ACM 100c

Irregular singular points - a brief introduction

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Irregular singular points

- So far we have categorized the three types of points we can encounter in linear ODE's
- The easiest is if the point is ordinary.
- In that case we can use Taylor series
- If a point is singular then we showed that under certain circumstances it could be a regular singular point
- In that case the behavior at the point is that of a general power law or a logarithmic singularity or some combination
- If the singular point is not a regular singular point, then it is an irregular singular point.

Irregular singular points

- In this case the nature of the singularity is like that of an essential singularity in the complex plane.
- You may recall from your work in complex analysis that the behavior near such a point can be very complicated.
- And that there is no general theory to help guide us.
- However, it is still possible to perform certain types of analyses that give us insight into the behavior near the singular point.
- Here we give an example of what is possible.

An example of an irregular singular point

- Consider the modified Bessel equation again:

$$y'' + \frac{y'}{x} - \left[1 - \frac{\nu^2}{x^2}\right] y = 0.$$

- Recall ν is a parameter that gives you the order of the Bessel function.
- It's generally an integer but it can be any kind of number - even complex.
- This ODE has an irregular singular point at ∞.
- To see this we simply make the substitution

$$x \to 1/t$$

- The ODE becomes

$$\frac{d^2y}{dt^2} + \left[\frac{2}{t} - \frac{1}{t^2}\right]\frac{dy}{dt} - \left[\frac{1}{t^4} - \frac{\nu}{t^2}\right] y = 0$$

- As $t \to 0$ we see there is an irregular singular point.

The modified Bessel function as $x \to \infty$

- To analyze the ODE we return to the original form in x

$$y'' + \frac{y'}{x} - \left[1 - \frac{\nu^2}{x^2}\right] y = 0.$$

- We will try to understand what the two solutions are doing as $x \to \infty$.
- First we'll choose $\nu = 0$ for a definite case.
- To analyze this we first make the following substitution:

$$y(x) = \frac{v(x)}{\sqrt{x}}.$$

- In terms of $v(x)$ the ODE becomes

$$v'' + \left[\frac{1}{4x^2} - 1\right] v = 0.$$

- We didn't have to do this but it makes our work a little easier by eliminating the v' term in the ODE.

The modified Bessel function as $x \to \infty$

- Now look at the resulting ODE:

$$v'' + \left[\frac{1}{4x^2} - 1\right] v = 0.$$

- Now note that as $x \to \infty$ the ODE becomes

$$v'' - v = 0,$$

- We know how to solve this.
- The solutions are $v(x) = \exp(\pm x)$.
- An exponential does indeed exhibit an irregular singular point as $x \to \infty$.
- So we might guess that the Bessel functions behave like this and this conclusion is basically correct.

The modified Bessel function as $x \to \infty$

- We will now write
$$y(x) = \frac{\exp(-x)}{\sqrt{x}} w(x)$$
for one of the solutions.
- What we are doing is "peeling away" the types of behavior the solutions have as we approach the irregular singular point
- The function $w(x)$ can be shown to satisfy
$$w'' + 2w' + \frac{w}{4x^2} = 0.$$
- By trial and error, we can show that one can get a series solution for this ODE as $x \to \infty$.
- But it is not a Frobenius expansion because the point $x \to \infty$ is still an irregular singular point.

The modified Bessel function as $x \to \infty$

- The following expansion can be derived:

$$w(x) = \sum_{n=0}^{\infty} a_n x^{-n}.$$

- As you will see we will get a recursion relation for the coefficients a_n.
- Note this is not a Frobenius expansion because it's in powers of $1/x$.
- We could argue that this is the Frobenius expansion about $t = 1/x$.
- But we will see that this idea is not quite right for reasons we uncover below.

The modified Bessel function as $x \to \infty$

- We can try to see if we can get a recursion relation for the coefficients.
- If we plug in the series we find that we are successful.
- We get the following recursion relation:

$$a_{n+1} = a_n \frac{(2n+1)^2}{4(n+1)},$$

with a_0 arbitrary.

- In this series, the coefficients are

$$\begin{aligned} a_1 &= a_0 \frac{3 \cdot 3}{4 \cdot 2} \\ a_2 &= a_0 \frac{(5 \cdot 5)(3 \cdot 3)}{(4 \cdot 4)(3 \cdot 2)} \\ a_3 &= a_0 \frac{(7 \cdot 7)(5 \cdot 5)(3 \cdot 3)}{(4 \cdot 4 \cdot 4)(4 \cdot 3 \cdot 2)} \end{aligned}$$

and so forth.

