ACM 100b

Variation of parameters for systems

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January 19, 2014

Variation of parameters for systems

- Recall from above

$$\boldsymbol{x} = \Phi^{-1}(z)\Phi(z_0)\boldsymbol{x}_0 + \Phi^{-1}(z)\int_{z_0}^{z} \Phi(t)\boldsymbol{f}(t)dt.$$

- We see that Φ and Φ^{-1} both appear.
- Even though they are related simply we will proceed as follows.
- Let

$$\Psi = \Phi^{-1}$$

- Then clearly

$$\Phi\Psi = I.$$

Variation of parameters for systems

- Differentiating the expression

$$\Phi\Psi = I.$$

and substituting into the ODE gives

$$\begin{aligned}\Psi' &= -\Phi^{-1}\Phi'\Psi \\ &= -\Psi\Phi'\Psi \\ &= -\Psi(-\Phi A\Psi) \\ &= (\Psi\Phi)A\Psi = A\Psi.\end{aligned}$$

- We see that this satisfies immediately the homogeneous version of the original ODE

$$\boldsymbol{x}' = A\boldsymbol{x} + \boldsymbol{f}.$$

Variation of parameters

- Suppose we know Ψ.
- We do know Ψ if we know all the homogeneous solutions.
- Now assume

$$\boldsymbol{x} = \Psi \boldsymbol{y}$$

- Plugging this into the original ODE gives

$$\Psi \boldsymbol{y}' = \boldsymbol{f} \quad \Longrightarrow \quad \boldsymbol{y}' = \Psi^{-1} \boldsymbol{f}.$$

- Combining this with the original homogeneous solutions gives us:

$$\boldsymbol{x} = \Psi(z)\boldsymbol{y}_0 + \Psi(z) \int_0^z \Psi^{-1}(t)\boldsymbol{f}(t)dt.$$

Variation of parameters

- The vector $\boldsymbol{y}_0$ in

$$\boldsymbol{x} = \Psi(z)\boldsymbol{y}_0 + \Psi(z)\int_0^z \Psi^{-1}(t)\boldsymbol{f}(t)dt.$$

  can be obtained by solving

$$\boldsymbol{x}_0 = \Psi\boldsymbol{y}_0$$

- You can see that this solution is completely analogous to the solution obtained via the adjoint method:

$$\boldsymbol{x} = \Phi^{-1}(z)\Phi(0)\boldsymbol{x}_0 + \Phi^{-1}(z)\int_0^z \Phi(t)\boldsymbol{f}(t)dt.$$

- This last expression shows we can get the full solution if we know the homogeneous solutions and is the formal expression of variation of parameters for systems.