Physics 106a — Classical Mechanics

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Lecture 14: Rotating Frames

Outline

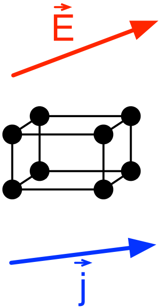
- What is a tensor?
- Relationship between time dependence of vectors in relatively rotating frames
- Application to Newtonian acceleration $\Rightarrow$ fictional/inertial forces
- Foucault pendulum
- Lagrangian and Hamiltonian approaches

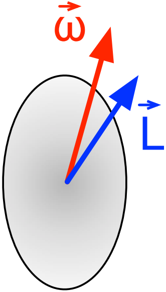
Review: vectors and rotations

- Many physical laws can be expressed in terms of vectors using the rotational symmetry of space.
- A vector can be considered as physical, geometric quantity independent of a choice of coordinates.
- A vector $\vec{a}$ can be represented by its components (a_1, a_2, a_3) with respect to a chosen orthonormal triad of basis vectors.
- Upon rotation, the components of a vector are transformed by an orthogonal rotation matrix U.
- Rotations through finite angles (and the corresponding rotation matrices) in general do not commute.
- Infinitesimal rotations can be represented by a vector $\vec{\delta\phi} = \delta\phi\hat{n}$. Infinitesimal rotation do commute.
- The rate of change of rotation can be represented by the angular velocity vector $\vec{\omega} = \vec{\delta\phi}/\delta t$.
- Solid body motion can be decomposed into rotation about an axis and translation along the axis.
- Rolling motion is pure rotation about the contact point or line.

Second rank tensors

Relates a vector (linearly) to another vector



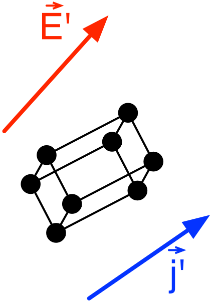


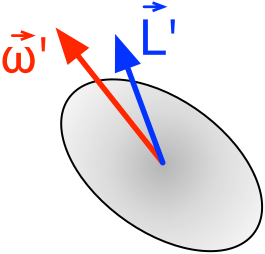
$$\vec{j} = \overleftrightarrow{\sigma} \cdot \vec{E}$$
$$P = \tfrac{1}{2}\vec{E} \cdot \overleftrightarrow{\sigma} \cdot \vec{E}$$

$$\vec{L} = \overleftrightarrow{I} \cdot \vec{\omega}$$
$$T = \tfrac{1}{2}\vec{\omega} \cdot \overleftrightarrow{I} \cdot \vec{\omega}$$

Second rank tensors

Relates a vector (linearly) to another vector





$$\vec{j}' = \overleftrightarrow{\sigma}' \cdot \vec{E}'$$
$$P = \tfrac{1}{2}\vec{E}' \cdot \overleftrightarrow{\sigma}' \cdot \vec{E}'$$

$$\vec{L}' = \overleftrightarrow{I}' \cdot \vec{\omega}'$$
$$T = \tfrac{1}{2}\vec{\omega}' \cdot \overleftrightarrow{I}' \cdot \vec{\omega}'$$

Second rank tensors

Component form

$$j_i = \sigma_{ij} E_j \qquad \text{2nd rank tensor has } 3^2 = 9 \text{ components}$$

Under a (passive) rotation the components transform as

$$\sigma'_{ik} = U_{i\alpha} U_{k\beta} \sigma_{\alpha\beta}$$

Each index of a tensor transforms like the component of a vector

nth rank tensors

nth rank tensor: linear vector function of $n-1$ vectors

$$T(\_,\_,\ldots,\_) \quad \text{such that} \quad T(\_\,,\vec{V}_1,\vec{V}_2\ldots\vec{V}_{n-1}) \Rightarrow \text{vector}$$

Example: elasticity tensor

$$\text{stress} = \text{elasticity tensor} \Leftarrow \text{strain}$$

Component form: 3^n components

$$\sigma_{ij} = \lambda_{ijkl} e_{kl} \quad \text{with} \quad e_{kl} = \frac{1}{2}\left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}\right)$$

Relatively rotating frames: general approach

My approach is based on the following principles:

- The rotation of frames relative to one another can be defined without using coordinate axes.
- Vectors may be defined physically/geometrically without reference to particular coordinate axes.
- We can understand and calculate the relationship between vectors in relatively rotating frames, and between their time dependences, without introducing components with respect to particular choices of axes.
- We can evaluate components of vectors with respect to any choice of coordinate axes (fixed in one or other frame of reference, rotating in some different way...).

