ACM 100b

Differentiation and integration of Fourier series

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Differentiation of Fourier series

- In many applications we want to differentiate and integrate a function expressed as a Fourier series.
- However, given that we just saw that Fourier series can sometimes converge in a nonuniform manner
- So we have to be concerned that taking the limits associated with a derivative and taking the number of terms in the series to $N \to \infty$ may lead to problems.
- As an example consider the Fourier sine transform of the function

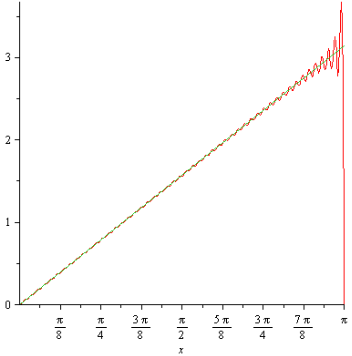
$$f(x) = x \qquad 0 \leq x \leq \pi$$

- The odd extension of this function shows us there is no problem at $x = 0$
- But there is a problem at $x = \pi$ because $f(\pi) \neq f(-\pi)$
- We expect a Gibbs phenomenon at the right endpoint.

- Let's compute the sine transform
- We get

$$x = \sum_{n=1}^{\infty} A_n \sin(nx) \quad \text{where} \quad A_n = (-1)^{n+1}\frac{2}{n}$$

- Indeed summing 100 terms we see the Gibbs phenomenon:



Differentiation of Fourier series

- Now we know the derivative of $f(x) = x$ is
$$f'(x) = 1$$
- So if we differentiate the Fourier sine series for x do we get the Fourier (cosine) series for 1?
- Differentiating term by term we get
$$f'(x) \stackrel{?}{=} \sum_{n=1}^{\infty} 2(-1)^n cos(nx)$$
- This doesn't look encouraging - the Fourier cosine series for 1 is 1
- This Fourier series doesn't seem to converge
- Also - it seems to give nonsensical values.
- At $x = \pi$ it seems to diverge.
- At $x = \pi/2$ it gives 0.
- In neither case is this the derivative of x which is just 1.

Differentiation of Fourier series

- In general you cannot differentiate a Fourier series terms by term if the series does not converge uniformly.
- Here is a formal statement that indicates when this can be done:

Theorem (Differentiation of Fourier series)

Suppose $f'(x)$ has a (not necessarily uniform) convergent Fourier expansion of the form

$$f'(x) = \beta_0/2 + \sum_{n=1}^{\infty} \beta_n \cos(nx) + \sum_{n=1}^{\infty} \alpha_n \sin(nx)$$

defined over the interval $0 < x < 2\pi$. If $f(x)$ is itself continuous in $0 \leq x \leq 2\pi$, and $f(0) = f(2\pi)$ then term by term differentiation of the Fourier series for $f(x)$ is valid.

Differentiation of Fourier series

- To see why this result holds consider the Fourier coefficients of the derivative $f'(x)$.

$$f'(x) = \beta_0/2 + \sum_{n=1}^{\infty} \beta_n \cos(nx) + \sum_{n=1}^{\infty} \alpha_n \sin(nx)$$

- The series coefficients are given by

$$\beta_0 = \frac{1}{2\pi} \int_0^{2\pi} f'(x)dx$$
$$\beta_n = \frac{1}{\pi} \int_0^{2\pi} f'(x) \cos(nx)dx \quad n \neq 0$$
$$\alpha_n = \frac{1}{\pi} \int_0^{2\pi} f'(x) \sin(nx)dx$$

Differentiation of Fourier series

- Now take the expression for β_n and integrate by parts once:

$$\begin{aligned}
\beta_n &= \frac{1}{\pi}\int_0^{2\pi} f'(x)\cos(nx)dx \\
&= \frac{1}{\pi}\left[f(x)\cos(nx)\right]_0^{2\pi} + \frac{n}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx \\
&= nA_n
\end{aligned}$$

where

$$A_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin(nx)dx$$

are the Fourier sine series coefficients for $f(x)$.

