Physics 106a 8 October, 2013

Lecture 3: Examples of the Lagrangian Approach

General approach

In class I go through a number of examples demonstrating the use of the Lagrangian:

1. Motion in an accelerating rail car (spaceship, Volkswagen, Prius ...)
2. A particle sliding around the interior of a frictionless cone (the rolling ball equivalent is often used in science museums to illustrate planetary orbits)
3. A bead on a rotating stiff frictionless wire (the wire traces out a conical surface)
4. Continuum elastic string (if time)

The general approach to a wide class of problems is

- Choose N generalized coordinates q_k that define the configuration and can be varied independently whilst maintaining any constraints.
- Evaluate the Lagrangian $L = T - V$ in some inertial frame in terms of these coordinates and their time derivatives. Here T is the kinetic energy and V is the potential energy.
- Solve the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_k}\right) - \frac{\partial L}{\partial q_k} = 0 \qquad \text{for } k = 1 \ldots N. \tag{1}$$

Some simplifying features often crop up, making it easier to solve these coupled differential equations. This is described in the next two subsections. These results also have general and profound importance, and will be discussed more later.

Ignorable coordinates and conserved momenta

Quite generally, we define the *conjugate momentum* p_k of the generalized coordinate q_k as

$$p_k = \frac{\partial L}{\partial \dot{q}_k}. \tag{2}$$

If a coordinate q_m does not explicitly appear in the Lagrangian, i.e.,

$$\frac{\partial L}{\partial q_m} = 0 \tag{3}$$

it is called *ignorable* or *cyclic*. Equation (1) immediately gives for this coordinate

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_m}\right) = 0. \tag{4}$$

Thus the momentum conjugate to an ignorable coordinate is a constant of the motion. This connects *symmetries* to *conserved quantities*. We will discuss this more later.

The Hamiltonian and time independent Lagrangians

Define the *Hamiltonian* as

$$H = \sum_k \dot{q}_k \frac{\partial L}{\partial \dot{q}_k} - L \equiv \sum_k \dot{q}_k p_k - L \,. \tag{5}$$

H is a *constant of the motion* if the Lagrangian does not explicitly depend on time, since straightforward algebra (take care on the difference between d/dt and $\partial/\partial t$) shows

$$\frac{dH}{dt} = -\frac{\partial L}{\partial t}\,, \tag{6}$$

i.e. the *total* time derivative of the Hamiltonian along the actual path of the dynamics given by $\{q_k(t)\}$ is given by the *explicit* time dependence of the Lagrangian function.

The Hamiltonian is often the energy $T + V$. It is so if both the potential energy is independent of velocities, as is usually needed to be able to define a potential, and if the kinetic energy is a quadratic form in $\dot{q}_k$. The latter condition means $T = \frac{1}{2}\sum_{kl} t_{kl}\dot{q}_k\dot{q}_l$ with t_{kl} possibly depending on coordinates $\{q_j\}$ and time t. Then $\sum_k \dot{q}_k \partial T/\partial \dot{q}_k = 2T$, which gives $H = T + V$ (see Hand and Finch, problem 1-9). These properties of the kinetic energy hold for time independent holonomic constraints (see Lecture 4).

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