ACM 100b

An example of the Sturm-Liouville problems - the Bessel equation

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Another example of the S-L ODE - the Bessel equation

- We will examine a Sturm-Liouville ODE that does not have simple solutions like sines and cosines
- This is the Bessel equation
- It comes up when we solve the heat equation in cylindrical coordinates as we will show later
- The Bessel ODE is given by

$$\frac{d^2y(x)}{dx^2} + \frac{1}{x}\frac{dy(x)}{dx} + \left(\lambda^2 - \frac{m^2}{x^2}\right) y(x) = 0 \qquad x_1 \leq x \leq x_2.$$

- Note as written it's not in the typical S-L form which is

$$-\frac{d}{dx}\left(p(x)\frac{d}{dx}y(x)\right) + q(x)y(x) = \lambda r(x)y(x)$$

The Bessel equation in S-L form

- But we can rewrite the ODE so that it is in S-L form
- Multiply each term of the ODE

$$\frac{d^2y(x)}{dx^2} + \frac{1}{x}\frac{dy(x)}{dx} + \left(\lambda^2 - \frac{m^2}{x^2}\right)y(x) = 0 \qquad x_1 \leq x \leq x_2.$$

by x:

$$x\frac{d^2y(x)}{dx^2} + \frac{dy(x)}{dx} + \left(\lambda^2 x - \frac{m^2}{x}\right)y(x) = 0 \qquad x_1 \leq x \leq x_2.$$

- We can write this in the form

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) - \frac{m^2}{x}y(x) + \lambda^2 xy(x) = 0$$

- Now we see this is in S-L form with

$$p(x) = x \quad q(x) = \frac{m^2}{x} \quad r(x) = x$$

- Note we replaced λ by λ^2 but this is just for convenience

Boundary conditions for the Bessel equation

- Because

$$p(x) = x \quad q(x) = \frac{m^2}{x} \quad r(x) = x$$

  we see that $p(x) > 0$ and $r(x) > 0$ *except* where $x = 0$ which is a singular point
- So for now we'll take our domain $x_1 \leq x \leq x_2$ to be $1 \leq x \leq 2$
- This is safely away from $x = 0$ so the ODE is nonsingular
- For boundary conditions we'll take

$$y(1) = 0 \quad y(2) = 0.$$

  which are of separable type.
- This problem is now a regular S-L ODE problem
- We'll take $m = 0$ for simplicity

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \lambda^2 xy(x) = 0. \qquad 1 \leq x \leq 2 \quad y(1) = 0 \quad y(2) = 0$$

The solutions of the Bessel equation

- The solutions of this ODE are new functions called *Bessel functions*
- They are not simple functions likes sines and cosines but we will see shortly that they share some important properties with sines and cosines.
- The ODE is second order so has two linearly independent solutions
- It has a regular singular point at $x = 0$.
- The Frobenius theory tells us there is one singular solution that blows up at $x = 0$ and one regular solution that does not blow up at the origin
- The regular solution is labeled $J_0(x)$
- The singular solution is labeled $Y_0(x)$
- Note that for our purposes since our domain is $1 \leq x \leq 2$ both of these linearly independent solutions are smooth over this interval.

Solution of the Bessel S-L problem

- For our purposes all we need to know is that J_0 and Y_0 are linearly independent
- The general solution to

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + \lambda^2 xy(x) = 0.$$

is

$$y(x) = c_1 J_0(\lambda x) + c_2 Y_0(\lambda x).$$

- Now in order to satisfy the boundary conditions we must have

$$C_1 J_0(\lambda x_1) + C_2 Y_0(\lambda x_1) = 0,$$
$$C_1 J_0(\lambda x_2) + C_2 Y_0(\lambda x_2) = 0.$$

or using that $x_1 = 1$ and $x_2 = 2$ we have

$$C_1 J_0(\lambda) + C_2 Y_0(\lambda) = 0,$$
$$C_1 J_0(2\lambda) + C_2 Y_0(2\lambda) = 0.$$

Homogeneous solutions of the Bessel S-L problem

- Note this system of equations is homogeneous
- So in general there is only the trivial solution $C_1 = C_2 = 0$
- But if for some special values of λ we could get the determinant

$$\begin{vmatrix} J_0(\lambda) & Y_0(\lambda) \\ J_0(2\lambda) & Y_0(2\lambda) \end{vmatrix}$$

  to vanish, we could get nontrivial special solutions.
- But because the system is homogeneous they would not be unique.
- So the question is - does this determinant

$$\Delta(\lambda) = \begin{vmatrix} J_0(\lambda) & Y_0(\lambda) \\ J_0(2\lambda) & Y_0(2\lambda) \end{vmatrix} = J_0(\lambda)Y_0(2\lambda) - Y_0(\lambda)J_0(2\lambda)$$

  vanish for some values of λ?

The determinant for our S-L problem

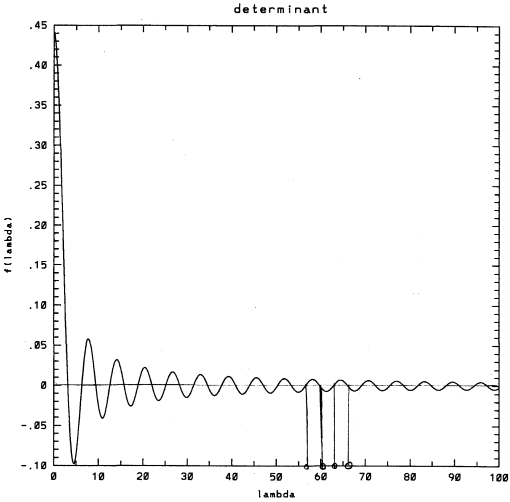
- It is not easy to see where this determinant

$$\Delta(\lambda) = J_0(\lambda) Y_0(2\lambda) - Y_0(\lambda) J_0(2\lambda)$$

  might vanish
- The Bessel functions are not simple things like sines and cosines.
- So we resort to computing it numerically and evaluating the determinant for various values of λ.

