Physics 106a — Classical Mechanics

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Lecture 4A: Derivation of the Generalized Equation of Motion

D'Alembert's Principle

We derived the result (d'Alembert's principle evaluating the virtual work for holonomic constraints)

$$\sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \mathcal{F}_k \equiv \frac{\delta W}{\delta q_k}$$

Task: evaluate the left hand side in terms of the kinetic energy

$$T = \sum_i \tfrac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i \to T(\{q_k\}, \{\dot{q}_k\}, t)$$

Lemma: Dot Cancellation

Lemma: *for holonomic constraints*

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} = \frac{\partial \vec{r}_i}{\partial q_k}$$

Proof:

- For $\vec{r}_i = \vec{r}_i(\{q_k\}, t)$

  $$\dot{\vec{r}}_i \equiv \frac{d\vec{r}_i}{dt} = \sum_k \frac{\partial \vec{r}_i}{\partial q_k}\dot{q}_k + \frac{\partial \vec{r}_i}{\partial t}$$

  so that $\dot{\vec{r}}_i = \dot{\vec{r}}_i(\{q_k\}, \{\dot{q}_k\}, t)$ and we can imagine changing q_k and $\dot{q}_k$ independently
- But $\partial \vec{r}_i / \partial q_k$ and $\partial \vec{r}_i / \partial t$ do not depend on $\{\dot{q}_l\}$ so

  $$\left(\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k}\right)_{\{\dot{q}_{l\neq k}\},\{q_l\},t} = \left(\frac{\partial \vec{r}_i}{\partial q_k}\right)_{\{q_{l\neq k}\},t}$$

- Be careful about what is held constant in partials!

Derivation of Generalized Equation of Motion

$$\sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \mathcal{F}_k, \qquad T = \sum_i \tfrac{1}{2} m_i \dot{\vec{r}}_i \cdot \dot{\vec{r}}_i \to T(\{q_k\}, \{\dot{q}_k\}, t)$$

Evaluate derivatives of T

$$\frac{\partial T}{\partial \dot{q}_k} = \sum_i m \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_k} = \sum_i \vec{p}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} \quad \text{(dot cancellation)}$$

$$\frac{\partial T}{\partial q_k} = \sum_i m \dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k} = \sum_i \vec{p}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k}$$

Differentiate first expression wrt t along path of dynamics $\{q_k(t)\}$

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) = \sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} + \sum_i \vec{p}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_k}\right)$$

First term is what we want, so look at the second term, remembering $\vec{r}_i = \vec{r}_i(\{q_k\}, t)$

Derivation of Generalized Equation of Motion

$$\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_k}\right) = \sum_l \frac{\partial^2 \vec{r}_i}{\partial q_l \partial q_k}\dot{q}_l + \frac{\partial^2 \vec{r}_i}{\partial t \partial q_k}$$

$$= \frac{\partial}{\partial q_k}\left(\sum_l \frac{\partial \vec{r}_i}{\partial q_l}\dot{q}_l + \frac{\partial \vec{r}_1}{\partial t}\right) = \frac{\partial \dot{\vec{r}}_i}{\partial q_k}$$

so that

$$\sum_i \vec{p}_i \cdot \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_k}\right) = \sum_i \vec{p}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial q_k} = \frac{\partial T}{\partial q_k}$$

This gives

$$\sum_i \dot{\vec{p}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_k} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k}$$

and so the result we want

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_k}\right) - \frac{\partial T}{\partial q_k} = \mathcal{F}_k \equiv \frac{\delta W}{\delta q_k}$$