ACM 100b

Application of the Fourier transform to ODE's

Dan Meiron

Caltech

February 23, 2014

Solving ODE's with Fourier transforms

- The Fourier transform is very useful for solving inhomogeneous constant coefficient ODE's over the fully infinite interval $-\infty < x < \infty$.
- We assume here that the solution $y(x) \to 0$ as $|x| \to \infty$.
- This is actually an implicit assumption.
- By using the transform to get the solution we are assuming the transform exists in the first place.
- We will consider the ODE

$$y'' - a^2 y = g(x) \qquad -\infty < x < \infty \qquad y \to 0 \text{ as } |x| \to \infty.$$

- We next apply the Fourier transform to both sides of the equation:

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [y'' - a^2 y] \exp(-ikx) dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) \exp(-ikx) dx.$$

Solving ODE's with Fourier transforms

- We define $G(k)$ to be the transform of $g(x)$
- We then use the properties of the transform for derivatives of $y(x)$ to convert the ODE to an algebraic equation for the transform of $y(x)$ which we call $Y(k)$:

$$[-k^2 - a^2]Y(k) = G(k)$$

- So we can solve for the transform $Y(k)$

$$Y(k) = \frac{-G(k)}{k^2 + a^2}.$$

- Note this is an algebraic equation.
- Using the inverse transform we would then have

$$y(x) = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{G(k)}{k^2 + a^2} \exp(ikx)dk$$

Solving ODE's with Fourier transforms

- At this point it doesn't look like we can go much further because we don't have an explicit expression for $G(k)$.
- However, note that the transform of the solution

$$Y(k) = \frac{-G(k)}{k^2 + a^2}.$$

is actually in the form of a product of two transforms:

$$Y_1(k) = G(k) \text{ and } Y_2(k) = -\frac{1}{k^2 + a^2},$$

- We can then apply the convolution theorem which states that

$$\mathcal{F}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\zeta)g(x-\zeta)d\zeta\right] = F(k)G(k).$$

- What this means is that our solution $y(x)$ must be a convolution that involves the inverse transforms of the two functions

$$Y_1(k) = G(k) \text{ and } Y_2(k) = -\frac{1}{k^2 + a^2}.$$

- The inverse transform for $Y_1(k)$ is simple since $G(k)$ is the transform for $g(x)$.
- The inverse transform of $Y_2(k)$ will be derived below using contour integration.
- If we call the inverse transform of $Y_2(k)$, $y_2(x)$ then our final answer is

$$y(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(\zeta) y_2(x - \zeta) d\zeta.$$

Solving ODE's with Fourier transforms

- We next focus on the inverse transform

$$y_2(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[-\frac{1}{k^2 + a^2} \right] \exp(ikx) dk$$

- We'll get the answer using contour integration in the complex k-plane.
- Let's consider $x > 0$.
- If this is the case we close the contour from $k = -\infty$ to $k = \infty$ by using a semicircular contour in the upper half complex k-plane.
- This allows us to show the contribution from the circle vanishes along the curved part of the semicircle.

- We then use the residue theorem and notice there is pole at the location

$$k = +i|a|$$

- The residue at this pole is

$$-\frac{1}{2i|a|}\exp(-|a|x),$$

- So we have that

$$y_2(x) = -\frac{1}{\sqrt{2\pi}}\frac{\pi}{|a|}\exp(-|a|x). \qquad x > 0$$

Solving ODE's with Fourier transforms

- If $x < 0$ we have to close in the lower half k-plane and pick up the other pole.
- Note that in this case the contour goes clockwise so there is an extra factor of (-1).
- We find that
$$y_2(x) = -\frac{1}{\sqrt{2\pi}}\frac{\pi}{|a|}\exp(+|a|x). \qquad x < 0$$
- Note there is a compact way to write both answers as one expression
$$y_2(x) = -\frac{\sqrt{\pi}}{\sqrt{2}|a|}\exp(-|a||x|).$$
- Now finally we use this answer in the convolution theorem to get
$$y(x) = -\frac{1}{2a}\int_{-\infty}^{\infty} g(\zeta)\exp(-|a||x-\zeta|)d\zeta.$$

Solving ODE's with Fourier transforms

- Note the form of the answer.
- This looks just like the expressions we have seen before when we calculated a Green's function for an inhomogeneous ODE
- Indeed we can get this answer in another way.
- Suppose we ask for the Greens function defined by

$$G'' - a^2 G = \delta(x - \zeta) \qquad -\infty < x < \infty.$$

- Since this is a problem over an infinite domain, all we can ask for is that the solution be finite as $|x| \to \infty$.
- Now to solve this problem we first get the matching conditions as $x \to \zeta$ for the Green's function.
- For simplicity, we'll set $\zeta = 0$ and do the matching there,
- Then we can infer the result for general ζ by letting $x \to x - \zeta$.

Solving ODE's with Fourier transforms

- The best way to do this is to integrate both sides of the equation over a small interval about the point $x = 0$:

$$\int_{-\epsilon}^{+\epsilon} G'' dx - a^2 \int_{-\epsilon}^{+\epsilon} G dx = \int_{-\epsilon}^{+\epsilon} \delta(x) dx = 1.$$

- Now we take the limit as $\epsilon \to 0$.
- The only way to make this work is to have

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} G(x) dx = 0,$$

- As a result G is continuous at $x = 0$.
- On the other hand we must have

$$\lim_{\epsilon \to 0} \int_{-\epsilon}^{+\epsilon} G'' dx = \lim_{\epsilon \to 0} G'(+\epsilon) - G'(-\epsilon) = 1.$$

Solving ODE's with Fourier transforms

- Now we note we can get the two homogeneous solutions of

$$y'' - a^2 y = 0$$

and these are

$$y(x) = A\exp(-|a|x) + B\exp(+|a|x).$$

- In order to make sure things are finite as $|x| \to \infty$ we must have

$$y(x) = \begin{cases} A\exp(-|a|x) & x > 0 \\ B\exp(+|a|x) & x < 0. \end{cases}$$

Solving ODE's with Fourier transforms

- Because we determined that G is continuous at $x = 0$ we must have
$$A = B$$
and we see that
$$\begin{aligned} G'(x) &\rightarrow -aA \qquad x \rightarrow 0^+ \\ G'(x) &\rightarrow aA \qquad x \rightarrow 0^-, \end{aligned}$$
- So we must have that
$$A = -\frac{1}{2a}.$$

Solving ODE's with Fourier transforms

- So we see that the Greens function for this equation is

$$G(x|\zeta) = -\frac{1}{2a}\exp(-|a||x-\zeta|),$$

- After we substitute $x - \zeta$ for x and we see that when we solve the ODE

$$y'' - a^2 y = g(x) \qquad -\infty < x < \infty$$

by means of Greens functions we get

$$y(x) = -\frac{1}{2a}\int_{-\infty}^{\infty} g(\zeta)\exp(-|a||x-\zeta|)d\zeta,$$

- This is exactly the same result as gotten using Fourier transforms and the convolution theorem.