XIII. ANGULAR MOMENTUM

We left the discussion of angular momentum on the back-burner in order to derive the spectrum of the Hydrogen atom. Now it is time to revisit it. We will do so by exploring the algebraic structure that arises with the angular momentum operators. Angular momentum is nasty - but at least the algebra is on our side.

A. z-direction angular momentum

Let's start simple. Let's only consider angular momentum in the z direction. Clasically, what is angular momentum in the z-direction measuring?

$$L_z = I_z \omega_z = I_z \dot{\phi} \tag{356}$$

where ϕ is again the azimuthal angle. For a particle moving on a ring (just as in the mid-term) this would simply be:

$$L_z = rm\dot{\phi}r = rp_{ring} \tag{357}$$

Now, taking the ring as our example, we see that in the same way that:

$$[\hat{p}_{ring}, \hat{x}_{ring}] = -i\hbar \tag{358}$$

we can write:

$$-i\hbar = [r\hat{p}_{ring}, \hat{x}_{ring}/r] = [\hat{L}_z, \phi]. \tag{359}$$

So indeed, we deduce what we already guessed: in quantum mechanics, the relation between a coordinate, and its associated momentum, is a 'unit' commutation relation. This also gives us the form of the angular momentum in real space:

$$\hat{L}_z = \frac{i}{\hbar}\frac{\partial}{\partial\phi}. \tag{360}$$

B. Commutation relations of $\hat{L}_z$

The next step is to figure out all the commutation relations of L_z, and also its companions, L_x and L_y, which is are less simply stated. Let's start simple. What is the commutation relation of L_z with the radius vector?

We need to write the radius vector, $\vec{r}$, in terms of spherical coordinates, but broken down into cartesian components. It'll be simpler that way. So we write:

$$\vec{r} = \hat{x}r\sin\theta\cos\phi + \hat{y}r\sin\theta\sin\phi + \hat{z}r\cos\theta \tag{361}$$

We need the commutator:

$$\left[\hat{L}_z, \vec{r}\right] = \hat{L}_z\vec{r}\cdot - \vec{r}\hat{L}_z\cdot \tag{362}$$

and I write a $\cdot$ to remind you that the commutator is written with an additional function in mind, to which the commutator may be applied. The only contribution will come from derivatives in L_z that operate on $\vec{r}$. Namely, the L_z pickes on ϕ. ϕ however, appears only in the x and y components, and is an easy target:

$$\left[\hat{L}_z, \vec{r}\right] = \frac{\hbar}{i}\frac{\partial}{\partial\phi}\left(\hat{x}r\sin\theta\cos\phi + \hat{y}r\sin\theta\sin\phi\right) \tag{363}$$

This becomes:

$$\frac{\hbar}{i}\left(-\hat{x}r\sin\theta\sin\phi + \hat{y}r\sin\theta\cos\phi\right) = i\hbar\hat{z}\times\vec{r} \tag{364}$$

So this is just as if we had rotated space, and asked - how much did the radius-vector change in response? Indeed, the z-component is not affected, and the x and y components just switch according to the cross product with $\hat{z}$.

C. Comutators and unitaries

There is something important to point out. This commutator is intimately related to the unitary rotation operator. When we do changes of bases between two orthonormal basis. What is the type of transforations we have? Unitary:

$$UU^\dagger = I \tag{365}$$

This makes sense since unitary transformations will keep our bra-ket inner product structure intact. Starting with two kets $|\psi\rangle$ and $|\phi\rangle$, transforming them using a unitary U into $U|\psi\rangle$ and $U|\phi\rangle$, and then taking the inner-product, we see:

$$\langle\psi| U^\dagger U |\phi\rangle = \langle\psi| I |\phi\rangle = \langle\psi |\phi\rangle . \tag{366}$$

BTW, what if we calculate matrix elements of some operator, say $\hat{A}$, after transforming? We get:

$$\langle\psi| U^\dagger A U |\phi\rangle \tag{367}$$

Which we could think of as a matrix element of another operator in the original basis:

$$= \langle\psi| \tilde{A} |\phi\rangle \tag{368}$$

with

$$\tilde{A} = U^\dagger A U \tag{369}$$

No surprise: operators transform as matrices.

