ACM 100b

Variation of parameters for linear second order ODE

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Inhomogeneous second order ODE

- Now consider the inhomogeneous linear ODE

$$y'' + p(x)y' + q(x)y = r(x)$$

- We know that the general solution of this equation must be of the form

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_{part}(x)$$

  where y_1 and y_2 are the two homogeneous solutions and y_{part} is the particular solution.
- Suppose you know the two homogeneous solutions
- Then the particular solution can be computed using this information.
- The method to do this is called *variation of parameters*

Variation of parameters

- Suppose we try a particular solution of the form

$$y_{part} = u_1(x)y_1(x) + u_2(x)y_2(x)$$

- Its derivatives are clearly given by

$$\begin{aligned} y_p'(x) &= u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2' \\ y_p''(x) &= [u_1''y_1 + 2u_1'y_1' + u_1y_1''] + [u_2''y_2 + 2u_2'y_2' + u_2y_2''] \end{aligned}$$

- Now we have two unknown functions u_1 and u_2 which seems like one too many.
- We really only need one so perhaps we can relate u_1 to u_2 to make life easier.
- Let's require that $u_1'y_1 + u_2'y_2 = 0$
- Then

$$\begin{aligned} y_p'(x) &= u_1y_1' + u_2y_2' \\ y_p''(x) &= [u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2'']. \end{aligned}$$

Variation of parameters

- Now substitute the derivatives into the ODE
- We get

$$[u_1'y_1' + u_1y_1''] + [u_2'y_2' + u_2y_2'']+$$
$$p(x)[u_1y_1' + u_2y_2'] + q(x)[u_1y_1 + u_2y_2] = r(x)$$

- But note y_1 and y_2 are homogeneous solutions
- So we have

$$u_1'y_1 + u_2'y_2 = 0$$
$$u_1'y_1' + u_2'y_2' = r(x).$$

- This is a linear system for the derivatives u' and v' and the determinant is the Wronskian $W(x)$
- So we can solve this system to get

$$u_1' = \frac{1}{\mathrm{W}(y_1, y_2)} \begin{vmatrix} 0 & y_2 \\ r & y_2' \end{vmatrix}, \quad u_2' = \frac{1}{\mathrm{W}(y_1, y_2)} \begin{vmatrix} y_1 & 0 \\ y_1' & r \end{vmatrix}$$

Variation of parameters

- This solution tells us

$$u_1' = \frac{-y_2 r}{\mathrm{W}}, \quad u_2' = \frac{y_1 r}{\mathrm{W}}.$$

- So we have the explicit solution

$$y_{part}(x) = -y_1(x) \int^x \frac{y_2(t) r(t)}{\mathrm{W(t)}} dt + y_2(x) \int^x \frac{y_1(t) r(t)}{\mathrm{W(t)}} dt.$$

- And our general solution is

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_{part}(x)$$

- This approach will always work if you have the two homogeneous solutions
- In fact it generalizes to n'th order ODE's too as we shall see

An example of variation of parameters

- As an example, consider

$$y'' + y = \sec(x), \quad 0 \leq x \leq \pi/2.$$

- The solutions to the homogeneous equation $y'' + y = 0$ and the corresponding Wronskian are given by

$$y_1 = \sin(x), \quad y_2 = \cos(x), \quad \mathrm{W} = \begin{bmatrix} \sin(x) & \cos(x) \\ \cos(x) & -\sin(x) \end{bmatrix} = -1.$$

- Thus, a particular solution is found as

$$\begin{aligned} y_p(x) &= \sin(x) \int^x \cos(t)\sec(t)dt - \cos(x) \int^x \sin(t)\sec(t)dt \\ &= x\sin(x) - \cos(x)\log(\cos(x)) \end{aligned}$$

- This gives the general solution

$$y(x) = c_1 \cos(x) + c_2 \sin(x) + x\sin(x) - \cos(x)\log(\cos(x)).$$