M.Sc. Environmental Geodesy – Geodetic Earth Observation

Mass Variations

Gravitational field of a radially layered spherical shell

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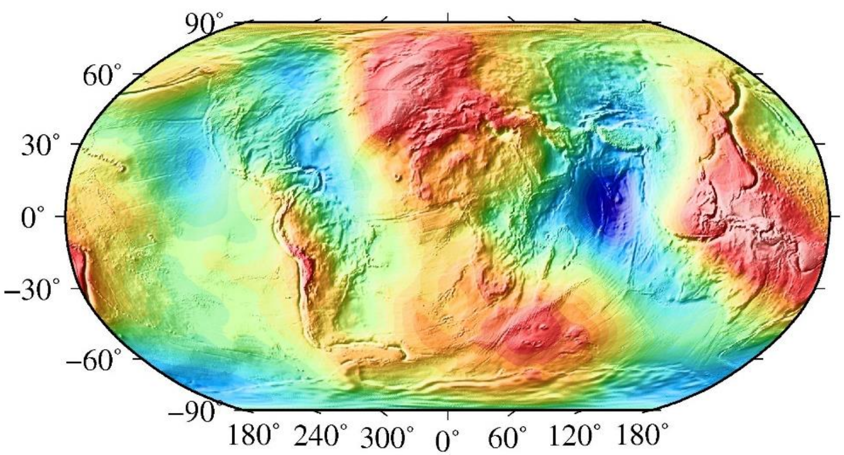
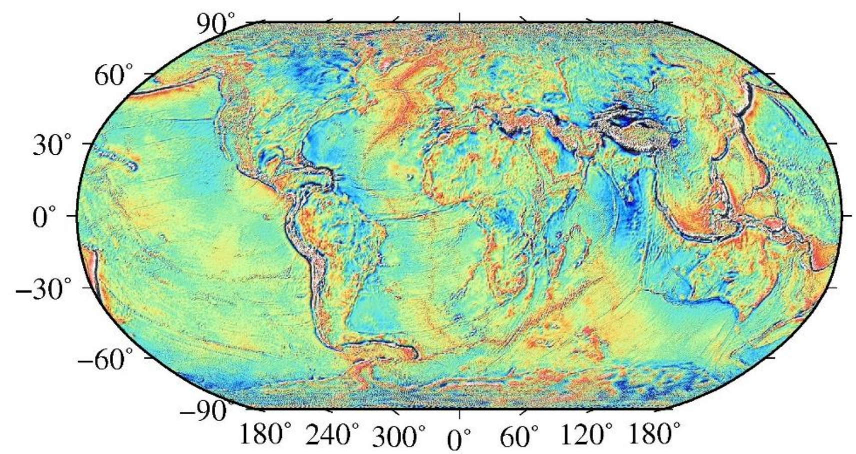
GEODETIC INSTITUTE – GEODETIC EARTH SYSTEM SCIENCE





Newton's integral

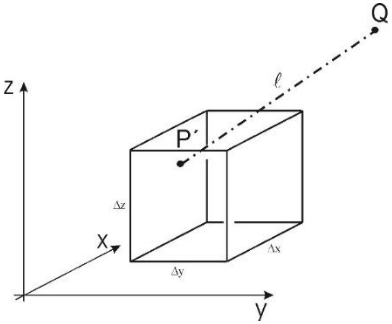
For a mass body, we obtain *Newton's integral*

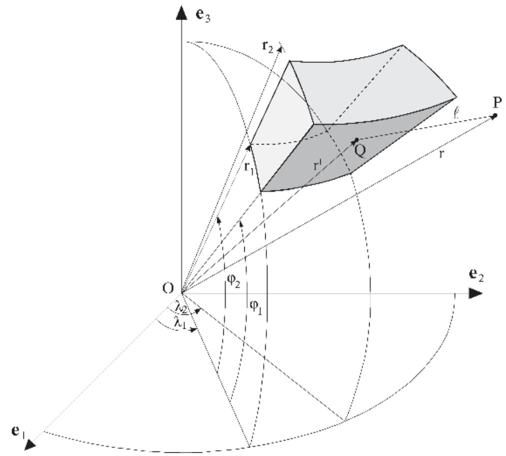
$$V(P) = G \iiint\limits_{\text{Body}} \frac{\varrho}{l}\, dv$$

where l is the distance between the volume element dv and the attracted point P.

