ACM95c Review Session for Midterm 2

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Introduction

This midterm will focus mainly on the heat equation, although Laplace's equation may also show up in the form of problems on the steady state of the heat equation. You should try to make sure that you feel comfortable with the following topics:

- Separation of variables and series solutions:
  - in two or more dimensions
  - with source terms
  - with nonhonogeneous boundary conditions
  - using unusual sets of orthogonal eigenfunctions such as Bessel functions
- Transforms:
  - Laplace transforms
  - Fourier transforms
  - Sine and cosine transforms

You should also have a sense of when to use each of these techniques: series solutions for finite intervals, Fourier transforms for infinite intervals, sine and cosine transforms for semi-infinite intervals, and Laplace transforms for the time variable.

For example, what technique(s) could you use to solve the following problem?

$$\begin{aligned} u_t &= u_{xx},\ 0 < x < \infty \\ u(0,t) &= g(x) \\ u(x,0) &= f(x) \end{aligned}$$

The usual way would be to use a sine transform, because we are on the half-line, and we have a Dirichlet BC at 0. (A Dirichlet BC is one where we are given the value of the function itself on the boundary, as opposed to a Neumann BC where the derivative is given). Then we could choose to solve the resulting ODE directly, or we could solve it using a Laplace transform if we so chose.

Alternatively, we could use a Laplace transform in time, and then solve the resulting ODE in x. We'd need another boundary condition in order for this to work, but we have one: we expect the solution to be bounded as $x \to \infty$ (and in fact, when we use the sine transform, we are assuming something even stronger than that, because implicit in the use of the transform is the notion that the transform is well defined and therefore finite!).

Example 1: Series Solutions

Consider the following problem:

$$\begin{aligned} u_t &= u_{xx} + u_x + u_{yy} + q(x,y,t),\ 0 < x < 1,\ 0 < y < 1 \\ u &= 0 \text{ on the boundaries} \\ u(x,y,0) &= f(x,y) \end{aligned}$$

It's on a finite domain, so the only method that we have learned so far for this situation is the series solution. At first glance, it might look like a simple sine series in both x and y could work, but, alas, the u_x term means that this won't work.

We'll have to use separation of variables to figure out a suitable set of Sturm-Liouville eigenfunctions with which to make a series solution.

So, ignoring $q(x,y,t)$ for the moment, we suppose $u = T(t)X(x)Y(y)$ and plug this into our PDE.

$$T'XY = TX''Y + TXY'' + TX'Y \tag{1}$$

$$\frac{T'}{T} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{X'}{X} \tag{2}$$

We want to pull out a spatial variable, since our plan is to solve for the t dependence with a full-blown series at the end which takes $q(x,y,t)$ into account and everything. So let's pull out $Y(y)$ first.

$$\frac{T'}{T} - \frac{X''}{X} - \frac{X'}{X} = \frac{Y''}{Y} = \lambda \tag{3}$$

So $Y'' = \lambda Y$, and from our boundary conditions we find that $Y(0) = Y(1) = 0$. This is a Sturm-Liouville problem that we saw many times in ACM95b, and it should come as no surprise that we obtain the solutions $Y_n(y) = \sin(n\pi y)$ with $\lambda = -n^2\pi^2$.

Now we solve for a set of eigenfunctions in x. We have

$$\frac{T'}{T} - \frac{X''}{X} - \frac{X'}{X} = -n^2\pi^2 \tag{4}$$

and then we write again

$$\frac{T'}{T} = \frac{X''}{X} = \frac{X'}{X} - n^2\pi^2 = -\mu \tag{5}$$

with $-\mu$ a new constant, and we solve for X which obeys the ODE

$$X'' + X' + (\mu - n^2\pi^2)X = 0 \tag{6}$$

subject to $X(0) = X(1) = 0$.

Let's put this in S-L form, that is $\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y + \lambda r(x)y = 0$. This won't actually help us to find solutions, but it will tell us something important about the solutions we will get. Using an integrating factor we obtain the form

$$\frac{d}{dx}\left(e^x X'\right) - (n\pi)^2 e^x X + \mu e^x X = 0 \tag{7}$$

with μ our eigenvalue. Note that $r(x) = e^x$. This will be important in the orthogonality relations that we must use later.