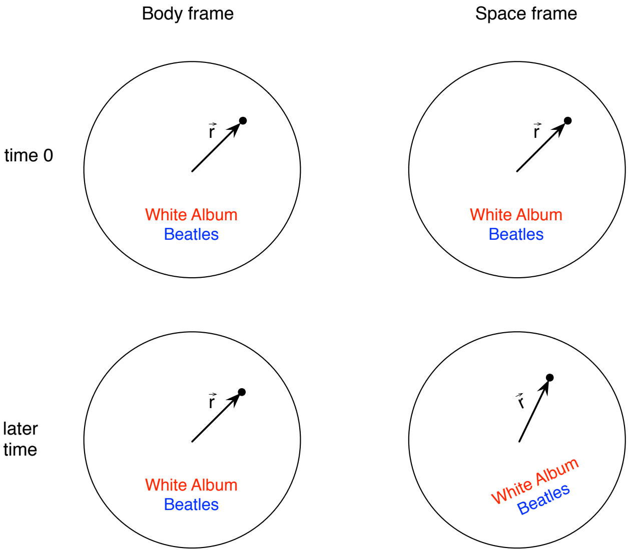
Hand and Finch prefer to relate the time dependences of vectors in relatively rotating frames by deriving relationships for the *components* of the vectors with respect to various choices of basis vectors. To facilitate the writing they also define a "bold vector" **a** representing the three components (a_1, a_2, a_3) of $\vec{a}$ relative to the unprimed basis. I will not use this notation.

Rotating frames: setup

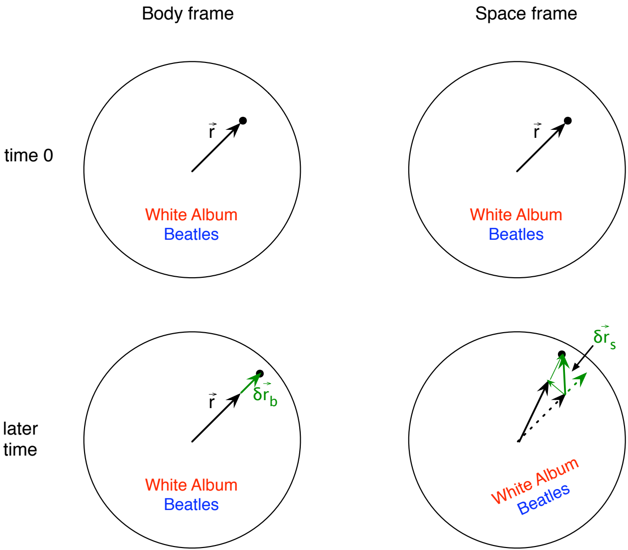
- Consider a “space” frame and a “body” frame
- Body frame is rotating at angular velocity $\vec{\omega}$ relative to space frame
- Space frame is rotating at angular velocity $-\vec{\omega}$ relative to body frame
- A vector that is time independent in one frame will be time dependent (rotating) in the other

$$\left.\frac{d\vec{a}}{dt}\right|_{\mathrm{s}} = \left.\frac{d\vec{a}}{dt}\right|_{\mathrm{b}} + \vec{\omega} \times \vec{a}$$

Relationship between time dependence in the two frames



Relationship between time dependence in the two frames



Relationship between time dependence in the two frames

Consider the change in a vector $\vec{a}$ in an infinitesimal time δt.

