ACM 100b

Green's functions - an introduction

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Recap

- In the previous lecture we considered the Sturm-Liouville ODE with an inhomogeneous term

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda r(x)y = r(x)f(x), \qquad a < x < b,$$

but with homogeneous boundary conditions

$$y(a) = y(b) = 0$$

- We showed this could be solved by using an expansion in terms of eigenfunctions of the associated Sturm-Liouville eigenvalue problem

$$\frac{d}{dx}\left(p(x)\frac{d\phi_n}{dx}\right) - q(x)\phi_n(x) + \lambda_n r(x)\phi_n(x) = 0, \qquad a < x < b$$

with boundary conditions $\phi_n(a) = 0 \quad \phi_n(b) = 0$

Recap

- We showed that the solution was

$$y(x) = \sum_{n=0}^{\infty} A_n \phi_n(x) \text{ with } A_n = \frac{f_n}{\lambda - \lambda_n}$$

and

$$f_n = \frac{\int_a^b r(x) f(x) \phi_n(x) dx}{\int_a^b r(x) \phi_n^2(x) dx}$$

- In this lecture we will investigate some further aspects of the structure of this solution

An integral representation of the solution

- Recall when we looked at convergence of Fourier series we “summed” the series to do the analysis.
- We can use the same idea here.
- First, let’s simplify things by redefining the normalization of the eigenfunctions so that they satisfy

$$\int_a^b r(x)\phi_n^2(x)dx = 1.$$

- In that case, we can rewrite the expression for the coefficients f_n:

$$f_n = \int_a^b r(x)f(x)\phi_n(x)dx,$$

- And we can rewrite the solution $y(x)$ as follows:

$$y(x) = \sum_{n=0}^{\infty} \frac{1}{\lambda - \lambda_n} \left[\int_a^b r(x')f(x')\phi_n(x')dx' \right] \phi_n(x).$$

An integral representation of the solution

- Rearranging the sum and integral we have

$$y(x) = \int_a^b f(x')r(x') \sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n(x')}{\lambda - \lambda_n} dx'.$$

- It's OK to do this interchange because we can assume the sum is uniformly convergent.
- Formally we can write

$$y(x) = \int_a^b f(x')r(x')G(x, x'; \lambda)dx',$$

where

$$G(x, x'; \lambda) = \sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n(x')}{\lambda - \lambda_n}.$$

An integral representation of the solution

- This expression

$$y(x) = \int_a^b f(x')r(x')G(x, x'; \lambda)dx',$$

where

$$G(x, x'; \lambda) = \sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n(x')}{\lambda - \lambda_n}.$$

is useful.

- It says that as long as λ is not an eigenvalue we only need to compute this function $G(x, x'; \lambda)$ once we are given the eigenfunctions.
- With this function in hand we can solve all inhomogeneous ODE's of the form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda r(x)y = r(x)f(x), \qquad a < x < b,$$

An integral representation of the solution

- The function $G(x, x'; \lambda)$

$$G(x, x'; \lambda) = \sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n(x')}{\lambda - \lambda_n}.$$

  is called a *Greens function* for the ODE.
- Note that it has nothing to do with the right hand side $r(x)f(x)$
- It's intrinsic only to the S-L ODE
- On the other hand recall that the functions $\phi_n(x)$ are all normalized and it turns out $\phi_n(x)$ varies in magnitude but as we have seen just oscillates like a sine function for all n
- So the terms decrease only like $1/n^2$ because $\lambda_n \sim n^2$
- What kind of function is this?
- We can tell that because of the decay in the coefficients it's a function that is continuous but has some problem with its first derivative.

An alternate approach

- To see what this function $G(x, x'; \lambda)$ is we approach the problem from another viewpoint.
- Let's consider a simpler problem

$$y'' + \lambda y = f(x) \qquad 0 \leq x \leq \pi \quad y(0) = 0 \quad y(\pi) = 0$$

- We solved this before using eigenfunctions but there is another approach we could use
- Recall if you know the homogeneous solutions of a linear ODE then you can use the method of *variation of parameters* to solve the inhomogeneous problem.

Variation of parameters

- Suppose you have a general second order homogeneous ODE

$$y'' + P(x)y' + Q(x)y = 0$$

- And suppose you know the homogeneous solutions $y_1(x)$ and $y_2(x)$
- Then you can construct a particular solution y_{part} to the inhomogeneous ODE

$$y'' + P(x)y' + Q(x)y = g(x)$$

by writing

$$y_{part}(x) = f_1(x)y_1(x) + f_2(x)y_2(x)$$

- Here $f_1(x)$ and $f_2(x)$ are functions which must be determined.