Plotting the determinant

- The determinant $\Delta(\lambda)$ is plotted below.



Locations of the roots of the determinant

- As can be seen, the determinant oscillates and crosses zero at various values of λ.
- At these values of λ, we expect we can get nontrivial solutions that satisfy the boundary conditions.
- While these are not simple values, like $\lambda = n\pi, n = 1, 2, 3 \ldots$, there is some structure that can be seen where the determinant vanishes.
- Here are the first 20 zeros and the distance between them:

1	3.1230309195957	0.
2	6.2734357139922	3.1504047943965
3	9.4182075422516	3.1447718282594
4	12.561423185525	3.1432156432738
5	15.703997892744	3.1425747072187
6	18.846248038288	3.1422501455443
7	21.988311475482	3.1420634371932
8	25.130257756407	3.1419462809251
9	28.272125734030	3.1418679776231
10	31.413938804238	3.1418130702080
11	34.555711892472	3.1417730882340
12	37.697454966930	3.1417430744580
13	40.839174937927	3.1417199709975
14	43.980876746472	3.1417018085451
15	47.122564018884	3.1416872724114
16	50.264239476645	3.1416754577608
17	53.405905201971	3.1416657253262
18	56.547562815056	3.1416576130850
19	59.689213595454	3.1416507803986
20	62.830858567117	3.1416449716621

The zeros take on a simple pattern as $n \to \infty$

- If we calculate the crossings of adjacent zeroes and see how far apart they are, we can see that the spacing of adjacent zeroes approaches a constant.
- That constant seems to be getting close to π.
- In fact the numbers themselves seem to be approaching $\lambda = n\pi$ for n large
- This is similar to the values of λ we calculated when we solved the heat equation.
- It will turn out this is not an accident.

The solutions look sort of like sines

- Additional similarities with sines and cosines can be seen when we look at the actual solutions.
- When the determinant crosses zero we can get a solution (unique up to some multiplicative constant).
- The values of $y(x)$ are plotted in the figures below for increasing values of λ.

The solution for the lowest value of λ

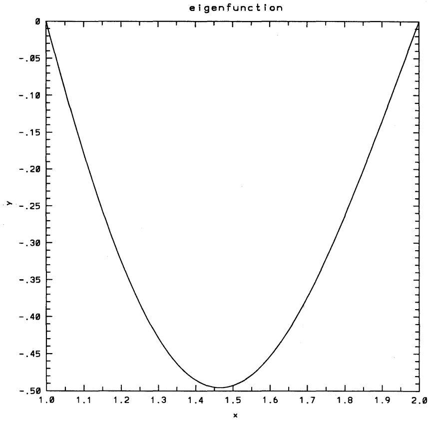


Figure : The solution corresponding to the first value of λ for which the determinant vanishes

The solution for the 6'th root

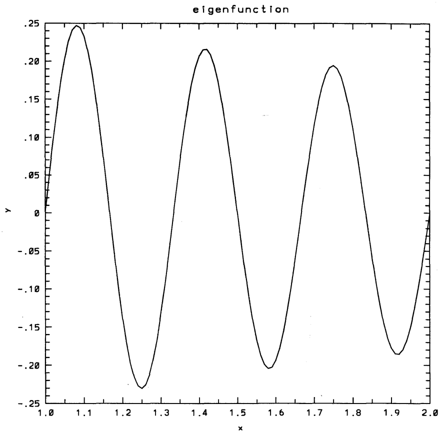


Figure : The solution corresponding to the sixth value of λ for which the determinant vanishes

The solution for the 11'th root

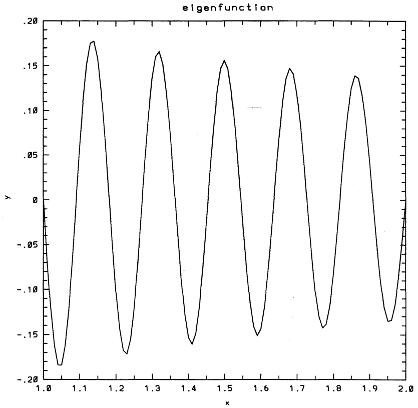


Figure : The solution corresponding to the 11'th value of λ for which the determinant vanishes

The solution for the 16'th root

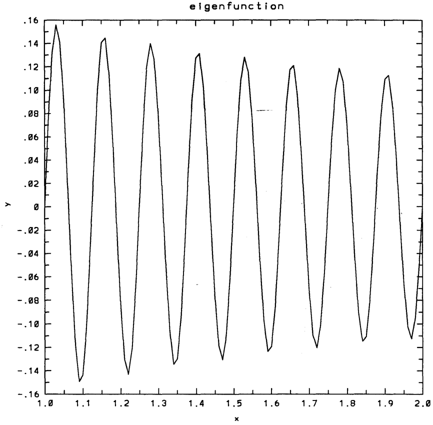


Figure : The solution corresponding to the 16'th value of λ for which the determinant vanishes

Overview of the solutions

- We can see that these solutions look increasingly sinusoidal
- Later in a quantitative sense which we will make precise, they do get increasingly close to $\sin m\pi x$ (times an envelope function which modulates the amplitude)
- Here m describes the $m+1$'th value of λ for which the determinant vanishes.
- This too is not an accident
- It's a general feature of the S-L problem which we will demonstrate.
- In order to see why this ODE is so special, we first have to derive an important identity which allows us to understand the properties of the solutions of this boundary value problem.