ACM 100b

Legendre functions

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The Legendre ODE

- We examine here an important case of a singular Sturm-Liouville problem called the Legendre ODE.
- The ODE arises when we try to use separation of variables to solve PDE problems involving the Laplacian in spherical coordinates.
- Spherical coordinates are given by the transformation

$$
\begin{aligned}
x &= r \sin\theta \cos\phi \\
y &= r \sin\theta \sin\phi \\
z &= r \cos\theta .
\end{aligned}
$$

- When we transform the Laplacian into these coordinates we get

$$\nabla^2 u(r,\theta,\phi) = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin\theta^2}\frac{\partial^2 u}{\partial\phi^2}.$$

The Legendre ODE

- Assume next that the function $u(r, \theta, \phi)$ has no variation in the $\phi-$ direction (meaning u is axi-symmetric).
- As we will see later it's possible to use separation of variables to solve the Laplace equation

$$\nabla^2 u(r, \theta) = 0$$

by trying separable solutions of the form

$$u(r, \theta) = r^\nu y(\cos \theta).$$

- Plugging this form of the solution into the equation allows you to get an equation of the form

$$\nu(\nu + 1) - 2 \cos \theta \frac{dy(\cos \theta)}{d\theta} - \sin^2 \theta \frac{d^2 y(\cos \theta)}{d\theta^2} = 0.$$

The Legendre ODE

- At this point it is customary to make the substitution

$$x = \cos\theta,$$

- And the result is *Legendre's differential equation:*

$$y'' - \frac{2x}{1-x^2} + \frac{\nu(\nu+1)}{1-x^2} y(x) = 0.$$

- This ODE can be put into the standard Sturm-Liouville form:

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + \nu(\nu+1)y(x) = 0 \qquad -1 \leq x \leq 1.$$

The Legendre equation

- Note in terms of the notation we have adopted previously we can make the following connections:
$$p(x) = 1 - x^2 \qquad q(x) = 0 \qquad r(x) = 1.$$
- And the eigenvalue λ can be written as
$$\lambda = \nu(\nu + 1).$$
- We see that the all-important function $p(x)$ is positive except at $x = -1$ and $x = 1$ where it vanishes
- So $x = \pm 1$ are singular points of the ODE and this is a singular Sturm-Liouville problem.
- The reason the singularity arises is because we used spherical coordinates.
- The points $x = \pm 1$ correspond to $\theta = \pi$ and $\theta = 0$.
- In spherical coordinates, these are the poles of the sphere and are singular points for this coordinate system.

Analyzing singular Sturm-Liouville problems

- In the previous slides we introduced a singular S-L problem associated with separation of variables from spherical coordinates

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+\nu(\nu+1)y(x)=0 \qquad -1\leq x\leq 1.$$

- This is the Legendre ODE
- Next we will analyze the solutions
- We will see the important differences between singular and regular S-L problems
- To start we noted there are singular points at $x=\pm 1$

Frobenius analysis of the Legendre ODE

- We next try to examine the type of singular points at $x = \pm 1$.
- The ODE can be written as

$$y'' - \frac{2x}{1-x^2}y' + \frac{\lambda}{1-x^2}y = 0.$$

- So we can see quite clearly it has regular singular points at $x = \pm 1$.
- When we have a singular Sturm-Liouville problem we know that it is possible that there may be discrete eigenvalues or eigenvectors but there is no guarantee.
- It turns out in this case there are discrete eigenvalues and eigenvectors but we will just take this as an assumption.
- When we have a regular singular point we can analyze the solutions of the ODE about the singular point via the Frobenius method.
- It turns out in this case, that approach tells us what the eigenvalues and eigenfunctions are.

Frobenius analysis of the Legendre ODE

- The first thing to do is see what the local behavior of the solutions is near the singular points $x = \pm 1$
- The way to do this is get the indicial equation for this ODE at both $x = \pm 1$ by examining a substitution like

$$y_{\pm}(x) = (x \pm 1)^{\alpha} \sum_{n=0}^{\infty} A_n (x \pm 1)^n.$$

- The indicial equation (at $x = +1$) for α gives

$$\alpha^2 = 0.$$

- From the Frobenius theory, this means one of the solutions at $x = 1$ is regular (that is, not singular)
- But the other one it turns out has a logarithmic singularity.
- If you do the analysis at $x = -1$ you get the same result for that point as well.

