# Physics 106b: Lecture 8 — 30 January 2014

## Hamiltonian Chaos: Theory

In the last lecture I used demonstrations and perturbation theory to show that the *invariant* tori corresponding to simple multiple-period motion with a rational or near rational relationship between the frequencies may break down on adding perturbations to an integrable system. In today's lecture I discuss how the breakdown to complex dynamics occurs. Hand and Finch give a good discussion of the ideas in §6.6-7 and §11.3-6.

### Fate of the Rational Tori – Poincaré-Birkoff Theory

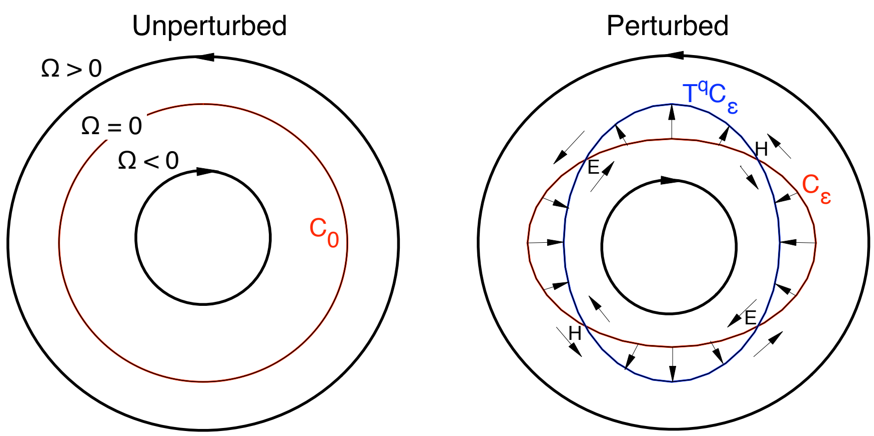


Figure 1: Behavior of the map $T^q$ in the unperturbed (integrable) and perturbed cases. For the unperturbed map there is a curve of fixed points $C_0$. In the perturbed case we can find a continuous curve $C_\varepsilon$ that is mapped without twist (i.e. in the radial direction). The mapping $T^qC_\varepsilon$ is shown by the arrows. There are an even number of intersections corresponding to alternating elliptic and hyperbolic fixed points, as can be seen combining the radial mapping with the twist, assumed to increase with increasing radii.

Consider an invariant curve with a winding number $p/q$ on the Poincaré section of the unperturbed system. This means that each point on the curve returns to the same point after $q$ iterations of the Poincaré map, i.e. each point is a fixed point of $T^q$. From now on we consider the effect of $T^q$. The curve is not necessarily a circle, but can be transformed to a circle by a smooth change of variables. For larger and smaller radii there will be invariant curves with winding numbers (under $T$) larger and smaller than $p/q$ and so with positive and negative twist under $T^q$.

Now perturb the system so that the Poincaré map becomes $T_\varepsilon$ and consider the effect of $T_\varepsilon^q$ in the vicinity of $C_0$. We look at the case where the continuation of $C_0$ is *not* invariant under the map. By continuity with the unperturbed case, for each $\theta$ we can find a radius at which the point is mapped purely in the radial direction (i.e. zero twist). Connecting these points gives a continuous curve $C_\varepsilon$ (a deformation of $C_0$) that is mapped only in the radial direction by $T_\varepsilon^q$. By the area preserving property of the map the image $T_\varepsilon^q C_\varepsilon$ of this curve under $T_\varepsilon^q$ must intersect $C_\varepsilon$ giving an *even* number of points that are fixed points of $T_\varepsilon^q$ (see Fig. 1). The sketch in Fig. 1, drawn with the insight of just small changes from the $\varepsilon = 0$ limit, shows that these must be alternating elliptic and hyperbolic fixed points.

I next show that there in fact at least $2q$ fixed points of $T_\varepsilon^q$. Let $P_\varepsilon$ be one of the fixed points. For the unperturbed map the points $P_0 = P_{\varepsilon\to 0}, T P_0, \ldots T^{q-1} P_0$ are distinct. For the perturbed system we have

$$T_\varepsilon^q T_\varepsilon P_\varepsilon = T_\varepsilon T_\varepsilon^q P_\varepsilon = T_\varepsilon P_\varepsilon \tag{1}$$

so that $T_\varepsilon P_\varepsilon$ is also a fixed point of $T_\varepsilon^q$ and by continuity in $\varepsilon$ will be distinct from $P_\varepsilon$, and similarly for $T_\varepsilon^2 P_\varepsilon$, etc. Therefore there are at least $q$ distinct fixed points of $T_\varepsilon^q$, mapped into one another by applying $T_\varepsilon$ some number of times. But elliptic fixed points cannot be mapped into hyperbolic fixed points, so there must be $2nq$ fixed points of $T_\varepsilon^q$ (with $n$ an integer) in the vicinity of $C_0$.

Thus the invariant circle on the Poincaré section with winding number $p/q$ has broken down into a sequence of $nq$ alternating elliptic and hyperbolic fixed points.

## Homoclinic Tangles

Chaos develops near the hyperbolic fixed points. The existence of complex dynamics is shown by the following argument.