- In the body frame $\vec{a}$ changes by $\delta\vec{a}|_{\mathrm{b}}$ if it is time dependent in this frame
- In the space frame there is an additional change $\delta\vec{\phi} \times \vec{a}$ due to the rotation of $\vec{a}$

  $$\delta\vec{a}|_{\mathrm{s}} = \delta\vec{a}|_{\mathrm{b}} + \delta\vec{\phi} \times \vec{a}$$

  with $\delta\vec{\phi}$ the infinitesimal rotation of the body frame in time δt
- Dividing by δt and taking $\delta t \to 0$ gives the relationship between the time derivatives in the two frames

  $$\left.\frac{d\vec{a}}{dt}\right|_{\mathrm{s}} = \left.\frac{d\vec{a}}{dt}\right|_{\mathrm{b}} + \vec{\omega} \times \vec{a}$$

- Equally we could write the inverse relation

  $$\left.\frac{d\vec{a}}{dt}\right|_{\mathrm{b}} = \left.\frac{d\vec{a}}{dt}\right|_{\mathrm{s}} + (-\vec{\omega}) \times \vec{a}$$

  This is the same form since space frame is rotating at angular velocity $-\vec{\omega}$ relative to the body frame

Example: velocity

Velocity

$$\vec{v}_{\mathrm{s}} = \left.\frac{d\vec{r}}{dt}\right|_{\mathrm{s}} = \left.\frac{d\vec{r}}{dt}\right|_{\mathrm{b}} + \vec{\omega} \times \vec{r} = \vec{v}_{\mathrm{b}} + \vec{\omega} \times \vec{r}$$

Note:

- Some physical objects and the vectors describing them are the *same* in relatively rotating frames, e.g. $\vec{r}$
- Some physical objects and the vectors describing them are *different* in relatively rotating frames, e.g. the velocity $\vec{v}$ measured for a particle at position $\vec{r}$, the angular velocity $\vec{\omega}$ measured for a rotating object

Example: acceleration

$$\vec{v}_{\mathrm{s}} = \vec{v}_{\mathrm{b}} + \vec{\omega} \times \vec{r}$$

Acceleration

$$\vec{a}_{\mathrm{s}} = \left.\frac{d\vec{v}_s}{dt}\right|_{\mathrm{s}} = \left.\frac{d}{dt}\right|_{\mathrm{s}} (\vec{v}_{\mathrm{b}} + \vec{\omega} \times \vec{r}) = \left.\frac{d\vec{v}_{\mathrm{b}}}{dt}\right|_{\mathrm{s}} + \vec{\omega} \times \left.\frac{d\vec{r}}{dt}\right|_{\mathrm{s}}$$

Calculate derivatives

$$\left.\frac{d\vec{v}_{\mathrm{b}}}{dt}\right|_{\mathrm{s}} = \left.\frac{d\vec{v}_{\mathrm{b}}}{dt}\right|_{\mathrm{b}} + \vec{\omega} \times \vec{v}_{\mathrm{b}}, \qquad \left.\frac{d\vec{r}}{dt}\right|_{\mathrm{s}} = \left.\frac{d\vec{r}}{dt}\right|_{\mathrm{b}} + \vec{\omega} \times \vec{r}$$

and then write $\vec{a}_{\mathrm{b}} = d\vec{v}_{\mathrm{b}}/dt|_{\mathrm{b}}$ and $\vec{v}_{\mathrm{b}} = d\vec{r}/dt|_{\mathrm{b}}$

$$\vec{a}_{\mathrm{s}} = \vec{a}_{\mathrm{b}} + 2\vec{\omega} \times \vec{v}_{\mathrm{b}} + \vec{\omega} \times (\vec{\omega} \times \vec{r}) + \dot{\vec{\omega}} \times \vec{r}$$

including a time dependent term with $\dot{\vec{\omega}}$ the time derivative of the angular velocity of the body frame relative to the space frame.