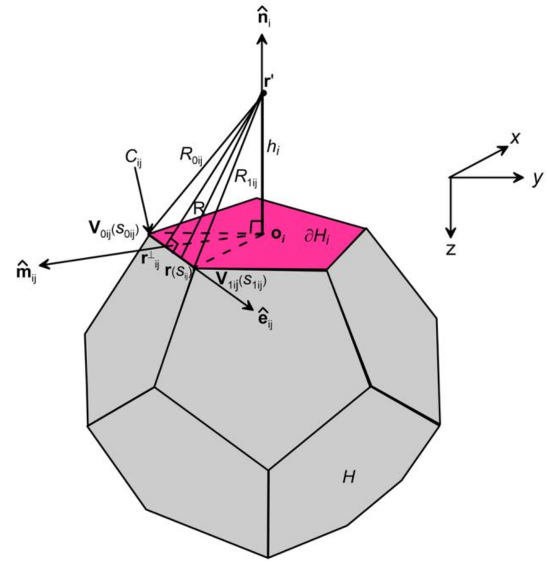
Remark: *If the density distribution in the body's interior and the boundary of the volume were known, then the problem of determining the body's gravitational potential field is solved by the volume integral.*

The prism, tesseroid and polyhedral body





$$v(r,\varphi,\lambda) = G\rho \int_{\lambda_1}^{\lambda_2} \int_{\varphi_1}^{\varphi_2} \int_{r_1}^{r_2} \frac{r'^2 \cos\varphi' \mathrm{d}r' \mathrm{d}\varphi' \mathrm{d}\lambda'}{\ell}$$



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Gravity potential W = V + Z

Gravitational potential V

Centrifugal potential Z

Gravitational attraction = grad V

- Caused by masses
- Masses have different densities
- Effects of masses play a central role in geodesy
- Mass discretization:
  - Point mass
  - Homogeneous sphere, spherical shell
  - Prism
  - Tesseroid
  - etc

Gravitational field of a radially layered spherical shell

In general, the following holds for the Earth's gravitational potential: $W = V + Z$

Apart from the centrifugal potential Z (caused by Earth's rotation)
the Newton integral applies to the gravitational potential V:

$$V = G \cdot \iiint\limits_{Earth} \frac{\rho \cdot d\Omega}{\ell}$$

The gravitational potential V depends on:
Distance ℓ between source point Q and computation point P

- the masses in the area Ω
- the density function $\rho(Q)$
  - The density function cannot be determined with sufficient accuracy for the entire Earth in order to determine V using the Newton integral

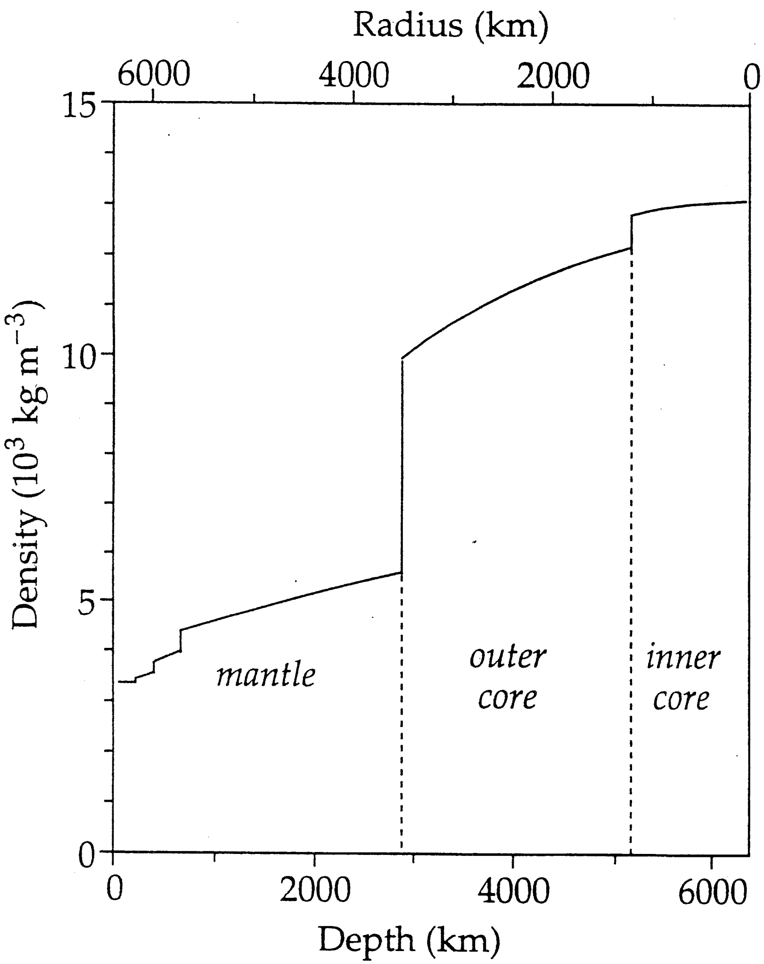
Gravitational field of a radially layered spherical shell

