ACM 100b

Classification of singular points

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Singular points of ODE's

- We have seen that as long as the coefficient functions of

  $$y'' + p(x)y' + q(x)y = 0$$

  are locally analytic about a given point $x = x_0$ it will be possible to derive a series solution for the ODE of the form

  $$y(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n,$$

- We have a guarantee that such a series will have a radius of convergence at least as large as the distance from x_0 to the nearest singularity in the complex plane of the coefficient functions.
- We next turn to the situation where $p(x)$ and/or $q(x)$ have singularities

Regular singular points

Definition

We call the point x_0 a *regular singular point* of

$$y'' + p(x)y' + q(x)y = 0$$

if

$$(x - x_0)^2 q(x) \text{ is analytic in the neighborhood of } x = x_0,$$

and

$$(x - x_0)p(x) \text{ is analytic in the neighborhood of } x = x_0.$$

Examples of singular points

- Below we give some examples:

$$
\begin{array}{ll}
y'' = y/(x-1) & \text{has a regular singular point at } x = 1 \\
y'' = y'/x + y/x^2 & \text{has a regular singular point at } x = 0 \\
y'' = y/x^3 & \text{does not have a regular singular point at } x = 0.
\end{array}
$$

- If a point is not an ordinary point or a regular singular point it's called an *irregular singular point*
- The last ODE has an irregular singular point at $x = 0$

Classification of singular points for higher order ODE's

- Consider an n'th order linear ODE given by

$$y^{(n)} + p_{n-1}(x)y^{(n-1)} + p_{n-2}(x)y^{(n-2)} + \ldots p_0(x)y = 0$$

- This ODE has a regular singular point at $x = x_0$ if all of

$$\begin{gathered}(x - x_0)^n p_0(x) \\ (x - x_0)^{n-1} p_1(x) \\ \vdots \\ (x - x_0) p_{n-1}(x)\end{gathered}$$

are all analytic in some neighborhood of $x = x_0$.