Variation of parameters

- To compute f_1 and f_2 we substitute

$$y_{part}(x) = f_1(x)y_1(x) + f_2(x)y_2(x)$$

into the ODE to get

$$\begin{aligned}&\left[f_1''y_1 + 2f_1'y_1' + f_1y_1'' + f_2''y_2 + 2f_2'y_2' + f_2y_2''\right]\\&+P(x)\left[f_1'y_1 + f_1y_1' + f_2'y_2 + f_2y_2'\right]\\&+Q(x)\left[f_1(x)y_1(x) + f_2(x)y_2(x)\right] = g(x)\end{aligned}$$

- Because y_1 and y_2 are homogeneous solutions this simplifies to

$$\left[f_1''y_1 + 2f_1'y_1' + f_2''y_2 + 2f_2'y_2'\right] + P(x)\left[f_1'y_1 + f_2'y_2\right] = g(x)$$

Variation of parameters

- Now we have one equation for two unknown functions
- We can simplify things even further by asking that

$$f_1' y_1 + f_2' y_2 = 0$$

- This just puts one constraint which turns out to be very convenient
- If we differentiate this constraint we find

$$f_1'' y_1 + f_2'' y_2 = -f_1' y_1' - f_2' y_2'$$

- So our equation

$$\left[f_1'' y_1 + 2f_1' y_1' + f_2'' y_2 + 2f_2' y_2'\right] + P(x)\left[f_1' y_1 + f_2' y_2\right] = g(x)$$

becomes

$$f_1' y_1' + f_2' y_2' = g(x)$$

Variation of parameters

- So we see we can compute f_1 and f_2 if we can solve the equations

$$
\begin{aligned}
f_1' y_1 + f_2' y_2 &= 0 \\
f_1' y_1' + f_2' y_2' &= g(x)
\end{aligned}
$$

- This a 2×2 linear system for f_1' and f_2':

$$
\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} f_1' \\ f_2' \end{pmatrix} = \begin{pmatrix} 0 \\ g(x) \end{pmatrix}
$$

- We recognize the determinant of the matrix as the Wronskian $W(x)$ of the system which for the ODE can always be determined and never vanishes as long as the coefficient functions are smooth.

Variation of parameters

- The solution is

$$f_1' = \frac{-y_2 g(x)}{W(x)} \qquad f_2' = \frac{y_1 g(x)}{W(x)}$$

  where

$$W(x) = y_1(x)y_2'(x) - y_2(x)y_1'(x)$$

- And the inhomogeneous solution is

$$y_{part}(x) = y_2(x)\int^x \frac{y_1(x')g(x')}{W(x')}dx' - y_1(x)\int^x \frac{y_2(x')g(x')}{W(x')}dx'$$

- We have written this using indefinite integration
- The full solution is

$$\begin{aligned} y(x) =& c_1 y_1(x) + c_2 y_2(x) + \\ & y_2(x)\int^x \frac{y_1(x')g(x')}{W(x')}dx' - y_1(x)\int^x \frac{y_2(x')g(x')}{W(x')}dx' \end{aligned}$$

  and c_1 and c_2 are determined from the boundary conditions

Application to the Sturm-Liouville ODE

- Now let's apply these ideas to

$$y'' + \lambda y = f(x)$$

- In this case the Wronskian of this ODE is just a nonzero constant
- And we know the homogeneous solutions to this ODE:

$$y_1(x) = \sin(\sqrt{\lambda}x) \qquad y_2(x) = \cos(\sqrt{\lambda}x)$$

- The Wronskian here is $W = -\sqrt{\lambda}$
- So we can write an inhomogeneous solution to our problem as follows:

$$\begin{aligned} y_{part} =& \cos(\sqrt{\lambda}x) \int^x \frac{\sin(\sqrt{\lambda}x')}{W} g(x')dx' \\ &- \sin(\sqrt{\lambda}x) \int^x \frac{\cos(\sqrt{\lambda}x')}{W} g(x')dx \end{aligned}$$

Application to the Sturm-Liouville problem

- In what follows it's a little easier to take a different set of homogeneous solutions:

$$y_1(x) = \sin(\sqrt{\lambda}x) \qquad y_2(x) = \sin(\sqrt{\lambda}(\pi - x))$$

- Note one satisfies the boundary condition at $x = 0$ and the other satisfies the boundary condition at $x = \pi$
- The Wronskian of these solutions is

$$W = -\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)$$

- So our general solution is

$$\begin{aligned} y(x) =& C_1 \sin(\sqrt{\lambda}x) + C_2 \sin(\sqrt{\lambda}(\pi - x)) \\ &+ \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)} \int_0^x \sin(\sqrt{\lambda}(\pi - x'))g(x')dx' \\ &- \frac{\sin(\sqrt{\lambda}(\pi - x))}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)} \int_0^x \sin(\sqrt{\lambda}x')g(x')dx' \end{aligned}$$

Application to the Sturm-Liouville problem

- Note that in the previous expression we are now using definite integration with the lower limit at $x = 0$
- Now applying the boundary conditions we get

$$C_2 = 0$$

$$C_1 = -\frac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)}\int_0^{\pi}\sin(\sqrt{\lambda}(\pi - x'))g(x')dx'$$

- Using these constants the solution can be written in the form

$$y(x) = \sin(\sqrt{\lambda}(x - \pi))\int_0^{x}\frac{\sin(\sqrt{\lambda}x')}{\sqrt{\lambda}\sin(\sqrt{\lambda})}g(x')dx' + \sin(\sqrt{\lambda}x)\int_x^{\pi}\frac{\sin(\sqrt{\lambda}(x' - \pi))}{\sqrt{\lambda}\sin(\sqrt{\lambda})}g(x')dx'$$