The series we got is divergent

- But look at the way the coefficients behave as $n \to \infty$
- Or use the ratio test on this series.
- But in a Frobenius expansion, we expect to find a finite radius of convergence
- Because the expansion is supposed to describe a locally analytic function.
- These coefficients do not decrease with increasing n.
- In fact they blow up factorially.
- This series is certainly not a Frobenius series.
- It has zero radius of convergence and is divergent!

Asymptotic expansions

- So we see that analysis of irregular singular points leads to essential singularities and divergent expansions
- Nevertheless, it turns out the divergent series is useful in describing the behavior of the Bessel function.
- It is called an *asymptotic expansion* and can actually give very accurate values of the Bessel function for x large when used properly.
- This topic is explored in detail in ACM 101.
- But we'll show that the series we got can actually be used to evaluate the Bessel function for large x.

Asymptotic expansion for $\nu = 5$

- A similar derivation as above will get you the asymptotic expansion for $\nu = 5$.
- We will look at the growing solution - the one that grows like $\exp(x)$
- We have the (divergent) series

$$I_5(x) = \frac{1}{\sqrt{2\pi}} \frac{\exp(x)}{x} \left[1 - \frac{100-1}{1!8x} + \frac{(100-1)(100-9)}{2!(8x)^2} - \cdots \right]$$

Evaluating the series

Table 3.1 Asymptotic approximations to $e^{-x}I_5(x)$ for five values of x using the series in (3.5.10)

Entries in the columns are the partial sums truncated after the x^{-N} term. Underlined partial sums are optimal asymptotic approximations. Notice that even when $x = 7$ the leading term in the asymptotic expansion gives a very poor approximation while the optimal asymptotic truncation is very accurate. The number in parentheses is the power of 10 multiplying the entry.

	x				
N	3.0	4.0	5.0	6.0	7.0
0	2.30324 (−1)	1.99471 (−1)	1.78412 (−1)	1.62868 (−1)	1.50786 (−1)
2	1.08147 (0)	4.59816 (−1)	2.39128 (−1)	1.45372 (−1)	1.00804 (−1)
4	2.01953 (−1)	4.74361 (−2)	2.52641 (−2)	2.35810 (−2)	2.61284 (−2)
6	2.11127 (−2)	1.14538 (−2)	1.49262 (−2)	1.98392 (−2)	2.45412 (−2)
7	1.16597 (−2)	1.03611 (−2)	1.47212 (−2)	1.97870 (−2)	2.45248 (−2)
8	5.50542 (−3)	9.82749 (−3)	1.46411 (−2)	1.97700 (−2)	2.45202 (−2)
9	1.20401 (−4)	9.47732 (−3)	1.45991 (−2)	1.97626 (−2)	2.45184 (−2)
10	−5.73580 (−3)	9.19172 (−3)	1.45717 (−2)	1.97585 (−2)	2.45176 (−2)
11	−1.33001 (−2)	8.91505 (−3)	1.45504 (−2)	1.97559 (−2)	2.45172 (−2)
12	−2.45677 (−2)	8.60595 (−3)	1.45314 (−2)	1.97540 (−2)	2.45169 (−2)
13	−4.35276 (−2)	8.21586 (−3)	1.45122 (−2)	1.97523 (−2)	2.45167 (−2)
14	−7.90210 (−2)	7.66817 (−3)	1.44907 (−2)	1.97508 (−2)	2.45166 (−2)
15	−1.52078 (−1)	6.82267 (−3)	1.44641 (−2)	1.97492 (−2)	2.45164 (−2)
20	−1.31437 (1)	−3.61663 (−2)	1.39178 (−2)	1.97329 (−2)	2.45155 (−2)
35	−3.12759 (10)	−1.24079 (6)	−4.90286 (2)	−8.13340 (−1)	2.06197 (−2)
	Exact value of $e^{-x}I_5(x)$				
	4.54090 (−3)	9.24435 (−3)	1.45403 (−2)	1.97519 (−2)	2.45164 (−2)
	Relative error in optimal asymptotic approximation, %				
	21.0	0.57	0.069	0.0024	0.000071

The asymptotic series can be very accurate for large x

- The table above shows what happens when we evaluate partial sums of this series
- Note that after a certain number of terms the answer starts to diverge (depending on the value of x)
- That's as it should be - it's a divergent series,
- But the value you get before it does diverge is quite accurate
- And this accuracy improves as x gets large.
- This is why such series are useful even if they are divergent.