Frobenius analysis of the Legendre ODE

- This raises the question of what kind of boundary conditions can we even place at $x = \pm 1$
- Recall these points correspond to the poles of the sphere.
- From physical considerations we do not want solutions which blow up at either point.
- But from the point of view of physical solutions there is nothing special about the poles of a sphere.
- So we could ask that the solution be regular at $x = -1$
- But then we have no control over what might happen at $x = 1$ where things could also blow up logarithmically.
- Similarly we could ask that a solution not blow up at $x = 1$
- But then how do we guarantee that it doesn't blow up at $x = -1$?
- In general, we'll get some linear superposition of the regular and irregular solution expanded about the opposite point.

Frobenius analysis of the Legendre ODE

- Our only hope is to see if perhaps there are special values of λ where a solution that starts out regular at say $x = -1$ stays regular when we examine $x = 1$.
- This is how the eigenvalue problem works in this case.
- Our boundary value problem is

$$y'' - \frac{2x}{1-x^2}y' + \frac{\lambda}{1-x^2}y = 0 \qquad -1 \leq x \leq 1$$

with boundary conditions

$$y(-1) \text{ finite} \qquad y(+1) \text{ finite.}$$

- Such considerations are typical of all singular Sturm-Liouville eigenfunctions.
- You cannot ask that the function vanish at a singular point
- You can hope the solution is finite at such points.
- We still don't know if there are any solutions for λ such that we can fulfill the conditions.

Eigenvalues and eigenfunctions of the Legendre ODE

- In this case, however, it turns out there are discrete values of λ where we get solutions that are finite at both points.
- In addition, the entire set of eigenvalues and the resulting set of functions form a complete set.
- We will not prove this here.
- Instead we'll derive the eigenvalues and eigenfunctions.
- To do this there is a clever trick which works for some other singular S-L problems as well.
- Look at the Taylor series for the solutions at $x = 0$.
- Now $x = 0$ is an *ordinary* point and this is a second order ODE so we can get two Taylor series solutions about $x = 0$
- We expect that both of them will have a radius of convergence of 1 because there are singularities at $x = \pm 1$.

Eigenvalues and eigenfunctions of the Legendre ODE

- It's convenient to return to the following form of the ODE:

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right]+\nu(\nu+1)y(x)=0 \qquad -1\leq x\leq 1.$$

- The results of a series expansion about $x=0$ are

$$y_1(x)=1+\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m)!}\left[\nu(\nu-2)(\nu-4)\cdots(\nu-2m+2)\right]\times$$
$$\left[(\nu+1)(\nu+3)\cdots(\nu+2m-1)\right]x^{2m}$$
$$y_2(x)=x+\sum_{m=1}^{\infty}\frac{(-1)^m}{(2m+1)!}\left[(\nu-1)(\nu-3)\cdots(\nu-2m+1)\right]\times$$
$$\left[(\nu+2)(\nu+4)\cdots(\nu+2m)\right]x^{2m+1}.$$

Eigenvalues and eigenfunctions of the Legendre equation

- Note we assumed initial conditions of the form

$$\begin{aligned} y_1(0) = 1, \quad y_1'(0) &= 0 \\ y_2(0) = 0, \quad y_2'(0) &= 1, \end{aligned}$$

  as a convenient way to show the general solution.
- If you perform the ratio test for convergence you find both series have a radius of convergence of 1 meaning there is a singularity at a radius of 1.
- From our Frobenius analysis we know where these singularities are - right on the x-axis.
- If you start with some generic initial condition at $x = 0$ as we did, your solution will most likely pick up the singularities at $x = \pm 1$.
- So it doesn't look too good for getting solutions that are regular at both points

Eigenvalues and eigenfunctions of the Legendre equation

- However, there is a way out.
- Suppose we take $\nu = 2n$ where $n = 0, 1, 2, \ldots$.
- Then the first series $y_1(x)$ terminates as a polynomial.
- This means it is entirely regular at $x = \pm 1$.
- The other solution $y_2(x)$ does not terminate.
- You get a full series for y_2 which we know means the solution for $y_2(x)$ for such values of ν must be singular at $x = \pm 1$.