To actually solve the ODE we return to form (6), which is a second order ODE with constant coefficients, which solves out easily (via ansatz $X = e^{mx}$)to

$$0 = m^2 + m - (n\pi)^2 + \mu \tag{8}$$

$$m_\pm = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + (n\pi)^2 - \mu} \tag{9}$$

$$X(x) = c_1 e^{m_+ x} + c_2 e^{m_- x} \tag{10}$$

We then apply BCs to this, $X(0) = X(1) = 0$. I will leave it to you to show that $m_\pm \in \mathbb{R}$ cannot satisfy this, so the square root must be imaginary, and the term inside the square root must be negative. This means our full solution looks like sines and cosines

$$X(x) = e^{-\frac{x}{2}}\left[c_3 \sin\left(x\sqrt{-\frac{1}{4} - (n\pi)^2 + \mu}\right) + c_4 \cos\left(x\sqrt{-\frac{1}{4} - (n\pi)^2 + \mu}\right)\right] \tag{11}$$

$X(0) = 0$ shows us quickly that the cosine term must be zero. The other boundary term can be shown to yield

$$m\pi = \sqrt{-\frac{1}{4} - (n\pi)^2 + \mu} \tag{12}$$

$$X_m(x) = e^{-\frac{x}{2}} \sin mx \tag{13}$$

Then finally, we know from the Sturm-Liouville form of the equations that these X_m are orthogonal such that $\int_0^1 X_m X_{m'} e^x \, dx = \delta_{mm'} \int_0^1 X_m^2 e^x \, dx$.

Finally we can execute the series expansion. We write

$$u(x, y, t) = \sum_{n,m=1}^{\infty} a_{nm}(t) X_m(x) Y_n(y) \tag{14}$$

$$q(x, y, t) = \sum_{n,m=1}^{\infty} q_{nm}(t) X_m(x) Y_n(y) \tag{15}$$

$$f(x, y) = \sum_{n,m=1}^{\infty} f_{nm} X_m(x) Y_n(y) \tag{16}$$

with the coefficients f_{mn}, q_{mn} determined by

$$f_{n,m} = \frac{\int_0^1 \int_0^1 f(x, y) e^x X_m(x) Y_n(y)\, dx\, dy}{\int_0^1 \int_0^1 \sin^2(m\pi x) \sin^2(n\pi y)\, dx\, dy} \tag{17}$$

and similarly for q. Note the factor of e^x goes away in the denominator because of the $e^{-\frac{x}{2}}$ in the X_m. Our equation then looks like

$$\sum_{n,m=1}^{\infty} a'_{nm}(t) X_m(x) Y_n(y) = \sum_{n,m=1}^{\infty} a_n m(t) \left(X''_m(x) + X'(x) \right) Y_n(y) \tag{18}$$

$$+ \sum_{n,m=1}^{\infty} a_n m(t) X_m(x) Y'_n(y) + \sum_{n,m=1}^{\infty} q_n m(t) X_m(x) Y_n(y) \tag{19}$$

But then recall that $X''_m + X'_m = ((n\pi)^2 - \mu) X_m = \left(-\frac{1}{4} - (m\pi)^2\right) X_m$. Convenient! This is why we wanted to use these eigenfunctions in the first place, after all. Using this and the fact that $Y''_n(y) = -n^2\pi^2 Y_n(y)$ we get

$$\sum_{n,m=1}^{\infty} a'_{nm}(t) X_m(x) Y_n(y) = \sum_{n,m=1}^{\infty} a_{nm}(t) \left(-\frac{1}{4} - (m\pi)^2\right) X_m(x) Y_n(y) \tag{20}$$

$$- \sum_{n,m=1}^{\infty} n^2\pi^2 a_{nm}(t) X_m(x) Y_n(y) + \sum_{n,m=1}^{\infty} q_{nm}(t) X_m(x) Y_n(y) \tag{21}$$

$$a'_{nm}(t) = a_{nm}(t) \left(-\frac{1}{4} - (m\pi)^2\right) - n^2\pi^2 a_{nm}(t) + q_{nm}(t) \tag{22}$$

This is a first-order ODE in time. We can solve it using an integrating factor, using the initial condition $a_{nm}(0) = f_{nm}$. Substituting this back into our series for u will then give us the solution.

