ACM 100b

Basic aspects of boundary value problems

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Boundary value problems vs initial value problems

- So far we have explored linear ODE's but we have focused on the initial value problems.
- That is we looked at equations of the form

$$\frac{d\boldsymbol{x}}{dz} = A(z)\boldsymbol{x}$$

  where A is an $n \times n$ matrix
- In order to consider some unique solution we applied n conditions at some initial point typically taken to be $z = 0$:

$$\boldsymbol{x}(z = 0) = \boldsymbol{x}_0.$$

- Under these conditions, we showed the existence and uniqueness of solutions provided $A(z)$ obeyed various mild smoothness criteria.

Boundary value problems vs initial value problems

- In many applications, we are asked to solve the same ODE
$$\frac{d\boldsymbol{x}}{dz} = A(z)\boldsymbol{x},$$
- But we want the solution over some fixed domain $z_0 < z < z_1$ rather than some neighborhood of z_0
- Most importantly not all n conditions are given at $z = z_0$.
- Instead, some are applied at $z = z_0$ and some at $z = z_1$.
- Of course, for any hope of uniqueness in the linear case we must have a total of n such conditions.
- Such problems are called *boundary value problems* in contrast to the initial value problems we have studied up till now.
- In the cases we will study we will write the system as an n'th order ODE where n is typically 2
- But the theory we will present holds for the system as well.

Existence and uniqueness is harder to prove

- The theory of existence and uniqueness for boundary value problems is considerably more complicated.
- This is to be expected because many of the guarantees we had for initial value problems are not present for boundary value problems.
- Consider a simple example of a second order homogeneous ODE

$$y'' + p(z)y' + q(z)y = 0 \qquad z_0 \leq z \leq z_1$$

- Suppose we know the general solution:

$$y(z) = c_1 y_1(z) + c_2 y_2(z).$$

- We'll assume that the coefficient functions are nice and smooth for any z
- So we can be assured the functions $y_1(z)$ and $y_2(z)$ are also similarly nice and smooth in the region $z_0 \leq z \leq z_1$.

An example

- Suppose we ask for a solution subject to the following conditions:

$$y(z = z_0) = a, \qquad y(z = z_1) = b.$$

- This is different from what we have done previously
- We are now asking that the solution satisfy two conditions as before.
- But they both involve the value of the solution at the two boundary end points.
- Plugging these conditions in, we get a 2×2 system to solve for c_1 and c_2:

$$\begin{aligned} c_1 y_1(z_0) + c_2 y_2(z_0) &= a, \\ c_1 y_1(z_1) + c_2 y_2(z_1) &= b. \end{aligned}$$

- Whether this system has a solution depends on the values the solutions take on at the boundary.

Example cont'd

- Now consider solving this 2×2 linear system:

$$
\begin{aligned}
c_1 y_1(z_0) + c_2 y_2(z_0) &= a, \\
c_1 y_1(z_1) + c_2 y_2(z_1) &= b.
\end{aligned}
$$

- For example, suppose neither a or b are zero.
- In that case we will have a solution as long as

$$
\begin{vmatrix} y_1(z_0) & y_2(z_0) \\ y_1(z_1) & y_2(z_1) \end{vmatrix} \neq 0,
$$

- In turn this clearly depends on what happens at the boundary and the values the solutions take on there.
- Suppose, on the other hand we had $a = b = 0$.
- Then, in general, we would expect the trivial solution $y(z) = 0$
- This is because if the above determinant did not vanish, we would have to take $c_1 = c_2 = 0$ which is just the trivial solution

Boundary value problems depend on global information

- On the other hand, it might be that in some cases we did get that

$$\begin{vmatrix} y_1(z_0) & y_2(z_0) \\ y_1(z_1) & y_2(z_1) \end{vmatrix} = 0.$$

- This might happen depending on the equation and the locations of the boundary
- In that case we would get nontrivial solutions but they would not be unique.
- We see then that such problems are harder to analyze.
- They seem to depend on matrices such as

$$\begin{pmatrix} y_1(z_0) & y_2(z_0) \\ y_1(z_1) & y_2(z_1). \end{pmatrix}$$

which are really about *global information* as regards the solution.

Much more is known about linear IVP's

- In contrast, linear initial value problems depend on the Wronskian determinant.
- For example for a second order ODE initial value problem

$$y'' + p(z)y' + q(z)y = 0 \qquad z \geq z_0$$

the Wronskian is given by

$$W(z) = \begin{vmatrix} y_1(z_0) & y_2(z_0) \\ y_1'(z_0) & y_2'(z_0) \end{vmatrix} = 0.$$

- Abel's theorem guarantees that this never vanishes as long as the matrix coefficients of a linear system are smooth.
- Boundary value problems (BVP) turn up in many applications and we will explore quite a few of these in ACM 100c.