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Lecture 14: Rotating Frames

In the last lecture we looked at rotations of coordinate axes or vectors. In the present lecture we use what we learned there to study the different description of dynamics in frames of reference that are *rotating* relative to one another. We will look at the application to the dynamics in a noninertial rotating frame by comparing with the dynamics in an inertial frame. In the next lecture we will use the ideas to study rigid body dynamics: in that case one frame is fixed to the rotating body, and the other is again an inertial frame.

My approach is based on the following principles:

- The rotation of frames relative to one another can be defined without using coordinate axes.
- Vectors may be defined physically/geometrically without reference to particular coordinate axes.
- Some physical quantities and the vectors that represent them are the same in relatively rotating frames, some are different

  For example although the vector displacement $\vec{r}$ from the origin (taken on the axis of rotation) to a point P fixed in the rotating frame is the *same* in the rotating frame and the nonrotating frame, the velocity of the point P is *different* ($\vec{0}$ in the rotating frame and $\vec{\omega} \times \vec{r}$ in the non-rotating frame, with $\vec{\omega}$ the angular velocity of the rotating frame with respect to the nonrotating frame) – the usual (local) Galilean transformation.
- We can understand and calculate the relationship between vectors in relatively rotating frames, and between their time dependences, without introducing components with respect to particular choices of axes.
- We can evaluate components of vectors (defined in either frame of reference, if there is a difference) with respect to any choice of coordinate axes (fixed in one or other frames of reference, rotating in some different way, or even (perversely) axes with accelerating rate of rotation...). Of course some may be more natural and give simpler expression than others (e.g. axes fixed in one or other frame).

Hand and Finch prefer to relate the time dependences of vectors in relatively rotating frames by deriving relationships for the *components* of the vectors with respect to various choices of basis vectors. I prefer to work directly with the vectors. To facilitate the writing they also define a "bold vector" **a** representing the three components (a_1, a_2, a_3) of $\vec{a}$ relative to the unprimed basis. I will not use this notation.

Time dependence of vectors in relatively rotating frames

In the applications we will consider a "rotating frame of reference" and an "inertial frame frame of reference". I will use notation derived from this, but for now, all that matters is that there are two frames, rotating relative to one another. We can establish this without defining a choice of basis: we know we are in a rotating frame on the Earth, without any orthogonal triad of basis vectors to aid us! We therefore consider two frames: a frame called the *space* frame, and a frame called the *body* frame rotating at angular velocity $\vec{\omega}$ with respect to this one. The space frame will be chosen as the inertial frame later on.

Consider a vector $\vec{a}$ [1]. For example $\vec{a}$ may be the vector $\vec{r}$ from the origin to a point P. We want to compare the time dependence of $\vec{a}$ in the two different frames. I emphasize that $\vec{a}$ is the *same* vector, representing the same physical object, in the two frames. For example at any time we would draw the *same* arrow representing the vector in the body frame and the space frame. The time dependence is different in the two frames however: for example the radius vector $\vec{r}$ to a point P that is fixed in the body frame is time independent in the body frame but time dependent (rotating) in the space frame. On the other hand the vector to a point Q fixed in the space frame is time independent in the space frame and time dependent in the body frame.

Consider an infinitesimal time δt. In this time, in the body frame $\vec{a}$ changes by $\delta\vec{a}|_{\mathrm{b}}$ if it is time dependent in the body frame. In the space frame there is an additional change $\delta\vec{\phi} \times \vec{a}$ due to the rotation of $\vec{a}$ with $\delta\vec{\phi}$ the infinitesimal rotation of the body frame in time δt, so that in the space frame

$$\delta\vec{a}|_{\mathrm{s}} = \delta\vec{a}|_{\mathrm{b}} + \delta\vec{\phi} \times \vec{a}\,. \tag{1}$$

Dividing by δt and taking $\delta t \to 0$ gives the relationship between the time derivatives in the two frames

$$\left.\frac{d\vec{a}}{dt}\right|_{\mathrm{s}} = \left.\frac{d\vec{a}}{dt}\right|_{\mathrm{b}} + \vec{\omega} \times \vec{a}\,. \tag{2}$$

This simply says that in the space frame there are two terms contributing to the rate of change of $\vec{a}$: one is just the rate of rotation of $\vec{a}$, present even if $\vec{a}$ is fixed (time independent) in the rotating frame, and the second is from the time derivative of $\vec{a}$ in the rotating frame.