Example 2: Fourier Transforms

Consider the problem

$$u_t = k u_{xx} + c u_x,\ -\infty < x < \infty \tag{23}$$

$$u(x, 0) = f(x). \tag{24}$$

It's on the entire real line, so we try a Fourier transform. Using $\hat{u}$ to denote the Fourier transform of u, we get

$$U_t = -\lambda^2 k U + ic\lambda U \tag{25}$$

recalling that the Fourier transform of a derivative is $\mathcal{F}(u_x) = i\lambda \mathcal{F}(u)$. Some of you may have missed that fact in the course of what I hear was a lightning-run through Fourier transforms. So here's the derivation (integration by parts carry hard[1]).

$$\mathcal{F}(f'(x)) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-i\lambda x}\, dx \tag{26}$$

$$= \frac{1}{\sqrt{2\pi}} \left[f(x)e^{-i\lambda x}\Big|_{-\infty}^{\infty} - (-i\lambda) \int_{-\infty}^{\infty} f(x) e^{-i\lambda x}\, dx \right] \tag{27}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} i\lambda f(x) e^{-i\lambda x}\, dx \tag{28}$$

where the boundary term vanishes because $f(\pm\infty) = 0$ is a necessary (though not sufficient) condition for f to have a well-defined Fourier transform in the first place. If f did not do this, the transform would be an infinite quantity.

Let's go back now to the ODE after having taken the Fourier transform, (25). This is an *easy* ODE, with solution

$$U(t,\lambda) = A(\lambda) e^{(-\lambda^2 k + ic\lambda)t} \tag{29}$$

Now we constrain $A(\lambda)$, via our ICs! We take the Fourier transform of the IC $u(x,0) = f(x) \Rightarrow U(x, \lambda = 0) = \hat{f}(\lambda)$, and this means

$$U(t,\lambda) = \hat{f}(\lambda) e^{(-\lambda^2 k + ic\lambda)t} \tag{30}$$

We must now take the inverse Fourier transform. The natural thing to do will be to use a convolution, since we don't know $\hat{f}$ explicitly, but we certainly do know what it is the Fourier transform of! Also, Gaussians Fourier transform to Gaussians. You (probably) haven't seen this before, but it's one of those facts that tends to come up pretty early on when you start using the Fourier transform, and there's a proof given in an appendix at the end of these notes. Specifically (and this is the actual correct formula, not the maybe-formula that I gave in class, which was off by a factor of $\sqrt{\frac{\pi}{2}}$, sorry about that) the transform of a Gaussian e^{-ax^2} is given by $\frac{1}{\sqrt{2a}} e^{-\frac{\lambda^2}{4a^2}}$.

We are almost there. The only part of the answer that we don't yet know how to deal with is the factor of $e^{ic\lambda t}$. This, however, is manageable using another nice little fact about Fourier transforms. Lets consider the Fourier transform of a shifted function $f(x-\beta)$.

$$\mathcal{F}\left(f(x-\beta)\right) = \int_{-\infty}^{\infty} f(x-\beta) e^{-i\lambda x} dx \tag{31}$$

Now take a change of variable $y = x - \beta$ to get

$$\mathcal{F}\left(f(x-\beta)\right) = \int_{-\infty}^{\infty} f(y) e^{-i\lambda(y+\beta)} dx \tag{32}$$

$$= e^{-i\lambda\beta} \mathcal{F}(f) \tag{33}$$

So $\mathcal{F}(f(x-\beta)) = e^{-i\lambda\beta}\mathcal{F}(f)$, and we can take the factor of $e^{ic\lambda t}$ into account by adding a shift of $-ct$ when we take the inverse Fourier transform of $\hat{f}$.

We are finally ready then to convolve. Let $g(x,t) = (kt)^{-\frac{1}{4}} e^{-\frac{x^2}{2\sqrt{kt}}}$ be the inverse Fourier transform of $e^{-\lambda^2 kt}$, then (note $*$ denotes convolution)

$$u = f(x+ct) * g(x,t) \tag{34}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(w+ct) g(x-w,t)\, dw \tag{35}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(w+ct)(kt)^{-\frac{1}{4}} e^{-\frac{(x-w)^2}{2\sqrt{kt}}}\, dw \tag{36}$$

Good luck!

