magnetic force implied by the motional emf appears. In Galilean relativity, forces are invariant upon change of inertial (fixed velocity) frame. This would imply that the magnetic force in the field-fixed frame is still present in the loop-fixed frame, but now we interpret it as an electric force because the loop is not moving.

In the case of changing magnetic fields, we simply have to invoke the expectation that the loop has no way of knowing whether it experiences a changing field because the current sourcing the field is moving or because it is changing: it only knows about the field that results, not the source of the field.

This Galilean relativity argument was, however, not recognized until after Faraday's observation.

Electromagnetic Induction (cont.)

We can make use of this argument to understand how electric and magnetic fields mix with each other under such Galilean (nonrelativistic) transformations. Let's assume we have written down our law, Equation 7.40, in both the rest frame of the loop and in the lab frame in which the loop is moving. The fields and position vectors in the loop rest frame are given ' symbols, the ones in the lab frame have no primes. The total emf can be measured explicitly using a voltmeter, and it is a scalar that is independent of frame (the reading on the voltmeter doesn't change if you see the voltmeter moving with the loop!). So we can equate the lab and rest frame expressions through \mathcal{E} :

$$\oint_{C'} d\vec{\ell}' \cdot \vec{E}' = \oint_{C(t)} d\vec{\ell} \cdot \left[\vec{E} + \frac{\vec{F}_{mag}}{q} \right] \quad (7.41)$$

Now, let's use our expression for the magnetic force term from our derivation of Equation 7.35, dropping the \vec{u} contribution that we had added in:

$$\oint_{C'} d\vec{\ell}' \cdot \vec{E}' = \oint_{C(t)} d\vec{\ell} \cdot \left[\vec{E} + \vec{v} \times \vec{B} \right] \quad (7.42)$$

Electromagnetic Induction (cont.)

Since the circuit is arbitrary, we may thus conclude

$$\vec{E}' = \vec{E} + \vec{v} \times \vec{B} \quad (7.43)$$

The equation can be taken to be completely general because adding a standard electrostatic field to both sides would leave the statement true while accounting for such electrostatic fields. Therefore, this is a rule for how electric fields transform from one frame to another under Galilean relativity, regardless of the source of the field. Electric fields are not the same in a fixed and a moving frame if magnetic fields are present, even before special relativity is considered! Special relativity then adds correction coefficients to the above equation.

It is important to note that the expectation that the electrostatic fields do not depend on frame has been an assertion so far, based on the assumption that Coulomb's Law is unaffected by whether the charges are moving or not. We will return to this point later in connection to Maxwell's Equations, as it will lead to a symmetrization of the above equation between \vec{E} and \vec{B} .

Electromagnetic Induction (cont.)

Example: A Stationary Alternating Current Generator

Recall the previous example of an AC generator that used a rotating square loop in a constant magnetic field. Instead, hold the loop fixed but assume that the magnetic field is being varied sinusoidally, $\vec{B}(t) = \vec{B}_0 \cos \omega t$. Then the flux is

$$\Phi(t) = A \vec{B}(t) \cdot \hat{n} = A B_0 \cos \omega t \quad (7.44)$$

Therefore, the emf generated is

$$\mathcal{E}(t) = -\frac{d\Phi}{dt} = A B_0 \omega \sin \omega t \quad (7.45)$$

just as before.

Electromagnetic Induction (cont.)

Differential Version of Faraday's Law

Consider the special case of an arbitrary circuit fixed in space. Equation 7.40 tells us

$$\oint_C d\vec{\ell} \cdot \vec{E} = -\frac{d}{dt} \int_{S(C)} da \hat{n}(\vec{r}) \cdot \vec{B}(\vec{r}, t) \quad (7.46)$$

Let's use Stokes' Theorem on the left side, and, since the contour is time-independent, we can move the time derivative inside the integral. We turn it into a partial derivative to make it clear that we do not need to worry about any possible time-dependence of \vec{r} (of which there is none here). This yields

$$\oint_{S(C)} da \hat{n}(\vec{r}) \cdot [\vec{\nabla} \times \vec{E}(\vec{r})] = - \int_{S(C)} da \hat{n}(\vec{r}) \cdot \frac{\partial \vec{B}(\vec{r}, t)}{\partial t} \quad (7.47)$$

Since the loop is arbitrary, the integrands must be equal:

$$\boxed{\vec{\nabla} \times \vec{E}(\vec{r}) = -\frac{\partial \vec{B}(\vec{r}, t)}{\partial t}} \quad (7.48)$$

This differential version of Faraday's Law is the generalization of $\vec{\nabla} \times \vec{E} = 0$ for time-dependent situations.

Electromagnetic Induction

Biot-Savart and Ampere's Law for the Induced Electric Field in the Absence of Charges

If we consider the special case of no charge density, then we have

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad (7.49)$$

This is mathematically identical to the equations of magnetostatics,

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad (7.50)$$

In magnetostatics, we saw that the above two equations, combined with the assumption $\vec{\nabla} \cdot \vec{A} = 0$, yielded Poisson's Equation for \vec{A} with $\mu_0 \vec{J}$ as the source (Equation 5.54). By correspondence, we can thus state

$$\boxed{\vec{E} = \vec{\nabla} \times \vec{A}_E \quad \nabla^2 \vec{A}_E = \frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla} \cdot \vec{A}_E = 0} \quad (7.51)$$

Electromagnetic Induction (cont.)

Now, if we assume appropriate boundary conditions — fields falling off at infinity, no surfaces on which the vector potential or field are specified — then we saw that we know what the Green Function is for this situation and that one could write an explicit formula for \vec{A} in terms of \vec{J} (Equation 5.54). The same holds here, so we may say

$$\boxed{\vec{A}_E(\vec{r}) = -\frac{1}{4\pi} \int_{\mathcal{V}} d\tau' \frac{\frac{\partial \vec{B}(\vec{r}')}{\partial t}}{|\vec{r} - \vec{r}'|}} \quad (7.52)$$

Finally, we may take the curl of the above expression to recover the analogue of the Biot-Savart Law. We did this backwards in the case of magnetostatics: we started with the Biot-Savart Law and derived that the field could be written as the curl of the form of the vector potential corresponding to the above. Nevertheless, that proof could be reversed, so we may conclude that the analogous Biot-Savart Law holds (compare to Equation 5.32)

$$\boxed{\vec{E}(\vec{r}) = -\frac{1}{4\pi} \int_{\mathcal{V}} d\tau' \frac{\frac{\partial \vec{B}(\vec{r}')}{\partial t} \times (\vec{r} - \vec{r}')}{| \vec{r} - \vec{r}' |^3} = -\frac{1}{4\pi} \frac{\partial}{\partial t} \int_{\mathcal{V}} d\tau' \frac{\vec{B}(\vec{r}') \times (\vec{r} - \vec{r}')}{| \vec{r} - \vec{r}' |^3}} \quad (7.53)$$

where we pulled the time derivative outside the integral under the assumption that the volume itself is time-independent.

Electromagnetic Induction (cont.)

We also note that, because \vec{E} satisfies the analogue of Ampere's Law, one can apply standard Ampere's Law techniques for finding \vec{E} when $\frac{\partial \vec{B}}{\partial t}$ is given.

Caution: We have made the *quasistatic* assumption, that all time derivatives are small enough that the propagation time for disturbances in the magnetic fields is much less than the timescales on which the field vary. This is what allows us to use the magnetostatic formulae in time-varying situations. If the time derivatives become large, then one needs the full formalism of electromagnetic waves, which we will develop later.

Electromagnetic Induction (cont.)

Example: Induced Electric Field for Coaxial Conductors (Griffiths 7.16, modified)

An alternating current $I = I_0 \cos \omega t$ flows down a long straight wire of radius a and returns along a coaxial conducting tube of inner radius b . (The outer radius will turn out to not matter.) Both conductors are assumed to be perfect conductors (infinite conductivity). We want to find the induced electric field as a function of the transverse radius s in cylindrical coordinates.

For reasons that we will be able to explain later when we discuss EM waves in the presence of conductors, the currents flow in sheets at the surfaces of the conductors because they have infinite conductivity.

The magnetic field inside the inner conductor vanishes because the field of a cylindrical sheet of current of this type is zero inside the cylindrical sheet. (One can show this by Ampere's Law with a loop in a plane containing the z -axis.) Outside the wire, the field is the usual $\vec{B}(s, t) = \hat{\phi} \mu_0 I(t) / 2\pi s$. The magnetic field of the return-current cylinder is zero inside (same reason as above). Outside the return-current sheet, its magnetic field is that of a wire carrying the total return current, which has the same magnitude but opposite sign of the field of the inner conductor. Thus, the total magnetic field is the inner conductor's magnetic field between the conductors and is zero inside the inner conductor and outside the inner wall of the outer conductor.

The system has azimuthal and z -translation symmetry, so the induced electric field must have the form $\vec{E} = E_s(s) \hat{s} + E_\phi(s) \hat{\phi} + E_z(s) \hat{z}$.

Electromagnetic Induction (cont.)

If we think about what kind of Amperian loop has a nonzero flux of $\partial \vec{B} / \partial t$, it is a loop in the sz plane with normal in the $\hat{\phi}$ direction. Let's first consider a loop of this kind with one z leg at infinity and the other at $s > b$. The contributions to the loop integral of the electric field along the two radial legs cancel, and the contribution from the leg at infinity vanishes assuming the fields fall off as $s \rightarrow \infty$, so this loop only gets a contribution from the z leg at finite radius, which picks out $E_z(s > b)$. The enclosed $\partial \vec{B} / \partial t$ vanishes, so we can conclude $E_z(s > b) = 0$.

Now, repeat with one z leg between a and b and one z leg outside the cylinder. The radial legs cancel and the z leg outside the cylinder contributes nothing. If the loop length in the z direction being ℓ , we thus have

$$E_z(a < s < b)\ell = - \int_0^\ell dz \int_s^b ds' \frac{\partial B_\phi(s', t)}{\partial t} = -\frac{\mu_0}{2\pi} \frac{\partial I}{\partial t} \ell \int_s^b \frac{ds'}{s'} \quad (7.54)$$

$$= \frac{\mu_0}{2\pi} \omega I_0 \ell \sin \omega t \ln \frac{b}{s} \quad (7.55)$$

$$\implies E_z(a < s < b) = \frac{\mu_0}{2\pi} \omega I_0 \sin \omega t \ln \frac{b}{s} \quad (7.56)$$

Note the sign: taking the loop normal to be $\hat{\phi}$ implies that the z leg with the nonzero contribution yields a positive contribution. Then the usual minus sign enters, which is cancelled by the derivative of $\cos \omega t$.

Electromagnetic Induction (cont.)

For $E_z(s < a)$, the same argument holds, except that the flux no longer increases as s is decreased, so we obtain the same E_z value obtained at $s = a$:

$$E_z(s < a) = E_z(s = a) = \frac{\mu_0}{2\pi} \omega I_0 \sin \omega t \ln \frac{b}{a} \quad (7.57)$$

It might be troubling that there is now an electric field inside a perfect conductor. The arguments we made in electrostatics that there can be no such field assumed a static situation, so there is no contradiction.

We can see $E_\phi(s)$ vanishes by using a loop in the $s\phi$ plane that has radial legs (ϕ constant) and azimuthal legs (s constant). One azimuthal leg can be taken to infinity so it yields no contribution, and the radial legs' contributions cancel, leaving only the contribution from the azimuthal leg at finite radius. But, unlike the E_z case, this loop has no magnetic flux through it, so $E_\phi(s) = 0$.

Finally, consider E_s , which we can show to vanish by both a heuristic and a mechanical argument. As we argued above, E_s can be a function of s only and must be independent of z . Suppose E_s points outward along \hat{s} at a particular s and consider $E_s(s, z = 0)$. If we rotate the system about this direction by 180° , then the current changes direction. But $E_s(s, z = 0)$ cannot change direction (sign) — it is tied to the current distribution. Yet the reversal of the direction of the current changes the sign of \vec{B} and thus $\partial\vec{B}/\partial t$. Then, by the Biot-Savart Law for \vec{E} , \vec{E} should change sign. We have a contradiction unless $E_s(s, z = 0) = 0$. Because of z -translation symmetry, the same must hold at any z .

Electromagnetic Induction (cont.)

More rigorously, consider the Biot-Savart integral for \vec{E} . Given that \vec{B} and $\partial\vec{B}/\partial t$ are both proportional to $\hat{\phi}$, the vector $\vec{r} - \vec{r}'$ must have a piece proportional to \hat{z} to yield a contribution to the \hat{s} component of \vec{E} . But \vec{B} is independent of z , while the \hat{z} component of $\vec{r} - \vec{r}'$ is odd about $z = z'$. So the integrand is odd about $z = z'$ and the integral vanishes.

Thus,

$$\vec{E}(s, t) = \hat{z} \frac{\mu_0}{2\pi} \omega I_0 \sin \omega t \ln \frac{b}{s} \quad (7.58)$$

One can easily see the sign makes sense. This \vec{E} tries to drive a current parallel to the current already flowing in the central conductor. When the current in the central conductor is decreasing, the electric field is increasing to try to drive a current in the same direction in which current is being lost by the decreasing current. It tries to generate a magnetic field that tries to compensate for the magnetic field that is being removed by the decreasing central conductor current. And vice versa for an increasing current.

Inductance

Mutual Inductance

We have so far considered magnetic fields and fluxes in the abstract, without any concern from where they come from. But they are generated by currents, so it is natural to want to connect the Faraday's Law emf to changing currents. We do that through the mutual inductance.

Consider two circuits \mathcal{C}_1 and \mathcal{C}_2 . Suppose a current I_1 is flowing in \mathcal{C}_1 . The magnetic flux at \mathcal{C}_2 is

$$\Phi_2 = \int_{\mathcal{S}(\mathcal{C}_2)} da_2 \hat{n}_2 \cdot \vec{B}_1(\vec{r}_2) = \int_{\mathcal{S}(\mathcal{C}_2)} da_2 \hat{n} \cdot \vec{\nabla} \times \vec{A}_1(\vec{r}_2) = \oint_{\mathcal{C}_2} d\vec{l}_2 \cdot \vec{A}_1(\vec{r}_2) \quad (7.59)$$

where we used the fact that \vec{B} is derived from a vector potential and also Stokes' Theorem. Now, let's use the relation between the current in \mathcal{C}_1 and \vec{A}_1 using the usual solution of the Poisson's Equation for \vec{A}_1 :

$$\Phi_2 = \frac{\mu_0}{4\pi} \oint_{\mathcal{C}_2} d\vec{l}_2 \cdot \oint_{\mathcal{C}_1} \frac{I_1 d\vec{l}_1}{|\vec{r}_2 - \vec{r}_1|} = \frac{\mu_0}{4\pi} I_1 \oint_{\mathcal{C}_2} \oint_{\mathcal{C}_1} \frac{d\vec{l}_2 \cdot d\vec{l}_1}{|\vec{r}_2 - \vec{r}_1|} \quad (7.60)$$

Inductance (cont.)

We rewrite this as follows:

$$\Phi_2 = M_{21} I_1 \quad M_{21} = \frac{\mu_0}{4\pi} \oint_{C_2} \oint_{C_1} \frac{d\vec{l}_2 \cdot d\vec{l}_1}{|\vec{r}_2 - \vec{r}_1|}$$

Neumann Formula (7.61)

where M_{21} is the mutual inductance between C_1 and C_2 and has units of Henries (volt-second/amp). Two important characteristics:

- ▶ $M_{21} = M_{12}$ because the definition is symmetric.
- ▶ M_{21} is a completely geometric quantity: it does not care about the amount of current flowing, just on the relative positions of the two contours.

The mutual inductance is interesting because we can now take the time derivative and calculate the emf at C_2 due to a change in I_1 :

$$\mathcal{E}_2 = -\frac{d\Phi_2}{dt} = -M_{21} \frac{dI_1}{dt}$$

(7.62)

If unclear, the sign should be chosen to satisfy Lenz's Law.

Inductance (cont.)

Self-Inductance

The above derivation works even when C_1 and C_2 are identical: a current loop induces an emf on itself. In practice, calculating the integral can be difficult because of the singularity at $\vec{r}_1 = \vec{r}_2$, but one can be assured that self-inductance exists and is not infinite. The symbol used is L and the corresponding equations are

$$\Phi = LI \quad L = \frac{\mu_0}{4\pi} \oint_C \oint_C \frac{d\vec{\ell}_2 \cdot d\vec{\ell}_1}{|\vec{r}_2 - \vec{r}_1|} \quad \mathcal{E} = -L \frac{dI}{dt} \quad (7.63)$$

In both the cases of mutual inductance and self-inductance, one rarely does the integral directly. Instead, one tries to find the field using Ampere's Law, then calculate the flux, and finally get M or L from Φ/I . This usually eliminates the singularity in the above integral.

Inductance (cont.)

Generalization to Volume Currents

It is straightforward to generalize the above to volume currents by using the usual relation between the vector potential and the volume current density

$$\vec{A}(\vec{r}) = \frac{1}{4\pi} \int_{\mathcal{V}} d\tau' \frac{\vec{J}(\vec{r}')}{|\vec{r} - \vec{r}'|} \quad (7.64)$$

which yields

$$M_{ij} = \frac{\mu_0}{4\pi} \frac{1}{I_1 I_2} \int_{\mathcal{V}_i} d\tau'_1 \int_{\mathcal{V}_j} d\tau'_2 \frac{\vec{J}(\vec{r}'_1) \cdot \vec{J}(\vec{r}'_2)}{|\vec{r} - \vec{r}'|} \quad (7.65)$$

$$L = \frac{\mu_0}{4\pi} \frac{1}{I^2} \int_{\mathcal{V}} d\tau'_1 \int_{\mathcal{V}} d\tau'_2 \frac{\vec{J}(\vec{r}'_1) \cdot \vec{J}(\vec{r}'_2)}{|\vec{r} - \vec{r}'|} \quad (7.66)$$

where \mathcal{V}_i is the volume of the i th inductor. We notice that the currents do not drop out as cleanly, but, assuming linear behavior of the current flow (the current does not flow differently as the overall magnitude of the current is changed), we expect $\vec{J} \propto \mathbf{I}$ and indeed the inductances are purely geometrical quantities as for the case of line currents.

Inductance (cont.)

Example: Self and Mutual Inductances of Solenoids

Let's first calculate the self-inductance of a solenoid of radius a . Recall that the field of a solenoid is only nonzero inside it and has value

$$\vec{B} = \mu_0 n I \hat{z} \quad (7.67)$$

where n is the number of turns per unit length, the solenoid axis is along \hat{z} , and the current flows along the $\hat{\phi}$ direction. The magnetic flux threading the solenoid and the self-inductance are therefore

$$\Phi = n \ell \pi a^2 B = \mu_0 n^2 \ell \pi a^2 I \quad \Rightarrow \quad L = \mu_0 n^2 \ell \pi a^2 \quad (7.68)$$

If we have two interpenetrating solenoids with turn densities n_1 and n_2 , radii $a_1 < a_2$, and lengths $\ell_1 < \ell_2$, then the flux into solenoid 1 of the field from solenoid 2 and the mutual inductance are

$$\Phi_2 = n_1 \ell_1 \pi a_1^2 B_2 = \mu_0 n_1 n_2 \ell_1 \pi a_1^2 I_1 \quad \Rightarrow \quad M = \mu_0 n_1 n_2 \ell_1 \pi a_1^2 \quad (7.69)$$

It is interesting and useful to note that we could calculate Φ_1 using M given the symmetry of M . This is very convenient, as calculating the field of solenoid 1 beyond its ends is nontrivial.

Magnetic Energy and Forces

Magnetic Energy in Terms of Currents

Let's consider the work that has been done to drive current against the back emf in an inductive object (e.g., a solenoid). That is, when current flows through an inductive object, it flows across a potential drop \mathcal{E} , but no energy is gained by the current, so the energy must go into the magnetic field of the inductor. The rate at which this work is being done is

$$\frac{dW}{dt} = \text{Power} = -I\mathcal{E} = LI \frac{dI}{dt} \quad (7.70)$$

We can integrate this over time to get the total work done and energy stored:

$$W = \frac{1}{2} LI^2 \quad (7.71)$$

Magnetic Energy and Forces (cont.)

We can rewrite the above using the vector potential:

$$LI = \Phi = \int_{S(C)} da \hat{n} \cdot \vec{B} = \oint_C d\vec{\ell} \cdot \vec{A} \quad (7.72)$$

$$\Rightarrow W = \frac{1}{2} LI^2 = \frac{I}{2} \oint_C d\vec{\ell} \cdot \vec{A} = \frac{1}{2} \oint_C d\ell \vec{I} \cdot \vec{A} \quad (7.73)$$

We can obviously generalize this for volume currents to

$$W = \frac{1}{2} \int_V d\tau \vec{J} \cdot \vec{A} \quad (7.74)$$

Magnetic Energy and Forces (cont.)

If we have a system of N inductive elements with inductance matrix M_{ij} ($M_{ii} \equiv L_i$, $M_{ij} = M_{ji}$), we can generalize the above as follows, imagining that we turn on the currents in the order $i = 1, 2, \dots, N$ so that we have to maintain the current I_i against the emf on inductor i felt due to its changing current as well as the changing currents of the inductors with $j > i$:

$$\begin{aligned}\frac{dW}{dt} &= \sum_{i=1}^N \frac{dW_i}{dt} = \sum_{i=1}^N (-I_i \mathcal{E}_i) = \sum_{i=1}^N \left[I_i M_{ii} \frac{dI_i}{dt} + \sum_{j>i}^N I_j M_{ij} \frac{dI_j}{dt} \right] \\ \implies W &= \sum_{i=1}^N \left[\frac{1}{2} M_{ii} I_i^2 + \sum_{j>i}^N M_{ij} I_i I_j \right] = \frac{1}{2} \sum_{i,j=1}^N M_{ij} I_i I_j\end{aligned}\quad (7.75)$$

If rewrite all our relations using matrix notation, with \underline{I} being a column vector of currents, $\underline{\Phi}$ being a column vector of fluxes, and $\underline{\underline{M}}$ being the matrix of mutual inductances, we have

$$\underline{\Phi} = \underline{\underline{M}} \underline{I} \quad W = \frac{1}{2} \underline{\Phi}^T \underline{I} = \frac{1}{2} \underline{I}^T \underline{\underline{M}} \underline{I} \quad (7.76)$$

The version in terms of the fields is applicable, unchanged for multiple inductors as long as the volume \mathcal{V} includes all of them.

Magnetic Energy and Forces (cont.)

Magnetic Energy in Terms of Magnetic Field

Let's manipulate the above to try to get it in terms of the magnetic field. We can use Ampere's Law to obtain

$$W = \frac{1}{2\mu_0} \int_V d\tau \vec{A} \cdot (\vec{\nabla} \times \vec{B}) \quad (7.77)$$

We use the product rule for the divergence of a cross-product,
 $\vec{\nabla} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{\nabla} \times \vec{a}) - \vec{a} \cdot (\vec{\nabla} \times \vec{b})$ to rewrite this as

$$W = \frac{1}{2\mu_0} \int_V d\tau \left[\vec{B} \cdot (\vec{\nabla} \times \vec{A}) - \vec{\nabla} \cdot (\vec{A} \times \vec{B}) \right] \quad (7.78)$$

$$= \frac{1}{2\mu_0} \int_V d\tau |\vec{B}|^2 - \frac{1}{2\mu_0} \oint_{S(V)} da \hat{n} \cdot (\vec{A} \times \vec{B}) \quad (7.79)$$

Now, the original volume integral was over only the region containing the current, but the volume integral could be extended to a larger region since there would be no additional contribution. So we do the usual thing and expand the volume to include all of space and take the surface term to infinity.

Magnetic Energy and Forces (cont.)

We assume that $\vec{A} \times \vec{B}$ falls off more quickly than $1/r^2$ (which they do for finite current distributions) so that the surface term goes to zero, or that the particulars of the configuration ensure the integral vanishes even if the current distribution is not finite and we expect a finite energy. Therefore,

$$W = \frac{1}{2\mu_0} \int d\tau |\vec{B}|^2 \quad (7.80)$$

Thus we see that the magnetic energy is just given by the integral of the square of the field. In this picture, we think of the magnetic energy as stored in the field rather than in the currents.

On the point about the surface term: For an infinite solenoid, the surface term only includes the endcaps of the solenoid, since \vec{B} vanishes outside the solenoid. The contributions of the two endcaps vanish because $\vec{A} \times \vec{B}$ points along \hat{s} in cylindrical coordinates, but the endcap's normal is along \hat{z} . For an infinite wire, even when calculated per unit length, all the terms are logarithmically infinite (even if one does the calculation using $\vec{J} \cdot \vec{A}$), so there is not much point in discussing that case.

It is interesting to think about how it is possible to store energy in a magnetic field given that the field can do no work. One has to think of this as the work done to drive against the induced electric field as the field was increased from zero to its final value.

Magnetic Energy and Forces (cont.)

Example: Magnetic Energy in a Solenoid

A solenoid of radius a with n turns per unit length and current I has a field $B = \mu_0 n I$. Therefore, the magnetic energy in a solenoid of length ℓ is

$$W = \frac{1}{2\mu_0} B^2 = \frac{1}{2} \mu_0 n^2 I^2 \pi a^2 \ell \quad (7.81)$$

Note that we can extract from this the self-inductance using $W = LI^2/2$, yielding $L = \mu_0 n^2 \pi a^2 \ell$ as we obtained by calculating the flux. To put some numbers on this, the LHC CMS experiment (<http://home.web.cern.ch/about/experiments/cms>) has a solenoid with a field of 4 T with radius $a = 3$ m and length 13 m. The stored energy is therefore about 2.5 gigajoules, an enormous number.

Example: Magnetic Energy in a Coaxial Cable

This is Griffiths Example 7.13. For a coaxial cable of length ℓ with inner and outer conductor radii a and b , the energy and resulting self-inductance are

$$W = \frac{\mu_0}{4\pi} I^2 \ell \ln \frac{b}{a} \quad L = \frac{\mu_0}{2\pi} \ell \ln \frac{b}{a} \quad (7.82)$$

Magnetic Energy and Forces (cont.)

Magnetic Energy of an Assembly of Free Currents in the Presence of Magnetizable Materials

This is done similarly to the electrostatic case. Consider the differential of work the battery must supply to change the *free currents*. Starting from Equation 7.74, dropping the $1/2$ that came from \vec{A} being proportional to \vec{J} in the integration, it is

$$\delta W = \int_V d\tau \vec{A} \cdot \delta \vec{J} = \int_V d\tau \vec{A} \cdot (\vec{\nabla} \times \delta \vec{H}) \quad (7.83)$$

Apply the same algebra and the same discarding of the surface term as in free space:

$$\delta W = \int_V d\tau \delta \vec{H} \cdot (\vec{\nabla} \times \vec{A}) = \int_V d\tau \delta \vec{H} \cdot \vec{B} \quad (7.84)$$

For nonlinear materials, we would need to apply the specific $\vec{B}(\vec{H})$ function go further. For linear materials, we use $\delta \vec{H} = \delta \vec{B}/\mu$ to do the integral and obtain the expected analogy to the free-space result:

$$W = \frac{1}{2\mu} \int_V d\tau |\vec{B}|^2 = \frac{\mu}{2} \int_V d\tau |\vec{H}|^2 = \frac{1}{2} \int_V d\tau \vec{H} \cdot \vec{B} \quad (7.85)$$

Magnetic Energy and Forces (cont.)

Magnetic Energy of a Magnetizable Material in an External Field

We can calculate this along the lines of the derivation we did for polarizable materials. Let's assume that we have a configuration of currents that generates fields \vec{B}_1 and \vec{H}_1 in a volume containing a permeable material μ_1 . Now, bring in a material of permeability μ_2 such that it occupies a volume \mathcal{V}_2 contained in \mathcal{V} while holding the source currents fixed. The fields change to \vec{B}_2 and \vec{H}_2 .

The energy difference we want to calculate is

$$U_2 - U_1 = \frac{1}{2} \int d\tau \left[\vec{B}_2 \cdot \vec{H}_2 - \vec{B}_1 \cdot \vec{H}_1 \right] \quad (7.86)$$

We can apply similar manipulations as we did for the electrostatic case. First, we rewrite the above as

$$U_2 - U_1 = \frac{1}{2} \int d\tau \left[\vec{B}_2 \cdot \vec{H}_1 - \vec{B}_1 \cdot \vec{H}_2 \right] + \frac{1}{2} \int d\tau \left[\vec{B}_1 + \vec{B}_2 \right] \cdot \left[\vec{H}_2 - \vec{H}_1 \right] \quad (7.87)$$

Magnetic Energy and Forces (cont.)

Since $\vec{\nabla} \cdot [\vec{B}_1 + \vec{B}_2] = 0$, it can be derived from a vector potential \vec{A} , allowing us to rewrite the second term as

$$\frac{1}{2} \int d\tau [\vec{H}_2 - \vec{H}_1] \cdot (\vec{\nabla} \times \vec{A}) \quad (7.88)$$

We use again the vector identity $\vec{\nabla} \cdot (\vec{a} \times \vec{b}) = \vec{b} \cdot (\vec{\nabla} \times \vec{a}) - \vec{a} \cdot (\vec{\nabla} \times \vec{b})$ to integrate by parts, and we turn the divergence into a surface term that we can discard because $\vec{H}_2 - \vec{H}_1$ should vanish as we go far from the permeable material, yielding for the second term

$$\frac{1}{2} \int d\tau \vec{A} \cdot \vec{\nabla} \times (\vec{H}_2 - \vec{H}_1) \quad (7.89)$$

The curl in the integrand vanishes because \vec{H}_2 and \vec{H}_1 are sourced by the same free currents. We are thus left with

$$U_2 - U_1 = \frac{1}{2} \int d\tau [\vec{B}_2 \cdot \vec{H}_1 - \vec{B}_1 \cdot \vec{H}_2] \quad (7.90)$$

Magnetic Energy and Forces (cont.)

Applying linearity $\vec{B} = \mu \vec{H}$, we then obtain

$$U_2 - U_1 = \frac{1}{2} \int d\tau (\mu_2 - \mu_1) \vec{H}_2 \cdot \vec{H}_1 \quad (7.91)$$

Finally, we recognize $\mu_2 - \mu_1 = 0$ except in the \mathcal{V}_2 , so

$$U_2 - U_1 = \frac{1}{2} \int_{\mathcal{V}_2} d\tau (\mu_2 - \mu_1) \vec{H}_2 \cdot \vec{H}_1 = \frac{1}{2} \int_{\mathcal{V}_2} d\tau \left(\frac{1}{\mu_1} - \frac{1}{\mu_2} \right) \vec{B}_2 \cdot \vec{B}_1 \quad (7.92)$$

This is the analogue of Equation 4.81 aside from a sign flip, which mechanically is due to the fact that $\vec{B} = \mu \vec{H}$ (rather than $\vec{H} = \mu \vec{B}$). If we take $\mu_1 = \mu_0$ and $\mu_2 = \mu$, we can use $\vec{M} = (\mu/\mu_0 - 1)\vec{H}$ to rewrite this as

$$W = U_2 - U_1 = \frac{1}{2} \int_{\mathcal{V}_2} d\tau \vec{M} \cdot \vec{B} \quad \iff \quad w = \frac{1}{2} \vec{M} \cdot \vec{B} \quad (7.93)$$

where \vec{M} is the magnetization density of the volume occupied by μ and \vec{B} is the magnetic field in the absence of the permeable material. There is a sign flip relative to the electrostatic case (Equation 4.82) that, mechanically, came from the sign flip in Equation 7.92.

Magnetic Energy and Forces (cont.)

How do we understand this sign flip conceptually? Trying to track the sign through the derivation is not illuminating. But it can be understood by comparing to our calculation of the energy of magnetic dipole in an external field, Equation 5.137, where we assumed that the magnetic field was given and held fixed without our having to account for how this was done. In that case, the potential energy of the configuration was $U = -\vec{m} \cdot \vec{B}$. (The factor of 1/2 comes from the linear relationship between \vec{m} and \vec{B} and the integration from zero field to \vec{B} , which is not important for this discussion). We see that we have a sign flip relative to that situation. It is sensible, then, to attribute the sign flip to the fact that, in deriving the expression $w = |\vec{B}|^2/2\mu_0$ that was the starting point for this derivation, we accounted for the work done by the batteries to maintain the free currents as the permeable material was brought in. No such work was required in the electrostatic case to hold the free charges fixed as a dielectric was brought in. The distinction is that the force applied to hold the free charges fixed is not associated with a displacement, so no work is done by those forces, while the force needed to keep the currents flowing does cause a displacement of the charges forming the currents and so work is done in the magnetic case.

Magnetic Energy and Forces (cont.)

Force and Torque on a Linear, Homogeneous Permeable Material in an External Field with Currents Fixed

Let's consider what happens if we have an infinitesimal generalized displacement of one of our inductors at fixed current, which causes changes in mutual inductance and thus field energy:

$$dW_{field}|_I = \frac{1}{2} \sum_{i,j=1}^N I_i I_j \frac{\partial M_{ij}}{\partial \xi} d\xi \quad (7.94)$$

If we assume fixed currents, we need to account for the work done by the batteries to keep these currents fixed under the displacement, which causes infinitesimal changes in magnetic fluxes (via changes in inductance) and thus emfs in the circuits for an infinitesimal time dt during which the displacement $d\xi$ has occurred:

$$dW_{bat}|_I = \sum_{i=1}^N I_i |\mathcal{E}_i| dt |_I = \sum_{i=1}^N I_i \sum_{j=1}^N d\Phi_{ij}|_I = \sum_{i=1}^N I_i \sum_{j=1}^N \left. \frac{\partial \Phi_{ij}}{\partial \xi} \right|_I d\xi = \sum_{i,j=1}^N I_i I_j \frac{\partial M_{ij}}{\partial \xi} d\xi \quad (7.95)$$

Note how we hold the currents fixed in the differentials, only inferring changes in flux from changes in inductance.

Magnetic Energy and Forces (cont.)

The total change in the energy of the system is then obtained by subtracting the work done by the battery from the field energy gained via the displacement:

$$dW_{tot}|_I = dW_{field}|_I - dW_{bat}|_I = dW_{field}|_I - 2 dW_{field}|_I = - dW_{field}|_I \quad (7.96)$$

Therefore, the generalized force in ξ is

$$F_\xi|_I = - \left(\frac{\partial W_{tot}}{\partial \xi} \right)_I = \left(\frac{\partial W_{field}}{\partial \xi} \right)_I = \frac{1}{2} \sum_{i,j=1}^N I_i I_j \frac{\partial M_{ij}}{\partial \xi} = \frac{1}{2} \mathbf{I}^T \left[\frac{\partial}{\partial \xi} \underline{\underline{M}} \right] \mathbf{I} \quad (7.97)$$

We have a result (imperfectly) analogous to Equations 4.93 and 4.94.

Magnetic Energy and Forces (cont.)

However, note one very important subtlety. In the expression for the force at fixed voltage in terms of the capacitance, we get the right result by simply holding the voltage fixed when we take the negative derivative of the field energy with respect to the displacement (Equation 4.94). Somehow the effect of the battery is taken into account properly in the field energy alone when it is written in terms of voltages. (You can check this with the simple parallel-plate capacitor example, with ξ being the plate separation. At fixed charge, the plates are attracted to each other; at fixed voltage, the plates are repelled because the field energy goes down by the plates moving further apart ($E^2 \propto 1/d^2$ at fixed V , volume only increases as d , so the energy stored in the field goes down as the plates move apart, and charge flows off the plates to reduce the field, so the battery also prefers for the plates to move apart. This is the sign of the force one gets from Equation 4.94). Here, to properly take into account the effect of the battery, we need to take the *positive derivative of the field energy with respect to the displacement*. This sign flip is related to Lenz's Law, for which there is no analogue in electrostatics.

Magnetic Energy and Forces (cont.)

Force and Torque on a Linear, Homogeneous Permeable Material in an External Field with Fluxes Fixed

For completeness, we can state what the analogy to holding charges fixed is: holding fluxes fixed. If $d\Phi/dt = 0$, then there are no emfs and there is no need for a battery to do work to drive currents against those emfs. So we only need to consider $dW_{field}|_\Phi$. We can directly calculate the generalized force from the energy holding the fluxes fixed:

$$F_\xi|_\Phi = - \left(\frac{\partial W_{field}}{\partial \xi} \right)_\Phi = - \frac{1}{2} \sum_{i,j=1}^N \Phi_i \Phi_j \frac{\partial M_{ij}^{-1}}{\partial \xi} = - \frac{1}{2} \Phi^T \left[\frac{\partial}{\partial \xi} \underline{\underline{M}}^{-1} \right] \Phi \quad (7.98)$$

which is the analogue of Equation 4.84.

It's not entirely clear at a microscopic level (*i.e.*, what has to happen to the currents) how one maintains fixed fluxes as inductors are moved around. But, certainly, one is assured that, if one sets up a system of inductors with currents and then disconnects them from their batteries, any movement of the loops must keep the fluxes fixed and change the currents accordingly since there are no batteries to work against the emfs and maintain the currents. This issue will be revisited in homework and is discussed in Griffiths Section 8.3 (4th edition).

Magnetic Energy and Forces (cont.)

To understand the relationship between the fixed currents and field flux cases, let's use the relationship between $\partial \underline{\underline{M}} / \partial \xi$ and $\partial \underline{\underline{M}}^{-1} / \partial \xi$:

$$\begin{aligned} 0 &= \frac{\partial}{\partial \xi} \underline{\underline{1}} = \frac{\partial}{\partial \xi} \left[\underline{\underline{M}}^{-1} \underline{\underline{M}} \right] = \frac{\partial}{\partial \xi} \left[\underline{\underline{M}}^{-1} \right] \underline{\underline{M}} + \underline{\underline{M}}^{-1} \left[\frac{\partial}{\partial \xi} \underline{\underline{M}} \right] \\ \implies \quad \frac{\partial}{\partial \xi} \left[\underline{\underline{M}}^{-1} \right] &= -\underline{\underline{M}}^{-1} \left[\frac{\partial}{\partial \xi} \underline{\underline{M}} \right] \underline{\underline{M}}^{-1} \end{aligned} \quad (7.99)$$

Inserting this into the expression for $F_\xi|_\Phi$, we obtain

$$\begin{aligned} F_\xi|_\Phi &= -\frac{1}{2} \underline{\Phi}^T \left[\frac{\partial}{\partial \xi} \underline{\underline{M}}^{-1} \right] \underline{\Phi} = \frac{1}{2} \underline{\Phi}^T \underline{\underline{M}}^{-1} \left[\frac{\partial}{\partial \xi} \underline{\underline{M}} \right] \underline{\underline{M}}^{-1} \underline{\Phi} \\ &= \frac{1}{2} \underline{I}^T \left[\frac{\partial}{\partial \xi} \underline{\underline{M}} \right] \underline{I} = F_\xi|_I \end{aligned} \quad (7.100)$$

which tells us the force at fixed flux is the *same* in direction to the force at fixed current, giving us an imperfect analogue of Equation 4.93. Again, we get a different result than in the electrostatic case because of Lenz's Law, which changed the sign on the force at fixed current force relative to its analogue force at fixed voltage (Equation 4.94).

Lecture 17: Maxwell's Equations

Conservation Laws, Poynting Vector, Stress and Torque Tensors

Date Revised: 2014/05/01 10:00

Minor modifications to derivation of Poynting's Theorem

Change sign convention for torque tensor

Corrections to typos in stress tensor example

Correction to typo in derivation of torque tensor

Date Given: 2014/04/24

Maxwell's Equations in Vacuum

The Inconsistency in our Equations

Let's write the full set of equations we have come to:

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad (7.101)$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} \quad (7.102)$$

Now, we know that the divergence of a curl vanishes: it's a vector identity. We should check that it holds! For the electric field, we have

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{E} = \vec{\nabla} \cdot -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{B} = 0 \quad (7.103)$$

If we repeat with \vec{B} , we obtain

$$\vec{\nabla} \cdot \vec{\nabla} \times \vec{B} = \mu_0 \vec{\nabla} \cdot \vec{J} = -\mu_0 \frac{\partial \rho}{\partial t} \neq 0 \quad \text{in general} \quad (7.104)$$

Maxwell's Equations in Vacuum (cont.)

There is a more physical way to see this by applying Ampere's Law to a circuit containing a parallel-plate capacitor. Construct an Ampere's Law loop around the wire carrying the current. Ampere's Law is satisfied because there is magnetic field in an azimuthal direction around the wire (giving a nonzero line integral of \vec{B}) and there is current passing through the disk-like surface whose boundary is the contour.

Now pick another surface that passes between the capacitor plates. This is an equally valid surface; nothing about our proof of Ampere's Law from the Biot-Savart Law assumed a particular choice of surface for the Ampere's Law surface integral. But this surface has no current intersecting it because it passes through the capacitor!

The reason this problem happens and we never noticed it before is because we have a case of non-steady currents here: charge piles up on the capacitor plates giving $\partial\rho/\partial t \neq 0$, exactly the conditions under which the differential version of Ampere's Law would be violated.

Maxwell's Equations in Vacuum (cont.)

The Displacement Current and Maxwell's Equations

In order to solve the above problem, we need something that will cancel

$$\mu_0 \vec{\nabla} \cdot \vec{J} = -\mu_0 \frac{\partial \rho}{\partial t} = -\mu_0 \frac{\partial}{\partial t} \left(\epsilon_0 \vec{\nabla} \cdot \vec{E} \right) = -\mu_0 \vec{\nabla} \cdot \left(\epsilon_0 \frac{\partial \vec{E}}{\partial t} \right) \quad (7.105)$$

Let's just add the necessary term to Ampere's Law:

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \quad (7.106)$$

The main question we have to ask is: was it ok to do this? Does it violate any of our previous conclusions? The only equation we have modified is the $\vec{\nabla} \times \vec{B}$ equation, so we only need to consider our study of magnetostatics, where we applied this equation. The addition preserves the usual behavior of $\vec{\nabla} \times \vec{B}$ for magnetostatics because $\partial \vec{E} / \partial t = 0$ in magnetostatics. Why? Two things can result in time dependence of \vec{E} . The first is time dependence in ρ . But in magnetostatics, we assume steady-state currents, explicitly requiring no buildup of charge and hence $\partial \rho / \partial t = 0$. The second is time dependence of \vec{B} , which can yield time dependence of \vec{E} via Faraday's Law. But magnetostatics assumes \vec{B} is constant in time, so there is no worry there.

Maxwell's Equations in Vacuum (cont.)

The added term is called the *displacement current density*,

$$\boxed{\vec{J}_d \equiv \epsilon_0 \frac{\partial \vec{E}}{\partial t}} \quad (7.107)$$

It is only called this because it appears in the Ampere's Law equation with the same units and form as a current. It is not a physical current carried by charges.

Maxwell's Equations in Vacuum (cont.)

By construction, \vec{J}_d solves the problem with $\vec{\nabla} \cdot \vec{\nabla} \times \vec{B}$. Let us see how it solves the problem with the integral version of Ampere's Law. The electric field in the capacitor is

$$E = \frac{\sigma}{\epsilon_0} = \frac{1}{\epsilon_0} \frac{Q}{A} \quad (7.108)$$

Therefore, the displacement current is

$$J_d = \epsilon_0 \frac{\partial E}{\partial t} = \frac{1}{A} \frac{dQ}{dt} = \frac{I}{A} \quad (7.109)$$

The integral form of Ampere's Law with the displacement current is therefore

$$\oint_C d\vec{l} \cdot \vec{B} = \mu_0 I_{encl} + \mu_0 \epsilon_0 \int_{S(C)} da J_d = \mu_0 I_{encl} + \mu_0 I \quad (7.110)$$

If we choose the first surface we discussed, the flat surface in the plane of the contour C , we get the first term, which gives $\mu_0 I$. If we choose the second surface between the capacitor plates, the first term vanishes but the second term gives $\mu_0 I$. Thus, the inconsistency seen earlier has been eliminated.

Maxwell's Equations in Vacuum (cont.)

Example: Displacement Current for Coaxial Conductors (Griffiths 7.36)

This is a continuation of the example from earlier. We want to calculate the displacement current density and the total displacement current and to compare quantitatively I and I_d .

The displacement current density is

$$\vec{J}_d = \epsilon_0 \frac{\partial \vec{E}}{\partial t} = \epsilon_0 \frac{\partial}{\partial t} \hat{z} \frac{\mu_0}{2\pi} \omega I_0 \sin \omega t \ln \frac{b}{s} = \mu_0 \epsilon_0 \frac{\omega^2 I_0}{2\pi} \cos \omega t \ln \frac{b}{s} \quad (7.111)$$

Let's integrate over s to get the total displacement current:

$$\begin{aligned} I_d &= \int_a^b s ds \int_0^{2\pi} d\phi \mu_0 \epsilon_0 \frac{\omega^2 I_0}{2\pi} \cos \omega t \ln \frac{b}{s} \\ &= \mu_0 \epsilon_0 \omega^2 I_0 \cos \omega t \int_a^b ds s \ln \frac{b}{s} = \mu_0 \epsilon_0 \omega^2 I_0 [\cos \omega t] b^2 \left[\frac{x^2}{2} \left(\ln x - \frac{1}{2} \right) \right] \Big|_{x=1}^{a/b} \\ &= \mu_0 \epsilon_0 \omega^2 I_0 \left[\frac{1}{4} (b^2 - a^2) + \frac{1}{2} a^2 \ln \frac{a}{b} \right] \cos \omega t \end{aligned} \quad (7.112)$$

Maxwell's Equations in Vacuum (cont.)

The ratio of the amplitudes of the displacement current and the true current, up to factors of order unity, is

$$\frac{I_d(t)}{I(t)} = \frac{\mu_0 \epsilon_0 \omega^2 b^2 I_0 \cos \omega t}{I_0 \cos \omega t} = \frac{\omega^2 b^2}{c^2} = \frac{b^2}{c^2/\omega^2} \quad (7.113)$$

The numerator of the final expression is the square of the length scale in the problem, a , the radius of the cylinder. The denominator, is up to a factor $(2\pi)^2$, the square of the distance light travels in one oscillation period. Thus, this quantity is a measure of how quasistatic the system is. We have mentioned before that, if $b \ll c/\omega$ is not satisfied, then our quasistatic approximation is invalid. This corroborates that: if the displacement current approaches or exceeds the real current, the system is not quasistatic and one should be applying a fully time-dependent formalism.

The ratio of the displacement current to the true current scales as ω^2 , so one must go to high frequency to notice it. Quantitatively, if we ask how high in frequency one must go to obtain $I_d/I = 0.01$ if we take $b = 2$ mm as the dimension of the coaxial conductor, we obtain

$$\nu = \frac{\omega}{2\pi} = \frac{1}{2\pi} \frac{c}{b} \sqrt{\frac{I_d}{I}} = \frac{1}{2\pi} \frac{3 \times 10^{11} \text{ mm/s}}{2 \text{ mm}} \sqrt{0.01} \approx 2 \text{ GHz} \quad (7.114)$$

GHz oscillators were not available in Faraday's time, so the fact that he did not observe the effects of the displacement current is not surprising.

Maxwell's Equations in Vacuum (cont.)

Maxwell's Equations in Vacuum

Putting it all together, we obtain *Maxwell's Equations*:

$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t}} \quad (7.115)$$

These, combined with the force law and continuity:

$$\boxed{\vec{F} = q (\vec{E} + \vec{v} \times \vec{B})} \quad \boxed{\vec{\nabla} \cdot \vec{J} = -\frac{\partial \rho}{\partial t}} \quad (7.116)$$

summarize classical electrodynamics in vacuum. We may rewrite the first set of equations in a way that emphasizes better the source terms, putting all the fields on the left side:

$$\boxed{\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \vec{\nabla} \times \vec{B} - \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J}} \quad (7.117)$$

Maxwell's Equations in Matter

Maxwell's Equations in Matter

Just as we found it convenient to rewrite the individual equations of electrostatics and magnetostatics using only the free charges and currents, it makes sense to do the same for Maxwell's Equations. The new twists we must take into account are the time-dependence of \vec{P} and \vec{M} . (We have already considered time dependence of ρ , \vec{J} , \vec{E} , and \vec{B} for quasistatic situations, corresponding to all length scales in the system small compared to c/ν .)

How to treat \vec{P} is motivated by the expression

$$\rho_b = -\vec{\nabla} \cdot \vec{P} \quad (7.118)$$

If \vec{P} is time-varying, we expect there to be a current \vec{J}_p associated with the resulting changes in ρ_b . In fact, the above expression suggests a good definition of \vec{J}_p :

$$\vec{J}_p = \frac{\partial \vec{P}}{\partial t} \iff \vec{\nabla} \cdot \vec{J}_p = -\vec{\nabla} \cdot \frac{\partial \vec{P}}{\partial t} = -\frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{P} = -\frac{\partial \rho_b}{\partial t} \quad (7.119)$$

That is, the definition on the left naturally gives the continuity relation between \vec{J}_p and ρ_b as one would like.

Maxwell's Equations in Matter (cont.)

Intuitively, think of \vec{J}_p as follows. Suppose one has a cylinder of polarizable material of length dz and cross-sectional area da and with polarization vector $\vec{P} = P\hat{z}$. The definition $\rho_b = -\vec{\nabla} \cdot \vec{P}$ implies that there is a bound surface charge $Q = \sigma da = \hat{n} \cdot \vec{P} da = \pm P da$ at each end. If, for example, we allow \vec{P} to vary sinusoidally, $\vec{P} = \vec{P}_0 \sin \omega t$, which corresponds to the surface charge obeying $Q(t) = P_0 da \sin \omega t$, then the current is

$$I_p = \vec{J}_p \cdot \hat{n} da = P_0 da \omega \cos \omega t = \frac{dQ}{dt} \quad (7.120)$$

as would be necessary to transfer charge back and forth between the two ends of the cylinder to yield the corresponding time-dependent surface charge.

Do we have to worry about time dependence of \vec{M} ? Recall that \vec{M} yields a bound current density

$$\vec{J}_b = \vec{\nabla} \times \vec{M} \quad (7.121)$$

Time dependence of \vec{M} yields time dependence of \vec{J}_b , which produces time dependence of \vec{B} and \vec{H} . As long as these variations are slow enough to satisfy the quasistatic limit (all length scales in problem much smaller than c/ν), then they are already accounted for by Ampere's Law and Faraday's Law.

Maxwell's Equations in Matter (cont.)

So now let's use all this to rewrite Maxwell's Equations. The charge and current densities have the following parts:

$$\rho = \rho_f + \rho_b = \rho_f - \vec{\nabla} \cdot \vec{P} \quad \vec{J} = \vec{J}_f + \vec{J}_b + \vec{J}_p = \vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \quad (7.122)$$

Using Gauss's law, $\epsilon_0 \vec{\nabla} \cdot \vec{E} = \rho_f - \vec{\nabla} \cdot \vec{P}$, and the definition of the displacement field, $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$, we obtain

$$\vec{\nabla} \cdot \vec{D} = \rho_f \quad (7.123)$$

Ampere's Law with the displacement current term is

$$\vec{\nabla} \times \vec{B} = \mu_0 \left(\vec{J}_f + \vec{\nabla} \times \vec{M} + \frac{\partial \vec{P}}{\partial t} \right) + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \quad (7.124)$$

We use $\vec{B} = \mu_0 (\vec{H} + \vec{M})$ as well as $\vec{D} = \epsilon_0 \vec{E} + \vec{P}$ to obtain

$$\vec{\nabla} \times \vec{H} = \vec{J}_f + \frac{\partial \vec{D}}{\partial t} \quad (7.125)$$

Now it is clear why the last term is called the *displacement current* — it is the apparent current due to the time variation of the displacement vector \vec{D} !

Maxwell's Equations in Matter (cont.)

Faraday's Law and $\vec{\nabla} \cdot \vec{B} = 0$ are not affected since they do not depend on the free and bound currents. Thus, Maxwell's Equations in matter are (again, putting all the fields on the left sides and the sources on the right):

$$\boxed{\vec{\nabla} \cdot \vec{D} = \rho_f} \quad \boxed{\vec{\nabla} \cdot \vec{B} = 0} \quad (7.126)$$

$$\boxed{\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0} \quad \boxed{\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}_f} \quad (7.127)$$

These equations must be supplemented by specific *constitutive relations* between \vec{E} and \vec{D} and between \vec{B} and \vec{H} to completely specify the behavior (and, of course, boundary conditions must be provided). One specific case is:

$$\boxed{\text{linear media: } \vec{P} = \chi_e \epsilon_0 \vec{E} \quad \vec{M} = \chi_m \vec{H} \quad \vec{D} = \epsilon \vec{E} \quad \vec{B} = \mu \vec{H}} \quad (7.128)$$

Maxwell's Equations in Matter (cont.)

Boundary Conditions for Maxwell's Equations

For both Maxwell's Equations in vacuum and in matter, we need to review how the fields change between regions that may be separated by surface charge and current densities and which may have different polarization and magnetization.

We recall from our prior calculations of this type that the discontinuity in the normal component of a field is determined by its divergence and the discontinuity in the tangential component by its curl. We also recall charge and current densities can become singular on a boundary. Thus:

- ▶ An integral of charge density over a volume containing a boundary reduces, as the height of the volume normal to the boundary is shrunk to zero thickness, to the surface charge density integrated over the intersection of the volume with the boundary. The volume component of the charge density yields zero contribution.
- ▶ An integral of a current density through an area reduces, as the width of the area normal to the boundary shrinks to zero, to the surface current density passing through the area. The area component of the current density yields zero contribution.

Fields themselves never have singularities like this, so any integral of a field vanishes as the volume or area is shrunk to zero. Hence, the addition of the displacement current does not modify the boundary conditions we have calculated in the past!

Section 8

Conservation Laws

Conservation of Charge

Motivation and Analogy: Conservation of Charge

Back when we first discussed the Lorentz Force, we discussed conservation of charge and the continuity equation:

$$\vec{\nabla} \cdot \vec{J}(\vec{r}) = -\frac{\partial \rho(\vec{r})}{\partial t} \quad (8.1)$$

This is an interesting equation because it enforces *local* conservation of charge: not only is there no creation or destruction of charge over the whole universe, there is also no creation or destruction of charge at a given point. Charge cannot jump from one place to another without a current flowing to move that charge.

In electrodynamics, we want to ask the same question for energy and momentum because we want to understand whether the fields we have constructed have true physical meaning or whether they are just mathematical constructs. Determining whether they carry momentum and energy is one way to answer that question, and such a consideration leads to the question of conservation of these quantities.

Conservation of Energy

Poynting's Theorem: Conservation of Energy

We have shown that the work required to set up distributions of charge or current is

$$W_e = \frac{\epsilon_0}{2} \int d\tau |\vec{E}|^2 \quad W_m = \frac{1}{2\mu_0} \int d\tau |\vec{B}|^2 \quad (8.2)$$

Recall that this is the work needed to move new charge in from infinity due to the repulsion from the charge already there, or the work that needs to be done to drive a current against a back emf. It is thus natural to expect that the total energy density in the electric and magnetic fields is

$$u = \frac{1}{2} \left(\epsilon_0 |\vec{E}|^2 + \frac{1}{\mu_0} |\vec{B}|^2 \right) \quad (8.3)$$

We will prove this by considering the work done to move charges as currents. We use the term *electromagnetic field* to reflect the fact that the fields influence each other and their energies are on the same footing.

Conservation of Energy (cont.)

Given a single particle of charge q acted on by the electromagnetic field, the work done on it as it moves by $d\vec{\ell}$ is

$$dW = \vec{F} \cdot d\vec{\ell} = q (\vec{E} + \vec{v} \times \vec{B}) dt = q \vec{E} \cdot \vec{v} dt \quad (8.4)$$

Now, if we consider a continuous distribution of charge and current, we may replace $q = \rho d\tau$ and $\rho \vec{v} = \vec{J}$, giving that the power is

$$\frac{dW}{dt} = \int d\tau \vec{E} \cdot \vec{J} \quad (8.5)$$

Let's manipulate the integrand using Ampere's Law:

$$\vec{E} \cdot \vec{J} = \frac{1}{\mu_0} \vec{E} \cdot (\vec{\nabla} \times \vec{B}) - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \quad (8.6)$$

One subtlety here: we started off talking about \vec{J} being acted upon by an electromagnetic field, and now it seems like we are writing \vec{J} as the source of that field. It is not the sole source of the field because, now with the displacement current term combined with Faraday's Law, there can be electric and magnetic fields that are sourced by each other's time variation rather than by physical currents. The above substitution is nevertheless valid because the total magnetic field must be the sum of that sourced by \vec{J} and that sourced by the displacement current.

Conservation of Energy (cont.)

Another subtlety is the issue of whether \vec{J} can create fields that do work on itself. This is entirely possible, as we saw in the example of the time-varying current in the coaxial conductor: a time-varying current generated a time-varying magnetic field that generated a time-varying electric field aligned with the original current. So we should not worry about this (except possibly the energetics, which can be addressed after we have obtained Poynting's Theorem).

Conservation of Energy (cont.)

Returning to our expression for $\vec{E} \cdot \vec{J}$, we can use the product rule $\vec{\nabla} \cdot (\vec{a} \times \vec{b}) = \vec{a} \cdot (\vec{\nabla} \times \vec{b}) - \vec{b} \cdot (\vec{\nabla} \times \vec{a})$ to rewrite it and then use Faraday's Law:

$$\vec{E} \cdot \vec{J} = \frac{1}{\mu_0} \vec{B} \cdot (\vec{\nabla} \times \vec{E}) - \frac{1}{\mu_0} \vec{\nabla} \cdot (\vec{E} \times \vec{B}) - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} \quad (8.7)$$

$$= -\frac{1}{\mu_0} \vec{B} \cdot \frac{\partial \vec{B}}{\partial t} - \epsilon_0 \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} - \frac{1}{\mu_0} \vec{\nabla} \cdot (\vec{E} \times \vec{B}) \quad (8.8)$$

$$= -\frac{\partial}{\partial t} \frac{1}{2} \left(\epsilon_0 |\vec{E}|^2 + \frac{1}{\mu_0} |\vec{B}|^2 \right) - \frac{1}{\mu_0} \vec{\nabla} \cdot (\vec{E} \times \vec{B}) \quad (8.9)$$

We thus obtain *Poynting's Theorem*:

$$\boxed{\frac{dW}{dt} = -\frac{d}{dt} \int_V d\tau \frac{1}{2} \left(\epsilon_0 |\vec{E}|^2 + \frac{1}{\mu_0} |\vec{B}|^2 \right) - \frac{1}{\mu_0} \oint_{S(V)} da \hat{n} \cdot (\vec{E} \times \vec{B})} \quad (8.10)$$

with the *Poynting vector* defined to be

$$\boxed{\vec{S} = \frac{1}{\mu_0} (\vec{E} \times \vec{B})} \quad (8.11)$$

Poynting's Theorem says that *the work done on the charges by the electromagnetic forces is equal to the decrease in the energy remaining in the fields minus the energy that flowed outward through the interface.* \vec{S} has units of energy per unit time per area and is considered the *energy flux density*.

Conservation of Energy (cont.)

Another useful form is given by putting the field energy density term on the left side:

$$\frac{dW}{dt} + \frac{d}{dt} \int_V d\tau \frac{1}{2} \left(\epsilon_0 |\vec{E}|^2 + \frac{1}{\mu_0} |\vec{B}|^2 \right) = - \frac{1}{\mu_0} \oint_{S(V)} da \hat{n} \cdot (\vec{E} \times \vec{B}) \quad (8.12)$$

$$\boxed{\frac{d}{dt} (E_{mech} + E_{field}) = - \oint_{S(V)} da \hat{n} \cdot \vec{S}} \quad (8.13)$$

The rate of change of the total energy in the volume is given by the flux of the Poynting vector through the boundary of the volume — this much more explicitly puts mechanical and field energy on the same footing and shows that both can be transported by the Poynting flux.

Note that this allays our fears about a current doing work on itself: while it may do so, energy remains conserved as long as one takes into account the field energy.

Conservation of Energy (cont.)

We may write both versions in a local form by recognizing that the volume being integrated over is arbitrary. If we define u_{mech} to be the density of mechanical energy ($W = \int_V d\tau u_{mech}$), then our two versions of Poynting's theorem yield the local relations (after converting the surface integral of \vec{S} back to a volume integral using the divergence theorem):

$$\boxed{\frac{\partial u_{mech}}{\partial t} = -\frac{\partial u_{field}}{\partial t} - \vec{\nabla} \cdot \vec{S}} \quad \Leftrightarrow \quad \boxed{\frac{\partial}{\partial t} (u_{mech} + u_{field}) = -\vec{\nabla} \cdot \vec{S}} \quad (8.14)$$

This is the kind of local conservation theorem we wanted, relating the rate of change of a density (here the energy density) to the divergence of a current density (here the Poynting vector).