Eigenvalues and eigenfunctions of the Legendre equation

- Let's explore the polynomial solutions we get for this choice of ν. The polynomials are

$$\begin{aligned} \nu &= 0 & y_1(x) &= 1 \\ \nu &= 2 & y_1(x) &= 1 - 3x^2, \end{aligned}$$

  and so forth.
- We see that all the polynomials are even.
- We notice that we can get more such polynomial solutions if we try $\nu = 2n + 1$.
- In that case the second series will terminate and give us

$$\begin{aligned} \nu &= 1 & y_1(x) &= x \\ \nu &= 3 & y_1(x) &= x - 5x^3/3. \end{aligned}$$

- These polynomials are all odd.

Eigenvalues and eigenfunctions of the Legendre ODE

- We also see that among these solutions we have used the values $\nu = 0, 1, 2, 3, 4 \ldots$.
- So when ν is an integer there are polynomial solutions.
- The eigenvalue λ for this S-L problem is given by

$$\lambda = \nu(\nu + 1) \qquad \nu = 0, 1, 2, 3, \ldots.$$

- And the eigenfunctions are polynomials which we can label by the values of ν:

$$\begin{aligned}
\nu &= 0 & \qquad P_0(x) &= 1 \\
\nu &= 1 & \qquad P_1(x) &= x \\
\nu &= 2 & \qquad P_2(x) &= 1 - 3x^2 \\
\nu &= 3 & \qquad P_3(x) &= x - 5x^3/3.
\end{aligned}$$

- These polynomials $P_\nu(x)$ are called the *Legendre polynomials*.

Properties of the Legendre polynomials

- It turns out the Legendre functions form a complete set of functions.
- They are also mutually orthogonal:

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \begin{cases} 0 & n \neq m \\ 2/(2n+1) & n = m, \end{cases}$$

- Any square integrable function can be expanded in terms of these polynomials:

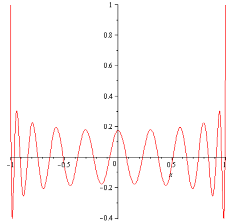
$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \qquad a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx.$$

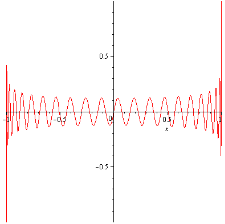
Properties of expansions in Legendre polynomials

- It turns out expansions in Legendre polynomials do not suffer from the Gibbs phenomenon as long as $f(x)$ is smooth between $-1 \leq x \leq 1$.
- In other words what $f(x)$ does at the boundaries (assuming it's regular there) does not affect the convergence of the Legendre series
- Of course if $f(x)$ has a discontinuity inside $-1 \leq x \leq 1$ then you will see a Gibbs phenomenon
- This should be contrasted to Fourier series where $f(x)$ must satisfy special conditions at the boundaries to avoid the Gibbs phenomenon.
- Expansions in Legendre polynomials converge uniformly over the interval $-1 \leq x \leq 1$ as long as $f(x)$ is smooth.
- We won't go further in to this but this behavior is associated with certain special properties of solutions of singular Sturm-Liouville ODE's.

Zeros of the polynomials

- It turns out that the Sturm comparison theorem can be modified for the Legendre polynomials
- This is the case even though they come from a singular problem.
- The comparison theorem is used to show that $P_n(x)$ has n distinct simple zeros and they all lie on the segment $-1 < x < 1$.
- Here are some plots of the Legendre polynomials





- The left hand plot is $P_{20}(x)$ and the right hand plot is $P_{41}(x)$
- Note the zeros are evenly spaced (like sines) in the interior but they get more dense near the boundaries.