- Note the manipulation of the limits of the integrals

Application to the Sturm-Liouville problem

- This can be compactly written in the following form:

$$y(x) = \int_0^{\pi} H(x, x'; \lambda) g(x') dx'$$

where

$$H(x, x'; \lambda) = \begin{cases} \dfrac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)} \sin(\sqrt{\lambda}x) \sin(\sqrt{\lambda}(x' - \pi)) & x < x' \\ \dfrac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)} \sin(\sqrt{\lambda}x') \sin(\sqrt{\lambda}(x - \pi)) & x > x' \end{cases}$$

The Greens function again

- Now recall we produced earlier a formal solution to the problem

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda r(x)y = r(x)f(x), \qquad a < x < b,$$

but with homogeneous boundary conditions

$$y(a) = y(b) = 0$$

- Suppose $\phi_n(x)$ are the normalized eigenfunctions satisfying

$$\frac{d}{dx}\left(p(x)\frac{d\phi_n(x)}{dx}\right) - q(x)\phi_n(x) + \lambda_n r(x)\phi_n(x) = 0, \qquad a < x < b,$$

with

$$\int_a^b \phi_n^2 dx = 1$$

The Greens function again

- Then we showed the solution to the problem

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda r(x)y = r(x)f(x), \qquad a < x < b,$$

is

$$y(x) = \int_a^b f(x')r(x')G(x,x';\lambda)dx',$$

where

$$G(x,x';\lambda) = \sum_{n=0}^{\infty} \frac{\phi_n(x)\phi_n(x')}{\lambda - \lambda_n}.$$

The Greens function again

- Now apply this to the problem

$$y'' - \lambda y = g(x) \qquad 0 \le x \le \pi \quad y(0) = 0 \quad y(\pi) = 0$$

- Here the normalized eigenfunctions are

$$\phi_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$$

- So we have for this problem

$$y(x) = \int_0^{\pi} G(x, x'; \lambda) g(x') dx',$$

where

$$G(x, x'; \lambda) = \frac{2}{\pi} \sum_{n=0}^{\infty} \frac{\sin(nx)\sin(nx'))}{\lambda - n^2}.$$

The Greens function again

- But we just solved the same problem a different way and showed

$$y(x) = \int_0^{\pi} H(x, x'; \lambda) g(x') dx'$$

where

$$H(x, x'; \lambda) = \begin{cases} \dfrac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)} \sin(\sqrt{\lambda}x)\sin(\sqrt{\lambda}(x' - \pi)) & x < x' \\ \dfrac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)} \sin(\sqrt{\lambda}x')\sin(\sqrt{\lambda}(x - \pi)) & x < x' \end{cases}$$

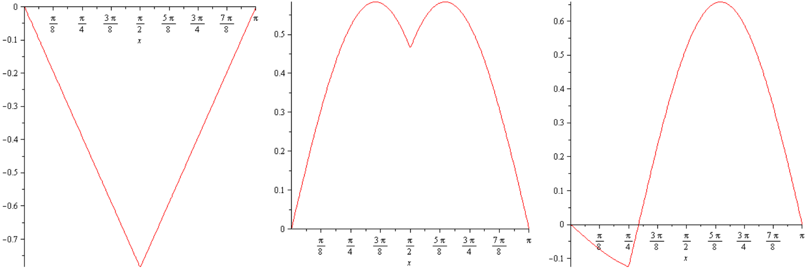
- So it must be that

$$\frac{2}{\pi}\sum_{n=0}^{\infty} \frac{\sin(nx)\sin(nx'))}{\lambda - n^2} =$$

$$\begin{cases} \dfrac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)} \sin(\sqrt{\lambda}x)\sin(\sqrt{\lambda}(x' - \pi)) & x < x' \\ \dfrac{1}{\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)} \sin(\sqrt{\lambda}x')\sin(\sqrt{\lambda}(x - \pi)) & x < x' \end{cases}$$

Properties of the Greens function

- We can use the summed version to get some feel for what the Greens function looks like and some of its properties
- First, the Greens function has a discontinuous derivative whenever $x = x'$
- This is shown below where we plot G for three cases



- Shown are $(\lambda = 0, x' = \pi/2)$,$(\lambda = 2, x' = \pi/2)$,$(\lambda = 2, x' = \pi/4)$,

Properties of the Greens' function

- Note also the Greens function obeys the homogeneous boundary conditions
- If you compute the derivative as you approach $x = x'$ from the right and then the left the difference between the two limits is always the same: 1.
- For $x < x'$ if you substitute the Greens function into the ODE you get

$$\frac{d^2G}{dx^2} + \lambda G = 0 \qquad x < x'$$

- Similarly, for $x > x'$ if you substitute the Greens function into the ODE you get

$$\frac{d^2G}{dx^2} + \lambda G = 0 \qquad x > x'$$

- However at $x = x'$ the second derivative of the Green's function is undefined because the first derivatives are discontinuous and so there is no limit when you try to compute the second derivative.