Conservation of Energy (cont.)

Example: Power Transported Down a Coaxial Cable (Griffiths Problem 8.1)

Consider a coaxial cable with central conductor diameter a and outer conductor of radius b and zero thickness. The current flows along $+\hat{z}$ on the central conductor and back along $-\hat{z}$ on the outer shell. We calculated something similar to this in a previous example. The inner conductor is held at voltage V and the outer conductor at $V = 0$ (ground) at one end of the cylinder, and there is a resistive sheet of conductivity σ_\square capping the other end.

Because the inner conductor is assumed to have infinite conductivity, there can be no electric field inside and thus all the current must flow on the surface (consequence of Ohm's Law: $\vec{J} = 0$ because $\vec{E} = 0$). Thus, this really looks the same as the prior example. The magnetic field between the conductors is

$$\vec{B}(s) = \frac{\mu_0 I}{2\pi s} \hat{\phi} \quad (8.15)$$

where I is the current due to V (value to be determined). In the prior example, we did not explicitly have a voltage on the inner conductor (effectively, the conductivity of the sheet at the end was infinite).

Conservation of Energy (cont.)

Here, since we have such a voltage, there is a line charge density on the inner conductor and a radial electric field. You are no doubt familiar with the Gauss' Law calculation of this configuration, which yields

$$\vec{E}(s) = \frac{\lambda}{2\pi\epsilon_0 s} \hat{s} \quad (8.16)$$

Let's find λ by matching to the applied voltage. The potential and field are

$$V(s) \propto \ln s \quad V = V(a) - V(b) = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{b}{a} \implies \vec{E}(s) = \frac{V}{s \ln \frac{b}{a}} \hat{s} \quad (8.17)$$

The Poynting vector is

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{I V}{2\pi s^2 \ln \frac{b}{a}} \hat{z} \quad (8.18)$$

The energy and energy current are between the conductors, not in them! The power flowing down the cable is found by integrating the Poynting vector over the cross-sectional area where the fields are:

$$P = \int_S da \hat{n} \cdot \vec{S}(s) = \int_a^b s ds \int_0^{2\pi} d\phi \frac{I V}{2\pi s^2 \ln \frac{b}{a}} = \frac{I V}{\ln \frac{b}{a}} \int_a^b \frac{ds}{s} = I V \quad (8.19)$$

Conservation of Energy (cont.)

Let's calculate the power dissipated in the resistive sheet at the end. The sheet does not disturb the potential because the sheet continues to satisfy Laplace's equation with the same boundary conditions in s : $V(a) = V$, $V(b) = 0$. Therefore, our electric field above is valid in the conducting sheet, and the surface current density and total current are

$$\vec{K}(s) = \sigma_{\square} \vec{E}(s) = \frac{\sigma_{\square} V}{s \ln \frac{b}{a}} \hat{s} \quad (8.20)$$

The total current is

$$I = \int_0^{2\pi} s d\phi K(s) = 2\pi s K(s) = \frac{2\pi \sigma_{\square} V}{\ln \frac{b}{a}} \quad (8.21)$$

Conservation of Energy (cont.)

We do not need it, but it is interesting to note that the resistance is

$$R = \frac{\ln \frac{b}{a}}{2\pi \sigma_{\square}} \quad (8.22)$$

The power dissipated in the resistor is

$$P = \int_S da \vec{K}(s) \cdot \vec{E}(s) = \int_a^b s ds \int_0^{2\pi} d\phi \sigma_{\square} \left(\frac{V}{\ln \frac{b}{a}} \right)^2 2\pi \int_a^b \frac{ds}{s} \quad (8.23)$$

$$= \frac{2\pi \sigma_{\square} V}{\ln \frac{b}{a}} = I V \quad (8.24)$$

as expected since this is the power coming down the central conductor and it cannot go beyond the resistive sheet since the fields go to zero out there (no current or charge density beyond the sheet).

Conservation of Linear Momentum

The Maxwell Stress Tensor

Griffiths gives an interesting example of how magnetic forces appear to disobey Newton's Third Law, which is based on conservation of momentum. Specifically, given two particles with momenta \vec{p}_1 and \vec{p}_2 and no external forces acting on them, we know $\vec{p}_1 + \vec{p}_2$ is constant. But if they are exerting forces on each other, then each one individually is not conserved. But the conservation of the sum implies

$$\vec{F}_{21} = \frac{d}{dt} \vec{p}_1 = -\frac{d}{dt} \vec{p}_2 = -\vec{F}_{12} \quad (8.25)$$

This is Newton's Third Law.

Conservation of Linear Momentum (cont.)

The natural question to ask is whether the electromagnetic fields carry momentum. The matter and fields are related by the fields exerting forces on the matter, so let's use these forces to connect the momentum of the matter and fields. The Lorentz Force Law is

$$\frac{d\vec{P}_{mech}}{dt} = \vec{F} = q(\vec{E} + \vec{v} \times \vec{B}) \quad (8.26)$$

Integrating this over a charge and current density gives

$$\frac{d\vec{P}_{mech}}{dt} = \int_V d\tau (\rho \vec{E} + \vec{J} \times \vec{B}) \quad (8.27)$$

Using Maxwell's Equations, we can write this in terms of the fields

$$\frac{d\vec{P}_{mech}}{dt} = \int_V d\tau \left(\epsilon_0 [\vec{\nabla} \cdot \vec{E}] \vec{E} + \left[\frac{1}{\mu_0} \vec{\nabla} \times \vec{B} - \epsilon_0 \frac{\partial \vec{E}}{\partial t} \right] \times \vec{B} \right) \quad (8.28)$$

Conservation of Linear Momentum (cont.)

After a remarkable amount of manipulation that we will not reproduce here — see Griffith Section 8.2.2 or Jackson Section 6.8 — one arrives at

$$\boxed{\frac{d\vec{P}_{mech}}{dt} = \int_{\mathcal{V}} d\tau \left[\vec{\nabla} \cdot \underline{\underline{\mathcal{T}}} - \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t} \right]} \quad (8.29)$$

where $\underline{\underline{\mathcal{T}}}$ is the *Maxwell Stress Tensor*

$$\boxed{\underline{\underline{\mathcal{T}}} = \sum_{i,j=1}^3 T_{ij} \hat{r}_i \hat{r}_j \quad T_{ij} = \epsilon_0 \left[E_i E_j - \frac{1}{2} \delta_{ij} E^2 \right] + \frac{1}{\mu_0} \left[B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right]} \quad (8.30)$$

and the vector dot products and divergence of $\underline{\underline{\mathcal{T}}}$ are given by

$$\vec{a} \cdot \underline{\underline{\mathcal{T}}} = \sum_{i=1}^3 a_i T_{ij} \hat{r}_j \quad \underline{\underline{\mathcal{T}}} \cdot \vec{a} = \sum_{j=1}^3 \hat{r}_i T_{ij} a_j \quad \vec{\nabla} \cdot \underline{\underline{\mathcal{T}}} = \sum_{i=1}^3 \hat{r}_j \frac{\partial}{\partial r_i} T_{ij} \quad (8.31)$$

Note that T_{ij} is symmetric in its indices. We are not terribly concerned in this course with the transformation properties of scalars, vectors, and tensors under coordinate system rotations, so we will not comment further on what a tensor is. Recall we encountered the quadrupole moment tensor earlier.

Conservation of Linear Momentum (cont.)

Using the divergence theorem on the first term and moving the time derivative in the second term outside the integral, we obtain

$$\boxed{\frac{d\vec{P}_{mech}}{dt} = \oint_{S(\mathcal{V})} da \hat{n}(\vec{r}) \cdot \underline{\underline{\mathcal{T}}}(\vec{r}) - \epsilon_0 \mu_0 \frac{d}{dt} \int_{\mathcal{V}} d\tau \vec{S}(\vec{r})} \quad (8.32)$$

This equation states that *the rate of change of the mechanical momentum in a volume \mathcal{V} is equal to the integral over the surface of the volume of the stress tensor's flux through that surface minus the rate of change of the volume integral of the Poynting vector.*

If we consider a static situation, then the second term vanishes and we are left with the flux of $\underline{\underline{\mathcal{T}}}$ over the surface. This justifies the naming of $\underline{\underline{\mathcal{T}}}$: it gives the force per unit area due to the electromagnetic fields, or the stress. T_{ij} is the force per unit area acting in the i th direction on an area element whose normal is in the j th direction. The diagonal elements are *pressures* and the off-diagonal forces are *shears*. More generally, the force per unit area in the \hat{n}_1 direction on an area element whose normal is in the \hat{n}_2 direction is

$$\frac{F(\hat{n}_1, \hat{n}_2)}{A} = \hat{n}_1 \cdot \underline{\underline{\mathcal{T}}} \cdot \hat{n}_2 \quad (8.33)$$

Conservation of Linear Momentum (cont.)

We may abstract out mechanical momentum and force densities \vec{p}_{mech} and \vec{f} — i.e., per unit volume expressions:

$$\vec{p}_{mech} \equiv \rho_m \vec{v}$$

where ρ_m is the *mass density*
and \vec{v} is the *velocity field*
of the mass density

$$\vec{f} \equiv \vec{\nabla} \cdot \underline{\underline{\mathcal{T}}} - \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t} \quad (8.34)$$

and we may conclude they are related locally because of the arbitrariness of the volume over which we are integrating:

$$\int_V d\tau \frac{\partial \vec{p}_{mech}}{\partial t} = \frac{d}{dt} \int_V d\tau \vec{p}_{mech} = \frac{d\vec{P}_{mech}}{dt} = \int_V d\tau \vec{f} \quad (8.35)$$

$$\Rightarrow \frac{\partial \vec{p}_{mech}}{\partial t} = \vec{f} = \vec{\nabla} \cdot \underline{\underline{\mathcal{T}}} - \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t} \quad (8.36)$$

This is the kind of conservation law we wanted to get to, a local one that relates the local momentum density to the divergence of the local stress tensor and the rate of change of the Poynting vector. It can also be viewed as a local force law, the specialization of Newton's Second Law to this situation.

Conservation of Linear Momentum (cont.)

As we did with the energy, and motivated by the appearance of a time derivative on the right side, we may rewrite the above as

$$\frac{d}{dt} \left(\vec{P}_{\text{mech}} + \epsilon_0 \mu_0 \int_{\mathcal{V}} d\tau \vec{S} \right) = \oint_{\mathcal{S}(\mathcal{V})} da \hat{n} \cdot \underline{\underline{\mathcal{T}}} \quad (8.37)$$

We are thus motivated to define the *linear momentum density* and *linear momentum* of the electromagnetic field as

$$\vec{p}_{\text{field}} \equiv \vec{g} \equiv \epsilon_0 \mu_0 \vec{S} = \epsilon_0 \vec{E} \times \vec{B} \quad \vec{P}_{\text{field}} = \int_{\mathcal{V}} d\tau \vec{g} \quad (8.38)$$

Conservation of Linear Momentum (cont.)

With that definition, we obtain

$$\boxed{\frac{d}{dt} (\vec{P}_{mech} + \vec{P}_{field}) = \oint_{S(V)} da \hat{n} \cdot \underline{\underline{\mathcal{T}}} \iff \frac{\partial}{\partial t} (\vec{p}_{mech} + \vec{g}) = \nabla \cdot \underline{\underline{\mathcal{T}}}} \quad (8.39)$$

We thus see that the rate of change of the total (mechanical + field) momentum in a volume is given by the integral of the stress tensor over the surface, or that the rate of change of the total momentum density at a point is given by the divergence of the stress tensor at that point. The stress tensor is thus seen to be the momentum current density in the same way that \vec{J} is the electric current density (up to a sign): both satisfy local continuity equations. The second equation can also be considered a generalized force law, where now we consider the rate of change of the momentum of both the particles and the fields, with $\nabla \cdot \underline{\underline{\mathcal{T}}}$ being the “force” that acts on both.

When there is no change in mechanical momentum — e.g., in empty space — then we can return to Equation 8.29 and, recognizing the volume is arbitrary, obtain the *continuity equation for the linear momentum of the electromagnetic field*:

$$\boxed{\nabla \cdot \underline{\underline{\mathcal{T}}} = \frac{\partial \vec{g}}{\partial t} = \epsilon_0 \mu_0 \frac{\partial \vec{S}}{\partial t}} \quad (8.40)$$

Conservation of Linear Momentum (cont.)

It is interesting to note that both \vec{S} and $\underline{\mathcal{T}}$ play two roles:

- ▶ \vec{S} is the power per unit area transported by the electromagnetic field, while $\epsilon_0 \mu_0 \vec{S}$ is the linear momentum per unit volume stored in the field. We will return later to this intimate connection between energy and momentum for the electromagnetic field, which, in quantum field theory, reflects the photon's masslessness.
- ▶ Similarly, $\underline{\mathcal{T}}$ plays two roles; both as a force per unit area (the stress) applied by the electromagnetic field as well as the momentum current density carried by the electromagnetic field (with a minus sign). This makes sense, as for the electromagnetic field to exert a force, it must provide momentum.

Note this issue of the sign. If we wanted $\underline{\mathcal{T}}$ to have a continuity equation like current and energy, where the rate of change of the conserved quantity is equal to the negative of the divergence of the current (loss of conserved quantity corresponds to outflow of current), we would have had to define $\underline{\mathcal{T}}$ with the opposite sign. But the sign given ensures that $\underline{\mathcal{T}}$ can be used to calculate forces without a sign flip. This makes sense: $\underline{\mathcal{T}}$ pointing into a volume should have a positive surface integral so that it indicates it is adding momentum to the volume. The only way out of this choice would be if we wanted to flip the sign and interpret $\underline{\mathcal{T}}$ as the force that the mechanical system exerts on the field (and then the continuity equation would behave the way we want), but that would be nonintuitive since we generally want to calculate the forces the field exerts on the mechanical system.

Conservation of Linear Momentum (cont.)

Example: Magnetic Force Between Two Spinning Charged Hemispheres (Griffiths Problem 8.3)

Given two hemispherical shells of radius R and uniform surface charge density σ spinning at angular frequency ω about the z axis, what is the magnetic force between the north and south hemispheres?

We have calculated the magnetic field for a similar configuration when we calculated the field of the uniformly magnetized sphere, which was

$$\vec{B}(r \leq R) = \frac{2}{3} \mu_0 M \hat{z} \quad (8.41)$$

$$\vec{B}(r \geq R) = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} \quad \text{with} \quad \vec{m} = \frac{4}{3}\pi R^3 M \hat{z} \quad (8.42)$$

The surface current was $\vec{K} = \hat{\phi} M \sin \theta$. In the new problem, the surface current is $\vec{K} = \hat{\phi} \sigma \omega R \sin \theta$, so we just need to replace M with $\sigma \omega R$, giving

$$\vec{B}(r \leq R) = \frac{2}{3} \mu_0 \sigma \omega R \hat{z} \quad (8.43)$$

$$\vec{B}(r \geq R) = \frac{\mu_0}{4\pi} \frac{3(\vec{m} \cdot \hat{r})\hat{r} - \vec{m}}{r^3} \quad \text{with} \quad \vec{m} = \frac{4}{3}\pi R^4 \sigma \omega \hat{z} \quad (8.44)$$

Conservation of Linear Momentum (cont.)

To calculate the force, we would nominally expect to calculate the flux of the stress tensor over the hemisphere (the plane at $z = 0$ for $r < R$ and the hemispherical shell $r = R$ at $z > 0$). However, any volume containing the charge on which we would like to calculate the force suffices for the calculation, given the derivation. So let's do the calculation more easily by setting the surface to be the $z = 0$ plane. The force will only be in the z direction by symmetry, so we need only the T_{3i} components. Moreover, because the plane we want to do the calculation for has a surface normal only in the z direction, we can restrict to the T_{33} component:

$$T_{33} = \frac{1}{2\mu_0} B_z^2 \implies T_{33}(r < R, z = 0) = \frac{2}{9} \mu_0 \sigma^2 \omega^2 R^2 \quad (8.45)$$

$$T_{33}(r > R, z = 0) = \frac{\mu_0 \sigma^2 \omega^2 R^8}{18 r^6} \quad (8.46)$$

We can do the area integral easily ($\hat{n} = -\hat{z}$ because we want the force on the upper half space and $-\hat{z}$ is the outward surface normal):

$$F_z = - \int_0^{2\pi} d\phi \left[\int_0^R r dr T_{33}(r < R, z = 0) + \int_R^\infty r dr T_{33}(r > R, z = 0) \right] \quad (8.47)$$

$$= -2\pi \left[\frac{R^2}{2} \frac{2}{9} \mu_0 \sigma^2 \omega^2 R^2 + \frac{\mu_0 \sigma^2 \omega^2 R^8}{72 R^4} \right] = -\frac{\pi}{4} \mu_0 \sigma^2 R^4 \omega^2 \quad (8.48)$$

Conservation of Angular Momentum

Angular Momentum of the Electromagnetic Field

One can go back and write analogues of everything we did for linear momentum for the case of angular momentum. The key point is that the manipulations that led us to Equation 8.29 did not rely on any transformations of integrals; we just needed to manipulate the integrand. Those manipulations remain valid, but now with a $\vec{r} \times$ in front inside the integral. Thus, we may also conclude

$$\frac{d\vec{L}_{mech}}{dt} = \int_V d\tau \left[\vec{r} \times (\vec{\nabla} \cdot \underline{\underline{\mathcal{I}}} - \epsilon_0 \mu_0 \frac{\partial}{\partial t} (\vec{r} \times \vec{S})) \right] \quad (8.49)$$

Conservation of Angular Momentum (cont.)

Let's manipulate the expression $\vec{r} \times (\vec{\nabla} \cdot \underline{\underline{\mathcal{T}}})$: we would obviously like to turn it into a pure divergence. Using Equation 8.31,

$$\vec{r} \times (\vec{\nabla} \cdot \underline{\underline{\mathcal{T}}}) = \vec{r} \times \sum_{i,j=1}^3 \hat{r}_j \frac{\partial T_{ij}}{\partial r_i} = \sum_{i,j,k=1}^3 \hat{r}_k \times \hat{r}_j r_k \frac{\partial T_{ij}}{\partial r_i} \quad (8.50)$$

$$= \sum_{i,j,k=1}^3 \hat{r}_k \times \hat{r}_j \left(\frac{\partial}{\partial r_i} r_k T_{ij} \right) - \sum_{i,j,k=1}^3 \hat{r}_k \times \hat{r}_j T_{ij} \frac{\partial r_k}{\partial r_i} \quad (8.51)$$

$$= \sum_{i,j,k=1}^3 \left(\frac{\partial}{\partial r_i} r_k \hat{r}_k \times T_{ji} \hat{r}_j \right) - \sum_{i,j,k=1}^3 \hat{r}_k \times \hat{r}_j T_{ij} \delta_{ik} \quad (8.52)$$

$$= \vec{\nabla} \cdot (\vec{r} \times \underline{\underline{\mathcal{T}}}) - \sum_{i,j,k=1}^3 \hat{r}_k \times \hat{r}_j T_{kj} \quad (8.53)$$

$$= \vec{\nabla} \cdot \underline{\underline{\mathcal{M}}} \quad \text{with } \underline{\underline{\mathcal{M}}} = \vec{r} \times \underline{\underline{\mathcal{T}}} \quad (8.54)$$

where the second term in the penultimate line has vanished because it is the product of a quantity that is asymmetric in j and k ($\hat{r}_k \times \hat{r}_j$) and one that is symmetric in i and j (T_{ij}). $\underline{\underline{\mathcal{M}}}$ is the analogue of the stress tensor, but now for torque.

Conservation of Angular Momentum (cont.)

Aside: From Ph106ab, you probably remember that angular momentum is perhaps more rigorously written as an asymmetric second-rank tensor, but, because such an object has only 3 independent quantities, it can be reduced to a vector (first-rank tensor) using cross-product notation. That applies here to both \vec{L}_{mech} and to $\vec{r} \times \vec{S}$. By extrapolation, $\underline{\underline{M}}$ may be written as a *third-rank* tensor. Since we don't use any of the transformation properties of these objects under rotations in this course, there is no need to use these higher-rank objects and so we stick with the less sophisticated vector notation for cross products.

Conservation of Angular Momentum (cont.)

We may thus write the analogue of Equation 8.29 for torque:

$$\boxed{\frac{d\vec{L}_{mech}}{dt} = \int_V d\tau \left[\vec{\nabla} \cdot \underline{\underline{M}} - \epsilon_0 \mu_0 \frac{\partial}{\partial t} (\vec{r} \times \vec{S}) \right]} \quad (8.55)$$

Using the divergence theorem, we may rewrite as we did the force equation

$$\boxed{\frac{d\vec{L}_{mech}}{dt} = \oint_{S(V)} da \hat{n}(\vec{r}) \cdot \underline{\underline{M}}(\vec{r}) - \epsilon_0 \mu_0 \frac{d}{dt} \int_V d\tau \vec{r} \times \vec{S}(\vec{r})} \quad (8.56)$$

We thus have a relation between the rate of change of mechanical angular momentum and the flux of the tensor $\underline{\underline{M}}$ into/out of the volume and the rate of the change of integral of the funny quantity containing the Poynting vector.

Conservation of Angular Momentum (cont.)

Let's turn this into a differential version. We need to define the mechanical momentum density and the torque density:

$$\vec{\ell}_{mech} \equiv \vec{r} \times \vec{p}_{mech} = \vec{r} \times \rho_m \vec{v}$$

$$\vec{n}_{torque} \equiv \vec{\nabla} \cdot \underline{\underline{\mathcal{M}}} - \epsilon_0 \mu_0 \frac{\partial}{\partial t} (\vec{r} \times \vec{S})$$

(8.57)

Then we have

$$\int_V d\tau \frac{\partial \vec{\ell}_{mech}}{\partial t} = \frac{d}{dt} \int_V d\tau \vec{\ell}_{mech} = \frac{d\vec{L}_{mech}}{dt} = \int_V d\tau \vec{n}_{torque} \quad (8.58)$$

$$\Rightarrow \frac{\partial \vec{\ell}_{mech}}{\partial t} = \vec{n}_{torque} = \vec{\nabla} \cdot \underline{\underline{\mathcal{M}}} - \epsilon_0 \mu_0 \frac{\partial}{\partial t} (\vec{r} \times \vec{S}) \quad (8.59)$$

Conservation of Angular Momentum (cont.)

As before, it is natural to define a *field angular momentum density* and move it to the left side of the above equations:

$$\vec{\ell}_{field} = \vec{r} \times \vec{g} = \epsilon_0 \mu_0 \vec{r} \times \vec{S} = \epsilon_0 \vec{r} \times (\vec{E} \times \vec{B}) \quad \vec{L}_{field} = \int_V d\tau \vec{\ell}_{field} \quad (8.60)$$

Therefore,

$$\frac{d}{dt} (\vec{L}_{mech} + \vec{L}_{field}) = \oint_{S(V)} da \hat{n} \cdot \underline{\underline{\mathcal{M}}} \quad \Leftrightarrow \quad \frac{\partial}{\partial t} (\vec{\ell}_{mech} + \vec{\ell}_{field}) = \vec{\nabla} \cdot \underline{\underline{\mathcal{M}}} \quad (8.61)$$

Again, we obtain a local conservation equation relating the rate of change of the total *angular momentum density* to the divergence of current density, here now the angular momentum current density. The second equation is a generalized “torque” equation, relating the rate of change of the angular momentum density to the generalized torque.

Note that the choice of sign for $\underline{\underline{\mathcal{M}}}$ follows the same convention as for the stress tensor: it gives a continuity equation with a sign flip but makes more sense for calculation of torque. Be aware that this sign convention is the opposite of Jackson's (his Problem 6.9).

Section 9

Electromagnetic Waves

Lecture 18:
Electromagnetic Waves in Vacuum

Date Revised: 2014/05/16 17:00

Correction to Equation 9.47 for circular polarization

Date Given: 2014/04/26

Electromagnetic Waves in Vacuum

Introduction and Study Guidelines

Maxwell's Equations have in them the seeds of self-propagating disturbances in the electromagnetic field: though time-varying charges and currents must generate the waves, they can propagate on their own once initiated. So, in this section, we will develop the theory of such waves propagating in either free space or linear dielectric media, without any free charges. Later on, we will discuss radiation, the process by which time-varying charges and currents generate electromagnetic waves.

We deviate from Griffiths' ordering of topics because you have seen the wave equation three times before, in Ph1c, Ph2/12a, and Ph106a, so we do not need to reintroduce it from scratch. Let's just launch into it and bring the formalism of waves in as we go.

Electromagnetic Waves in Vacuum (cont.)

From Maxwell's Equations to the Wave Equation

As noted earlier, we will consider Maxwell's Equations in free space with no sources:

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \vec{\nabla} \times \vec{B} - \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = 0 \quad (9.1)$$

These equations couple \vec{E} and \vec{B} , so let's try to decouple them by eliminating \vec{B} from the $\vec{\nabla} \times \vec{E}$ equation and \vec{E} from the $\vec{\nabla} \times \vec{B}$ equation by taking the curl again and using one of our standard vector identities:

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \vec{\nabla} \times \left(-\frac{\partial \vec{B}}{\partial t} \right) \quad \vec{\nabla} \times (\vec{\nabla} \times \vec{B}) = \epsilon_0 \mu_0 \vec{\nabla} \times \left(-\frac{\partial \vec{E}}{\partial t} \right)$$

$$\vec{\nabla} (\vec{\nabla} \cdot \vec{E}) - \nabla^2 \vec{E} = -\epsilon_0 \mu_0 \frac{\partial}{\partial t} \vec{\nabla} \times \vec{B} \quad \vec{\nabla} (\vec{\nabla} \cdot \vec{B}) - \nabla^2 \vec{B} = -\epsilon_0 \mu_0 \frac{\partial}{\partial t} \vec{\nabla} \times \vec{E}$$

$$\nabla^2 \vec{E} = \epsilon_0 \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad \nabla^2 \vec{B} = \epsilon_0 \mu_0 \frac{\partial^2 \vec{B}}{\partial t^2}$$

where $\vec{\nabla} \cdot \vec{E} = 0$ because there is no charge density. These are copies of the wave equation.

Electromagnetic Waves in Vacuum (cont.)

Specifically, these are component-by-component versions of the equation

$$\nabla^2 f(\vec{r}, t) = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} f(\vec{r}, t) \quad (9.2)$$

One can see by substitution that any function of the form

$$f(\vec{r}, t) = g(w) \quad \text{with} \quad w = \vec{k} \cdot \vec{r} - \omega t \quad (9.3)$$

satisfies the wave equation. First, we need to calculate the derivatives:

$$\nabla^2 f(\vec{r}, t) = \sum_{i=1}^3 \frac{\partial^2 f}{\partial r_i^2} = \sum_{i=1}^3 \frac{\partial}{\partial r_i} \frac{dg}{dw} \frac{\partial w}{\partial r_i} = \sum_{i=1}^3 \frac{\partial}{\partial r_i} \frac{dg}{dw} k_i \quad (9.4)$$

$$= \sum_{i=1}^3 k_i \frac{d^2 g}{dw^2} \frac{\partial w}{\partial r_i} = \sum_{i=1}^3 k_i^2 \frac{d^2 g}{dw^2} = |\vec{k}|^2 \frac{d^2 g}{dw^2} \quad (9.5)$$

$$\frac{\partial^2}{\partial t^2} f(\vec{r}, t) = \frac{\partial}{\partial t} \frac{dg}{dw} \frac{\partial w}{\partial t} = \frac{\partial}{\partial t} \frac{dg}{dw} (-\omega) \quad (9.6)$$

$$= -\omega \frac{d^2 g}{dw^2} \frac{\partial w}{\partial t} = \omega^2 \frac{d^2 g}{dw^2} \quad (9.7)$$

Electromagnetic Waves in Vacuum (cont.)

The wave equation is satisfied by the assumed form if

$$|\vec{k}|^2 = \frac{\omega^2}{v^2} \quad (9.8)$$

This condition says we can rewrite the argument, eliminating either ω or $|\vec{k}|$ as a free variable, as

$$\omega = \pm \vec{k} \cdot \vec{r} - |\vec{k}| v t = \frac{\omega}{v} \left(\pm \hat{k} \cdot \vec{r} - v t \right) = \omega \left(\pm \frac{\hat{k}}{v} \cdot \vec{r} - t \right) \quad (9.9)$$

where we have chosen ω and v to be always nonnegative while \vec{k} is allowed to take on any sign and direction. If we define $\vec{v} = v \hat{k}$ (so far, v is a scalar, not a vector), then we can also write

$$\omega = \pm \vec{k} \cdot \vec{r} - \vec{k} \cdot \vec{v} t = \frac{\omega}{v} \hat{k} \cdot (\pm \vec{r} - \vec{v} t) \quad (9.10)$$

Electromagnetic Waves in Vacuum (cont.)

In all cases, we can see that surfaces of constant w are given by

$$\delta w = 0 \quad \Rightarrow \quad \omega (\pm \hat{k} \cdot \delta \vec{r} - v \delta t) = 0 \quad (9.11)$$

$$\pm \frac{\hat{k} \cdot \delta \vec{r}}{\delta t} = v \quad (9.12)$$

That is, the surfaces of constant w propagate in space along the direction $\pm \hat{k}$ at speed v . This implies that the “shape function” $g(w)$ propagates at this speed. The time dependence at a given point in space has *angular frequency* ω , *frequency* $\nu = \omega/2\pi$, and *period* $T = 1/\nu$. The quantity k is the *propagation constant*, and the spatial dependence implies a *wavelength* $\lambda = 2\pi/k = v/\nu$.

Returning to our electromagnetic wave equations, we thus see that these waves in the electric and magnetic fields propagate at speed $v = 1/\sqrt{\epsilon_0 \mu_0}$ which is now, by definition, the *speed of light*, denoted by c .

Electromagnetic Waves in Vacuum (cont.)

General Properties of Solutions to the EM Wave Equations

We can use Maxwell's Equations to derive some general properties about electromagnetic waves. Many of these connect to the fact that \vec{E} and \vec{B} are vector quantities. We will begin by assuming the waves are sinusoidal solutions of the most general form allowed so far

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \cos \left(\vec{k}_E \cdot \vec{r} - \omega_E t + \delta_E \right) \quad (9.13)$$

$$\vec{B}(\vec{r}, t) = \vec{B}_0 \cos \left(\vec{k}_B \cdot \vec{r} - \omega_B t + \delta_B \right) \quad (9.14)$$

where we have allowed different ω , propagation directions $\hat{\vec{k}}$, and phase shifts δ because nothing has restricted that freedom yet. (The sign freedom on $\hat{\vec{k}}$ has been absorbed into $\hat{\vec{k}}$.) We have assumed sinusoidal solutions because they form a complete basis, so any solution can be decomposed in terms of them. We can demonstrate the following:

Electromagnetic Waves in Vacuum (cont.)

► *Transversality*

We can rewrite the divergence equations:

$$0 = \vec{\nabla} \cdot \vec{E} = \frac{d\vec{E}}{dw} \cdot \vec{\nabla} w = -\frac{\omega_E}{c} \hat{k}_E \cdot \vec{E}_0 \sin(\vec{k}_E \cdot \vec{r} - \omega_E t + \delta_E) \quad (9.15)$$

$$0 = \vec{\nabla} \cdot \vec{B} = \frac{d\vec{B}}{dw} \cdot \vec{\nabla} w = -\frac{\omega_B}{c} \vec{k}_B \cdot \vec{B}_0 \sin(\vec{k}_E \cdot \vec{r} - \omega_E t + \delta_E) \quad (9.16)$$

For the above equations to hold at all points in space, it is necessary for \vec{E}_0 and \vec{B}_0 to be perpendicular to their respective propagation directions. They are thus *transverse* waves: the field disturbance is in the direction perpendicular to propagation.

Electromagnetic Waves in Vacuum (cont.)

► Orthogonality and Equality of \vec{k} , ω , and δ

Let's write the curl equations. First, we take the necessary derivatives:

$$\begin{aligned}\vec{\nabla} \times \vec{E} &= \sum_{i,j,k=1}^3 \epsilon_{ijk} \hat{r}_i \frac{\partial E_k}{\partial r_j} = \sum_{i,j,k=1}^3 \epsilon_{ijk} \hat{r}_i \frac{dE_k}{dw} \frac{\partial w}{\partial r_j} \\ &= \sum_{i,j,k=1}^3 \epsilon_{ijk} \hat{r}_i \frac{dE_k}{dw} k_{E,j} = \vec{k}_E \times \frac{d\vec{E}}{dw} = \frac{\omega_E}{c} \hat{k}_E \times \frac{d\vec{E}}{dw} \\ &= -\frac{\omega_E}{c} \hat{k}_E \times \vec{E}_0 \sin(\vec{k}_E \cdot \vec{r} - \omega_E t + \delta_E)\end{aligned}\tag{9.17}$$

$$\frac{\partial \vec{E}}{\partial t} = \frac{d\vec{E}}{dw} \frac{\partial w}{\partial t} = -\omega_E \frac{d\vec{E}}{dw} = \omega_E \vec{E}_0 \sin(\vec{k}_E \cdot \vec{r} - \omega_E t + \delta_E)\tag{9.18}$$

\vec{B} has similar derivatives. Plugging the above into the curl equations, we obtain

$$-\frac{\omega_E}{c} \hat{k}_E \times \vec{E}_0 \sin(\vec{k}_E \cdot \vec{r} - \omega_E t + \delta_E) = -\omega_B \vec{B}_0 \sin(\vec{k}_E \cdot \vec{r} - \omega_E t + \delta_E)\tag{9.19}$$

$$\frac{\omega_B}{c} \hat{k}_B \times \vec{B}_0 \sin(\vec{k}_E \cdot \vec{r} - \omega_E t + \delta_E) = -\frac{\omega_E}{c^2} \vec{E}_0 \sin(\vec{k}_E \cdot \vec{r} - \omega_E t + \delta_E)\tag{9.20}$$

Electromagnetic Waves in Vacuum (cont.)

We can conclude two things from these equations: 1) in order for the equality to hold at all points in space and at all times, the arguments of the sin functions on the two sides must be the same, $\vec{k}_E = \vec{k}_B$, $\omega_E = \omega_B$, and $\delta_E = \delta_B$; and 2) \vec{k} , \vec{E}_0 , and \vec{B}_0 form a mutually orthogonal set of vectors.

In the end, we therefore have the following relation between \vec{E} and \vec{B} :

$$\vec{E}(\vec{r}, t) = \vec{E}_0 \cos(\vec{k} \cdot \vec{r} - \omega t + \delta) \quad (9.21)$$

$$\vec{B}(\vec{r}, t) = \vec{B}_0 \cos(\vec{k} \cdot \vec{r} - \omega t + \delta) \quad (9.22)$$

$$\vec{B}_0 = \frac{1}{c} \hat{\vec{k}} \times \vec{E}_0 \quad (9.23)$$

Electromagnetic Waves in Vacuum (cont.)

One note on the behavior of independent polarizations. Given a propagation direction \hat{k} , we can pick two directions, \hat{n}_1 and \hat{n}_2 , to form a basis for \vec{E} . A natural choice is to require $\hat{n}_1 \times \hat{n}_2 = \hat{k}$, which also implies $\hat{k} \times \hat{n}_1 = \hat{n}_2$. Then $\vec{E}_0 = E_1 \hat{n}_1 + E_2 \hat{n}_2$ and $\vec{B}_0 = B_1 \hat{n}_1 + B_2 \hat{n}_2$. E_1 and E_2 are the two possible *polarizations* of the electric field. (Of course, we can pick any \hat{n}_1 we want; once \hat{n}_1 has been picked, then the polarization directions are set.) The curl of \vec{E} relation then implies

$$B_2 = \vec{B}_0 \cdot \hat{n}_2 = \frac{1}{c} \vec{E}_0 \cdot \hat{n}_1 = \frac{E_1}{c} \quad B_1 = \vec{B}_0 \cdot \hat{n}_1 = -\frac{1}{c} \vec{E}_0 \cdot \hat{n}_2 = -\frac{E_2}{c} \quad (9.24)$$

We have assumed the two complementary polarizations have the same \vec{k} , so they have the same ω , but there is no condition connecting E_1 and E_2 , or B_1 and B_2 , so there is no requirement that the complementary polarizations have matching δ .

Electromagnetic Waves in Vacuum (cont.)

Rewriting EM Waves using the Auxiliary Field

It is interesting to note at this point that \vec{B} is not the most natural field quantity to work with: it is smaller than \vec{E} by a factor c , which is large. If we instead use $\vec{H} = \vec{B}/\mu_0$, then we obtain

$$\vec{H}_0 = \frac{1}{Z_0} \hat{k} \times \vec{E}_0 \quad Z_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \quad (9.25)$$

The quantity $Z_0 \approx 377 \Omega$ is known as the *impedance of free space* and has units of resistance (impedance). We see that \vec{H} is now only a factor of 377 smaller than \vec{E} . We also recall that \vec{H} has units of surface current density. This foreshadows the way \vec{H} will be related to the surface currents that the electric field drives.

Electromagnetic Waves in Vacuum (cont.)

Energy and Momentum in Electromagnetic Waves

The energy density in an electromagnetic wave, now using $|\vec{B}| = |\vec{E}|/c = |\vec{E}|\sqrt{\epsilon_0 \mu_0}$, is

$$u = \frac{1}{2} \left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2 \right) = \frac{1}{2} \left(\epsilon_0 E^2 + \frac{\epsilon_0 \mu_0}{\mu_0} E^2 \right) = \epsilon_0 E^2 \quad (9.26)$$

The energy flux per unit area is the Poynting vector:

$$\vec{S} = \frac{1}{\mu_0} \vec{E} \times \vec{B} = \frac{E^2}{c \mu_0} \hat{k} = c \epsilon_0 E^2 \hat{k} = c u \hat{k} \quad (9.27)$$

Thus, we see that the energy transported by the electromagnetic wave travels at the speed of light, just as the wave does. The momentum density vector is

$$\vec{g} = \epsilon_0 \mu_0 \vec{S} = \frac{u}{c} \hat{k} = \frac{\epsilon_0 E^2}{c} \hat{k} \quad (9.28)$$

Electromagnetic Waves in Vacuum (cont.)

We can also write down the stress tensor. First, consider the special case $\hat{k} = k \hat{z}$ so $\hat{n}_1 = \hat{x}$ and $\hat{n}_2 = \hat{y}$. Consider a wave polarized along \hat{x} . Then the fields are

$$\vec{E} = E \hat{x} \quad \vec{B} = B \hat{y} = \frac{E}{c} \hat{y} = \sqrt{\mu_0 \epsilon_0} E \hat{y} \quad (9.29)$$

The stress tensor is

$$T_{11} = \epsilon_0 \left(E^2 - \frac{1}{2} E^2 \right) + \frac{1}{\mu_0} \left(-\frac{1}{2} B^2 \right) = 0 \quad (9.30)$$

$$T_{22} = \epsilon_0 \left(-\frac{1}{2} E^2 \right) + \frac{1}{\mu_0} \left(B^2 - \frac{1}{2} B^2 \right) = 0 \quad (9.31)$$

$$T_{33} = \epsilon_0 \left(-\frac{1}{2} E^2 \right) + \frac{1}{\mu_0} \left(-\frac{1}{2} B^2 \right) = -u \quad (9.32)$$

$$T_{12} = T_{13} = T_{23} = 0 \quad (9.33)$$

$$\Rightarrow \underline{\underline{T}} = -u \hat{z} \hat{z} \quad (9.34)$$

The stress tensor for the complementary polarization is the same.

Electromagnetic Waves in Vacuum (cont.)

It is reasonable to extrapolate from the above that the generic stress tensor (now making the time dependence explicit) is

$$\underline{\underline{\mathcal{I}}} = -\hat{k}\hat{k}E_0^2 \cos^2(\vec{k} \cdot \vec{r} - \omega t + \delta) \quad (9.35)$$

One explanation of the reason for the negative sign is that $\underline{\underline{\mathcal{I}}}$ is the *negative* of the momentum current density.

From the stress tensor, we can calculate the *radiation pressure*, the force per unit area that would be applied to an object that absorbs the electromagnetic wave. Recall that $\hat{n}_1 \cdot \underline{\underline{\mathcal{I}}} \cdot \hat{n}_2$ gives the force acting in the \hat{n}_1 direction on a surface element whose normal is in the \hat{n}_2 direction. Since $\underline{\underline{\mathcal{I}}} \propto -\hat{k}\hat{k}$, the force is only nonzero (and positive) in the \hat{k} direction on an area element whose *outward* normal is in the $-\hat{k}$ direction. (Recall how, in our example of using the stress tensor to calculate the force between the two spinning charged hemispheres, the surface normal was in the $-\hat{z}$ direction for calculating the force on the hemisphere in the upper half-space.) The radiation pressure in the \hat{k} direction is then

$$P = \hat{k} \cdot \underline{\underline{\mathcal{I}}} \cdot -\hat{k} = E_0^2 \cos^2(\vec{k} \cdot \vec{r} - \omega t + \delta) = u \quad (9.36)$$

We will see later that, if the wave is not absorbed but reflected, the wave maintains its amplitude $|\vec{E}|$ but its \hat{k} reverses sign, implying that the momentum transfer and thus the pressure are increased by a factor of 2.

Electromagnetic Waves in Vacuum (cont.)

It is not particularly useful to write down the angular momentum density and the angular momentum tensor for a plane wave.

Time Averaging

For sinusoidally oscillating plane waves, it is standard to take time averages of quantities. Obviously, the fields themselves time average to zero. But energy and momentum do not:

$$\langle u(\vec{r}_0) \rangle = \left\langle \epsilon_0 E_0^2 \cos^2 (\vec{k} \cdot \vec{r}_0 - \omega t + \delta) \right\rangle = \frac{1}{2} \epsilon_0 E_0^2 \quad (9.37)$$

$$\langle \vec{S}(\vec{r}_0) \rangle = \frac{1}{2} c \epsilon_0 E_0^2 \hat{k} \quad \langle g(\vec{r}_0) \rangle = \frac{1}{2} \frac{\epsilon_0 E_0^2}{c} \hat{k} \quad \langle \underline{\underline{T}}(\vec{r}_0) \rangle = -\frac{1}{2} \epsilon_0 E_0^2 \hat{k} \hat{k} \quad (9.38)$$

The average power per unit area transported by the wave is the *intensity*

$$I = \langle |\vec{S}| \rangle = \frac{1}{2} c \epsilon_0 E_0^2 \quad (9.39)$$

Electromagnetic Waves in Vacuum (cont.)

Complex Notation for Plane Waves

For the sake of convenience in manipulation, we will from now on use *complex notation* for plane waves,

$$\vec{E}(\vec{r}, t) = \hat{n} \mathcal{R} \left[\tilde{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right] \quad \vec{B}(\vec{r}, t) = \frac{\hat{k} \times \hat{n}}{c} \mathcal{R} \left[\tilde{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \right] \quad (9.40)$$

$$\tilde{E}_0 = E_0 e^{i\delta} \quad (9.41)$$

where \tilde{E}_0 is now a complex number into which we have absorbed the phase factor $e^{i\delta}$. We will not carry along tildes on the vectors \vec{E} and \vec{E}_0 because it would be too cumbersome for the notation. It will be clear from context whether we mean the complex or real fields.

When calculating time averages of quadratic quantities in complex notation, we have to take a complex conjugate of one factor so that the result is real and throw in a $1/2$ to get the right time-average factor. For example,

$$\langle u \rangle = \frac{1}{2} \epsilon_0 \left(\langle \vec{E}^* \cdot \vec{E} \rangle \right) = \frac{1}{2} \epsilon_0 \left(\tilde{E}_0^* \tilde{E}_0 \right) = \frac{1}{2} \epsilon_0 E_0^2 \quad (9.42)$$

The same would apply for calculating $\underline{\mathcal{T}}$.

Electromagnetic Waves in Vacuum (cont.)

Types of Polarization

So far, we have only discussed linear polarization, wherein the direction of \vec{E} at a particular point in space stays constant over time (aside from sign flips). However, by combining two orthogonal linear polarization with appropriate coefficients, one can obtain more complex behavior.

The simplest extension is to consider what happens when you add two orthogonal polarizations of the same amplitude but with a possible phase shift:

$$\vec{E}(\vec{r}, t) = \frac{\tilde{E}_0}{\sqrt{2}} \left(\hat{n}_1 + \hat{n}_2 e^{i\delta} \right) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (9.43)$$

If $\delta = 0$ or $\delta = \pi$, then one just obtains a different linear polarization in the direction $(\hat{n}_1 \pm \hat{n}_2) / \sqrt{2}$.

Electromagnetic Waves in Vacuum (cont.)

But if $\delta = \pm\pi/2$, then the wave polarized along \hat{n}_2 is $\pi/2$ out of phase with the wave polarized along \hat{n}_1 : when the \hat{n}_1 mode has zero amplitude, the \hat{n}_2 mode has maximum amplitude and vice versa. If we take the real part, it is clear what is going on:

$$\mathcal{R} [\vec{E}(\vec{r}, t)] = \frac{E_0}{\sqrt{2}} \left[\hat{n}_1 \cos(\vec{k} \cdot \vec{r} - \omega t + \delta_0) + \hat{n}_2 \cos(\vec{k} \cdot \vec{r} - \omega t + \delta_0 \pm \frac{\pi}{2}) \right] \quad (9.44)$$

$$= \frac{E_0}{\sqrt{2}} \left[\hat{n}_1 \cos(\vec{k} \cdot \vec{r} - \omega t + \delta_0) \mp \hat{n}_2 \sin(\vec{k} \cdot \vec{r} - \omega t + \delta_0) \right] \quad (9.45)$$

The polarization vector maintains an amplitude E_0 but it sweeps around in a circle with period $T = 2\pi/\omega$: this is *circular polarization*. If $\delta = +\pi/2$, then the rotation at a particular point in space, looking *into* the wave (*i.e.*, looking toward $-\hat{k}$), is counterclockwise and the wave is called *left circularly polarized*. Conversely, $\delta = -\pi/2$ yields clockwise rotation and *right circular polarization*. One also speaks in terms of helicity, in which case one considers the rotation of the polarization relative to the direction of motion using the right-hand rule. The left circularly polarized wave has *positive helicity* because the polarization vector rotates around $+\hat{k}$ according to the right-hand rule (thumb along $+\hat{k}$). The right circularly polarized wave has negative helicity because it obeys the left-hand rule.

Electromagnetic Waves in Vacuum (cont.)

The next possibility is to allow unequal coefficients:

$$\vec{E}(\vec{r}, t) = \tilde{E}_0 (\alpha \hat{n}_1 + \beta \hat{n}_2 e^{i\delta}) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad \alpha^2 + \beta^2 = 1 \quad (9.46)$$

When $\delta = 0$ or $\delta = \pi$, we again obtain linear polarization, but now making an angle $\theta = \tan^{-1}(\pm \beta / \alpha)$ with the \hat{n}_1 axis (sign is the same as sign of $e^{i\delta}$).

If we now consider $\delta = \pm\pi/2$ and $\alpha \neq \beta$, we obtain an *elliptically polarized* wave: at a fixed point \vec{r}_0 , the polarization vector sweeps out an ellipse whose semimajor and semiminor axes are along \hat{n}_1 and \hat{n}_2 . If δ is an arbitrary value, then the semimajor and semiminor axes are rotated from the \hat{n}_1 - \hat{n}_2 system by an angle related to δ .

It turns out that elliptically polarized waves are easier to analyze if they are rewritten in terms of the two helicities of circular polarization. That is, if we take as our polarization basis and field decomposition

$$\hat{n}_{\pm} = \frac{1}{\sqrt{2}} (\hat{n}_1 \pm e^{i\pi/2} \hat{n}_2) \quad \vec{E}(\vec{r}, t) = (\tilde{E}_+ \hat{n}_+ + \tilde{E}_- \hat{n}_-) e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (9.47)$$

then the parameters of the ellipse traced out by the polarization vector are:

$$r e^{i\theta} = \frac{\tilde{E}_-}{\tilde{E}_+} \quad \frac{\text{semiminor axis}}{\text{semimajor axis}} = \left| \frac{1-r}{1+r} \right| \quad \text{angle wrt } \hat{n}_1 = \frac{\theta}{2} \quad (9.48)$$

where the angle is measured *looking into the wave* (i.e., looking in the $-\hat{k}$ direction).

Electromagnetic Waves in Vacuum (cont.)

It is interesting to note that a circularly polarized plane wave is no different in terms of angular momentum than a linearly polarized plane wave according to the definition of the angular momentum density, Equation 8.60:

$$\vec{\ell}_{\text{field}} = \epsilon_0 \vec{r} \times (\vec{E} \times \vec{B}) = \epsilon_0 E_0^2 \vec{r} \times \hat{k} \quad (9.49)$$

One can see that, the angular momentum has to do with the relative orientation of the propagation direction and the position vector than with the nature of the polarization. This reflects the fact that, in quantum mechanics, the helicity of the wave becomes the intrinsic spin angular momentum of the photon, while the quantity calculated above is the *orbital angular momentum* of the photon and has to do with the spatial distribution of the EM wave.

Lecture 19: Electromagnetic Waves in Perfectly Nonconducting Matter

Date Revised: 2014/05/01 13:00

Fixed a number of instances of use of $\vec{k} \times$ instead of $\hat{k} \times$

Date Given: 2014/05/01

Electromagnetic Waves in Perfectly Nonconducting Matter

Propagation in Linear Media

Maxwell's Equations in matter in the absence of free charges and currents are

$$\vec{\nabla} \cdot \vec{D} = 0 \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = 0 \quad (9.50)$$

As noted earlier, we need relations between \vec{D} and \vec{E} and between \vec{H} and \vec{B} to make use of these. If we assume linear media

$$\vec{D} = \epsilon \vec{E} \quad \vec{B} = \mu \vec{H} \quad (9.51)$$

then the equations reduce to

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \vec{\nabla} \times \vec{B} - \epsilon \mu \frac{\partial \vec{E}}{\partial t} = 0 \quad (9.52)$$

These are the same as our equations in vacuum, leading to the same kinds of waves, but with the modification

$$v = \frac{1}{\sqrt{\epsilon \mu}} = \frac{c}{n} \quad \text{with} \quad n = \sqrt{\frac{\epsilon \mu}{\epsilon_0 \mu_0}} = \text{index of refraction} \quad (9.53)$$

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

This mathematical transformation reflects a remarkable fact: the complicated polarization and magnetization of the medium occurring as the wave passes through it do nothing except change its speed and, we shall see, affect the wave amplitude. This is a consequence of the linearity of the medium we assume.

Most materials in which waves can propagate (as we will see, this means materials that do not have high conductivities) have $\mu \approx \mu_0$, so $n \approx \sqrt{\epsilon/\epsilon_0} = \sqrt{\epsilon_r}$. Since $\epsilon_r > 1$ in general (there are very few paraelectric materials that enhance the field rather than act to decrease it), light generally goes more slowly in dielectrics.

The relation between \vec{B} and \vec{E} , Equation 9.23, is modified in the obvious manner:

$$\vec{B}(\vec{r}, t) = \frac{1}{\nu} \hat{k} \times \vec{E}(\vec{r}, t) \quad (9.54)$$

As we did for free space, we also have that \vec{H} and \vec{E} are related by an impedance, $Z = \sqrt{\mu/\epsilon}$, which we now call the *wave impedance*. With it, we have

$$\vec{H}(\vec{r}, t) = \frac{\vec{B}(\vec{r}, t)}{\mu} = \frac{1}{Z} \hat{k} \times \vec{E}(\vec{r}, t) \quad (9.55)$$

Recall that \vec{H} carries units of surface current density.

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

Energy density, the Poynting vector, the stress tensor, and intensity take on unsurprising forms given the above modification:

$$u = \frac{1}{2} \left(\epsilon |E|^2 + \frac{1}{\mu} |B|^2 \right) = \epsilon |E|^2 \quad \vec{S} = \frac{1}{\mu} \vec{E}^* \times \vec{B} = v \epsilon |E|^2 \hat{k} \quad \underline{\underline{T}} = -\epsilon |E|^2 \hat{k} \hat{k} \quad (9.56)$$

The time averages for a sinusoidal wave of (real) amplitude E_0 are:

$$\langle u \rangle = \frac{1}{2} \epsilon E_0^2 \quad \langle \vec{S} \rangle = \frac{1}{2\mu} \langle \vec{E}^* \times \vec{B} \rangle = \frac{1}{2} v \epsilon E_0^2 \hat{k} \quad (9.57)$$

$$\langle \underline{\underline{T}} \rangle = -\frac{1}{2} \epsilon E_0^2 \hat{k} \hat{k} \quad I = \langle |\vec{S}| \rangle = \frac{1}{2} v \epsilon E_0^2 \quad (9.58)$$

Boundary Conditions

Recalling our boundary conditions for linear media (Equations 2.49, 4.23, 6.40, 6.42) and applying our assumption of no free currents, we have (\hat{n} = normal from 1 to 2, \hat{s} = any tangential vector at interface):

$$\hat{n} \cdot \epsilon_1 \vec{E}_1 = \hat{n} \cdot \epsilon_2 \vec{E}_2 \quad \hat{n} \cdot \vec{B}_1 = \hat{n} \cdot \vec{B}_2 \quad \hat{s} \cdot \vec{E}_1 = \hat{s} \cdot \vec{E}_2 \quad \hat{s} \cdot \frac{\vec{B}_1}{\mu_1} = \hat{s} \cdot \frac{\vec{B}_2}{\mu_2} \quad (9.59)$$

We will apply these to calculate the reflection and refraction of EM waves.

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

Reflection and Refraction

We will skip past the case of normal incidence, which you studied in Ph1c, but we will consider it as a special case of our generic results.

Assume we have a wave with propagation vector \vec{k}_i propagating in medium 1 and incident on an interface with normal \hat{n} . We expect there to be reflected and transmitted waves. We write these all as

$$\vec{E}_i(\vec{r}, t) = \vec{E}_{0,i} e^{i(\vec{k}_i \cdot \vec{r} - \omega t)} \quad \vec{B}_i(\vec{r}, t) = \frac{1}{v_1} \hat{k}_i \times \vec{E}_i \quad (9.60)$$

$$\vec{E}_r(\vec{r}, t) = \vec{E}_{0,r} e^{i(\vec{k}_r \cdot \vec{r} - \omega t)} \quad \vec{B}_r(\vec{r}, t) = \frac{1}{v_1} \hat{k}_r \times \vec{E}_r \quad (9.61)$$

$$\vec{E}_t(\vec{r}, t) = \vec{E}_{0,t} e^{i(\vec{k}_t \cdot \vec{r} - \omega t)} \quad \vec{B}_t(\vec{r}, t) = \frac{1}{v_2} \hat{k}_t \times \vec{E}_t \quad (9.62)$$

We have already applied the condition that the frequencies of the three waves are identical. They have to be in order to for any boundary conditions connecting them to be applicable at all time. Then

$$k_i v_1 = k_r v_1 = k_t v_2 = \omega \implies k_r = k_i \quad k_t = \frac{v_1}{v_2} k_i = \frac{n_2}{n_1} k_i \quad (9.63)$$

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

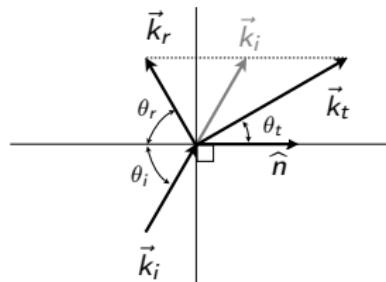
Now, consider the kinds of matching conditions we will write down. They must always hold over all \vec{r} in the interface. For example, if $\hat{n} = \hat{z}$ then the matching conditions must hold at all x and y and are of the form

$$(\) e^{i(\vec{k}_i \cdot (\hat{x}\hat{x} + \hat{y}\hat{y}) - \omega t)} + (\) e^{i(\vec{k}_r \cdot (\hat{x}\hat{x} + \hat{y}\hat{y}) - \omega t)} = (\) e^{i(\vec{k}_t \cdot (\hat{x}\hat{x} + \hat{y}\hat{y}) - \omega t)} \quad (9.64)$$

In order for these to hold at arbitrary x and y , it must be that

$$\hat{x} \cdot \vec{k}_i = \hat{x} \cdot \vec{k}_r = \hat{x} \cdot \vec{k}_t \quad (9.65)$$

$$\hat{y} \cdot \vec{k}_i = \hat{y} \cdot \vec{k}_r = \hat{y} \cdot \vec{k}_t \quad (9.66)$$



These conditions imply that if you project all three propagation vectors into the plane of the interface (whose normal is \hat{n}), then their projections in that plane are equal. Furthermore, there is a plane formed by this common xy projection of the propagation vectors and the normal \hat{n} (which is normal to the projection plane) and that all three vectors lie in this plane.

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

Each propagation vector makes an angle with the interface normal. We label them θ_i , θ_r , and θ_t . The projection of a given wavevector parallel to the normal is therefore $|\vec{k}| \cos \theta$ while the projection perpendicular to the normal — i.e., in the plane of the interface — is $|\vec{k}| \sin \theta$. Since we have argued that these projections into the plane of the interface are equal, we thus have

$$k_i \sin \theta_i = k_r \sin \theta_r = k_t \sin \theta_t \quad (9.67)$$

Now, using our relations $k_r = k_i$ and $k_t = \frac{n_2}{n_1} k_i$, we may conclude:

$$\text{law of reflection:} \quad \theta_i = \theta_r \quad (9.68)$$

$$\text{law of refraction (Snell's Law):} \quad n_1 \sin \theta_i = n_2 \sin \theta_t \quad (9.69)$$

Snell's Law tells us that, if $n_2 > n_1$, the light ray bends toward the normal, and it bends away from the normal if $n_1 > n_2$.

Total internal reflection occurs if $n_1 > n_2$ and $\sin \theta_i > n_2/n_1$, then $\sin \theta_t > 1$ and there is no solution for k_t . This happens because, when $n_2 < n_1$, then the magnitude $k_t < k_i$, so then the projection $k_i \sin \theta_i$ can be too large for k_t to match.

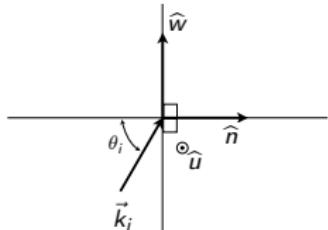
Note that none of these results depending on knowing anything about Maxwell's Equations or boundary conditions: these are generic properties of any waves, which is why they hold for sound waves, phonons, etc.

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

Reflected and Transmitted Field Relations

Now, let's apply our electromagnetic boundary conditions to figure out how the amplitudes and energies are related. We may drop all the exponential factors because we have established that they are identical at the interface. We need to define two new vectors:

$$\hat{u} = \frac{\hat{n} \times \vec{k}_i}{|\hat{n} \times \vec{k}_i|} = \frac{\hat{n} \times \vec{k}_r}{|\hat{n} \times \vec{k}_r|} = \frac{\hat{n} \times \vec{k}_t}{|\hat{n} \times \vec{k}_t|}$$
$$\hat{w} = \hat{u} \times \hat{n} \quad (9.70)$$



It doesn't matter which \hat{k} we use to define \hat{u} because we argued earlier that all of them have the same projection in the plane of the interface, which is the piece that matters for the above cross product. \hat{u} is also normal to all the \vec{k} , so it is perpendicular to the plane in which the propagation vectors lie. \hat{w} is then the obvious third direction, and it and \hat{n} define the plane that the \vec{k} live in.

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

With these definitions, our boundary conditions can be written as

$$\hat{n} \cdot \epsilon_1 (\vec{E}_{0,i} + \vec{E}_{0,r}) = \hat{n} \cdot \epsilon_2 \vec{E}_{0,t} \quad \left[\begin{array}{c} \hat{u} \\ \hat{w} \end{array} \right] \cdot (\vec{E}_{0,i} + \vec{E}_{0,r}) = \left[\begin{array}{c} \hat{u} \\ \hat{w} \end{array} \right] \cdot \vec{E}_{0,t} \quad (9.71)$$

$$\hat{n} \cdot (\vec{B}_{0,i} + \vec{B}_{0,r}) = \hat{n} \cdot \vec{B}_{0,t} \quad \left[\begin{array}{c} \hat{u} \\ \hat{w} \end{array} \right] \cdot \frac{1}{\mu_1} (\vec{B}_{0,i} + \vec{B}_{0,r}) = \left[\begin{array}{c} \hat{u} \\ \hat{w} \end{array} \right] \cdot \frac{1}{\mu_2} \vec{B}_{0,t} \quad (9.72)$$

where the stacking of \hat{u} and \hat{w} is just meant to indicate that those equations apply with either \hat{u} on both sides or \hat{w} on both sides.