Density from models:

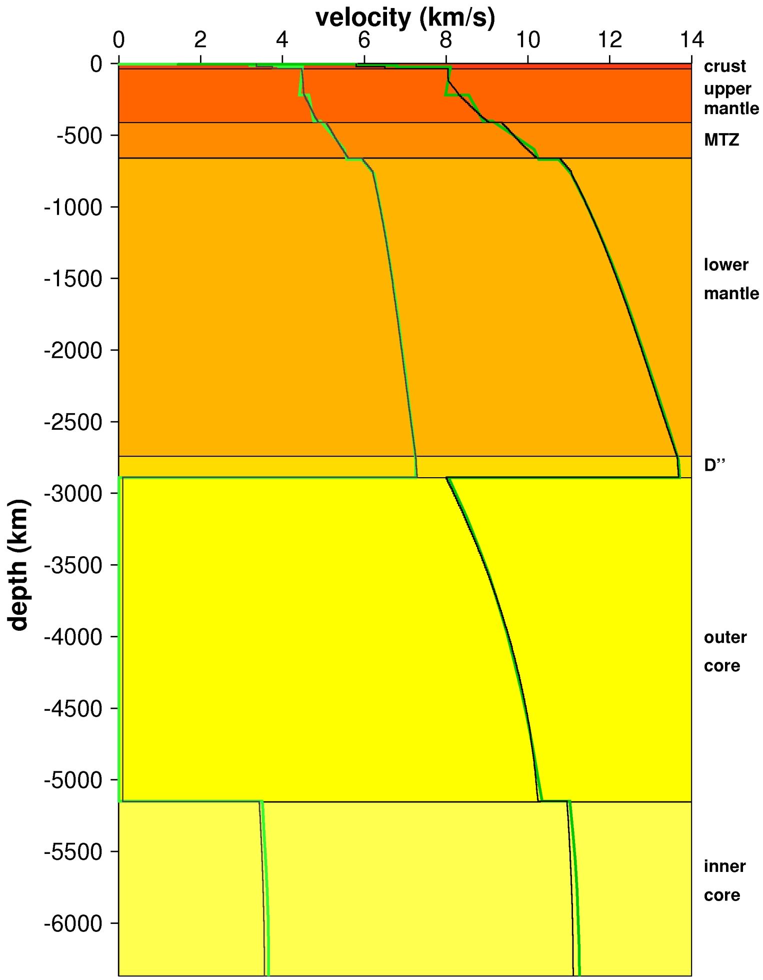
Using seismological measurements of the velocity of known compression and shear waves, the relationship between the pressure created by waves, relative volume change that a body experiences due to the pressure, and the density can be established.

Various models (radially density functions, discretized models):
- PREM (Preliminary Reference Earth Model)
- Wiechert (two-layer/two-density model)
- …

