Physics 106b — Classical Mechanics

Michael Cross

California Institute of Technology

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Hamiltonian Chaos I

Hamiltonian Chaos

Trajectories in the two dimensional phase space for a time independent, one degree of freedom system are necessarily simple, since they cannot cross

Thus we will look at:

- 2 coupled nonlinear Hamiltonian oscillators
- a single periodically driven Hamiltonian oscillator

Outline

Today

- Integrable systems and invariant tori
- Double pendulum example (demo)
- Periodically driven rotor (numerical demo)
- Perturbation theory for weak perturbations
  - problem of *small divisors*

Next lecture

- Behavior for large perturbations, chaos etc.

Where are we?

Changing the Hamiltonian from integrable to nonintegrable:

Invariant tori with rational winding numbers break down

- Why? — Look at by perturbation theory
  - For weak enough perturbations tori with sufficiently irrational winding numbers survive
  - Which ones and how many? — KAM theory (hard math, briefly describe results)

- How? — Look at geometrically (next lecture)

Integrable Sytems

An N degree of freedom is *integrable* if there is a canonical transformation to action angle variables $(\mathbf{q}, \mathbf{p}) \to (\boldsymbol{\theta}, \mathbf{I})$ such that the Hamiltonian $H_0 \to H_0(\mathbf{I})$.

The action variables $\mathbf{I} = (I_1, I_2 \ldots I_N)$ are constants of the motion and the angles evolve as $\dot{\boldsymbol{\theta}} = \boldsymbol{\omega}_0(\mathbf{I})$ i.e.

$$\dot{\theta}_j = \omega_{0,j}(I_1, \ldots I_N) = \partial H_0 / \partial I_j$$

This is motion on an N-torus.

Compare

- solvable mechanical system: integrable motion is on a surface of dimension N
- statistical mechanics: system experiences the surface of dimension $2N - 1$ consistent with fixed total energy

Question: What happens to integrable dynamics if a small general perturbation is added to the Hamiltonian?

Perturbation Theory

Zeroth order: integrable

The canonical transformation $(\mathbf{q}, \mathbf{p}) \to (\boldsymbol{\theta}, \mathbf{I})$ is given by the type-II generating function $S_0(\mathbf{I}, \mathbf{q})$ with the Hamilton-Jacobi equation for S_0

$$H_0\left(\frac{\partial S_0}{\partial \mathbf{q}}, \mathbf{q}\right) = H_0(\mathbf{I}) \qquad \text{independent of } \boldsymbol{\theta}$$

The θ_j evolve at constant rates $\omega_{0,j}$ with

$$\boldsymbol{\omega}_0 = \frac{\partial H_0}{\partial \mathbf{I}} \qquad \text{or in component form} \quad \omega_{0,j} = \frac{\partial H_0}{\partial I_j}$$

Express the perturbation in terms of the action angle variables of H_0, so that

$$H(\mathbf{I}, \boldsymbol{\theta}) = H_0(\mathbf{I}) + \varepsilon H_1(\mathbf{I}, \boldsymbol{\theta}) \quad \text{with } \varepsilon \text{ small}$$

Try to solve by a canonical transformation $(\boldsymbol{\theta}, \mathbf{I}) \to (\boldsymbol{\theta}', \mathbf{I}')$ to new action $\mathbf{I}'$ and angle $\boldsymbol{\theta}'$ variables so that the new Hamiltonian H' does not depend on $\boldsymbol{\theta}'$.

The canonical transformation is given by the generating function $S(\mathbf{I}', \boldsymbol{\theta})$ satisfying the Hamilton-Jacobi equation

$$H\left(\frac{\partial S}{\partial \boldsymbol{\theta}}, \boldsymbol{\theta}\right) = H(\mathbf{I}')$$

The generating function is expanded in a power series in ε

$$S(\mathbf{I}', \boldsymbol{\theta}) = \mathbf{I}' \cdot \boldsymbol{\theta} + \varepsilon S_1(\mathbf{I}', \boldsymbol{\theta}) + \cdots \Rightarrow \boldsymbol{I} = \partial S / \partial \boldsymbol{\theta} = \boldsymbol{I}' + \varepsilon \partial S_1 / \partial \boldsymbol{\theta} + \cdots$$

Substitute in and expand to first order in ε

$$\left[H_0(\mathbf{I}') + \varepsilon \boldsymbol{\omega}_0(\mathbf{I}') \cdot \frac{\partial S_1(\mathbf{I}', \boldsymbol{\theta})}{\partial \boldsymbol{\theta}} + \cdots \right] + \varepsilon H_1(\mathbf{I}', \boldsymbol{\theta}) + \cdots = H(\mathbf{I}')$$

$$\varepsilon\left[\boldsymbol{\omega}_0(\mathbf{I})\cdot\frac{\partial S_1(\mathbf{I},\boldsymbol{\theta})}{\partial\boldsymbol{\theta}}+H_1(\mathbf{I},\boldsymbol{\theta})\right]+\cdots=H(\mathbf{I})-H_0(\mathbf{I})$$