Now, we must consider two cases for the polarization of the incoming wave: *in parallel to the plane of incidence* or *perpendicular to the plane of incidence*:

in (parallel to) the plane of incidence $\vec{E}_{0,i} \cdot \hat{u} = 0$ (9.73)

perpendicular to the plane of incidence $\vec{E}_{0,i} \cdot \left[\begin{array}{c} \hat{n} \\ \hat{w} \end{array} \right] = 0$ (9.74)

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

In (parallel to) the plane of incidence:

When $\vec{E}_{0,i}$ is in the plane of incidence, we can decompose it into pieces along \hat{w} and along \hat{n} . There is freedom on the sign convention, and we choose

$$\vec{E}_{0,i} = \tilde{E}_{0,i} (\hat{w} \cos \theta_i - \hat{n} \sin \theta_i) \quad (9.75)$$

With these choices, we then have (remember, $\vec{B} = \hat{k} \times \vec{E}/v$):

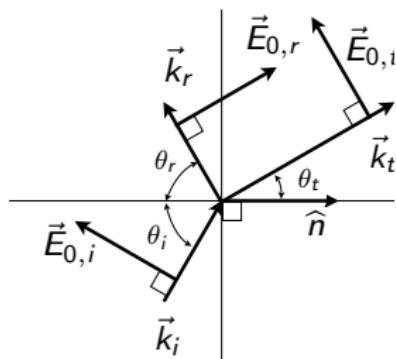
$$\vec{E}_{0,r} = \tilde{E}_{0,r} (\hat{w} \cos \theta_r + \hat{n} \sin \theta_r) \quad (9.76)$$

$$\vec{E}_{0,t} = \tilde{E}_{0,t} (\hat{w} \cos \theta_t - \hat{n} \sin \theta_t) \quad (9.77)$$

$$\vec{B}_{0,i} = \frac{\tilde{E}_{0,i}}{v_1} \hat{u} \quad (9.78)$$

$$\vec{B}_{0,r} = -\frac{\tilde{E}_{0,r}}{v_1} \hat{u} \quad (9.79)$$

$$\vec{B}_{0,t} = \frac{\tilde{E}_{0,t}}{v_2} \hat{u} \quad (9.80)$$



Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

So, restricting to the boundary conditions with information, we obtain

$$\hat{n} : \quad \epsilon_1 \left(-\tilde{E}_{0,i} \sin \theta_i + \tilde{E}_{0,r} \sin \theta_r \right) = -\tilde{E}_{0,t} \sin \theta_t \quad (9.81)$$

$$\hat{w} : \quad \tilde{E}_{0,i} \cos \theta_i + \tilde{E}_{0,r} \cos \theta_r = \tilde{E}_{0,t} \cos \theta_t \quad (9.82)$$

$$\hat{u} : \quad \frac{1}{\mu_1 v_1} \left(\tilde{E}_{0,i} - \tilde{E}_{0,r} \right) = \frac{1}{\mu_2 v_2} \tilde{E}_{0,t} \quad (9.83)$$

If we do the algebra to find $\tilde{E}_{0,r}$ and $\tilde{E}_{0,t}$ in terms of $\tilde{E}_{0,i}$, we obtain *Fresnel's Equations in (parallel to) the plane of incidence*:

$$\alpha = \frac{\cos \theta_t}{\cos \theta_i} \quad \beta = \frac{\mu_1 n_2}{\mu_2 n_1} = \frac{Z_1}{Z_2} \quad (9.84)$$

$$\frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}} = \left(\frac{\alpha - \beta}{\alpha + \beta} \right) \quad \frac{\tilde{E}_{0,t}}{\tilde{E}_{0,i}} = \left(\frac{2}{\alpha + \beta} \right)$$

*Fresnel's
Equations
in (parallel
to) the plane
of incidence*

(9.85)

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

Perpendicular to the plane of incidence:

When $\vec{E}_{0,i}$ is perpendicular to the plane of incidence, it must be parallel to \hat{u} because \hat{u} defines the normal to that plane (again, up to a sign choice.) Now $\vec{B}_{0,i}$ is in the plane of incidence. We have

$$\vec{E}_{0,i} = \tilde{E}_{0,i} \hat{u} \quad (9.86)$$

With these choices, we then have (again, $\vec{B} = \hat{k} \times \vec{E}/v$):

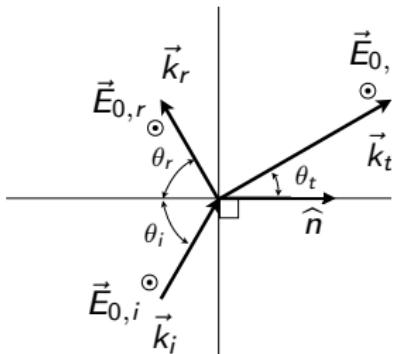
$$\vec{E}_{0,r} = \tilde{E}_{0,r} \hat{u} \quad (9.87)$$

$$\vec{E}_{0,t} = \tilde{E}_{0,t} \hat{u} \quad (9.88)$$

$$\vec{B}_{0,i} = \frac{\tilde{E}_{0,i}}{v_1} (-\hat{w} \cos \theta_i + \hat{n} \sin \theta_i) \quad (9.89)$$

$$\vec{B}_{0,r} = \frac{\tilde{E}_{0,r}}{v_1} (\hat{w} \cos \theta_r + \hat{n} \sin \theta_r) \quad (9.90)$$

$$\vec{B}_{0,t} = \frac{\tilde{E}_{0,t}}{v_2} (-\hat{w} \cos \theta_t + \hat{n} \sin \theta_t) \quad (9.91)$$



Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

Restricting to the boundary conditions with information, we obtain

$$\hat{n} : \frac{1}{\nu_1} (\tilde{E}_{0,i} \sin \theta_i + \tilde{E}_{0,r} \sin \theta_r) = \frac{1}{\nu_2} \tilde{E}_{0,t} \sin \theta_t \quad (9.92)$$

$$\hat{w} : \frac{1}{\mu_1 \nu_1} (\tilde{E}_{0,i} \cos \theta_i - \tilde{E}_{0,r} \cos \theta_r) = \frac{1}{\mu_2 \nu_2} \tilde{E}_{0,t} \cos \theta_t \quad (9.93)$$

$$\hat{u} : \tilde{E}_{0,i} + \tilde{E}_{0,r} = \tilde{E}_{0,t} \quad (9.94)$$

We repeat the algebra to obtain *Fresnel's Equations perpendicular to the plane of incidence*:

$$\frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}} = \left(\frac{1 - \alpha \beta}{1 + \alpha \beta} \right) \quad \frac{\tilde{E}_{0,t}}{\tilde{E}_{0,i}} = \left(\frac{2}{1 + \alpha \beta} \right) \quad \begin{matrix} \textit{Fresnel's} \\ \textit{Equations} \\ \textit{perpendicular} \\ \textit{to the plane} \\ \textit{of incidence} \end{matrix} \quad (9.95)$$

with α and β as defined earlier.

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

Implications for Signs and Magnitudes for Electric Field

- ▶ *Sign of transmitted wave*

To understand the sign of the transmitted wave, we just need to notice that α and β are always positive numbers. α is always positive because θ_i and θ_r are restricted to the first quadrant. Therefore, all the quantities in the expressions for $\tilde{E}_{0,t}$ are positive, and thus the transmitted wave always has the same sign electric field as the incident wave.

Since $\vec{k}_i \cdot \hat{n}$ and $\vec{k}_t \cdot \hat{n}$ have the same sign and $\vec{B} \propto \vec{k} \times \vec{E}$, we may also conclude that the sign of the magnetic field of the transmitted wave is unchanged.

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

► Sign of electric field of reflected wave, general case

The sign of the reflected wave depends on the sizes of α and β . What general statements can we make?

We know that, for a given pair of media, either $\sin \theta_t / \sin \theta_i < 1$ or $\sin \theta_t / \sin \theta_i > 1$ is true for all angles because this ratio is set by Snell's Law, $\sin \theta_t / \sin \theta_i = n_1 / n_2$. Since sin and cos are monotonic, we can conclude that $\alpha = \cos \theta_t / \cos \theta_i$ is also either smaller than or greater than 1 for all angles, with the case being determined by n_1 / n_2 :

$$\frac{n_1}{n_2} > 1 \iff \frac{\sin \theta_t}{\sin \theta_i} > 1 \iff \frac{\cos \theta_t}{\cos \theta_i} < 1 \iff \alpha < 1 \quad (9.96)$$

$$\frac{n_1}{n_2} < 1 \iff \frac{\sin \theta_t}{\sin \theta_i} < 1 \iff \frac{\cos \theta_t}{\cos \theta_i} > 1 \iff \alpha > 1 \quad (9.97)$$

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

However, $\beta = \frac{\mu_1}{\mu_2} \frac{n_2}{n_1} = \frac{Z_1}{Z_2}$, so the size of n_2/n_1 relative to unity does not determine the size of β relative to unity. No generic statement can be made about the relative size of α , β , and $1/\beta$. We can only make statements conditional on their relative sizes:

- ▶ *Parallel incidence*

$\alpha > \beta$ gives the same sign and $\alpha < \beta$ gives opposite signs. *If* there is a switchover, it occurs at

$$\frac{\cos \theta_t}{\cos \theta_i} = \frac{\mu_1}{\mu_2} \frac{n_2}{n_1} = \frac{Z_1}{Z_2} \iff \frac{\sin 2\theta_t}{\sin 2\theta_i} = \frac{\mu_1}{\mu_2}$$

If the switchover occurs, the reflection vanishes at the switchover angle.

- ▶ *Perpendicular incidence*

In this case, $\alpha\beta < 1$ gives the same sign and $\alpha\beta > 1$ gives opposite signs. *If* there is a switchover, it occurs at

$$\frac{\cos \theta_t}{\cos \theta_i} = \frac{\mu_2}{\mu_1} \frac{n_1}{n_2} = \frac{Z_2}{Z_1} \iff \frac{\tan \theta_t}{\tan \theta_i} = \frac{\mu_2}{\mu_1}$$

Again, if the switchover occurs, the reflection vanishes at the switchover angle.

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

- ▶ *Sign of electric field of reflected wave for $\mu_1 = \mu_2$, and Brewster's Angle*

This case is typical for everyday experience; there are few light-transmitting yet also $\mu \neq \mu_0$ materials. We can be much more specific in this case because now $\beta = n_2/n_1$ and there is a clear relationship between α and β , which we may now rewrite as

$$\frac{n_1}{n_2} > 1 \iff \beta = \frac{n_2}{n_1} < 1 \quad \text{and} \quad \alpha = \frac{\cos \theta_t}{\cos \theta_i} < 1 \quad (9.98)$$

$$\frac{n_1}{n_2} < 1 \iff \beta = \frac{n_2}{n_1} > 1 \quad \text{and} \quad \alpha = \frac{\cos \theta_t}{\cos \theta_i} > 1 \quad (9.99)$$

Considering the two cases separately:

- ▶ *Parallel incidence, $\mu_1 = \mu_2$*

Since α and β are both either < 1 or > 1 , it is possible for $\alpha = \beta$ to be true and therefore the sign may depend on the angle. At $\theta_i = \theta_t = 0$, we have $\alpha = 1$ identically, so the sign of the reflected wave is $1 - \beta$, which is positive if $\beta < 1$ and negative if $\beta > 1$. We then have to ask whether it is possible for the polarity of $\alpha - \beta$ to change, which happens if there is a zero in $\alpha - \beta$.

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

To solve for this angle, called *Brewster's Angle*, θ_B , we can square the ratio of cosines so it can be written in terms of sines, then use Snell's Law to obtain:

$$\frac{1 - \frac{n_1^2}{n_2^2} \sin^2 \theta_B}{1 - \sin^2 \theta_B} = \frac{n_2^2}{n_1^2} \iff \sin^2 \theta_B = \frac{n_2^2}{n_1^2 + n_2^2} \quad (9.100)$$

$$\iff \tan \theta_B = \frac{n_2}{n_1} = \beta = \frac{Z_1}{Z_2} \quad \begin{matrix} \text{Brewster's} \\ \text{Angle for} \\ \mu_1 = \mu_2 \end{matrix} \quad (9.101)$$

Note that $0 < \theta_B < \pi/4$ for $n_2 < n_1$ and $\pi/4 < \theta_B < \pi/2$ for $n_2 > n_1$. Therefore, we may summarize this case as:

sign of reflected wave for parallel incidence and $\mu_1 = \mu_2$:

$$0 < \theta_i < \theta_B : \quad \text{sign} \left(\frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}} \right) = \text{sign} \left(\frac{n_1 - n_2}{n_1} \right) \quad (9.102)$$

$$\theta_B < \theta_i < \theta_{max} : \quad \text{sign} \left(\frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}} \right) = \text{sign} \left(\frac{n_2 - n_1}{n_1} \right) \quad (9.103)$$

$$\theta_{max} = \sin^{-1} \left(\min \left(1, \frac{n_2}{n_1} \right) \right) \quad (9.104)$$

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

- ▶ *Perpendicular incidence, $\mu_1 = \mu_2$*

The perpendicular incidence case is easier to analyze. At normal incidence, $\alpha = 1$ again and the formulae become identical to the parallel incidence case, giving us the same behavior: the reflected wave is positive if $\beta < 1$ and negative if $\beta > 1$. This common behavior must hold, as the parallel and perpendicular cases are degenerate for normal incidence.

To see if there is an angle at which the sign of the reflected wave can flip, we need to know if there is an angle at which the reflected wave vanishes. It is easy to see there is not: for $1 - \alpha\beta$ to vanish, we require $\alpha = 1/\beta$. For the case $\mu_1 = \mu_2$, we thus require $\alpha = \frac{n_1}{n_2}$. But we saw above that, when $\frac{n_1}{n_2} > 1$ we have $\alpha < 1$ and when $\frac{n_1}{n_2} < 1$ we have $\alpha > 1$. Thus, there is never a zero in the reflected wave, and the sign can never flip.

Therefore, we may summarize this case as:

sign of reflected wave for perpendicular incidence and $\mu_1 = \mu_2$:

$$0 < \theta_i < \theta_{max} : \quad \text{sign} \left(\frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}} \right) = \text{sign} \left(\frac{n_1 - n_2}{n_1} \right) \quad (9.105)$$

$$\theta_{max} = \sin^{-1} \left(\min \left(1, \frac{n_2}{n_1} \right) \right) \quad (9.106)$$

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

- ▶ *Practical implications of Brewster's Angle*

Because the reflected amplitude for parallel incidence goes through a zero at θ_B , it is small for angles near θ_B . For the everyday materials water ($n = 1.33$) and glass ($n \approx 1.5$), this angle is 53° and 56° , implying this is a typical viewing angle and explaining why polarized sunglasses reduce glare: by blocking the reflected polarization that is perpendicular to the plane of incidence, they block the only component of the reflected wave that has appreciable amplitude. They are designed to pass the parallel component because it has no reflection near Brewster's angle. Be sure to get the orientation right: if the interface is a horizontal surface, then that perpendicular plane is actually parallel to the surface for light coming from above, so the plane of parallel incidence is vertical for the viewer.

- ▶ *Sign of magnetic field of reflected wave*

For any of these cases, we can obtain the sign of the *magnetic field* of the reflected wave by applying the rule $\vec{B} \propto \vec{k} \times \vec{E}$ to the reflected wave. From this, and from the sign flip of $\vec{k}_r \cdot \hat{n}$ relative to $\vec{k}_i \cdot \hat{n}$, we can conclude that the magnetic field of the reflected wave has the opposite behavior as the electric field in both the parallel and perpendicular cases: if the electric field receives a sign flip, the magnetic field does not, and vice versa.

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

Reflected and Transmitted Energy

The energy flux (intensity) at a particular point is

$$I_j = \left\langle |\vec{S}_j| \right\rangle = \frac{1}{2} \epsilon_j v_j E_j^2 \cos \theta_j \quad (9.107)$$

We can calculate from this the reflected and transmitted energy or power ratios:

$$\mathcal{R} = \frac{I_r}{I_i} = \left(\frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}} \right)^2 = \left(\frac{\alpha - \beta}{\alpha + \beta} \right)^2 \quad \text{parallel} \quad (9.108)$$

$$= \left(\frac{1 - \alpha \beta}{1 + \alpha \beta} \right)^2 \quad \text{perpendicular} \quad (9.109)$$

$$\mathcal{T} = \frac{I_t}{I_i} = \frac{\epsilon_2 v_2}{\epsilon_1 v_1} \left(\frac{\tilde{E}_{0,t}}{\tilde{E}_{0,i}} \right)^2 \frac{\cos \theta_t}{\cos \theta_i} = \alpha \beta \left(\frac{2}{\alpha + \beta} \right)^2 \quad \text{parallel} \quad (9.110)$$

$$= \alpha \beta \left(\frac{2}{1 + \alpha \beta} \right)^2 \quad \text{perpendicular} \quad (9.111)$$

By calculating $\mathcal{R} + \mathcal{T}$ explicitly, one can see that $\mathcal{R} + \mathcal{T} = 1$ always in both cases.

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

Normal Incidence

Let's summarize the results for normal incidence: $\theta_i = \theta_r = \theta_t = 0$. This implies $\alpha = 1$, for which the parallel and perpendicular cases are equivalent, yielding:

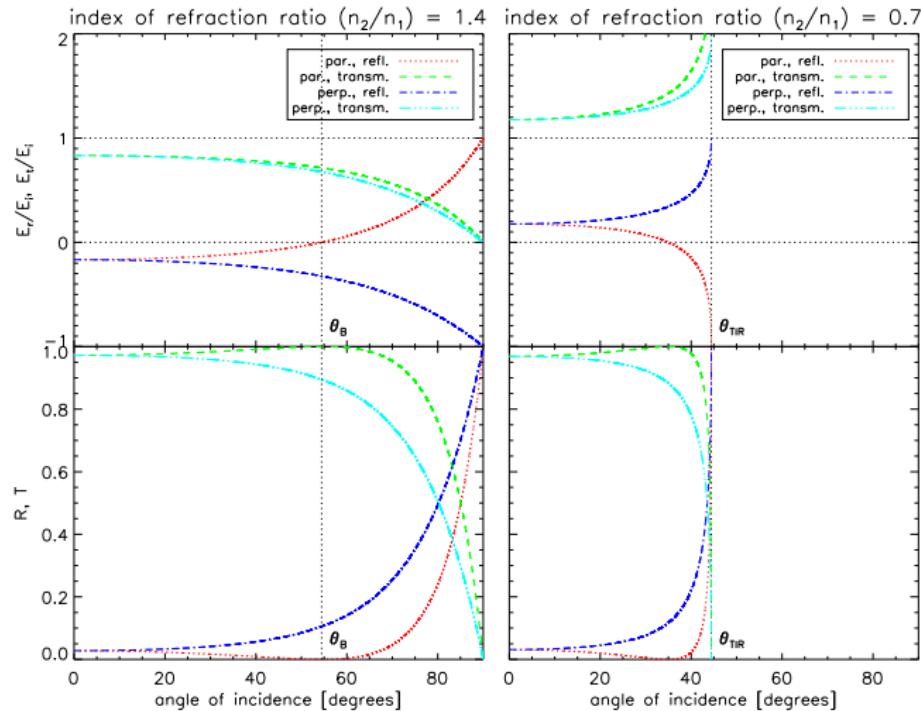
$$\frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}} = \frac{1 - \beta}{1 + \beta} = \frac{1 - \frac{Z_1}{Z_2}}{1 + \frac{Z_1}{Z_2}} \quad \mathcal{R} = \left(\frac{1 - \beta}{1 + \beta} \right)^2 = \left(\frac{1 - \frac{Z_1}{Z_2}}{1 + \frac{Z_1}{Z_2}} \right)^2 \quad (9.112)$$

$$\frac{\tilde{E}_{0,t}}{\tilde{E}_{0,i}} = \frac{2}{1 + \beta} = \frac{2}{1 + \frac{Z_1}{Z_2}} \quad \mathcal{T} = \beta \left(\frac{2}{1 + \beta} \right)^2 = \frac{Z_1}{Z_2} \left(\frac{2}{1 + \frac{Z_1}{Z_2}} \right)^2 \quad (9.113)$$

Electromagnetic Waves in Perfectly Nonconducting Matter (cont.)

Typical Behavior

Notice the nonmonotonic behavior in \mathcal{R} and \mathcal{T} near θ_B for the parallel incidence case: this is necessary to yield the zero in the reflected electric field and in \mathcal{R} at θ_B . The same behavior is not present in the transmitted field amplitude; $\mathcal{T} = 1$ comes about because of the α factor in \mathcal{T} . Notice also how the transmitted field amplitude appears to diverge but the transmitted energy vanishes at $\theta_{TIR} = \sin^{-1} \frac{n_2}{n_1}$; again, the α factor in \mathcal{T} explains this.



Lecture 20:
Electromagnetic Waves in Conductors
Electromagnetic Waves in Dispersive Matter

Date Revised: 2014/05/10 06:00

Revised calculation of Joule power dissipation for
EM wave in a good conductor

2014/06/06 Correct missing $\cdot \hat{s}$ in Eqn. 9.150

Date Given: 2014/05/06

Electromagnetic Waves in Conducting Matter

Maxwell's Equations and the Wave Equation for Conductors; Plane-Wave Solutions

The primary distinction between a conducting medium and a nonconducting medium is that Ohm's Law is now obeyed, $\vec{J}_f = \sigma \vec{E}$. Incorporating it into Maxwell's Equations for a linear medium gives

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho_f}{\epsilon} \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \vec{\nabla} \times \vec{B} - \mu \sigma \vec{E} - \epsilon \mu \frac{\partial \vec{E}}{\partial t} = 0 \quad (9.114)$$

We can quickly see that we do not need to worry about free charge. If we combine the continuity equation, $\vec{\nabla} \cdot \vec{J}_f = -\partial \rho_f / \partial t$ with Gauss's Law, we obtain

$$\frac{\partial \rho_f}{\partial t} = -\sigma (\vec{\nabla} \cdot \vec{E}) = -\frac{\sigma}{\epsilon} \rho_f \quad (9.115)$$

The solution to this equation (with $\sigma/\epsilon > 0$) is a decaying exponential in time, $\rho_f(t) = \rho_f(t=0) \exp(-t/\tau)$ with $\tau = \epsilon/\sigma$. Thus, any free charge quickly decays away. In practice, it flows out to the edges of the conductor to make the DC field in the conductor vanish. The time constant for the decay as compared to the relevant timescale (e.g., the wave oscillation period $T = 2\pi/\omega$) is a measure of how good the conductor is at the frequencies of interest: for a good conductor, one must have $\tau \ll T$.

Electromagnetic Waves in Conducting Matter (cont.)

Assuming this condition is satisfied, we then have again a homogeneous set of Maxwell's Equations:

$$\vec{\nabla} \cdot \vec{E} = 0 \quad \vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \vec{\nabla} \times \vec{B} - \mu \sigma \vec{E} - \epsilon \mu \frac{\partial \vec{E}}{\partial t} = 0 \quad (9.116)$$

We may do as before to obtain wave equations:

$$\nabla^2 \vec{E} = \epsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2} + \sigma \mu \frac{\partial \vec{E}}{\partial t} \quad \nabla^2 \vec{B} = \epsilon \mu \frac{\partial^2 \vec{B}}{\partial t^2} + \sigma \mu \frac{\partial \vec{B}}{\partial t} \quad (9.117)$$

As we did for nonconducting matter, we assume plane wave solutions because any solution can be built from them by linearity:

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad \vec{B}(\vec{r}, t) = \vec{B}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (9.118)$$

We dispense with repeating the proof that \vec{k} , ω , and δ are the same for \vec{E} and \vec{B} (the proof is similar in form) and simply assume it and check it below. We determine the relation between \vec{E}_0 and \vec{B}_0 later.

Electromagnetic Waves in Conducting Matter (cont.)

Plugging in, one obtains an equation for \vec{k} :

$$\vec{k} \cdot \vec{k} = \epsilon \mu \omega^2 + i \sigma \mu \omega \quad (9.119)$$

We take the square root, yielding

$$\sqrt{\vec{k} \cdot \vec{k}} \equiv k + i \kappa \quad \nu_{\epsilon\mu} = \frac{1}{\sqrt{\epsilon \mu}} \quad k_{\epsilon\mu} = \frac{\omega}{\nu_{\epsilon\mu}} \quad \lambda_{\epsilon\mu} = \frac{2\pi}{k_{\epsilon\mu}} \quad \tau = \frac{\epsilon}{\sigma} \quad (9.120)$$

$$k = k_{\epsilon\mu} \left[\frac{\sqrt{1 + \frac{1}{\omega^2 \tau^2}} + 1}{2} \right]^{1/2} \quad \kappa = k_{\epsilon\mu} \left[\frac{\sqrt{1 + \frac{1}{\omega^2 \tau^2}} - 1}{2} \right]^{1/2} \equiv \frac{1}{\delta} \quad (9.121)$$

The wave thus takes the form

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{-(\hat{k} \cdot \vec{r} / \delta)} e^{i(k \hat{k} \cdot \vec{r} - \omega t)} \quad \vec{B}(\vec{r}, t) = \vec{B}_0 e^{-(\hat{k} \cdot \vec{r} / \delta)} e^{i(k \hat{k} \cdot \vec{r} - \omega t)} \quad (9.122)$$

which is a plane wave propagating in the direction \hat{k} with propagation constant k , wavelength $\lambda = 2\pi/k$, speed $v = \omega/k = \lambda\nu$, and index of refraction $n = c/v$, but with decaying amplitude. The decay length is the *skin depth*, $\delta = 1/\kappa$. Notice that the time dependence is unchanged and given by ω and ν still while the spatial dependence has been modified.

Electromagnetic Waves in Conducting Matter (cont.)

Poor Conductor Limit

It is instructive to take the *poor conductor limit*, $\omega\tau \gg 1$. The second term under the inner square root is small and one can Taylor expand to obtain

$$k \xrightarrow{\omega\tau \gg 1} \frac{k_{\epsilon\mu}}{\sqrt{2}} \left[\left(1 + \frac{1}{2} \frac{1}{\omega^2 \tau^2} \right) + 1 \right]^{1/2} \approx k_{\epsilon\mu} \quad (9.123)$$

$$\kappa \xrightarrow{\omega\tau \gg 1} \frac{\omega}{v_{\epsilon\mu} \sqrt{2}} \left[\left(1 + \frac{1}{2} \frac{1}{\omega^2 \tau^2} \right) - 1 \right]^{1/2} \approx \frac{k_{\epsilon\mu}}{2\omega\tau} = \frac{\sigma}{2} \sqrt{\frac{\mu}{\epsilon}} = \frac{Z_{\epsilon\mu} \sigma}{2} \quad (9.124)$$

$$\delta \xrightarrow{\omega\tau \gg 1} \frac{2}{\sigma} \sqrt{\frac{\epsilon}{\mu}} = \frac{2}{Z_{\epsilon\mu} \sigma} \quad (9.125)$$

When the conductivity is low, meaning that the response time is large compared to the oscillation frequency, the wave propagation speed and wavelength are the same as in a nonconducting medium. The form for the skin depth is particularly interesting because it combines the wave impedance with the conductivity to obtain the skin depth. We can write the skin depth another way, now using the wavelength in the medium:

$$\delta \xrightarrow{\omega\tau \gg 1} \frac{2\omega\tau}{k_{\epsilon\mu}} = \frac{2\tau}{T} \lambda_{\epsilon\mu} \gg \lambda_{\epsilon\mu} \quad (9.126)$$

$2\tau/T \gg 1$ counts the number of wavelengths over which the amplitude decays: in a poor conductor, the decay of the wave happens over many wavelengths.

Electromagnetic Waves in Conducting Matter (cont.)

Good Conductor Limit

Similarly, we can take the *good conductor* limit, $\omega\tau \ll 1$. Here, the $\frac{1}{\omega^2\tau^2}$ term dominates the square root and k and κ converge to the same value:

$$\text{good conductor: } k, \kappa \xrightarrow{\omega\tau \ll 1} \frac{\omega}{v_{\epsilon\mu}} \left[\frac{1}{2} \sqrt{\frac{1}{\omega^2\tau^2}} \right]^{1/2} = \sqrt{\frac{\omega}{2v_{\epsilon\mu}^2\tau}} = \sqrt{\frac{\mu\sigma\omega}{2}} \quad (9.127)$$

$$\delta \xrightarrow{\omega\tau \ll 1} \sqrt{\frac{2}{\mu\sigma\omega}} \quad (9.128)$$

There is no propagating wave because the decay constant and the propagation constant are the same; put another way, $\delta = 1/\kappa = 1/k = \lambda_{\epsilon\mu}/2\pi$, indicating the wave decays in about 1/6 of a wavelength!

Electromagnetic Waves in Conducting Matter (cont.)

General Results

Coming back to our general plane-wave solution, we write (with \hat{n}_1 defining the polarization of \vec{E})

$$\vec{E}(\vec{r}, t) = \tilde{E}_0 \hat{n}_1 e^{-(\hat{k} \cdot \vec{r} / \delta)} e^{i(k \hat{k} \cdot \vec{r} - \omega t)} \quad (9.129)$$

and we use the $\vec{\nabla} \times \vec{E}$ equation to find

$$\vec{B}(\vec{r}, t) = \frac{k + i\kappa}{\omega} \tilde{E}_0 \hat{k} \times \hat{n}_1 e^{-(\hat{k} \cdot \vec{r} / \delta)} e^{i(k \hat{k} \cdot \vec{r} - \omega t)} \quad (9.130)$$

We may write

$$k + i\kappa = K e^{i\phi} \quad K = \sqrt{k^2 + \kappa^2} = k_{\epsilon\mu} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega} \right)^2} \right]^{1/2} \quad (9.131)$$

$$\tan \phi = \frac{\kappa}{k} = \left[\frac{\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega} \right)^2} - 1}{\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega} \right)^2} + 1} \right]^{1/2} \quad (9.132)$$

Electromagnetic Waves in Conducting Matter (cont.)

With the above definition, we can more simply write the relation between the phases and amplitude of \vec{E} and \vec{B} :

$$\frac{\tilde{B}_0}{\tilde{E}_0} = \frac{k + i\kappa}{\omega} = \frac{Ke^{i\phi}}{\omega} = e^{i\phi} \frac{k_{\epsilon\mu}}{k_{\epsilon\mu} v_{\epsilon\mu}} \left[\sqrt{1 + \left(\frac{\sigma}{\epsilon\omega} \right)^2} \right]^{1/2} = \frac{e^{i\phi}}{v_{\epsilon\mu}} \left[1 + \left(\frac{\sigma}{\epsilon\omega} \right)^2 \right]^{1/4} \quad (9.133)$$

Thus, we see that there is now a phase shift between electric and magnetic fields, with the magnetic field lagging behind the electric field. The amplitudes are related in a somewhat complicated manner that depends on the wave velocity as before but now with a modification factor that depends on the conductivity. Again, taking some limits:

$$\text{poor conductor : } \frac{1}{\omega\tau} = \frac{\sigma}{\epsilon\omega} \ll 1 \quad \Rightarrow \quad \frac{\tilde{B}_0}{\tilde{E}_0} \rightarrow \frac{1}{v_{\epsilon\mu}} \quad (9.134)$$

$$\text{good conductor : } \frac{1}{\omega\tau} = \frac{\sigma}{\epsilon\omega} \gg 1 \quad \Rightarrow \quad \frac{\tilde{B}_0}{\tilde{E}_0} \rightarrow \frac{e^{i\pi/4}}{v_{\epsilon\mu}} \sqrt{\frac{1}{\omega\tau}} \gg \frac{1}{v_{\epsilon\mu}} \quad (9.135)$$

In a poor conductor, the relation approaches the perfect nonconductor case (as we expect). In contrast, in a perfect conductor, the magnetic field is enhanced (because of the currents flowing) and the magnetic field lags the electric field by exactly $\pi/4$ (not $\pi/2$ as in an inductor!).

Electromagnetic Waves in Conducting Matter (cont.)

Energy Density, Poynting Vector, and Intensity

We calculate these using our standard formulae:

$$\langle u \rangle = \frac{1}{4} \left(\epsilon \left| \tilde{E}_0 \right|^2 + \frac{1}{\mu} \left| \tilde{B}_0 \right|^2 \right) e^{-(2 \hat{k} \cdot \vec{r} / \delta)} \quad (9.136)$$

$$= \frac{1}{4} \left(\epsilon + \frac{1}{\mu} \frac{1}{v_{\epsilon\mu}^2} \left[\left(\frac{\sigma}{\epsilon \omega} \right)^2 + 1 \right]^{1/2} \right) \left| \tilde{E}_0 \right|^2 e^{-(2 \hat{k} \cdot \vec{r} / \delta)} \quad (9.137)$$

$$= \frac{1}{4} \epsilon \left| \tilde{E}_0 \right|^2 \left(1 + \left[\left(\frac{\sigma}{\epsilon \omega} \right)^2 + 1 \right]^{1/2} \right) e^{-(2 \hat{k} \cdot \vec{r} / \delta)} \quad (9.138)$$

$$= \frac{1}{2} \epsilon \frac{k^2}{k_{\epsilon\mu}^2} \left| \tilde{E}_0 \right|^2 e^{-(2 \hat{k} \cdot \vec{r} / \delta)} \quad (9.139)$$

$$\langle \vec{S} \rangle = \frac{1}{2\mu} \mathcal{R} (\langle \vec{E}^* \times \vec{B} \rangle) = \frac{\hat{k}}{2\mu} \left| \tilde{E}_0 \right|^2 \frac{e^{-(2 \hat{k} \cdot \vec{r} / \delta)}}{v_{\epsilon\mu}} \left[1 + \left(\frac{\sigma}{\epsilon \omega} \right)^2 \right]^{1/4} \cos \phi \quad (9.140)$$

$$= \frac{\hat{k}}{2} v_{\epsilon\mu} \epsilon \left| \tilde{E}_0 \right|^2 \left[1 + \left(\frac{\sigma}{\epsilon \omega} \right)^2 \right]^{1/4} e^{-(2 \hat{k} \cdot \vec{r} / \delta)} \cos \phi \quad (9.141)$$

$$= \frac{\hat{k}}{2} v_{\epsilon\mu} \epsilon \frac{K}{k_{\epsilon\mu}} \left| \tilde{E}_0 \right|^2 e^{-(2 \hat{k} \cdot \vec{r} / \delta)} \cos \phi = \hat{k} \frac{k_{\epsilon\mu}}{k} v_{\epsilon\mu} \langle u \rangle \quad (9.142)$$

$$I = \langle |\vec{S}| \rangle \quad (9.143)$$

Electromagnetic Waves in Conducting Matter (cont.)

Let's take the poor conductor and good conductor limits:

$1/\omega\tau = \sigma/\epsilon\omega \ll 1$, poor conductor:

$$\langle u \rangle \rightarrow \frac{1}{2} \epsilon |\tilde{E}_0|^2 e^{-(2\hat{k} \cdot \vec{r}/\delta)} \quad (9.144)$$

$$\langle \vec{s} \rangle \rightarrow \frac{\hat{k}}{2} v_{\epsilon\mu} \epsilon |\tilde{E}_0|^2 e^{-(2\hat{k} \cdot \vec{r}/\delta)} \cos \phi = v_{\epsilon\mu} \langle u \rangle \hat{k} \quad I \rightarrow v_{\epsilon\mu} \langle u \rangle \quad (9.145)$$

The poor conductor expressions match our expressions for a wave in a nonconducting medium except that they decay with depth as one would expect.

Electromagnetic Waves in Conducting Matter (cont.)

$1/\omega\tau = \sigma/\epsilon\omega \gg 1$, good conductor:

$$\langle u \rangle \rightarrow \frac{1}{4} \epsilon |\tilde{E}_0|^2 \frac{\sigma}{\epsilon\omega} e^{-(2\hat{k}\cdot\vec{r}/\delta)} = \frac{1}{4} \frac{|\tilde{J}_0| |\tilde{E}_0|}{\omega} e^{-(2\hat{k}\cdot\vec{r}/\delta)} \quad (9.146)$$

$$\langle \tilde{S} \rangle \rightarrow \frac{\hat{k}}{2} \nu_{\epsilon\mu} \epsilon |\tilde{E}_0|^2 e^{-(2\hat{k}\cdot\vec{r}/\delta)} \sqrt{\frac{\sigma}{\epsilon\omega}} \frac{1}{\sqrt{2}} = \frac{1}{4} |\tilde{J}_0| |\tilde{E}_0| e^{-(2\hat{k}\cdot\vec{r}/\delta)} \delta \hat{k} \quad (9.147)$$

$$= \omega \delta \langle u \rangle \hat{k} \quad I \rightarrow \omega \delta \langle u \rangle \quad (9.148)$$

From the above, we can calculate the total power dissipated in the conductive medium per unit area. We just need to know how much power flows in per unit area at the $\hat{k} \cdot \vec{r} = 0$ plane: the wave is propagating along \hat{k} , and there is no power flowing backward along \hat{k} . This power flow at $\hat{k} \cdot \vec{r} = 0$ is $I(\hat{k} \cdot \vec{r} = 0)$ because the plane we are considering has surface normal \hat{k} . Therefore, the power dissipated is

$$P_{diss} = I(\hat{k} \cdot \vec{r} = 0) = \frac{1}{4} |\tilde{J}_0| |\tilde{E}_0| \delta = \frac{1}{2} |\tilde{J}_0| |\tilde{E}_0| \frac{\delta}{2}$$

which is as one would expect: $|\tilde{J}_0| |\tilde{E}_0| / 2$ is the Joule power dissipation per unit volume, and $\delta/2$ is the energy decay length and so provides the thickness over which that power is dissipated to give a power dissipation per unit area.

Electromagnetic Waves in Conducting Matter (cont.)

Reflection at a Conducting Surface

Because free charges and currents are now possible, our boundary conditions differ from those for an interface between two nonconducting media:

$$\hat{n} \cdot [\epsilon_1 \vec{E}_1 - \epsilon_2 \vec{E}_2] = \sigma_f \quad \hat{n} \cdot [\vec{B}_1 - \vec{B}_2] = 0 \quad (9.149)$$

$$\hat{s} \cdot [\vec{E}_1 - \vec{E}_2] = 0 \quad \hat{s} \cdot \left[\frac{\vec{B}_1}{\mu_1} - \frac{\vec{B}_2}{\mu_2} \right] = (\vec{K}_f \times \hat{n}) \cdot \hat{s} \quad (9.150)$$

If we restrict to conductors that obey Ohm's Law, $\vec{J} = \sigma \vec{E}$, then we may conclude $\vec{K}_f = 0$ because the current singularity of a surface current requires a singularity in the field, which cannot happen.

The general case is very complicated to analyze; a measure of the difficulty is that not even Jackson tries to do it! Let's consider only the case of normal incidence. Since $\hat{n} \cdot \vec{E} = 0$, we have $\sigma_f = 0$ and the situation is similar to analyzing the case of nonconducting media.

Electromagnetic Waves in Conducting Matter (cont.)

Recall our vectors \hat{w} and \hat{u} in the plane of the interface. Our three expressions for the fields are

$$\vec{E}_i(\vec{r}, t) = \tilde{E}_{0,i} \hat{w} e^{i(k_i \hat{n} \cdot \vec{r} - \omega t)} \quad \vec{B}_i(\vec{r}, t) = \frac{1}{v_1} \tilde{E}_{0,i} \hat{u} e^{i(k_i \hat{n} \cdot \vec{r} - \omega t)} \quad (9.151)$$

$$\vec{E}_r(\vec{r}, t) = \tilde{E}_{0,r} \hat{w} e^{i(-k_i \hat{n} \cdot \vec{r} - \omega t)} \quad \vec{B}_r(\vec{r}, t) = -\frac{1}{v_1} \tilde{E}_{0,r} \hat{u} e^{i(-k_i \hat{n} \cdot \vec{r} - \omega t)} \quad (9.152)$$

$$\begin{aligned} \vec{E}_t(\vec{r}, t) &= \tilde{E}_{0,t} \hat{w} e^{i(k_t \hat{n} \cdot \vec{r} - \omega t)} \\ \vec{B}_t(\vec{r}, t) &= \frac{k_t + i \kappa_t}{\omega} \tilde{E}_{0,t} \hat{u} e^{i(k_t \hat{n} \cdot \vec{r} - \omega t)} \\ &\quad \times e^{-(\kappa_t \hat{n} \cdot \vec{r})} \end{aligned} \quad (9.153)$$

where $v_1 = 1/\sqrt{\epsilon_1 \mu_1}$. Inserting these expressions into the boundary condition equations, we obtain

$$\tilde{E}_{0,i} + \tilde{E}_{0,r} = \tilde{E}_{0,t} \quad (9.154)$$

$$\frac{1}{\mu_1 v_1} [\tilde{E}_{0,i} - \tilde{E}_{0,r}] = \frac{k_t + i \kappa_t}{\mu_2 \omega} \tilde{E}_{0,t} \quad (9.155)$$

Electromagnetic Waves in Conducting Matter (cont.)

Solving, we obtain

$$\frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}} = \left(\frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right) \quad \frac{\tilde{E}_{0,t}}{\tilde{E}_{0,i}} = \left(\frac{2}{1 + \tilde{\beta}} \right) \quad (9.156)$$

$$\text{with } \tilde{\beta} \equiv \frac{\mu_1}{\mu_2} \frac{v_1}{\omega} (k_t + i \kappa_t) = \frac{\mu_1}{\mu_2} \frac{v_1}{v_2} \frac{k_t + i \kappa_t}{\omega/v_2} = \beta \frac{k_t + i \kappa_t}{\omega/v_2} = \frac{Z_1}{Z_2} \frac{k_t + i \kappa_t}{k_2} \quad (9.157)$$

where $v_2 = 1/\sqrt{\epsilon_2 \mu_2}$, $k_2 = \omega/v_2$, $Z_1 = \sqrt{\mu_1/\epsilon_1}$, $Z_2 = \sqrt{\mu_2/\epsilon_2}$, and $\beta = \mu_1 n_2 / \mu_2 n_1 = Z_1/Z_2$ was defined in Equation 9.84. The reflected power is

$$\mathcal{R} = \left| \frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}} \right|^2 = \left| \frac{1 - \tilde{\beta}}{1 + \tilde{\beta}} \right|^2 \quad (9.158)$$

The expression for the transmitted power, if calculated directly from the Poynting vector in the conductor, is rather complicated. Of course, one can just use $T = 1 - \mathcal{R}$ in that case.

These expressions are very similar in spirit to the normal incidence equations for nonconducting media, Equations 9.112 and 9.113, except that β is replaced by $\tilde{\beta}$. Of course, this causes the reflected and transmitted waves to acquire nontrivial phase shifts relative to the incident wave.

Electromagnetic Waves in Conducting Matter (cont.)

As usual, let's take the poor conductor and good conductor limits:

$$\frac{1}{\omega\tau} = \frac{\sigma}{\epsilon\omega} \ll 1 \quad \text{poor conductor} \quad \frac{1}{\omega\tau} = \frac{\sigma}{\epsilon\omega} \gg 1 \quad \text{good conductor} \quad (9.159)$$

$$\tilde{\beta} \rightarrow \beta \quad \tilde{\beta} \rightarrow \frac{\mu_1}{\mu_2} \frac{v_1}{\omega} \frac{1+i}{\delta} \rightarrow \infty \quad (9.160)$$

$$\frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}} \rightarrow \left(\frac{1-\beta}{1+\beta} \right) \quad \frac{\tilde{E}_{0,r}}{\tilde{E}_{0,i}} \rightarrow -1 \quad (9.161)$$

$$\frac{\tilde{E}_{0,t}}{\tilde{E}_{0,i}} \rightarrow \left(\frac{2}{1+\beta} \right) \quad \frac{\tilde{E}_{0,t}}{\tilde{E}_{0,i}} \ll 1 \quad (9.162)$$

$$\mathcal{R} \rightarrow \left(\frac{1-\beta}{1+\beta} \right)^2 \quad \mathcal{R} \rightarrow 1 \quad (9.163)$$

$$\mathcal{T} \rightarrow \beta \left(\frac{2}{1+\beta} \right)^2 \quad \mathcal{T} \ll 1 \quad (9.164)$$

The poor conductor limit is the normal incidence case for two nonconducting media, Equations 9.112 and 9.113, and the good conductor limit gives perfect reflection with a sign flip for the reflected wave.

The case of a wave not at normal incidence becomes more algebraically challenging because three of the boundary condition equations are required and two of them involve the angles of incidence and transmission, as we saw in the nonconducting case. The complex nature of $k + i\kappa$ further complicates the situation.

Electromagnetic Waves in Dispersive Matter

Classical Model for Frequency Dependence of Permittivity – Dispersive Matter

All of the parameters we have been working with — σ , μ , and ϵ — are generally frequency-dependent. The one that is usually most obvious is the frequency dependence of the dielectric constant; this is, for example, how a prism works to disperse optical light, via the frequency dependence of ϵ and thus of $n \approx \sqrt{\epsilon/\epsilon_0}$. This dependence of the speed of light on frequency is called *dispersion*.

We will build a simple model for dispersion. The key elements of the model are:

- ▶ Electrons in a nonconducting medium are bound to their locations and can move around the minimum of the potential. Any (reasonable) potential looks quadratic near its minimum because the first nonzero term in a Taylor expansion of the potential near a minimum ($dU/dx = 0$) is the quadratic term. So we assume the electron moves in a quadratic potential and hence feels a binding force

$$F_{binding} = -m\omega_0^2 x \quad (9.165)$$

where x is the displacement from the equilibrium position and ω_0 is the natural frequency of the oscillation. The assumed quadratic potential is
$$U(x) = m\omega_0^2 x^2/2.$$

Electromagnetic Waves in Dispersive Matter (cont.)

- ▶ There is some damping force by which the electron can lose energy as it oscillates about the minimum of the potential. In most materials, this damping force is from emission of phonons (quantized crystal acoustic vibrations). The damping force is assumed to follow a standard damping force law

$$F_{damping} = -m\gamma \frac{dx}{dt} \quad (9.166)$$

The m is extracted from the damping coefficient for simplicity later.

- ▶ There is an incident electromagnetic wave with electric field polarization such that it drives the electron in the $\pm x$ direction:

$$F_{driving} = qE = qE_0 \cos \omega t \quad (9.167)$$

The wave angular frequency ω and the natural frequency of oscillation ω_0 are not assumed to be the same.

Electromagnetic Waves in Dispersive Matter (cont.)

The resulting equation of motion for the electron is

$$m \frac{d^2x}{dt^2} = F = F_{binding} + F_{damping} + F_{driving} \quad (9.168)$$

$$\Rightarrow \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \omega_0^2 x = \frac{q}{m} E_0 \cos \omega t \quad (9.169)$$

You've seen this equation, the *driven, damped simple harmonic oscillator*, in Ph1a and Ph106a. You know that we switch to complex notation so that the right side becomes $\tilde{E}_0 e^{-i\omega t}$ and we assume a harmonic solution $\tilde{x}(t) = \tilde{x}_0 e^{-i\omega t}$. Plugging in this solution gives an algebraic equation that can be solved for \tilde{x}_0 , which gives the amplitude and phase of the solution:

$$\tilde{x}_0 = \frac{q/m}{\omega_0^2 - \omega^2 - i\gamma\omega} \tilde{E}_0 \quad (9.170)$$

From this, we can calculate the dipole moment of the electron, which will lead us to the polarization density of the medium:

$$\tilde{p}(t) = q \tilde{x}(t) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega} \tilde{E}_0 e^{-i\omega t} \quad (9.171)$$

Electromagnetic Waves in Dispersive Matter (cont.)

We can obtain the polarization density by the usual formula of summing over all the electrons. Suppose there are multiple electrons per atom or molecule with different binding and damping forces. If there are f_j electrons per site with natural frequency ω_j and damping γ_j , and that there are N sites per unit volume, then the polarization density is

$$\vec{P} = \frac{N q^2}{m} \left(\sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i \gamma_j \omega} \right) \vec{E} \quad (9.172)$$

where \vec{E} and \vec{P} are complex vectors. Clearly, there is a proportionality between \vec{E} and \vec{P} , and we may define a *complex susceptibility*, $\tilde{\chi}_e$, *complex permittivity*, $\tilde{\epsilon}$, and *complex dielectric constant*, $\tilde{\epsilon}_r$,

$$\vec{P} = \tilde{\chi}_e \epsilon_0 \vec{E} \implies \tilde{\epsilon}_r = \frac{\tilde{\epsilon}}{\epsilon_0} = 1 + \tilde{\chi}_e = 1 + \frac{N q^2}{m \epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i \gamma_j \omega} \quad (9.173)$$

Electromagnetic Waves in Dispersive Matter

Wave Solutions for Dispersive Matter

With the complex permittivity, the wave equation becomes

$$\nabla^2 \vec{E} = \tilde{\epsilon} \mu_0 \frac{\partial^2 \vec{E}}{\partial t^2} \quad (9.174)$$

with $\tilde{\epsilon}$ being a complex number. If we assume plane wave solution, this becomes very much like the wave equation in conductive matter:

$$\text{complex permittivity: } \vec{k} \cdot \vec{k} \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} = \tilde{\epsilon} \mu_0 \omega^2 \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (9.175)$$

$$\text{conducting matter: } \vec{k} \cdot \vec{k} \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} = (\epsilon \mu \omega^2 + i \sigma \mu \omega) \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (9.176)$$

So, as with the conducting matter, we define a complex propagation vector

$$\vec{k} \cdot \vec{k} = \tilde{\epsilon} \mu_0 \omega \iff \sqrt{\vec{k} \cdot \vec{k}} = k + i \kappa \quad (9.177)$$

and the wave obeys

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{-(\kappa \hat{k} \cdot \vec{r})} e^{i(k \hat{k} \cdot \vec{r} - \omega t)} \quad (9.178)$$

The wave speed is $v = \omega/k$, the index of refraction is $n = c k / \omega$, and the intensity attenuation constant is $\alpha = 2 \kappa$.

Electromagnetic Waves in Dispersive Matter (cont.)

Now, let's figure out what k and κ are. Since there is a square root involved, it is not trivial. Let's specialize to gases where the density and thus the N -proportional term in $\tilde{\epsilon}_r$ is small so we can Taylor expand the square root:

$$k + i\kappa = \sqrt{\tilde{\epsilon}\mu_0\omega^2} = \frac{\omega}{c} \sqrt{\tilde{\epsilon}_r} \approx \frac{\omega}{c} \left[1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2 - i\gamma_j\omega} \right] \quad (9.179)$$

We separate the real and imaginary parts to obtain

$$n = \frac{ck}{\omega} \approx 1 + \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{f_j (\omega_j^2 - \omega^2)}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \quad (9.180)$$

$$\frac{\alpha}{2} = \kappa = \frac{\omega^2}{c} \frac{Nq^2}{2m\epsilon_0} \sum_j \frac{\gamma_j f_j}{(\omega_j^2 - \omega^2)^2 + \gamma_j^2 \omega^2} \quad (9.181)$$

Our assumption of the second term being small is very similar to the poor conductor case: the wave propagates into the medium, with $\tilde{B}_0 = \tilde{E}_0/v = \tilde{E}_0 n/c$ and with the magnetic and electric fields in phase, and with the wave decaying over a large number of wavelengths given by the ratio $\delta/\lambda = k/(2\pi\kappa)$.

Lecture 21:
Electromagnetic Waves in Dispersive Matter, cont.
Transmission Lines

Date Revised: 2014/05/23 5:30

Had $v = \sqrt{\epsilon\mu}$ instead of $v = 1/\sqrt{\epsilon\mu}$ in Eqns 9.206 and 9.208.

Date Given: 2014/05/08

Electromagnetic Waves in Dispersive Matter

Low-Frequency Behavior, $\omega \ll \omega_j$

If the frequency of the EM wave is much lower than any of the resonant frequencies, then we can ignore the damping term in the denominator, yielding

$$n \xrightarrow{\omega \ll \omega_j} 1 + \frac{N q^2}{2 m \epsilon_0} \sum_j \frac{f_j}{\omega_j^2 - \omega^2} = 1 + \frac{N q^2}{2 m \epsilon_0} \sum_j \frac{f_j}{\omega_j^2 \left(1 - \frac{\omega^2}{\omega_j^2}\right)} \quad (9.182)$$

$$\approx 1 + \frac{N q^2}{2 m \epsilon_0} \sum_j \frac{f_j}{\omega_j^2} \left(1 + \frac{\omega^2}{\omega_j^2}\right) = 1 + \frac{N q^2}{2 m \epsilon_0} \sum_j \frac{f_j}{\omega_j^2} + \omega^2 \frac{N q^2}{2 m \epsilon_0} \sum_j \frac{f_j}{\omega_j^4} \quad (9.183)$$

We see the behavior consists of a frequency-independent offset of n from unity combined with a quadratic dependence on frequency. This can be rewritten as

$$n = 1 + A \left(1 + \frac{B}{\lambda^2}\right) \quad (9.184)$$

This is known as Cauchy's Formula, with A being the *coefficient of refraction* (the offset from unity) and B being the *coefficient of dispersion* (the normalization of the frequency-dependent, dispersive term). The rise in the index of refraction with frequency reflects the increasing ability of the resonances to exchange energy with the field as the frequency approaches them.

Electromagnetic Waves in Dispersive Matter (cont.)

The reason that the expression asymptotes to a constant at low frequency is that, in this limit, the index of refraction is just the square root of the static dielectric constant, which depends on the effective spring constant for static distortions of the electron distribution about the sites. We can see this by going back to the dielectric constant, dropping the quadratic term, and rewriting in terms of the spring constants $k_j = m\omega_j^2$:

$$\frac{\tilde{\epsilon}}{\epsilon_0} = 1 + \frac{Nq^2}{\epsilon_0} \sum_j \frac{f_j}{k_j} \quad (9.185)$$

Electromagnetic Waves in Dispersive Matter (cont.)

Conductor Behavior, $\omega_0 = 0, \omega \ll \omega_j$

If we set $\omega_0 = 0$ but assume $\omega \ll \omega_j$ for $j > 0$, we obtain (going back to Equation 9.179 and looking at $\tilde{\epsilon}/\epsilon_0$ not n , so no factor of 1/2):

$$\frac{\tilde{\epsilon}}{\epsilon_0} = 1 + \frac{N q^2}{\epsilon_0} \sum_j \frac{f_j}{k_j} + i \frac{N_e q^2}{\epsilon_0 m} \frac{1}{\gamma_0 - i \omega} \frac{1}{\omega} + \mathcal{O}(\omega^2) \quad (9.186)$$

where $N_e = N f_0$ is the number of free conduction electrons (no restoring force, hence $\omega_j = \omega_0 = 0$ for them). Recalling our analogy between dielectrics with complex permittivity and conductors,

$$\vec{k} \cdot \vec{k} = [\mathcal{R}(\tilde{\epsilon}) + i \mathcal{I}(\tilde{\epsilon})] \mu_0 \omega^2 \longleftrightarrow \vec{k} \cdot \vec{k} = (\epsilon \mu \omega^2 + i \sigma \mu \omega) \quad (9.187)$$

$$= \left(\epsilon + i \frac{\sigma}{\omega} \right) \mu \omega^2 \quad (9.188)$$

Thus, we may identify

$$\sigma = \omega \mathcal{I} \left(\tilde{\epsilon} \Big|_{\omega \ll \omega_j} \right) = \frac{N_e q^2}{m} \frac{1}{\gamma_0 - i \omega} \xrightarrow{\omega/\gamma_0 \ll 1} \frac{N_e q^2}{m} \frac{1}{\gamma_0} \quad (9.189)$$

($\omega/\gamma_0 \ll 1$ implies the damping time $1/\gamma_0$ is much less than the wave period T .) That is, a conductor can be treated like a dispersive medium with only one resonance, $\omega_0 = 0$. Nonconductors have their first resonance at $\omega_1 > 0$. In this way, we can view conductors and nonconductors in a unified way depending on whether there is a resonant frequency at $\omega = 0$.

Electromagnetic Waves in Dispersive Matter (cont.)

We can tie these two models together at a deeper level. Recall our definition of the damping force, $|F_{damping}| = m \gamma v$, which we can rewrite as

$$\frac{1}{\gamma} = \frac{v}{|F_{damping}/m|} \quad (9.190)$$

which has units of time: $1/\gamma$ is a characteristic damping time. Recall also our Drude model for DC conductivity, with

$$\sigma = \frac{N_e q^2 \lambda}{2 m v_{thermal}} \quad (9.191)$$

Tying the two together, we have

$$\frac{v}{|F_{damping}|/m} = \frac{1}{\gamma_0} = \frac{m}{N_e q^2} \sigma = \frac{\lambda}{2 v_{thermal}} = \frac{\tau}{2} \quad (9.192)$$

where τ is the time between scatters in the Drude model. That the two times are related makes perfect sense: at zero frequency, the timescale on which acceleration by a DC electric field is damped is the scattering time, and thus this scattering time should set the resonance amplitude damping time $1/\gamma_0$.

Electromagnetic Waves in Dispersive Matter (cont.)

Plasma Behavior, $\omega \gg \omega_j$

Again, we can ignore the damping behavior because now $\omega \gg \omega_j$ (the $\omega^2 - \omega_j^2$ term dominates and we can neglect the ω_j^2 piece), providing

$$n \xrightarrow{\omega \gg \omega_j} 1 - \frac{N q^2}{2 m \epsilon_0} \sum_j \frac{f_j}{\omega^2} \equiv 1 - \frac{1}{2} \frac{\omega_p^2}{\omega^2} \quad \omega_p^2 \equiv \frac{N q^2}{m \epsilon_0} \sum_j f_j = \frac{N Z q^2}{m \epsilon_0} \quad (9.193)$$

($Z = \sum_j f_j$ is the number of electrons per site.) This is the same relationship one would have obtained if one had ignored $F_{binding}$ and $F_{damping}$: this behavior arises only from the force being given by the driving electric field. At these high frequencies, the oscillation occurs much faster than any binding ($\omega \gg \omega_j$) or damping ($\omega \gg \gamma_j$) forces can have effect and the electrons can be considered to be free. The asymptotic return of the index of refraction back toward unity reflects the fact that, due to the electrons' inertia, the distance they can move during one cycle of the EM wave, and hence the polarizability of the medium, decreases as $1/m \omega^2$ at high frequency. ω_p is called the *plasma frequency* because it defines the behavior of a plasma of free electrons.

Notice that $n < 1$ is possible in the plasma limit, yielding wave speeds that are larger than c . Griffiths discusses how the group velocity of waves prevents a violation of special relativity.

Electromagnetic Waves in Dispersive Matter (cont.)

Strangely enough, the plasma behavior can hold at much lower frequencies in tenuous plasmas where the electrons are free and binding forces are negligible. In these cases, one has to return to the full permittivity (Equation 9.173) and set ω_j^2 and γ_j to zero because the approximation that the dispersive term in the refractive index is small compared to unity fails. In such cases, one has a permittivity

$$\frac{\tilde{\epsilon}}{\epsilon_0} = 1 - \frac{\omega_p^2}{\omega^2} \quad (9.194)$$

It is possible for this relation to hold even for $\omega < \omega_p$, in which case, $\tilde{\epsilon}$ becomes negative. Note that this is *not* equivalent to the perfect conductor limit: in that case, $\tilde{\epsilon}$ acquires a significant imaginary component, as we just saw in the previous section. The implications of the purely negative $\tilde{\epsilon}$ are that there is reflection at an interface into such a medium (total internal reflection for all angles) and the field falls off exponentially into the medium with skin depth and decay constant

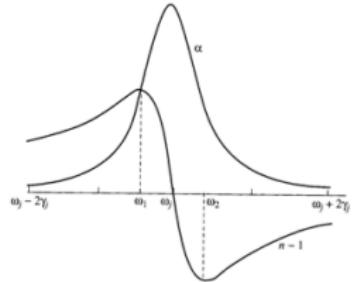
$$\frac{1}{\delta} = \kappa = \frac{\omega_p}{c} \quad (9.195)$$

It is particularly interesting to note that it is this plasma frequency phenomenon, *not the perfect conductor limit*, that explains the optical reflectivity of metals. There are resonances in the ultraviolet in metals that then bring $\tilde{\epsilon} > 0$ and allow metals to be ultraviolet transparent. This phenomenon also explains how the ionosphere reflects AM (500–1600 kHz) and shortwave (2.3–26 MHz) radio waves. AM radio reflection off the ionosphere is conditions-dependent and is better at night than during day.

Electromagnetic Waves in Dispersive Matter (cont.)

Near-Resonance Behavior

The behavior of the refractive index and skin depth near a resonance are shown below.



The index of refraction takes on its continuum (ω far from ω_j) value on-resonance but displays a characteristic negative slope behavior through the resonance. The decay constant displays a maximum (the skin depth displays a minimum) on resonance because the system is best able to absorb (and dissipate) energy on resonance. This negative slope of the refractive index is termed *anomalous dispersion* because it deviates from the smooth behavior seen well below ($\omega \ll \omega_j$) or above ($\omega \gg \omega_j$) the resonances.

