ACM 100b

Frobenius theory for regular singular points

Dan Meiron

Caltech

January 13, 2014

What happens near a regular singular point?

- A solution of an ODE with a regular singular point may actually be analytic there.
- If it is not analytic at the singular point then the type of singularity it can exhibit is either a pole of some order or an algebraic or logarithmic branch singularity.
- It can be shown that there is always one solution of the form

$$(x - x_0)^{\alpha} A(x),$$

- Here α is in general a complex number and is called the *indicial exponent*.
- In addition, $A(x)$ is guaranteed to be analytic at $x = x_0$
- And if you expand $A(x)$ in a Taylor series about the point $x = x_0$ then that series will be convergent with a radius of convergence at least as large as the distance to the singularity nearest to $x = x_0$.
- The combination of the Taylor series for $A(x)$ combined with the leading singular factor $(x - x_0)^{\alpha}$ is called a *Frobenius series*

Example of a regular singular point

- As an example, consider the function defined by the ODE

$$y' = y/\sinh(x)$$

- This ODE has a regular singular point at $x = 0$.
- This is because near $x = 0$, the function $1/\sinh(x)$ blows up like $1/x$.
- So our definition tells us that $x = 0$ is a regular singular point.
- But we can solve this ODE to get

$$y = c\tanh(x)$$

- Note that this is actually analytic at $x = 0$ as we said could happen.
- But note too the Taylor series has a radius of convergence of π because of the vanishing of $\sinh(x)$ at $x = i\pi$.

Behavior of solutions near regular singular points

- We stated earlier that near a regular singular point at $x = x_0$ the solution behaves like
$$(x - x_0)^\alpha A(x),$$
- If the order of the ODE is $n \geq 2$, then there is a second linearly independent solution of the form
$$y = (x - x_0)^\beta B(x),$$
or
$$y = (x - x_0)^\alpha A(x) \ln(x - x_0) + C(x)(x - x_0)^\beta.$$

Behavior of solutions near regular singular points

- Such solutions
$$y = (x - x_0)^\beta B(x),$$
or
$$y = (x - x_0)^\alpha A(x)\ln(x - x_0) + C(x)(x - x_0)^\beta.$$
arise because what is happening is that, near $x = x_0$, the ODE becomes similar to a Euler type ODE of the form
$$y'' + \frac{a}{(x - x_0)}y' + \frac{b}{(x - x_0)^2}y = 0.$$
- We showed earlier that ODE's have power law or logarithm solutions.
- The functions $A(x)$, $B(x)$, and $C(x)$ are all analytic at $x = x_0$
- And their series have radii of convergence at least as large as the distance to the nearest singularity of the coefficient functions.

Behavior of solutions near regular singular points

- In general, for each new linearly independent solution there is a new analytic function of x and a new indicial exponent or another power of $\ln(x - x_0)$.
- For an n'th order solution we can expect solutions like

$$y(x) = (x - x_0)^{\gamma} \sum_{i=0}^{n-1} \ln(x - x_0)^i A_i(x),$$

- Here A_i are analytic at $x = x_0$.
- Conversely if all the solutions at a given point $x = x_0$ have this form, then the point is a regular singular point.

If it's not an ordinary point or a regular singular point...

- If the point x_0 is not an ordinary point or a regular singular point, it is an *irregular singular point*.
- There is no rigorous theory to guide us in this case.
- As we will see later, such points correspond generally to essential singularities in the complex plane.
- At least one of the solutions in this case is *not* of the form of a Taylor series or a Frobenius series.