ACM 100b

Transforms of more general functions

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Recap

- In our previous lecture we discussed using the Fourier transform to solve linear constant coefficient ODE's on the interval $-\infty < x < \infty$
- We also introduced the cosine transform

$$
\begin{aligned}
F_c(k) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos(kx) dx \\
f(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty F_c(k) \cos(kx) dk.
\end{aligned}
$$

- And the sine transform

$$
\begin{aligned}
F_s(k) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin(kx) dx \\
f(x) &= \sqrt{\frac{2}{\pi}} \int_0^\infty F_s(k) \sin(kx) dk.
\end{aligned}
$$

- These can be used to solve ODE's on the interval $0 \le x \le \infty$.

Recap

- Either transform could be used for ODE's on the interval $0 < x < \infty$
- But they are mostly used for ODE's with even derivative terms
- The sine transform is convenient when the value of the solution is given at $x = 0$
- The cosine transform is useful when the derivative of the solution is given at $x = 0$

Transforming a wider class of functions

- It often happens that the function $f(x)$ you want to transform does not vanish sufficiently quickly as $|x| \to \infty$ so that the integral

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ikx) dx$$

  does not technically exist.
- There are several ways to cope with this
- One way is to use convergence factors and delta functions
- Another way is to use contour deformation.
- We'll illustrate these approaches.

Use of a convergence factor

- Suppose we wanted to compute the full Fourier transform of the function $f(x) = 1$
- This is
$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-ikx)dx$$
- This integral is not convergent.
- However, we can make it convergent by instead considering
$$F_a(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(-|a|x)\exp(-ikx)dx$$
- The idea is to compute the result for finite a and then take the limit as $a \to 0$

Use of a convergence factor

- If we perform the integral we would get

$$F_a(k) = \frac{1}{\sqrt{2\pi}} \frac{2a}{a^2 + k^2}$$

- It is not hard to show that the area under this function is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{2a}{a^2 + k^2} dk = \sqrt{2\pi}$$

- On the other hand unless $k = 0$ the limit of this function as $a \to 0$ is 0.
- So this must be the function

$$\sqrt{2\pi}\delta(k) \text{ as } a \to 0$$

- And we can confirm that the inverse transform of this function gives us back the function 1.

Use of a convergence factor

- From here we can see that if

$$F(k) = \frac{\sqrt{2\pi}}{2}\left[\delta(k-1) + \delta(k+1)\right]$$

  then the inverse transform gives us

$$f(x) = \cos(x)$$

- In this way we can get the transforms of any trigonometric function even if the integrals don't converge.
- This technique works for other transforms as well.
- For example consider the cosine transform of $\sin(x)$
- Using instead the transform with a convergence factor we get

$$F_c(k) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \exp(-ax)\sin(x)\cos(kx)dx \qquad a > 0$$

Use of a convergence factor

- Doing the integral we get

$$F_c(k) = \frac{a^2 - k^2 + 1}{\left(a^2 + k^2 + 2\,k + 1\right)\left(a^2 + k^2 - 2\,k + 1\right)}$$

- You can now work with this in any ODE problem
- When the time comes to get the final answer you can then take the limit $a \to 0$
- You do need to check that the result will not depend on how this limit is taken

Use of contour deformation

- Suppose we want to transform the function

$$g(x) = \begin{cases} x^{+1/2} & x > 0 \\ 0 & x < 0 \end{cases}$$

- Now if we write the integral for the transform to k space we get

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty x^{1/2} \exp(-ikx) dx$$

- This integral also will not converge if we consider k real.
- But notice it will converge if $\mathrm{Im}(k) < 0$.
- In fact you can show by setting $k = \xi - i\epsilon$ where now ξ is real and $\epsilon > 0$ that

$$F(k) = \frac{1}{2\sqrt{2}} \exp(-3\pi i/4) k^{-3/2}$$

- This is valid as long as $-\pi < \arg(k) < 0$

Use of contour deformation

- You can see that by analytic continuation you can say that

$$F(k) = \frac{1}{2\sqrt{2}} \exp(-3\pi i/4) k^{-3/2}$$

  for all complex k as long as $k \neq 0$
- Now suppose we write $k = k_1 - ik_2$ and hold $k_2 > 0$ and constant and look at the variation of F with k_1:

$$G(k_1) = F(k_1 - ik_2)$$

- $G(k_1)$ is the transform of the function

$$g(x) = \exp(-k_2 x) f(x)$$

- This function could have been transformed using the usual Fourier transform because it does decay.

Use of contour deformation

- So we can write

$$g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(+ik_1 x) G(k_1) dk_1$$

- But this means

$$f(x) = \exp(k_2 x) g(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp(+i(k_1 - ik_2)x) F(k_1 - ik_2) dk_1$$

- But this is

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty - ik_2}^{\infty - ik_2} \exp(+ikx) F(k) dk$$

- And now using Cauchy's integral theorem we can write this as

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\Gamma_1} \exp(+ikx) F(k) dk$$

where Γ_1 is the real axis except for an indentation about the point $k = 0$.