Math 1b - Notes

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Contents

Chapter 1

Introduction/Vector Spaces - January 7

Getting over 90% on all homework sets and on the midterm gives exemption from the final. The grade will be split evenly between midterm, final, and eight homework sets.

Know notation and terminology:

- Sets - Part 2 of Intro to Vol I of Apostol
- $\forall$ - For all
- $\exists$ - There exists
- Functions - A function f mapping a set X to a set Y such that f maps $\forall x \in X$ to exactly one element of Y called the image of x under f.
- $f : X \to Y$ - Denotes a function mapping X to Y
- $f(x) = y, f : x \mapsto y$ - Denotes the image of x as y under f.
- $X \times Y$ - Set product: $\{(x, y) : x \in X, y \in Y\}$.
- (x, y) - Ordered pair characterised by $(x_1, y_1) = (x_2, y_2)$ if and only if $x_1 = x_2, y_1 = y_2$.

A vector space is a set V with some operations on V both of which satisfy axioms given in the text. We first discuss binary operations, which is a mapping $A \times A \to A$. Given $a, b \in A$, call $a * b$ the "product" or "sum" of a and b. An example then of a vector space is $\mathbb{R}$, over which addition is a binary operation, or $\mathbb{Z}$ over which multiplication is a binary operator.

If we then let X be a set and A be the set of all functions $X \to X$, then note that $f \circ g$ is also a function that maps $X \to X$ and thus $\circ \in A$.

We then give the field $F = \mathbb{R}$ or $F = \mathbb{C}$. Let m, n be positive integers. We can then create an $m \times n$ matrix over F, where elements are indexed by ordered pair (i, j) where $1 \leq i \leq m$ and $1 \leq j \leq n$. This yields a rectangular array for our matrix! Yay. The rows and columns are indexed by i and j respectively. We can then define $M_{m,n}$ as the set of all $m \times n$ matricies. We can then add two matricies from $M_{m,n}$ by adding element-wise.

We then look at matrix products, where we examine square matricies $m = n$ and so $M_n = M_{n,n}$. We then define the product of $a_{ij} \cdot b_{ij} = \sum a_{ik} b_{kj}$.

We can then construe the following axioms for binary operations for some operation $*$ on A. We then define:

- Associativity - $a * (b * c) = (a * b) * c$. All examples thusfar are associative. Note that associativity is critical for expressions like $a*b*c$ to make sense, because while $*$ is only defined as a binary operator, we can write $(a*b)*c$.

Chapter 2

- January 11

Let $F = \mathbb{R}$ or $F = \mathbb{C}$. Let $m, n \in \mathbb{N}$, and $M_{m,n}$ to be the set of all $m \times n$ matricies. Matrix addition and scalar multiplication follow intuitively, and $M_{m,n}$ is thus a vector space over F.

We know that for any vector space V that the 0 vector is unique, as is the inverse $-v$ of any element $v \in V$. We then discuss another theorem:

Theorem 1.3: Let $a, b \in F$ and $u, v \in V$. Then:

1. $av = 0$ if and only if $a = 0$ or $v = 0$.
2. $(-a)b = -(ab)$.
3. If $v \neq 0$ and $av = bv$ then $a = b$.
4. If $a \neq 0$ and $av = au$ then $v = u$.
5. For $n > 0$ we have $nv = v + v + ... + v$ where the sum is carried on n times.

We now discuss subspaces. A subspace U of any vector space V is a nonempty subset of V that is closed under addition and multiplication. We then have the following theorem:

Theorem 1.4: If U is a subspace of V then the restriction of the addition and scalar multiplication on V to U makes U a vector space, that is, U and V are closed under the same operations.

We then note that for some $u \in U$, then by Theorem 1.3 $0 \cdot u = 0$, and so $0 \in U$, because U must be closed under multiplication by scalars.

We now discuss linear span. Let $S \leq V$ (note just subset, not necessarily space). The **linear span** of S is $L(S)$ which is the set of all linear combinations of all vectors in S. By convention, $L(\emptyset) = 0$, which is a subspace. As an example, we note that for $u, v \in V$ that $L(v) = Fv$ where $F = \mathbb{R}$ OR $\mathbb{C}$, and $L(u, v) = au + bv, a, b \in F$. We now arrive at a lemma:

Lemma 1D: For $S \leq V$, $L(S)$ is the smallest subspace of V containing S.

Note that this lemma cannot be directly cited when asked to prove said lemma, because this will be a future homework problem. We now discuss linear independence; a subset S of V is **linearly dependent** if there exists a nonempty finite subset $\{S_1, ...S_n\} \in S$ and scalars $a_1, ...a_n \in F$ such that not all scalars are 0 and $a_1S_1 + ... + a_nS_n = 0$. A set is linearly independent if it is not dependent. For example, the empty set is linearly independent (note definition specifies nonempty). On the other hand, $\{0\}$ is linearly dependent. Lastly, $\{x\}, x \in V_{\neq 0}$ is linearly independent.

Chapter 3

A very long hiatus later...matricies - January 28

We have a typical setup, $F = \mathbb{R}, \mathbb{C}$, either. We have a vector space V of n dimensions, F-space with ordered basis $X = x_i$. We have $L = L(V)$ which is the spaceof linear maps, and M_n. Lastly, for $f \in L, m_x(f) \in M_n$ is the matrix of f with respect to x.

We introduce (okay, maybe they've been introduced already, but they're new to me) two small theorems. Th 2.15 - $m_x : L \to M_n$ is an isomorphism. Th 2.16: For $f, g \in L, m_x(f \circ g) = m_x(f) \cdot m_x(g)$.

We then discuss Th 2H: Let $m = m_x : L \to M_n, f \in L, A = m(f)$. We then have $m(id_v) = I$ the identity matrix, f has an inverse iff A has an inverse, and if f has an inverse then $m(f^{-1}) = m(f)^{-1} = A^{-1}$, and m^{-1} is a linear operator. We then discuss a few unnoteworthy proofs of these theorems.

Note that 2.15 tells us that m is an isomorphism and a 1-1 correspondence, and so there must exist an inverse $m^{-1} : M_n \to L$, which still obeys $m^{-1}(A) \cdot m^{-1}(B) = m^{-1}(AB)$.

We now discuss a change of coordinates. Since m_x was a map in X coordinates, what happenes if we use a second basis Y in V? What is the relationship between $m_x(f), m_y(f)$. This is given by Th 4.6: Let g be the unique member of L such that $g(x_i) = y_i$. Then $g : V \to V$ is an isomorphism, so $B = m_x(g)$ has a unique inverse B^{-1}, and $m_y(f) = B^{-1}m_x(f)B$. More proofs come, but one noteworthy aspect is that Th 2.12 tells us that g is unique.

We call matricies $A, C \in M_n$ similar if there exists an invertible matrix $B \in M_n$ such that $A = B^{-1}CB$. Th 4.8 then says that the two statements A, C are similar and that there exist X, Y, f such that $m_x(f) = A, m_y(f) = C$. More proofs follow. Zzz...

We now discuss non-square matricies. Define the transpose of $A = a_{i,j} \in M_{m,n}$ to be $A^T = a^t_{i,j} = a_{j,i}$, e.g. transpose of a row vector is a column vector and vice versa. Notationally, define A_i to be the i-th row vector and $A^{(j)}$ to be the transpose of the j-th column vector. The column space of A is then $L(A^{(1)}, ...A^{(n)})$. The rank is then defined as the dimension of the column space, denoted as (A).