ACM 100b

Solving Sturm-Liouville boundary value problems with Sturm-Liouville eigenfunctions

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Solving boundary value problems with S-L eigenfunctions

- So far we have focused a lot on the S-L ODE and its eigenfunctions
- While these have some interesting properties we would like to see some applications
- We will apply S-L eigenfunctions to solving boundary value problems for linear ODE's
- We will do this in steps:
  - Using S-L eigenfunctions to solve inhomogeneous S-L ODE problems but with homogeneous boundary conditions
  - Using S-L eigenfunctions to solve inhomogeneous S-L ODE problems but with inhomogeneous boundary conditions
  - Using S-L eigenfunctions to solve general linear ODE boundary value problems

Solving inhomogeneous S-L problems with S-L eigenfunctions

- Consider the Sturm-Liouville ODE with homogeneous boundary conditions but this time with an inhomogeneous term:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) - q(x)y(x) + \lambda r(x)y = r(x)f(x), \qquad a < x < b,$$

- We will assume for now that we have homogeneous boundary conditions $y(a) = y(b) = 0$.
- For now assume that λ is just a general scalar and not an eigenvalue.
- We assume $p(x), r(x) > 0$
- The $r(x)$ function is sitting on the right hand side for convenience only

Solving inhomogeneous S-L problems

- How might we solve this ODE boundary value problem?
- We note that this ODE has special solutions $\phi_n(x)$ that satisfy

$$\frac{d}{dx}\left(p(x)\frac{d\phi_n}{dx}\right) - q(x)\phi_n + \lambda_n r(x)\phi_n = 0, \qquad a < x < b,$$

  that satisfy the homogeneous boundary conditions
- In addition we just showed that such sets of functions were *complete* in that any square integrable function could be expanded in terms of these functions
- So it makes sense to think about writing the solution as

$$y(x) = \sum_{n=1}^{\infty} A_n \phi_n(x)$$

  and see if we can compute A_n
- Of course once we get the A_n we better make sure the series converges.

Solving inhomogeneous S-L problems

- If we substitute the series of S-L ODE's into the boundary value problem we get

$$\frac{d}{dx}\left(p(x)\frac{d}{dx}\sum_{n=1}^{\infty}A_n\phi_n(x)\right) -$$
$$q(x)\sum_{n=1}^{\infty}A_n\phi_n(x) +$$
$$\lambda r(x)\sum_{n=1}^{\infty}A_n\phi_n(x) = r(x)f(x), \qquad a < x < b,$$

- Now we just went though a big song and dance about how one must be careful differentiating Fourier series term by term
- The sum $\sum_{n=1}^{\infty} A_n\phi_n(x)$ is just a glorified Fourier series so is it OK to differentiate the sum term by term (twice no less)?

Solving inhomogeneous S-L problems

- The answer we'll see is yes but for now let's just assume it is OK, solve the problem and see if we get some contradiction.
- Recall that the functions ϕ_n are eigenfunctions
- So the result of all the terms in

$$\frac{d}{dx}\left(p(x)\frac{d}{dx}\sum_{n=1}^{\infty}A_n\phi_n(x)\right) -$$

$$q(x)\sum_{n=1}^{\infty}A_n\phi_n(x)+$$

$$\lambda r(x)\sum_{n=1}^{\infty}A_n\phi_n(x) = r(x)f(x), \qquad a < x < b,$$

is just

$$\sum_{n=1}^{\infty}(\lambda - \lambda_n)r(x)A_n\phi_n(x) = r(x)f(x)$$

Solving inhomogeneous S-L problems

- Now to solve this problem

$$\sum_{n=1}^{\infty}(\lambda - \lambda_n)r(x)A_n\phi_n(x) = r(x)f(x)$$

  use orthogonality.

- Since the eigenfunctions are complete we can expand $f(x)$ as well:

$$f(x) = \sum_{n=1}^{\infty} f_n\phi_n(x)$$

  where

$$f_n = \frac{\int_a^b r(x)f(x)\phi_n(x)dx}{\int_a^b r(x)\phi_n^2(x)dx}$$

Solving inhomogeneous S-L problems

- So we see by orthogonality that

$$(\lambda - \lambda_n)A_n = f_n(x) \qquad n = 0, 1, 2, \cdots ,$$

- And so

$$A_n = \frac{f_n}{\lambda - \lambda_n}$$

- And the solution is

$$y(x) = \sum_{n=0}^{\infty} \frac{f_n}{\lambda - \lambda_n} \phi_n(x)$$

- Note that as long as λ is not an eigenvalue everything is actually fine.
- The series for $y(x)$ actually converges faster than the series for $f(x)$ because we know that $\lambda_n \to \infty$ as $n \to \infty$

Solving inhomogeneous S-L problems

- In fact we know the λ_n increase typically like n^2
- This is because eventually the eigenfunctions eventually resemble sine functions and the eigenvalues become proportional to the eigenvalues for the ODE

$$\frac{d^2}{dx^2}y + \lambda y = 0.$$

- So as long as $\lambda \neq \lambda_n$ the series is uniformly convergent
- In fact it's sufficiently convergent that it was OK to differentiate it twice because the terms vanish *faster than* $1/n^2$.
- So our solution makes sense.

The Riemann-Lebesgue lemma again

- One question that arises in the previous argument is how do you know the Fourier series coefficients for $f(x)$ don't do something crazy.
- If we had some function that had Fourier series coefficients f_n that *grow* with increasing n then we could have a problem in justifying the convergence of the series

$$y(x) = \sum_{n=0}^{\infty} \frac{f_n}{\lambda - \lambda_n} \phi_n(x)$$

- Recall that for any absolutely integrable function there is a result called the *Riemann-Lebesgue lemma* that says

$$\lim_{n \to \infty} \int_a^b f(x) \phi_n(x) dx \to 0$$

- In other words the coefficients decrease to zero as $n \to \infty$
- The lemma won't tell you how fast this happens - just that it does.

The Fredholm alternative

- Suppose we were unlucky enough to ask for a solution when $\lambda = \lambda_m$, one of the eigenvalues?
- Then we see that in general there is no solution because the m'th term in the series

$$y(x) = \sum_{n=0}^{\infty} \frac{f_n}{\lambda - \lambda_n} \phi_n(x)$$

  will blow up if $\lambda = \lambda_m$
- But...if we were lucky enough to have $f_m = 0$ then the function $f(x)$ has none of the offending eigenfunction and the series solution can be fixed:

$$y(x) = \sum_{\substack{n=0 \\ n \neq m}}^{\infty} \frac{f_n}{\lambda - \lambda_n} \phi_n(x)$$

- But in this case the solution is not unique.

The Fredholm alternative

- You can always add an arbitrary amount of $\phi_m(x)$.
- So we have

$$y(x) = \sum_{\substack{n=0 \\ n\neq m}}^{\infty} \frac{f_n}{\lambda - \lambda_n}\phi_n(x) + C\phi_m(x)$$

  where C is arbitrary.
- This is called the *Fredholm alternative*
- If $\lambda = \lambda_m$ then one of two things happens
  - For most inhomogeneous terms $f(x)$ there is no solution
  - But for some special $f(x)$ that have no component of the offending eigenfunction $\phi_m(x)$ there is a solution but it is not unique.