Note that nothing so far depends on the space frame being inertial: the result applies between one frame (space) and another frame (body) rotating at angular velocity $\vec{\omega}$ relative to it. Indeed we could equally well write Eq. (2) as the inverse relation

$$\left.\frac{d\vec{a}}{dt}\right|_{\mathrm{b}} = \left.\frac{d\vec{a}}{dt}\right|_{\mathrm{s}} + (-\vec{\omega}) \times \vec{a} \tag{3}$$

which is exactly the same form since the space frame is rotating at angular velocity $-\vec{\omega}$ relative to the body frame.

As an example consider $\vec{a}$ to be $\vec{r}$, the vector between the origin and a moving point in the body frame. Then Eq. (2) becomes a relationship between the velocities of the point in the two frames

$$\vec{v}_{\mathrm{s}} = \vec{v}_{\mathrm{b}} + \vec{\omega} \times \vec{r}\,. \tag{4}$$

Of course we expect the velocities of the point P in the two frames to be different – Eq. (4) is just the local Galilean boost adding the local velocity of the body frame at the point $\vec{r}$, coming from the rotation, to the motion with respect to the body frame. The vectors $\vec{v}_{\mathrm{s}}$ and $\vec{v}_{\mathrm{b}}$ are *different* vectors corresponding to different physical objects. From the perspective of the space frame we would say $\vec{v}_{\mathrm{b}}$ is the velocity of a point relative to the local velocity of the rotating frame, and $\vec{v}_{\mathrm{s}}$ is the "actual" velocity of the point.

[1] Note that here $\vec{a}$ is a general vector, not the acceleration.

Component notation

As I said above, Hand and Finch derive the results by first expressing the vectors in component form: refer the vector $\vec{a}$ to some choice of axes, e.g. ones fixed in the body frame, consider the time dependent relationship of such axes relative to the space axes to give components along axes fixed in the space frame, and re-express in vector notation. Here's a simplified version of this approach (from Thornton and Marion, example 10.1).

Consider a vector $\vec{a}$ with components $a_i(t)$ with respect to axes $\hat{\imath} \equiv \hat{e}_1$, $\hat{\jmath} \equiv \hat{e}_2$, $\hat{k} \equiv \hat{e}_3$ fixed in the body frame

$$\vec{a} = a_i \hat{e}_i \tag{5}$$

(remember the Einstein summation convention). In the space frame

$$\left.\frac{d\vec{a}}{dt}\right|_{\mathrm{s}} = \frac{da_i}{dt}\hat{e}_i + a_i \left.\frac{d\hat{e}_i}{dt}\right|_{\mathrm{s}} \tag{6}$$

where in the space frame the unit vectors $\hat{e}_i$ are time dependent due to the rotation of the body frame relative to the space frame

$$\left.\frac{d\hat{e}_i}{dt}\right|_{\mathrm{s}} = \vec{\omega} \times \hat{e}_i \tag{7}$$

(you can see this result by drawing some pictures, and considering infinitesimal rotations about the each axis, etc.). Therefore

$$\left.\frac{d\vec{a}}{dt}\right|_{\mathrm{s}} = \frac{da_i}{dt}\hat{e}_i + \vec{\omega} \times (a_i \hat{e}_i) \tag{8}$$

$$= \left.\frac{d\vec{a}}{dt}\right|_{\mathrm{b}} + \vec{\omega} \times \vec{a} \tag{9}$$

the same result as before.