Expand the $\boldsymbol{\theta}$ dependence of H_1 and S_1 in Fourier series (absorb constant in H_0)

$$H_1=\sum_{\mathbf{m}\neq\mathbf{0}}H_{1,\mathbf{m}}(\mathbf{I})e^{i\mathbf{m}\cdot\boldsymbol{\theta}}$$

$$S_1=\sum_{\mathbf{m}\neq\mathbf{0}}S_{1,\mathbf{m}}(\mathbf{I})e^{i\mathbf{m}\cdot\boldsymbol{\theta}}$$

where the sum is over vectors of integers $(m_1, m_2, \ldots m_N)$. Substituting gives

$$\varepsilon\sum_{\mathbf{m}\neq\mathbf{0}}\left[i\mathbf{m}\cdot\boldsymbol{\omega}_0(\mathbf{I})S_{1,\mathbf{m}}(\mathbf{I})+H_{1,\mathbf{m}}(\mathbf{I},\boldsymbol{\theta})\right]e^{i\mathbf{m}\cdot\boldsymbol{\theta}}+\cdots=0$$

Since the $e^{i\mathbf{m}\cdot\boldsymbol{\theta}}$ are linearly independent functions of $\boldsymbol{\theta}$ this gives

$$S_{1,\mathbf{m}}(\mathbf{I})=i\,\frac{H_{1,\mathbf{m}}(\mathbf{I},\boldsymbol{\theta})}{\mathbf{m}\cdot\boldsymbol{\omega}_0(\mathbf{I})}$$

$$S(\mathbf{I},\boldsymbol{\theta}) = \mathbf{I}\cdot\boldsymbol{\theta} + i\varepsilon \sum_{\mathbf{m}\neq\mathbf{0}} \frac{H_{1,\mathbf{m}}(\mathbf{I},\boldsymbol{\theta})}{\mathbf{m}\cdot\boldsymbol{\omega}_0(\mathbf{I})} e^{i\mathbf{m}\cdot\boldsymbol{\theta}} + \cdots$$

Clearly there is a problem if any denominator in the sum is zero

$$\mathbf{m}\cdot\boldsymbol{\omega}_0(\mathbf{I}) = 0 \qquad \text{for any } \mathbf{m}$$

i.e. for those values of the action for which there is a rational relationship between the unperturbed frequencies (rational winding number for $N = 2$)

$$m_1\omega_{0,1} + m_2\omega_{0,2} + \cdots + m_N\omega_{0,N} = 0$$

with m_j integers

Perturbation theory diverges for the tori with a rational relationship between the frequencies: these will be destroyed by an arbitrarily small perturbation.

$$S(\mathbf{I},\boldsymbol{\theta}) = \mathbf{I}\cdot\boldsymbol{\theta} + i\varepsilon \sum_{\mathbf{m}\neq\mathbf{0}} \frac{H_{1,\mathbf{m}}(\mathbf{I},\boldsymbol{\theta})}{\mathbf{m}\cdot\boldsymbol{\omega}_0(\mathbf{I})} e^{i\mathbf{m}\cdot\boldsymbol{\theta}} + \cdots$$

- There will also be problems if the denominator becomes too small, i.e. if the frequencies are such that the rational relationship is almost satisfied.
- Increasing $|\mathbf{m}|$ gives an increasing flexibility in choosing the m_j so that $\mathbf{m}\cdot\boldsymbol{\omega}_0(\mathbf{I})$ is close to zero.
- On the other hand for increasing $|\mathbf{m}|$ the numerator $H_{1,\mathbf{m}}(\mathbf{I},\boldsymbol{\theta})$ becomes smaller (the size of very high frequency Fourier components decreases for a smooth function).
- Also if the first term in the expansion is becoming dangerously large, we must also look at higher order terms and check them too.
- May be able to resum ∞ number of terms to give "renormalized" quantities e.g. $\boldsymbol{\omega}_0(\mathbf{I}) \to \boldsymbol{\omega}(\mathbf{I})$

This leads to a hard mathematical problem — the problem of small divisors — corresponding to the difficulty of treating resonances in Hamiltonian systems.

KAM Theory

Kolmogorov (1954), Arnold (1963), and Mosur (1967)

Version for $N = 2$ (due to Mosur, cf. Hand and Finch):

The tori of H_0 survive a small enough smooth perturbation if the winding number Ω is sufficiently irrational, i.e. if for any integers r, s

$$\left|\Omega - \frac{r}{s}\right| > \frac{C(\varepsilon)}{s^{2.5}} \qquad \text{with } C(\varepsilon) \to 0 \text{ as } \varepsilon \to 0\,.$$

Systematic rational approximation to irrational

$$\text{Continued fraction:} \quad \Omega = a_0 + \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} \equiv (a_0, a_1, a_2, \ldots)$$

$$\text{e.g.} \quad \pi \equiv (3, 7, 15, 1, 292, 1, 1, 1 \ldots) \Rightarrow \frac{3}{1}, \ \frac{22}{7}, \ \frac{333}{106}, \ \frac{355}{113}, \ \frac{103993}{33102}, \ldots$$

KAM Theory

Thus we end up with an intricate picture:

- Tori with winding numbers over a range $\propto s^{-2.5}$ about every rational r/s are destroyed by the perturbation.
- Although there are an infinite number of rationals in the unit interval , the sum of all these ranges is finite, and goes to zero as $C(\varepsilon) \to 0$. This is because there are of order s rationals in the unit interval with denominator s (r runs from 1 to $s-1$ but some have already been counted, e.g. $2/4 \equiv 1/2$) and the sum

$$\sum_{s=1}^{\infty} s \frac{1}{s^{2.5}}$$

  converges.
- Tori with winding numbers outside these windows survive the perturbation.
- An infinite number of tori are destroyed, but most survive!