ACM 100b

Sine and cosine transforms

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The sine and cosine transforms

- Consider the forward Fourier transform:

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ikx) dx.$$

- Suppose we have that $f(x)$ is an even function.
- That is $f(-x) = f(x)$.
- In that case

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) \exp(-ikx) dx = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos(kx) dx.$$

- Notice that this automatically implies that $F(k)$ is then an even function of k when we consider real values of k.

The sine and cosine transforms

- If we then write the reverse transform

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(k) \exp(ikx),$$

we see that

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F(k) \cos(kx) dk.$$

- We see that there is then a new transform pair for functions defined on the interval $0 < x < \infty$:

$$\begin{aligned} F_c(k) &= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos(kx) dx \\ f(x) &= \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c(k) \cos(kx) dk. \end{aligned}$$

- This transform pair defines what we call the *Fourier cosine transform.*

The sine and cosine transforms

- Note that again you can use any normalization you want as long as the appropriate overall factors appear when you transform forward and back.
- So for example you could define

$$
\begin{aligned}
F_c(k) &= \frac{2}{\pi}\int_0^\infty f(x)\cos(kx)dx \\
f(x) &= \int_0^\infty F_c(k)\cos(kx)dk.
\end{aligned}
$$

  and this would be a perfectly good transform pair as well.
- There is no standard normalization.
- In a similar vein, we can define the Fourier sine transform:

$$
\begin{aligned}
F_s(k) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin(kx)dx \\
f(x) &= \sqrt{\frac{2}{\pi}}\int_0^\infty F_s(k)\sin(kx)dk.
\end{aligned}
$$

The sine and cosine transforms

- Under differentiation, the sine and cosine transforms behave a bit differently from the Fourier transform.
- For example the cosine transform of the derivative of a function is related to the sine transform of that function and vice versa.
- However, the cosine transform of the second derivative is related to the cosine transform of the function in a simple way.
- Using integrating by parts you can show that

$$\sqrt{\frac{2}{\pi}} \int_0^\infty y''(x) \cos(kx) dx = -k^2 Y_c(k) - \sqrt{\frac{2}{\pi}} y'(0)$$

- Similarly for sine transforms

$$\sqrt{\frac{2}{\pi}} \int_0^\infty y''(x) \sin(kx) dx = -k^2 Y_s(k) + \sqrt{\frac{2}{\pi}} k y(0)$$

where Y_c and Y_s are, respectively, the Fourier cosine and sine transforms of $y(x)$.

Solving ODE's with sine and cosine transforms

- Because of these properties for derivatives these transforms are useful for second order problems defined on the semi-infinite interval $0 < x < \infty$.
- For example suppose we want to solve the boundary value problem

$$y'' - a^2 y = g(x) \qquad y'(0) = B \qquad 0 < x < \infty.$$

- We can then apply the cosine transform to both sides and notice that the boundary condition appears in a natural way in the transform:

$$[-k^2 - a^2]Y_c(k) - \sqrt{\frac{2}{\pi}} y'(0) = G_c(k),$$

- So we have

$$Y_c(k) = -\frac{G_c(k) + \sqrt{\frac{2}{\pi}} B}{a^2 + k^2}.$$

Solving ODE's with sine and cosine transforms

- An inverse cosine transform then allows you to compute the final result.

$$y(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty Y_c(k) \cos(kx) dk.$$

- So we then have to do the integral

$$y(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty -\frac{G_c(k) + \sqrt{\frac{2}{\pi}} B}{a^2 + k^2} \cos(kx) dk.$$

- You would then use the convolution theorem for cosine transforms to progress further.
- The sine transform is used similarly for problems where the value of the solution is given as a boundary condition at $x = 0$.
- For both transforms there are appropriate convolution theorems, Parseval identities etc.