Mass Variations
Exercise Sheet 2

Elaboration of:

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Mass Variations

Exercise 2: Earth model of Wiechert

Wiechert's (1897) two-layer Earth model consists of a homogeneous spherical Earth core with radius R_K and a homogeneous Earth mantle and is described by the following parameters:

- Radius of the Earth's core: R_K = 3100 km
- Constant density of the Earth's core: ρ_K = 12,3 g·cm^{-3}
- Outer radius of the mantle: R_M = 6371 km (= Earth's surface)
- Constant density of the mantle: ρM = 4,8 g·cm^{-3}

Newton's gravitational constant is: $G = 6{,}672 \cdot 10^{-11}$ m^3·kg^{-1}·s^{-2}.

Calculate:

a) the total $M = M_K + M_M$ of the Earth in Wiechert's Earth model,

b) the constant mean density ρ of a homogeneous Earth with the same total mass M,

c) the gravitational potential and its first derivative in steps of 1000 km for $0 \le r \le 10000$ km and at the density limits R_K and R_M in each case for

c.1) the two-layer Earth model according to Wiechert and

c.2) the homogeneous Earth model with the mean density ρ.

State the results of subtasks a) and b) to five significant digits, those of subtask c) with a calculation accuracy of eight significant digits!

Plot the results of c.1) and c.2) (label the graphs clearly!).

Plot the potential curves of both earth models in a joint graph.

Also create a graph in which the first derivatives of both models are shown.

Indicate the core-mantle boundary and the Earth's surface (spherical boundary) in the diagrams.

Note and state the signs and units!

Discuss your results. In particular, discuss the locations of the density jumps and the quantitative behavior of the potential and first derivative in the radial direction.

a) Total mass of the earth in Wiechert's Earth model

Emil Johann Wiechert presented the first verifiable model of a layered structure of the Earth. The two-layer Earth model consists of a homogeneous spherical Earth core with radius RK and a homogeneous Earth mantle.

The total mass of the earth can be obtained by adding mass of core and mantle (1.1),

$$M = M_K + M_M \tag{1.1}$$

and the mass can get from eq(1.2)

$$M = \rho V \tag{1.2}$$

The volume of two parts can get from the following formula (1.3) and (1.4)

$$V_K = \frac{4\pi {R_K}^3}{3} \tag{1.3}$$

$$V_M = \frac{4\pi({R_M}^3 - {R_K}^3)}{3} \tag{1.4}$$

All the values of constant in the formula above is shown as:

Radius of the Earth's core: RK = 3100 km

Constant density of the Earth's core: ρK = 12,3 g·cm-3

Outer radius of the mantle: RM = 6371 km (= Earth's surface)
Constant density of the mantle: ρM = 4,8 g·cm-3

Finally, the Mass of the two-layer Earth model is 6.1353e+24 kg.

b) the constant mean density ρ of a homogeneous Earth with the same total mass M

The mean density of the homogeneous Earth can be calculated by eq(1.2) and (1.3), and the answer is $5.6640*10^3$ kg/m^3

c.1) two-layer Earth model

c.1.1) gravitational potential

It can be divided into 3 parts: core layer, mantle layer and outside space

c.1.1.1) core layer

In this part, it's influenced by two parts. The first part can be seen as inside of a homogeneous solid sphere, with the radius as R_K, which can be calculated as eq(3.1)

$$V = 2\pi G\rho(R^2 - \frac{1}{3}r^2) \tag{3.1}$$

In the eq(3.1), *G* = 6,672 ·10-11 m3·kg-1·s-2, R is the radius of earth core R_K.

The second part can be seen as inside of a homogeneous spherical shell, it can be calculated as eq(3.2)

$$V = 2\pi G\rho({R_M}^2 - {R_K}^2) \tag{3.2}$$

which is a constant value.

c.1.1.2) mantle layer

In the mantle layer, it can also be obtained by two parts, inside of a homogeneous spherical shell (3.3) and outside of a homogeneous solid sphere(3.4).

$$V = 2\pi G\rho({R_M}^2 - \frac{2}{3r}{R_K}^3 - \frac{1}{3}r^2) \tag{3.3}$$

$$V = \frac{GM}{r} \tag{3.4}$$

c.1.1.3) outside space

In the outside space, we can imagine the earth as a mass point, the mass of it is calculated in (1.1). The gravitational potential also can get from eq(3.4).

The result of the gravitational potential of two-layer Earth model is shown in table 1 and fig 1.