[1]For this piece of phrasing, you *may* blame Yubo Su. He was kind enough to give me his notes to use as a first draft and I felt I had to leave that comment in.

Appendix: Fourier Transform of a Gaussian

We look into the Fourier transform of a Gaussian. Consider $u(x) = e^{-ax^2}$, then we wish to compute its Fourier transform

$$\mathcal{F}[u(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ax^2} e^{-i\lambda x} \, dx \tag{37}$$

$$= \int_{-\infty}^{\infty} e^{-ax^2 - i\lambda x} \, dx \tag{38}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{4a^2}} \int_{-\infty}^{\infty} e^{-a\left(x^2 + \frac{i\lambda x}{a} - \frac{\lambda^2}{4a^2}\right)} \, dx \tag{39}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{4a^2}} \int_{-\infty}^{\infty} e^{-a\left(x + \frac{i\lambda}{2}\right)^2} \, dx \tag{40}$$

where we have completed the square in the exponent and pulled the term out of the exponent as it is not x-dependent. Then we want to evaluate the integral. We make change of variables $y = x + \frac{i\lambda}{2}$, which gives us

$$U(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{4a^2}} \int_{-\infty - \frac{i\lambda}{2}}^{\infty - \frac{i\lambda}{2}} e^{-ay^2} \, dy \tag{41}$$

It then looks a bit tricky how to resolve this integral with complex bounds! Thankfully complex analysis gives us this answer pretty easily. Consider the following contour $\mathcal{C}$ in Figure 1

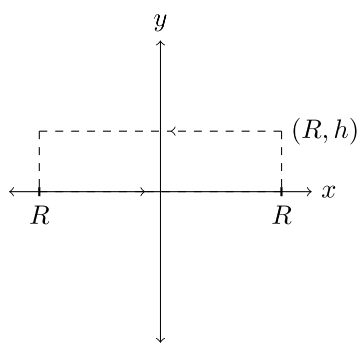

Figure 1: Contour $\mathcal{C}$ in question

As usual, we take the limit as $R \to \infty$. Since the integrand e^{-az^2} is entire over the inside of $\mathcal{C}$, it is clear that the contour integral along $\mathcal{C}$ vanishes (Cauchy-Goursat from 95a). There are then four segments to examine

- We note that the contour integral vanishes on both ends of height h, because $\left|e^{-az^2}\right| = e^{-a(\mathrm{Re}\, z)^2 + a(\mathrm{Im}\, z)^2}$ and since $\mathrm{Im}\, z < h$ while $\mathrm{Re}\, z = R \to \infty$, so by the ML bound ($L = h$) the contributions at the end vanish.

- The horizontal segments are just equal to $\int_{-\infty}^{\infty} e^{-az^2} \, dz - \int_{-\infty+ih}^{\infty+ih} e^{-az^2} \, dz$ and since everything else vanishes the contribution by the horizontal components must also vanish.

Therefore we find that $\int_{-\infty}^{\infty} e^{-az^2} \, dz = \int_{-\infty+ih}^{\infty+ih} e^{-az^2} \, dz$. (Optional: The astute reader might wonder why, since the Gaussian is entire, the integral along the real line doesn't vanish if we try closing the real line by a semicircular arc. This is because the contribution along the arc can't be made to vanish by the ML bound because the maximum value on the arc goes with $e^{a(\mathrm{Im}\, z)^2}$ which is not bounded as $R \to \infty$)

Let's get back to business now. We've shown that we can translate the bounds of our integral by some arbitrary imaginary constant. Thus, we proceed

$$U(\lambda) = \frac{1}{\sqrt{2\pi}} e^{-\frac{\lambda^2}{4a^2}} \int_{-\infty}^{\infty} e^{-ay^2}\, dy \tag{42}$$

Then any table of integrals will tell us that the Gaussian integral above is $\sqrt{\frac{\pi}{a}}$, so we finally obtain

$$U(\lambda) = \frac{1}{\sqrt{2a}} e^{-\frac{\lambda^2}{4a^2}} \tag{43}$$

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