Rotational Contribution of Permanently Polarized Molecules

One can show (we will not) that the rotational degrees of freedom of liquid or gas molecules with *permanent* dipole moments give a contribution to the permittivity

$$\tilde{\epsilon}_{rot} = 1 + \frac{\chi_{rot}}{1 - i\omega\tau_{rot}} \quad (9.196)$$

where χ_{rot} is the zero-frequency contribution to the susceptibility and τ is the time constant for relaxation of an imposed polarization due to thermal fluctuations.

Electromagnetic Waves in Dispersive Matter (cont.)

Rotational/Conduction and Anomalous Dispersion Regimes

The generic behavior of the real and imaginary parts of $\tilde{\epsilon}$ (usually denoted by ϵ' and ϵ''), in the rotational/conduction limits and through the anomalous dispersion features, is shown here.

Notice how, after each resonance is passed, the continuum value of ϵ' decreases a bit: this is because the numerator of each term in the sum for $n (= \sqrt{\epsilon'})$ changes sign after ω passes through ω_j . Eventually, not shown here, ϵ' goes to the plasma limit and decreases toward unity as $\omega \rightarrow \infty$.

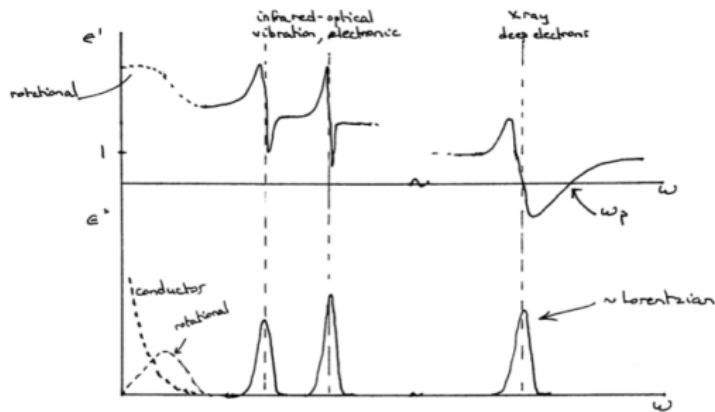


Figure courtesy of M. Cross

Transmission Lines

Introduction and Study Guide

We are going to undertake a more in-depth treatment of guided waves and transmission lines than Griffiths because it is an interesting and very useful topic in modern physics research. We will first consider transmission lines from the circuit theory perspective, then consider Maxwell's Equations in confined regions and the propagating solutions.

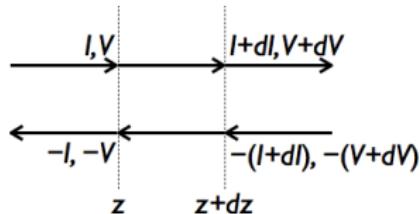
The material on transmission lines can be found in Heald and Marion §7.1. The material on waveguides is found in Heald and Marion §7.3–7.5 and Jackson §8.1–8.5.

Transmission Lines (cont.)

Circuit Transmission Line Theory

Here, we consider a specialized arrangement consisting of two electrodes with a voltage between them and currents flowing on them. We allow for the voltage and current to be position-dependent, which is a necessity for a propagating wave. We shall see that the critical ingredients necessary for propagating solutions are inductance and capacitance.

We start with the figure below, which consists of two wires on which there is a position-dependent voltage difference and a position-dependent current. We treat the two wires completely (anti)symmetrically: if there is a voltage $V(z)$ on one wire, there is a voltage $-V(z)$ on the other wire at the same point. Thus, the voltage difference between the wires is $\Delta V(z) = 2 V(z)$, which is the quantity we will be primarily concerned with. There is a current flowing on each wire, again asymmetric, with values $I(z)$ and $-I(z)$. There may be a net charge $Q(z)$ and $-Q(z)$ on the wires. We assume the system has an inductance per unit length for the pair of wires of \mathcal{L} and a capacitance per unit length between the wires of \mathcal{C} .



Transmission Lines (cont.)

We write down equations connecting the various quantities. An incremental accumulation of charge on each wire is related to the capacitance and the voltage between them

$$dQ = C dz d(\Delta V) \quad (9.197)$$

Now, if there is an accumulation of charge at a point z , it is because, during a time interval dt , the current leaving that point z is less than the current entering. This difference in current is $-dI$ (get the polarity correct: if $dI < 0$, then less current leaves the point z on the top wire than enters, so there is a net gain of positive charge on the top wire at point z), so

$$-dI dt = dQ = d(\Delta V) C dz \quad (9.198)$$

Dividing by both differentials gives

$$\frac{\partial I}{\partial z} = -C \frac{\partial \Delta V}{\partial t} \quad (9.199)$$

Transmission Lines (cont.)

The voltage drop along the wire over a length dz is due to the current flowing through the inductance:

$$\frac{1}{2} d(\Delta V) = dV = -\frac{\mathcal{L}}{2} dz \frac{\partial I}{\partial t} \quad (9.200)$$

The factor of $1/2$ on the left side comes from the fact that the voltage difference is $\Delta V = 2V$.

Why the factor of $1/2$ on the right side? Because \mathcal{L} is the inductance per unit length for the two wires combined, so it gives the magnetic flux per unit length divided by the current. Since the system is (anti)symmetric, the current in each wire contributes half the magnetic flux, hence the factor of $1/2$. Another way of looking at it is that, because \mathcal{L} is the inductance per unit length of the two wires together, the voltage drop that would appear due to $\mathcal{L} dz \frac{\partial I}{\partial t}$ would appear around the loop formed from z to $z + dz$ on the top wire to $z + dz$ on the bottom wire to z on the bottom wire, and thus only half the voltage drop appears between z and $z + dz$ along each wire individually.

Note also that the sign makes sense: The voltage induced has the polarity needed to drive the current in the direction that opposes the flux change due to the current change.

Transmission Lines (cont.)

Canceling the $1/2$ factors and moving the dz to the left side gives

$$\frac{\partial \Delta V}{\partial z} = -\mathcal{L} \frac{\partial I}{\partial t} \quad (9.201)$$

We can take derivatives of our two equations (Equations 9.199 and 9.201) to obtain

$$\frac{\partial^2 I}{\partial z^2} = \mathcal{L} \mathcal{C} \frac{\partial^2 I}{\partial t^2} \quad \frac{\partial^2 \Delta V}{\partial z^2} = \mathcal{L} \mathcal{C} \frac{\partial^2 \Delta V}{\partial t^2} \quad (9.202)$$

We thus have wave equations describing waves in one dimension moving with speed $v = 1/\sqrt{\mathcal{L}\mathcal{C}}$. We saw in Problem Set 4 that $\mathcal{L}\mathcal{C} = \epsilon\mu$ and thus the propagation speed is the same as for EM waves in a medium of the same ϵ, μ .

Transmission Lines (cont.)

In one dimension and for a fixed frequency ω , there are two solutions with velocity $+v$ and $-v$ and they have the general form (as we derived earlier for EM waves)

$$I_{\pm}(z, t) = F \left(\pm \frac{\omega}{v} z - \omega t \right) \quad \Delta V(z, t) = V_{\pm}(z, t) = Z_{\pm} I_{\pm} F \left(\pm \frac{\omega}{v} z - \omega t \right) \quad (9.203)$$

where \pm indicates the direction of propagation and Z_{\pm} to be determined defines the ratio $(\Delta V)/I$. We of course assume the same form for the current and voltage because they are tied to each other through derivatives. We can determine Z_{\pm} , which gives their ratio, by plugging into $\frac{\partial I}{\partial z} = -C \frac{\partial \Delta V}{\partial t}$:

$$\pm I_{\pm} \frac{\omega}{v} \frac{\partial F}{\partial u} = C Z_{\pm} I_{\pm} \omega \frac{\partial F}{\partial u} \implies Z_{\pm} = \pm \frac{1}{v C} = \pm \sqrt{\frac{L}{C}} \equiv \pm Z_{LC} \quad (9.204)$$

We see that the wave propagating to $+z$ has current flow to the right in the top wire when the top wire has a positive voltage while the $-z$ mode has current flow to the left when the top wire has a positive voltage. We will consider these two modes to have the same (*characteristic*) impedance, Z_{LC} , remembering that the currents flow in opposite directions. While Z_{LC} has units of ohms because it is the ratio of a voltage to a current, realize that it has nothing to do with resistivity in the wires!

If μ and ϵ have frequency dependence, then so do L , C , v , and Z_{LC} .

Transmission Lines (cont.)

Examples of Transmission Lines

► Coaxial cable

For coaxial cable with an inner solid conductor of radius a and outer shell conductor of radius b , we know from prior calculations that the capacitance and inductance per unit length are

$$C = \frac{2\pi\epsilon}{\ln \frac{b}{a}} \quad L = \frac{\mu \ln \frac{b}{a}}{2\pi} \quad (9.205)$$

Therefore, the transmission line properties are:

$$v = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\epsilon\mu}} = \frac{c}{n} \quad Z_{LC} = \sqrt{\frac{L}{C}} = \frac{\ln \frac{b}{a}}{2\pi} \sqrt{\frac{\mu}{\epsilon}} \quad (9.206)$$

For $\mu \approx \mu_0$, this gives $Z_{LC} \approx (60\Omega) \ln \frac{b}{a} / \sqrt{\epsilon_r}$. Teflon and polyethylene are frequently used as the dielectric, having $\epsilon_r = 2.0$ for teflon and $\epsilon_r = 2.3$ for high-density polyethylene. Standard coaxial cable impedances are 50Ω and 75Ω . 75Ω is a typical cable impedance for television antennas because it is a good match to the antenna impedance (which we will discuss later).

Transmission Lines (cont.)

► *Stripline*

This structure consists of two thin metal ribbons of width w and separation $h \ll w$. The capacitance is just that of a parallel plate capacitor. The inductance is easily obtained by calculating the field between two infinite sheets of current flowing in opposite directions ($B = \mu K$) and then calculating from that the flux per unit length ($\Phi/\ell = \mu K h$) and dividing by the current ($I = K w$). Thus, we have

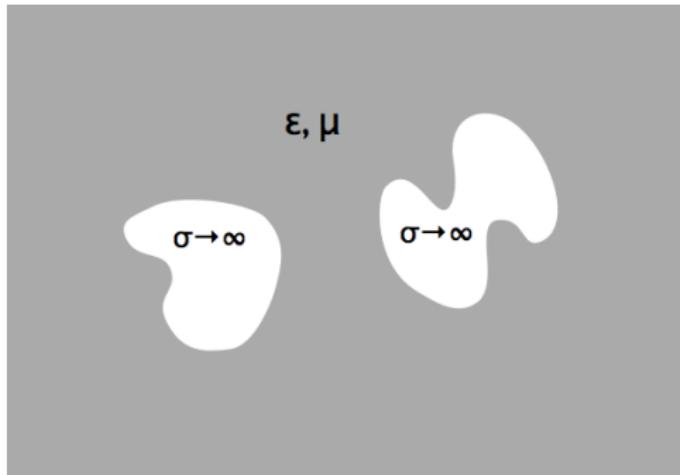
$$C = \frac{\epsilon w}{h} \quad L = \frac{\mu h}{w} \quad (9.207)$$

$$\nu = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{\epsilon \mu}} = \frac{c}{n} \quad Z_{LC} = \sqrt{\frac{L}{C}} = \frac{h}{w} \sqrt{\frac{\mu}{\epsilon}} \quad (9.208)$$

Transmission Lines (cont.)

Generalization to Arbitrary Electrode Shapes via Wave Formulation

We derived the above for the case of two wires, but we can extend to two electrodes of arbitrary shape that are z-translation independent, as indicated below.



Transmission Lines (cont.)

To do this, let's think about the fields and the charge and current densities rather than just voltages and currents. We aim to find a solution to Maxwell's Equations that takes the form of a wave propagating in the z -direction and that has transverse fields:

$$\vec{E}(\vec{r}, t) = [\hat{x} E_x(x, y) + \hat{y} E_y(x, y)] e^{i(k z - \omega t)} \quad \rho(\vec{r}, t) = \rho_0(x, y) e^{i(k z - \omega t)} \quad (9.209)$$

$$\vec{B}(\vec{r}, t) = [\hat{x} B_x(x, y) + \hat{y} B_y(x, y)] e^{i(k z - \omega t)} \quad \vec{J}(\vec{r}, t) = \hat{z} J_z(x, y) e^{i(k z - \omega t)} \quad (9.210)$$

ρ_0 and J_z are assumed to be surface charge and current densities, so they have δ functions in them, but we don't write them out explicitly because it is unnecessary for our discussion here.

We will show that, under the assumptions made above, in which the charge and current densities are the same in form and have a wave dependence on z , then the propagating wave solution is just a static solution in (x, y) multiplied by the $e^{i(k z - \omega t)}$ dependence, where the static solution is that sourced by ρ_0 and $J_z = v \rho_0$.

When we get to waveguides, we will see that this special case yields the TEM (transverse-electric-magnetic) modes, while more realistic current distributions yield other, more complicated modes.

Transmission Lines (cont.)

To do this, let's write out Maxwell's Equations assuming the above form. The four equations involving x and y derivatives are

$$[\vec{\nabla} \times \vec{E}]_z = -\frac{\partial \vec{B}}{\partial t} \cdot \hat{z} \implies \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0 \quad (9.211)$$

$$[\vec{\nabla} \cdot \vec{E}] = \frac{\rho}{\epsilon} \implies \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} = \frac{\rho_0(x, y)}{\epsilon} \quad (9.212)$$

$$[\vec{\nabla} \times \vec{B}]_z = \mu \vec{J}_f \cdot \hat{z} + \epsilon \mu \frac{\partial \vec{E}}{\partial t} \cdot \hat{z} \implies \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = \mu J_z(x, y) \quad (9.213)$$

$$[\vec{\nabla} \cdot \vec{B}] = 0 \implies \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \quad (9.214)$$

These four equations are satisfied by the form we have chosen; the exponential factor multiplies both sides of every equation and thus cancels out. They are a static solution because they are sourced by the normalizations ρ_0 and J_z , which are not time-dependent.

Transmission Lines (cont.)

The four equations in z and t derivatives are

$$[\vec{\nabla} \times \vec{E}]_x = -\frac{\partial \vec{B}}{\partial t} \cdot \hat{x} \implies -\frac{\partial E_y}{\partial z} = -\frac{\partial B_x}{\partial t} \quad (9.215)$$

$$[\vec{\nabla} \times \vec{E}]_y = -\frac{\partial \vec{B}}{\partial t} \cdot \hat{y} \implies \frac{\partial E_x}{\partial z} = -\frac{\partial B_y}{\partial t} \quad (9.216)$$

$$[\vec{\nabla} \times \vec{B}]_x = \mu \vec{J}_f \cdot \hat{x} + \epsilon \mu \frac{\partial \vec{E}}{\partial t} \cdot \hat{x} \implies -\frac{\partial B_y}{\partial z} = \epsilon \mu \frac{\partial E_x}{\partial t} \quad (9.217)$$

$$[\vec{\nabla} \times \vec{B}]_y = \mu \vec{J}_f \cdot \hat{y} + \epsilon \mu \frac{\partial \vec{E}}{\partial t} \cdot \hat{y} \implies \frac{\partial B_x}{\partial z} = \epsilon \mu \frac{\partial E_y}{\partial t} \quad (9.218)$$

These equations are satisfied by the exponential factor along with the relation

$$\vec{B} = \frac{1}{\nu \epsilon \mu} \hat{k} \times \vec{E} \implies \hat{x} B_x + \hat{y} B_y = \frac{1}{\nu \epsilon \mu} (-\hat{x} E_y + \hat{y} E_x) \quad (9.219)$$

Transmission Lines (cont.)

Let's relate the charge and current densities by applying continuity, $\vec{\nabla} \cdot \vec{J} + \partial \rho / \partial t = 0$:

$$\frac{\omega}{v_{\epsilon\mu}} J_0(x, y) = \omega \rho_0(x, y) \implies J_0(x, y) = v_{\epsilon\mu} \rho_0(x, y) \quad (9.220)$$

In addition to relating the two densities, this equation now justifies the assumption we made in Problem Set 4 about the current in a perfect conductor flowing only on the surfaces: since we know there is no charge density inside a perfect conductor (the decay constant for the charge density was $\tau = \epsilon/\sigma \rightarrow 0$, in the language of our solutions for EM waves in conductors), the above equation tells us there is no current density inside the conductor either.

This, along with the fact that $\hat{x} E_x + \hat{y} E_y$ and $\hat{x} B_x + \hat{y} B_y$ are sourced by the constant charge density ρ_0 and current density J_z , shows what we set out to prove, that a transmission line has propagating wave solutions in z whose transverse field configuration is identical to that of the static problem source by charge density ρ_0 and current density $J_z = v_{\epsilon\mu} \rho_0$. Note that these are *not the only* solutions, as we will show later.

Transmission Lines (cont.)

Now let's recover a transmission line formulation from the wave formulation. Given the wave formulation solution, we can calculate the voltage and current via

$$\Delta V(z, t) = \int_{\vec{r}_+}^{\vec{r}_-} d\vec{\ell} \cdot \vec{E} \quad (9.221)$$

$$I(z, t) = \pm \oint_{C_\pm} d\ell K_z = \pm v_{\epsilon\mu} \oint_{C_\pm} d\ell \sigma_0 = \pm \frac{1}{Z_{\epsilon\mu}} \oint_{C_\pm} d\ell \hat{n} \cdot \vec{E} \quad (9.222)$$

where we used $\epsilon v_{\epsilon\mu} = \sqrt{\epsilon/\mu} = 1/Z_{\epsilon\mu}$. K_z is the surface current density, with J_z and K_z related by a δ function that defines the conductors' surfaces, and σ_0 is the surface charge density, with σ_0 and ρ_0 also related by this δ function. The voltage integral is from any point on the + electrode to any point on the - electrode (since each is an equipotential in (x, y)) and the current (charge) integral is over the entire + or - electrode.

Transmission Lines (cont.)

We can then determine the characteristic impedance from

$$Z = \frac{\Delta V(z, t)}{I(z, t)} = Z_{\epsilon\mu} \frac{\pm \oint_{C_\pm} d\ell \hat{n} \cdot \vec{E}}{\int_{\vec{r}_+}^{\vec{r}_-} d\ell \cdot \vec{E}} \quad (9.223)$$

We see that the characteristic impedance is proportional to $Z_{\epsilon\mu}$ with the proportionality determined by the line integral of the electric field and the contour integral of the normal component of the electric field. This is what we found earlier in our examples, that the transmission line impedance Z_{LC} was $Z_{\epsilon\mu}$ times a factor determined by the geometry. In fact, it holds generally that

$$Z = Z_{LC} = \sqrt{\frac{\mathcal{L}}{C}} = \mathcal{L} \sqrt{\frac{1}{LC}} = \mathcal{L} \sqrt{\frac{1}{\epsilon\mu}} = \frac{\mathcal{L}_0}{\mu_0} \sqrt{\frac{\mu}{\epsilon}} = \frac{\mathcal{L}_0}{\mu_0} Z_{\epsilon\mu} \quad (9.224)$$

where $\mathcal{L}_0 = \mathcal{L}\mu_0/\mu$ is the inductance per unit length obtained for the same geometry with $\mu = \mu_0$. \mathcal{L}_0/μ_0 is thus a purely geometric quantity. We do not prove this because the proof is very closely related to that done in Problem Set 4 to show $LC = \epsilon\mu$.

Lecture 22:
Transmission Lines (cont.)
Waveguides

Date Revised: 2014/05/20 17:00

Corrected some v to $v_{\epsilon\mu}$

Sign error in derivation of BC for waveguides

Added explicit relations between $\vec{E}_{0,\perp}$ and $\vec{B}_{0,\perp}$

Date Given: 2014/05/13

Transmission Lines

Energy and Power

The power density transported by the EM wave is given by the Poynting vector:

$$\langle \vec{S} \rangle = \frac{1}{2\mu} \mathcal{R} (\langle \vec{E}^* \times \vec{B} \rangle) = \hat{k} \frac{1}{2\mu \nu_{\epsilon\mu}} |\tilde{E}_0|^2 = \hat{k} \frac{1}{2} \nu_{\epsilon\mu} \epsilon |\tilde{E}_0|^2 = \hat{k} \nu_{\epsilon\mu} \langle u \rangle \quad (9.225)$$

The Poynting vector has units of power/area. Therefore, if we integrate it over the xy-plane, we get the total power flowing past a given point:

$$P = \int_S da \hat{z} \cdot \langle \vec{S} \rangle = \frac{1}{2} \nu_{\epsilon\mu} \int_S da \epsilon |\tilde{E}_0|^2 \quad (9.226)$$

The integral (with the 1/2) is the energy per unit per unit length in the electric field, which can be rewritten using the capacitance per unit length and the voltages (recall, this is how we derived $u_e = \frac{1}{2} \epsilon E^2$). Thus, we have

$$P = \frac{1}{2} \nu_{\epsilon\mu} C |V_0|^2 = \frac{1}{2} \frac{C}{\sqrt{\mathcal{L}\mathcal{C}}} |V_0|^2 = \frac{1}{2} \frac{|V_0|^2}{Z_{LC}} = \frac{1}{2} \mathcal{R} (I_0^* V_0) \quad (9.227)$$

where V_0 is amplitude of the wave giving ΔV , $I_0 = V_0/Z_{LC}$ is the amplitude of the wave giving I , and we take the real part so the result is generalizable (so far, I and ΔV are in phase). We thus have an expression for the power flowing down the transmission line in terms of the natural transmission line parameters, the current and voltage.

Transmission Lines (cont.)

Reflection and Transmission at a Transmission Line Junction

Now, let's consider a wave incident on a junction between two transmission lines (at $z = 0$) of characteristic impedances Z_1 and Z_2 . We assume a solution consisting of an incident and reflected wave at $z < 0$ and a transmitted wave at $z > 0$. The voltage must be continuous as always, and the current must be continuous to avoid a buildup of charge at the boundary. (If we choose to match fields instead: \vec{E} and \vec{B} are transverse and there are no free charges or currents at the boundary, so their normal components vanish and their transverse components are continuous, yielding the same conditions on V and I .) The matching equations at the boundary then are

$$z < 0 : \quad V(z, t) = e^{i(k_1 z - \omega t)} + \tilde{r} e^{-i(k_1 z + \omega t)} \quad (9.228)$$

$$I(z, t) = \frac{1}{Z_1} e^{i(k_1 z - \omega t)} - \frac{\tilde{r}}{Z_1} e^{-i(k_1 z + \omega t)} \quad (9.229)$$

$$z > 0 : \quad V(z, t) = \tilde{t} e^{i(k_2 z - \omega t)} \quad (9.230)$$

$$I(z, t) = \frac{\tilde{t}}{Z_2} e^{i(k_2 z - \omega t)} \quad (9.231)$$

where \tilde{r} and \tilde{t} are to be determined. Note that we know $k_2 v_2 = \omega = k_1 v_1$. As usual, we have assumed the same time dependence because the matching conditions must be satisfied at all times. Notice the sign on the left-going reflected current term.

Transmission Lines (cont.)

Setting $z = 0$, we obtain

$$1 + \tilde{r} = \tilde{t} \quad \frac{1 - \tilde{r}}{Z_1} = \frac{\tilde{t}}{Z_2} \implies \tilde{r} = \frac{Z_2 - Z_1}{Z_2 + Z_1} \quad \tilde{t} = \frac{2Z_2}{Z_1 + Z_2} \quad (9.232)$$

We see that these equations are identical to Equations 9.112 and 9.113 for the fields in the case of normal incidence of an EM wave on the interface between two different perfectly non-conducting materials.

We may calculate the power reflection and transmission coefficients based on Equation 9.227:

$$\mathcal{R} = \frac{\frac{1}{2}|\tilde{r}|^2/Z_1}{\frac{1}{2}|1|^2/Z_1} = \left(\frac{Z_2 - Z_1}{Z_2 + Z_1} \right)^2 \quad \mathcal{T} = \frac{\frac{1}{2}|\tilde{t}|^2/Z_2}{\frac{1}{2}|1|^2/Z_1} = \frac{Z_1}{Z_2} \left(\frac{2Z_2}{Z_1 + Z_2} \right)^2 \quad (9.233)$$

One can easily see $\mathcal{R} + \mathcal{T} = 1$ and that these expressions also match what we derived for an EM wave at normal incidence.

Transmission Lines (cont.)

Reflection and Transmission at a Load Impedance

If, instead of a second transmission line section, we have a junction to a lumped-element load with complex impedance Z_L , then we just have

$$z > 0 : \quad V(t) = \tilde{t} \quad I(t) = \frac{\tilde{t}}{Z_1} \quad (9.234)$$

This doesn't change the matching conditions, so we have the same relations as Equations 9.232 and 9.233 with Z_2 replaced by Z_L . In particular, there is full transmission of the power from the transmission line into the load if $Z_L = Z_1$: this is what is known as "impedance matching."

Input Impedance of a Terminated Line

Another interesting and frequently considered question is what happens at the interface between transmission lines when the second line is terminated in load impedance after a length ℓ . We can phrase this in terms of an effective impedance looking into the loaded transmission line.

Transmission Lines (cont.)

Let's assume that there are left- and right-going waves on this length ℓ of transmission line with amplitudes V_+ and V_- to be determined. (We won't write the $e^{-i\omega t}$ time dependence for now; think of it as considering everything at $t = 0$.) Define the "complex-wave ratio" as the ratio of the left-going and right-going amplitudes:

$$r(z) = \frac{V_-(z)}{V_+(z)} \quad (9.235)$$

The voltage and current at any point on the line are

$$V(z) = V_+(z) + V_-(z) = V_+(r) [1 + r(z)] \quad (9.236)$$

$$I(z) = I_+(z) - I_-(z) = \frac{V_+(z)}{Z_{LC}} - \frac{V_-(z)}{Z_{LC}} = \frac{V_+(z)}{Z_{LC}} [1 - r(z)] \quad (9.237)$$

We can then calculate an impedance at any point on the line:

$$Z(z) = \frac{V(z)}{I(z)} = Z_{LC} \frac{1 + r(z)}{1 - r(z)} \quad (9.238)$$

Transmission Lines (cont.)

We know or can now calculate the functions $V_+(z)$, $V_-(z)$, $r(z)$, and $Z(z)$:

$$V_+(z) = V_+(0) e^{ikz} \quad r(z) = r(0) e^{-2ikz} \quad (9.239)$$

$$V_-(z) = V_-(0) e^{-ikz} \quad Z(z) = Z_{LC} \frac{1 + r(0) e^{-2ikz}}{1 - r(0) e^{-2ikz}} \quad (9.240)$$

Now, to find $Z_i = Z(0)$, the effective input impedance of the transmission line, we simply set $Z(\ell) = Z_L$, find $r(0)$ from the relation between $Z(z)$ and $r(0)$, and then from that find $Z_i = Z(0)$ from $r(0)$:

$$Z(\ell) = Z_L \implies r(0) = e^{2ik\ell} \frac{\frac{Z_L}{Z_{LC}} - 1}{\frac{Z_L}{Z_{LC}} + 1} \quad (9.241)$$

$$\implies Z_i = Z(0) = Z_{LC} \frac{1 + r(0)}{1 - r(0)} = Z_{LC} \frac{Z_L - i Z_{LC} \tan k\ell}{Z_{LC} - i Z_L \tan k\ell} \quad (9.242)$$

Transmission Lines (cont.)

Based on this formula, some examples of the behavior are given below. It is crucial to recognize that the behavior depends on k so is frequency-dependent!

► Short-circuit termination

$$Z_L = 0 \implies Z_i = -i Z_{LC} \tan(k\ell) \quad (9.243)$$

The line acts purely reactive (no resistance). It is capacitive or inductive depending on the length in numbers of quarter-wavelengths:

$$(2n) \frac{\lambda}{4} < \ell < (2n+1) \frac{\lambda}{4} \implies \mathcal{I}(Z_i) < 0 \implies \text{inductive} \quad (9.244)$$

$$(2n-1) \frac{\lambda}{4} < \ell < (2n) \frac{\lambda}{4} \implies \mathcal{I}(Z_i) > 0 \implies \text{capacitive} \quad (9.245)$$

Note that the sign conventions for inductive and capacitive reactances are the opposite of what is used in Ph1c and in engineering because we use here a $e^{-i\omega t}$ time dependence instead of the usual $e^{i\omega t}$ dependence.

► Open-circuit termination

$$Z_L = \infty \implies Z_i = i Z_{LC} \cot(k\ell) \quad (9.246)$$

Again, capacitive or inductive depending on the length.

Transmission Lines (cont.)

- ▶ Quarter-wavelength

$$\ell = \frac{\lambda}{4} \quad Z_i = \frac{Z_L^2 c}{Z_L} \quad (9.247)$$

Acts like a transformer of the load impedance.

- ▶ Half-wavelength

$$\ell = \frac{\lambda}{2} \quad Z_i = Z_L \quad (9.248)$$

No change in impedance.

Waveguides

Definition of a Waveguide

There are two critical ingredients in the concept of a *waveguide*:

- ▶ First, that an EM wave is propagating along the z direction and is confined by (for now) lossless metal boundaries in the xy -plane. The space need not be simply connected (e.g., a coaxial cable).
- ▶ Second, that voltages appear and currents flow sinusoidally on those walls to support the propagating wave. The sinusoidal flow eliminates the need for an explicit return conductor because the time average of the current at a given point vanishes: at any point, the current goes in the $+z$ direction half the time, then in the $-z$ direction half the time. We assume that the waveguide goes on forever or is terminated in an impedance that matches the waveguide impedance (this will be explained below) so we can neglect the end effects.

Waveguides (cont.)

Eigenvector-Eigenvalue Equation for Waveguide Solutions

So, we will assume that the electric and magnetic fields have propagating wave solutions in the z direction with $e^{i(kz-\omega t)}$ dependences (k positive or negative):

$$\vec{E}(\vec{r}, t) = \vec{E}_0(\vec{r}_\perp) e^{i(kz-\omega t)} \quad \vec{B}(\vec{r}, t) = \vec{B}_0(\vec{r}_\perp) e^{i(kz-\omega t)} \quad (9.249)$$

where \vec{E}_0 and \vec{B}_0 are, as usual, complex, and $\vec{r}_\perp = x\hat{x} + y\hat{y}$. We calculated earlier what happens when you plug into Maxwell's Equations such a solution, but now we may not assume that the waves are transverse, we must allow for z components.

Plugging into the two curl equations yields

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i \omega B_z \quad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -i \frac{\omega}{v_{\epsilon\mu}^2} E_z \quad (9.250)$$

$$\frac{\partial E_z}{\partial y} - i k E_y = i \omega B_x \quad \frac{\partial B_z}{\partial y} - i k B_y = -i \frac{\omega}{v_{\epsilon\mu}^2} E_x \quad (9.251)$$

$$i k E_x - \frac{\partial E_z}{\partial x} = i \omega B_y \quad i k B_x - \frac{\partial B_z}{\partial x} = -i \frac{\omega}{v_{\epsilon\mu}^2} E_y \quad (9.252)$$

We allow for the medium in the waveguide to have uniform ϵ, μ via use of $v_{\epsilon\mu}$ instead of c in the above equations.

Waveguides (cont.)

We may solve these algebraic equations for the transverse components in terms of the z , or *longitudinal* components. (e.g., plug the equation with B_y on the RHS into the equation for E_x on the RHS and solve for E_x and one gets an equation for E_x in terms of E_z and B_z .) Defining any vector with a \perp subscript to be the vector's component in the xy -plane, and defining $k_{\epsilon\mu} = \omega/v_{\epsilon\mu}$, we obtain

$$\vec{B}_{0,\perp} = \frac{1}{k_{\epsilon\mu}^2 - k^2} \left[i k \vec{\nabla}_\perp B_{0,z} + i \frac{k_{\epsilon\mu}}{v_{\epsilon\mu}} \hat{z} \times \vec{\nabla}_\perp E_{0,z} \right] \quad (9.253)$$

$$\vec{E}_{0,\perp} = \frac{1}{k_{\epsilon\mu}^2 - k^2} \left[i k \vec{\nabla}_\perp E_{0,z} - i k_{\epsilon\mu} v_{\epsilon\mu} \hat{z} \times \vec{\nabla}_\perp B_{0,z} \right] \quad (9.254)$$

Note that the transverse components end up out of phase with the longitudinal component by $\pi/2$ due to the factor i . Note that one cannot find a generic, simple relationship between $\vec{E}_{0,\perp}$ and $\vec{B}_{0,\perp}$ like we had before for transverse waves. We will see below that we can obtain such a relationship, but differently for different modes.

For propagation in the $-\hat{z}$ direction, one can work through the algebra to see that only the sign on k in the first terms changes; the second terms are unchanged. This is because the prefactors in the second terms (which come from $\omega/v_{\epsilon\mu}^2$), as well as the sign of \hat{z} , do not depend on k .

We note that the above relations could be written in terms of \vec{H} by replacing $v_{\epsilon\mu}$ with $Z_{\epsilon\mu}$. The relations do not become more symmetric by doing so.

Waveguides (cont.)

Next, let's use the Maxwell divergence equations by taking the divergence of the above. We note that

$$\vec{\nabla} \cdot (\hat{z} \times \vec{\nabla}_{\perp}) = \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \cdot \left(\hat{y} \frac{\partial}{\partial x} - \hat{x} \frac{\partial}{\partial y} \right) = 0 \quad (9.255)$$

Therefore,

$$0 = \vec{\nabla} \cdot \vec{E} = e^{i(kz - \omega t)} \left(\vec{\nabla}_{\perp} \cdot \vec{E}_{0,\perp} + i k E_{0,z} \right) \quad (9.256)$$

$$= \frac{i k}{k_{\epsilon\mu}^2 - k^2} \nabla_{\perp}^2 E_{0,z} + i k E_{0,z} \quad (9.257)$$

Rewriting, and doing the same with $\vec{\nabla} \cdot \vec{B}$, we obtain

$$\boxed{\nabla_{\perp}^2 E_{0,z} + (k_{\epsilon\mu}^2 - k^2) E_{0,z} = 0 \quad \nabla_{\perp}^2 B_{0,z} + (k_{\epsilon\mu}^2 - k^2) B_{0,z} = 0 \quad k_{\epsilon\mu} = \frac{\omega}{v_{\epsilon\mu}}} \quad (9.258)$$

These are clearly eigenvalue-eigenvector equations, where the action of an operator on the solution is equal to a constant times the solution.

Waveguides (cont.)

If we choose $E_{0,z} = 0$, then the waves are called *transverse electric (TE)* modes, and if we choose $B_{0,z} = 0$, they are called *transverse magnetic (TM)* modes. By linearity, any arbitrary mixture of TE and TM modes can be obtained by summing TE and TM modes. Note that, because $\vec{E}_{0,\perp}$ depends on $E_{0,z}$ and $B_{0,z}$, setting $E_{0,z} = 0$ still allows for nonzero $\vec{E}_{0,\perp}$ for TE modes, and similarly for $\vec{B}_{0,\perp}$ for TM modes. Note that the mode wavelength, set by k , is not necessarily the same as it is in unconfined space, set by $k_{\epsilon\mu}$. One can check that, for these modes,

$$\text{TE: } \vec{B}_{0,\perp} = \frac{1}{v_{\epsilon\mu}} \frac{k}{k_{\epsilon\mu}} \hat{z} \times \vec{E}_{0,\perp} \quad \text{TM: } \vec{B}_{0,\perp} = \frac{1}{v_{\epsilon\mu}} \frac{k_{\epsilon\mu}}{k} \hat{z} \times \vec{E}_{0,\perp} \quad (9.259)$$

Waveguides (cont.)

There are also *transverse electric and magnetic (TEM)* modes. These are not obtained by setting $E_{0,z} = B_{0,z} = 0$ in the above wave equations. Rather, one must go back to the curl equations and set the z components to be zero there. This results in simplified equations for $\vec{E}_{0,\perp}$ and $\vec{B}_{0,\perp}$ as we derived in our transmission line theory discussion (Equations 9.209 and 9.210): static solutions in two dimensions give the transverse field behavior for the propagating TEM mode (with $\vec{B}_{0,\perp} = \hat{z} \times \vec{E}_{0,\perp} / v_{\epsilon\mu}$). In fact, “transmission line” and “TEM mode” should be considered synonymous, especially given what we will prove soon, which is that hollow waveguides cannot have TEM modes. This makes sense, as our definition of transmission lines required two (and exactly two) electrodes carrying complementary currents and voltages, and because we used a quasistatic formalism that required the definition of a \mathcal{L} and \mathcal{C} to define transmission lines.

Waveguides (cont.)

Boundary Conditions in a Waveguide

Now, let's consider the boundary conditions for a waveguide. Since we have reduced the problem to one of finding the z components of the fields, we need boundary conditions on $E_{0,z}$ and $B_{0,z}$ on the walls, which we will define by a contour \mathcal{C} .

There will be free charges and currents flowing in the walls to generate the fields, so we may not assume anything about $\hat{n} \cdot \vec{E}$ and $\hat{t} \cdot \vec{B}$ at the walls.

For a perfect conductor, the skin depth vanishes. Therefore, there is no magnetic or electric field in the conductor. This, along with $\nabla \cdot \vec{B} = 0$, implies $\hat{n} \cdot \vec{B} = 0$ at the walls. So, in any mode, $\vec{B}_{0,\perp}$ must vanish at the walls. However, this is not yet useful because it is a condition on $\vec{B}_{0,\perp}$ while our wave equation is for $B_{0,z}$.

The vanishing of the fields in the conductor along with Faraday's Law tells us $\hat{t} \cdot \vec{E} = 0$. (Remember, fields cannot have a δ -function spatial dependence, so the right side of Faraday's Law yields no contribution to the surface integral that is related to the usual loop integral we do to determine tangential boundary conditions like this.) One allowed direction of \hat{t} is $\hat{t} = \hat{z}$, which gives us one of the boundary conditions we will use, $\hat{z} \cdot \vec{E} = 0$, or

$$E_{0,z}|_{\mathcal{C}} = 0$$

(9.260)

Note that this holds even for non-TE modes.

Waveguides (cont.)

We can also apply Faraday's law with the orthogonal tangent vector, $\hat{t}_\perp = \hat{z} \times \hat{n}$. Using $\hat{t}_\perp \cdot \vec{E} = 0$ and Equation 9.254:

$$0 = \hat{t}_\perp \cdot \vec{E} = \hat{t}_\perp \cdot \vec{E}_{0,\perp} = \hat{t}_\perp \cdot \frac{1}{k_{\epsilon\mu}^2 - k^2} \left[i k \vec{\nabla}_\perp E_{0,z} - i v_{\epsilon\mu} k_{\epsilon\mu} \hat{z} \times \vec{\nabla}_\perp B_{0,z} \right]$$

The first term on the right side vanishes because it calculates the gradient of $E_{0,z}$ along \hat{t}_\perp , which vanishes because the boundary is an equipotential. The second term can be cleaned up using the vector identity $\vec{a} \cdot (\vec{b} \times \vec{\nabla}) = (\vec{a} \times \vec{b}) \cdot \vec{\nabla}$ to yield another boundary condition on the z components of the fields.:.

$$0 = \hat{t}_\perp \cdot (\hat{z} \times \vec{\nabla}_\perp) B_{0,z} = (\hat{t}_\perp \times \hat{z}) \cdot \vec{\nabla}_\perp B_{0,z} \implies \boxed{\hat{n} \cdot \vec{\nabla}_\perp B_{0,z} \Big|_C = 0} \quad (9.261)$$

We may also try to use $\hat{n} \cdot \vec{B} = 0$ combined with Equation 9.253:

$$0 = \hat{n} \cdot \vec{B} = \hat{n} \cdot \vec{B}_{0,\perp} = \hat{n} \cdot \frac{1}{k_{\epsilon\mu}^2 - k^2} \left[i k \vec{\nabla}_\perp B_{0,z} + i \frac{k_{\epsilon\mu}}{v_{\epsilon\mu}} \hat{z} \times \vec{\nabla}_\perp E_{0,z} \right]$$

Both terms vanish already: the first term because $\hat{n} \cdot \vec{\nabla}_\perp B_{0,z} = 0$, the second because $\hat{n} \cdot (\hat{z} \times \vec{\nabla}_\perp) E_{0,z} = (\hat{n} \times \hat{z}) \cdot \vec{\nabla}_\perp E_{0,z} = -\hat{t}_\perp \cdot \vec{\nabla}_\perp E_{0,z}$, which vanishes because the waveguide surface is an equipotential. So no further boundary conditions are available.

Lecture 23: Waveguides (cont.)

Date Revised: 2014/05/27 7:00

Correct some v to $v_{\epsilon\mu}$ and some clarification
in discussion of loss in waveguides

Be more careful about time-average symbols (Eqn. 9.289 onward)

Clarify the coordinate system used in evaluating curls
for imperfect conductor waveguides (Eqn. 9.298)

2014/06/03: Correct sign error in Eqn. 9.278

Date Given: 2014/05/20

No TEM Mode in Hollow Waveguide

These boundary conditions let us immediately prove that *hollow (simply connected in the xy-plane) waveguides have no TEM modes*. Returning to Maxwell's equations for the waveguide and using $E_{0,z} = B_{0,z} = 0$ for the TEM mode, we obtain

$$\vec{\nabla} \cdot \vec{E} = 0 \implies \frac{\partial E_{0,x}}{\partial x} + \frac{\partial E_{0,y}}{\partial y} = 0 \quad [\vec{\nabla} \times \vec{E}]_z = 0 \implies \frac{\partial E_{0,y}}{\partial x} - \frac{\partial E_{0,x}}{\partial y} = 0$$

The other curl equation terms vanish because $\vec{E}_0(x, y)$ has no z dependence and $E_{0,z}$ vanishes. Thus, \vec{E}_0 has no curl, so it is the gradient of a scalar potential $V_0(x, y)$. The vanishing of $\vec{\nabla} \cdot \vec{E}$ tells us that this potential $V_0(x, y)$ satisfies Laplace's (instead of Poisson's) Equation. The boundary condition on \vec{E} implies any connected boundary of the waveguide is an equipotential for V_0 . If there is only one boundary, then we may use the fact that solutions to Laplace's Equation have no local minima or maxima to conclude that the potential must be constant across the (x, y) -plane and thus that $\vec{E}_{0,\perp} = 0$. This, with $B_{0,z} = 0$, implies $\vec{B}_{0,\perp} = 0$ also. Thus, there is no TEM mode.

This argument does not apply for a non-hollow waveguide because there are at least two disconnected boundaries that may not be at the same potential.

Waveguides (cont.)

Generic Properties of TM Modes

For a TM mode, we set $B_{0,z} = 0$ and drop the $B_{0,z}$ wave equation in Equation 9.258 and focus on the $E_{0,z}$ equation. The relevant boundary condition is $E_{0,z} = 0$ because the $B_{0,z}$ condition is useless. Rewriting the $E_{0,z}$ piece of Equation 9.258 in a generic form gives

$$\nabla_{\perp}^2 \psi + \gamma^2 \psi = 0 \quad \gamma^2 = k_{\epsilon\mu}^2 - k^2 \quad \psi|_{\text{boundary}} = 0 \quad (9.262)$$

This is an eigenvalue-eigenvector problem (like the Schrodinger equation in two dimensions). The boundary conditions result in a discrete set of allowed γ_n and thus solutions ψ_n . From the solutions ψ_n , one can obtain

$$\text{TM modes: } E_{0,z} = \psi_n \quad \vec{E}_{0,\perp} = \frac{i k}{\gamma_n^2} \vec{\nabla}_{\perp} \psi_n \quad \vec{B}_{0,\perp} = \frac{1}{\nu_{\epsilon\mu}} \frac{i k_{\epsilon\mu}}{\gamma_n^2} \hat{z} \times \vec{\nabla}_{\perp} \psi_n$$

(9.263)

The specific solutions depend on the shape of the boundary chosen.

Waveguides (cont.)

Generic Properties of TE Modes

For a TE mode, we set $E_{0,z} = 0$ and drop the $E_{0,z}$ wave equation in Equation 9.258 and focus on the $B_{0,z}$ equation. The relevant boundary condition is $\hat{n} \cdot \vec{\nabla}_{\perp} B_{0,z} = 0$ because the $E_{0,z}$ condition is useless. Rewriting the $B_{0,z}$ piece of Equation 9.258 in a generic form gives

$$\nabla_{\perp}^2 \psi + \gamma^2 \psi = 0 \quad \gamma^2 = k_{\epsilon\mu}^2 - k^2 \quad \hat{n} \cdot \vec{\nabla} \psi \Big|_{\text{boundary}} = 0 \quad (9.264)$$

Again, we have an eigenvalue-eigenvector problem with a discrete set of solutions γ_n due to the boundary conditions, yielding

$$\text{TE modes : } B_{0,z} = \psi_n \quad \vec{B}_{0,\perp} = \frac{i k}{\gamma_n^2} \vec{\nabla}_{\perp} \psi_n \quad \vec{E}_{0,\perp} = -v_{\epsilon\mu} \frac{i k_{\epsilon\mu}}{\gamma_n^2} \hat{z} \times \vec{\nabla}_{\perp} \psi_n$$

(9.265)

Again, the specific solutions depend on the shape of the boundary chosen.

Waveguides (cont.)

Complementarity and Completeness of TM and TE Modes

It should be clear that the TM and TE solutions are complementary in that they solve the same eigenvalue-eigenvector equation but with complementary boundary conditions. These two types of boundary conditions are the only types of boundary conditions implied by Maxwell's Equations, so the solutions form a complete set to describe any propagating field in the waveguide. There is of course no reason for the γ_n 's of the TE and TM modes to be the same.

Waveguides (cont.)

Dispersion Relations, Wave Speed, Propagation Constant, and Wave Impedance

The TEM modes are simple: the transverse electric and magnetic fields just solve the static Maxwell's Equations in two dimensions. There is no eigenvalue-eigenvector equation to be solved. The propagation constant is always $k = k_{\epsilon\mu} = \omega/v_{\epsilon\mu}$, the wavelength is $\lambda = \lambda_0$, the wave speed is $v_{\epsilon\mu}$, and the wave impedance is $Z_{\epsilon\mu}$ independent of frequency, with no cutoff frequency.

On the other hand, the TE and TM modes satisfy the condition (from $\gamma^2 = k_{\epsilon\mu}^2 - k^2$ and $k_{\epsilon\mu} = \omega/v_{\epsilon\mu}$)

$$\omega^2 = v_{\epsilon\mu}^2 (k^2 + \gamma_n^2) \quad (9.266)$$

Thus, there is a cutoff frequency $\omega_{c,n} = v_{\epsilon\mu} \gamma_n$ such that, for $\omega < \omega_{c,n}$, we find $k^2 < 0$. Therefore, we have a decaying mode with decay constant

$$\boxed{\omega_{c,n} = v_{\epsilon\mu} \gamma_n \quad : \quad \kappa_n(\omega) = -i k_n(\omega) = \sqrt{\gamma_n^2 - \frac{\omega^2}{v_{\epsilon\mu}^2}} = \frac{\omega_{c,n}}{v_{\epsilon\mu}} \sqrt{1 - \frac{\omega^2}{\omega_{c,n}^2}}} \quad (9.267)$$

Note that the decay constant depends on which type (TM or TE) and number n of mode one considers! We see that the decay length $d_n(\omega) = 1/\kappa_n(\omega) \rightarrow \infty$ as $\omega \rightarrow \omega_{c,n}$ from below.

Waveguides (cont.)

For the propagating modes with $\omega > \omega_{c,n}$, the propagation constant, wavelength, and wave speed are

$$\boxed{\begin{aligned} \omega_{c,n} = v_{\epsilon\mu} \gamma_n \quad : \quad k_n(\omega) &= \sqrt{\frac{\omega^2}{v_{\epsilon\mu}^2} - \gamma_n^2} = \frac{\omega}{v_{\epsilon\mu}} \sqrt{1 - \frac{\omega_{c,n}^2}{\omega^2}} = k_{\epsilon\mu} \sqrt{1 - \frac{\omega_{c,n}^2}{\omega^2}} \\ \omega > \omega_{c,n} \end{aligned} \quad (9.268)}$$

$$v_n(\omega) = \frac{\omega}{k_n(\omega)} = \frac{v_{\epsilon\mu}}{\sqrt{1 - \frac{\omega_{c,n}^2}{\omega^2}}} \quad \lambda_n(\omega) = \frac{2\pi}{k_n(\omega)} = \frac{2\pi}{\sqrt{1 - \frac{\omega_{c,n}^2}{\omega^2}}} \quad (9.269)$$

We see that the propagation constant, wavelength, and the wave speed go to the unguided values $k_{\epsilon\mu} = \omega/v_{\epsilon\mu}$, $\lambda_0 = 2\pi/k_{\epsilon\mu} = v_{\epsilon\mu}/\nu$, and $v_{\epsilon\mu}$, respectively, as $\omega \rightarrow \infty$. As $\omega \rightarrow \omega_{c,n}$, the propagation constant vanishes, the wavelength becomes infinite, and the wave speed becomes infinite. (The latter two must become infinite together because their ratio is ν , which remains finite. Similarly, because $k_n(\omega) v_n(\omega) = \omega$, $k_n(\omega) \rightarrow 0$ as $v_n(\omega) \rightarrow \infty$ so ω can remain constant.)

Waveguides (cont.)

We recall from our discussion of EM waves in matter that the *wave impedance* is implicitly defined via the relation between \vec{H} and \vec{E} , Equation 9.55. The generalization in the case of waveguides uses the *transverse components* of \vec{H} and \vec{E} (following the relation between the transverse components of \vec{B} and \vec{E} , Equations 9.259):

$$\vec{H}_{0,\perp} = \frac{\vec{B}_{0,\perp}}{\mu} = \frac{1}{Z} \hat{k} \times \vec{E}_{0,\perp} \quad (9.270)$$

Using our relations between $\vec{B}_{0,\perp}$ and $\vec{E}_{0,\perp}$ for the TM and TE modes and the expression for $k_n(\omega)$ we just derived, we obtain (with $Z_{\epsilon\mu} = \sqrt{\mu/\epsilon}$)

$$Z_n^{\text{TM}}(\omega) = \frac{k_n^{\text{TM}}(\omega)}{k_{\epsilon\mu}} \sqrt{\frac{\mu}{\epsilon}} = Z_{\epsilon\mu} \sqrt{1 - \frac{\omega_{c,n}^2}{\omega^2}} \quad Z_n^{\text{TE}}(\omega) = \frac{k_{\epsilon\mu}}{k_n^{\text{TE}}(\omega)} \sqrt{\frac{\mu}{\epsilon}} = \frac{Z_{\epsilon\mu}}{\sqrt{1 - \frac{\omega_{c,n}^2}{\omega^2}}} \quad (9.271)$$

Strangely enough, the wave impedance behaves differently as $\omega \rightarrow \omega_{c,n}$ for the TM and TE modes. This is explained by the current flow. In the TE mode, $\vec{B}_\perp \propto k_n^{\text{TE}}(\omega) \rightarrow 0$ as $\omega \rightarrow \omega_{c,n}$ (while $\vec{E}_\perp \propto k_{\epsilon\mu} v_{\epsilon\mu}$), and we need $Z_n^{\text{TE}}(\omega) \rightarrow \infty$ to obtain this. On the other hand, in the TM mode, $\vec{E}_\perp \propto k_n^{\text{TM}}(\omega) \rightarrow 0$ as $\omega \rightarrow \omega_{c,n}$ (while $\vec{B}_\perp \propto k_{\epsilon\mu}/v_{\epsilon\mu}$), and we need $Z_n^{\text{TM}}(\omega) \rightarrow 0$ to obtain this.

Waveguides (cont.)

Rectangular Waveguide Modes

Let's now specialize to a particular geometry, which will let us find the modes that solve the previously mentioned eigenvector-eigenvalue problem. Our first example will be rectangular waveguide of height a in x and width b in y . One corner of the waveguide is at $(x, y) = 0$ and the diagonal corner is at $(x, y) = (a, b)$.

As one might expect, we try to solve the eigenvalue-eigenvector equation by separation of variables. Let's first try the TE mode, so $B_{0,z}(x, y)$ is what we will solve for. We assume the usual separation of variables form $B_{0,z}(x, y) = X(x) Y(y)$, which yields:

$$\frac{1}{X(x)} \frac{d^2 X}{dx^2} + \frac{1}{Y(y)} \frac{d^2 Y}{dy^2} + (k_{\epsilon\mu}^2 - k^2) = 0 \quad (9.272)$$

As usual, assume that the first two terms are constants:

$$\frac{1}{X(x)} \frac{d^2 X}{dx^2} = -k_x^2 \quad \frac{1}{Y(y)} \frac{d^2 Y}{dy^2} = -k_y^2 \quad k^2 = k_{\epsilon\mu}^2 - k_x^2 - k_y^2 \quad (9.273)$$

Waveguides (cont.)

The solutions to the differential equations are clearly exponentials, and they have imaginary arguments for $k_x^2 \geq 0$:

$$X(x) = A \sin k_x x + B \cos k_x x \quad (9.274)$$

Our boundary condition is $\hat{n} \cdot \vec{\nabla} B_{0,z} \Big|_{\mathcal{C}} = 0$. Explicitly,

$$\frac{dX}{dx} \Big|_{x=0,a} = 0 \implies A = 0 \quad k_x = \frac{m\pi}{a} \quad X(x) \propto \cos \frac{m\pi x}{a} \quad (9.275)$$

Similarly, $\frac{dY}{dy} \Big|_{y=0,b} = 0 \implies k_y = \frac{n\pi}{b} \quad Y(y) \propto \cos \frac{n\pi y}{b}$ (9.276)

Thus

rectangular
waveguide
TE_{mn} mode

$$B_{0,z} \propto \cos \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \quad \gamma_{mn}^2 = k_x^2 + k_y^2 = \frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}$$

(9.277)

This is called the TE_{mn} mode. From $B_{0,z}$, we can obviously obtain $\vec{B}_{0,\perp}$ and $\vec{E}_{0,\perp}$ via Equations 9.265.

On the next few slides, we show some visualizations of the fields and surface charge and current densities for the TE₁₀ mode.

Waveguides (cont.)

The propagation constant can now be written

$$k = \sqrt{k_{\epsilon\mu}^2 - k_x^2 - k_y^2} = \sqrt{k_{\epsilon\mu}^2 - \gamma_{mn}^2} = \sqrt{k_{\epsilon\mu}^2 - \frac{\pi^2 m^2}{a^2} - \frac{\pi^2 n^2}{b^2}} \quad (9.278)$$

and thus the cutoff frequency for the TE_{mn} mode is

rectangular
waveguide
 TE_{mn} mode

$$\omega_{c,mn} = c \pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \quad (9.279)$$

The rest of our generic discussion of dispersion relations applies.

Griffiths has a nice discussion of interpreting these modes as the result of a free EM wave propagating through the wave guide with an infinite number of reflections off the walls. It can help to build some intuition for how these modes arise. The actual calculation clearly can only be applied in the simple case of rectangular waveguide.

Waveguides (cont.)

TE₁₀ mode, xz plane. The mode is independent of y.

Lines: \vec{B} ; Shading: E_y, σ

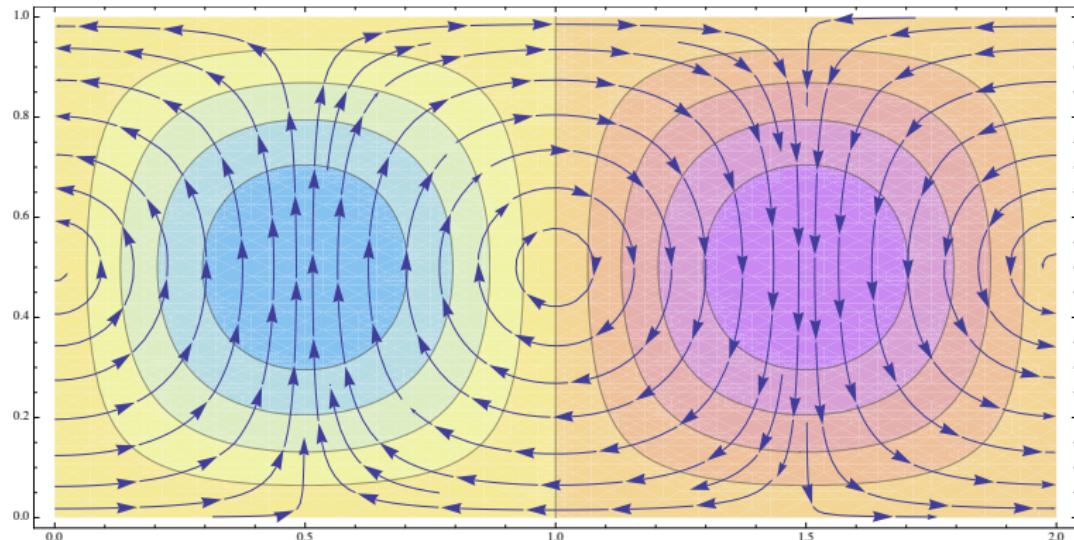


Figure courtesy of M. Cross

Waveguides (cont.)

TE₁₀ mode, xz plane. The mode is independent of y.

Lines: \vec{K} ; Shading: E_y, σ

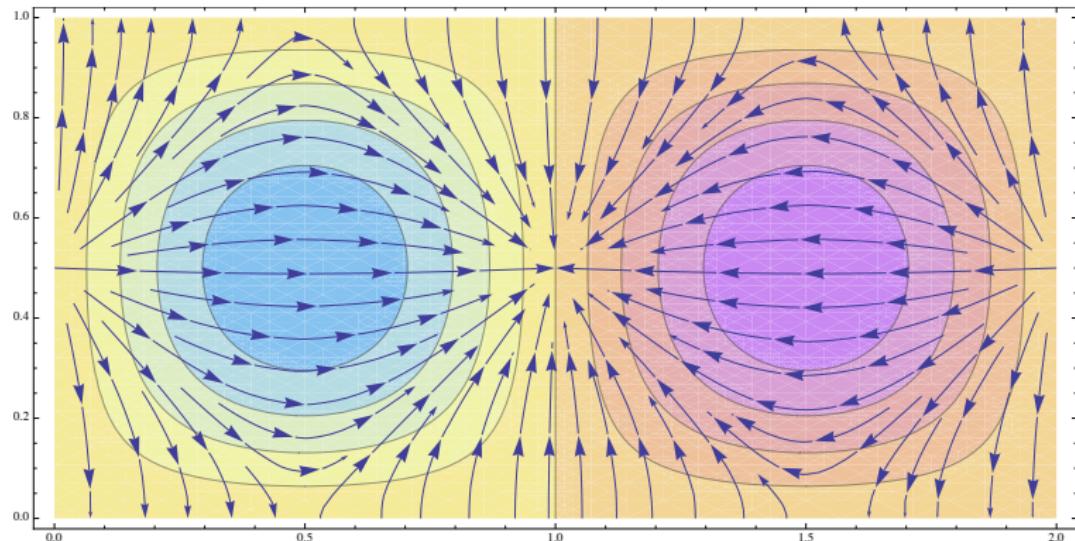


Figure courtesy of M. Cross

Waveguides (cont.)

TE₁₀ mode. The mode is independent of y .

This figure displaced by $\lambda/4$ to the left relative to cross sections.

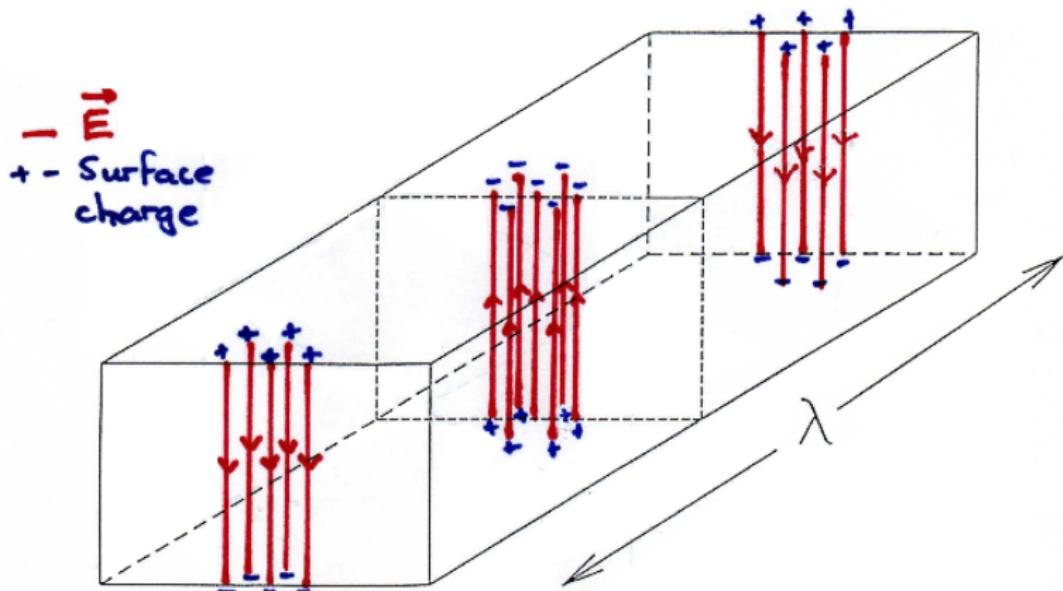


Figure courtesy of M. Cross

Waveguides (cont.)