Differentiation of Fourier series

- A similar relation exists between the coefficients α_n and the Fourier cosine coefficients B_n.
- We can therefore write

$$\frac{d}{dx}\sum_{n=1}^{\infty}(A_n\cos(nx)+B_n\sin(nx))=\sum_{n=1}^{\infty}(nB_n\cos(nx)-nA_n\sin(nx))$$

- But note the results we just got by integration by parts are only correct if $f(x)$ is continuous in $0<x<2\pi$ and also $f(0)=f(2\pi)$.
- We can also translate these results into results about the rate of convergence of Fourier series.
- If the Fourier series (for $f(x)$) converges uniformly it's OK to differentiate term by term (once)
- To differentiate the resulting series again you must again check if the differentiated Fourier series is uniformly convergent

Integration of Fourier series

- We saw that term by term differentiation of a Fourier series is not always allowed
- You have to check the uniform convergence of the series
- In contrast term by term integration of Fourier series is *always allowed* as long as the function you are integrating is piecewise continuous, and integrable.
- That is, the series gotten by integrating a Fourier series term by term is a series representation of

$$\int^{x} f(x')dx'$$

regardless of whether the Fourier series for $f(x)$ is uniformly convergent.

Integration of Fourier series

- However, it's important to note that the resulting integral is not always a Fourier series
- For example, suppose we had the Fourier series

$$f(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n} \cos(nx)$$

- This series is not uniformly convergent
- But it's OK to integrate term by term:

$$\int^{x} f(x')dx' = C_0 + x + \sum_{n=1}^{\infty} \frac{1}{n^2} \sin(nx)$$

  where C_0 is a constant of integration
- Note the extra presence of x in the result

Integration of Fourier series

- Why is term by term integration always allowed?
- What we are claiming is if $f(x)$ is piecewise continuous and has Fourier series

$$\frac{b_0}{2} + \sum_{n=1}^{\infty} \left(b_n \cos(nx) + a_n \sin(nx) \right)$$

- Then the integral is

$$F(x) = \int_0^x f(x')dx' = \frac{b_0 x}{2} + \sum_{n=1}^{\infty} \left[\frac{b_n}{n} \sin(nx) + \frac{a_n}{n}(1 - \cos(nx)) \right]$$

Integration of Fourier series

- To show this recall that since $F'(x) = f(x)$ is piecewise continuous then
$$g(x) = F(x) - \frac{b_0 x}{2}$$
has a Fourier series given by
$$g(x) = \frac{B_0}{2} + \sum_{n=1}^{\infty} (B_n \cos(nx) + A_n \sin(nx))$$
- What are the coefficients A_n etc.?

Integration of Fourier series

- Look at the coefficients B_n and integrate by parts:

$$
\begin{aligned}
B_n &= \frac{1}{\pi}\int_0^{2\pi} g(x)\cos(nx)dx \\
&= \frac{1}{\pi}\left[\frac{g(x)\sin(nx)}{n}\right]_0^{2\pi} - \frac{1}{\pi}\int_0^{2\pi} g'(x)\frac{\sin(nx)}{n}dx \\
&= -\frac{1}{\pi}\int_0^{2\pi}\left(f(x) - \frac{b_0}{2}\right)\frac{\sin(nx)}{n}dx \\
&= -\frac{a_n}{n} \qquad \text{if } n \geq 1
\end{aligned}
$$

Integration of Fourier series

- Similarly for the A_n

$$A_n = \frac{1}{\pi}\int_0^{2\pi} g(x)\sin(nx) = \frac{\pi b_0 - F(2\pi)}{\pi n} + \frac{b_n}{n} = \frac{b_n}{n}$$

since

$$b_0 = \frac{1}{\pi}\int_0^{2\pi} f(x)dx = F(2\pi)/\pi$$

- So we see that there is never any problem integrating by parts and getting the relationship between the Fourier series and its integral
- Note however the analysis requires us to subtract out the linear piece in $g(x)$.
- After isolating that piece and putting it back in we do see the result of integrating a Fourier series is a Fourier series plus possibly a linear function.