Another thing - what is a unitary operator? How do you make it? One way is to exponentiate a Hermitian operator. For instance:

$$U = e^{-i\hat{\mathcal{H}}t/\hbar} \tag{370}$$

Is a really important unitary operator. It evolves wave functions in time.

Which operator rotates space about the z-axis? It is:

$$R(\delta\phi) = e^{i\hat{L}_z\delta\phi/\hbar} \tag{371}$$

You can get the feeling for this by expanding the exponent in a power series of L_z - it'll look like a Taylor series. The connection between these exponents and the commutator is described in the notes, but I'm going to skip it in class.

D. More commutators with L_z

We saw that $[L_z, \vec{r}] = i\hbar\hat{z}\times\vec{r}$. How do you think that would change if I put another vector operator, say $\hat{v}$, or $\hat{\vec{L}}$, instead of $\vec{r}$? Well essentially any vector, I can describe in terms of spherical coordinates:

$$\vec{v} = \hat{x}v\sin\theta\cos\phi + \hat{y}v\sin\theta\sin\phi + \hat{z}v\cos\theta \tag{372}$$

where $v = |\vec{v}|$, and θ and ϕ are still directions in 3d space, but they do not describe the location of a particle. They may describe its velocity, or something else, but that doesn't matter. When we differentiate with respect to ϕ, it's really like moving the x axis. It has the sense of a rotation. That should also be intuitive: when there is is angular momentum, it means that the system rotates. And with it, rotate all its vectors - by definition.

So what do we conclude from this verbiage? That we can replace $\vec{r}$ in the commutator with L_z with any vector, and have:

$$[L_z, \vec{v}] = i\hbar\hat{z}\times\vec{v}. \tag{373}$$

Particularly important to me right now, are the commutation relations with the other components of $\vec{L}$. I'll write them split to components:

$$[L_z, L_x] = i\hbar L_y, \text{ and: } [L_z, L_y] = -i\hbar L_x \tag{374}$$

It's nicer to think of these cyclically, with the ordered triad xyz:

$$[L_y, L_z] = i\hbar L_x, [yz\to x] \quad [L_z, L_x] = i\hbar L_y, [zx\to y] \tag{375}$$

Now that I write the relations this way, what is the remaining one?

$$[L_x, L_y] = i\hbar L_z, [xy\to z]. \tag{376}$$

That was really not that hard. We don't really know what the angular momentum operators are, except for L_z, but we do know all of their commutation relations! That will be enough.

E. Ladder operators

Our next job is to construct what in the jargon is known as the 'representation' of the angular momentum operators. By this we mean - understand how we can write the angular momentum operators as finite-size matrices.

The first thing to do is to again realize that the total angular momentum does not change due to space rotations. This implies that it commutes with all the angular momentum operators:

$$[L^2, L_\alpha] \tag{377}$$

to think about the commutator, we may well think of the question - how does L^2 change if we rotate the system about the α axis. L^2 is a scalar, so the answer is - not at all. Therefore:

$$[L^2, L_\alpha] = 0 \tag{378}$$

But this means that we can choose one axis, which we quickly take advantage of to define α as the z exis , and simultaneaously find the eigenstates of both operators (Matrices that commute can be simultaneously diagonalized).

The next question is a bit more complicated. Can we construct the matrices L_x and L_y in terms of the a basis which consist of eigenstates of L_z and L^2? The answer is a rather inspiring yes. In the case of the harmonic oscillators we constructed the whole Hilbert space in terms of eigenstates of $\hat{\mathcal{H}}$ by figuring out ladder operators:

$$[\hat{\mathcal{H}}, a^+] = \hbar\omega a^+ \tag{379}$$

and

$$[\hat{\mathcal{H}}, a^-] = -\hbar\omega a^- \tag{380}$$

Can we do something like that here, but for L_Z?