Table 1 Gravitational potential of two-layer Earth model

Distance(km)	**Gravitational potential(m^2/s^{-2})**	**Distance(km)**	**Gravitational potential(m^2/s^{-2})**
0	111890460	6000	67936210
1000	110171680	**6371**	64251696
2000	105015350	7000	58478222
3000	96421461	8000	51168444
3100	95373006	9000	45483062
4000	86554732	10000	40934755
5000	77395847		

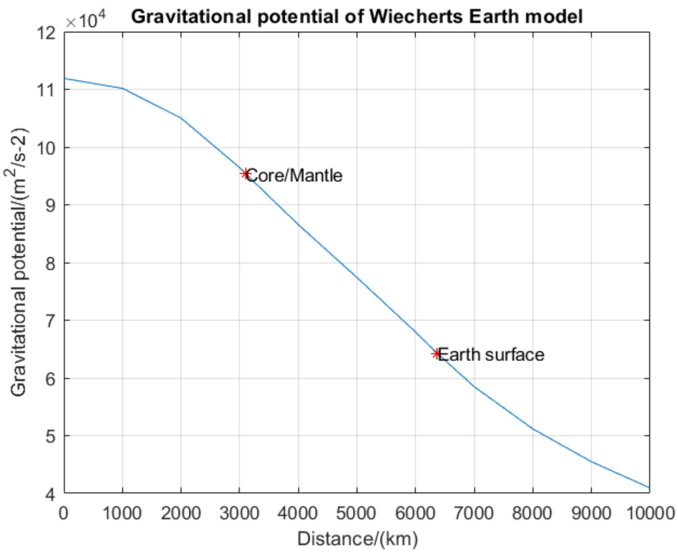

Figure 1 Gravitational potential of two-layer Earth model

c.1.2) first derivative

c.1.2.1) core layer

The first derivative of gravitational potential in the core layer can be calculated from two parts. On the one hand, it's in interior of a homogeneous solid sphere, the formula is below (3.5)

$$\frac{\partial V}{\partial r} = -\frac{4\pi G\rho}{3} \cdot r \tag{3.5}$$

on the other hand, it's in the cavity of the homogeneous spherical shell (3.6)

$$\frac{\partial V}{\partial r} = 0 \tag{3.6}$$

Considering the influence of the spherical shell is 0, we can get the answer only by the eq(3.5).

c.1.2.2) mantle layer

In this layer, it's also influenced by two parts. Firstly, it's outside of a homogeneous solid sphere (3.7)

$$\frac{\partial V}{\partial r} = -\frac{GM}{r^2} \tag{3.7}$$

What's more, it's inside of a homogeneous spherical shell, with the radius of R_K and R_M. The formula is shown as (3.8)

$$\frac{\partial V}{\partial r} = -\frac{4\pi G\rho}{3} \cdot \left(r - \frac{{R_K}^3}{r^2}\right) \tag{3.8}$$

Finally, add the two parts up and we can get the results.

c.1.2.3) outside space

In the outside of the earth, we can process it as (3.7), even there're two layers, we can still consider it as point with all the mass at the centre of the earth.

The result of the first derivative of gravitational potential of two-layer Earth model is shown in table 2 and fig 2.

Table 2 First derivative of gravitational potential of two-layer Earth model

Distance(km)	First derivative (m/s^{-2})	Distance(km)	First derivative (m/s^{-2})
0	0	6000	-9.7834678
1000	-3.4375558	**6371**	-10.085025
2000	-6.8751116	7000	-8.3540317
3000	-10.312667	8000	-6.3960555
3100	-10.656423	9000	-5.0536735
4000	-9.2686933	10000	-4.0934755
5000	-9.2051876		

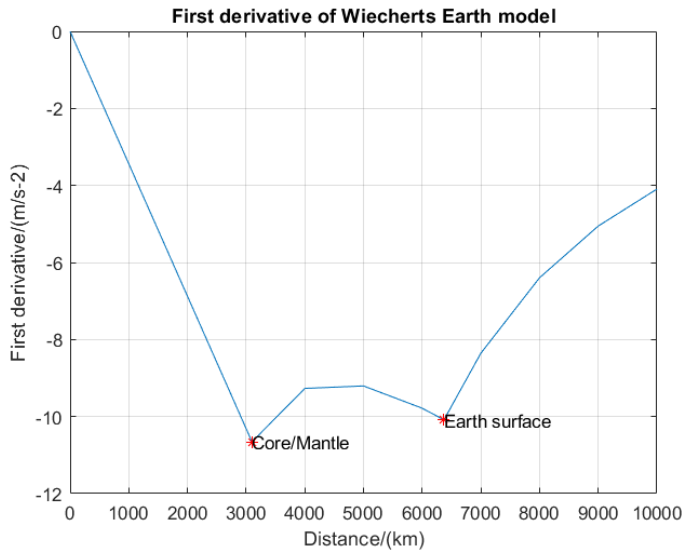

Figure 2 First derivative of gravitational potential of two-layer Earth model

c.2) homogeneous Earth model

c.2.1) gravitational potential

It can be divided into 2 parts: inside of Earth and outside space

c.2.1.1) inside of Earth

It's can be seen as inside of a homogeneous solid sphere, which can get from eq(3.1).

c.2.1.2) outside space

In the outside space, we can also imagine the earth as a mass point, the mass of it is calculated in (1.1). The gravitational potential also can get from eq(3.4).