TE_{10} mode. The mode is independent of y .

This figure displaced by $\lambda/4$ to the left relative to cross sections.

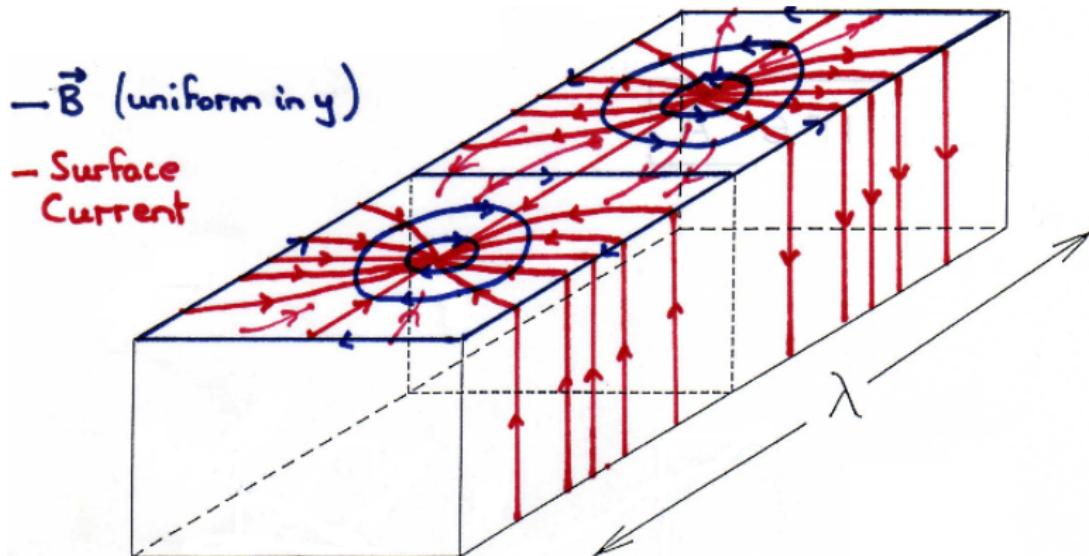


Figure courtesy of M. Cross

Waveguides (cont.)

It is straightforward to obtain the TM modes. The same math applies except the boundary condition is $X(x)|_{x=0,a} = 0$ and $Y(y)|_{y=0,b} = 0$, so one obtains sines instead of cosines:

rectangular
waveguide
TM_{mn} mode

$$E_{0,z} \propto \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \quad \omega_{c,mn} = c\pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}$$

(9.280)

These sine modes are of course necessary to ensure $E_{0,z}$ vanishes at the walls as required by our boundary conditions. From $E_{0,z}$, we can obviously obtain $\vec{B}_{0,\perp}$ and $\vec{E}_{0,\perp}$ via Equations 9.263.

In general, the TM modes for any system require less work to determine because the boundary condition is easier to calculate and enforce. Whenever you do a problem that asks for both the TE and TM modes, do the TM modes first.

Waveguides (cont.)

TEM Modes in a Coaxial Cable

The simplest example of TEM modes is the coaxial cable, to which our “no TEM mode in hollow waveguide” theorem does not apply. We learned that the TEM modes have fields that are the same as the static fields for the configuration with the relation Equation 9.219 between the electric and magnetic fields:

$$\vec{B} = \frac{1}{\nu_{\epsilon\mu}} \hat{k} \times \vec{E} \quad (9.281)$$

We thus know the electrostatic and magnetostatic fields for this configuration have the form

coaxial waveguide TEM mode	$\vec{E}_{0,\perp} = \frac{A}{s} \hat{s}$	$\vec{B}_{0,\perp} = \frac{A}{\nu_{\epsilon\mu} s} \hat{\phi}$	(9.282)
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where the relative normalization has been determined from the aforementioned relation between the fields.

Waveguides (cont.)

Energy, Poynting Vector, and Group Velocity

Let's calculate the Poynting Vector for a waveguide:

$$\langle \vec{S} \rangle = \frac{1}{2\mu} \mathcal{R} (\langle \vec{E}^* \times \vec{B} \rangle) \quad (9.283)$$

This gets complicated because the fields are not fully transverse and, moreover, the transverse and longitudinal fields are out of phase by $\pi/2$. Let's evaluate $\vec{E}^* \times \vec{B}$:

$$\begin{aligned} \vec{E}^* \times \vec{B} &= E_{0,z}^* \hat{z} \times \vec{B}_{0,\perp} + \vec{E}_{0,\perp}^* \times B_{0,z} \hat{z} + \vec{E}_{0,\perp}^* \times \vec{B}_{0,\perp} \\ &\stackrel{\text{TM}}{=} E_{0,z}^* \hat{z} \times \left(\frac{1}{\nu_{\epsilon\mu}} \frac{i k_{\epsilon\mu}}{\gamma_n^2} \hat{z} \times \vec{\nabla}_\perp E_{0,z} \right) + \left(\frac{i k_n^{\text{TM}}(\omega)}{\gamma_n^2} \vec{\nabla}_\perp E_{0,z} \right)^* \times \frac{i k_{\epsilon\mu}}{\nu_{\epsilon\mu} \gamma_n^2} \hat{z} \times \vec{\nabla}_\perp E_{0,z} \end{aligned}$$

where we have specialized to TM modes. Using the *BAC – CAB* rule for the triple-vector product, and also $\hat{z} \cdot \vec{\nabla}_\perp = 0$, we obtain (now showing the TE analogue)

$$\vec{E}^* \times \vec{B} \stackrel{\text{TM}}{=} -\frac{1}{\nu_{\epsilon\mu}} \frac{i k_{\epsilon\mu}}{\gamma_n^2} E_{0,z}^* \vec{\nabla}_\perp E_{0,z} + \hat{z} \frac{k_n^{\text{TM}}(\omega) k_{\epsilon\mu}}{\nu_{\epsilon\mu} \gamma_n^4} \left| \vec{\nabla}_\perp E_{0,z} \right|^2 \quad (9.284)$$

$$\vec{E}^* \times \vec{B} \stackrel{\text{TE}}{=} \nu_{\epsilon\mu} \frac{i k_{\epsilon\mu}}{\gamma_n^2} B_{0,z}^* \vec{\nabla}_\perp B_{0,z} + \hat{z} \nu_{\epsilon\mu} \frac{k_n^{\text{TE}}(\omega) k_{\epsilon\mu}}{\gamma_n^4} \left| \vec{\nabla}_\perp B_{0,z} \right|^2 \quad (9.285)$$

Waveguides (cont.)

The first term points transverse to the direction of propagation. Since the walls are perfectly conducting, there can be no time-averaged, net energy flow in that direction. So, when we time-average, we neglect it. That leaves us with:

$$\left\langle \vec{S}_{\text{TM}} \right\rangle = \frac{\hat{z}}{2} \frac{\omega k_n^{\text{TM}}(\omega)}{\gamma_n^4} \epsilon \left| \vec{\nabla}_{\perp} E_{0,z} \right|^2 \quad \left\langle \vec{S}_{\text{TE}} \right\rangle = \frac{\hat{z}}{2} \frac{\omega k^{\text{TE}}(\omega)}{\gamma_n^4} \frac{1}{\mu} \left| \vec{\nabla}_{\perp} B_{0,z} \right|^2 \quad (9.286)$$

Let's integrate $\left\langle \vec{S} \right\rangle \cdot \hat{z}$ over the waveguide cross-sectional area to get the total power. Using ψ_n to represent $E_{0,z}$ or $B_{0,z}$ as appropriate, we can manipulate the area integral using the two-dimensional analogue of Green's First Identity (Equation 3.12):

$$\int_S da \left| \vec{\nabla}_{\perp} \psi_n \right|^2 = \oint_{C(S)} d\ell \psi^* \hat{n} \cdot \vec{\nabla}_{\perp} \psi - \int_S da \psi^* \nabla_{\perp}^2 \psi \quad (9.287)$$

The first term vanishes due to our boundary conditions: either ψ or $\hat{n} \cdot \vec{\nabla}_{\perp} \psi$ always vanishes. The second term can be transformed using the eigenvector-eigenvalue equation that yield the solutions ψ_n , yielding:

$$\int da \left| \vec{\nabla}_{\perp} \psi_n \right|^2 = \gamma_n^2 \int_S da |\psi|^2 \quad (9.288)$$

Waveguides (cont.)

Thus, the time-averaged power flow is

$$\langle P_{\text{TM}} \rangle = \frac{1}{2} \frac{\omega k_n^{\text{TM}}(\omega)}{\gamma_n^2} \epsilon \int_S da |E_{0,z}|^2 = \frac{1}{2} \nu_{\epsilon\mu} \frac{\omega^2}{\omega_{c,n}^2} \sqrt{1 - \frac{\omega_{c,n}^2}{\omega^2}} \epsilon \int_S da |E_{0,z}|^2 \quad (9.289)$$

$$\langle P_{\text{TE}} \rangle = \frac{1}{2} \frac{\omega k_n^{\text{TE}}(\omega)}{\gamma_n^2} \frac{1}{\mu} \int_S da |B_{0,z}|^2 = \frac{1}{2} \nu_{\epsilon\mu} \frac{\omega^2}{\omega_{c,n}^2} \sqrt{1 - \frac{\omega_{c,n}^2}{\omega^2}} \frac{1}{\mu} \int_S da |B_{0,z}|^2 \quad (9.290)$$

If we do similar integrals to calculate the time-averaged energy per unit length in the waveguide, and we obtain

$$\langle U_{\text{TM}} \rangle = \frac{1}{2} \frac{\omega^2}{\omega_{c,n}^2} \epsilon \int_S da |E_{0,z}|^2 \quad \langle U_{\text{TE}} \rangle = \frac{1}{2} \frac{\omega^2}{\omega_{c,n}^2} \frac{1}{\mu} \int_S da |B_{0,z}|^2 \quad (9.291)$$

Waveguides (cont.)

What is interesting here is that we do not have $\langle P \rangle = v_{\epsilon\mu} \langle U \rangle$ as we had in free space, or even $\langle P \rangle = v_n(\omega) \langle U \rangle$ as we might expect given the guided wave speed. This makes sense, as $v_n(\omega) > v_{\epsilon\mu}$, so $v_n(\omega)$ can exceed the speed of light. The quantity relating $\langle P \rangle$ and $\langle U \rangle$ is

$$v_{\epsilon\mu} \sqrt{1 - \frac{\omega_{c,n}^2}{\omega^2}} = \frac{d\omega}{dk_n(\omega)} \quad (9.292)$$

which one can see by differentiating the dispersion relation $\omega^2 = v_{\epsilon\mu}^2 ([k_n(\omega)]^2 + \gamma_n^2)$:

$$2\omega d\omega = v_{\epsilon\mu}^2 (2k_n dk_n + 0) \implies \frac{d\omega}{dk_n} = v_{\epsilon\mu}^2 \frac{k_n(\omega)}{\omega} = v_{\epsilon\mu} \sqrt{1 - \frac{\omega_{c,n}^2}{\omega^2}} \quad (9.293)$$

This speed is the *group velocity*, $v_{g,n}(\omega)$. You are aware from Ph12 how the group velocity is the speed at which a wave packet propagates. The group velocity here can never exceed $v_{\epsilon\mu}$, thus preventing the power flow from going faster than the speed of light. So, we have

$$\boxed{\langle P \rangle = v_{g,n}(\omega) \langle U \rangle \quad v_{g,n}(\omega) = v_{\epsilon\mu} \sqrt{1 - \frac{\omega_{c,n}^2}{\omega^2}} \quad v_{g,n}(\omega) v_n(\omega) = v_{\epsilon\mu}^2} \quad (9.294)$$

Waveguides (cont.)

Waveguides with Finite Conductivity

Before considering imperfect conductors, let's pull together everything we know about perfect conductors:

- ▶ The conductivity is infinite, so, even when \vec{E} is varying, the surface charge redistributes itself so that the \vec{E}_c , the field inside the conductor, vanishes. Our boundary conditions from Gauss's Law and Faraday's Law imply that there is a discontinuity in $\hat{n} \cdot \vec{E}$ due to this surface charge density while the $\hat{t} \cdot \vec{E}$ is continuous, so the electric field outside the conductor is normal to its surface. Our TE and TM mode boundary conditions are consistent with these conditions.
- ▶ What about the magnetic field inside the conductor? Since \vec{E}_c vanishes, we know there can be no displacement current sourcing a magnetic field. It is difficult to rigorously argue that there can be no \vec{J}_f in the conductor (since $\sigma \rightarrow \infty$, $\vec{E}_c = 0$ is consistent with this), but we know EM waves die out over a skin depth in a good conductor, and we know the skin depth vanishes as $\sigma \rightarrow \infty$, so it is physically reasonable to expect that there is no \vec{J}_f in a perfect conductor assuming fields are incident from the outside of the conductor. Certainly, unless someone is driving a \vec{J}_f in the conductor, it will not be there.

Waveguides (cont.)

Since there is no current density in the conductor, and also based on our EM wave argument, the field inside the conductor vanishes, $\vec{B}_c = 0$. With $\vec{B}_c = 0$, $\vec{\nabla} \cdot \vec{B} = 0$ implies that $\hat{n} \cdot \vec{B} = 0$ outside the conductor. A discontinuity in $\hat{t} \cdot \vec{H}$ is allowed if there is a surface current density \vec{K} at the surface such that $\hat{n} \times \vec{H} = \vec{K}$. This implies the magnetic field outside the conductor is perfectly tangential.

- ▶ Let's revisit the continuity of the tangential component of \vec{E} in light of the surface current density. This continuity continues to hold: Faraday's Law tells us that there will only be a discontinuity in $\hat{t} \cdot \vec{E}$ if the magnetic field in the \hat{s} direction ($\hat{s} = \hat{t} \times \hat{n}$) has δ -function behavior at the surface. While the surface current density does have δ -function behavior, \vec{H} is finite outside the conductor and zero inside, so it does not have that kind of behavior.
- ▶ Finally, the existence of the surface current density \vec{K} is consistent with the lack of a tangential \vec{E} because $\sigma \rightarrow \infty$.
- ▶ This surface current density's magnetic field cancels that of the external field to ensure no magnetic field propagates into the interior of the conductor (much the same way as the surface charge density does this for the electric field).

The above fact that magnetic fields vanish in perfect conductors when there is no free current source in the conductor has been hinted at before in homework and in our discussion of TEM modes for transmission lines, but we have never discussed it in as general a setting as this before. For our waveguide mode derivation, we only required $\hat{n} \cdot \vec{\nabla}_{\perp} B_{0,z}$ vanish at the boundary.

Waveguides (cont.)

Now, let's consider a waveguide with walls having *good* conductivity. There are two adjustments to be made:

- ▶ First, our boundary conditions no longer hold exactly because we may not assume that the electric and magnetic fields vanish perfectly inside the conductor: as we know, they decay over a skin depth, and the applicability of Ohm's Law now implies that a δ -function-like surface current density is not physically allowed (it would require a δ -function-like electric field for finite σ).
- ▶ Second, there is power dissipation in the walls due to the currents flowing in a resistive material and thus attenuation of the wave.

It is quite difficult to include these effects exactly, but we can derive some basic results in the case of a good conductor.

Specifically, we assume a successive approximation scheme where the fields outside the conductor may now have $\hat{\mathbf{t}} \cdot \vec{\mathbf{E}}$ and $\hat{\mathbf{n}} \cdot \vec{\mathbf{B}}$ components, but they are small compared to $\hat{\mathbf{n}} \cdot \vec{\mathbf{E}}$ and $\hat{\mathbf{t}} \cdot \vec{\mathbf{B}}$. We obtain the dominant components of the field from a boundary value problem solution for perfectly conductive walls (e.g., our waveguide solutions), then we use the boundary conditions and Maxwell's Equations to find the small fields inside the conductor and the correction fields outside.

Waveguides (cont.)

We know, in fact, that the solutions must be those we obtained for EM waves in good conductors. We just need to know how to normalize them. The normalization is found by using the fact that $\hat{t} \cdot \vec{H}$ is continuous and $\hat{n} \cdot \vec{H} \approx 0$, so the EM wave in the conductor looks like a wave that is at normal incidence on the interface. If ξ is the coordinate normal to the interface heading into the conductor, we have

$$\hat{t} \cdot \vec{H}_c(\xi, t) = \hat{t} \cdot \vec{H}(0) e^{-\xi/\delta} e^{i(\xi/\delta - \omega t)} \quad (9.295)$$

where $\delta = \sqrt{2/\mu_c \sigma \omega}$ is the skin depth in the conductor. The corresponding component of the magnetic field is then

$$\hat{t} \cdot \vec{B}_c(\xi, t) = \frac{\mu_c}{\mu} \hat{t} \cdot \vec{B}(\xi = 0) e^{-\xi/\delta} e^{i(\xi/\delta - \omega t)} \quad (9.296)$$

where μ and μ_c are the permeabilities outside and inside the conductor. We can obtain the component of \vec{E} parallel to the interface from the relation between \vec{B} and \vec{E} for good conductors, Equations 9.129, 9.130, and 9.133, yielding

$$(\hat{n} \times \hat{t}) \cdot \vec{E}_c(\xi, t) = \frac{\omega \delta}{1 + i} \hat{t} \cdot \vec{B}_c = \frac{\omega \delta}{1 + i} \frac{\mu_c}{\mu} \hat{t} \cdot \vec{B}(\xi = 0) e^{-\xi/\delta} e^{i(\xi/\delta - \omega t)} \quad (9.297)$$

where \hat{n} points outward from the conductor into the incident medium. We thus have the dominant components of the fields inside the conductor.

Waveguides (cont.)

We may now calculate the correction terms to the fields outside the conductor. The tangential electric field outside the conductor is given by $\hat{n} \times \hat{t} \cdot \vec{E}_c(0, t)$ by continuity of this component. The normal component of \vec{B} outside the conductor is found using Faraday's Law (for evaluating the curl, consider a temporary primed coordinate system, distinct from the one we have been using from the waveguide, with $\hat{n} = \hat{z}'$, $\hat{t} = \hat{x}'$, and $\hat{n} \times \hat{t} = \hat{y}'$):

$$\begin{aligned} i\omega\hat{n}\cdot\vec{B}(\xi=0,t) &= \left(\hat{t}\cdot\vec{\nabla}\right)\left[(\hat{n}\times\hat{t})\cdot\vec{E}(\xi=0,t)\right] - \left(\hat{n}\times\hat{t}\cdot\vec{\nabla}\right)\left[\hat{t}\cdot\vec{E}(\xi=0,t)\right] \\ \hat{n}\cdot\vec{B}(\xi=0,t) &= -\frac{i}{\omega}\left(\hat{t}\cdot\vec{\nabla}\right)\left[\frac{\omega\delta}{1+i}\frac{\mu_c}{\mu}\hat{t}\cdot\vec{B}(\xi=0)\right] \\ &= -\frac{\delta}{1-i}\frac{\mu_c}{\mu}\left(\hat{t}\cdot\vec{\nabla}\right)\left[\hat{t}\cdot\vec{B}(\xi=0)\right] \end{aligned} \tag{9.298}$$

Since $\hat{n}\cdot\vec{B}$ is continuous, this also gives the normal component of \vec{B}_c at the interface, which we will ignore since it is small compared to the tangential component.

Waveguides (cont.)

In the good conductor approximation, we expect the wave in the waveguide to maintain its functional form but to decay with some decay length that is large compared to the wavelength. We can determine this decay constant by considering the (mean) Joule dissipation in the conductor. This power dissipated per unit volume in the conductor is

$$\frac{d\langle P_J \rangle}{d\tau} = \left\langle \vec{J}^* \cdot \vec{E} \right\rangle = \frac{\sigma}{2} \left| \hat{n} \times \hat{t} \cdot \vec{E}_c(\xi, t) \right|^2 = \frac{\sigma}{2} \frac{\omega^2 \delta^2}{2} \frac{\mu_c^2}{\mu^2} \left[\hat{t} \cdot \vec{B}(\xi = 0, t) \right]^2 e^{-2\xi/\delta} \quad (9.299)$$

When we integrate this over depth to get the Joule dissipation per unit area, $e^{-2\xi/\delta}$ is replaced by $\delta/2$. After some algebra (recall, $\delta = \sqrt{2/\mu_c \sigma \omega}$), we find

$$\frac{d\langle P_J \rangle}{da} = \frac{1}{2\sigma\delta} \frac{1}{\mu^2} \left[\hat{\mathbf{t}} \cdot \vec{B}(\xi=0, t) \right]^2 = \frac{1}{2\sigma\delta} |\hat{\mathbf{n}} \times \hat{\mathbf{t}} \cdot \vec{K}|^2 \quad (9.300)$$

where we have now made use of the fact that, for the perfect conductor waveguide solution, the tangential component of the magnetic field gives the surface current density. Integrating over the boundary, the rate of power loss per unit length along the waveguide is therefore

$$-\frac{d\langle P \rangle}{dz} = \oint_{C(S)} d\ell \frac{1}{2} \frac{|\hat{n} \times \hat{t} \cdot \vec{K}|^2}{\sigma \delta} \quad (9.301)$$

Waveguides (cont.)

Now, assuming a TM mode (so $\vec{K} = K_z \hat{z}$ and $B_{0,z} = 0$) and using the tangential boundary condition on the magnetic field:

$$K_{0,z} = \frac{1}{\mu} \vec{B} \cdot \hat{\mathbf{t}}_{\perp} = \frac{1}{\mu} \frac{i k_{\epsilon\mu}}{\nu_{\epsilon\mu} \gamma_n^2} \hat{\mathbf{t}}_{\perp} \cdot (\hat{z} \times \vec{\nabla}_{\perp} E_{0,z}) = \frac{1}{\mu} \frac{i k_{\epsilon\mu}}{\nu_{\epsilon\mu} \gamma_n^2} (\hat{\mathbf{t}}_{\perp} \times \hat{z}) \cdot \vec{\nabla}_{\perp} E_{0,z} \quad (9.302)$$

$$= \frac{1}{\mu} \frac{i k_{\epsilon\mu}}{\nu_{\epsilon\mu} \gamma_n^2} \hat{n} \cdot \vec{\nabla}_{\perp} E_{0,z} \quad (9.303)$$

where we used the cyclicity of the triple vector product and $\hat{\mathbf{t}}_{\perp} = \hat{z} \times \hat{n}$. Plugging into the power loss expression and using $k_{\epsilon\mu} = \omega / \nu_{\epsilon\mu}$ and $\omega_{c,n} = \nu_{\epsilon\mu} \gamma_n$, we obtain

$$\boxed{-\frac{d\langle P_{\text{TM}} \rangle}{dz} = \frac{1}{2\sigma\delta} \frac{\omega^2}{\omega_{c,n}^2} \frac{1}{\mu^2 \omega_{c,n}^2} \oint_{C(S)} d\ell |\hat{n} \cdot \vec{\nabla}_{\perp} E_{0,z}|^2} \quad (9.304)$$

One can obtain the corresponding expression for TE modes:

$$\boxed{-\frac{d\langle P_{\text{TE}} \rangle}{dz} = \frac{1}{2\sigma\delta} \frac{\omega^2}{\omega_{c,n}^2} \frac{1}{\mu^2} \oint_{C(S)} d\ell \left[\frac{\nu_{\epsilon\mu}^2}{\omega_{c,n}^2} \left(1 - \frac{\omega_{c,n}^2}{\omega^2} \right) |\hat{n} \times \vec{\nabla}_{\perp} B_{0,z}|^2 + \frac{\omega_{c,n}^2}{\omega^2} |B_{0,z}|^2 \right]} \quad (9.305)$$

Waveguides (cont.)

To calculate the power loss per unit length exactly, we must evaluate the above integrals using the known waveguide solutions. However, we can determine the approximate size of the above expressions. Recall that, for perfectly conducting walls, the eigenvalue-eigenvector equation implies (see Equations 9.287 and 9.288)

$$\int da \left| \vec{\nabla}_{\perp} \psi_n \right|^2 = \gamma_n^2 \int_S da |\psi|^2 \quad (9.306)$$

The above exact equality implies the approximate equality

$$\left\langle \left| \hat{n} \cdot \vec{\nabla}_{\perp} E_{0,z} \right|^2 \right\rangle_{C(S)} \approx \gamma_n^2 \left\langle |E_{0,z}|^2 \right\rangle_S \quad \left\langle \left| \hat{n} \times \vec{\nabla}_{\perp} B_{0,z} \right|^2 \right\rangle_{C(S)} \approx \gamma_n^2 \left\langle |B_{0,z}|^2 \right\rangle_S$$

where the averages are over the boundary or the surface, not time. Thus, we can relate the line integral over the boundary to an area integral over the cross section:

$$\oint_{C(S)} \frac{1}{\gamma_n^2} \left| \hat{n} \cdot \vec{\nabla}_{\perp} E_{0,z} \right|^2 = \xi_n \frac{C}{A} \int_S da |E_{0,z}|^2 \quad \text{with} \quad C = \oint_{C(S)} d\ell \quad A = \int_S da \quad (9.307)$$

where ξ_n is a fudge factor that has to be calculated for the specific geometry and is defined by the above equations. There is a similar expression for the TE modes. The area integral on the right is related to the power flow down the waveguide, Equation 9.289.

Waveguides (cont.)

In fact, using Equations 9.289 and Equation 9.290 and the analogue of the above equation for TE modes, one can show that the *field* decay constant is

$$\kappa_n = -\frac{1}{2\langle P \rangle} \frac{d\langle P \rangle}{dz} = \frac{1}{Z_{\epsilon\mu} \sigma \delta_n} \frac{C}{2A} \frac{\sqrt{\frac{\omega}{\omega_{c,n}}}}{\sqrt{1 - \frac{\omega_{c,n}^2}{\omega^2}}} \left[\xi_n + \eta_n \frac{\omega_{c,n}^2}{\omega^2} \right] \quad (9.308)$$

where δ_n is the skin depth δ evaluated at $\omega_{c,n}$, ξ_n is the order unity quantity defined above for TM modes (and can be defined analogously for TE modes), and η_n is an analogous quantity that is nonzero only for TE modes (it corresponds to the second term in the TE power loss expression that is proportional to $\omega_{c,n}^2/\omega^2$). We do not present generic exact formulae for ξ_n and η_n ; for any given waveguide geometry, one can find κ_n by evaluating the explicit $d\langle P \rangle/dz$ and $\langle P \rangle$ formulae.

The useful and interesting thing about the above expression is that it exhibits all the frequency dependence. The ξ_n and η_n factors depend on the solution to the eigenvector-eigenvalue equation, which is geometry- but not frequency-dependent.

Section 10

Potentials and Radiation

Lecture 24: Potential Formulation

Date Revised: 2014/05/27 18:00

Corrections to Equations 10.66 and 10.67

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Revisiting Scalar and Vector Potentials

Recall how we arrived at Maxwell's Equations. We first developed Faraday's Law by incorporating both empirical information (Faraday's observations) and the requirement of the Lorentz Force being consistent with Galilean relativity. We then found an inconsistency that required the introduction of the displacement current, yielding the full set of Maxwell's Equations.

However, we have not revisited the static potentials that we developed in the static field cases. Let's do that.

We still have that $\vec{\nabla} \cdot \vec{B} = 0$, so the Helmholtz Theorem guarantees that we may continue to write

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad (10.1)$$

Potential Formulation (cont.)

However, Faraday's Law implies that $\vec{\nabla} \times \vec{E} \neq 0$ when \vec{B} has time dependence. Therefore, we cannot assume $\vec{E} = -\vec{\nabla}V$. However, using $\vec{B} = \vec{\nabla} \times \vec{A}$, we see that

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} = -\frac{\partial}{\partial t} (\vec{\nabla} \times \vec{A}) \quad \Rightarrow \quad \vec{\nabla} \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0 \quad (10.2)$$

Thus, the Helmholtz Theorem implies

$$\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla}V \quad \Rightarrow \quad \boxed{\vec{E} = -\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t}} \quad (10.3)$$

By design, these new definitions of V and \vec{A} satisfy the homogeneous Maxwell Equations (the ones that have no sources):

$$\vec{\nabla} \cdot \vec{B} = 0 \quad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad (10.4)$$

Potential Formulation (cont.)

Let's check whether Gauss's Law and Ampere's Law can be satisfied. In terms of the potentials, Gauss's Law becomes

$$\frac{\rho}{\epsilon_0} = \vec{\nabla} \cdot \vec{E} = -\nabla^2 V - \frac{\partial}{\partial t} (\vec{\nabla} \cdot \vec{A}) \quad (10.5)$$

and Ampere's Law becomes

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{J} - \epsilon_0 \mu_0 \vec{\nabla} \frac{\partial V}{\partial t} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{A}}{\partial t^2} \quad (10.6)$$

which we may rewrite using the usual $\vec{\nabla} \times (\vec{\nabla} \times \vec{a}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{a}) - \nabla^2 \vec{a}$ identity:

$$\left(\nabla^2 \vec{A} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{A}}{\partial t^2} \right) - \vec{\nabla} \left(\vec{\nabla} \cdot \vec{A} + \epsilon_0 \mu_0 \frac{\partial V}{\partial t} \right) = -\mu_0 \vec{J} \quad (10.7)$$

These are second order coupled partial differential equations for V and \vec{A} and, so far, we see no reason Equations 10.5 and 10.7 cannot be solved.

Potential Formulation (cont.)

We can write Gauss' Law (Equation 10.5) and Ampere's Law (Equation 10.7) in the symmetrical form (defining \square^2 with the opposite sign convention as Griffiths to be consistent with our later discussion of special relativity):

$$\square^2 V - \frac{\partial L}{\partial t} = \frac{\rho}{\epsilon_0} \quad \square^2 \vec{A} + \vec{\nabla} L = \mu_0 \vec{J} \quad (10.8)$$

$$\text{with} \quad \square^2 \equiv \epsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} - \nabla^2 \quad L \equiv \vec{\nabla} \cdot \vec{A} + \epsilon_0 \mu_0 \frac{\partial V}{\partial t} \quad (10.9)$$

Potential Formulation (cont.)

Gauge Freedom and Transformations

The new equations for the potentials reduce the number of degrees of freedom from six to four. There remains what is called a *gauge freedom* that we will first describe and then make choices for. This is the generalization of the freedom we had to set the constant offset for the potential in electrostatics and to choose the value of $\vec{\nabla} \cdot \vec{A}$ in magnetostatics.

Gauge freedom answers the question: how much freedom in \vec{A} and V yield the same fields? That is, let us assume two sets of potentials (V, \vec{A}) and (V', \vec{A}') that differ by functions $\vec{\alpha}$ and β :

$$\vec{A}'(\vec{r}, t) = \vec{A}(\vec{r}, t) + \vec{\alpha}(\vec{r}, t) \quad V'(\vec{r}, t) = V(\vec{r}, t) + \beta(\vec{r}, t) \quad (10.10)$$

The requirement that these two sets of potentials yield the same fields gives the equations:

$$0 = \vec{\nabla} \times [\vec{A}'(\vec{r}, t) - \vec{A}(\vec{r}, t)] = \vec{\nabla} \times \vec{\alpha}(\vec{r}, t) \quad (10.11)$$

$$\begin{aligned} 0 &= -\vec{\nabla} [V'(\vec{r}, t) - V(\vec{r}, t)] - \frac{\partial}{\partial t} [\vec{A}'(\vec{r}, t) - \vec{A}(\vec{r}, t)] \\ &= -\vec{\nabla} \beta(\vec{r}, t) - \frac{\partial \vec{\alpha}(\vec{r}, t)}{\partial t} \end{aligned} \quad (10.12)$$

Potential Formulation (cont.)

The first equation implies that $\vec{\alpha}$ can be written as the gradient of a scalar

$$\vec{\alpha}(\vec{r}, t) = \vec{\nabla}\lambda(\vec{r}, t) \quad (10.13)$$

which we then plug into the other equation

$$\vec{\nabla} \left[\beta(\vec{r}, t) + \frac{\partial\lambda(\vec{r}, t)}{\partial t} \right] = 0 \quad (10.14)$$

Therefore, the quantity in parentheses must be position-independent, which yields

$$\beta(\vec{r}, t) + \frac{\partial\lambda(\vec{r}, t)}{\partial t} = k(t) \quad (10.15)$$

We can absorb $k(t)$ into λ : add to λ the position-independent term $\int_0^t dt' k(t')$, whose gradient vanishes and thus does not affect $\alpha(\vec{r}, t)$. Thus $\beta(\vec{r}, t) = -\partial\lambda(\vec{r}, t)/\partial t$ and we have the relations

$$\vec{A}'(\vec{r}, t) = \vec{A}(\vec{r}, t) + \vec{\nabla}\lambda(\vec{r}, t) \quad V'(\vec{r}, t) = V(\vec{r}, t) - \frac{\partial\lambda(\vec{r}, t)}{\partial t} \quad (10.16)$$

This kind of change in (V, \vec{A}) that has no effect on the fields is called a *gauge transformation*. The choice of λ (perhaps implicit through specifying properties of \vec{A} and V) is called the choice of *gauge*.

Potential Formulation (cont.)

Coulomb Gauge

This is the gauge we chose earlier for magnetostatics,

$$\boxed{\text{Coulomb Gauge} \quad \vec{\nabla} \cdot \vec{A} = 0} \quad (10.17)$$

In this gauge, Poisson's Equation simplifies to the electrostatic form

$$\boxed{\text{Coulomb Gauge} \quad \nabla^2 V = -\frac{\rho}{\epsilon_0}} \quad (10.18)$$

That is, the charge density sets the potential in the same way as in electrostatics, so changes in charge density propagate into the potential *instantaneously*. Of course, you know from relativity that this is not possible. We will see that there are corrections from $\partial \vec{A} / \partial t$ that prevent \vec{E} from responding instantaneously to such changes. The differential equation for \vec{A} becomes

$$\boxed{\text{Coulomb Gauge} \quad \nabla^2 \vec{A} - \epsilon_0 \mu_0 \frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0 \vec{J} + \epsilon_0 \mu_0 \vec{\nabla} \left(\frac{\partial V}{\partial t} \right)} \quad (10.19)$$

which is difficult to use if V has time dependence. If V is time-independent (e.g., no free charge density), then this is a wave equation with a source term.

Potential Formulation (cont.)

Lorenz Gauge

Here the choice is

$$\boxed{\text{Lorenz Gauge} \quad \vec{\nabla} \cdot \vec{A} = -\epsilon_0 \mu_0 \frac{\partial V}{\partial t}} \quad (10.20)$$

The reason this choice is made is that it sets $L = 0$ in Equation 10.9 so that we obtain symmetric, inhomogeneous (have source terms) wave equations for V and \vec{A} :

$$\boxed{\text{Lorenz Gauge} \quad \square^2 V = \frac{\rho}{\epsilon_0} \quad \square^2 \vec{A} = \mu_0 \vec{J} \quad \square^2 \equiv \epsilon_0 \mu_0 \frac{\partial^2}{\partial t^2} - \nabla^2} \quad (10.21)$$

with, again, the opposite sign convention as Griffiths for \square^2 . This will be the natural gauge for calculating radiation of EM waves from moving charges and currents, especially in a manner that is manifestly consistent with relativity.

Potential Formulation (cont.)

The Lorentz Force Law in Potential Form and Classical Mechanics

Let's write the Lorentz Force Law in terms of the potentials:

$$\vec{F} = \frac{d\vec{p}}{dt} = q \left[-\vec{\nabla}V - \frac{\partial \vec{A}}{\partial t} + \vec{v} \times (\vec{\nabla} \times \vec{A}) \right] \quad (10.22)$$

Using the *BAC – CAB* rule for the triple vector product and recognizing \vec{v} is not a function of position, we may rewrite as

$$\frac{d\vec{p}}{dt} = -q \left[\frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{A} + \vec{\nabla} (V - \vec{v} \cdot \vec{A}) \right] \quad (10.23)$$

We may write

$$\frac{\partial \vec{A}}{\partial t} + (\vec{v} \cdot \vec{\nabla}) \vec{A} = \frac{d\vec{A}}{dt} \quad (10.24)$$

where d/dt is the *total derivative* of \vec{A} , taking into account both the explicit time dependence of \vec{A} (the $\partial \vec{A}/\partial t$ term) and the time dependence of \vec{A} to which the particle is subject due to its motion (the $(\vec{v} \cdot \vec{\nabla}) \vec{A}$ term). This kind of derivative should be familiar to you from classical mechanics when considering the total time derivative of the Lagrangian. The quantity on the left side of the above equation is called the *convective derivative* of \vec{A} .

Potential Formulation (cont.)

Let's be explicit about why the convective derivative correctly gives the total derivative of \vec{A} . Consider a time interval dt in which the particle moves $d\vec{r} = \vec{v} dt$:

$$d\vec{A} = \vec{A}(\vec{r} + \vec{v} dt, t + dt) - \vec{A}(\vec{r}, t) \quad (10.25)$$

$$= v_x dt \frac{\partial \vec{A}}{\partial x} + v_y dt \frac{\partial \vec{A}}{\partial y} + v_z dt \frac{\partial \vec{A}}{\partial z} + dt \frac{\partial \vec{A}}{\partial t} = dt \left[(\vec{v} \cdot \vec{\nabla}) \vec{A} + \frac{\partial \vec{A}}{\partial t} \right] \quad (10.26)$$

With the appropriate definition of the total derivative of \vec{A} , we may rewrite the Lorentz Force Law as

$$\frac{d}{dt} (\vec{p} + q \vec{A}) = -\vec{\nabla} [q (V - \vec{v} \cdot \vec{A})] \quad (10.27)$$

which may be rewritten using the *canonical momentum*, $\vec{\pi}$, and a *velocity-dependent potential*, $U(\vec{r}, \vec{v}, t)$:

$$\boxed{\frac{d\vec{\pi}}{dt} = -\vec{\nabla} U \quad \vec{\pi} = \vec{p} + q \vec{A} \quad U(\vec{r}, \vec{v}, t) = q [V(\vec{r}, t) - \vec{v} \cdot \vec{A}(\vec{r}, t)]} \quad (10.28)$$

Potential Formulation (cont.)

You may have seen this idea of a velocity-dependent potential in Ph106a. Recall that the Lagrangian formulation of mechanics is derivable as a special case for conservative forces of the *generalized equation of motion*:

$$\sum_i \vec{F}_i^{(nc)} \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \mathcal{F}_k = \frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_k} \right) - \frac{\partial T}{\partial q_k} \quad (10.29)$$

where the $\vec{F}_i^{(nc)}$ are the set of *non-constraint forces*, the q_k are generalized coordinates, and T is the kinetic energy in terms of the generalized coordinates and velocities. For a conservative force, the force term on the left side can be written as a gradient of a potential energy U , and that term can be moved to the right side and $L = T - U$ defined to obtain the Euler-Lagrange Equation

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = 0 \quad (10.30)$$

For nonconservative, velocity-dependent forces like the Lorentz Force, it is clear that, if one can construct the appropriate function U such that its partial derivatives with respect to q_k and \dot{q}_k generate the appropriate force terms on the left side, then those derivatives can be moved to the right side just as in the conservative case and one can apply the Euler-Lagrange Equations to $L = T - U$ where U is velocity-dependent.

Potential Formulation (cont.)

We can find a similar equation relating the time derivative of the particle energy to the potentials. This will become useful when we consider the relativistic generalization of the Lorentz Force Law, which will include an energy component. We obtain this by calculating the work done per unit time by the Lorentz Force:

$$\frac{dT}{dt} = \vec{v} \cdot \vec{F} = q \left[\vec{v} \cdot (-\vec{\nabla} V) - \vec{v} \cdot \frac{\partial \vec{A}}{\partial t} + \vec{v} \cdot \vec{v} \times (\vec{\nabla} \times \vec{A}) \right] \quad (10.31)$$

The last term vanishes because the triple cross product is perpendicular to \vec{v} , reflecting the fact that *static* magnetic fields can do not work. Now, let's add the total derivative of $q V$,

$$\frac{d}{dt} (q V) = (\vec{v} \cdot \vec{\nabla}) (q V) + \frac{\partial}{\partial t} (q V) \quad (10.32)$$

to both sides, yielding

$$\boxed{\frac{d}{dt} (T + q V) = \frac{\partial}{\partial t} [q (V - \vec{v} \cdot \vec{A})]} \quad (10.33)$$

This gives us an expression for conservation of energy in terms of the potentials (rather than the fields) for fully time-dependent (but non-relativistic) situations. The latter term on the RHS does not imply magnetic fields do work, as \vec{B} is always perpendicular to \vec{A} , so $\vec{v} \cdot \vec{A}$ has no contribution from \vec{B} .

Potential Formulation (cont.)

Retarded Potentials

We would like to obtain solutions to the inhomogeneous wave equations, Equation 10.21. Those equations have the structure of Poisson's Equation aside from the addition of the time derivative term. In the case of static fields, they in fact reduce to Poisson's Equation exactly. We know from the homogeneous wave equation that this time derivative implies that the solution propagates at speed $c = 1/\sqrt{\epsilon_0 \mu_0}$. It is thus plausible to believe that the solutions are given by generalizing the static equations to account for the propagation time:

retarded time	$t_r = t - \frac{ \vec{r} - \vec{r}' }{c}$	(10.34)
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retarded potentials	$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int_V d\tau' \frac{\rho(\vec{r}', t_r)}{ \vec{r} - \vec{r}' }$	$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_V d\tau' \frac{\vec{j}(\vec{r}', t_r)}{ \vec{r} - \vec{r}' }$	(10.35)
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We check below that these forms in fact solve the inhomogeneous wave equations.

Potential Formulation (cont.)

We would like to act with \square^2 on integrals of the following type:

$$\phi(\vec{r}, t) = \int_V d\tau' \frac{f(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \quad (10.36)$$

Since \square^2 acts on the \vec{r} coordinate, we can pass it through the integral to act on the integrand. The time term of \square^2 is easy:

$$\frac{\partial^2}{\partial t^2} \left[\frac{f(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right] = \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial^2}{\partial t^2} f(\vec{r}', t_r) = \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial^2}{\partial t_r^2} f(\vec{r}', t_r) \quad (10.37)$$

because $\frac{\partial t_r}{\partial t} = 1$. For the space term in \square^2 , we will need $\nabla^2 f(\vec{r}', t_r)$. Calculating this by brute force is difficult because $\vec{\nabla}$ is with respect to \vec{r} but t_r depends on $|\vec{r} - \vec{r}'|$. We'll employ a similar trick as we did when doing such manipulations in electrostatics, which is to define $\vec{s} = \vec{r} - \vec{r}'$ (so that $t_r = t - s/c$) and recognize that

$$\nabla_{\vec{r}}^2 f(\vec{r}', t_r)|_{\vec{r}} = \nabla_{\vec{s}}^2 f(\vec{r}', t_r)|_{\vec{s}=\vec{r}-\vec{r}'} \quad (10.38)$$

because the expression is a scalar and thus cannot depend on the coordinate system origin aside from evaluating it at the correct physical location (the business with the arguments in the subscripts).

Potential Formulation (cont.)

Let's evaluate the easier expression:

$$\nabla_{\vec{s}}^2 f(\vec{r}', t_r) = \frac{1}{s^2} \frac{\partial}{\partial s} \left(s^2 \frac{\partial}{\partial s} f(\vec{r}', t_r) \right) = \frac{1}{s^2} \frac{\partial}{\partial s} \left(s^2 \frac{\partial f}{\partial t_r} \frac{\partial t_r}{\partial s} \right) \quad (10.39)$$

$$= \frac{1}{s^2} \frac{\partial}{\partial s} \left(s^2 \frac{\partial f}{\partial t_r} \left(-\frac{1}{c} \right) \right) = \frac{1}{c^2} \frac{\partial^2 f}{\partial t_r^2} - \frac{2}{s c} \frac{\partial f}{\partial t_r} \quad (10.40)$$

We note that $\partial/\partial s$ did not act on the \vec{r}' dependence of $f(\vec{r}', t_r)$ because the transformation of variables was from (\vec{r}, \vec{r}', t) to (\vec{s}, \vec{r}', t) and so the independence of \vec{r} and \vec{r}' translates to independence of \vec{s} and \vec{r}' . From the above, we obtain the expression we will need below:

$$\frac{\nabla_{\vec{r}}^2 f(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} = \frac{1}{c^2} \frac{\partial^2 f}{\partial t_r^2} \frac{1}{|\vec{r} - \vec{r}'|} - \frac{2}{c} \frac{\partial f}{\partial t_r} \frac{1}{|\vec{r} - \vec{r}'|^2} \quad (10.41)$$

In a similar fashion, again because it is a scalar, we may evaluate an expression we will need below:

$$2 \left[\vec{\nabla}_{\vec{r}} f(\vec{r}', t_r) \right] \cdot \vec{\nabla}_{\vec{r}} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) = 2 \left[\vec{\nabla}_{\vec{s}} f(\vec{r}', t_r) \right] \cdot \vec{\nabla}_{\vec{s}} \left(\frac{1}{s} \right) \quad (10.42)$$

$$= \frac{2}{c s^2} \frac{\partial f}{\partial t_r} = \frac{2}{c} \frac{\partial f}{\partial t_r} \frac{1}{|\vec{r} - \vec{r}'|^2} \quad (10.43)$$

Potential Formulation (cont.)

Now, to proceed with the second term involving ∇^2 . Let's work on it using the product rule (this is a bit cleaner than the way Griffiths does it):

$$\nabla^2 \left[\frac{f(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right] = \left[\vec{\nabla} \cdot \frac{\vec{\nabla} f(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} + \vec{\nabla} \cdot \left(f(\vec{r}', t_r) \vec{\nabla} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \right) \right] \quad (10.44)$$

$$\begin{aligned} &= \left[\frac{\nabla^2 f(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} + f(\vec{r}', t_r) \nabla^2 \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \right. \\ &\quad \left. + 2 \left[\vec{\nabla} f(\vec{r}', t_r) \right] \cdot \vec{\nabla} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \right] \end{aligned} \quad (10.45)$$

$$= \left[\frac{1}{|\vec{r} - \vec{r}'|} \frac{1}{c^2} \frac{\partial^2 f}{\partial t_r^2} + f(\vec{r}', t_r) [4\pi\delta(\vec{r} - \vec{r}')] \right] \quad (10.46)$$

where we plugged in the expressions we derived above for the first and third terms and discarded the cancelling terms, and we used the usual relation between the delta function and the Laplacian of the inverse distance. In the end result we have, we see that the first term will cancel the $(1/c^2) \partial^2/\partial t^2$ term. The second term with the delta function can be easily evaluated (note that doing so gives $t_r = t$, also).

Potential Formulation (cont.)

Combining the two terms and applying the integral, we have

$$\square^2 \phi(\vec{r}, t) = \int_V d\tau' \left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right] \left(\frac{f(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right) \quad (10.47)$$

$$= 4\pi \int_V d\tau' f(\vec{r}', t_r) \delta(\vec{r} - \vec{r}') = 4\pi f(\vec{r}, t) \quad (10.48)$$

Applying this to our proposed expressions for $V(\vec{r}, t)$ and $\vec{A}(\vec{r}, t)$, we obtain

$$\square^2 V(\vec{r}, t) = \frac{\rho(\vec{r}, t)}{\epsilon_0} \quad \square^2 \vec{A}(\vec{r}, t) = \mu_0 \vec{J}(\vec{r}, t) \quad (10.49)$$

as desired.

Potential Formulation (cont.)

We also need to check that the Lorenz gauge condition is satisfied so that these solutions are in the desired gauge. To do so, let's calculate the divergence of the integrand in the expression for \vec{A} . First, we remind you that

$$\vec{\nabla}_{\vec{r}} \frac{1}{|\vec{r} - \vec{r}'|} = -\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} = \frac{\vec{r}' - \vec{r}}{|\vec{r} - \vec{r}'|^3} = -\vec{\nabla}_{\vec{r}'} \frac{1}{|\vec{r} - \vec{r}'|} \quad (10.50)$$

Therefore,

$$\vec{\nabla}_{\vec{r}} \cdot \left(\frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right) = \frac{\vec{\nabla}_{\vec{r}} \cdot \vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} + \vec{J}(\vec{r}', t_r) \cdot \vec{\nabla}_{\vec{r}} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \quad (10.51)$$

$$= \frac{\vec{\nabla}_{\vec{r}} \cdot \vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} - \vec{J}(\vec{r}', t_r) \cdot \vec{\nabla}_{\vec{r}'} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \quad (10.52)$$

$$= \frac{\vec{\nabla}_{\vec{r}} \cdot \vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} - \vec{\nabla}_{\vec{r}'} \cdot \left(\frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right) + \frac{\vec{\nabla}_{\vec{r}'} \cdot \vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \quad (10.53)$$

$$= -\vec{\nabla}_{\vec{r}'} \cdot \left(\frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right) + \frac{1}{|\vec{r} - \vec{r}'|} \left[\frac{\partial \vec{J}}{\partial t_r} \cdot \vec{\nabla}_{\vec{r}} t_r - \frac{\partial \rho}{\partial t} + \frac{\partial \vec{J}}{\partial t_r} \cdot \vec{\nabla}_{\vec{r}'} t_r \right] \quad (10.54)$$

where the continuity equation was used when evaluating $\vec{\nabla}_{\vec{r}'} \cdot \vec{J}(\vec{r}', t_r)$ but not for $\vec{\nabla}_{\vec{r}} \cdot \vec{J}(\vec{r}', t_r)$ because the argument of \vec{J} must match the divergence's variable for continuity to apply. Note that the $\frac{\partial}{\partial t_r}$ are evaluated at (\vec{r}', t_r) .

Potential Formulation (cont.)

$$\text{Now, } \vec{\nabla}_{\vec{r}} t_r = -\frac{1}{c} \vec{\nabla}_{\vec{r}} |\vec{r} - \vec{r}'| = \frac{1}{c} \vec{\nabla}_{\vec{r}'} |\vec{r} - \vec{r}'| = -\vec{\nabla}_{\vec{r}'} t_r \quad (10.55)$$

Therefore, those terms cancel each other and we have

$$\vec{\nabla}_{\vec{r}} \cdot \left(\frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right) = -\vec{\nabla}_{\vec{r}'} \cdot \left(\frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right) - \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial \rho}{\partial t} \quad (10.56)$$

We may now calculate the divergence of \vec{A} :

$$\vec{\nabla} \cdot \vec{A}(\vec{r}, t) = \vec{\nabla} \cdot \frac{\mu_0}{4\pi} \int_V d\tau' \frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \quad (10.57)$$

$$= \frac{\mu_0}{4\pi} \int_V d\tau' \left[-\vec{\nabla}_{\vec{r}'} \cdot \left(\frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right) - \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial \rho}{\partial t} \right] \quad (10.58)$$

$$= -\epsilon_0 \mu_0 \frac{\partial}{\partial t} \frac{1}{4\pi \epsilon_0} \int_V d\tau' \frac{\rho(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} = -\epsilon_0 \mu_0 \frac{\partial}{\partial t} V(\vec{r}, t) \quad (10.59)$$

where the first term has been transformed into a surface integral at infinity that vanishes because the current distribution is assumed to be finite in extent. We thus see the Lorenz condition is satisfied.

Potential Formulation (cont.)

Fields from Retarded Potentials: Jefimenko's Equations

The natural next step is to calculate the fields from the above retarded potentials. This is straightforward using calculations similar to what we did above (easier, in fact, because we are only evaluating first derivatives rather than \square^2). First, let's write down a generic expression for a first derivative:

$$\frac{\partial}{\partial r_i} \left(\frac{f(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right) = \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial f}{\partial t_r} \frac{\partial t_r}{\partial r_i} + f(\vec{r}', t_r) \frac{\partial}{\partial r_i} \left(\frac{1}{|\vec{r} - \vec{r}'|} \right) \quad (10.60)$$

$$= \left[-\frac{1}{c} \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial f}{\partial t_r} - \frac{f(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|^2} \right] \frac{(\vec{r} - \vec{r}') \cdot \hat{r}_i}{|\vec{r} - \vec{r}'|} \quad (10.61)$$

Then, $\vec{E}(\vec{r}, t) = -\vec{\nabla} V(\vec{r}, t) - \frac{\partial}{\partial t} \vec{A}(\vec{r}, t)$ (10.62)

$$= -\frac{1}{4\pi} \int_V d\tau' \left[\frac{1}{\epsilon_0} \vec{\nabla} \left(\frac{\rho(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right) + \mu_0 \frac{\partial}{\partial t} \left(\frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right) \right] \quad (10.63)$$

$$= \frac{1}{4\pi\epsilon_0} \int_V d\tau' \left[\rho(\vec{r}', t_r) \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} + \frac{1}{c} \frac{\partial \rho}{\partial t_r} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^2} - \frac{1}{c^2} \frac{\partial \vec{J}}{\partial t_r} \frac{1}{|\vec{r} - \vec{r}'|} \right] \quad (10.64)$$

where, as usual, ρ and \vec{J} are evaluated at (\vec{r}', t_r) .

Potential Formulation (cont.)

Calculating the magnetic field is straightforward:

$$\vec{B}(\vec{r}, t) = \vec{\nabla}_{\vec{r}} \times \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} d\tau' \vec{\nabla}_{\vec{r}} \times \left(\frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right) \quad (10.65)$$

$$= \frac{\mu_0}{4\pi} \int_{\mathcal{V}} d\tau' \sum_{i,j,k=1}^3 \epsilon_{ijk} \hat{r}_i \frac{\partial}{\partial r_j} \left(\frac{J_k(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|} \right) \quad (10.66)$$

$$= \frac{\mu_0}{4\pi} \int_{\mathcal{V}} d\tau' \sum_{i,j,k=1}^3 \epsilon_{ijk} \hat{r}_i \left[-\frac{1}{c} \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial J_k}{\partial t_r} - \frac{J_k(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|^2} \right] \frac{(\vec{r} - \vec{r}') \cdot \hat{r}_j}{|\vec{r} - \vec{r}'|} \quad (10.67)$$

$$= \frac{\mu_0}{4\pi} \int_{\mathcal{V}} d\tau' \left[\frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|^2} + \frac{1}{c} \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial \vec{J}}{\partial t_r} \right] \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|} \quad (10.68)$$

where we obtained sign flips in the last step by exchanging the j and k indices.

Potential Formulation (cont.)

We thus have Jefimenko's Equations:

$$\vec{E}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int_V d\tau' \left[\rho(\vec{r}', t_r) \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} + \frac{1}{c} \frac{\partial \rho}{\partial t_r} \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^2} - \frac{1}{c^2} \frac{\partial \vec{J}}{\partial t_r} \frac{1}{|\vec{r} - \vec{r}'|^2} \right] \quad (10.69)$$

$$\vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_V d\tau' \left[\frac{\vec{J}(\vec{r}', t_r)}{|\vec{r} - \vec{r}'|^2} + \frac{1}{c} \frac{1}{|\vec{r} - \vec{r}'|} \frac{\partial \vec{J}}{\partial t_r} \right] \times \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|} \quad (10.70)$$

We see that, in both expressions, there is a first term that consists of the standard expression for the field but with the source term evaluated at the retarded time and then a second term that responds to the time dependence of the usual source term. For the electric field, there is a third term that is not so easily explained because it arises from the inherently time dependent $\partial \vec{A}/\partial t$ term. These equations are not incredibly useful because it is usually easier to calculate the retarded potential and differentiate to find the field, but they give a clear physical understanding of how disturbances in the source distributions determine the fields at a distance.

Potential Formulation (cont.)

From Jefimenko's Equations, one sees how the quasistatic approximation can be formally derived. Let τ_s be the characteristic timescale on which a source term varies, as defined by

$$\frac{1}{\tau_s} = \frac{1}{f} \frac{\partial f}{\partial t_r} \quad (10.71)$$

where f is any source term (ρ or a component of \vec{J}). Then the second terms are smaller than the first terms by the factor

$$\alpha = \frac{|\vec{r} - \vec{r}'|}{c} \frac{1}{f} \frac{\partial f}{\partial t_r} = \frac{|\vec{r} - \vec{r}'|/c}{\tau_s} \quad (10.72)$$

which is the ratio of the light travel time to the source term variation timescale. The quasistatic approximation consists of assuming that $\alpha \ll 1$, which justifies neglecting these terms and setting $t_r = t$. The third term in the \vec{E} equation is smaller than the leading terms by a factor of α^2 .

We also note that the second terms in each equation fall off like $1/|\vec{r} - \vec{r}'|$ while the other terms fall off like $1/|\vec{r} - \vec{r}'|^2$, so it is these second terms that dominate at large distances: these are most important for calculating radiation from time-varying sources. We won't do these calculations using the above expressions (we will use the retarded potentials), but it is interesting to recognize this fact.

Lecture 25:
Fields of a Moving Point Charge

Date Revised: 2014/05/27 18:00

Clarified notation for derivatives of \vec{w} and $\vec{\beta}$
with respect to their arguments

Date Given: 2014/05/27

Potentials and Fields of a Moving Point Charge

Introduction and Study Guide

Our practical goal is to get to the point where we can calculate the power radiated by an accelerated point charge. To get there, we need to calculate the potentials of a moving point charge, then the fields, and from the fields the power.

We proceed in a somewhat different manner than Griffiths in Section 10.3.2, who brute-force differentiates the potential to get the field. Like Griffiths, we first calculate the potentials of a moving point charge (valid with or without acceleration) — the Lienard-Wiechert potentials — in terms of the retarded time. Though we do it in a more mathematically rigorous way, followed by a heuristic derivation. Then, rather than directly differentiating them, we rewrite them in terms of the projected current position of the particle. This makes it easier to carry out the differentiations to get the fields. We do this for a fixed-velocity particle first. We can rewrite the result in terms of the retarded time. Then, we return to the Lienard-Wiechert potentials and incorporate the effect of acceleration through possible time-dependence of the velocity, by direct differentiation allowing the velocity to vary, which is much easier now that the derivatives of all the other terms have been taken. Finally, from the fields we calculate the radiated power pattern for the general case, specializing to slowly moving particles at the end to obtain the Larmor Formula.

Potentials and Fields of a Moving Point Charge (cont.)

We calculate the power radiated by an accelerated charge before moving on to the general topic of dipole and multipole radiation. Griffiths proceeds in the opposite order for reasons that are not clear, first doing dipole/multipole radiation in Section 11.1 before doing the radiation of a moving point charge in Section 11.2.1.

Our treatment of the calculation of the fields follows that of M. Cross's Ph106c lecture notes, which do not appear to follow any specific textbook. The treatment of radiation follows that of Heald and Marion Sections 8.7 and 8.8.

We will not cover the topic of radiation reaction (Griffiths 11.2.2 and 11.2.3) — it's interesting, but it is a side topic of primarily academic interest.

Potentials and Fields of a Moving Point Charge (cont.)

Potentials of a Moving Point Charge: Lienard-Wiechert Potentials

Let's consider a point charge q moving with trajectory $\vec{w}(t)$. The charge density is $\rho(\vec{r}', t_r) = q \delta(\vec{r}' - \vec{w}(t_r))$. The scalar potential is, by Equation 10.35

$$V(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \int d\tau' \frac{\delta(\vec{r}' - \vec{w}(t_r))}{|\vec{r} - \vec{r}'|} = \frac{q}{4\pi\epsilon_0} \int d\tau' \frac{\delta\left(\vec{r}' - \vec{w}\left(t - \frac{|\vec{r} - \vec{r}'|}{c}\right)\right)}{|\vec{r} - \vec{r}'|} \quad (10.73)$$

The subtlety here is that the argument of \vec{w} depends on the variable of integration, \vec{r}' . Normally, a delta function has as its argument the variable being integrated over minus a constant so that the integral picks out the value of the integrand at that constant. To resolve this subtlety, let's insert a delta function in time:

$$V(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \int d\tau' \int_{-\infty}^{\infty} dt' \frac{\delta(\vec{r}' - \vec{w}(t'))}{|\vec{r} - \vec{r}'|} \delta\left(t' - t + \frac{|\vec{r} - \vec{r}'|}{c}\right) \quad (10.74)$$

$$= \frac{q}{4\pi\epsilon_0} \int_{-\infty}^{\infty} dt' \frac{\delta\left(t' - t + \frac{|\vec{r} - \vec{w}(t')|}{c}\right)}{|\vec{r} - \vec{w}(t')|} \quad (10.75)$$

We still have a delta function with a subtlety in the argument, but at least now the integral is only over one dimension.