Generating function

- The function
$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}}$$
has a Taylor series expansion in t of the form
$$G(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n.$$
- In other words the coefficients are the Legendre polynomials
- This means that we can get any Legendre polynomial by taking an appropriate derivative of $G(x,t)$:
$$P_n(x) = \frac{1}{n!}\frac{d^n}{dx^n} G(x,t)$$
- We call such functions whose series coefficients give us an entire set of results like this a *generating function*.
- We will see more of these generating functions in other applications.

The Rodrigues formula

- There is a compact expression that also allows one to get further properties of all the Legendre polynomials.
- It can be shown that

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n.$$

- This is known as the *Rodrigues formula.*
- This is often useful when we want to get properties for the entire set of Legendre polynomials
- There are Rodrigues formulas for other sets of Sturm-Liouville eigenfunctions as well.

Recursion relations

- Either the generating function or the Rodrigues formula can be used to infer some recursion relations among the Legendre polynomials.
- Recall that the generating function

$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} P_n(x)t^n.$$

- From this we can easily see that

$$(1-2xt+t^2)G^2 = 1.$$

- Now differentiate both sides with respect to t to get

$$(1-2xt+t^2)\frac{\partial G}{\partial t} + (t-x)G = 0.$$

Recursion relations

- Now we know that

$$G(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n,$$

- And if we differentiate with respect to t we get

$$\frac{\partial G}{\partial t} = \sum_{n=0}^{\infty} n P_n(x) t^{n-1}.$$

- Plug these series into the relation

$$(1 - 2xt + t^2)\frac{\partial G}{\partial t} + (t - x)G = 0,$$

- And matching powers of t
- This gives the following relation:

$$(n+1)P_{n+1} - 2xnP_n + (n-1)P_{n-1} + P_{n-1} - xP_n(x) = 0,$$

Recursion relations

- This expression can be simplified to relate the polynomials in the following way:

$$(2n+1)xP_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \qquad n = 1, 2, \ldots.$$

- This is an example of a *recursion relationship*
- We can use this relationship to compute the next Legendre polynomial if we know the previous two:

$$P_{n+1}(x) = \left(\frac{2n+1}{n+1}\right) xP_n(x) - \frac{n}{n+1} P_{n-1}(x)$$

- To get this sequence started you use the known values of P_0 and P_1:

$$P_0(x) = 1 \qquad P_1(x) = x.$$

- Using this we can compute P_2, P_3, etc.
- In fact it gives us a way to evaluate the n'th Legendre polynomial in n steps at a specific point using $2n$ multiplications and n additions.

Recursion relations

- Suppose we differentiate the relation

$$(1-2xt+t^2)G^2 = 1$$

with respect to x

- Then we get a recursion relation that relates the derivatives of the polynomials to the polynomials themselves.
- In particular we get

$$-tG + (1-2xt+t^2)G_x = 0,$$

- Using the same type of manipulations as before we come up with

$$P_n(x) = P'_{n+1}(x) - 2xP'_n(x) + P'_{n-1}(x).$$

- This means we can compute all the derivatives of the Legendre polynomials recursively if we know the polynomials themselves.
- These recurrence relations are useful in that they make it very easy to get the Legendre polynomials and their derivatives quickly.

Other families of orthogonal polynomials

- The Legendre polynomials are just one of a class of orthogonal polynomials that come from singular Sturm-Liouville problems.
- There are several others that are important and are used in applications.
- All of them obey orthogonality relations over their respective intervals.
- In other words if $Q_n(x)$ is one of these polynomials then we have

$$\int_a^b r(x)Q_n(x)Q_m(x)dx = 0 \qquad n \neq m.$$

Other families of orthogonal polynomials

- Some other important families are

Polynomial name	a	b	$r(x)$
Hermite	$-\infty$	∞	$\exp(-x^2/2)$
Laguerre	0	∞	$\exp(-x)$
Chebyshev	-1	1	$(1-x^2)^{-1/2}$

- And there are other families as well.
- All of them have a generating function, Rodrigues formula and recursion relations.