The result of the gravitational potential of homogeneous Earth model is shown in table 3 and fig 3.

Table 3 Gravitational potential of homogeneous Earth model

Distance(km)	Gravitational potential(m^2/s^{-2})	Distance(km)	Gravitational potential(m^2/s^{-2})
0	96377544	6000	67884300
1000	95586065	**6371**	64251696
2000	93211628	7000	58478222
3000	89254233	8000	51168444
4000	83713880	9000	45483062
5000	76590569	10000	40934755

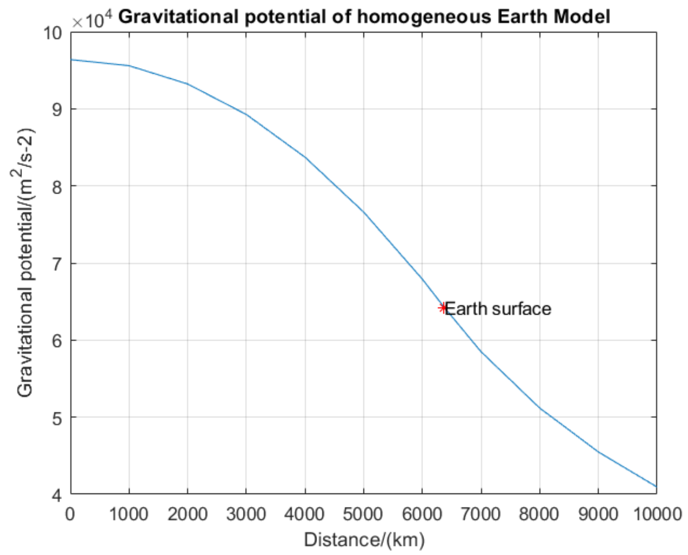

Figure 3 Gravitational potential of homogeneous Earth model

c.2.2) first derivative

c.2.2.1) inside of Earth

It's can be seen as inside of a homogeneous solid sphere and calculated by eq(3.5)

c.2.2.2) outside space

We can also consider the earth as a point and get the result by eq(3.7).

The result of the first derivative of gravitational potential of homogeneous Earth model is shown in table 4 and fig 4.

Table 4 First derivative of gravitational potential of homogeneous Earth model

Distance(km)	**First derivative (m/s^{-2})**	**Distance(km)**	**First derivative (m/s^{-2})**
0	0	6000	-9.4977478
1000	-1.5829580	**6371**	-10.085025
2000	-3.1659159	7000	-8.3540317
3000	-4.7488739	8000	-6.3960555
4000	-6.3318319	9000	-5.0536735
5000	-7.9147899	10000	-4.0934755