Potentials and Fields of a Moving Point Charge (cont.)

To evaluate the above integral, we note the following property of delta functions

$$\delta(f(x)) = \sum_i \delta(x - z_i) \left| \frac{df}{dx} \right|_{z_i}^{-1} \quad (10.76)$$

where the z_i are all the zeros of $f(x)$, $f(z_i) = 0$. That this expression is correct can be seen by doing an integral over an interval in x containing z_i , Taylor expanding $f(x)$ around z_i , and doing a change of variables:

$$\int_{z_i-\epsilon}^{z_i+\epsilon} dx \delta(f(x)) = \int_{z_i-\epsilon}^{z_i+\epsilon} dx \delta\left((x - z_i) \left| \frac{df}{dx} \right|_{z_i}\right) = \frac{\int_{u_i-\epsilon_u}^{u_i+\epsilon_u} du \delta(u - u_i)}{\left| \frac{df}{dx} \right|_{z_i}} \quad (10.77)$$

$$\text{with } u = x \left| \frac{df}{dx} \right|_{z_i} \quad u_i = z_i \left| \frac{df}{dx} \right|_{z_i} \quad \epsilon_u = \epsilon \left| \frac{df}{dx} \right|_{z_i} \quad (10.78)$$

where we have used the absolute value of the derivative to avoid sign flip issues (direction of integration stays the same). The above holds at every zero z_i , so any interval that includes a particular zero receives a term of the above type. Thus the desired property is proven.

Potentials and Fields of a Moving Point Charge (cont.)

With the above in hand, let us rewrite the delta function. First, we need the derivative with respect to the variable of integration of the argument of the delta function:

$$\frac{\partial}{\partial t'} \left(t' - t + \frac{|\vec{r} - \vec{w}(t')|}{c} \right) = 1 + \frac{1}{c} \frac{\vec{r} - \vec{w}(t')}{|\vec{r} - \vec{w}(t')|} \cdot \vec{w}'(t') \quad (10.79)$$

We use \vec{w}' to indicate the derivative of \vec{w} with respect to its argument so that there is no confusion about whether we are taking a derivative with respect to t or t_r . We then indicate that it should be evaluated at t' by the notation $\vec{w}'(t')$. We need to evaluate the above quantity at the zero of the aforementioned function, which is at

$$t' - t + \frac{|\vec{r} - \vec{w}(t')|}{c} = 0 \iff t' = t_r \quad (10.80)$$

The integral for the voltage thus becomes

$$V(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \int_{-\infty}^{\infty} dt' \frac{\delta(t' - t_r)}{1 - \frac{1}{c} \frac{\vec{r} - \vec{w}(t')}{|\vec{r} - \vec{w}(t')|} \cdot \vec{w}'(t')} \frac{1}{|\vec{r} - \vec{w}(t')|} \quad (10.81)$$

$$= \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{w}(t_r)|} \frac{1}{1 - \vec{\beta}(t_r) \cdot \frac{\vec{r} - \vec{w}(t_r)}{|\vec{r} - \vec{w}(t_r)|}} \quad (10.82)$$

$$\text{with } \vec{\beta}(t_r) = \frac{\vec{w}'(t_r)}{c} \quad (10.83)$$

Potentials and Fields of a Moving Point Charge (cont.)

Let's define the *retarded relative position vector*

$$\vec{R}_r(\vec{r}, t) = \vec{r} - \vec{w}(t_r) = \vec{r} - \vec{w}\left(t - \frac{|\vec{R}_r(\vec{r}, t)|}{c}\right) \quad (10.84)$$

Notice the implicitness inherent in the above: \vec{R}_r depends on t_r , but t_r depends on \vec{R}_r . We'll deal with this later.

Using the above definition, rewriting the above, and performing a similar analysis on each component of the vector potential, we obtain the *Lienard-Wiechert potentials*:

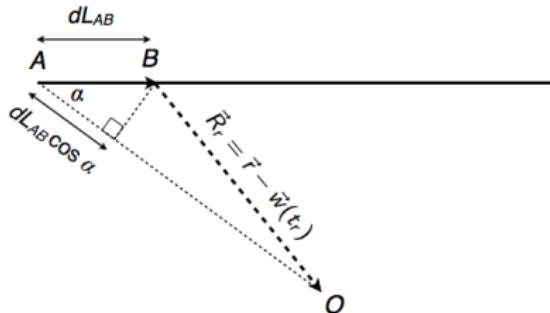
Lienard-Wiechert scalar potential	$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{R_r(\vec{r}, t)} \frac{1}{1 - \vec{\beta}(t_r) \cdot \hat{R}_r(\vec{r}, t)}$	(10.85)
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Lienard-Wiechert vector potential	$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{q c \vec{\beta}(t_r)}{R_r(\vec{r}, t)} \frac{1}{1 - \vec{\beta}(t_r) \cdot \hat{R}_r(\vec{r}, t)}$	(10.86)
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We see that these potentials are what one might expect based on the retarded time, but with a correction factor related to the angle between the direction of motion and the position vector between the moving particle and the observer. Note that, while $\vec{\beta} \cdot \hat{R}_r < 0$ is possible, the correction factor is always nonnegative because $\beta < 1$. Note also that the potentials are stronger than the steady-state case in the half-space ahead of the particle and weaker in the half-space behind it, with the steady-state value obtained on the boundary.

Potentials and Fields of a Moving Point Charge (cont.)

The term in the denominator is related to the finite travel time of light and the fact that the particle is moving. To explain this, consider a particle moving on a straight line path between the points A and B , separated by an infinitesimal length dL_{AB} and with A further than B from the observer at O , as in the following diagram:



The length dL_{AB} is exaggerated for visibility. The particle takes a time $dt_{AB} = dL_{AB}/c\beta$ to travel from A to B . Because A is farther from O than B , the information about the particle's charge and speed at point A has to travel a larger distance than the information at point B by an amount

$$dL = dL_{AB} \cos \alpha = dL_{AB} \hat{\beta} \cdot \hat{R}_r = c dt_{AB} \hat{\beta} \cdot \hat{R}_r \quad (10.87)$$

where $\vec{R}_r(t) = \vec{r} - \vec{w}(t_r)$ is the vector from the particle's position at t_r to the observer at \vec{r} , $\hat{R}_r = \vec{R}_r/|\vec{R}_r|$, and $\hat{\beta} = \vec{\beta}/\beta$.

Potentials and Fields of a Moving Point Charge (cont.)

Thus, rather than the information from the two points arriving at the observer separated by a time dt_{AB} , the information arrives with separation by a shorter time

$$dt'_{AB} = dt_{AB} - \frac{dL}{c} = dt_{AB} \left(1 - \vec{\beta} \cdot \hat{R}_r\right) \quad (10.88)$$

One can think this factor taking effect via a “piling up” effect. In the case of a stationary charge, the information about the particle’s position and charge being radiated outward reaches the observer with the same time separation as it is emitted: two bits of information emitted with time separation dt arrive at the observer with time separation dt . At this arrival frequency, the potential strength is maintained at the distant point at its electrostatic value. But, when the charge is moving, the arrival time separation is reduced by the factor dt'_{AB}/dt_{AB} , resulting in an intensification of the potential by the inverse of that factor:

$$\text{electrodynamic potential} = \text{electrostatic potential} \times \left(\frac{dt_{AB}}{dt'_{AB}}\right) \quad (10.89)$$

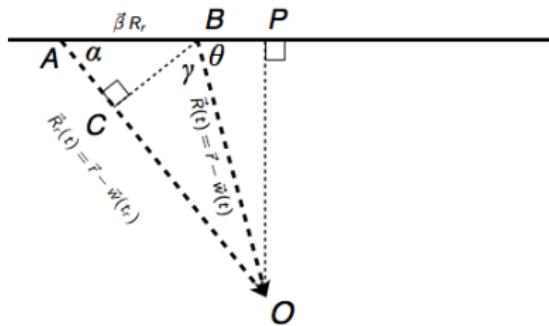
$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{R_r} \frac{1}{1 - \vec{\beta} \cdot \hat{R}_r} \quad (10.90)$$

as we derived more formally above. The dt_{AB}/dt'_{AB} can be interpreted as the Jacobian change-of-variables factor needed to go from time system to the other.

Potentials and Fields of a Moving Point Charge (cont.)

Lienard-Wiechert Potentials in Terms of Current Position for Uniform Motion

We may write this another way. We make use of the following figure and assume $\vec{\beta}$ is constant:



The point O is the position \vec{r} at which we want to calculate the potential. The points A and B are the position of the particle at the retarded time t_r and the current time t . The vectors AO and BO are given by

$$AO : \quad \vec{R}_r(t) = \vec{r} - \vec{w}(t_r) \qquad \qquad BO : \quad \vec{R}(t) = \vec{r} - \vec{w}(t) \quad (10.91)$$

where we have defined $\vec{R}(t)$ consistently with Griffiths.

Potentials and Fields of a Moving Point Charge (cont.)

The denominator of Equation 10.85 may be rewritten

$$R_r \left(1 - \vec{\beta} \cdot \hat{R}_r\right) = R_r - \vec{\beta} R_r \cdot \hat{R}_r = |AO| - |AC| = |CO| \quad (10.92)$$

which is obtained as follows. The first term is the length of the vector AO . The first factor in the second term is a vector along the particle's velocity vector $\vec{v} = c \vec{\beta}$, which points from A to B . Its length is βR_r , which is the distance the particle travels in the time that light propagates from A to O . Therefore, the first factor is the vector AB . The dot product with the second factor projects AB along AO , giving the length AC . Thus, the difference of the lengths $|AO|$ and $|AC|$ given by the above expression above is the length $|CO|$. From the figure, we also see

$$|CO| = |BO| \sin \gamma = R \sqrt{1 - \cos^2 \gamma} \quad (10.93)$$

We can relate γ to α and θ :

$$\gamma = \pi - \left(\theta + \left(\frac{\pi}{2} - \alpha\right)\right) = \frac{\pi}{2} - (\theta - \alpha) \quad (10.94)$$

$$\implies \cos \gamma = \cos \left[\frac{\pi}{2} - (\theta - \alpha)\right] = \sin(\theta - \alpha) = \sin \theta \cos \alpha - \cos \theta \sin \alpha \quad (10.95)$$

Potentials and Fields of a Moving Point Charge (cont.)

Now, from the figure,

$$R_r \sin \alpha = R \sin \theta \quad (10.96)$$

because the right triangles with \vec{R}_r and \vec{R} as hypotenuses share the same vertical side, PO . We also have

$$|AP| = |AB| + |BP| \quad (10.97)$$

$$\implies R_r \cos \alpha = \beta R_r + R \cos \theta \quad (10.98)$$

We use the above two relations to substitute for $\cos \alpha$ and $\sin \alpha$ in the expression for $\cos \gamma$:

$$\cos \gamma = \sin \theta \left[\sin \theta \frac{\beta R_r + R \cos \theta}{R_r} - \cos \theta \frac{R \sin \theta}{R_r} \right] = \beta \sin \theta \quad (10.99)$$

Thus, we can rewrite the denominator of Equation 10.85 as

$$R_r \left(1 - \vec{\beta} \cdot \hat{R}_r \right) = R \sqrt{1 - \beta^2 \sin^2 \theta} \quad (10.100)$$

Potentials and Fields of a Moving Point Charge (cont.)

Inserting this into Equations 10.85 and 10.86, we obtain

Lienard-Wiechert scalar potential using current position	$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{R(\vec{r}, t)} \frac{1}{\sqrt{1 - \beta^2 \sin^2 \theta}}$	(10.101)
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Lienard-Wiechert vector potential using current position	$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{q c \vec{\beta}}{R(\vec{r}, t)} \frac{1}{\sqrt{1 - \beta^2 \sin^2 \theta}}$	(10.102)
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These are versions of the Lienard-Wiechert potentials that use the *current position* of the particle. It has been assumed that the particle has constant velocity between t_r and t . The $\beta \rightarrow 0$ limit is also clear here: the correction term becomes unity.

The above form is quite useful because the retarded time no longer appears. The retarded time complicates calculation of spatial derivatives because t_r depends on \vec{r} . With the above dependence on the current time, the derivatives are easier to calculate.

Note how the potentials are now forward-backward symmetric in magnitude, yet they were not when written in terms of the retarded time – strange! Also, they are now enhanced along the $\theta = \pi/2$ direction and return to the $\beta = 0$ limit at $\theta = 0$ or π , along the particle's trajectory.

Potentials and Fields of a Moving Point Charge (cont.)

Fields of a Uniformly Moving Point Charge

With the potentials written in terms of the position of the particle at the current time $\vec{w}(t)$, it is now straightforward to calculate the fields from the potentials. We *first do this with $\vec{\beta}$ assumed to be fixed*. The calculation can be simplified by writing everything in terms of $\vec{R} = \vec{r} - \vec{w}(t)$ because we can write both the space and time derivatives in terms of derivatives with respect to \vec{R} :

$$\frac{\partial}{\partial r_i} = \frac{\partial R_i}{\partial r_i} \frac{\partial}{\partial R_i} = \frac{\partial}{\partial R_i} \quad \Rightarrow \quad \vec{\nabla}_{\vec{r}} = \vec{\nabla}_{\vec{R}} \quad (10.103)$$

$$\frac{\partial}{\partial t} = \frac{\partial R_i}{\partial t} \frac{\partial}{\partial R_i} = -w'_i \frac{\partial}{\partial R_i} = -c \vec{\beta} \cdot \vec{\nabla}_{\vec{R}} \quad (10.104)$$

(Recall, w'_i is the derivative of w_i with respect to its argument.) So, let's write the position dependence more clearly: there is dependence on \vec{R} both explicitly and via θ . Letting $\vec{R} = R_{||} \hat{\beta} + \vec{R}_{\perp}$, we have

$$|\vec{r} - \vec{w}(t)|^2 (1 - \beta^2 \sin^2 \theta) = R^2 \left(1 - \beta^2 \frac{R_{\perp}^2}{R^2} \right) = R_{||}^2 + (1 - \beta^2) R_{\perp}^2 \quad (10.105)$$

Potentials and Fields of a Moving Point Charge (cont.)

With the above, we can calculate the electric field (assuming $\vec{\beta}$ fixed):

$$\vec{E}(\vec{r}, t) = -\vec{\nabla}_{\vec{r}} V - \frac{\partial \vec{A}}{\partial t} \quad (10.106)$$

$$= \frac{q}{4\pi\epsilon_0} \left[-\vec{\nabla}_{\vec{R}} + \epsilon_0 \mu_0 c^2 \vec{\beta} \vec{\beta} \cdot \vec{\nabla}_{\vec{R}} \right] \frac{1}{\sqrt{R_{||}^2 + (1 - \beta^2) R_{\perp}^2}} \quad (10.107)$$

(The $\epsilon_0 \mu_0$ factor arises because \vec{A} is proportional to $\mu_0/4\pi$, not $1/4\pi\epsilon_0$.) We need to treat the components parallel and perpendicular to $\vec{\beta}$ separately (and use $\epsilon_0 \mu_0 c^2 = 1$):

$$E_{||} = -\frac{q}{4\pi\epsilon_0} [1 - \beta^2] \frac{\partial}{\partial R_{||}} \frac{1}{\sqrt{R_{||}^2 + (1 - \beta^2) R_{\perp}^2}} \quad (10.108)$$

$$= -\frac{q}{4\pi\epsilon_0} [1 - \beta^2] \frac{-R_{||}}{\left[R_{||}^2 + (1 - \beta^2) R_{\perp}^2 \right]^{3/2}} \quad (10.109)$$

$$\vec{E}_{\perp} = -\frac{q}{4\pi\epsilon_0} \vec{\nabla}_{\vec{R}_{\perp}} \frac{1}{\sqrt{R_{||}^2 + (1 - \beta^2) R_{\perp}^2}} \quad (10.110)$$

$$= -\frac{q}{4\pi\epsilon_0} \frac{-(1 - \beta^2) \vec{R}_{\perp}}{\left[R_{||}^2 + (1 - \beta^2) R_{\perp}^2 \right]^{3/2}} \quad (10.111)$$

Potentials and Fields of a Moving Point Charge (cont.)

Therefore,

$$\vec{E} = E_{||} \hat{\beta} + \vec{E}_{\perp} = \frac{q}{4\pi\epsilon_0} \frac{\vec{R}}{R^3} \frac{1 - \beta^2}{[1 - \beta^2 \sin^2 \theta]^{3/2}} \quad (10.112)$$

For the magnetic field, we have, using the identity $\vec{\nabla} \times f \vec{a} = f \vec{\nabla} \times \vec{a} - (\vec{a} \times \vec{\nabla})f$ along with the fact that $\vec{\beta}$ is a constant and $\vec{\beta} \times R_{||} \hat{\beta} = 0$:

$$\vec{B} = \vec{\nabla}_{\vec{r}} \times \vec{A} = \vec{\nabla}_{\vec{R}} \times \frac{\mu_0}{4\pi} \frac{q c \vec{\beta}}{\sqrt{R_{||}^2 + (1 - \beta^2) R_{\perp}^2}} \quad (10.113)$$

$$= -\frac{1}{c} \frac{q}{4\pi\epsilon_0} \vec{\beta} \times \vec{\nabla}_{\vec{R}} \frac{1}{\sqrt{R_{||}^2 + (1 - \beta^2) R_{\perp}^2}} \quad (10.114)$$

$$= \frac{1}{c} \frac{q}{4\pi\epsilon_0} \frac{\vec{\beta} \times \vec{R}}{R^3} \frac{1 - \beta^2}{[1 - \beta^2 \sin^2 \theta]^{3/2}} \quad (10.115)$$

$$= \frac{1}{c} \vec{\beta} \times \vec{E} \quad (10.116)$$

Potentials and Fields of a Moving Point Charge (cont.)

Summarizing, we obtain Griffiths Equation 10.75 and part of Equation 10.76

Electric field
of moving point charge
using current position

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{\vec{R}}{R^3} \frac{1 - \beta^2}{[1 - \beta^2 \sin^2 \theta]^{3/2}} \quad (10.117)$$

Magnetic field
of moving point charge
using current position

$$\vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{q c \vec{\beta} \times \vec{R}}{R^3} \frac{1 - \beta^2}{[1 - \beta^2 \sin^2 \theta]^{3/2}} \quad (10.118)$$

Relation between electric and
magnetic fields of moving point charge
using current position

$$\vec{B} = \frac{1}{c} \vec{\beta} \times \vec{E} \quad (10.119)$$

When written in terms of the particle's *current* position, the fields have radial dependence similar to the $\beta = 0$ versions, but there are factors that impose additional angular and β dependence. The field is enhanced by a factor $1/\sqrt{1 - \beta^2}$ (the Lorentz factor of the particle, γ) in the plane transverse to the direction of motion (at $\theta = \pi/2$) and reduced by a factor $1 - \beta^2$ along the axis of motion ($\theta = 0$ or $\theta = \pi$). Note that these enhancements are consistent with those determined for the current position potentials but not with those determined for the retarded position potentials. Otherwise, the dependences on q , $c\beta$, and \vec{R} are unchanged from the $\beta = 0$ limit.

We also note that, even if the particle has stopped between t_r and t , the field at \vec{r} acts as if the particle has continued moving and has reached the "current position."

Potentials and Fields of a Moving Point Charge (cont.)

We can use the above form to obtain the fields in terms of the retarded position. Recall the relation that we used to convert the Lienard-Wiechert potentials from retarded position to current position:

$$R \sqrt{1 - \beta^2 \sin^2 \theta} = R_r \left(1 - \vec{\beta} \cdot \hat{R}_r \right) \quad (10.120)$$

Revisiting the geometry, we also have

$$\vec{R} = \vec{R}_r - \vec{\beta} R_r = R_r \left(\hat{R}_r - \vec{\beta} \right) \quad (10.121)$$

In addition, we note that, because $\vec{E} \propto \vec{R} = R_r(\hat{R}_r - \vec{\beta})$, then $(\hat{R}_r - \vec{\beta}) \times \vec{E} = 0$ and therefore

$$\vec{B} = \frac{1}{c} \vec{\beta} \times \vec{E} = \frac{1}{c} \left[\vec{\beta} + (\hat{R}_r - \vec{\beta}) \right] \times \vec{E} = \frac{1}{c} \hat{R}_r \times \vec{E} \quad (10.122)$$

Potentials and Fields of a Moving Point Charge (cont.)

Inserting the above relations into the equations for the fields in terms of the current position, we have

Electric field
of moving point charge
using retarded position

$$\vec{E}(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{(\vec{R}_r - \vec{\beta} R_r)}{R_r^3} \frac{(1 - \beta^2)}{\left[1 - \vec{\beta} \cdot \hat{R}_r\right]^3}$$

(10.123)

Magnetic field
of moving point charge
using retarded position

$$\vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{q c \vec{\beta} \times \vec{R}_r}{R_r^3} \frac{(1 - \beta^2)}{\left[1 - \vec{\beta} \cdot \hat{R}_r\right]^3}$$

(10.124)

Relation between electric and
magnetic fields of moving point charge
using retarded position

$$\vec{B} = \frac{1}{c} \vec{\beta} \times \vec{E} = \frac{1}{c} \hat{R}_r \times \vec{E}$$

(10.125)

These equations match Griffiths Equations 10.72 and 10.73 with zero acceleration ($\vec{a} = 0$).

We note that these fields, unlike the current position fields and potentials, exhibit the same forward-backward asymmetry as the retarded position potentials. There is also global reduction from the $\beta = 0$ case by the factor $1 - \beta^2 < 1$.

Potentials and Fields of a Moving Point Charge (cont.)

Fields of an Accelerating Point Charge

In calculating the fields from the Lienard-Wiechert potentials, we assumed the charge was moving uniformly — that $\vec{\beta}$ was constant — by passing it through all derivatives. Let's now drop this assumption. Since our geometrical derivations assumed that $\vec{\beta}$ was the velocity vector at the retarded time, we will replace $\vec{\beta}$ with $\vec{\beta}(t_r)$ in all our prior expressions.

We will find that it is straightforward to use the retarded position expressions for the Lienard-Wiechert potentials.

Potentials and Fields of a Moving Point Charge (cont.)

First, let's consider the partial derivatives of $\vec{\beta}(t_r)$. Evaluating these is difficult because of the somewhat circular dependence of $\vec{w}(t_r)$ and t_r . We start with the definition of t_r , $t_r = t - |\vec{r} - \vec{w}(t_r)|/c$, rewriting it as

$$c^2(t - t_r)^2 = (\vec{r} - \vec{w}(t_r)) \cdot (\vec{r} - \vec{w}(t_r)) \quad (10.126)$$

Let's take the partial derivatives of both sides. First, with respect to t holding \vec{r} fixed, and recalling that $\vec{\beta} = \vec{w}'/c$,

$$2c^2(t - t_r) \left(1 - \frac{\partial t_r}{\partial t} \Big|_{\vec{r}} \right) = 2(\vec{r} - \vec{w}(t_r)) \cdot (-\vec{w}'(t_r)) \frac{\partial t_r}{\partial t} \Big|_{\vec{r}} \quad (10.127)$$

$$- \left[cR_r - \vec{R}_r \cdot c\vec{\beta}(t_r) \right] \frac{\partial t_r}{\partial t} \Big|_{\vec{r}} = -cR_r \quad (10.128)$$

$$\frac{\partial t_r}{\partial t} \Big|_{\vec{r}} = \frac{1}{1 - \vec{\beta}(t_r) \cdot \hat{R}_r} \quad (10.129)$$

Potentials and Fields of a Moving Point Charge (cont.)

We repeat the same procedure with $\partial/\partial r_i$, holding t fixed:

$$-2c^2(t - t_r) \frac{\partial t_r}{\partial r_i} \Big|_t = 2 \left[(r_i - w_i(t_r)) + (\vec{r} - \vec{w}(t_r)) \cdot (-\vec{w}'(t_r)) \frac{\partial t_r}{\partial r_i} \Big|_{t_r} \right] \quad (10.130)$$

$$- \left[cR_r - \vec{R}_r \cdot c\vec{\beta}(t_r) \right] \frac{\partial t_r}{\partial r_i} \Big|_t = R_{r,i} \quad (10.131)$$

$$\frac{\partial t_r}{\partial r_i} \Big|_t = -\frac{1}{c} \frac{R_{r,i}/R_r}{1 - \vec{\beta}(t_r) \cdot \hat{R}_r} \quad (10.132)$$

$$\Rightarrow \vec{\nabla}_{\vec{r}} t_r \Big|_t = -\frac{1}{c} \frac{\hat{R}_r}{1 - \vec{\beta}(t_r) \cdot \hat{R}_r} \quad (10.133)$$

Finally, we will actually be taking derivatives of $\vec{\beta}$, so we note, defining $\vec{\beta}' = \vec{w}''$ as the first derivative of $\vec{\beta}$ or second derivative of \vec{w} with respect to its argument:

$$\frac{\partial \vec{\beta}}{\partial t} \Big|_{\vec{r}} = \vec{\beta}'(t_r) \frac{\partial t_r}{\partial t} \quad \frac{\partial \beta_i}{\partial r_j} \Big|_t = \beta'_i(t_r) \frac{\partial t_r}{\partial r_j} \Big|_t \Rightarrow \vec{\nabla}_{\vec{r}} \vec{\beta} \Big|_t = \vec{\beta}'(t_r) \vec{\nabla}_{\vec{r}} t_r \Big|_t \quad (10.134)$$

where the partial derivatives of t_r are what we just calculated and the rightmost quantity is a tensor (an inner product will be taken with it to recover a vector).

Potentials and Fields of a Moving Point Charge (cont.)

Next, to calculate \vec{E} , we start from Equations 10.85 and 10.86 but now allow $\vec{\beta}$ to be differentiated also. We do not begin with Equations 10.108 and 10.110, even though that's how we did the calculation before, because the dependence on $\vec{\beta}$ in the first set of equations is simpler, and we know that the fixed- $\vec{\beta}$ calculation takes care of everything except the variation in $\vec{\beta}$. We only show the new terms, understanding that $\vec{\beta}$ and $\vec{\beta}'$ are evaluated at t_r :

$$-\vec{\nabla}_{\vec{r}} V = -\vec{\nabla}_{\vec{r}} V \Big|_{\vec{\beta}} - \frac{1}{4\pi\epsilon_0} \frac{q}{R_r} \frac{-1}{[1 - \vec{\beta} \cdot \hat{R}_r]^2} \left(-\vec{\nabla}_{\vec{r}} [\vec{\beta} \cdot \hat{R}_r] \Big|_{t_r} \right) \quad (10.135)$$

$$= -\vec{\nabla}_{\vec{r}} V \Big|_{\vec{\beta}} + \frac{q}{4\pi\epsilon_0} \frac{\hat{R}_r}{R_r} \frac{1}{[1 - \vec{\beta} \cdot \hat{R}_r]^3} \frac{\hat{R}_r}{c} \cdot \vec{\beta}' \quad (10.136)$$

$$-\frac{\partial \vec{A}}{\partial t} = -\frac{\partial \vec{A}}{\partial t} \Big|_{\vec{\beta}} - \frac{\mu_0}{4\pi} \frac{q c}{R_r} \left[\frac{-\vec{\beta}}{[1 - \vec{\beta} \cdot \hat{R}_r]^2} \left(-\frac{\partial}{\partial t} [\vec{\beta} \cdot \hat{R}_r] \Big|_{\vec{r}} \right) + \frac{1}{1 - \vec{\beta} \cdot \hat{R}_r} \frac{\partial \vec{\beta}}{\partial t} \Big|_{\vec{r}} \right] \quad (10.137)$$

$$= -\frac{\partial \vec{A}}{\partial t} \Big|_{\vec{\beta}} - \frac{\mu_0}{4\pi} \frac{q c}{R_r} \frac{1}{[1 - \vec{\beta} \cdot \hat{R}_r]^3} \left[(1 - \vec{\beta} \cdot \hat{R}_r) \vec{\beta}' + \vec{\beta} (\hat{R}_r \cdot \vec{\beta}') \right] \quad (10.138)$$

where we used our formulae from the previous page for $\partial \vec{\beta} / \partial t \Big|_{\vec{r}}$ in the last step.

Potentials and Fields of a Moving Point Charge (cont.)

We can combine the three terms, recognizing that $\epsilon_0 \mu_0 = 1/c^2$, using the *BAC – CAB* rule, and adding powers of R_r to turn \hat{R}_r into \vec{R}_r :

$$\vec{E} = \vec{E}\Big|_{\vec{\beta}} + \frac{q}{4\pi\epsilon_0} \frac{1}{c R_r} \frac{1}{\left[1 - \vec{\beta} \cdot \hat{R}_r\right]^3} \left[(\hat{R}_r - \vec{\beta}) (\hat{R}_r \cdot \vec{\beta}') + (1 - \vec{\beta} \cdot \hat{R}_r) \vec{\beta}' \right] \quad (10.139)$$

$$\vec{E}(\vec{r}, t) = \vec{E}(\vec{r}, t)\Big|_{\vec{\beta}} + \frac{q}{4\pi\epsilon_0} \frac{1}{R_r^3} \frac{1}{c} \frac{\vec{R}_r \times \left[(\vec{R}_r - \vec{\beta} R_r) \times \vec{\beta}' \right]}{\left[1 - \vec{\beta} \cdot \hat{R}_r\right]^3} \quad \begin{matrix} \text{Electric field} \\ \text{of accelerated} \\ \text{point charge} \\ \text{using retarded} \\ \text{position} \end{matrix} \quad (10.140)$$

($\vec{\beta}$ and $\vec{\beta}'$ are evaluated at t_r .) We see that the expression is very similar to the fixed- $\vec{\beta}$ version, Equation 10.123, except that $(\vec{R}_r - \vec{\beta} R_r)(1 - \beta^2)$ is replaced with the triple cross product. We also see that this term is smaller than the fixed- $\vec{\beta}$ term by a factor $(R_r/c) \vec{\beta}'$, which is sensible: this factor is the ratio of the light travel time to the characteristic acceleration timescale, where the latter is normalized to c , not to $c\beta$.

Potentials and Fields of a Moving Point Charge (cont.)

Similarly,

$$\vec{B} = \vec{B}\Big|_{\vec{\beta}} + \frac{\mu_0}{4\pi} \frac{q c}{R_r} \left[\frac{\vec{\nabla}_{\vec{r}} \times \vec{\beta}}{1 - \vec{\beta} \cdot \hat{R}_r} \Big|_t - \vec{\beta} \times \frac{\vec{\nabla}_{\vec{r}} [\hat{R}_r \cdot \vec{\beta}] \Big|_t}{[1 - \vec{\beta} \cdot \hat{R}_r]^2} \right] \quad (10.141)$$

$$= \vec{B}\Big|_{\vec{\beta}} + \frac{\mu_0}{4\pi} \frac{q c}{R_r} \left[-\frac{1}{c} \frac{\hat{R}_r \times \vec{\beta}'}{[1 - \vec{\beta} \cdot \hat{R}_r]^2} + \frac{\vec{\beta} \times \hat{R}_r}{[1 - \vec{\beta} \cdot \hat{R}_r]^3} \frac{\hat{R}_r}{c} \cdot \vec{\beta}' \right] \quad (10.142)$$

$$\boxed{\vec{B}(\vec{r}, t) = \vec{B}(\vec{r}, t)\Big|_{\vec{\beta}} + \frac{\mu_0}{4\pi} \frac{q c}{R_r^3} \frac{1}{c} \frac{(R_r - \vec{\beta} \cdot \vec{R}_r) \vec{\beta}' \times \vec{R}_r + (\vec{R}_r \cdot \vec{\beta}') \vec{\beta} \times \vec{R}_r}{[1 - \vec{\beta} \cdot \hat{R}_r]^3}} \quad (10.143)$$

$$\boxed{\vec{B} = \frac{1}{c} \hat{R}_r \times \vec{E} \neq \frac{1}{c} \vec{\beta} \times \vec{E} \quad \text{Magnetic field of accelerated point charge using retarded position}} \quad (10.144)$$

($\vec{\beta}$ and $\vec{\beta}'$ are evaluated at t_r .) As with \vec{E} , we see a form similar to the fixed- $\vec{\beta}$ case with replacement of $\vec{\beta} \times \vec{R}_r(1 - \beta^2)$ by the complicated expression in the numerator. This term is also smaller than the fixed- $\vec{\beta}$ term by a factor $(R_r/c) \vec{\beta}'$. Only one of the relations between \vec{E} and \vec{B} is preserved.

Lecture 26:
Power Radiated by a Moving Point Charge
Dipole Radiation

Date Revised: 2014/05/29 14:00

Correct erroneous R^2 factors in various power formulae

2014/06/05 Correct Equation 10.152 to match what text says

2014/06/07 Modify text around Equation 10.152 to clarify

Date Given: 2014/05/29

Potentials and Fields of a Moving Point Charge

Power Radiated by a Point Charge

With the above detailed formulae for the fields due to an accelerated charge, we can make our first study of *radiation*, which consists of the self-propagating fields created by accelerated charges. We will of course calculate the power radiated using the Poynting vector, $\vec{S} = \vec{E} \times \vec{B}/\mu_0$ (no need for complex conjugations or taking real parts because all fields are real here; also, we don't time average because we are not yet considering sinusoidal behavior). We can quickly see that only the terms involving the acceleration $\vec{a} = c \vec{\beta}'$ are important at large distances by extracting the dependences of various terms from the full expressions:

$$\vec{E}\Big|_{\vec{\beta}} : \quad \frac{q}{\epsilon_0} \frac{1}{\gamma^2} \frac{1}{R_r^2} \quad \vec{B}\Big|_{\vec{\beta}} : \quad \frac{q}{\epsilon_0} \frac{1}{c} \frac{\beta}{\gamma^2} \frac{1}{R_r^2} \quad (10.145)$$

$$\vec{E} - \vec{E}\Big|_{\vec{\beta}} : \quad \frac{q}{\epsilon_0} \vec{\beta}' \frac{1}{R_r} \quad \vec{B} - \vec{B}\Big|_{\vec{\beta}} : \quad \frac{q}{\epsilon_0} (1 + \beta) \vec{\beta}' \frac{1}{R_r} \quad (10.146)$$

The $|_{\vec{\beta}}$ terms are called the *velocity* terms and the others are called the *acceleration* terms. Only the latter are important at large distances, the so-called *far field*.

Potentials and Fields of a Moving Point Charge (cont.)

We note two important facts about the fields:

- ▶ The acceleration electric field is perpendicular to \hat{R}_r (because it is the result of a cross-product including \hat{R}_r), which points along the line-of-sight from the retarded position of the point charge to the point at which we want to know the fields. (Let's use the term "field point" for this point, \vec{r} .) This may be easier to visualize if you imagine looking at the retarded position from the perspective of the field point: \hat{R}_r is the vector pointing at you along that line-of-sight.
- ▶ The acceleration magnetic field is perpendicular to both this line-of-sight and the acceleration electric field (because the magnetic field is proportional to the cross product of \hat{R}_r and the electric field).

Let's now calculate the Poynting vector:

$$\vec{S} \approx \frac{1}{\mu_0} \left(\vec{E} - \vec{E}|_{\vec{\beta}} \right) \times \left(\vec{B} - \vec{B}|_{\vec{\beta}} \right) = \frac{1}{2\mu_0} \left(\vec{E} - \vec{E}|_{\vec{\beta}} \right) \times \frac{1}{c} \left[\hat{R}_r \times \left(\vec{E} - \vec{E}|_{\vec{\beta}} \right) \right]$$

$$\boxed{\vec{S} \approx \frac{1}{c\mu_0} \hat{R}_r \left| \vec{E} - \vec{E}|_{\vec{\beta}} \right|^2}$$

(10.147)

where the last step used the *BAC – CAB* rule and $\hat{R}_r \cdot (\vec{E} - \vec{E}|_{\vec{\beta}}) = 0$. We see that the Poynting vector is along the line-of-sight from the retarded position to the field point.

We'll consider the cases of the acceleration parallel and perpendicular to the velocity separately and then add the two.

Potentials and Fields of a Moving Point Charge (cont.)

Let's first do the parallel acceleration case. A standard application of this case is to calculate *bremsstrahlung* ("braking radiation"), the radiation an electron gives off as it is decelerated by interaction with matter (primarily the Coulomb force from the positive charge of nuclei.) We set $\vec{\beta} \times \vec{\beta}' = 0$. The relevant piece of Equation 10.140 is

$$\vec{E}_{\vec{\beta}', \parallel \vec{\beta}} = \frac{q}{4\pi\epsilon_0} \frac{1}{R_r} \frac{\hat{R}_r \times (\hat{R}_r \times \frac{1}{c} \vec{\beta}')}{{[1 - \vec{\beta} \cdot \hat{R}_r]}^3} = -\frac{q}{4\pi\epsilon_0} \frac{1}{R_r} \frac{\frac{1}{c} \vec{\beta}'_\perp}{{[1 - \vec{\beta} \cdot \hat{R}_r]}^3} \quad (10.148)$$

with $\vec{\beta}'_\perp = \vec{\beta}' - \hat{R}_r (\hat{R}_r \cdot \vec{\beta}')$ acceleration perpendicular to line-of-sight from the retarded position (\hat{R}_r) (10.149)

or $\vec{a}_\perp = \vec{a} - \hat{R}_r (\hat{R}_r \cdot \vec{a})$ (10.150)

where we used the *BAC – CAB* rule to evaluate the triple cross product. The result tells us that only the acceleration perpendicular to the line-of-sight is responsible for the far field radiation. (Be sure not to confuse this with the case of acceleration perpendicular to the velocity! In fact, in this case, the acceleration is parallel to the velocity.) A bit strange — velocity and acceleration in one direction only yields radiation at angles away from that direction, and the radiation is strongest in the direction perpendicular to the acceleration and velocity.

Potentials and Fields of a Moving Point Charge (cont.)

Next, if we define the \vec{a} to be the z -axis of a spherical coordinate system and θ the polar angle, then we have $\vec{a} = a\hat{z}$, $|\vec{a}_\perp| = a \sin \theta$, and $\vec{\beta} \cdot \hat{R}_r = \beta \cos \theta$, so

$$\vec{S} = \frac{\mu_0 q^2}{16 \pi^2} \frac{1}{c R_r^2} \frac{a^2 \sin^2 \theta}{[1 - \beta \cos \theta]^6} \hat{R}_r \quad (10.151)$$

(We rewrote ϵ_0 in terms of c and μ_0 .) The energy radiated into a unit area per unit field point time interval is $d^2 U/dt dA = \hat{R}_r \cdot \vec{S}$.

However, we will usually be interested in knowing the energy *lost by the particle* per unit time, so we need to convert the dt in the above to a dt_r by dividing by $\partial t_r / \partial t$, which we calculated in Equation 10.129. The distinction between the two is the same “piling up” effect that we discussed earlier in the context of calculating the Lienard-Wiechert potential. Therefore (multiplying by R_r^2 to convert from power per unit area to power per unit solid angle):

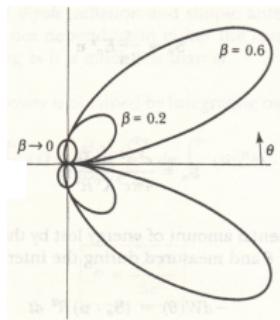
$$\begin{aligned} \frac{dP}{d\Omega} &= \frac{d^2 U}{dt d\Omega} = \frac{d^2 U}{dt d\Omega} \left(\frac{\partial t_r}{\partial t} \Bigg|_{\vec{\beta}} \right)^{-1} = R_r^2 \hat{R}_r \cdot \vec{S} \left(\frac{\partial t_r}{\partial t} \Bigg|_{\vec{\beta}} \right)^{-1} \\ &= \frac{\mu_0 q^2}{16 \pi^2} \frac{1}{c} \frac{a^2 \sin^2 \theta}{[1 - \beta \cos \theta]^6} \left(1 - \vec{\beta} \cdot \hat{R}_r \right) \end{aligned} \quad (10.152)$$

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2}{16 \pi^2} \frac{a^2}{c} \frac{\sin^2 \theta}{[1 - \beta \cos \theta]^5} \quad \text{power per solid angle radiated by a point charge accelerated parallel to direction of motion}$$

(10.153)

Potentials and Fields of a Moving Point Charge (cont.)

The following figure has polar plots that illustrate the shape of this function for a particle moving from left to right. When $\beta \ll 1$, one obtains a distribution symmetric between the forward and the reverse direction. As β increases, the denominator begins to take effect and the radiation is strongly directed into the forward hemisphere (though the radiation along the direction of motion continues to vanish).



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One can integrate the above over all angles to obtain the total radiated power

$$P = \frac{\mu_0 q^2}{6\pi} \frac{a^2}{c(1-\beta^2)^3} = \frac{\mu_0 q^2}{6\pi} \frac{a^2}{c} \gamma^6 \quad \text{total power radiated by a point charge accelerated parallel to direction of motion}$$
 (10.154)

where $\gamma = (1 - \beta^2)^{-1/2}$ is the usual relativistic Lorentz factor.

Potentials and Fields of a Moving Point Charge (cont.)

Now, let's do the perpendicular acceleration case. The archetypal example of this kind of radiation is synchrotron radiation emitted by a particle being accelerated while in circular motion, such as in a circular particle accelerator or in an astrophysical or laboratory magnetic field. Unfortunately, in this case we have to keep both terms in the accelerated electric field Equation 10.140 and the expression is not very simple. If we set $\vec{\beta} = \beta \hat{z}$ and $\vec{a} = a \hat{x}$, then one can work out the vector algebra (see, e.g., Heald and Marion Section 8.8) to show

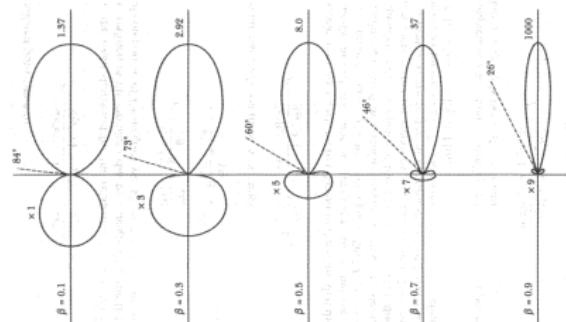
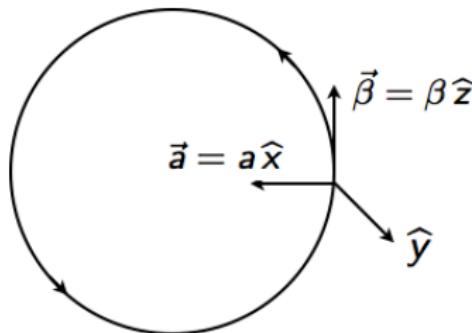
$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2}{16 \pi^2} \frac{a^2}{c} \frac{(1 - \beta \cos \theta)^2 - (1 - \beta^2) \sin^2 \theta \cos^2 \phi}{[1 - \beta \cos \theta]^6}$$

power per solid angle
radiated by a
point charge
moving along \hat{z} and
accelerated along \hat{x}

(10.155)

Potentials and Fields of a Moving Point Charge (cont.)

The radiation pattern is shown below for a particle with velocity in the vertical direction and acceleration in the horizontal direction. As $\beta \rightarrow 1$, the radiation becomes “beamed” in the direction of motion. Synchrotron light sources use this principle to produce light in the direction of the particle’s motion.



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The illegible dashed lines indicate the angle at which the radiated power vanishes.

The total radiated power is

$$P = \frac{\mu_0 q^2}{6\pi} \frac{a^2}{c (1 - \beta^2)^2} = \frac{\mu_0 q^2}{6\pi} \frac{a^2}{c} \gamma^4$$

total power radiated by a
point charge accelerated
perpendicular to
direction of motion

(10.156)

Potentials and Fields of a Moving Point Charge (cont.)

By extrapolation and returning to Equation 10.140, we can write the general formula for the power per unit solid angle:

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2}{16 \pi^2} \frac{a^2}{c} \frac{\left| \hat{R}_r \times [(\hat{R}_r - \vec{\beta}) \times \hat{a}] \right|^2}{[1 - \beta \cos \theta]^5}$$

power per solid angle
radiated by a
point charge
with velocity $\vec{\beta}$ and
acceleration \vec{a}

(10.157)

Combining the two total power formulae, which only differ by a factor of γ^2 , yields the total power radiated for an arbitrary angle between velocity and acceleration:

$$P = \frac{\mu_0 q^2}{6 \pi} \frac{a^2}{c} \frac{1 - |\vec{\beta} \times \hat{a}|^2}{(1 - \beta^2)^3} = \frac{\mu_0 q^2}{6 \pi} \frac{a^2}{c} \gamma^6 \left(1 - |\vec{\beta} \times \hat{a}|^2 \right)$$

total power radiated
by an accelerated
point charge
Lienard's Formula

(10.158)

Potentials and Fields of a Moving Point Charge (cont.)

Finally, we obtain the “slowly moving charge” limit by letting $\beta \rightarrow 0$. Since the parallel acceleration derivation dropped the $\beta \times \vec{a}$ term, that is the easiest one to work from; we just let $\beta \rightarrow 0$ in those expressions:

$$\frac{dP}{d\Omega} = \frac{\mu_0 q^2}{16\pi^2} \frac{a^2 \sin^2 \theta}{c} \quad P = \frac{\mu_0 q^2}{6\pi} \frac{a^2}{c}$$

power per solid angle and total power radiated by a slowly moving point charge with acceleration \vec{a}
Larmor's Formula

(10.159)

The angular dependence is simply $\sin^2 \theta$ and is given by the $\beta \rightarrow 0$ limits of the figures shown previously. The main aspect is that the angular dependence is now symmetric because the direction of motion does not break the symmetry.

Dipole Radiation

Introduction and Study Guide

As usual, we don't follow Griffiths closely. We repeat his examples of electric and magnetic dipole radiation. We then go to the full radiation calculation via the technique of Fourier Transforms. This makes it easier to interpret the radiation field in terms of spherical outgoing waves.

Dipole Radiation (cont.)

Ideal Electric Dipole Radiation

Let us calculate the radiation from an oscillating ideal electric dipole. We assume it to consist of two charges q_0 and $-q_0$ whose positions are varying with time $\vec{r}_+(t)$ and $\vec{r}_-(t)$ with $\vec{r}_-(t) = -\vec{r}_+(t)$. The distance vector from the negative to the positive charge is thus $\vec{d}(t) = 2\vec{r}_+(t)$ and the dipole moment is $\vec{p}(t) = q_0 \vec{d}(t)$. We let $d \rightarrow 0$ while keeping $\vec{p}(t)$ nonzero so that the dipole becomes pointlike. We can calculate the radiated (acceleration) component of the electric and magnetic fields from Equations 10.140 and 10.144, but we can make a number of immediate simplifications that make the calculation easier:

- ▶ $\vec{\beta} \rightarrow 0$: Since we will let $d \rightarrow 0$, there is no velocity in this limit. This lets us take the “slowing moving charge” limit, which eliminates the $\vec{\beta}$ term in the numerator and the $\vec{\beta} \cdot \hat{R}_r$ term in the denominator.
- ▶ $\vec{R}_r^+ = \vec{R}_r^- = \vec{r}$: Since we let $d \rightarrow 0$, the dipole can be considered to be a pointlike object at the origin. The vectors from the charges to the field point \vec{r} are the same, $\vec{R}_r^+ = \vec{R}_r^- = \vec{R}_r$. Moreover, we can place the dipole at the origin so $\vec{R}_r = \vec{r}$.
- ▶ $q \ddot{\vec{a}} \rightarrow \ddot{\vec{p}}$: With the above two approximations, all the factors in the electric field formula are the same for the two particles except for the $q \ddot{\vec{a}}$ factors. But $q \ddot{\vec{a}} = q_0 \ddot{\vec{r}}_+ = q_0 \ddot{\vec{d}}/2$ for the positive charge and $q \ddot{\vec{a}} = -q_0 \ddot{\vec{r}}_- = q_0 \ddot{\vec{d}}/2$ for the negative charge, so the two terms just sum to $q_0 \ddot{\vec{d}} = \ddot{\vec{p}}$.

Dipole Radiation (cont.)

With the above, the radiated electric and magnetic fields are

$$\vec{E}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{\hat{r} \times (\hat{r} \times \ddot{\vec{p}})}{c^2 r} \quad \vec{B}(\vec{r}, t) = \frac{1}{c} \hat{r} \times \vec{E} \quad (10.160)$$

(Remember, $\ddot{\vec{p}}$ is evaluated at t_r !) If we further assume the special case $\vec{p} = p \hat{z}$, then we may use

$$\hat{r} \times (\hat{r} \times \hat{z}) = \hat{r} \times (-\hat{\phi} \sin \theta) = \hat{\theta} \sin \theta \quad (10.161)$$

$$\hat{r} \times [\hat{r} \times (\hat{r} \times \hat{z})] = \hat{r} \times \hat{\theta} \sin \theta = \hat{\phi} \sin \theta \quad (10.162)$$

(We did this same calculation using the *BAC – CAB* rule to obtain \vec{a}_\perp in the accelerating point charge case. This result can be seen to be the same because $\hat{\theta} \sin \theta = -\vec{p}_\perp / |\vec{p}|$ for $\vec{p} = p \hat{z}$.) So then

$$\vec{E}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{\ddot{p} \sin \theta}{c^2 r} \hat{\theta} \quad \vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{\ddot{p} \sin \theta}{c r} \hat{\phi} \quad (10.163)$$

Therefore, the radiated power (again, $\beta = 0$ limit) is

$$\frac{dP}{d\Omega} = r^2 \hat{r} \cdot \vec{S} = \frac{\mu_0}{16\pi^2} \frac{\ddot{p}^2 \sin^2 \theta}{c} \quad P = \int d\Omega \frac{dP}{d\Omega} = \frac{\mu_0}{6\pi} \frac{\ddot{p}^2}{c} \quad (10.164)$$

which is just Equations 10.159 with $q a$ replaced by \ddot{p} .

Dipole Radiation (cont.)

If we consider the special case of harmonic motion, $p(t) = p_0 \cos \omega t$, then we have

$$\vec{E}(\vec{r}, t) = -\frac{1}{4\pi\epsilon_0} \frac{p_0 \omega^2 \sin \theta}{c^2 r} \hat{\theta} \cos \left[\omega \left(t - \frac{r}{c} \right) \right] \quad (10.165)$$

$$\vec{B}(\vec{r}, t) = -\frac{\mu_0}{4\pi} \frac{p_0 \omega^2 \sin \theta}{c r} \hat{\phi} \cos \left[\omega \left(t - \frac{r}{c} \right) \right] \quad (10.166)$$

Here we see the explicit evaluation of $\ddot{\vec{p}}$ at t_r . The *time-averaged* radiated power is (all fields are real, no need for complex conjugation or taking real parts):

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{1}{2} \left. \frac{dP}{d\Omega} \right|_{t=r/c} = \frac{\mu_0}{32\pi^2} \frac{p_0^2 \omega^4 \sin^2 \theta}{c} \quad (10.167)$$

$$\langle P \rangle = \frac{1}{2} P|_{t=r/c} = \frac{\mu_0}{12\pi} \frac{p_0^2 \omega^4}{c} \quad (10.168)$$

(We didn't have to worry about the $\partial t_r / \partial t$ factor because we have taken $\beta \rightarrow 0$.)

Note that we don't need to specify approximations as Griffiths does because we have assumed a perfect point dipole. When we do the calculation for an arbitrary charge distribution, we will be more careful.

We dispense with the ideal magnetic dipole because there is no way to make it pointlike and use our results for the accelerating point charge. We might as well just get on with the derivation for an arbitrary charge/current distribution.

Dipole Radiation (cont.)

Digression on Fourier Transforms

We are going to use Fourier techniques to simplify the derivations in the case of radiation from an arbitrary source distribution, so we need to define the formalism.

We use the concept of orthonormal functions that we developed in connection to separation of variables. We state without proof that any function of time can be written in the form ($\omega = 2\pi f$):

$$g(t) = \int_{-\infty}^{\infty} df \tilde{g}(f) e^{-i\omega t} \quad \tilde{g}(f) = \int_{-\infty}^{\infty} dt g(t) e^{i\omega t} \quad (10.169)$$

$$\delta(t) = \int_{-\infty}^{\infty} df e^{-i\omega t} \quad \delta(f) = \int_{-\infty}^{\infty} dt e^{i\omega t} = 2\pi \delta(\omega) \quad (10.170)$$

The function $\tilde{g}(f)$ is the *Fourier Transform* of the function $g(t)$. It has the units of g divided by frequency. For $g(t)$ to which no boundary conditions have been applied (typical BC would be to assume periodicity with a particular frequency f_0 or to assume Dirichlet or Neumann BC at two times t_a and t_b), there is no restriction on the values of f — any value must be allowed. (In real life, the rate at which you can sample $g(t)$ sets an upper limit on the values of f for which information is available — the so-called Nyquist criterion — but we won't worry about real life here...).

Dipole Radiation (cont.)

Note that our Fourier Transform sign convention for the argument of the exponential is the opposite of that used in electrical engineering. As long as one is self-consistent, you can choose whichever convention you like. But be careful when comparing to other texts.

The value of the above lies in the linearity of electrodynamics. Specifically, define

$$\rho(\vec{r}, t) = \rho_0(\vec{r}) e^{-i\omega t} \quad \vec{J}(\vec{r}, t) = \vec{J}_0(\vec{r}) e^{-i\omega t} \quad (10.171)$$

and suppose the resulting fields are

$$\vec{E}(\vec{r}, t) = \vec{\mathcal{E}}[\rho_0, \vec{J}_0](\vec{r}) e^{-i\omega t} \quad \vec{B}(\vec{r}, t) = \vec{\mathcal{B}}[\rho_0, \vec{J}_0](\vec{r}) e^{-i\omega t} \quad (10.172)$$

where $\vec{\mathcal{E}}[](\vec{r})$ and $\vec{\mathcal{B}}[](\vec{r})$ indicate functional dependence: put in the spatial dependences $\rho_0(\vec{r})$, $\vec{J}_0(\vec{r})$ and what you get out are the spatial dependences $\vec{\mathcal{E}}(\vec{r})$, $\vec{\mathcal{B}}(\vec{r})$. (We are guaranteed the time dependence is unchanged by the linearity of Maxwell's Equations.) Then we can obtain the fields for arbitrary charge distribution by summing using the Fourier Transform.

Dipole Radiation (cont.)

More generally, define

$$\tilde{\rho}(\vec{r}, f) = \int_{-\infty}^{\infty} dt \rho(\vec{r}, t) e^{i\omega t} \quad \tilde{\vec{J}}(\vec{r}, f) = \int_{-\infty}^{\infty} dt \vec{J}(\vec{r}, t) e^{i\omega t} \quad (10.173)$$

The motivation for these expressions is that multiplication by the $e^{i\omega t}$ factor followed by the integration would pick out the $\rho_0(\vec{r})$ and $\vec{J}_0(\vec{r})$ shown on the previous page. Then, from the above solution that determines fields from sources, we know

$$\tilde{\vec{E}}(\vec{r}, f) = \vec{\mathcal{E}}[\tilde{\rho}(\vec{r}, f), \tilde{\vec{J}}(\vec{r}, f)] \quad \tilde{\vec{B}}(\vec{r}, f) = \vec{\mathcal{B}}[\tilde{\rho}(\vec{r}, f), \tilde{\vec{J}}(\vec{r}, f)] \quad (10.174)$$

and then, by linearity,

$$\vec{E}(\vec{r}, t) = \int_{-\infty}^{\infty} df \tilde{\vec{E}}(\vec{r}, f) e^{-i\omega t} \quad \vec{B}(\vec{r}, t) = \int_{-\infty}^{\infty} df \tilde{\vec{B}}(\vec{r}, f) e^{-i\omega t} \quad (10.175)$$

Therefore, we only need determine the functions $\vec{\mathcal{E}}[](\vec{r})$ and $\vec{\mathcal{B}}[](\vec{r})$ by determining the fields for a harmonic time dependence $e^{i\omega t}$ of the sources and then we can use Fourier Transforms to calculate the fields for arbitrary source time dependence.

Dipole Radiation (cont.)

Linearity also holds, surprisingly, for quadratic quantities like energy and radiation if one time averages. Specifically, let's calculate the time-averaged Poynting vector:

$$\left\langle \vec{S}(\vec{r}, t) \right\rangle = \frac{1}{2\mu_0} \left\langle \mathcal{R} \left(\vec{E}^*(\vec{r}, t) \times \vec{B}(\vec{r}, t) \right) \right\rangle \quad (10.176)$$

$$= \frac{1}{2\mu_0} \left\langle \mathcal{R} \left(\int_{-\infty}^{\infty} df_1 \int_{-\infty}^{\infty} df_2 e^{i(\omega_1 - \omega_2)t} \tilde{\vec{E}}^*(\vec{r}, f_1) \times \tilde{\vec{B}}(\vec{r}, f_2) \right) \right\rangle \quad (10.177)$$

$$= \frac{1}{2\mu_0} \mathcal{R} \left(\int_{-\infty}^{\infty} df_1 \int_{-\infty}^{\infty} df_2 \left\langle e^{i(\omega_1 - \omega_2)t} \right\rangle \tilde{\vec{E}}^*(\vec{r}, f_1) \times \tilde{\vec{B}}(\vec{r}, f_2) \right) \quad (10.178)$$

Now,

$$\left\langle e^{i\omega t} \right\rangle = \lim_{T \rightarrow \infty} \frac{1}{T} \int_{-T/2}^{T/2} dt e^{i\omega t} = \lim_{T \rightarrow \infty} \frac{\int_{-T/2}^{T/2} dt e^{i\omega t}}{\int_{-T/2}^{T/2} dt} = \frac{\delta(f)}{\delta(f = 0)} \quad (10.179)$$

Dipole Radiation (cont.)

So,

$$\left\langle \vec{S}(\vec{r}, t) \right\rangle = \frac{1}{2\mu_0} \mathcal{R} \left(\int_{-\infty}^{\infty} df_1 \int_{-\infty}^{\infty} df_2 \frac{\delta(f_1 - f_2)}{\delta(f=0)} \tilde{\vec{E}}^*(\vec{r}, f_1) \times \tilde{\vec{B}}^*(\vec{r}, f_2) \right) \quad (10.180)$$

$$= \frac{1}{\delta(f=0)} \frac{1}{2\mu_0} \mathcal{R} \left(\int_{-\infty}^{\infty} df_1 \tilde{\vec{E}}^*(\vec{r}, f_1) \times \tilde{\vec{B}}^*(\vec{r}, f_1) \right) \quad (10.181)$$

There is a funny normalizing factor that corrects for the fact that the Fourier Transforms of the fields have the units of field divided by frequency; since there is only one integral over frequency left, the $\delta(f=0)$ provides another unit of frequency in the numerator as needed. More importantly, the above tells us that the *time-averaged* Poynting vector is obtained by summing up the contributions from each frequency in a linear way: the power at different frequencies just adds up, so we can calculate the power for a given frequency and then do sums to get total power. The cross-terms drop away in the time average.

A final point to make is that time derivatives become powers of ω for Fourier transforms. This is seen by differentiating the expression for $g(t)$:

$$\frac{dg}{dt} = \frac{d}{dt} \int_{-\infty}^{\infty} df \tilde{g}(f) e^{-i\omega t} = \int_{-\infty}^{\infty} df \tilde{g}(f) \frac{d}{dt} e^{-i\omega t} = \int_{-\infty}^{\infty} df (-i\omega) \tilde{g}(f) e^{-i\omega t} \quad (10.182)$$

So in what we do below, a factor of ω can be interpreted as a time derivative.

Lecture 27:
Radiation from an Arbitrary Source Distribution
Review of Special Relativity

Date Revised: 2014/05/31 15:00

2014/06/03 10:00: Drop $d^2/\lambda \ll r$ requirement when redundant
with other requirements

Date Given: 2014/05/31

Dipole Radiation

Radiation from an Arbitrary Source Distribution

Let's now consider the radiation from an arbitrary configuration of time-varying charges and currents. It will be no surprise that we need to apply techniques similar to those used for calculating the potentials and fields of electric and magnetic multipole configurations. As motivated by our discussion of Fourier Transforms, we assume

$$\rho(\vec{r}, t) = \rho_0(\vec{r}) e^{-i\omega t} \quad \vec{J}(\vec{r}, t) = \vec{J}_0(\vec{r}) e^{-i\omega t} \quad (10.183)$$

We assume that ρ_0 and \vec{J}_0 are zero outside some volume \mathcal{V} near the origin. The retarded scalar and vector potentials are (Equations 10.35; recall, $t_r = t - |\vec{r} - \vec{r}'|/c$)

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int_{\mathcal{V}} d\tau' \frac{\rho(\vec{r}, t_r)}{|\vec{r} - \vec{r}'|} \quad \vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \int_{\mathcal{V}} d\tau' \frac{\vec{J}(\vec{r}, t_r)}{|\vec{r} - \vec{r}'|} \quad (10.184)$$

When $e^{-i\omega t}$ is evaluated at $t = t_r$, we get

$$e^{-i\omega t_r} = e^{-i\omega t} e^{ik|\vec{r} - \vec{r}'|} \quad \text{with} \quad k = \frac{\omega}{c} \quad (10.185)$$

Dipole Radiation (cont.)