Fictitious or inertial forces

Now suppose the space frame is inertial, and the body frame noninertial

- In the space frame Newton tells us

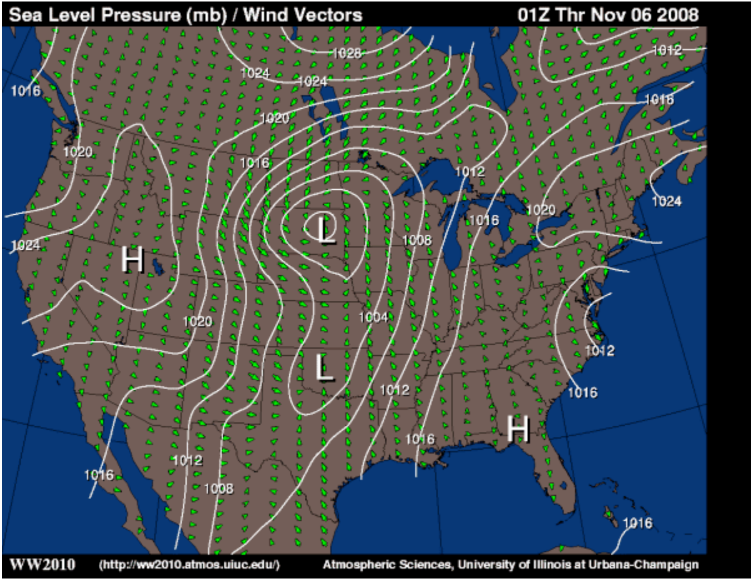
$$\vec{F} = m\vec{a}_s$$

- The relationship between the acclerations then gives

$$m\vec{a}_{\mathrm{b}} = \vec{F} - 2m\vec{\omega} \times \vec{v}_{\mathrm{b}} - m\vec{\omega} \times (\vec{\omega} \times \vec{r}) - m\dot{\vec{\omega}} \times \vec{r}$$

- We have moved the purely kinematic "correction" terms to the acceleration onto the right hand side, and we describe them as *fictitious* forces: Coriolis, centrifugal, and Euler.
- On the other hand, particularly for linear acceleration, it sometimes seems perverse to separate the effects into "real" and "inertial" (cf. Einstein and gravity), and so the less pejorative term *inertial* forces is often used.

Weather map


Sea Level Pressure (mb) / Wind Vectors
01Z Thr Nov 06 2008
1028
1024
1024
1020
1016
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H
L
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1020
1012
1008
1004
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1012
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1024
1012
1016
1016
WW2010
(http://ww2010.atmos.uiuc.edu/)
Atmospheric Sciences, University of Illinois at Urbana-Champaign

Hurricane Sandy

Lagrangian approach

Polar coordinates (r, ϕ, z) with respect to axes fixed in the body frame with z along the rotation axis

$$\text{Lagrangian} \quad L = T - V \quad \text{with} \quad T = \frac{1}{2}m\left[\dot{r}^2 + r^2(\dot{\phi} + \omega)^2 + \dot{z}^2\right]$$

Euler-Lagrange equations

$$\begin{aligned}
\frac{d}{dt}(m\dot{r}) - mr(\dot{\phi} + \omega)^2 + \frac{\partial V}{\partial r} &= 0 \\
\frac{d}{dt}[mr^2(\dot{\phi} + \omega)] + \frac{\partial V}{\partial \phi} &= 0 \\
\frac{d}{dt}(m\dot{z}) + \frac{\partial V}{\partial z} &= 0
\end{aligned}$$

Rearrange

$$\begin{aligned}
m\ddot{r} - mr\dot{\phi}^2 &= 2m\omega r\dot{\phi} + mr\omega^2 + F_r \\
mr\ddot{\phi} + 2m\dot{r}\dot{\phi} &= -2m\omega\dot{r} + F_\phi \\
m\ddot{z} &= F_z
\end{aligned}$$

Do not add inertial forces by hand to potential in Lagrangian

Hamiltonian approach

The Hamiltonian is

$$H = p_r\dot{r} + p_\phi\dot{\phi} + p_z\dot{z} - L$$

The interesting term is $p_\phi\dot{\phi}$ with

$$p_\phi = \frac{\partial L}{\partial\dot{\phi}} = mr^2(\dot{\phi} + \omega)$$

so that

$$p_\phi\dot{\phi} = mr^2(\dot{\phi} + \omega)\dot{\phi} = mr^2(\dot{\phi} + \omega)^2 - mr^2(\dot{\phi} + \omega)\omega$$

Using this, and the simple expressions for p_r, p_z gives the Hamiltonian

$$H = \frac{p_r^2 + p_z^2}{2m} + \frac{p_\phi^2}{2mr^2} + V - \omega p_\phi$$

which is

$$H = H_{\omega=0} - \omega p_\phi$$

and only change is the extra term $-\vec{\omega}\cdot\vec{l}$ with $\vec{l}$ the angular momentum