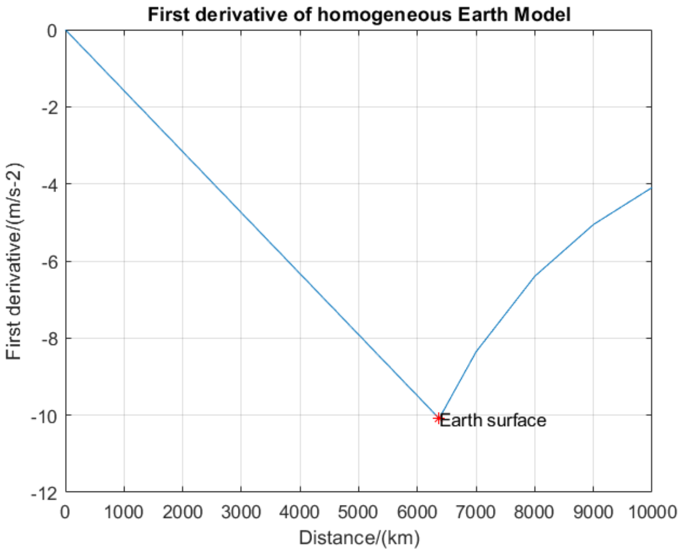

Figure 4 First derivative of gravitational potential of homogeneous Earth model

d) figures and discussion

d.1) *Gravitational potential of two models*

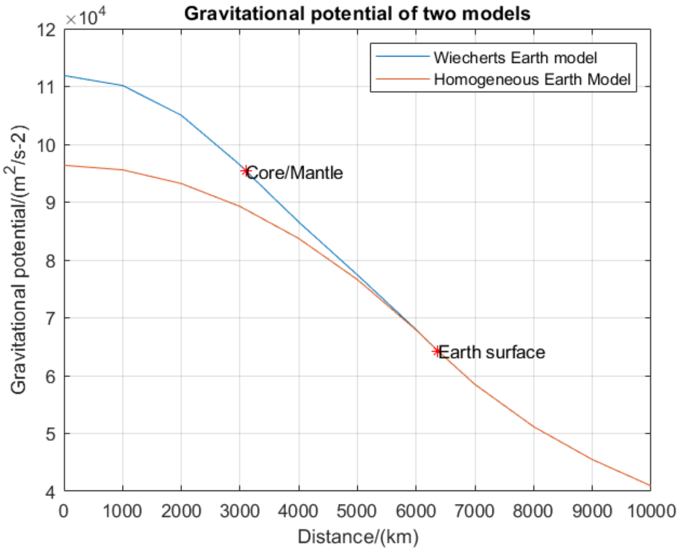

Figure 5 Gravitational potential of two models

Two different models' gravitational potential have the same descending trend. For the Wiechert's Earth model, it starts as 111890460 m^2/s^{-2} at the center of the earth, which is larger than homogeneous earth model. The reason of that is even two models have the same mass, the different mass distribution also causes the variation of gravitational potential.

The descending speed of the curve of Wiechert's Earth model turns faster then slower. The change points are two surfaces: core/mantle and mantle/outer space. It's caused by the components of values in different layers.

In the core layer, it decreases proportionally with r^2, which is also the same to the first layer of homogeneous Earth model.

At the edge of the Earth surface, two models have the same results and figures. The reason is we can consider the earth sphere as a mass point in the center of the earth as the density of it is homogeneous. In the outer space, the gravitational potential is inversely proportional to distance according to the formula.

In the mantle layer of Wiechert's Earth model, the formula of gravitational potential is combined with two parts above, which is because it can be seen as inside of a homogeneous spherical shell and outside of a homogeneous solid sphere.

d.2) *First derivative of gravitational potential of two models*

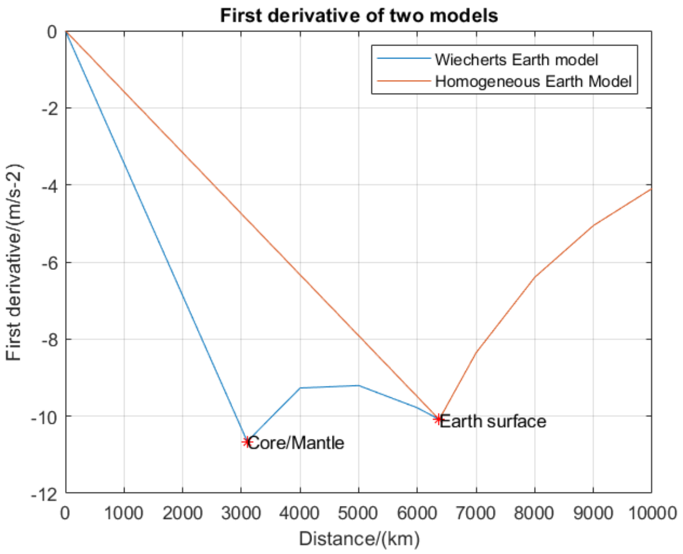

Figure 6 First derivative of gravitational potential of two models

In general, the two parts of homogeneous of earth model have the similar changing behaviors as the first and third part of Wiechert's Earth model. Also, the first derivative of two models all start at 0 in the center of the earth.

One important thing is a homogeneous sphere shell has no influence in the cavity of it. So, in the core layer, the first derivative decreases proportionally with r. And due to the density, the value of core layer changes faster than it in the first land part of homogeneous Earth model.

Let's skip the mantle layer and focus on the outer space first. For the reason of the mass of two model is the same, and both models are homogeneous in separate layers, we can consider the Earth as a mass point in the center of it. So they increase proportionally with r2.

Back to the mantle layer, it's influenced by two parts. Firstly, it's outside of a homogeneous solid sphere. What's more, it's inside of a homogeneous spherical shell. That's a combination of two changing rules mentioned above.

Feedback in data processing:

From my point of view, I think the hardest part of this exercise is to think about each layer of each model in turn, then try to convert the earth model to math model. In the first time, I tried to do the project with multiple integrals and functions. It takes much time and didn't work well. Then I just use the formulas at the last of the slides which have been simplified and get the answer easily.

I didn't state the results of subtasks a) and b) to five significant digits, those of subtask c) with a calculation accuracy of eight significant digits in the matlab code. Because I found I can only change the way of representation of the numbers in the IDE, or I have to transform all the numbers into string type then do the rounding. So I just copy the answer to the excel then do the rounding, which is not shown in the code.

All in all, it's a good exercise to train the skills of problem transformation and classification discussion.