ACM 100b

Fundamental matrices

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Fundamental matrices

- Once we know the $\boldsymbol{x}_i$, with $i = 1, \ldots, n$ we know that the general solution is given by

$$\boldsymbol{x} = \sum_{i=1}^{n} c_i \boldsymbol{x}_i$$

- In order to solve the initial value problem $\boldsymbol{x}(z_0) = \boldsymbol{x}_0$ we would have to solve the linear system

$$\begin{bmatrix} \vdots & \vdots & & \vdots \\ \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \boldsymbol{x}_0$$

- Once we do this we have a solution that satisfies the initial condition.
- We can also solve the problem by developing a fundamental matrix which we describe next.

Fundamental matrices

- Suppose instead of solving the linear system

$$\begin{bmatrix} \vdots & \vdots & & \vdots \\ \boldsymbol{x}_1 & \boldsymbol{x}_2 & \cdots & \boldsymbol{x}_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \boldsymbol{x}_0$$

  we solve for a linear superposition that satisfies the following initial condition

$$\sum_{i=1}^{n} g_{i1} \boldsymbol{x}_i(z_0) = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

- Let's call this superposition ϕ_1:

$$\phi_1(z) = \sum_{i=1}^{n} g_{i1} \boldsymbol{x}_i(z)$$

Fundamental matrices

- Then we solve for another linear superposition

$$\sum_{i=1}^{n} g_{i2} \boldsymbol{x}_i = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

- Call this superposition ϕ_2:

$$\phi_2(z_0) = \sum_{i=1}^{n} g_{i12} \boldsymbol{x}_i(z)$$

- We can do this up to n times by placing a 1 in the right hand side and zero everywhere else and form the vectors $\phi_3 \ldots \phi_n$

Fundamental matrices

- We can construct a matrix $\Phi(z)$ using the ϕ_i as columns defined by

$$\Phi(z) = \begin{bmatrix} \vdots & \vdots & & \vdots \\ \phi_1 & \phi_2 & \cdots & \phi_n \\ \vdots & \vdots & & \vdots \end{bmatrix}$$

- Note that this matrix satisfies

$$\Phi' = A\Phi$$

because every column of the matrix is a solution of the ODE.
- But also we see that

$$\Phi(z_0) = I \qquad \text{where } i \text{ is the } n \times n \text{ identity matrix}$$

- This matrix $\Phi(z)$ is called a *fundamental matrix* for the system

Using the fundamental matrix

- A lot of work went into getting the fundamental matrix.
- Why bother with this?
- Because we can now solve for any initial condition.
- The solution to

  $$\boldsymbol{x}' = A\boldsymbol{x} \quad \boldsymbol{x}(z_0) = \boldsymbol{x}_0$$

  is

  $$\boldsymbol{x} = \Phi \boldsymbol{x}_0.$$

- Note that the matrix Φ is totally independent of the initial condition $\boldsymbol{x}_0$.
- Once you solve for it you can solve for any initial condition.
- This technique will be used repeatedly in the case of linear systems in this course.
- The idea is known generally as the method of *Green's functions.*