ACM 100b

Reduction of order for second order linear ODE's

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January 12, 2014

Reduction of order

- As we stated before, there is no explicit expression for the general solution of a second order linear ODE.
- However, if we are lucky enough to find one solution of the homogeneous equation there is a procedure to find the other solution.
- Recall a second order linear ODE has two linearly independent solutions.
- So if we can get the second one we can get the general solution.
- There is a method to do this called *reduction of order*

Reduction of order

- Suppose we know one nontrivial solution $y_1(x)$ of the homogeneous ODE.

$$y'' + p(x)y' + q(x)y = 0.$$

- We then set the second solution to be

$$y_2(x) = v(x)y_1(x)$$

  where $v(x)$ is unknown.
- If we substitute the expression for y_2 in the ODE we get

$$v[y_1'' + p(x)y_1' + qy] + v'(x)(2y_1' + py_1') + v''y_1 = 0.$$

- But note that because y_1 is a solution of the ODE the first term above is zero and so we get

$$v'(2y_1' + py_1') + v''y_1 = 0$$

Reduction of order

- But if we set $u = v'$ in

$$v'(2y_1' + py_1') + v''y_1 = 0$$

we see this is really a first order equation for $u(x)$:

$$u(2y_1' + py_1') + u'y_1 = 0$$

- We can readily solve this get

$$\frac{dv}{dx} = d\exp\left[-\int^x \left(p(s) + \frac{2y_1'}{y_1}\right)ds\right] = \frac{d}{y_1^2}\exp\left[-\int^x p(s)ds\right],$$

where d is some arbitrary constant.

Reduction of order

- And so
$$v(x) = d \int^x \frac{1}{y_1^2(t)} \exp\left[-\int^t p(s)ds \right] dt.$$
- The general solution is
$$y = c_1 y_1 + c_2 v y_1.$$
- It is easy to show that the second solution is linearly independent of the first provided $p(x)$ is continuous.