Physics 106a — 29 October, 2013

Lecture 9: Hamiltonian Mechanics

Hamiltonian equations of motion

In the Hamiltonian formulation of dynamics each second order ODE given by the Euler-Lagrange equation in terms of the Lagrangian is replaced by two first order ODEs for the coordinate and the conjugate momentum given by derivatives of the *Hamiltonian*.

The Hamiltonian is defined (as we have seen before) by

$$H(\{q_k\},\{p_k\},t)=\sum_k p_k\dot{q}_k - L(\{q_k\},\{\dot{q}_k\},t), \qquad p_k=\frac{\partial L}{\partial \dot{q}_k}, \tag{1}$$

where the last equation is inverted to give $\dot{q}_k$ in terms of other coordinates and momentum so that the Hamiltonian is investigated as a function of the independent variables $\{q_k\},\{p_k\},t$. This is an important point: the *Lagrangian* is considered to be a function of *coordinates* and *velocities* (and maybe time) whereas the *Hamiltonian* is considered to be a function of *coordinates* and *momenta* (and maybe time).

Forming the total differential dH from Eq. (1) gives

$$\frac{\partial H}{\partial p_k}=\dot{q}_k, \qquad \frac{\partial H}{\partial q_k}=-\frac{\partial L}{\partial q_k}, \qquad \frac{\partial H}{\partial t}=-\frac{\partial L}{\partial t}. \tag{2}$$

Remember in these partials the other independent variables are held fixed, so that $\partial L/\partial t$ means $(\partial L/\partial t)_{\{q_k\},\{\dot{q}_k\}}$ whereas $\partial H/\partial t$ means $(\partial H/\partial t)_{\{q_k\},\{p_k\}}$, etc. Together with the Euler-Lagrange equation, Eq. (2) gives us the *Hamiltonian equations of motion*

$$\dot{q}_k=\frac{\partial H}{\partial p_k} \tag{3}$$

$$\dot{p}_k=-\frac{\partial H}{\partial q_k} \tag{4}$$

as well as

$$\frac{dH}{dt}=\frac{\partial H}{\partial t}. \tag{5}$$

Since $\partial H/\partial t=-\partial L/\partial t$ we see (again) that the Hamiltonian is a constant of the motion if L (and so H) is not an explicit function of time.

Phase space

Plotting the $(\{q_k(t)\},\{p_k(t)\})$ *trajectories* in the $2N$ dimensional *phase space* for N degrees of freedom q_k gives a nice picture of the dynamics. A simple example is the θ, p_θ phase space of the pendulum — you should sketch trajectories for different energies, from small amplitude oscillations to high energy "running" solutions. Because trajectories cannot cross for a time independent Hamiltonian except at fixed points there are strong constraints on the possible dynamics: just fixed points, periodic orbits, and homo/heteroclinic orbits) in a two dimensional phase space — the *Poincaré-Bendixson theorem*.

Legendre transformations

The transformation $L(\{q_k\}, \{\dot{q}_k\}, t) \Rightarrow H(\{q_k\}, \{p_k\}, t)$ is an example of a *Legendre transformation* for changing independent variables. Most simply for a function $f(x)$ the Legendre transformation $f(x) \to B(s)$ takes the form

$$B(s) = xs - f(x) \qquad \text{with} \qquad s = \frac{df}{dx}\,, \tag{6}$$

where the last equation is inverted to calculate $x(s)$. (This is always possible if $f(x)$ is a *convex function*, i.e. the curvature is everywhere of the same sign.) Note the geometrical interpretation: s is the slope of the tangent to $f(x)$ at x, and $-B$ is the intercept of this tangent with the ordinate — see Hand and Finch, Fig. 5.3. Thus $B(s)$ specifies the same curve in terms of the tangents defined in slope-intercept form. The inverse transformation takes exactly the same form

$$f(x) = sx - B(s) \qquad \text{with} \qquad x = \frac{dB}{ds}\,. \tag{7}$$

where the last relation can be seen by explicitly differentiating $B(s)$ in Eq. (6). A Legendre transformation (rather than say $C(z) = f(x(z))$ for some arbitrary choice of $z(x)$) has the advantage that for a convex function the inverse can always be found and no information about the function is lost. Legendre transformations are used in thermodynamics, e.g. from Helmholtz to Gibbs free energies

$$F(T, V) \Rightarrow H(T, P) = F + PV \qquad \text{with } P = -\partial F/\partial V, \tag{8}$$

(the sign differences from Eq. (6) are inessential differences due to conventions in the definitions of the various quantities).

