ACM 100c

High order ODE's as systems

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January 12, 2014

Conversion of an n'th order ODE to a first order system

- In many applications we deal with a system of ODE's of one independent variable and more than one dependent variable.
- For example consider

$$
\begin{aligned}
F_1(z, y_1, y_1', y_1'', y_2, y_2') &= 0 \\
F_2(z, y_1, y_1', y_2, y_2', y_2'', y_2''') &= 0
\end{aligned}
$$

- This system has two equations in two dependent variables.
- This system is second order in y_1 and first order in y_2 for the first equation, first order in y_1 and third order in y_2 for the second equation.
- It is more convenient to rewrite such a system as a system of first order equations.

Converting to a first order system

- Assume we can rewrite the equations mentioned above

$$F_1(z, y_1, y_1', y_1'', y_2, y_2') = 0$$
$$F_2(z, y_1, y_1', y_2, y_2', y_2'', y_2''') = 0$$

as the system

$$\begin{aligned} y_1'' &= G_1(z, y_1, y_1', y_2, y_2') \\ y_2''' &= G_2(z, y_1, y_1', y_2, y_2', y_2''). \end{aligned}$$

- All we did was isolate the highest derivative on the left hand side
- We'll show we can convert this to a first order system

Converting to a first order system

- Now we can introduce five new dependent variables

$$x_1 = y_1, \quad x_2 = y_1', \quad x_3 = y_2, \quad x_4 = y_2', \quad x_5 = y_2''$$

so that the original system

$$\begin{aligned} y_1'' &= G_1(z, y_1, y_1', y_2, y_2') \\ y_2''' &= G_2(z, y_1, y_1', y_2, y_2', y_2''). \end{aligned}$$

becomes

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= G_1(z, x_1, x_2, x_3, x_4) \\ x_3' &= x_4 \\ x_4' &= x_5 \\ x_5' &= G_2(z, x_1, x_2, x_3, x_4, x_5). \end{aligned}$$

- This is a system of five first order equations.

Converting to a first order system

- In general, any system can be written as a first order system

$$x_i' = G_i(z, x_1, x_2, ..., x_n), \quad i = 1, 2, ..., n$$

- Now introduce vector notation

$$\boldsymbol{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \mathbf{h}(z, \boldsymbol{x}) = \begin{pmatrix} G_1(z, x_1, ..., x_n) \\ \vdots \\ G_n(z, x_1, x_2, ..., x_n) \end{pmatrix}.$$

- We can write this compactly as $\boldsymbol{x}' = \mathbf{h}(z, \boldsymbol{x})$.

A system can always be converted back to a high order ODE

- For example the linear system

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 \\ x_2' &= a_{21}x_1 + a_{22}x_2 \end{aligned}$$

  can be reduced to a second order (linear) ODE.
- Simply use the first equation to get x_2:

$$x_2 = \frac{x_1' - a_{11}x_1}{a_{12}}$$

- And then substitute in the second equation:

$$x_2' = \frac{x_1'' - a_{11}x_1' - a_{11}'x_1}{a_{12}} - \frac{a_{12}'(x_1' - a_{11}x_1)}{a_{12}^2}$$

A system can always be converted back to a high order ODE

- The result is

$$x_1'' - \left[a_{11} + a_{22} + \frac{a_{12}'}{a_{12}}\right] x_1' + \left[a_{11}a_{22} - a_{12}a_{21} - \left(\frac{a_{12}a_{11}' - a_{12}'a_{11}}{a_{12}}\right)\right] x_1 = 0$$

- This is equivalent to

$$x_1'' + p(z)x_1' + q(z)x_1 = 0.$$

- Note that many choices of the a_{ij}'s lead to same p, q