Physics 106b — Classical Mechanics

Michael Cross

California Institute of Technology

Winter Term, 2014

Parametrically Driven Oscillators

Outline

- Parametric drive: what is it?
- Square wave drive – exact solution
- Sinusoidal drive (Mathieu equation) – harmonic analysis
- Applications

Parametrically driven oscillators

Hill equation: A *parameter* of an oscillator is periodically modulated. The general form for a linear oscillator is the *Hill equation*

$$\ddot{q} + a(t)\dot{q} + b(t)q = 0$$

with $a(t), b(t)$ periodic functions with some period T.

Floquet's theorem: For a linear ODE with coefficients that are periodic in time with period T we can find solutions in the form

$$q(t) = e^{\sigma t} P(t)$$

with σ a constant, complex in general, and $P(t)$ a periodic function with the same period T: $P(t+T) = P(t)$.

Stability: $\mathrm{Re}\,\sigma$ plays the role of a stability parameter: $\mathrm{Re}\,\sigma > 0$ indicates an exponentially growing solution.

Hand and Finch write σ as $i\mu$.

Parametric drive: square wave

- Equation of motion

$$\ddot{\theta} + \gamma\dot{\theta} + F(t)\theta = 0 \quad \text{with} \quad F(t) = \begin{cases} 1 + r & \text{for} \quad 0 \le t < T/2 \\ 1 - r & \text{for} \quad T/2 \le t < T \end{cases}$$

- General solutions for $F(t) = 1 \pm r$

$$\theta_\pm(t) = e^{-\gamma t/2}[A_\pm \cos\omega_\pm t + B_\pm \sin\omega_\pm t] \quad \text{with } \omega_\pm = \sqrt{1 \pm r - (\gamma/2)^2}$$

- The "cosine" and "sine" like solutions over each half cycle

$$\theta_{c\pm}(t) = e^{-\gamma t/2}\left[\cos\omega_\pm t + \frac{\gamma}{2\omega_\pm}\sin\omega_\pm t\right]$$

$$\theta_{s\pm}(t) = e^{-\gamma t/2}\frac{1}{\omega_\pm}\sin\omega_\pm t$$

- Calculate the monodromy matrix M matrix by multiplying M_+ and M_- for the two half cycles

$$M = \begin{bmatrix} \theta_{c-}(T/2) & \theta_{s-}(T/2) \\ \dot{\theta}_{c-}(T/2) & \dot{\theta}_{s-}(T/2) \end{bmatrix}\begin{bmatrix} \theta_{c+}(T/2) & \theta_{s+}(T/2) \\ \dot{\theta}_{c+}(T/2) & \dot{\theta}_{s+}(T/2) \end{bmatrix}$$

Parametric drive: sinusoidal

Sinusoidal drive gives the damped *Mathieu equation*

$$\ddot{\theta} + \gamma\dot{\theta} + [1 + h\cos\omega_d t]\theta = 0$$

Study primary $n = 1$ tongue $\omega_d \approx 2$ using perturbation theory for small γ, h

Floquet's theorem: look for a solution

$$\theta(t) = e^{\sigma t}\left[Ae^{i\omega_d t/2} + \text{c.c.}\right] + \text{harmonics}$$

Use method of Harmonic Analysis: substitute in to the EOM and set the coefficient of $e^{i\omega_d t/2}$ term to zero.

Tricky term: $\cos\omega_d t\ \theta = \frac{1}{2}(e^{i\omega_d t} + e^{-i\omega_d t})(Ae^{i\omega_d t/2} + A^* e^{-i\omega_d t/2})e^{\sigma t}$

$$\left[-\left(\frac{i\omega_d}{2} + \sigma\right)^2 + \gamma\left(\frac{i\omega_d}{2} + \sigma\right) + 1\right]A + \frac{h}{2}A^* = 0$$