Ignorable coordinates and the Routhian

The Hamiltonian formulation is particularly simple for ignorable coordinates. For an ignorable coordinate q_m we know $\partial L/\partial q_m = 0$ which implies, using Eq. (2), $\partial H/\partial q_m = 0$. The equations of motion for this degree of freedom are

$$\dot{p}_m = -\frac{\partial H}{\partial q_m} = 0 \qquad p_m \text{ a constant of the motion,} \tag{9}$$

$$\dot{q}_m = \frac{\partial H}{\partial p_m}. \tag{10}$$

This simplicity can be exploited by doing a partial Legendre transformation over the ignorable coordinates to form the *Routhian* which effectively reduces the number of degrees of freedom that have to be studied. Arrange the order of coordinates so that $q_{s+1} \dots q_N$ are ignorable and define

$$\mathcal{R}(q_1, q_2, \dots q_s; \dot{q}_1, \dot{q}_2, \dots \dot{q}_s, p_{s+1}, \dots p_N) = \sum_{i=s+1}^{N} p_i \dot{q}_i - L. \tag{11}$$

Note there is no dependence on $q_{s+1} \dots q_N$ since these are ignorable, and the momenta appearing in $\mathcal{R}$ are constant. Thus the Routhian $\mathcal{R}$ acts as the Lagrangian for the remaining s variables, giving the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{R}}{\partial \dot{q}_k} - \frac{\partial \mathcal{R}}{\partial q_k} = 0, \quad k = 1 \dots s. \tag{12}$$

As an example, consider the Kepler problem. We reduced the problem to the Lagrangian for the coordinates r, ϕ in the plane perpendicular to the angular momentum, with the Lagrangian

$$L = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\phi}^2) + \frac{k}{r}, \tag{13}$$

and $p_\phi = \mu r^2\dot{\phi}$ is constant. Remember from Lecture 7 that I said it is *incorrect* to eliminate $\dot{\phi}$ from the Lagrangian in favor of the constant p_ϕ: we now know that we should do a Legendre transformation to allow this substitution, i.e. form the Routhian $\mathcal{R} = p_\phi\dot{\phi} - L$ and *then* eliminate $\dot{\phi}$ using p_ϕ. This gives

$$\mathcal{R}(r, \dot{r}, p_\phi) = -\left[\frac{1}{2}\mu\dot{r}^2 + V_{\text{eff}}\right] \quad \text{with} \quad V_{\text{eff}} = \frac{k}{r} - \frac{p_\phi^2}{2\mu r^2}, \tag{14}$$

which *can* now be used as the effective Lagrangian for the r coordinate. Applying Eq. (12) to the radial coordinate gives

$$\mu\ddot{r} = -\frac{dV_{\text{eff}}}{dr} = \frac{p_\phi^2}{\mu r^3} - \frac{k}{r^2}. \tag{15}$$

The Hamiltonian equations for ϕ, p_ϕ are

$$\dot{p}_\phi = 0, \quad \dot{\phi} = \frac{p_\phi}{\mu r^2}. \tag{16}$$

The procedure is not much different from what we used before, but perhaps makes the calculation more natural, e.g. the effective potential appears in a straightforward way in Eqs. (14,15).

Hand and Finch mention the Routhian on p23, and then in a number of problems (see the index).

Hamilton's principle

Hamilton's principle can be expressed as finding the stationary value of the action written in terms of H

$$S = \int \sum_k [p_k\dot{q}_k - H(\{q_k\}, \{p_k\}, t)]\, dt \tag{17}$$

for *independent variations* of $\{q_k\}, \{p_k\}$ with δq_k zero at the beginning and end as usual, but *no* restriction on δp_k at the beginning and end (see slides).

Michael Cross, October 28, 2013