Physics in rotating frames

We now apply these ideas by deriving the equations of motion in a rotating frame in vector form starting from Newton's law of motion in a non-rotating (inertial) frame.

Newton's law of acceleration is true in inertial frames. It is not true in frames rotating with respect to the inertial frames. It is not understood what physics picks out one particular frame (and all frames moving at a fixed velocity relative to it) out of all the relatively rotating frames as the inertial set. We know that the Earth is a rotating noninertial frame, and that the frame of the "fixed stars" or 4° background radiation remanent from the Big Bang can be considered to be nonrotating to an excellent approximation. But we (or at least I) don't understand any causal reason for this. See Hand and Finch §8.12 for a discussion of this question, and one idea (Mach's Principle), not yet verified, for answering it.

Acceleration

To find the relationship of the accelerations in the two frames we take the time derivative of Eq. (4) using Eq. (2). Let's first assume $\vec{\omega}$ is constant. Then

$$\vec{a}_{\mathrm{s}} = \left.\frac{d\vec{v}_s}{dt}\right|_{\mathrm{s}} = \left.\frac{d}{dt}\right|_{\mathrm{s}} (\vec{v}_{\mathrm{b}} + \vec{\omega}\times\vec{r}) = \left.\frac{d\vec{v}_{\mathrm{b}}}{dt}\right|_{\mathrm{s}} + \vec{\omega}\times\left.\frac{d\vec{r}}{dt}\right|_{\mathrm{s}} . \tag{10}$$

Now use Eq. (2) on the two time derivatives:

$$\left.\frac{d\vec{v}_{\mathrm{b}}}{dt}\right|_{\mathrm{s}} = \left.\frac{d\vec{v}_{\mathrm{b}}}{dt}\right|_{\mathrm{b}} + \vec{\omega}\times\vec{v}_{\mathrm{b}} \tag{11}$$

$$\left.\frac{d\vec{r}}{dt}\right|_{\mathrm{s}} = \left.\frac{d\vec{r}}{dt}\right|_{\mathrm{b}} + \vec{\omega}\times\vec{r} \tag{12}$$

and then $\vec{a}_{\mathrm{b}} = d\vec{v}_{\mathrm{b}}/dt|_{\mathrm{b}}$ and $\vec{v}_{\mathrm{b}} = d\vec{r}/dt|_{\mathrm{b}}$. This gives

$$\vec{a}_{\mathrm{s}} = \vec{a}_{\mathrm{b}} + 2\vec{\omega}\times\vec{v}_{\mathrm{b}} + \vec{\omega}\times(\vec{\omega}\times\vec{r}). \tag{13}$$

For a time dependent $\vec{\omega}$ there is an additional term $\dot{\vec{\omega}}\times\vec{r}$ on the right hand side with $\dot{\vec{\omega}}$ the time derivative of the angular velocity of the body frame relative to the space frame.

Fictitious or inertial forces

Now we implement the assumption that the space frame is inertial, and the body frame noninertial. In the space frame Newton tells us

$$\vec{F} = m\vec{a}_s . \tag{14}$$

Equation (13) then gives

$$m\vec{a}_{\mathrm{b}} = \vec{F} - 2m\vec{\omega}\times\vec{v}_{\mathrm{b}} - m\vec{\omega}\times(\vec{\omega}\times\vec{r}) - m\dot{\vec{\omega}}\times\vec{r} \tag{15}$$

We have moved the purely kinematic "correction" terms to the acceleration onto the right hand side, and we describe them as *fictitious forces*: Coriolis, centrifugal, and Euler. On the other hand, remembering Einstein taught us the equivalence of gravity and acceleration, it may seem perverse to separate effects into "real" and "fictitious", and so the term *inertial forces* is also used.