Sure thing. check out:

$$L^+ = L_x + iL_y \tag{381}$$

consider:

$$[L_z, L_x + iL_y] = i\hbar(L_y - iL_x) = \hbar(L_x + iL_y) \tag{382}$$

Indeed a ladder operator! It raises the angular momentum by - wait - just an $\hbar$. What about the step down operator? Easy, just the adjoint:

$$L^- = (L_x + iL_y)^\dagger = L_x - iL_y. \tag{383}$$

and

$$[L_z, L_x - iL_y] = -[L_z, L_x + iL_y]^\dagger = i\hbar(L_y + iL_x) = -\hbar(L_x - iL_y). \tag{384}$$

We have the ladder operators, so we know that eigenvalues of L_z must be of the form:

$$L_z = m\hbar + m_0\hbar \tag{385}$$

with m being integer, and this way the algebra connects all states to each other. But what could m_0 be? Could it be any number? Let's think for a bit about the possible answer. First, in the midterm, we had a particle in a ring. The solutions for the particle in the ring were:

$$\psi(x) = e^{ikx} \tag{386}$$

such that $2\pi rk = 2\pi m$. This implies:

$$rp = m\hbar \tag{387}$$

so clearly $m_0 = 0$ is allowed, and seems to make sense here. Could it be other things? When we turn our system upside down, since the angular momentum is a vector, it seems that this makes:

$$L_z \to -L_z \tag{388}$$

and $m_0 \rightarrow -m_0$. If we want to stay within the same Hilbert subspace, what must m_0 be? Zero works, since $-0 = 0$. So $L_z = m\hbar$ seems to be okay, with m being an integer. We would mape integers to integers. Any other values? $m_0 = 1/2$ is a tempting possibility.

Now let's think a bit. We're thinking about the angular momentum of some particle. What could the total angular momentum be classically? Anything! just like regular momentum. We could have a particle not moving - angular momentum zero. we can have a particle orbiting a nuclei - possibly a lot of angular momentum! What is the total angular momentum? The square root of L^2. Can we say what L^2 is with our Algebra-guerilla attack?

We can, again, in a manner very similar to what we did with the harmonic oscillator. Start with:

$$L^2 = L_x^2 + L_y^2 + L_z^2 \tag{389}$$

the L_z squared we have down - this is the so-called axis of quantization, and we know that $L_z = m\hbar$. Does the rest look familiar? Doesn't it tickle you to think about it as an harmonic oscillator? There we tried:

$$L_x^2 + L_y^2 = (L_x - iL_y)(L_x + iL_y) - i[L_x, L_y] \tag{390}$$

Which coincides with our ladder operators:

$$= L^- L^+ + L_z \tag{391}$$

cool. This means that:

$$L^2 = L^- L^+ + L_z + L_z^2. \tag{392}$$

Now comes the only complicated part. Consider a particle, or a top, in a state with all angular momentum in the L_z direction: $\langle L_z \rangle = \ell\hbar$ with ℓ a positive integer. What would be the result of applying the raising operator?

$$L^+ |\ell\rangle = |\ell + 1\rangle??? \tag{393}$$

But the L^+ consists of generators of rotation. They cannot change the total angular momentum. The putative state $|\ell + 1\rangle$ must have more angular momentum in the z-direction than its antecedent, but this would contradict the above result! So we must conclude:

$$L^+ |\ell\rangle = 0 \tag{394}$$

Looking back at our equation for the total angular momentum squared:

$$L^2 = L^- L^+ + L_z(I + L_z) \tag{395}$$

implies:

$$L^2 |\ell\rangle = \hbar^2 \ell(\ell + 1) |\ell\rangle \tag{396}$$

An eigenstate of L^2.

This requires us to modify our notation a bit. Let's call this state $|\ell, \ell\rangle$ why two entries? One for the integer that determines L^2, and the other for the integer that describes L_Z.

Note that the total angular momentum squared is a bit more then $\ell^2\hbar^2$. It has an extra $\ell\hbar^2$. This implaies that even when the entire angular momentum is pointing in the z direction ($L_z = \ell\hbar$, $L_z^2 = \ell^2\hbar^2$) there is some extra angular momentum in the x-y plane, $L_x^2 + L_y^2 = \hbar^2\ell$. This is the result of uncertainty. We know that when two operators do not commute, they can not both have zero uncertainty. Here it is L_x and L_y that have uncertainty.