For $\omega_d \approx 2$ and leaving out terms in σ^2 and $\sigma\gamma$ this can be approximated

$$[(2 - \omega_d) + i\,(2\sigma + \gamma)]\,A = -\tfrac{1}{2}hA^*$$

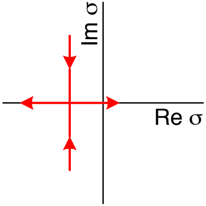
Parametric drive: sinusoidal

$$[(2-\omega_d)+i\,(2\sigma+\gamma)]\,A=-\tfrac{1}{2}hA^*$$

Magnitude-squared gives the growth rate

$$\sigma=-\frac{\gamma}{2}\pm\frac{1}{4}\sqrt{h^2-(\omega_d-2)^2}$$

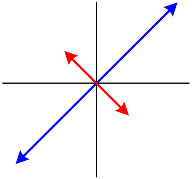
For $\omega_d=2$ instability occurs for $h>2\gamma$



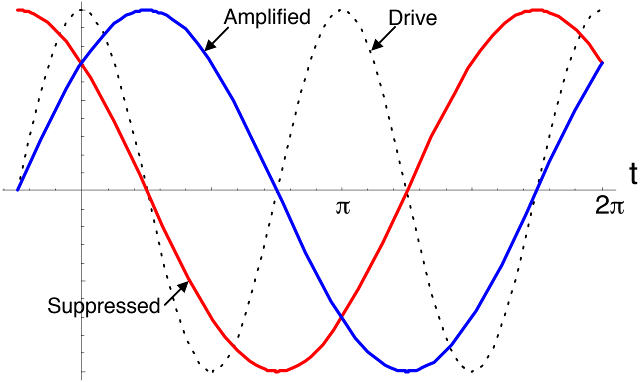
On resonance and at instability, $A=|A|e^{i\delta}$ with δ the phase relative to the drive

$$\sigma=0\Rightarrow iA=-A^* \quad \text{so that} \quad \delta=\frac{\pi}{4}$$

$$\sigma=-\gamma\Rightarrow -iA=-A^* \quad \text{so that} \quad \delta=-\frac{\pi}{4}$$



Parametric drive: sinusoidal



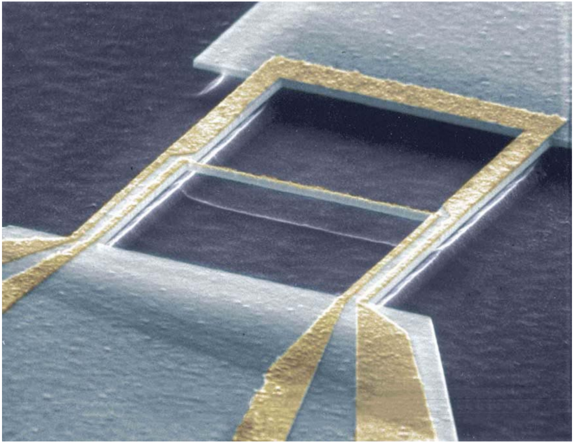
Application

Nanoscale mechanical amplifier

- Darrell Harrington, Caltech Thesis (2002)
- Karabalin, Feng, and Roukes, Nano Letters **9**, 3116 (2009)
- Karabalin, Masmanidis, and Roukes, Appl. Phys. Lett. **97**, 183101 (2010)