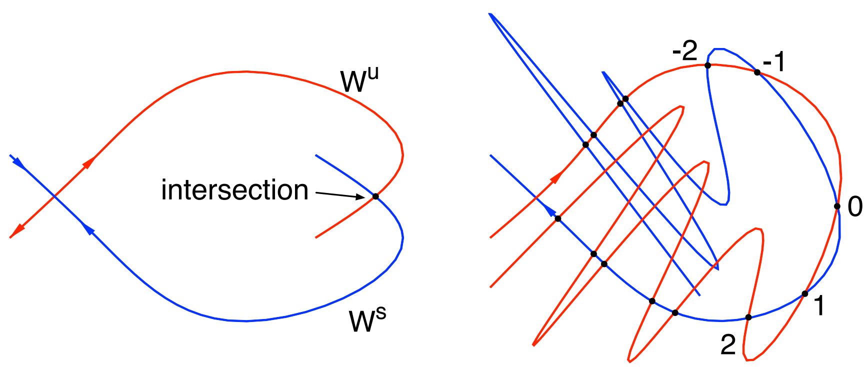


Figure 2: A homoclinic tangle

Consider a hyperbolic fixed point $\vec{x}_f$ of a map $M$ (e.g. $T_\varepsilon^q$). Linearizing the behavior near the fixed point would show one stable and one unstable direction. The *stable manifold* $W^s$ of $x_f$ is the set of points $\vec{x}$ such that $M^n\vec{x} \to \vec{x}_f$ as $n \to \infty$; the *unstable manifold* $W^u$ of $x_f$ is the set of points $\vec{x}$ such that $M^{-n}\vec{x} \to \vec{x}_f$ as $n \to \infty$. The stable and unstable manifolds are the nonlinear extension of the stable and unstable eigenvectors of the linear analysis around $\vec{x}_f$. Although the manifolds are curves, the dynamics of the map from one initial condition will, of course, jump between discrete points on the curve.

It is possible that the stable and unstable manifold coincide. This is what happens in integrable systems such as the undriven Hamiltonian pendulum. However it is not likely to occur in the perturbed, nonintegrable system. The stable manifold cannot cross itself or the stable manifold of another fixed point (since then there would be a point without a unique inverse). Similarly an unstable manifold cannot cross itself or another unstable manifold. However an unstable manifold can cross a stable manifold. When this occurs through a transverse intersection rather than a tangency, a "homoclinic tangle" (or "heteroclinic tangle" if the manifolds belong to different fixed points) occurs, and it can be shown that there is chaotic dynamics nearby.

The construction is shown in Fig. 2. Suppose the manifolds intersect at the point $\vec{x}_0$, which therefore lies both on the stable manifold $W^s$ and the unstable manifold $W^u$. The map $\vec{x}_1 = M(\vec{x}_0)$ of $\vec{x}_0$ must also lie on both $W^u$ and $W^s$ and so is another intersection of the manifolds. There are an infinite number of points $M^n(\vec{x}_0)$ before the fixed point $\vec{x}_f$ is reached, and so there are an infinite number of intersections. As the point $M^n(\vec{x}_0)$ approaches $\vec{x}_f$ the unstable manifold is stretched along the unstable direction, but in a way that cannot lead to self intersections. Similarly under the inverse mapping $M^{-n}$ an infinite number of intersections are produced. This gives the wild type of behavior shown in Fig. 2.

The existence of chaos can be shown through demonstrating the existence of a Smale horseshoe map. This goes beyond the level of the class, but here's a brief description if you are interested. You can find more details in Lecture 23 of the notes I made for a Ph161 course on chaos a number of years ago that you can find on my Caltech website.

### Smale Horseshoe Map*

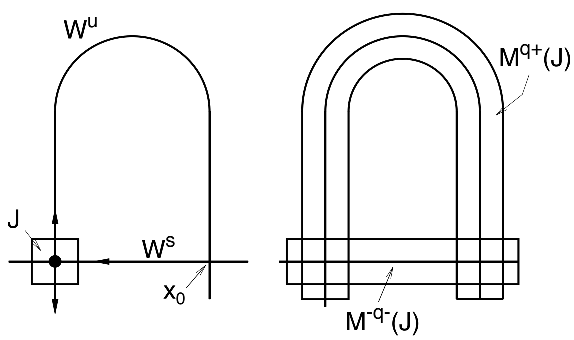


Figure 3: Smale horseshoe map due to a homoclinic intersection. (Note that thinning of the forward and backwards iterates of $J$ to preserve the area is not depicted in the figure.)

Consider the effect of the map $M$ on a small square $J$ centered on the fixed point. The map will stretch the square along $W^u$ (using continuity and the fact that the fixed point remains fixed). After a sufficient number $q_+$ of forward iterations $J^+ = M^{q_+}(J)$ will include the intersection point $\vec{x}_0$. Similarly under inverse iterations $J$ is stretched along $W^s$ and after a sufficient number $q_-$ of iterations [1] $J^- = M^{-q_-}(J)$ will include the point $\vec{x}_0$. Now we find that the map $\bar{M} = M^{q_+ + q_-}$ maps the rectangle $J^-$ into the horseshoe $J^+$. This map is called the Smale horseshoe, and iterations of points starting in $J$ have been shown to have chaotic dynamics (I leave you to research this topic if you are interested). Note the stretching-and-folding nature of the map. This is an essential feature of chaotic dynamics, where trajectories in the phase space cannot cross, and volumes in phase space are preserved, but the long time dynamics are nonrepeating. Thus the transverse intersection of $W^u$ and $W^s$ implies the existence of complex dynamics.

I have developed these arguments pictorially. Poincaré, Birkoff, Smale etc. were mathematiciancs, and the arguments can be made rigorous.

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[1] This $q$ is not related to the $q$ of the winding number expression.