See Hand and Finch §7.8-10 for the Foucault pendulum, tower of Pisa, and hurricane examples.

Lagrangian and Hamiltonian approach

It may be worth remembering at this point that we can use the Lagrangian or Hamiltonian approach to avoid these subtleties of rotating vectors. We evaluate the kinetic energy in the inertial frame in terms of the coordinates defined in the rotating frame. Equation (4) is sometimes useful to calculate v_s^2 in terms of the time dependence of coordinates in the body frame, but this can be avoided by using polar coordinates, for example. These approaches have been used in various assignments, and in class earlier, but I pull together the results here.

Let's set up generalized coordinates as the cylindrical polar coordinates (r, ϕ, z) in the body frame with the z-axis along the direction of the angular velocity of the body frame relative to the space frame. The kinetic energy must be evaluated in an inertial frame and is

$$T = \frac{1}{2}m\left[\dot{r}^2 + r^2(\dot{\phi}+\omega)^2 + \dot{z}^2\right], \tag{16}$$

and then the Lagrangian is $L = T - V$. The Euler-Lagrange equations are

$$\frac{d}{dt}(m\dot{r}) - mr(\dot{\phi}+\omega)^2 + \frac{\partial V}{\partial r} = 0\,, \tag{17}$$

$$\frac{d}{dt}[mr^2(\dot{\phi}+\omega)] + \frac{\partial V}{\partial \phi} = 0\,, \tag{18}$$

$$\frac{d}{dt}(m\dot{z}) + \frac{\partial V}{\partial z} = 0\,. \tag{19}$$

Rearrange the terms to put the ω dependent terms and the forces on the right hand side:

$$m\ddot{r} - mr\dot{\phi}^2 = 2m\omega r\dot{\phi} + mr\omega^2 + F_r\,, \tag{20}$$

$$mr\ddot{\phi} + 2m\dot{r}\dot{\phi} = -2m\omega\dot{r} + F_\phi\,, \tag{21}$$

$$m\ddot{z} = F_z\,. \tag{22}$$

with $\vec{F} = -\vec{\nabla}V$ the force with polar components F_r, F_ϕ, F_z, with for example $F_\phi = -r^{-1}\partial V/\partial\phi$. The terms on the left hand side are the acceleration as would be evaluated in polar coordinates in the absence of rotation. The extra terms on the right hand side come from the kinetic energy, and can be seen to be the Coriolis and centrifugal fictitious forces in polar coordinates. Note in particular that the fictitious forces appear naturally in the Lagrangian approach, where is is easy to evaluate the kinetic energy in an inertial frame in terms of coordinates defined in a noninertial frame. Since the Lagrangian is constructed in an inertial frame, the centrifugal and Coriolis forces *must not* be added by hand as terms in the potential in this approach.

The Hamiltonian is

$$H = p_r \dot{r} + p_\phi \dot{\phi} + p_z \dot{z} - L. \tag{23}$$

The interesting term is $p_\phi \dot{\phi}$ with

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = mr^2(\dot{\phi} + \omega) \tag{24}$$

so that

$$p_\phi \dot{\phi} = mr^2(\dot{\phi} + \omega)\dot{\phi} = mr^2(\dot{\phi} + \omega)^2 - mr^2(\dot{\phi} + \omega)\omega. \tag{25}$$

Using this, and the simple expressions for p_r, p_z gives the Hamiltonian

$$H = \frac{p_r^2 + p_z^2}{2m} + \frac{p_\phi^2}{2mr^2} + V - \omega p_\phi \tag{26}$$

$$= H_{\omega=0} - \omega p_\phi \tag{27}$$

so that the only change to the Hamiltonian from using coordinates in the rotating frame is the extra term $-\vec{\omega} \cdot \vec{l}$ with $\vec{l}$ the angular momentum (e.g. $l_z = p_\phi = mr^2(\dot{\phi} + \omega)$).

Michael Cross, November 13, 2013