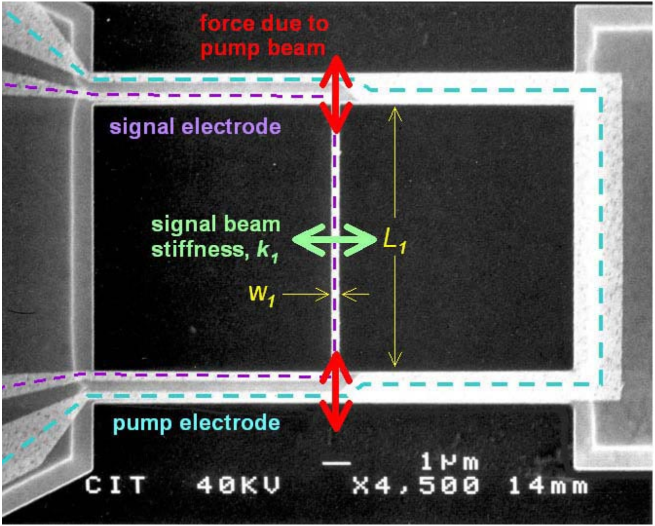
Nanomechanical Beam Resonator

Roukes Group, Caltech, 2002



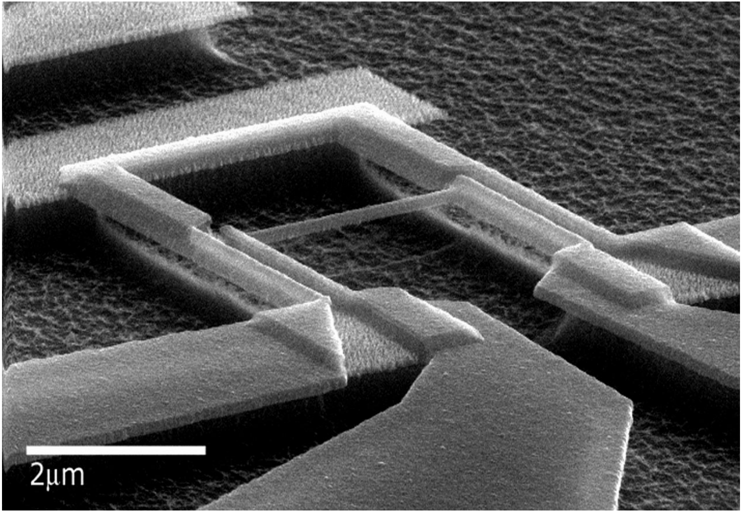
Nanomechanical Beam Resonator

Roukes Group, Caltech, 2002



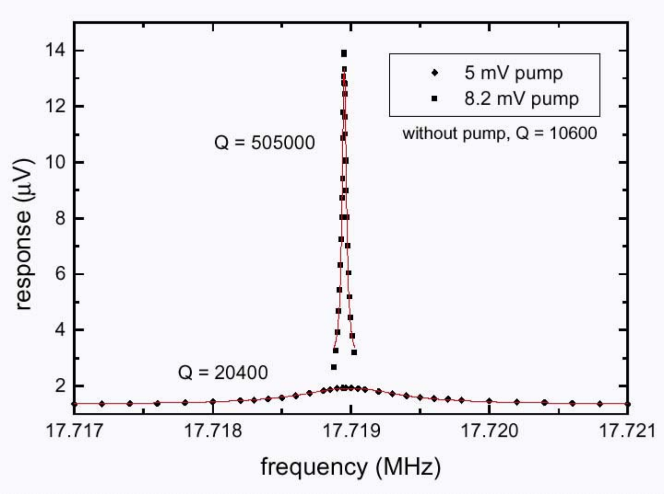
Nanomechanical Beam Resonator

Roukes Group, Caltech, 2008



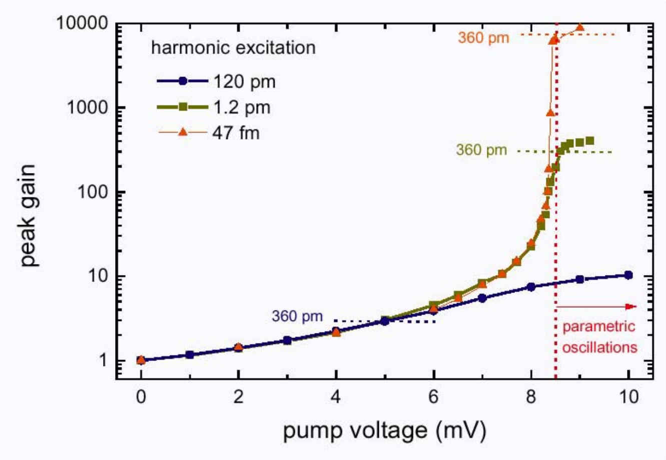
Parametric Drive: Enhanced Q

Roukes Group, Caltech, 2002



Parametric Amplification

Roukes Group, Caltech, 2002



Squeezing of Thermal Noise

Roukes Group, Caltech, 2002