Let's rewrite the retarded potentials using this information. With the assumption of harmonic time dependence and no sources outside \mathcal{V} ($\rho = 0$, $\vec{J} = 0$ outside \mathcal{V}), Ampere's Law tells us that, outside \mathcal{V} ,

$$\epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B} \quad \Rightarrow \quad \vec{E} = c^2 \frac{i}{\omega} \vec{\nabla} \times \vec{B} \quad (10.186)$$

Therefore, we only need to calculate \vec{A} . It becomes

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} e^{-i\omega t} \int_{\mathcal{V}} d\tau' \vec{J}_0(\vec{r}') \frac{e^{i k |\vec{r} - \vec{r}'|}}{|\vec{r} - \vec{r}'|} \quad (10.187)$$

Now, for large distances, $|\vec{r}| \gg |\vec{r}'|$ for \vec{r}' inside \mathcal{V} , we can Taylor expand the denominator and the argument of the exponential. We have an expansion of the denominator from our electric and magnetic multipole expansions. We start with:

$$|\vec{r} - \vec{r}'| = \sqrt{r^2 - 2\vec{r} \cdot \vec{r}' + (r')^2} = r \sqrt{1 - 2\hat{r} \cdot \frac{\vec{r}'}{r} + \left(\frac{r'}{r}\right)^2} \quad (10.188)$$

$$= r - \hat{r} \cdot \hat{r}' r' + \mathcal{O}\left(\frac{d^2}{r}\right) \quad (10.189)$$

where d is the characteristic size of the source distribution.

Dipole Radiation (cont.)

Putting these two together, we have

$$\frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|} \approx \frac{1}{r} \left[1 + \hat{r} \cdot \hat{r}' \frac{r'}{r} + \mathcal{O}\left(\frac{d^2}{r^2}\right) \right] \exp\left[i k \left(r - \hat{r} \cdot \hat{r}' r' + \mathcal{O}\left(\frac{d^2}{r}\right)\right)\right] \quad (10.190)$$

$$= \frac{e^{ikr}}{r} \left[1 + \hat{r} \cdot \hat{r}' \frac{r'}{r} + \mathcal{O}\left(\frac{d^2}{r^2}\right) \right] \exp\left(-ikr' \hat{r} \cdot \hat{r}' + ik \mathcal{O}\left(\frac{d^2}{r}\right)\right) \quad (10.191)$$

Now that the dependences are clear, let's figure out what we need to keep. The first factor is independent of r' and multiplies the whole expression, so no issue there. In the second factor, there are power law dependences on r' and r , and there are no direct power-law dependences elsewhere, so we may look at this factor alone. In the limit $d \ll r$, we may keep the first term and discard the second and remaining terms because they fall off as higher power of r . (We kept the second term for the electric and magnetic dipole expansions because the first term, the monopole term, vanished.) Now, looking at the exponential, it has a completely imaginary argument, which causes the phase of the exponent to vary. The ratio of the second term to the first term of this phase factor is $(kd^2/r)/(kd) = d/r$, so it varies much less quickly than the first term, so the first one will dominate the phase variation of the argument of the exponential. However, we also want to check that the second term in the argument is not large in an absolute sense: if it is of order unity, then it also causes fast variations in the phase of the argument of the exponential. We thus require $k d^2/r \ll 2\pi$, or $d^2/\lambda \ll r$: this is the so-called "far-field" approximation, with d^2/λ being the "far-field distance." So, with these approximations, we can drop the $\mathcal{O}()$ terms.

Dipole Radiation (cont.)

Inserting the above into our expression for the vector potential without the higher-order terms, we have

radiation from arbitrary

source distribution

vector potential

$d \ll r, d^2/\lambda \ll r$

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{i(kr - \omega t)}}{r} \int_V d\tau' \vec{J}_0(\vec{r}') e^{-ikr' \hat{r} \cdot \hat{r}'} \quad (10.192)$$

It may seem strange that we dropped the $\hat{r} \cdot \hat{r}' r'$ term from the expansion of $|\vec{r} - \vec{r}'|^{-1}$, the one that yielded the electric dipole and magnetic dipole potentials.

The reason we don't need it here is because of the exponential — its variation over the scale of the source distribution (note that we did not make the approximation $k d = 2\pi d/\lambda \ll 1!$) prevents the cancellation that occurred for the monopole terms in the static case.

This expression makes very explicit the physical picture that the harmonic dependence of the source current drives a spherical outgoing wave — the $e^{i(kr - \omega t)}/r$ factor.

From the above, we can calculate the fields, time-averaged Poynting vector, radiation pattern, and total power radiated (for harmonic time dependence!):

$$\vec{B} = \vec{\nabla} \times \vec{A} \quad \vec{E} = c^2 \frac{i}{\omega} \vec{\nabla} \times \vec{B}$$

$$\left\langle \vec{S} \right\rangle = \frac{1}{2\mu_0} \mathcal{R} \left(\left\langle \vec{E}^* \times \vec{B} \right\rangle \right) \quad \left\langle \frac{dP}{d\Omega} \right\rangle = r^2 \hat{r} \cdot \left\langle \vec{S} \right\rangle \quad \langle P \rangle = \int d\Omega \frac{dP}{d\Omega}$$

Dipole Radiation (cont.)

If we make the further approximation $d \ll \lambda$, then the exponential can be Taylor expanded to obtain

radiation from
arbitrary
source distribution $\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{i(kr - \omega t)}}{r} \sum_{m=0}^{\infty} \frac{(-ik)^m}{m!} \int_V d\tau' \vec{J}_0(\vec{r}') (r' \hat{r} \cdot \hat{r}')^m$
vector potential
 $d \ll r, d \ll \lambda$

(10.193)

We have dropped the $d^2/\lambda \ll r$ requirement because it is implied by other two requirements. This is now the *multipole expansion for radiation* — successive terms probe finer and finer structure of the source distribution. We see next how they give electric dipole, magnetic dipole, and electric quadrupole radiation.

Dipole Radiation (cont.)

Electric Dipole Radiation

We take the $m = 0$ term, which is

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{i(kr - \omega t)}}{r} \int_{\mathcal{V}} d\tau' \vec{J}_0(\vec{r}') \quad (10.194)$$

We dealt with this same expression in our study of magnetic multipoles, Equations 5.89 and 5.90, where there we saw it vanished because we assumed $\vec{\nabla} \cdot \vec{J}_0 = -\partial \rho_0 / \partial t = 0$ for magnetostatics. Here, we do not make that assumption, so

$$\begin{aligned} J_{0,i} e^{-i\omega t} &= \vec{\nabla} \cdot (r_i \vec{J}_0 e^{-i\omega t}) - r_i \frac{\partial}{\partial t} (\rho_0 e^{-i\omega t}) \\ &= \vec{\nabla} \cdot (r_i \vec{J}_0 e^{-i\omega t}) - i\omega r_i \rho_0 e^{-i\omega t} \end{aligned} \quad (10.195)$$

When we do the integral, the first term vanishes: it can be turned into a surface integral of $r_i \vec{J}$ at the boundary of \mathcal{V} and, since the sources are contained in \mathcal{V} , there can be no current flowing through the boundary of \mathcal{V} . That leaves the second term, so the vector potential becomes

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{i(kr - \omega t)}}{r} \int_{\mathcal{V}} d\tau' (-i\omega) \rho_0(\vec{r}') \vec{r}' = -i \frac{\mu_0}{4\pi} \frac{\omega \vec{p}_0}{r} e^{i(kr - \omega t)} \quad (10.196)$$

where we have used the definition of the dipole moment, Equation 3.201. We see that, in the static limit ($\omega \rightarrow 0$), the expression vanishes as we expect.

Dipole Radiation (cont.)

To calculate the magnetic field, we need to take the curl. The curl will act on both the $1/r$ and the phase factor in the exponent. But the action of the curl on $1/r$ will yield $1/r^2$, which we should drop for consistency with our prior approximations. So we calculate

$$\vec{\nabla} \times (\vec{p} e^{ikr}) = -\vec{p}_0 \times \vec{\nabla} e^{ikr} = -\vec{p}_0 \times e^{ikr} \vec{\nabla}(ikr) = -\vec{p}_0 \times e^{ikr} ik\hat{r} \quad (10.197)$$

Therefore

$$\vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{\omega^2 \hat{r} \times \vec{p}_0}{cr} e^{i(kr - \omega t)} \quad d \ll r, d \ll \lambda \quad (10.198)$$

where the factor of ω/c came from k . We obtain the electric field from

$\vec{E} = c^2(i/\omega) \vec{\nabla} \times \vec{B} = c^2(-k/\omega)\hat{r} \times \vec{B} = -c\hat{r} \times \vec{B}$, where we used a similar technique for the calculation of the curl. This yields (using $\mu_0 = 1/(c^2 \epsilon_0)$)

$$\vec{E}(\vec{r}, t) = -\frac{1}{4\pi\epsilon_0} \frac{\omega^2 \hat{r} \times (\hat{r} \times \vec{p}_0)}{c^2 r} e^{i(kr - \omega t)} \quad d \ll r, d \ll \lambda \quad (10.199)$$

The geometry of the fields is as follows. The magnetic field is normal to both the line-of-sight to the dipole at the origin (\hat{r}) and to the dipole's direction \vec{p}_0 . So, for a dipole along \hat{z} , \vec{B} oscillates in the $\hat{\phi}$ direction. The electric field is normal to the line-of-sight and to \vec{B} , so it oscillates in the $\hat{\theta}$ direction. Both $\hat{\theta}$ and $\hat{\phi}$ are in the plane normal to \hat{r} .

Dipole Radiation (cont.)

Let's now generalize these using our discussion of Fourier Transforms. Let's assume an arbitrary time dependence of the charge density $\rho(\vec{r}, t)$ and the dipole moment $\vec{p}(t)$. The former has a Fourier decomposition (Equation 10.173), which we can use to Fourier decompose the latter:

$$\tilde{\rho}(\vec{r}, f) = \int_{-\infty}^{\infty} dt \rho(\vec{r}, t) e^{i\omega t} \iff \rho(\vec{r}, t) = \int_{-\infty}^{\infty} df \tilde{\rho}(\vec{r}, f) e^{-i\omega t}$$

$$\implies \vec{p}(\vec{r}, t) = \int_V d\tau' \rho(\vec{r}', t) \vec{r}' = \int_V d\tau' \vec{r}' \int_{-\infty}^{\infty} df \tilde{\rho}(\vec{r}, f) e^{-i\omega t} \quad (10.200)$$

$$= \int_{-\infty}^{\infty} df \tilde{\vec{p}}(f) e^{-i\omega t} \quad \text{with} \quad \tilde{\vec{p}}(f) = \int_V d\tau' \tilde{\rho}(\vec{r}, f) \vec{r}' \quad (10.201)$$

Now, the quantity $\tilde{\vec{p}}(f) e^{-i\omega t}$ is just like $\vec{p}_0 e^{-i\omega t}$ since both \vec{p}_0 and $\tilde{\vec{p}}(f)$ have no spatial dependence. Therefore, our expressions for \vec{A} , \vec{B} , and \vec{E} for harmonic dependence apply to it. Since Maxwell's Equations are linear, we can just sum up the contribution to the fields from each frequency f using Fourier Transforms.

Dipole Radiation (cont.)

That is, we may write

$$\tilde{\vec{A}}(\vec{r}, f) e^{-i\omega t} = -i \frac{\mu_0}{4\pi} \frac{\omega \tilde{\vec{p}}(f)}{r} e^{i(kr - \omega t)} \quad (10.202)$$

where we have factored the $e^{-i\omega t}$ dependence out of the vector potential and added the f argument to indicate that each $\tilde{\vec{p}}(f)$ generates its own $\tilde{\vec{A}}(\vec{r}, f)$. Now, sum up over frequency components using the Fourier Transform:

$$\vec{A}(\vec{r}, t) = \int_{-\infty}^{\infty} df \tilde{\vec{A}}(\vec{r}, f) e^{-i\omega t} = \int_{-\infty}^{\infty} df \left[-i \frac{\mu_0}{4\pi} \frac{\omega \tilde{\vec{p}}(f)}{r} e^{i kr} \right] e^{-i\omega t} \quad (10.203)$$

Next, recall how time derivatives are related to factors of $-i\omega$ in Fourier Transforms, Equation 10.182, and also recall Equation 10.185, which tells us that $e^{i(kr - \omega t)} = e^{-i\omega t_r}$ (taking $|\vec{r}| \gg |\vec{r}'|$). These let us rewrite the above as

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{1}{r} \int_{-\infty}^{\infty} df \left[-i\omega \tilde{\vec{p}}(f) \right] e^{-i\omega t_r} = \frac{\mu_0}{4\pi} \frac{\dot{\vec{p}}(t_r)}{r} \quad (10.204)$$

where $\dot{\vec{p}}$ indicates the derivative of \vec{p} with respect to its argument. One can do similar Fourier analyses for \vec{B} and \vec{E} . This is easier than taking the curl and time derivative of the above expression directly because one does not have to worry about the complication of taking the derivatives of expressions involving retarded time.

Dipole Radiation (cont.)

The result is the full generalization of our harmonic time dependence expressions:

electric dipole radiation vector potential $d \ll r, d \ll \lambda$	$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{\dot{\vec{p}}(t_r)}{r}$	(10.205)
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electric dipole radiation magnetic field $d \ll r, d \ll \lambda$	$\vec{B}(\vec{r}, t) = -\frac{\mu_0}{4\pi} \frac{\hat{r} \times \ddot{\vec{p}}(t_r)}{c r}$	(10.206)
---	--	----------

electric dipole radiation electric field $d \ll r, d \ll \lambda$	$\vec{E}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{\hat{r} \times (\hat{r} \times \ddot{\vec{p}}(t_r))}{c^2 r}$	(10.207)
---	--	----------

which now match Equation 10.160, the ideal dipole expressions. Note that the dipole moment derivatives are evaluated at t_r ! These equations also match Equations 11.54, 11.56, and 11.57 of Griffiths.

Dipole Radiation (cont.)

Since the fields of the electric dipole radiation term are identical to the ones we found for the perfect electric dipole example, the radiation pattern and total power will be the same. If we define a coordinate system with $\vec{p} \propto \hat{z}$ and θ the spherical coordinate polar angle, then the vector algebra we applied in deriving Equations 10.164 holds. Thus, for an arbitrary time dependence, we recover those equations (using $dP/d\Omega = r^2 \hat{r} \cdot \vec{S}$, no complex conjugations or real parts necessary because these formulae do not assume harmonic time dependence):

electric dipole radiation pattern $d \ll r, d \ll \lambda$	$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2} \frac{\ddot{p}^2 \sin^2 \theta}{c}$	$P = \frac{\mu_0}{6\pi} \frac{\ddot{p}^2}{c}$	(10.208)
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If we assume harmonic time dependence and time-average (also now taking the necessary complex conjugations and real parts if we use complex notation), we obtain Equations 10.167:

electric dipole radiation pattern harmonic time dependence $d \ll r, d \ll \lambda$	$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0}{32\pi^2} \frac{p_0^2 \omega^4 \sin^2 \theta}{c}$	$\langle P \rangle = \frac{\mu_0}{12\pi} \frac{p_0^2 \omega^4}{c}$
---	--	--

(10.209)

Dipole Radiation (cont.)

Magnetic Dipole and Electric Quadrupole Radiation

Now, let's look at the $m = 1$ term:

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{i(kr - \omega t)}}{r} (-i k) \int_V d\tau' \vec{J}_0(\vec{r}') \hat{r} \cdot \hat{r}' r' \quad (10.210)$$

This term is a factor of order $k d = 2\pi d/\lambda$ smaller than the electric dipole term. We also calculated this term in the process of deriving the vector potential due to a magnetic dipole, starting at Equation 5.93. Again, we now have to allow $\vec{\nabla} \cdot \vec{J}_0 = -\partial \rho_0 / \partial t \neq 0$. This yields (many steps left out!)

$$\vec{J}_0(\vec{r}') \hat{r} \cdot \hat{r}' r' = \frac{1}{2} \left[(\vec{r}' \times \vec{J}_0) \times \hat{r} \right] + \frac{1}{2} r' \left[(\hat{r} \cdot \hat{r}') \vec{J}_0 + (\hat{r} \cdot \vec{J}_0) \hat{r}' \right] \quad (10.211)$$

The second two terms involves quantities similar to those we dealt with for the electric dipole term. We can integrate these terms by parts to obtain expressions involving $\vec{\nabla} \cdot \vec{J}_0$ (and therefore bringing in one more power of r' to cancel the dimensions of $\vec{\nabla}$), which can be evaluated using continuity, $\vec{\nabla} \cdot \vec{J}_0 = -\partial \rho_0 / \partial t = i\omega \rho_0$. They therefore result in terms containing $i\omega \rho_0$ and two powers of r' . These are electric quadrupole terms. They are of the same order of magnitude as the terms we will keep, but they are complicated and so we will drop them for this study.

Dipole Radiation (cont.)

The first term recalls the definition of the magnetic dipole moment (Equation 5.105)

$$\vec{m}_0 = \int_V d\tau' \frac{1}{2} \vec{r}' \times \vec{j}_0(\vec{r}') \quad (10.212)$$

Using this definition, we may rewrite the vector potential as

$$\vec{A}(\vec{r}, t) = i \frac{\mu_0}{4\pi} \frac{\omega \hat{r} \times \vec{m}_0}{c r} e^{i(kr - \omega t)} \quad d \ll r, d \ll \lambda \quad (10.213)$$

This expression is smaller than the analogous expression for electric dipole radiation (Equation 10.196) by a factor $(m_0/c)/p$, which we can see is $k d$ as expected:

$$\frac{1}{p} \frac{m_0}{c} = \frac{1}{\rho d} \frac{J d}{c} = \frac{1}{\rho d} \frac{\omega d \rho d}{c} = k d \quad (10.214)$$

where $J \propto \omega d \rho$ follows from our evaluation of \vec{J} in the electric dipole radiation case. The fields are easily derived using the same procedure as for the electric dipole term to evaluate the curls, yielding

$$\vec{B}(\vec{r}, t) = -\frac{\mu_0}{4\pi} \frac{\omega^2 \hat{r} \times (\hat{r} \times \vec{m}_0)}{c^2 r} e^{i(kr - \omega t)} \quad d \ll r, d \ll \lambda \quad (10.215)$$

Dipole Radiation (cont.)

For the electric field, we can use the harmonic relation $\vec{E} = -c\hat{r} \times \vec{B}$ (from $\vec{E} = c^2(i/\omega)\vec{\nabla} \times \vec{B}$) to find

$$\vec{E}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{\omega^2 \hat{r} \times (\hat{r} \times (\hat{r} \times \vec{m}_0))}{c^3 r} e^{i(kr - \omega t)} \quad (10.216)$$

In the electric dipole case, we had $\vec{B} \propto (1/c)\hat{r} \times \vec{E} \propto \hat{r} \times (\hat{r} \times (\hat{r} \times \vec{p}))$, but we also derived $\vec{B} \propto \hat{r} \times \vec{p}$. We thus suspect that the quadruple vector product in \vec{E} for magnetic dipole radiation reduces to $\hat{r} \times \vec{m}_0$. Let's prove that explicitly using the *BAC – CAB* rule for an arbitrary vector \vec{a} :

$$\hat{r} \times [\hat{r} \times (\hat{r} \times \vec{a})] = \hat{r}[\hat{r} \cdot (\hat{r} \times \vec{a})] - (\hat{r} \times \vec{a})(\hat{r} \cdot \hat{r}) = -\hat{r} \times \vec{a} \quad (10.217)$$

Thus, we may rewrite \vec{E} in the magnetic dipole case as

$$\vec{E}(\vec{r}, t) = -\frac{1}{4\pi\epsilon_0} \frac{\omega^2 \hat{r} \times \vec{m}_0}{c^3 r} e^{i(kr - \omega t)} \quad d \ll r, d \ll \lambda \quad (10.218)$$

Griffiths derives the magnetic dipole fields for the special case $\vec{m}_0 \propto \hat{z}$. We saw for the ideal electric dipole that $\hat{r} \times (\hat{r} \times \hat{z}) = \hat{\theta} \sin \theta$ and $\hat{r} \times (\hat{r} \times (\hat{r} \times \hat{z})) = \hat{\phi} \sin \theta$.

Applying that here yields $\vec{B} \propto -\hat{\theta} \sin \theta$ and $\vec{E} \propto \hat{\phi} \sin \theta$, thus matching Griffiths Equations 11.36 and 11.37.

Dipole Radiation (cont.)

As we did with the electric dipole radiation field, we can generalize these expressions using Fourier Transforms, yielding

magnetic dipole radiation
vector potential
 $d \ll r, d \ll \lambda$

$$\vec{A}(\vec{r}, t) = -i \frac{\mu_0}{4\pi} \frac{\hat{r} \times \dot{\vec{m}}(t_r)}{c r} \quad (10.219)$$

magnetic dipole radiation
magnetic field
 $d \ll r, d \ll \lambda$

$$\vec{B}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{\hat{r} \times (\hat{r} \times \ddot{\vec{m}}(t_r))}{c^2 r} \quad (10.220)$$

magnetic dipole radiation
electric field
 $d \ll r, d \ll \lambda$

$$\vec{E}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{\hat{r} \times \ddot{\vec{m}}(t_r)}{c^3 r} \quad (10.221)$$

where again all the dipole moments and derivatives thereof are evaluated at the retarded time. There is no analogue of these generic results in Griffiths, who only considers the special case of $\vec{m} \propto \hat{z}$ and harmonic time dependence, as we derived above.

Dipole Radiation (cont.)

To determine the power pattern, we can use the calculations we did for the electric dipole case because the dependences are similar: up to signs and normalization, \vec{B} for the electric case matches \vec{E} for the magnetic case and \vec{E} for the electric case matches \vec{B} for the magnetic case. In both cases, $\vec{E} \times \vec{B} \propto +\hat{r}$ as is necessary for outgoing radiation, so we can dispense with the signs. The normalization is set by the replacement $p \rightarrow m/c$. So, we have (again, $dP/d\Omega = r^2 \hat{r} \cdot \vec{S}$, all fields real)

magnetic dipole radiation pattern $d \ll r, d \ll \lambda$	$\frac{dP}{d\Omega} = \frac{\mu_0}{16\pi^2} \frac{\ddot{m}^2 \sin^2 \theta}{c^3}$	$P = \frac{\mu_0}{6\pi} \frac{\ddot{m}^2}{c^3}$	(10.222)
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If we assume harmonic time dependence and time-average (taking complex conjugations and real parts as necessary for complex notation), we obtain the analogue of Equations 10.167:

magnetic dipole radiation pattern harmonic time dependence $d \ll r, d \ll \lambda$	$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0}{32\pi^2} \frac{m_0^2 \omega^4 \sin^2 \theta}{c^3}$	$\langle P \rangle = \frac{\mu_0}{12\pi} \frac{m_0^2 \omega^4}{c^3}$
---	--	--

(10.223)

Given the replacement $p \rightarrow m/c$, the magnetic dipole radiation power is down by a factor of $[(m/c)/p]^2 = (k d)^2 = (2\pi d/\lambda)^2$ relative to electric dipole radiation.

Dipole Radiation (cont.)

Postscript

Griffiths is unfortunately lacking in examples on the topic of radiation. The reason for this is that, with the above formalism in hand, the basic examples consist largely of calculating electric and magnetic dipole moments and then plugging into the above formulae. Rather than provide much in the way of such plug-and-chug, it makes sense to move on to classical scattering theory and antennas, which provide more meaty and meaningful applications of the above.

If it seems like the above discussion was a bit tortured — the term-by-term manipulation of Equation 10.193, the by-hand splitting of the $m = 1$ term into magnetic dipole and electric quadrupole radiation, the reappearance of the same electric and magnetic field dependences — that impression is correct. There is a more unified way of treating the multipole expansion, which is given in Jackson Chapter 16, but which we do not have time to present.

Section 11

Relativity and Electrodynamics

Review of Special Relativity

Study Guide

Since you already studied the fundamentals of special relativity in Ph106b, we will skip Sections 12.1 and 12.2 of Griffiths and move directly to the applications to electrodynamics, albeit with some review of notation, definitions, etc.

Also, Griffiths' discussion in Sections 12.3.1 and 12.3.2 is largely a duplication of what you saw in Ph1bc. We will not reproduce that discussion here. Section 12.3.1 demonstrates, using the movement of a charged particle near a current-carrying wire, that a force that we consider magnetic in one frame of reference is purely electric in the rest frame of the charged particle. Section 12.3.2 derive the transformation of the fields by applying appropriate Lorentz contractions and time dilations to specific cases like capacitor plates and solenoids. Of course, you should review these discussions, which show that the electric and magnetic fields are tied to each other not just through Maxwell's Equations but also through relativistic transformations.

Instead, we will jump right into a fully relativistic formulation starting with sources and potentials. We will recover results we previously derived without the aid of relativity.

Review of Special Relativity (cont.)

Sign Convention

As with our definition of \square^2 , we deviate from Griffiths and follow the more standard sign convention for relativity where the inner product of a four-vector is of the form $(\text{time component})^2 - (\text{space components})^2$.

Reference Frames

As you know, special relativity is concerned with how physical quantities behave when we look at them in different *inertial reference frames* — frames that are moving at fixed velocity. A frame that moves at fixed velocity is simply one in which Newton's First Law is obeyed — an object's velocity is unchanged if no forces act on it. We will denote such frames by F , \tilde{F} , etc.

The term we use for the way a quantity behaves under such a reference frame transformation is *transformation properties*: some quantities are *invariant* under a change of reference frame, some quantities turn into linear combinations of one another (vectors and tensors) and some quantities simply don't follow any rules. Our goal is to formulate electrodynamics in such a way that the transformation properties of the various quantities we deal with — source distributions, potentials, fields, etc. — are made clear. We will see that electrodynamics is deeply related to relativity, a realization that in fact initiated the development of special relativity by Poincaré, Lorentz, Einstein, and others. In fact, Einstein's first paper on special relativity is titled "On the Electrodynamics of Moving Bodies" (*Annalen der Physik*, **17**: 891 (1905)).

Review of Special Relativity (cont.)

Four-Vectors in Special Relativity

A four-vector \vec{r} is an object whose *coordinate representation* in a given reference frame F consists of four numbers r^μ for which the transformation of the coordinate representation between inertial references frames F and \tilde{F} with F having relative velocity $c\beta = c\beta \hat{x}$ with respect to \tilde{F} is given by the Lorentz transformation law

$$\tilde{r}^\mu = \sum_{\mu, \nu=0}^3 \Lambda_\nu^\mu r^\nu \equiv \Lambda_\nu^\mu r^\nu \quad \text{with} \quad \Lambda_\nu^\mu = \begin{bmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (11.1)$$

The prototypical four-vector is the space-time position vector

$$r^\mu = (r^0, r^1, r^2, r^3) = (ct, x, y, z) \quad (11.2)$$

We introduce Greek indices to indicate 0, 1, 2, 3, and the *Einstein summation convention* wherein a repeated index implies a sum over that index. The term *contraction* is used to refer to matching up an index and summing over it.

Review of Special Relativity (cont.)

By application of three-dimension rotation matrices, one can show that, for an arbitrary direction of motion $\vec{\beta}$, the Lorentz transformation matrix has the form

$$\Lambda_0^0 = \gamma \quad \Lambda_i^0 = \Lambda_0^i = \gamma \beta_i \quad \Lambda_j^i = \delta_{ij} + (\gamma - 1) \frac{\beta_i \beta_j}{\beta^2} \quad i, j = 1, 2, 3 \quad (11.3)$$

where δ_{ij} is the usual Kronecker delta. We use Latin indices to refer to the indices 1, 2, 3 and/or summation over only those indices.

Review of Special Relativity (cont.)

The Invariant Norm of a Four-Vector and the Metric

We would like to define a quantity connected to \vec{r} that is invariant under Lorentz transformations (change of reference frame). Since we know how \vec{r} transforms, it is just a matter of constructing the right combination of the components of \vec{r} . It turns that such a quantity, the *invariant norm*, is given by

$$|\vec{r}|^2 = |r^\mu|^2 = g_{\mu\nu} r^\mu r^\nu \quad \text{with} \quad g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \quad (11.4)$$

g is the *metric*. It has coordinate representation $g_{\mu\nu}$ as given by the above in any frame (i.e., $\tilde{g}_{\mu\nu} = g_{\mu\nu}$ for any F and \tilde{F}). One can check by explicit calculation that this particular definition of g and the resulting definition of $|\vec{r}|^2$ is indeed independent of reference frame by working out the algebra to see

$$g_{\mu\nu} \tilde{r}^\mu \tilde{r}^\nu = g_{\mu\nu} r^\mu r^\nu \quad (11.5)$$

using the relation $\tilde{r}^\mu = \Lambda_\nu^\mu r^\nu$ and the definition of Λ above. It is important to recognize the chain of logic here: g cannot be *derived* from some more fundamental principle; it has this form because this form provides the invariant quantity we wanted.

Review of Special Relativity (cont.)

Tensors

More generally, a *n-th rank tensor* \mathcal{T} is an object that has 4^n components organized with n indices, $T^{\mu_1 \dots \mu_n}$ and that transforms under a change of reference frame via the rule

$$\tilde{T}^{\mu_1 \dots \mu_n} = \Lambda^{\mu_1}_{\nu_1} \dots \Lambda^{\mu_n}_{\nu_n} T^{\nu_1 \dots \nu_n} \quad (11.6)$$

A four-vector is a first-rank tensor. We will see second-rank tensors soon enough. A *four-scalar* (or just *scalar*) is a zeroth-rank tensor and thus is invariant under Lorentz transformations; conversely, any quantity that is invariant under Lorentz transformations is a scalar. From now on, when we talk about tensors, we also include four-vectors in the category. We will see that the metric is a second-rank tensor.

Review of Special Relativity (cont.)

Covariant and Contravariant Indices

With $g_{\mu\nu}$ defined as it is, we introduce the concept of “lowering” and “raising” indices. For tensors, we have only defined “raised” indices. We *define* “lowered indices” via contraction with the metric:

$$r_\mu = g_{\mu\nu} r^\nu \quad T^{\mu_1 \cdots \mu_{j-1}}{}_\nu^{\mu_{j+1} \cdots \mu_n} = g_{\nu\mu_j} T^{\mu_1 \cdots \mu_{j-1}\mu_j\mu_{j+1} \cdots \mu_n} \quad (11.7)$$

It is straightforward to see that $r_0 = r^0$ and $r_i = -r^i$ for $i = 1, 2, 3$.

If we are going to lower indices, we need a way to raise them, so we define in any frame

$$g^{\mu\nu} g_{\nu\sigma} = g_{\mu\nu} g^{\nu\sigma} = \delta_\sigma^\mu = \delta_\mu^\sigma \iff g^{\mu\nu} \equiv (g^{-1})_{\mu\nu} \quad (11.8)$$

where the $^{-1}$ implies the matrix inversion operation and where both versions of δ are the identity matrix with ones along the diagonal in any reference frame. We can check that this definition of $g^{\mu\nu}$ raises indices in a self-consistent way:

$$g^{\mu\nu} r_\nu = g^{\mu\nu} g_{\nu\sigma} r^\sigma = \delta_\sigma^\mu r^\sigma = r^\mu \quad (11.9)$$

That is, we recover the raised-index four-vector we started with if we use the above definitions.

Review of Special Relativity (cont.)

Now, because g is so simple, it turns out that $g^{\mu\nu} = g_{\mu\nu}$ (just try it out in any given reference frame). Note also that

$$g^\mu_\nu = g^{\mu\sigma} g_{\sigma\nu} = \delta^\mu_\nu \quad (11.10)$$

Moreover, it doesn't make much sense to write $\delta^{\mu\nu}$ or $\delta_{\mu\nu}$ because they are the metric,

$$\delta_{\mu\sigma} = g_{\mu\nu} \delta^\nu_\sigma = g_{\mu\sigma} \quad \delta^{\mu\sigma} = g^{\mu\nu} \delta_\nu^\sigma = g^{\mu\sigma} \quad (11.11)$$

We can derive Lorentz transformation properties for "lowered" indices by using the known properties of the raised indices and the metric:

$$\tilde{r}_\mu = \tilde{g}_{\mu\nu} \tilde{r}^\nu = g_{\mu\nu} \Lambda^\nu_\sigma r^\sigma = g_{\mu\nu} \Lambda^\nu_\sigma g^{\sigma\lambda} r_\lambda \quad (11.12)$$

$$\Rightarrow \quad \boxed{\tilde{r}_\mu = \tilde{g}_{\mu\nu} \tilde{r}^\nu = \Lambda_\mu^\lambda r_\lambda \quad \text{with} \quad \Lambda_\mu^\lambda = g_{\mu\nu} g^{\sigma\lambda} \Lambda^\nu_\sigma} \quad (11.13)$$

where we used $\tilde{g}_{\mu\nu} = g_{\mu\nu}$ (definition of metric) in the first step, the definition of the raising operation in the second step, and then we applied a self-consistent use of raising and lower operations on Λ^ν_σ in the third step. If one works it out in detail, one can see that the components Λ_μ^ν are given by Λ^μ_ν with a sign flip on the velocity vector.

Review of Special Relativity (cont.)

Raised indices are called *contravariant* indices and lowered indices are called *covariant*. The rationale for this is a bit tortured.

For the sake of developing intuition, consider a coordinate system rotation instead of Lorentz transformation. The rotation changes the coordinate axes. But a point in space remains fixed — the rotation is just a relabeling of points in space. So its coordinates in the new coordinate system are different than those in the old coordinate system. The amount by which the coordinates must change is exactly the opposite of the amount that the axes changed. For example, suppose one rotates about the z-axis by $+30^\circ$ so that the new x-axis is in the first quadrant of the old coordinate system. Then a point that was on the x-axis of the old coordinate system is now in the fourth quadrant of the new coordinate system. When the unit vectors are rotated in the positive direction, the coordinates of points rotate in the negative direction.

There is an analogous discussion for Lorentz transformations, leading us to conclude that the coordinates r^μ of a space-time point $\overset{\Rightarrow}{r}$ transform in a manner opposite to that by which the coordinate axes transform. Hence the term *contravariant* — “contra” meaning “opposite to”. Lower indices transform like the coordinate axes, so they are called *covariant*.

A four-vector can be covariant or contravariant because it has only one index. It is more complicated for rank- n tensors — index can be covariant or contravariant.

Review of Special Relativity (cont.)

Covariant Gradient

The next obvious tool we need is four-vector version of the gradient operator. The natural thing to want is for this gradient operator to do the right thing for Taylor expansions. Given a scalar function $S(\vec{r})$, we would like

$$S(\vec{r} + d\vec{r}) - S(\vec{r}) = (\nabla S)_\mu dr^\mu \quad (11.14)$$

Note that the left side is a coordinate-system-independent quantity — invariant under Lorentz transformation — while the right side is made of coordinate-system-dependent — contravariant and covariant under Lorentz transformation — quantities. We know by the form of the Taylor expansion that, in a particular reference frame,

$$(\nabla S)_\mu = \frac{\partial S}{\partial r^\mu} \equiv \partial_\mu S \quad \Rightarrow \quad \boxed{\partial_\mu \equiv \left(\frac{\partial}{\partial r^0}, \frac{\partial}{\partial r^1}, \frac{\partial}{\partial r^2}, \frac{\partial}{\partial r^3} \right)} \quad (11.15)$$

would yield the desired result. Thus, we use the above as the definition of the *four-gradient*.

Review of Special Relativity (cont.)

Note that the contravariant version is

$$\partial^\mu \equiv \left(\frac{\partial}{\partial r^0}, -\frac{\partial}{\partial r^1}, -\frac{\partial}{\partial r^2}, -\frac{\partial}{\partial r^3} \right) \quad (11.16)$$

The Lorentz-invariant second derivative is called the *d'Alembertian*, which we introduced earlier:

$$\square^2 = \partial_\mu \partial^\mu = \frac{\partial}{\partial(r^0)^2} - \frac{\partial}{\partial(r^1)^2} - \frac{\partial}{\partial(r^2)^2} - \frac{\partial}{\partial(r^3)^2} \quad (11.17)$$

$$\Rightarrow \boxed{\square^2 = \partial_\mu \partial^\mu = \frac{1}{c^2} \frac{\partial}{\partial t^2} - \vec{\nabla}^2} \quad (11.18)$$

Lecture 28: Relativity and Electrodynamics

Date Revised: 2014/06/03 10:15

2014/06/05 07:00 Correct errors in Eqn 11.46-11.50

2014/06/05 15:00 Correct error in definition of angular momentum
tensor

Date Given: 2014/06/03

Relativity and Electrodynamics

The Covariant Source Density

The natural way to define the charge/current source four-vector is to consider a charge distribution in its rest frame F and Lorentz transform it. If ρ_0 is the rest-frame charge density, and there is no current in this rest frame because no charges are moving, then, going to a new frame \tilde{F} relative to which F is moving at velocity $c\vec{\beta}$, we know the charge density increases due to Lorentz contraction and a current appears given by the charge density in \tilde{F} and the velocity $c\vec{\beta}$:

$$\rho = \gamma \rho_0 \quad \vec{J} = \rho \vec{v} = \rho_0 \gamma \vec{v} \quad (11.19)$$

This is easily summarized using the *four-velocity* $\overset{\Rightarrow}{v}$:

$$v^\mu = \gamma(c, \vec{v}) \quad J^\mu = \rho_0 v^\mu = \gamma(\rho_0 c, \rho_0 \vec{v}) = (\rho c, \rho \vec{v}) \quad (11.20)$$

Since $\overset{\Rightarrow}{v}$ is known to be a four-vector, and we are multiplying it by the invariant quantity ρ_0 , we are assured $\overset{\Rightarrow}{J}$ is also a four-vector, which we call the *covariant source density*. Note: ρ itself is not an invariant quantity, but the rest-frame charge density ρ_0 is invariant because the total amount of charge cannot depend on the reference frame. The continuity equation is then written:

$$0 = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = \partial_\mu J^\mu \quad (11.21)$$

Relativity and Electrodynamics (cont.)

The Covariant Potential

Recall Equations 10.21, our inhomogeneous wave equations in the Lorenz gauge:

$$\square^2 V = \frac{\rho}{\epsilon_0} \quad \square^2 = \mu_0 \vec{J} \quad \square^2 \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \quad (11.22)$$

If we define the *covariant potential* \vec{A} by

$$A^\mu \equiv \left(\frac{V}{c}, \vec{A} \right) \quad (11.23)$$

then the above equations can be rewritten as the *covariant wave equation*

$$\square^2 \vec{A} = \mu_0 \vec{J} \quad (11.24)$$

Since \vec{J} is a four-vector and \square^2 and μ_0 are scalars, we are assured that \vec{A} is a four-vector as desired. Of course, the above equations only apply in Lorenz gauge, the condition for which can be written in a relativistic form:

$$0 = \epsilon_0 \mu_0 \frac{\partial V}{\partial t} + \vec{\nabla} \cdot \vec{A} = \partial_\mu A^\mu = \nabla \cdot \vec{A} \quad (11.25)$$

The Lorenz gauge condition is invariant under change of reference frame because it is a four-scalar by definition. Note how it is the natural extension of $\nabla \cdot \vec{A} = 0$.

Relativity and Electrodynamics (cont.)

Lienard-Wiechert Potential via Lorentz Transformation

The definition of \vec{A} makes derivation of the Lienard-Wiechert potentials trivial. In the rest frame F of a point charge q , the covariant potential has components is

$$V(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{[x^2 + y^2 + z^2]^{1/2}} \quad \vec{A}(\vec{r}, t) = 0 \quad (11.26)$$

Now, to have the charge move with velocity $\vec{v} = c\vec{\beta}$, we simply need to obtain \vec{A} in a lab frame \tilde{F} relative to which the charge's rest frame F moves at \vec{v} and Lorentz transform the covariant potential and the coordinates. That is, we use the Lorentz transformation Equation 11.1

$$\frac{1}{c}\tilde{V} = \gamma \left[\frac{1}{c}V + \beta A_x \right] = \frac{\gamma}{c}V \quad \tilde{A}_x = \gamma \left[A_x + \beta \frac{1}{c}V \right] = \frac{\gamma\beta}{c}V \quad (11.27)$$

and $\tilde{A}_y = A_y = 0$ and $\tilde{A}_z = A_z = 0$.

Relativity and Electrodynamics (cont.)

This is half the work. V is of course written in terms of the rest frame coordinates $r^\mu = (c t, x, y, z)$, so we need to rewrite it in terms of the lab frame coordinates $\tilde{r}^\mu = (c \tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$. These coordinates are also related by Lorentz transformation, but we need the one going in the opposite direction (because we are now writing F coordinates in terms of \tilde{F} coordinates)

$$c t = \gamma [c \tilde{t} - \beta \tilde{x}] \quad x = \gamma [-\beta c \tilde{t} + \tilde{x}] \quad (11.28)$$

and $y = \tilde{y}$ and $z = \tilde{z}$. Combining the Lorentz transformation of the potential with the above transformation of the coordinates, we obtain:

$$\tilde{V}(\tilde{r}^\mu) = \frac{1}{4 \pi \epsilon_0} \frac{\gamma q}{\left[\gamma^2 (\beta c \tilde{t} - \tilde{x})^2 + \tilde{y}^2 + \tilde{z}^2 \right]^{1/2}} \quad (11.29)$$

$$= \frac{1}{4 \pi \epsilon_0} \frac{q}{\left[\left\{ (\tilde{x} - v \tilde{t})^2 + \tilde{y}^2 + \tilde{z}^2 \right\} - \beta^2 (\tilde{y}^2 + \tilde{z}^2) \right]^{1/2}} \quad (11.30)$$

The coordinates in the above are the current position of the charge in \tilde{F} , so we should be able to match this expression to Equation 10.101, the Lienard-Wiechert scalar potential in terms of the current position of the particle.

Relativity and Electrodynamics (cont.)

If we define

$$R(\tilde{r}^\mu) = \left[(\tilde{x} - v \tilde{t})^2 + \tilde{y}^2 + \tilde{z}^2 \right]^{1/2} \quad \sin^2 \theta = \frac{\tilde{y}^2 + \tilde{z}^2}{\left[(\tilde{x} - v \tilde{t})^2 + \tilde{y}^2 + \tilde{z}^2 \right]} \quad (11.31)$$

then we have

$$\tilde{V}(\tilde{r}^\mu) = \frac{1}{4 \pi \epsilon_0} \frac{q}{R(\tilde{r}^\mu)} \frac{1}{[1 - \beta^2 \sin^2 \theta]^{1/2}} \quad (11.32)$$

$$\tilde{A}_x(\tilde{r}^\mu) = \frac{1}{c^2} \frac{1}{4 \pi \epsilon_0} \frac{q v}{R(\tilde{r}^\mu)} \frac{1}{[1 - \beta^2 \sin^2 \theta]^{1/2}} \quad (11.33)$$

which matches Equations 10.101 and 10.102 (remember, $c^2 \epsilon_0 = 1/\mu_0$).

The above calculation is an excellent example of how enmeshed relativity and electrodynamics are. What was previously a painful calculation, involving complicated derivatives of retarded time and extreme care with delta function properties, became trivial via Lorentz transformation.

Relativity and Electrodynamics (cont.)

Electric and Magnetic Fields

We define the *electromagnetic field tensor* or the *Faraday tensor* \mathcal{F} by

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \iff -F^{0j} = F^{j0} = \frac{1}{c} E_j \quad F^{ij} = -\epsilon_{ijk} B_k \quad (11.34)$$

where i, j, k run over 1, 2, 3 as explained earlier and ϵ_{ijk} is the completely antisymmetric Levi-Civita symbol. (We define $E_j \equiv E^j$ and $B_j \equiv B^j$). From the above,

$$\iff F^{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} \quad (11.35)$$

Thanks to the definition of \mathcal{F} in terms of the covariant gradient and the covariant potential, we are assured it is a second-rank tensor. \mathcal{F} is clearly the Lorentz-covariant way to represent the electric and magnetic fields: its second-rank tensor transformation properties provide all the information we need. Given the above, the field tensor's explicit transformation law is

$$\tilde{F}^{\mu\nu} = \Lambda^\mu_\lambda \Lambda^\nu_\sigma F^{\lambda\sigma} \quad (11.36)$$

Relativity and Electrodynamics (cont.)

Let's write these transformation laws in terms of the fields explicitly so we can see the transformation rules more clearly. We have (where we *define* lowered roman indices by $E_j \equiv E^j$, $B_j \equiv B^j$, $\beta_i \equiv \beta^i$, $v_i \equiv v^i = c\beta^i$, and $\delta_{jk} \equiv \delta^j_{~k}$):

$$\tilde{E}_j = -c \tilde{F}^{0j} = -c \Lambda^0_{~\lambda} \Lambda^j_{~\sigma} F^{\lambda\sigma} \quad (11.37)$$

$$= -c \left[\Lambda^0_{~0} \Lambda^j_{~0} F^{00} + \Lambda^0_{~0} \Lambda^j_{~k} F^{0k} + \Lambda^0_{~k} \Lambda^j_{~0} F^{k0} + \Lambda^0_{~k} \Lambda^j_{~\ell} F^{k\ell} \right] \quad (11.38)$$

where we have expanded out the sums into space and time components. We use $F^{00} = 0$ and the generic forms of the components of Λ , Equation 11.3,

$$\tilde{E}_j = -c \left[\gamma \left(\delta_{jk} + (\gamma - 1) \frac{\beta_j \beta_k}{\beta^2} \right) F^{0k} + \gamma \beta_k \gamma \beta_j F^{k0} + \gamma \beta_k \left(\delta_{j\ell} + (\gamma - 1) \frac{\beta_j \beta_\ell}{\beta^2} \right) F^{k\ell} \right] \quad (11.39)$$

Next, we use the antisymmetry of \mathcal{F} : $F^{k0} = -F^{0k}$ and $\beta_k \beta_\ell F^{k\ell} = 0$ in the last term of the last term. This yields

$$\tilde{E}_j = -c \left[\gamma F^{0j} + \frac{\beta_j \beta_k}{\beta^2} [\gamma^2 - \gamma - \beta^2 \gamma^2] F^{0k} + \gamma \beta_k F^{kj} \right] \quad (11.40)$$

Relativity and Electrodynamics (cont.)

Next, we use $F^{0j} = -E_j/c$ and $F^{kj} = -F^{jk} = \epsilon_{jkl} B_\ell$:

$$\tilde{E}_j = \gamma E_j + (1 - \gamma) \frac{\beta_j \beta_k}{\beta^2} E_k - \gamma v_k \epsilon_{jkl} B_\ell \quad (11.41)$$

We will reduce this a simpler form soon, but let's obtain the analogous result for the magnetic field first. To extract \vec{B} from \mathcal{F} , we need the identity $\epsilon_{ijk} \epsilon_{ijm} = 2 \delta_{km}$. With this,

$$-\frac{1}{2} \epsilon_{jkl} F^{kl} = \frac{1}{2} \epsilon_{jkl} \epsilon_{klm} B_m = \frac{1}{2} \epsilon_{k\ell j} \epsilon_{k\ell m} B_m = \delta_{jm} B_m = B_j \quad (11.42)$$

Therefore,

$$\tilde{B}_j = -\frac{1}{2} \epsilon_{jkl} \tilde{F}^{kl} = -\frac{1}{2} \epsilon_{jkl} \Lambda^\lambda_\lambda \Lambda^\ell_\sigma F^{\lambda\sigma} \quad (11.43)$$

$$\begin{aligned} &= -\frac{1}{2} \epsilon_{jkl} \left[\gamma \beta_k \left(\delta_{\ell m} - (\gamma - 1) \frac{\beta_\ell \beta_m}{\beta^2} \right) F^{0m} + \left(\delta_{km} - (\gamma - 1) \frac{\beta_k \beta_m}{\beta^2} \right) \gamma \beta_\ell F^{m0} \right. \\ &\quad \left. + \left(\delta_{km} - (\gamma - 1) \frac{\beta_k \beta_m}{\beta^2} \right) \left(\delta_{\ell n} - (\gamma - 1) \frac{\beta_\ell \beta_n}{\beta^2} \right) F^{mn} \right] \quad (11.44) \end{aligned}$$

where we have used $F^{00} = 0$ already.

Relativity and Electrodynamics (cont.)

Using $F^{m0} = -F^{0m}$, $F^{m\ell} = -F^{\ell m}$, $F^{kn} = -F^{nk}$, $\epsilon_{jkl} \beta_k \beta_\ell = 0$ and $\beta_m \beta_n F^{mn} = 0$:

$$\tilde{B}_j = -\frac{1}{2} \epsilon_{jkl} \left[\gamma \beta_k F^{0\ell} - \gamma \beta_\ell F^{0k} + F^{k\ell} - \frac{\gamma - 1}{\beta^2} (\beta_k \beta_m F^{\ell m} + \beta_\ell \beta_n F^{nk}) \right] \quad (11.45)$$

then, using $\epsilon_{jkl} = -\epsilon_{j\ell k}$, $F^{0a} = -E_a/c$, $-\epsilon_{jkl} F^{k\ell}/2 = B_j$ and $F^{ab} = -\epsilon_{abc} B_c$:

$$\tilde{B}_j = \frac{1}{2c} \gamma \left[\epsilon_{jkl} \beta_k E^\ell + \epsilon_{j\ell k} \beta_\ell E^k \right] + B_j - \frac{1}{2} \epsilon_{jkl} \frac{\gamma - 1}{\beta^2} [\beta_k \beta_m \epsilon_{\ell mn} B_n + \beta_\ell \beta_n \epsilon_{nkm} B_m] \quad (11.46)$$

and, finally, using the cyclicity and antisymmetry of ϵ_{abc} in its indices followed by the identity $\epsilon_{abc} \epsilon_{dec} = \delta_{ad} \delta_{be} - \delta_{ae} \delta_{bd}$ and then apply the δ 's and rearranging:

$$\tilde{B}_j = \frac{\gamma}{c^2} \epsilon_{jkl} v_k E_\ell + B_j - \frac{1}{2} \frac{\gamma - 1}{\beta^2} [\beta_k \beta_m B_n \epsilon_{jkl} \epsilon_{mn\ell} + \beta_\ell \beta_n B_m \epsilon_{j\ell k} \epsilon_{nmk}] \quad (11.47)$$

$$= \frac{\gamma}{c^2} \epsilon_{jkl} v_k E_\ell + B_j - \frac{\gamma - 1}{\beta^2} \beta_k \beta_m B_n [\delta_{jm} \delta_{kn} - \delta_{jn} \delta_{km}] \quad (11.48)$$

$$= \frac{\gamma}{c^2} \epsilon_{jkl} v_k E_\ell + B_j - \frac{\gamma - 1}{\beta^2} [\beta_j \beta_k B_k - \beta^2 B_j] \quad (11.49)$$

$$= \gamma B_j + (1 - \gamma) \frac{\beta_j \beta_k}{\beta^2} B_k + \frac{\gamma}{c^2} \epsilon_{jkl} v_k E_\ell \quad (11.50)$$

which we see is a form very similar to the one we obtained for the transformation of the electric field, Equation 11.41.

Relativity and Electrodynamics (cont.)

We may write Equations 11.41 and 11.50 in three-vector form:

$$\tilde{\vec{E}} = \gamma \vec{E} + (1 - \gamma) \hat{\beta} (\hat{\beta} \cdot \vec{E}) - \gamma \vec{v} \times \vec{B} \quad (11.51)$$

$$\tilde{\vec{B}} = \gamma \vec{B} + (1 - \gamma) \hat{\beta} (\hat{\beta} \cdot \vec{B}) + \frac{\gamma}{c^2} \vec{v} \times \vec{E} \quad (11.52)$$

Let us resolve the fields into pieces parallel and perpendicular to the velocity:

$$\begin{aligned} \tilde{\vec{E}}_{||} &= \hat{\beta} \hat{\beta} \cdot \vec{E} & \vec{E}_{\perp} &= \vec{E} - \hat{\beta} \hat{\beta} \cdot \vec{E} & \tilde{\vec{B}}_{||} &= \hat{\beta} \hat{\beta} \cdot \vec{B} & \vec{B}_{\perp} &= \vec{B} - \hat{\beta} \hat{\beta} \cdot \vec{B} \end{aligned} \quad (11.53)$$

Thus, we have

$$\tilde{E}_{||} = E_{||} \quad \tilde{\vec{E}}_{\perp} = \gamma [\vec{E}_{\perp} - \vec{v} \times \vec{B}_{\perp}] \quad (11.54)$$

$$\tilde{B}_{||} = B_{||} \quad \tilde{\vec{B}}_{\perp} = \gamma \left[\vec{B}_{\perp} + \frac{1}{c^2} \vec{v} \times \vec{E}_{\perp} \right] \quad (11.55)$$

Thus, we recover the results in Griffiths §12.3.2 where the transformation properties of the fields are derived. (These are the equivalent of Equations 12.109 in Griffiths, accounting for the fact that those equations give the field in F in terms of the field in \tilde{F} and thus have a sign flip on the velocity.) We see that these properties have fallen out of the Lorentz transformation of the \mathcal{F} tensor.

Relativity and Electrodynamics (cont.)

Fields of a Moving Point Charge

Let's do the obvious and check that the field transformation properties we have derived yield the same fields for the moving point charge as we derived (more painfully) by differentiating the Lienard-Wiechert potentials. The fields in the charge's rest frame are

$$\vec{E}(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} \vec{r} \quad \vec{B}(\vec{r}, t) = 0 \quad (11.56)$$

We take $\vec{v} = v \hat{x}$. Applying the Lorentz transformation to the fields, we obtain

$$\tilde{E}_x(r^\mu) = \tilde{E}_{||} = E_{||} = E_x = \frac{1}{4\pi\epsilon_0} \frac{q}{r^3} x \quad (11.57)$$

$$\tilde{B}_x(r^\mu) = \tilde{B}_{||} = B_{||} = B_x = 0 \quad (11.58)$$

$$\tilde{E}_{yz}(r^\mu) = \tilde{\vec{E}}_\perp = \gamma \left[\vec{E}_\perp - \vec{v} \times \vec{B}_\perp \right] = \frac{1}{4\pi\epsilon_0} \frac{\gamma q}{r^3} (y \hat{y} + z \hat{z}) \quad (11.59)$$

$$\tilde{B}_{yz}(r^\mu) = \tilde{\vec{B}}_\perp = \gamma \left[\vec{B}_\perp + \frac{1}{c^2} \vec{v} \times \vec{E}_\perp \right] = \frac{1}{c^2} \vec{v} \times \tilde{\vec{E}} \quad (11.60)$$

These fields are still in terms of the F -frame coordinates, so they are not very useful as is.

Relativity and Electrodynamics (cont.)

As with the Lienard-Wiechert potential example, we use the Lorentz transformation of the space-time vector to rewrite r^μ in terms of \tilde{r}^μ ,

$$c t = \gamma [c \tilde{t} - \beta \tilde{x}] \quad x = \gamma [-\beta c \tilde{t} + \tilde{x}] \quad (11.61)$$

and $y = \tilde{y}$ and $z = \tilde{z}$. Therefore,

$$\tilde{E}_x = \frac{1}{4 \pi \epsilon_0} \frac{\gamma q}{[\gamma^2 (\tilde{x} - v \tilde{t})^2 + \tilde{y}^2 + \tilde{z}^2]^{3/2}} (\tilde{x} - v \tilde{t}) \quad (11.62)$$

$$\tilde{E}_{yz} = \frac{1}{4 \pi \epsilon_0} \frac{\gamma q}{[\gamma^2 (\tilde{x} - v \tilde{t})^2 + \tilde{y}^2 + \tilde{z}^2]^{3/2}} (\tilde{y} \hat{y} + \tilde{z} \hat{z}) \quad (11.63)$$

which may write as

$$\tilde{\vec{E}}(\tilde{r}^\mu) = \frac{1}{4 \pi \epsilon_0} \frac{\gamma q}{[\gamma^2 (\tilde{x} - v \tilde{t})^2 + \tilde{y}^2 + \tilde{z}^2]^{3/2}} \vec{R}(\tilde{r}^\mu) \quad (11.64)$$

$$= \frac{1}{4 \pi \epsilon_0} \frac{q}{\gamma^2 [\{(\tilde{x} - v \tilde{t})^2 + \tilde{y}^2 + \tilde{z}^2\} - \beta^2 (\tilde{y}^2 + \tilde{z}^2)]^{3/2}} \vec{R}(\tilde{r}^\mu) \quad (11.65)$$

where $\vec{R} = \tilde{r} - \vec{v} \tilde{t}$.

Relativity and Electrodynamics (cont.)

Just as we did for the Lienard-Wiechert potentials, we may write this in a form closer to the current-position version we obtained for the electric field, Equations 10.117:

$$\tilde{\vec{E}} = \frac{1}{4\pi\epsilon_0} \frac{q \vec{R}(\tilde{r}^\mu)}{[R(\tilde{r}^\mu)]^3} \frac{1 - \beta^2}{[1 - \beta^2 \sin^2 \theta]^{3/2}} \quad (11.66)$$

The expression for the magnetic field may be summarized as

$$\tilde{\vec{B}}(\tilde{r}^\mu) = \frac{1}{c} \vec{\beta} \times \tilde{\vec{E}}(\tilde{r}^\mu) \quad (11.67)$$

which matches Equation 10.119.

Relativity and Electrodynamics (cont.)

Electromagnetic Field Tensor Invariant

We can construct an invariant quantity by contracting \mathcal{F} over both indices, $F_{\mu\nu} F^{\mu\nu}$. To see what value this takes on, let's first calculate $F_{\mu\nu}$:

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix} \quad (11.68)$$

$$\Rightarrow F_{\mu\nu} = \begin{bmatrix} 0 & E_x/c & E_y/c & E_z/c \\ -E_x/c & 0 & -B_z & B_y \\ -E_y/c & B_z & 0 & -B_x \\ -E_z/c & -B_y & B_x & 0 \end{bmatrix} \quad (11.69)$$

Then, $F_{\mu\nu} F^{\mu\nu}$ consists of multiplying the two matrices not with standard matrix multiplication but rather element by element and then summing over all elements. That yields the Lorentz invariant:

$$F_{\mu\nu} F^{\mu\nu} = -\frac{2}{c^2} \vec{E} \cdot \vec{E} + 2 \vec{B} \cdot \vec{B} \quad \Rightarrow \quad \boxed{-\frac{c^2}{2} F_{\mu\nu} F^{\mu\nu} = E^2 - c^2 B^2} \quad (11.70)$$

Relativity and Electrodynamics (cont.)

The Dual Electromagnetic Field Tensor and a Second Field Tensor Invariant

We may define the *dual tensor* \mathcal{G} using the completely antisymmetric four-index Levi-Civita symbol $\epsilon^{\mu\nu\lambda\sigma}$:

$$G^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} F_{\lambda\sigma} \quad \implies \quad G^{00} = 0 \quad G^{jj} = 0 \quad (11.71)$$

and

$$-G^{0j} = G^{j0} = \frac{1}{2} \epsilon^{0j}_{\lambda\sigma} F^{\lambda\sigma} \quad G^{ij} = \frac{1}{2} \epsilon^{ij}_{\lambda\sigma} F^{\lambda\sigma} \quad (11.72)$$

$$= \frac{1}{2} \epsilon^{0j}_{ik} F^{ik} \quad = \frac{1}{2} (\epsilon^{ij}_{0k} F^{0k} + \epsilon^{ij}_{k0} F^{k0}) \quad (11.73)$$

$$= -\frac{1}{2} \epsilon_{jik} \epsilon_{ik\ell} B_\ell \quad = \epsilon^{ij}_{k0} F^{k0} \quad (11.74)$$

$$= \frac{1}{2} \epsilon_{jik} \epsilon_{\ell ik} B_\ell \quad = \frac{1}{c} \epsilon_{ijk} E_k \quad (11.75)$$

$$= \delta_{j\ell} B_\ell = B_j \quad (11.76)$$

Relativity and Electrodynamics (cont.)

We may write \mathcal{G} out component by component:

$$G^{\mu\nu} = \begin{bmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z/c & -E_y/c \\ B_y & -E_z/c & 0 & E_x/c \\ B_z & E_y/c & -E_x/c & 0 \end{bmatrix} \quad (11.77)$$

which is obtained from $F^{\mu\nu}$ by the replacement $\vec{E}/c \rightarrow \vec{B}$ and $\vec{B} \rightarrow -\vec{E}/c$.

