ACM 100b

Review of ODE's - part 6

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Series for general second order linear ODE's

- We next turn to the general second order linear ODE
$$y'' + p(x)y' + q(x)y = 0$$
- In the example above
$$y'' = xy$$
the coefficient functions for the ODE are
$$p(x) = 0 \qquad q(x) = x.$$
- It is easy to see how we can use the series approach if the coefficient function are polynomials.
- But if the coefficient functions are more general it is still possible to get series.
- But it's more complicated - we have to expand the coefficient functions in terms of power series.
- Since we assumed that $p(x)$ and $q(x)$ are smooth enough in some neighborhood of $z = x_0$, this is always possible.

Series for general second order ODE's

- For the ODE

$$y'' + p(x)y' + q(x)y = 0$$

we can then write

$$p(x) = \sum_{n=0}^{\infty} p_n(x - x_0)^n \qquad p_n = \left.\frac{d^n p(x)}{dx^n}\right|_{x=x_0}$$

$$q(x) = \sum_{n=0}^{\infty} q_n(x - x_0)^n \qquad q_n = \left.\frac{d^n p(x)}{dx^n}\right|_{x=x_0}.$$

Series for general second order ODE's

- Now inserting the series for $p(x)$, $q(x)$ along with our expansion for $y(x)$

$$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

into the ODE we get

$$\sum_{n=0}^{\infty} (n+1)(n+2) a_{n+2} (x - x_0)^n +$$
$$\sum_{n=0}^{\infty} p_n (x - x_0)^n \sum_{n=0}^{\infty} (n+1) a_{n+1} (x - x_0)^n +$$
$$\sum_{n=0}^{\infty} q_n (x - x_0)^n \sum_{n=0}^{\infty} a_n (x - x_0)^n = 0.$$

Series for general second order ODE's

- We can then expand term by term,
- Then match like powers of $(x - x_0)^n$
- And finally derive a recursion relation.
- We can write this in general as follows:

$$(n+1)(n+2)a_{n+2} + \sum_{k=0}^{n}(n-k+1)p_k a_{n-k+1} + \sum_{k=0}^{n} q_k a_{n-k} = 0. \qquad n = 0, 1, 2, \ldots$$

- Recall a_0 and a_1 come from the IVP
- We can determine the remaining a_n and develop a series solution.
- But are these series useful? We discuss this next.