What invariants can be formed with \mathcal{G} ? Well, $G_{\mu\nu} G^{\mu\nu} = F_{\mu\nu} F^{\mu\nu}$ because the resulting contraction of the Levi-Civita symbol with itself yields an identity operator. (Or, just make the \vec{E} and \vec{B} replacements in $F_{\mu\nu} F^{\mu\nu} = E^2 - c^2 B^2$ and the result is the same.) A new invariant is obtained by

$$F_{\mu\nu} G^{\mu\nu} = -\frac{4}{c} \vec{E} \cdot \vec{B} \quad \Rightarrow \quad -\frac{c}{4} F_{\mu\nu} G^{\mu\nu} = \vec{E} \cdot \vec{B} \quad (11.78)$$

Relativity and Electrodynamics (cont.)

Maxwell's Equations

Finally, we can rewrite Maxwell's Equations in a very simple form using the electromagnetic field tensor and its dual. The inhomogeneous equations (the ones with source terms on the right side, Gauss's Law and Ampere's Law) are:

$$\partial_\mu F^{\mu\nu} = \partial_\mu \partial^\mu A^\nu - \partial_\mu \partial^\nu A^\mu = \square^2 A^\nu - \partial^\nu \partial_\mu A^\mu = \mu_0 J^\nu - 0 \quad (11.79)$$

$$\boxed{\partial_\mu F^{\mu\nu} = \mu_0 J^\nu} \quad (11.80)$$

where the second term vanishes due to the Lorenz gauge condition (Equation 11.25). The homogenous equations are

$$\partial_\mu G^{\mu\nu} = \frac{1}{2} \partial_\mu \epsilon^{\mu\nu\lambda\sigma} (\partial_\lambda A_\sigma - \partial_\sigma A_\lambda) = \partial_\mu \epsilon^{\mu\nu\lambda\sigma} \partial_\lambda A_\sigma = 0 \quad (11.81)$$

$$\boxed{\partial_\mu G^{\mu\nu} = 0} \quad (11.82)$$

where the second step is possible because $\epsilon^{\mu\nu\lambda\sigma}$ are both antisymmetric under exchange of λ and σ and the third step holds because $\partial_\mu \partial_\lambda$ is symmetric under exchange of these indices. It is interesting that physics statements — Faraday's law and the divergencelessness of \vec{B} — reduce to a mathematical identity via the definition of \mathcal{G} . This is a sign that we have defined \mathcal{G} well!

Relativity and Electrodynamics (cont.)

It is straightforward to see that these equations yield the standard Maxwell Equations:

$$\partial_\mu F^{\mu 0} = \mu_0 J^0 \quad \partial_\mu F^{\mu i} = \mu_0 J^i \quad (11.83)$$

$$\partial_0 F^{00} + \partial_i F^{i0} = \mu_0 \rho c \quad \frac{1}{c} \partial_t F^{0i} + \partial_j F^{ji} = \mu_0 J^i \quad (11.84)$$

$$0 + \frac{\partial}{\partial r^i} \frac{E_i}{c} = \frac{1}{c \epsilon_0} \rho \quad - \frac{1}{c^2} \frac{\partial E^i}{\partial t} - \epsilon_{jik} \frac{\partial B_k}{\partial r^j} = \mu_0 J^i \quad (11.85)$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \quad \vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \epsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} \quad (11.86)$$

Since the structure of \mathcal{G} parallels that of \mathcal{F} with the replacement $E^i/c \rightarrow B^i$ and $B^i \rightarrow -E^i/c$, we may reuse the above arithmetic to see that:

$$\partial_\mu G^{\mu 0} = 0 \quad \partial_\mu G^{\mu i} = 0 \quad (11.87)$$

$$\vec{\nabla} \cdot \vec{B} = 0 \quad - \frac{1}{c} \vec{\nabla} \times \vec{E} = c \epsilon_0 \mu_0 \frac{\partial \vec{B}}{\partial t} \quad (11.88)$$

$$\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t} \quad (11.89)$$

Relativity and Electrodynamics (cont.)

Lorentz Force

We can also rewrite the Lorentz Force in a relativistically covariant way. As with the definition of the covariant source density, we use the rest-frame relation and the Lorentz transformation to figure out what the correct form is. In the rest frame of a particle of charge q , we know the nonrelativistic force is

$$\frac{d}{dt} \left(m \frac{dx^i}{dt} \right) = q E^i = q c F^{i0} \quad (11.90)$$

where $m dx^i/dt$ is the nonrelativistic momentum (we don't use v^i or p^i to avoid confusion with what will follow). How do we make this relativistic? The quantity being differentiated on the left side is $m dx^i/dt$; its natural generalization is, as you know, the covariant momentum $p^\mu = m v^\mu$. On the right side, c is the rest-frame time component of v^μ , so $c F^{i0}$ looks like the time component of $v_\nu F^{\nu i}$. We are still stuck with a d/dt on the left side, but that becomes relativistically invariant if we replace it with $d/d\tau$ where τ is the proper time of the particle. Thus, we are motivated to write

$$\boxed{\frac{dp^\mu}{d\tau} = q F^{\mu\nu} v_\nu} \quad (11.91)$$

This expression is now relativistically covariant because both sides of the equation have a four-vector multiplying a scalar.

Relativity and Electrodynamics (cont.)

Of course, we need to check that this generalization yields the correct Lorentz Force in an arbitrary frame, so let's evaluate it in a frame in which the charged particle is not at rest. We have for the space components (using $\vec{v} \cdot \hat{r}_i$ for the i th component of the nonrelativistic \vec{v} to avoid confusion with the space components of the relativistic v^μ)

$$\frac{dp^i}{d\tau} = q \left(F^{i0} v_0 + F^{ij} v_j \right) \quad (11.92)$$

$$\frac{dt}{d\tau} \frac{dp^i}{dt} = q \left(\frac{E_i}{c} \gamma c - \epsilon_{ijk} B_k \gamma (-\vec{v} \cdot \hat{r}_j) \right) \quad (11.93)$$

$$\gamma (\vec{F} \cdot \hat{r}_i) = \gamma q \left(E_i + (\vec{v} \times \vec{B})_i \right) \quad (11.94)$$

$$\implies \vec{F} = q \left(\vec{E} + \vec{v} \times \vec{B} \right) \quad (11.95)$$

where $v_i = -v^i = -\vec{v} \cdot \hat{r}_i$ by the index-lowering rules and we note the subtlety that, in relativity, Newton's second law is (e.g., Equation 12.59 of Griffiths),

$$\vec{F} \cdot \hat{r}_i = \frac{dp^i}{dt} = \frac{d}{dt} (\gamma m \vec{v} \cdot \hat{r}_i) \quad (11.96)$$

We note that Griffiths defines $K^\mu = dp^\mu/d\tau$ as the *Minkowski force* because it is a relativistically covariant (transforms by the Lorentz transformation rule) quantity; dp^μ/dt is not.

Relativity and Electrodynamics (cont.)

Let's look at the time component that comes along for free in the above. It is

$$\frac{d}{d\tau} \gamma m c = q \left(F^{00} v_0 + F^{0i} v_i \right) \quad (11.97)$$

$$\frac{dt}{d\tau} \frac{d}{dt} \gamma m c = q \left[(0)(c) + \left(-\frac{E_i}{c} \right) (-\gamma \vec{v} \cdot \hat{\vec{r}}_i) \right] \quad (11.98)$$

$$\gamma \frac{d}{dt} \gamma m c = \gamma q \frac{E_i}{c} \vec{v} \cdot \hat{\vec{r}}_i \quad (11.99)$$

$$\frac{dU}{dt} = q \vec{v} \cdot \vec{E} \quad (11.100)$$

where $U = \gamma m c^2$ is the energy of the particle and we again used $v_i = -v^i = -\vec{v} \cdot \hat{\vec{r}}_i$. This is just conservation of energy: the rate of change of the particle energy is given by the work being done by the electric field (the magnetic field does no work).

Relativity and Electrodynamics (cont.)

Maxwell Energy-Momentum Tensor and Conservation of Energy-Momentum

It is natural to generalize the Maxwell Stress Tensor that we defined some time ago. We define the *Maxwell Energy-Momentum Tensor* \mathcal{T} as

$$\boxed{T^{\mu\nu} = \frac{1}{\mu_0} \left[F_{\lambda}^{\mu} F^{\lambda\nu} + \frac{1}{4} g^{\mu\nu} F_{\lambda\sigma} F^{\lambda\sigma} \right]} \quad (11.101)$$

The components of \mathcal{T} are (recall, we found the relativistic invariant $F_{\mu\nu} F^{\mu\nu}$ before):

$$T^{00} = \frac{1}{\mu_0} \left[F_0^0 F^{00} + g_{ij} F^{0j} F^{i0} + \frac{1}{4} g^{00} \left(-\frac{2}{c^2} \right) (E^2 - c^2 B^2) \right] \quad (11.102)$$

$$= \frac{1}{\mu_0} \left[(-1) \left(-\frac{E^2}{c^2} \right) - \frac{1}{2} \left(\frac{E^2}{c^2} - B^2 \right) \right] = \frac{\epsilon_0}{2} (E^2 + c^2 B^2) = u \quad (11.103)$$

where u is the EM field energy density.

Relativity and Electrodynamics (cont.)

The space-space components are

$$T^{ij} = \frac{1}{\mu_0} \left[F^{i\lambda} g_{\lambda\sigma} F^{\sigma j} - \frac{g^{ij}}{2} \left(\frac{E^2}{c^2} - B^2 \right) \right] \quad (11.104)$$

$$= \frac{1}{\mu_0} \left[g_{00} F^{i0} F^{0j} + g_{k\ell} (F^{ik} F^{\ell j}) + \frac{\delta^{ij}}{2} \left(\frac{E^2}{c^2} - B^2 \right) \right] \quad (11.105)$$

$$= \frac{1}{\mu_0} \left[F^{i0} F^{0j} - (F^{ik} F^{kj}) + \frac{\delta^{ij}}{2} \left(\frac{E^2}{c^2} - B^2 \right) \right] \quad (11.106)$$

$$= \frac{1}{\mu_0} \left[-\frac{1}{c^2} E_i E_j - \epsilon_{ik\ell} B_\ell \epsilon_{kjm} B_m + \frac{\delta^{ij}}{2} \left(\frac{E^2}{c^2} - B^2 \right) \right] \quad (11.107)$$

$$= \epsilon_0 \left[-E_i E_j + c^2 \epsilon_{i\ell k} B_\ell \epsilon_{jm k} B_m + \frac{\delta^{ij}}{2} (E^2 - c^2 B^2) \right] \quad (11.108)$$

$$= \epsilon_0 \left[-E_i E_j + c^2 (\delta_{ij} \delta_{\ell m} - \delta_{im} \delta_{\ell j}) B_\ell B_m + \frac{\delta^{ij}}{2} (E^2 - c^2 B^2) \right] \quad (11.109)$$

$$= \epsilon_0 \left[-E_i E_j - c^2 B_i B_j + \frac{\delta^{ij}}{2} (E^2 + c^2 B^2) \right] \quad (11.110)$$

which is the *negative* of the Maxwell Stress Tensor we defined in Equation 8.30. Note that we allow space-space (roman) indices to be contracted without requiring one raised and one lowered index.

Relativity and Electrodynamics (cont.)

The space-time components are

$$T^{0i} = \frac{1}{\mu_0} F^{0\lambda} g_{\lambda\sigma} F^{\sigma i} = \frac{1}{\mu_0} F^{0j} g_{jj} F^{ji} \quad (11.111)$$

where we used $F^{00} = 0$ and g being diagonal to drop the $\lambda = 0$ and $\sigma = 0$ terms and to eliminate the sum over σ . Then, we have

$$T^{0i} = -\frac{1}{\mu_0} \left(-\frac{E_j}{c} \right) \epsilon_{jik} B_k = -\frac{1}{\mu_0} \frac{1}{c} \epsilon_{jik} E_j B_k = -\frac{1}{c} S_i \quad (11.112)$$

where $\vec{S} = \vec{E} \times \vec{B}/\mu_0$ is the Poynting vector.

Relativity and Electrodynamics (cont.)

With the energy-momentum tensor in hand, it is natural to rewrite our energy and linear momentum conservation laws using it. If we take the four-divergence of the energy-momentum tensor, we find

$$\partial_\mu T^{\mu\nu} = \frac{1}{\mu_0} \left[(\partial_\mu F^\mu_\lambda) F^{\lambda\nu} + F^\mu_\lambda \partial_\mu F^{\lambda\nu} + \frac{1}{2} \partial^\nu (F_{\lambda\sigma} F^{\lambda\sigma}) \right] \quad (11.113)$$

We use the inhomogeneous Maxwell Equation, $\partial_\mu F^{\mu\nu} = \mu_0 J^\nu$, to rewrite the first term and move it to the left side, and also rewrite the right side:

$$\partial_\mu T^{\mu\nu} - J_\lambda F^{\lambda\nu} = \frac{1}{2\mu_0} F_{\mu\lambda} \left[\partial^\mu F^{\lambda\nu} + \partial^\mu F^{\lambda\nu} + \partial^\nu F^{\mu\lambda} \right] \quad (11.114)$$

We may rewrite the last two terms using the homogeneous Maxwell Equation, which we rewrite as:

$$0 = \partial_\sigma G^{\sigma\lambda} = \frac{1}{2} \partial_\sigma \epsilon^{\sigma\lambda\nu\mu} F_{\nu\mu} = \partial^\lambda F^{\nu\mu} + \partial^\mu F^{\lambda\nu} + \partial^\nu F^{\mu\lambda} \quad (11.115)$$

$$-\partial^\lambda F^{\nu\mu} = \partial^\mu F^{\lambda\nu} + \partial^\nu F^{\mu\lambda} \quad (11.116)$$

(the second step in the first line is obtained by just writing out all the terms and combining the ones that differ by a single flip of two indices).

Relativity and Electrodynamics (cont.)

Using this in the equation for the divergence of the energy-momentum tensor, and then reordering the indices on the second term on each side (and picking up minus signs), we have

$$\partial_\mu T^{\mu\nu} - J_\lambda F^{\lambda\nu} = \frac{1}{2\mu_0} F_{\mu\lambda} [\partial^\mu F^{\lambda\nu} - \partial^\lambda F^{\nu\mu}] \quad (11.117)$$

$$\partial_\mu T^{\mu\nu} + F^{\nu\lambda} J_\lambda = \frac{1}{2\mu_0} F_{\mu\lambda} [\partial^\mu F^{\lambda\nu} + \partial^\lambda F^{\mu\nu}] \quad (11.118)$$

The quantity in brackets on the right side is now symmetric in μ and λ , while $F_{\mu\lambda}$ is antisymmetric, so the right side vanishes. Moving the field-current term to the right side, we obtain

$$\partial_\mu T^{\mu\nu} = -F^{\nu\lambda} J_\lambda \quad (11.119)$$

If we write out the time and space components of this four-vector equation, we obtain

$$-\frac{\partial u_{field}}{\partial t} - \vec{\nabla} \cdot \vec{S} = \vec{J} \cdot \vec{E} = -\frac{\partial u_{mech}}{\partial t} \quad (11.120)$$

$$-\frac{\partial \vec{g}}{\partial t} + \vec{\nabla} \cdot \underline{\underline{T}} = -[\rho \vec{E} + \vec{J} \times \vec{B}] = -\frac{\partial \vec{p}_{mech}}{\partial t} \quad (11.121)$$

which are Equations 8.14 and 8.36 from our discussion of conservation laws.

Relativity and Electrodynamics (cont.)

Angular Momentum Tensor

The relativistic angular momentum tensor is

$$M^{\mu\nu\sigma} = r^\mu T^{\nu\sigma} - r^\nu T^{\mu\sigma} \quad (11.122)$$

which is the natural generalization of the three-dimension angular momennum tensor we defined earlier, Equation 8.54; its space-space-space components recover that definition. However, in three dimensions, we could write those as a cross-product. That is not possible in four dimensions. Instead, we use the analogous construction, the antisymmetrized product of \vec{r} and T .

Section 12

Applications of Radiation

Lecture 29:
Applications of Radiation:
Classical Scattering Theory, Antennas

Date Revised: 2013/05/30

Date Given: There was not time to present this material in class,
but last year's notes are provided "as-is."

Classical Scattering Theory

What is Scattering?

The interaction of an electromagnetic wave with matter consists of two components, coherent polarization and scattering, which we can define in a bit more detail:

- ▶ *coherent polarization*

This is the effect we considered when we modeled a material as having a permittivity ϵ and a permeability μ . The medium electrically and magnetically polarizes *coherently* with the incoming electromagnetic wave and energy is exchanged *coherently* between the wave and the medium. We made the implicit assumption in that discussion that energy could only be dissipated either via unspecified loss (imaginary component of the index of refraction) or by driving of electrical currents with Joule power dissipation in the resistive medium (an actual conductive term in Maxwell's Equations). We neglected any capability of the medium to radiate in directions other than that of the wave propagation. Effectively, we considered only the near-field, quasistatic response of the medium, wherein the movement timescales of the charged particles responsible for the permittivity and permeability were assumed to be small compared to the light travel time to the observation point.

- ▶ *scattering*

We now have the tools to consider this component, which consists of the incoherent re-radiation of power absorbed by the medium as an outgoing electromagnetic wave in a direction different from the direction of propagation of the incoming wave. This is the power-loss mechanism we did not consider in the above.

Classical Scattering Theory (cont.)

Scattering of Polarized Light by a Small Object, $d \ll \lambda$

Consider a linearly polarized EM wave incident

$$\vec{E}(\vec{r}, t) = \vec{E}_0 e^{i(\vec{k} \cdot \vec{r} - \omega t)} \quad (12.1)$$

on a single particle of size $d \ll \lambda$ ($k d \ll 1$) and of frequency-dependent polarizability $\alpha(\omega)$. Then the dipole moment of the particle is (in complex notation)

$$\vec{p} = \alpha(\omega) \vec{E} = \alpha(\omega) \vec{E}_0 e^{-i\omega t} \quad (12.2)$$

This is an oscillating, accelerating electric dipole. We can calculate its radiated power pattern easily using Equation 10.209:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0}{32\pi^2} \frac{\alpha^2 |\vec{E}_0|^2 \omega^4 \sin^2 \theta}{c} \quad (12.3)$$

where θ is the polar angle relative to the polarization vector of the incoming wave. Note that the emitted power pattern has no dependence on ϕ , the azimuthal angle around the polarization vector.

Classical Scattering Theory (cont.)

We define the particle's *differential* and *total scattering cross section* as

$$\frac{d\sigma}{d\Omega} = \frac{\left\langle \frac{dP}{d\Omega} \right\rangle}{\text{incident power flux}} \quad \sigma = \frac{\langle P \rangle}{\text{incident power flux}} \quad (12.4)$$

With the incident power flux $I = \langle |\vec{S}| \rangle = c \epsilon_0 |\vec{E}_0|^2 / 2$, we obtain

linearly polarized cross section
 $d \ll r, d^2/\lambda \ll r, d \ll \lambda$

$$\frac{d\sigma_{pol}}{d\Omega} = \left(\frac{\alpha}{4\pi\epsilon_0} \right)^2 \frac{\omega^4}{c^4} \sin^2 \theta \quad \sigma_{pol} = \frac{8\pi}{3} \left(\frac{\alpha}{4\pi\epsilon_0} \right)^2 \frac{\omega^4}{c^4} \quad (12.5)$$

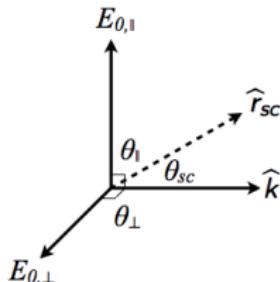
Because $\alpha/4\pi\epsilon_0$ carries units of volume, σ carries units of area and $d\sigma/d\Omega$ units of area/solid angle. The reason for this is that we assume an incoming plane wave of infinite transverse extent, so we cannot calculate the fraction of its total power scattered — that would vanish because the power is infinite. Rather, we calculate the power scattered given an input power per unit area, so we have to multiply by an area to get the right output units.

Classical Scattering Theory (cont.)

Scattering of Unpolarized Light by a Small Object, $d \ll \lambda$

It is in general useful to calculate the corresponding results for unpolarized incoming light — light with equal amounts of the two complementary linear polarizations. Since the two polarization vectors are orthogonal, we need to write their scattering cross-sections in the same coordinate system in order to sum their contributions. So far, we have written the scattering cross-section in terms of angles relative to the polarization vector. Instead, let's write it in angles relative to the incoming wave's propagation vector \hat{k} and the viewing direction (the outgoing scattered wave direction) \hat{r}_{sc} .

Consider a coordinate system whose z-axis is the incident wave direction \hat{k} and whose xz plane is defined by the scattered direction (viewing direction) \hat{r}_{sc} . Let θ_{sc} be the polar angle of \hat{r}_{sc} . The incoming light has components polarized parallel to and perpendicular to the scattering plane.



Classical Scattering Theory (cont.)

For the polarization component perpendicular to the scattering plane, the scattering plane makes an angle $\pi/2$ with the polarization vector. The angle θ_{sc} is the ϕ angle in a coordinate system whose z-axis is the incoming polarization vector. Thus, for any θ_{sc} , the angle θ in the differential cross section formula is $\pi/2$. Taking half of the incoming power to be in this polarization component, we obtain

$$\frac{d\sigma_{\perp}}{d\Omega} = \frac{1}{2} \left. \frac{d\sigma}{d\Omega} \right|_{\theta=\pi/2} = \frac{1}{2} \left(\frac{\alpha}{4\pi\epsilon_0} \right)^2 \frac{\omega^4}{c^4} \quad (12.6)$$

For the polarization component parallel to the scattering plane, the scattering plane lies in the xz-plane of the coordinate system whose z-axis is the incoming polarization vector (and whose x-axis is the incident direction \hat{k}), and the plane formed by the incoming polarization vector and \hat{k} is the xz-plane of the scattering coordinate system. Thus, the polar angle in the scattering coordinate system and the polar angle in the incident polarization coordinate system are complementary, so $\sin\theta_{||} = \cos\theta_{sc}$ and therefore

$$\frac{d\sigma_{||}}{d\Omega} = \frac{1}{2} \left. \frac{d\sigma}{d\Omega} \right|_{\theta=\pi/2-\theta_{sc}} = \frac{1}{2} \left(\frac{\alpha}{4\pi\epsilon_0} \right)^2 \frac{\omega^4}{c^4} \cos^2 \theta_{sc} \quad (12.7)$$

Classical Scattering Theory (cont.)

Adding the two together and integrating over all angles to get the total power gives

unpolarized cross section
 $d \ll r, d^2/\lambda \ll r, d \ll \lambda$

$$\frac{d\sigma_{unpol}}{d\Omega} = \left(\frac{\alpha}{4\pi\epsilon_0}\right)^2 \frac{\omega^4}{c^4} \frac{1 + \cos^2\theta_{sc}}{2} \quad \sigma_{unpol} = \frac{8\pi}{3} \left(\frac{\alpha}{4\pi\epsilon_0}\right)^2 \frac{\omega^4}{c^4} \quad (12.8)$$

The total scattering cross section is unchanged because the polarized cross section has no dependence on polarization angle.

Classical Scattering Theory (cont.)

Application: Rayleigh Scattering, or Why the Sky is Blue

Recall our discussion of the permittivity and index of refraction of dispersive materials. At frequencies well below the atomic or molecular transition frequencies, the permittivity becomes a constant (Equation 9.185):

$$\frac{\epsilon}{\epsilon_0} = 1 + \frac{N q^2}{\epsilon_0} \sum_j \frac{f_j}{k_j} \quad (12.9)$$

This implies the polarizability α becomes constant. Thus, the frequency dependence of the scattering cross section comes entirely from the ω^4 term. This strong dependence on frequency implies that the blue portion of the solar spectrum scatters much more than the red portion — seven times comparing 400 nm to 650 nm.

Classical Scattering Theory (cont.)

The above strong dependence leads to a number of interesting phenomena:

- ▶ When we look away from the sun, the light we are seeing is primarily scattered light, which is dominantly blue.
- ▶ When we look at the sun at sunrise or sunset, when the path length through the atmosphere is large, the blue light has been dominantly scattered out of the line-of-sight and we see a red sunrise or sunset.
- ▶ Because of the angular dependence in Equation 12.8, when we look in a direction normal to the sun's rays (e.g., straight up when the sun is low in the sky, but not so low that all the blue light has been lost), we can only see scattered light in the polarization perpendicular to the scattering plane. So, if we look in this direction with polarized sunglasses, we can make the sky appear bluer by selecting the scattered polarization (the other polarization consists of light primarily reflected from our surroundings) or less blue by selecting the unscattered polarization.

Classical Scattering Theory (cont.)

Application: Thomson Scattering

A simple situation in which to apply our theory of scattering quantitatively is the case of Thomson scattering, which is scattering of an electromagnetic wave off of a free electron. Recall that we calculated the relation between a bound, damped electron's dipole moment and the incident electric field to be (Equation 9.171)

$$\tilde{p}(t) = e \tilde{x}(t) = \frac{q^2/m}{\omega_0^2 - \omega^2 - i\gamma\omega} \tilde{E}_0 e^{-i\omega t} \quad (12.10)$$

If we take the limit $\omega \gg \omega_0$ and $\omega \gg \sqrt{\gamma\omega}$, we obtain

$$\alpha_{free} = \frac{\tilde{p}_{free}(t)}{\tilde{E}_0 e^{-i\omega t}} = -\frac{e^2}{m\omega^2} \quad (12.11)$$

Classical Scattering Theory (cont.)

Plugging in to our total scattering cross section equation, Equation 12.8, and rearranging (canceling powers of ω , moving c^2 around), we obtain the differential and total scattering cross section of a free electron:

free-electron (Thomson) cross section

$$\frac{d\sigma_{free}}{d\Omega} = \left(\frac{e^2}{4 \pi \epsilon_0 m_e c^2} \right)^2 \frac{1 + \cos^2 \theta_{sc}}{2} \quad \sigma_{Thomson} = \frac{8 \pi}{3} \left(\frac{e^2}{4 \pi \epsilon_0 m_e c^2} \right)^2 \quad (12.12)$$

The total cross section is called the *Thomson cross section* and is ubiquitous because it sets the scale of scattering of EM waves off of electrons, even when quantum mechanical effects are taken into account.

The quantity in parentheses is called the classical electron radius

$$r_{classical} = \frac{e^2}{4 \pi \epsilon_0 m_e c^2} \quad (12.13)$$

because it is the radius one obtains by equating the rest mass energy of the electron, $m_e c^2$, to the Coulomb energy required to assemble of sphere of uniform charge density of total charge e and radius $r_{classical}$.

Classical Scattering Theory (cont.)

Application: Small ($d \ll \lambda$) Dielectric Sphere Scattering

Another simple application is to consider scattering off of a dielectric sphere of radius $d \ll \lambda$ and permittivity ϵ . The polarization of such a sphere was calculated in Griffiths Example 4.7, which we quoted in Equation 4.68, with:

$$\vec{p} = 4\pi\epsilon_0 d^3 \frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \vec{E}_0 \quad (12.14)$$

α is easily read off the above expression, yielding

dielectric sphere scattering cross section
 $d \ll r, d^2/\lambda \ll r, d \ll \lambda$

$$\frac{d\sigma_{\text{diel. sph.}}}{d\Omega} = \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \right)^2 (k d)^4 d^2 \frac{1 + \cos^2 \theta_{sc}}{2} \quad (12.15)$$

$$\sigma_{\text{diel. sph.}} = \frac{8\pi}{3} \left(\frac{\epsilon - \epsilon_0}{\epsilon + 2\epsilon_0} \right)^2 (k d)^4 (\pi d^2) \quad (12.16)$$

where $(k d)^4$ comes from $(\omega/c)^4 d^4$.

Classical Scattering Theory (cont.)

Application: Scattering in a Gas

For a gas, because the atoms are in random positions, we may apply our scattering theory to the medium as a whole. Let's determine the rate at which incoming power is attenuated due to scattering to off-axis angles. The decay rate of the transmitted power is given by

$$P(z) = P(0) e^{-\gamma z} \quad \gamma = n \sigma \quad (12.17)$$

where n is the density of atoms or molecules and σ is the scattering cross section per atom or molecule. To find σ , we need the polarizability, α . Recalling the definition of the dielectric constant ϵ_r , we have

$$\vec{P} = n \vec{p} = n \alpha \vec{E} \quad \text{and} \quad \vec{P} = \chi_e \epsilon_0 \vec{E} = (\epsilon - \epsilon_0) \vec{E} \quad \Rightarrow \quad \alpha = \frac{\epsilon - \epsilon_0}{n} \quad (12.18)$$

Inserting this into our expression for σ , we obtain

$$\gamma_{dilute} = n \sigma = n \frac{8 \pi}{3} \left(\frac{\epsilon - \epsilon_0}{4 \pi \epsilon_0 n} \right)^2 \frac{\omega^4}{c^4} = \frac{8 \pi^3}{3} \left(\frac{\epsilon - \epsilon_0}{\epsilon_0} \right)^2 \frac{1}{n \lambda^4} \quad (12.19)$$

In a denser material, we need to use the *Clausius-Mossoti relation* to relate the permittivity and the polarizability. We derived this relation in a problem set in Ph106b, and we just applied a related relation in the previous example of dielectric sphere scattering.

Classical Scattering Theory (cont.)

Application: Scattering in a Dense Medium

In a dense medium (e.g., a liquid), individual scatterers can no longer be treated as totally independent. Instead, we treat the medium as consisting of cells of size d such that $d \ll \lambda$ while also $d \gg \xi$ where ξ is the correlation length of fluctuations in the medium's density. The number of scatterers per unit volume is $1/d^3$ because that is the number of cells per unit volume. The "polarizability per scatterer", which is α in the case of a single scatterer, is given by $\alpha \delta N_d$ where δN_d is the rms fluctuation in the number of scatterers in cells of size d . Thus, the attenuation length becomes (using $\alpha = (\epsilon - \epsilon_0)/n$ from the previous page)

$$\gamma_{dense} = (\# \text{ of cells per unit volume}) \times (\text{cross section per cell}) \quad (12.20)$$

$$= \frac{1}{d^3} \frac{8\pi}{3} \left(\frac{\alpha \delta N_d}{4\pi \epsilon_0} \right)^2 = \frac{8\pi^3}{3} \left(\frac{\epsilon - \epsilon_0}{\epsilon_0} \right)^2 \frac{1}{n \lambda^4} \frac{\langle \delta N_d^2 \rangle}{n d^3} \quad (12.21)$$

In an ideal gas (not what we are dealing with), δN_d is a Poisson-fluctuating variable with $\langle \delta N_d^2 \rangle = n d^3$, so we recover the prior dilute expression.

Classical Scattering Theory (cont.)

More generally, one can derive from thermodynamics that

$$\frac{\langle \delta N_d^2 \rangle}{n d^3} = n k_B T \beta_T \quad \text{with} \quad \beta_T = -\frac{1}{V} \left(\frac{\partial P}{\partial V} \right)_T \quad (12.22)$$

where β_T is the *isothermal compressibility* (basically, how hard it is stuff more atoms or molecules into a given volume). With this, we have

$$\gamma = \frac{8 \pi^3}{3} \left(\frac{\epsilon - \epsilon_0}{\epsilon_0} \right)^2 \frac{1}{n \lambda^4} k_B T \beta_T \quad (12.23)$$

This kind of Rayleigh scattering is quite important for light propagation in liquids; scattering, rather than absorption, is the primary reason for light attenuation in liquids.

Interestingly, β_T can diverge at the liquid-gas critical point, giving rise to extremely large scattering called *critical opalescence*, which provides a useful way to determine if a system is at this point.

Antennas

Introduction and Study Guide

Antennas are our second major application of the theory of radiation we have developed. This material is not covered in Griffiths, it is largely from Heald and Marion Sections 9.4, 9.5, and 9.7.

Antennas (cont.)

An $d \ll \lambda$ Electric Dipole Antenna

Let us consider an electric dipole given by a line current along the z -axis $I(t) = I_0 e^{-i\omega t}$ that has length d . This is the current that flows in the case of a perfect electric dipole, with $\int I(t) d = \dot{p} = q_0 \dot{d}$. Plugging this into the electric dipole radiation formulae for harmonic time dependence and $\vec{p} \propto \hat{z}$, Equations 10.209, we obtain

$$\vec{B} = -i \frac{\mu_0}{4\pi} \omega I_0 d \frac{e^{i(kr-\omega t)}}{cr} \hat{\phi} \sin \theta \quad \vec{E} = -i \frac{1}{4\pi \epsilon_0} \omega I_0 d \frac{e^{i(kr-\omega t)}}{c^2 r} \hat{\theta} \sin \theta \quad (12.24)$$

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0}{32\pi^2} \frac{\omega^2 I_0^2 d^2 \sin^2 \theta}{c} \quad \langle P \rangle = \frac{\mu_0}{12\pi} \frac{\omega^2 I_0^2 d^2}{c} = \frac{(kd)^2}{6\pi} \langle I^2 \rangle Z_f \quad (12.25)$$

with $Z_f = \sqrt{\mu_0/\epsilon_0}$, the impedance of free space, and $\langle I^2 \rangle = I_0^2/2$.

We see the first hint of a concept of an antenna's *radiation impedance* or *radiation resistance*, which gives the relation between mean square current and the radiated power. For an antenna radiating into free space, power is radiated most efficiently when $P/\langle I^2 \rangle = Z_f$. This short line antenna does not do a good job of radiating into free space because of the prefactor $(kd)^2/6\pi \ll 1$. We immediately see that, to match free-space well, an antenna must have a size of order the wavelength it is radiating. The approximations we made to obtain the electric and magnetic dipole radiation fields fail and we must go back to the more general expressions.

Antennas (cont.)

General Setup

Given the above, let us recall a number of results from our general discussion of radiating systems. The form for the vector potential under the assumption $d \ll r$ and $d^2/\lambda \ll r$, Equation 10.192, is

$$\vec{A}(\vec{r}, t) = \frac{\mu_0}{4\pi} \frac{e^{i(kr - \omega t)}}{r} \int_{\mathcal{V}} d\tau' \vec{J}_0(\vec{r}') e^{-ikr' \hat{r} \cdot \hat{r}'} \quad (12.26)$$

We did not derive expressions for the fields in this general case, so we do so here. We recall that, with the above assumed harmonic time dependence, the application of the curl to get the magnetic and electric fields yields $i k \hat{r} \times$ factors from the argument of the exponential (neglecting the term of higher order in r that comes from differentiating the $1/r$ dependence). Therefore,

antenna magnetic field	$\vec{B}(\vec{r}, t) = i \frac{\mu_0}{4\pi} \frac{\omega}{c} \frac{e^{i(kr - \omega t)}}{r} \hat{r} \times \int_{\mathcal{V}} d\tau' \vec{J}_0(\vec{r}') e^{-ikr' \hat{r} \cdot \hat{r}'} \quad (12.27)$
------------------------------	--

antenna magnetic field	$\vec{E}(\vec{r}, t) = \frac{i c^2}{\omega} \vec{\nabla} \times \vec{B} = -c \hat{r} \times \vec{B} \quad (12.28)$
------------------------------	--

We are now going to consider antennas that can be treated as carrying line currents, allows us to make the replacement $d\tau' \vec{J}_0(\vec{r}') \rightarrow I d\vec{l}'$.

Antennas (cont.)

Antenna Gain or Efficiency

An antenna is fundamentally a device for either radiating power outward in a desired pattern, usually with the goal of radiating into a particular direction with high efficiency, or for receiving power from a particular direction with high efficiency. In the transmission sense, "efficiency" means that the fraction of power emitted in a particular direction is large. This can be quantified using the idea of *antenna gain*:

antenna gain

$$G(\hat{r}) = \frac{\left\langle \frac{dP}{d\Omega}(\hat{r}) \right\rangle}{\left\langle \frac{P}{4\pi} \right\rangle} \quad (12.29)$$

which is just the ratio of the power per unit solid angle into a particular direction divided by the total power into all solid angles. If this number is large in a particular direction, then the antenna efficiently transmits in that direction. We shall see that this same number quantifies how efficient an antenna is in reception mode.

Antennas (cont.)

Antenna Impedance

Since the antenna may not be small compared to λ , the mean-square current $\langle I^2 \rangle$ may not be the same everywhere in the antenna. For specificity, we choose to evaluate $\langle I^2 \rangle$ at the point where the antenna is fed by wires or a waveguide, the so-called “gap” drive point. The overall *antenna impedance* and *radiation resistance* are then defined to be

antenna
impedance

$$Z_{ant} = \frac{V_g}{I_g}$$

radiation
resistance

$$R_{rad} = \frac{\langle P \rangle}{\langle I_g^2 \rangle}$$

(12.30)

For an ideal antenna, $Z_{ant} = R_{rad}$ (we use R instead of Z for the radiation component because it is always real by definition), but there may be additional resistive or reactive components.

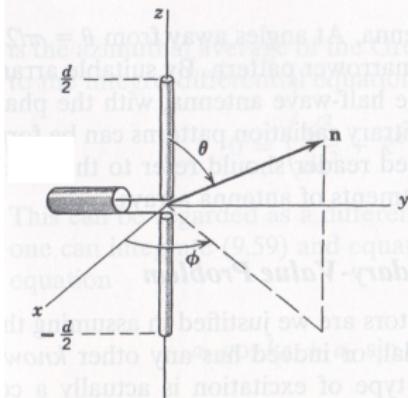
Of particular importance is that Z_{ant} defines the relation between power flowing in on a transmission line and power received by the antenna (which can then be radiated). If $Z_{ant} = Z_{line}$, then the antenna is maximally efficient at receiving power from the transmission line. If this is not true, as may be the case because transmission lines have a set of standard impedances, then a transformer can be used to impedance match such a transmission line to an antenna. We discussed earlier how appropriate lengths of transmission line could be used as transformers. These can work quite well for antennas intended for single-frequency use. Broadband impedance matching — matching over a large fractional bandwidth $\Delta\nu/\nu$ — is more difficult.

Antennas (cont.)

Center-Driven Line Antennas

Line antennas consist of antennas with a line current in the z -direction, $J_0(\vec{r}') d\tau' = I(\vec{r}') d\ell \hat{z}$. In general, solving for the vector potential and current even approximately is nontrivial. Jackson §9.4B demonstrates that, for a thin-wire antenna that is perfectly conducting and with wire radius $a \ll \lambda$, d and has azimuthal symmetry about the antenna's long direction d , the vector potential and the current are sinusoidally dependent on the distance along the antenna. We will assume this behavior in the following.

We assume a configuration as illustrated below, where the antenna consists of a long wire of length d in the z -direction, symmetrically placed about the origin, with a gap of negligible extent at $z = 0$. The two pieces of the antenna are connected to the two electrodes of a transmission line.



Antennas (cont.)

The current in the wire is assumed to follow

$$I(z) = I_0 \sin \left[k \left(\frac{d}{2} - |z| \right) \right] \quad (12.31)$$

This form is sinusoidal as desired and also satisfies the boundary condition that the current vanish at the end. The derivative is not continuous at $z = 0$ except for certain values of d . The current at the gap drive point is $I_g = I_0 \sin \frac{k d}{2}$. Given the current distribution, we may calculate the magnetic field, noting that $\hat{r} \times d\ell \hat{z} = -\hat{\phi} \sin \theta$:

$$\vec{B}(\vec{r}, t) = -i \hat{\phi} \frac{\mu_0}{4\pi} \frac{\omega}{c} I_0 \frac{e^{i(kr - \omega t)}}{r} \sin \theta \int_{-d/2}^{d/2} dz' e^{-ikz' \cos \theta} \sin \left[k \left(\frac{d}{2} - |z'| \right) \right] \quad (12.32)$$

Let us rewrite this as the product of a function of λ and a function of d/λ and θ :

center-driven line antenna magnetic field	$\vec{B}(\vec{r}, t) = -i \hat{\phi} \frac{\mu_0}{4\pi} \frac{\omega}{c} I_0 \frac{\lambda}{2} \frac{e^{i(kr - \omega t)}}{r} f \left(\frac{d}{\lambda}, \theta \right)$	(12.33)
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center-driven line antenna magnetic field pattern	$f \left(\frac{d}{\lambda}, \theta \right) = \frac{\sin \theta}{\pi} \int_{-\pi d/\lambda}^{\pi d/\lambda} dZ e^{-i Z \cos \theta} \sin \left(\frac{\pi d}{\lambda} - Z \right)$	(12.34)
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Antennas (cont.)

Comparing with our calculation of the radiation field due to a line current electric dipole, we see that if we make the correspondence,

$$I_0 d \leftrightarrow I_0 \frac{\lambda}{2} \quad \sin \theta \leftrightarrow f \left(\frac{d}{\lambda}, \theta \right) \quad (12.35)$$

and note that the relation between \vec{E} and \vec{B} is the same as for the line current electric dipole, then the expressions calculated for the line current electric dipole apply, giving

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{\mu_0}{32\pi^2} \frac{\omega^2 I_0^2 \frac{\lambda^2}{4}}{c} \left| f \left(\frac{d}{\lambda}, \theta \right) \right|^2 = \frac{Z_f}{32} I_0^2 \left| f \left(\frac{d}{\lambda}, \theta \right) \right|^2 \quad (12.36)$$

The gain and radiation resistance are

antenna gain pattern	$G(\theta) = \frac{\left f \left(\frac{d}{\lambda}, \theta \right) \right ^2}{\langle f ^2 \rangle}$ with $\langle f ^2 \rangle = \frac{1}{4\pi} \int d\Omega \left f \left(\frac{d}{\lambda}, \theta \right) \right ^2$	(12.37)
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antenna radiation resistance	$R_{rad} = \frac{P}{\langle I_g^2 \rangle} = \frac{\frac{Z_f}{32} I_0^2 4\pi \langle f ^2 \rangle}{\frac{1}{2} I_{g,0}^2} = \frac{\pi}{4} \frac{I_0^2}{I_{g,0}^2} \langle f ^2 \rangle Z_f$	(12.38)
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Antennas (cont.)

For the sinusoidal current we have assumed, we can evaluate this expression for arbitrary d . This is done by using the trigonometric product identities

$$\cos A \sin B = \frac{1}{2} [\sin (A + B) - \sin (A - B)] \quad (12.39)$$

$$\sin A \sin B = \frac{1}{2} [\cos (A - B) - \cos (A + B)] \quad (12.40)$$

and also using the Euler formula for $e^{-i k z' \cos \theta}$. This yields

center-driven
line antenna
radiation
field pattern

$$f\left(\frac{d}{\lambda}, \theta\right) = \frac{2}{\pi} \frac{\cos\left(\frac{k d}{2} \cos \theta\right) - \cos\frac{k d}{2}}{\sin \theta} \quad (12.41)$$

Antennas (cont.)

We can calculate these quantities for some specific antenna lengths. A plot comparing the radiation patterns of our idealized electric dipole and half-wave and full-wave antennas is given at the end.

- ▶ *half-wave antenna, $d = \lambda/2$*

The current is

$$I(z) = I_0 \sin\left(\frac{\pi}{2} - k|z'|\right) = I_0 \cos kz \quad (12.42)$$

The quantity $kd/2 = \pi/2$, so

$$f\left(\frac{d}{\lambda} = \frac{1}{2}, \theta\right) = \frac{2 \cos\left(\frac{\pi}{2} \cos \theta\right)}{\pi \sin \theta} \quad \langle |f|^2 \rangle = 0.247 \quad (12.43)$$

$$I_{g,0} = I_0 \quad R_{rad} \approx 73 \Omega \quad (12.44)$$

This type of antenna is well-matched to 75Ω coaxial cable.

Antennas (cont.)

- ▶ full-wave antenna, $d = \lambda$

The current is

$$I(z) = I_0 \sin(\pi - k|z'|) = I_0 \sin k|z'| = I_0 |\sin kz'| \quad (12.45)$$

The current at the drive point formally vanishes and the antenna's radiation resistance is formally infinite — basically, no matter what voltage the transmission line has on it, it cannot drive a current at the antenna drive point. Clearly, not a very good antenna to drive with a transmission line! It is the corrections we ignored in making the assumption of perfectly sinusoidal current will enable $I_g \neq 0$. However, we can still calculate the radiation pattern using our formula for f with $k d/2 = \pi$, giving

$$f\left(\frac{d}{\lambda} = 1, \theta\right) = \frac{2}{\pi} \frac{\cos(\pi \cos \theta) - \cos \pi}{\sin \theta} = \frac{4}{\pi} \frac{\cos^2\left(\frac{\pi}{2} \cos \theta\right)}{\sin \theta} \quad (12.46)$$

$$\langle |f|^2 \rangle = 0.672 \quad (12.47)$$

Antennas (cont.)

- ▶ *folded half-wave antenna*

The idea here is to fold a full-wave antenna at $z = \pm\lambda/4$ and connect the two ends together back at the center. This joining changes the boundary condition so that the joined point should be a point of peak current and the fold points are now current zeros. If s is the distance along the antenna from the drive point, so that $s = \pm\lambda/2$ sit at $z = 0$, the current becomes $I(s) = I_0 \cos ks$ instead of $I(s) = I_0 |\sin ks|$. The current is negative at $s = \pm\lambda/2 = d/2$, but, because the antenna has been folded so this piece of the antenna is pointed in the opposite direction as the unfolded piece, this current that is negative relative to s is positive relative to z . The antenna now looks like two half-wave antennas right next to each other. the radiation pattern is the same as a half-wave antenna, but the effective current I_0 for a given gap drive voltage $I_{g,0}$ is enhanced by a factor of two relative to the half-wave antenna, so the radiation resistance increases by a factor of $(I_0/I_{g,0})^2 = 4$:

$$I_{g,0} = \frac{1}{2} I_0 \quad R_{rad} = 4 \times 73 \Omega = 292 \Omega \quad (12.48)$$

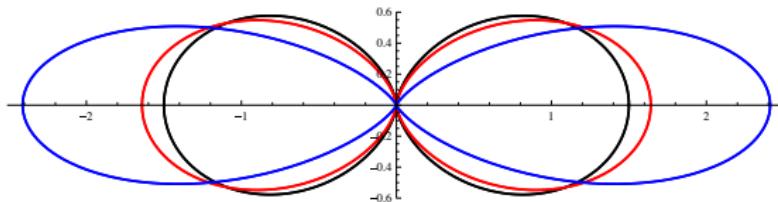
This higher impedance is a better match to flat-pair or twisted-pair transmission line, which have impedances of about 300Ω .

Antennas (cont.)

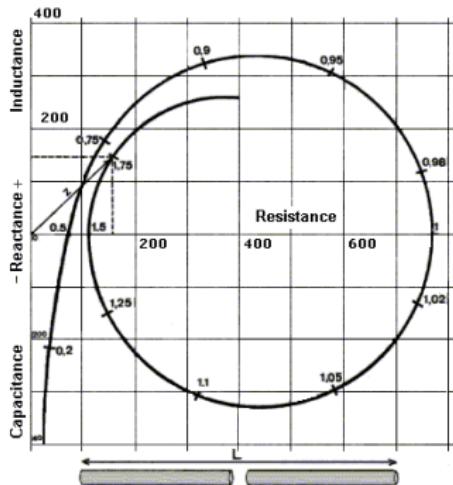
► *Arbitrary Length*

Given on the following slide is a calculation of the *antenna impedance* as a function of length in units of λ . The length is marked with tickmarks on the trajectory. The impedance is a complex number whose resistive (real) part is the horizontal axis and whose reactive (imaginary) part is the vertical axis. The real part incorporates both the *radiation resistance*, which is always real by definition, as well as any actual resistive component. The reactive component simply indicates that the current and voltage at the drive point (which are supplied by the feeding transmission line) are out of phase, with the sign of the phase defining whether the antenna has a capacitive or inductive reactance. The right side of the trajectory, at $d/\lambda = 1$, would go off to infinity for an ideal full-wave antenna; nonidealities keep it finite. At $d/\lambda = 3/2$, the antenna looks similar to a half-wave antenna but with a somewhat higher impedance. One can use a plot like this to tune the antenna length so its impedance best matches the transmission line feeding it.

Antennas (cont.)



Radiation patterns of ideal electric dipole (black), half-wave antenna (red), and full-wave antenna (blue). The \cos^2 dependence of the full-wave antenna is clear from the greater antenna gain. Figure courtesy of M. Cross.



Antenna impedance as a function of antenna length in units of λ . Tickmarks indicate d/λ . The axes are in Ω and give the complex Z_{ant} . **NOTE: This plot follows the engineering sign convention. Our sign convention is the opposite, so the curve should be mirrored through the horizontal axis. $d < \lambda/2$ will still be the capacitive portion and $\lambda/2 < d < \lambda$ the inductive section (and so on for $d > \lambda$).**

<http://www.astrosurf.com/luxorion/qsl-swr.htm>

Lecture 26:
Applications of Radiation: Antennas Ct'd
Relativity Review

Date Revised: 2013/06/06

Date Given: 2013/05/30

Driven Arrays of Line Antennas

Suppose one has an array of N antennas, each displaced from the origin by $\vec{\Delta}_j$, $j = 1$ to N . Let \vec{E}_0 be the radiation field pattern of an individual antenna situated at the origin. Then the radiation pattern of the array is easily calculated by recognizing that each antenna's contribution to the total electric field is the same up to the phase offsets due to the $e^{-i k \hat{r}' \cdot \vec{\Delta}_j}$ factor and the displacement in \vec{r}' between antennas:

$$\vec{E} = \sum_{j=1}^N \frac{i c^2}{\omega} \vec{\nabla} \times [\vec{\nabla} \times \vec{A}_j(\vec{r}, t)] \quad (12.49)$$

$$= \sum_{j=1}^N \frac{i c^2}{\omega} \vec{\nabla} \times \left[\vec{\nabla} \times \frac{\mu_0}{4\pi} \frac{e^{i(kr - \omega t)}}{r} \int_{C_j} d\vec{\ell} I(\vec{r}') e^{-i k \hat{r} \cdot \vec{\ell}} \right] \quad (12.50)$$

$$= \sum_{j=1}^N \frac{i c^2}{\omega} \vec{\nabla} \times \left[\vec{\nabla} \times \frac{\mu_0}{4\pi} \frac{e^{i(kr - \omega t)}}{r} \int_{C_0} d\vec{\ell} I(\vec{r}') e^{-i k \hat{r} \cdot (\vec{\Delta}_j + \vec{r}')} \right] \quad (12.51)$$

$$= \vec{E}_0 \sum_{j=1}^N e^{-i k \hat{r} \cdot \vec{\Delta}_j} \quad (12.52)$$

where we factored $e^{-i k \hat{r} \cdot \vec{\Delta}_j}$ out of the integral and recognized \vec{E}_0 in what remains.

Antennas (cont.)

You hopefully recognize this sum over phase factors of this type as the same kind of sum present in the calculation of the interference pattern due to an array of slits in a screen. Let us consider an analogue of that, an array of N center-fed antennas oriented with their currents flowing along the z -axis and displaced from the origin with uniform spacing Δ . Let the first antenna be at \vec{r}_0 with $r_0 \ll r$. The radiation field pattern is

$$\vec{E} = \vec{E}_0 \sum_{j=1}^N e^{-i k \hat{r} \cdot \vec{\Delta}_j} = \vec{E}_0 \sum_{j=1}^N e^{-i k \hat{r} \cdot (\vec{r}_0 + (j-1) \vec{\Delta})} \quad (12.53)$$

$$= \vec{E}_0 e^{-i k \hat{r} \cdot \vec{r}_0} \sum_{j=1}^N e^{-i k \hat{r} \cdot \vec{\Delta}(j-1)} = \vec{E}_0 e^{-i k \hat{r} \cdot \vec{r}_0} \frac{1 - e^{-i k N \hat{r} \cdot \vec{\Delta}}}{1 - e^{-i k \hat{r} \cdot \vec{\Delta}}} \quad (12.54)$$

where we have used the standard result $\sum_{j=1}^N r^{j-1} = (1 - r^N)/(1 - r)$ for geometric series. Thus, if $f_0(d/\lambda, \theta)$ is the radiation pattern corresponding to \vec{E}_0 , we have

$$\left| f\left(\frac{d}{\lambda}, \theta\right) \right|^2 = \left| f_0\left(\frac{d}{\lambda}, \theta\right) \right|^2 \left| \frac{1 - e^{-i k N \hat{r} \cdot \vec{\Delta}}}{1 - e^{-i k \hat{r} \cdot \vec{\Delta}}} \right|^2 \quad (12.55)$$

driven-array
antenna
radiation pattern

$$\left| f\left(\frac{d}{\lambda}, \theta\right) \right|^2 = \left| f_0\left(\frac{d}{\lambda}, \theta\right) \right|^2 \frac{\sin^2 \frac{N k \hat{r} \cdot \vec{\Delta}}{2}}{\sin^2 \frac{k \hat{r} \cdot \vec{\Delta}}{2}}$$

$$(12.56)$$

Antennas (cont.)

The factor at the end is the same factor one finds in the calculation of the interference pattern from an array of slits. It has the following properties:

$$\text{peak position at } k \hat{r} \cdot \vec{\Delta} = 2n\pi \quad (12.57)$$

$$\text{peak height } N^2 \quad (12.58)$$

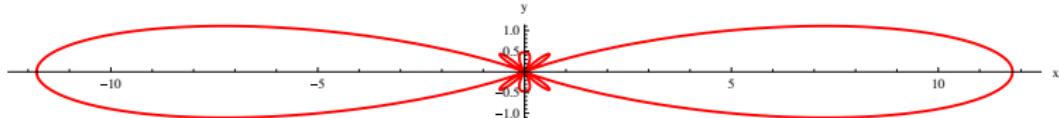
$$\text{peak width } \delta(k \hat{r} \cdot \vec{\Delta}) \approx 2\sqrt{6}/N \quad (12.59)$$

$$\text{secondary peaks at } k \hat{r} \cdot \vec{\Delta} = m\pi/N \quad (m \neq 2nN\pi) \quad (12.60)$$

$$\text{secondary peak height } \mathcal{O}(1) \quad (12.61)$$

These results for the peak height and width are obtained by Taylor expanding the sines near $k \hat{r} \cdot \vec{\Delta} = 2n\pi$ to first and third order, while the secondary maxima are found by requiring $k \hat{r} \cdot \vec{\Delta} = m\pi/N \neq 2n\pi$.

This structure is easier to understand if we set $\vec{\Delta} = \Delta \hat{y}$, which yields $k \hat{r} \cdot \vec{\Delta} = k \Delta \sin \theta \sin \phi$. If we additionally assume $\Delta < \lambda$, this ensures there are at most two primary peaks at $\phi = 0, \pi$ because $k \Delta \sin \theta \sin \phi < 2\pi$ for all (θ, ϕ) . The figure below shows the gain in the xy -plane ($\theta = \pi/2$) for $\Delta = \lambda/2$, $N = 5$. Off this plane, there are no primary peaks, only secondary peaks. Figure courtesy of M. Cross.



Antennas (cont.)

Driven arrays of antennas are pervasive. Here are some examples:

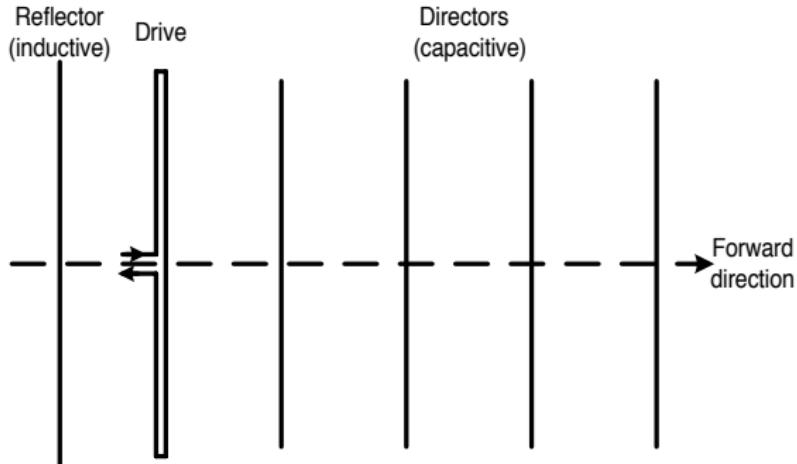
- ▶ The radar in the nose cone of most airplanes is a driven array antenna. The phase of the current driving the individual elements is varied, which is the equivalent of changing the orientation of $\vec{\Delta}$ so that the emitted (and received) radiation beam can be pointed away from the direction of motion of the plane without steering a physical antenna.
- ▶ Much of radio astronomy is done with driven antenna arrays. They are of course operated in reception mode (see below). The signals from the individual antennas are routed through cables to a central “correlator.” In principle, one should sum the signals from all the antennas. But the interesting behavior comes from the cross-terms in the sum, so, instead all independent pairs of voltage signals (proportional to electric field) are multiplied, which gives all the cross-terms, from which one can reconstruct the cross-terms in the above sum. The advantage of this approach is the very narrowly peaked gain function thus obtained — the width of the gain function is roughly the same as that of a single radio telescope whose diameter is the extent of the array. These are called “radio interferometers”. The Jansky Very Large Array in New Mexico and the Atacama Large Millimeter Array in Chile are national observatories consisting in which the individual antennas are themselves radio telescopes. Caltech is part operator of a similar array, the Combined Array for Millimeter Astronomy, in the Inyo Mountains. At longer wavelengths, a number of projects are using arrays of simple antennas to obtain very large collecting areas (appreciable fractions of a square kilometer) that view a large fraction of the sky at one time and for which different resolution functions can be obtained by digitizing and then doing the multiplications in a computer.

Antennas (cont.)

Passive Arrays of Line Antennas

Many television or radio antennas are arrays of linear antennas with one driven element and many passive elements. These passive elements are driven by radiation received from the driven element, and then they radiate with a definite phase relative to the driven antenna. The combined radiation pattern of the driven and passive antennas can be designed to be peaked in a particular direction. As we will see below, this forward peaking of the emitted radiation pattern also ensures that the antenna is a good receiver in that direction.

Here, we consider a *Yagi-Uda* antenna, shown below.



Antennas (cont.)

The important features of this design are as follows:

- ▶ There is one driven antenna, which is a folded half-wave antenna (for reasons discussed earlier).
- ▶ There is one antenna placed behind the the drive antenna that is intended as a back-reflector: it is spaced $\lambda/4$ behind the driven antenna so that the wave that reflects from it is in phase in the forward direction (remember the extra π phase shift due to a reflection of a conductor). This is the equivalent of saying that the wave that is radiated by the back-reflector, which is excited by the driven antenna, is in-phase with the driven antenna's wave in the forward direction and out of phase in the backward direction.
- ▶ There are multiple “director” antennas placed ahead of the driven antenna. They are intended to constructively interfere with the driven antenna in the forward direction and destructively in the backward direction. If there were just one, it would be placed $\lambda/2$ in front of the driven antenna (so the reflected wave in the reverse direction is out of phase). With multiple directors, the optimal spacing turns out to be about $\lambda/3$.

Antennas (cont.)

- ▶ The length of the passive antennas is also important. Recall the figure showing the antenna impedance as a function of size: antennas that have $d < \lambda/2$ are capacitive and those with $\lambda/2 < d < \lambda$ are inductive. In the forward direction, the phase factor $-i k \hat{r} \cdot \vec{\Delta}_j$ is negative imaginary. With our $-i \omega t$ time dependence, a capacitive element has a positive imaginary component to its impedance (positive phase factor), so these elements should be capacitive to cancel this negative imaginary phase and yield constructive interference in the forward direction (and, conversely, destructive interference in the reverse direction). Thus, the directors should have $d_d < \lambda/2$. Conversely, the back-reflector should be inductive and thus have $\lambda/2 < d_r < d$. Usually, d_d and d_r are not much different from d .

Antennas (cont.)

Antenna Reciprocity Theorem

One can prove a theorem that the power received by an antenna from a plane wave in direction \hat{k} is

$$P_r(\hat{k}) = \frac{\lambda^2}{4\pi} G(-\hat{k}) \times \text{incident flux} \quad (12.62)$$

The quantity $(\lambda^2/4\pi) G(-\hat{k})$ is the *effective area of the antenna* in the direction \hat{k} (sort of like the cross section for scattering we discussed earlier.) By its definition, $\int d\Omega G(\hat{r}) = 4\pi$, so the above also tells us that the total effective area of the antenna integrated over all possible incoming angles is λ^2 . This is an important theorem in antenna theory because it provides a normalizing factor for measuring antenna efficiency: if you surround an antenna with a blackbody radiator, you know the incident flux. The antenna should receive a power λ^2 times the flux incident from the blackbody. You can then measure the power exiting the antenna onto a transmission line and take the ratio of observed to expected power to determine the antenna's overall efficiency. This could also be used, for example, to determine the antenna's radiation resistance.

The proof of this theorem is interesting because it involves the *Lorentz Reciprocity Theorem* and can be found in Drabowich *et al.*, *Modern Antennas*, §4.1.