Gravitational field of a radially layered spherical shell

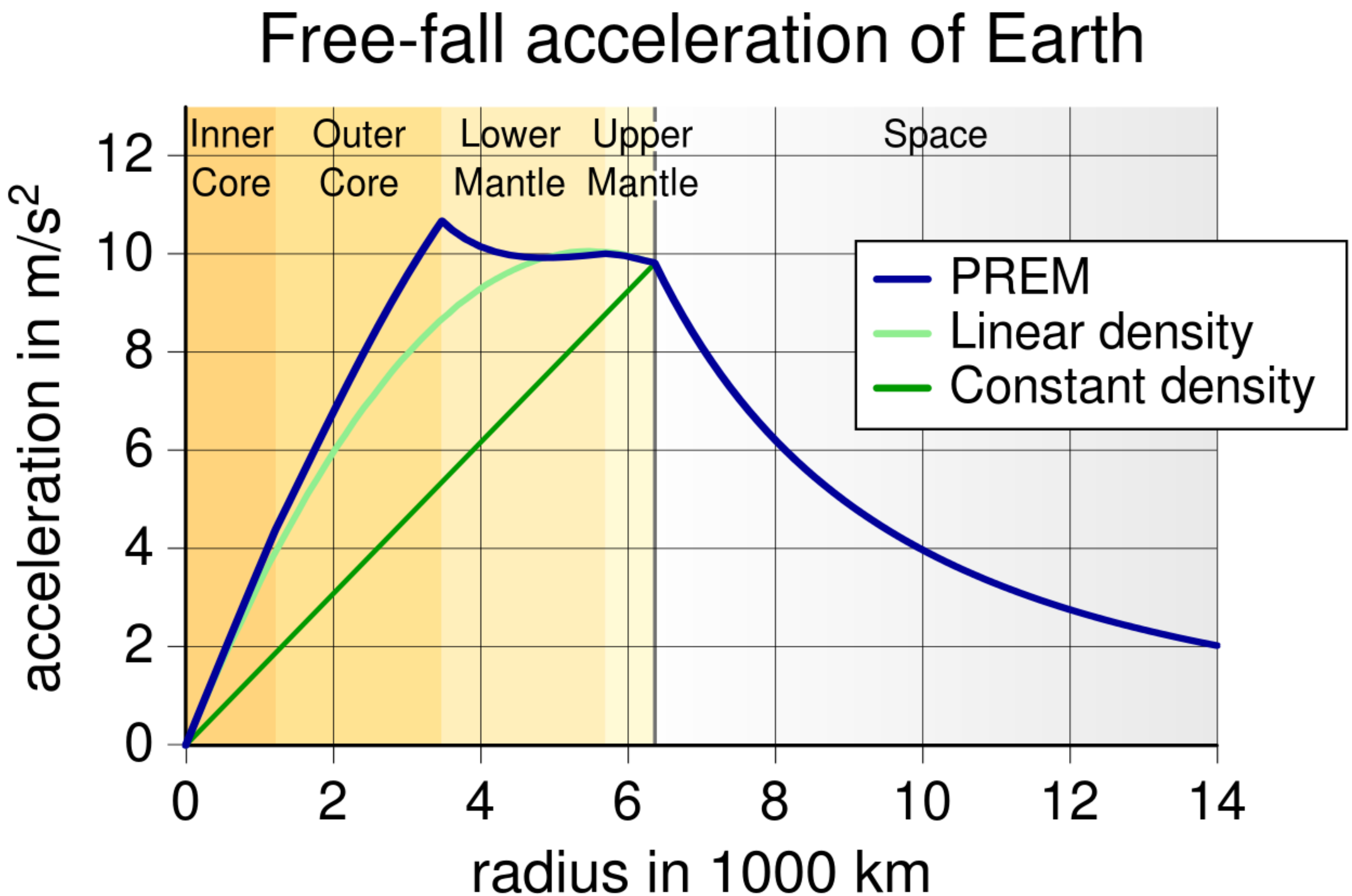


Radial density distribution in the Earth according to the PREM model (Dziewonski, 1989)



Seismic velocities for **P-** and **S-**waves (https://de.wikipedia.org/wiki/PREM)

Gravitational field of a radially layered spherical shell



(https://de.wikipedia.org/wiki/PREM)

Gravitational field of a radially layered spherical shell

Density unknown,
But gravity values measured on the Earth's surface are known

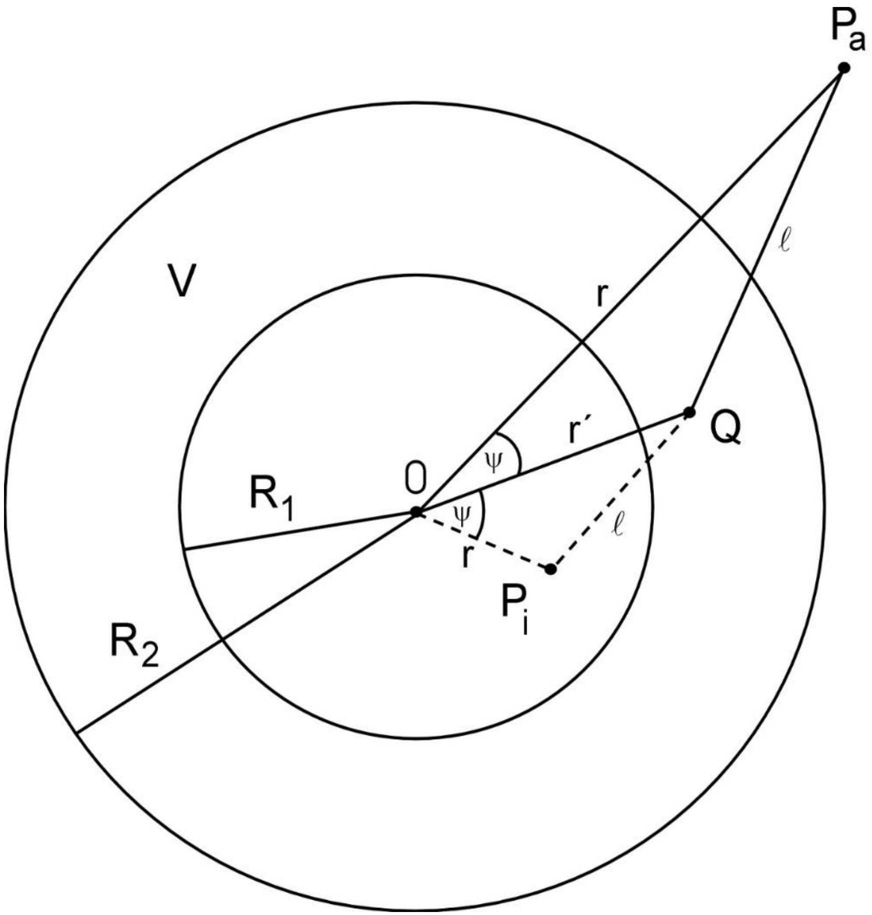
- Approximation of the gravity potential by normal potential U of a level ellipsoid (Somigliana-Pizetti normal field)
- Symmetry to the axis of rotation and equatorial plane.
- Definition of the disturbing potential: $T = W - U$
- Using the fundamental equation of physical geodesy the relationship between gravity anomalies and disturbing potential T can be established $\boxed{\Delta g = g_0 - \gamma_0}$ $\boxed{\Delta g = -\frac{2}{r} \cdot T - \frac{\partial T}{\partial r}}$
- The spherical function development for the disturbance potential can be established:

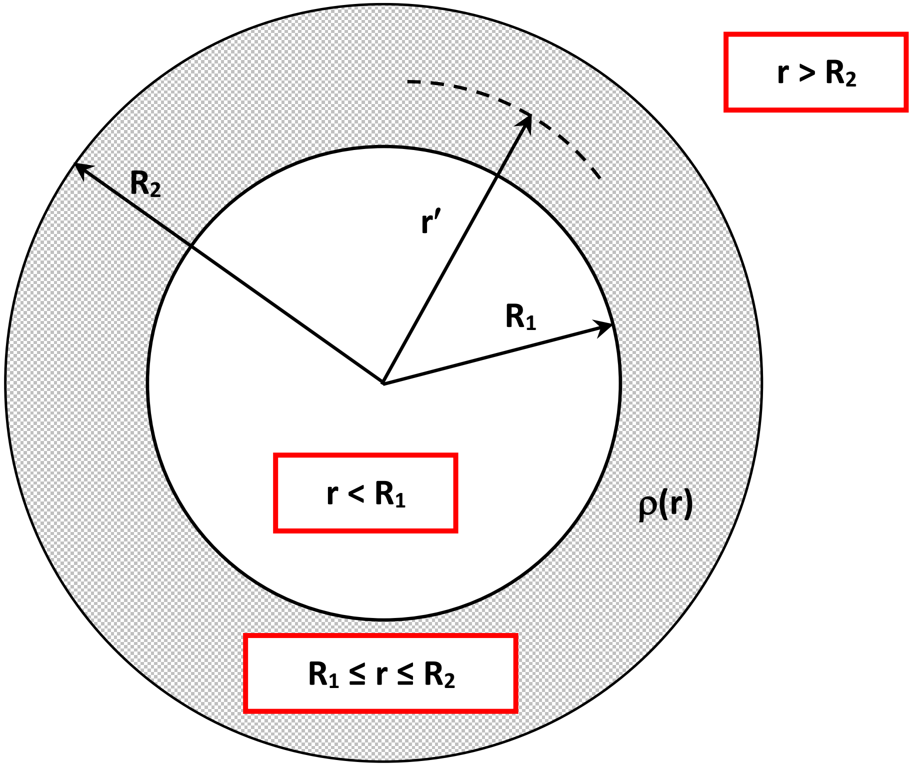
$$\boxed{T = \frac{GM}{r} \sum_{n=2}^{\infty} \left(\frac{a}{r}\right)^n \sum_{m=0}^{n} \left(c_{nm}^T \cdot \cos m\lambda + s_{nm}^T \cdot \sin m\lambda\right) \cdot P_{nm}(\cos\vartheta)}$$

- Each value of a gravity anomaly then provides a determination (observation) equation for the spherical function coefficients c_{nm}^T, s_{mn}^T
- Finally the geoid undulation is obtained by Bruns' equation: (distance along the ellipsoid normal between ellipsoid and geoid). $\boxed{N = \frac{T}{\gamma}}$

Gravitational field of a radially layered spherical shell

Calculation of the potential and the 1st derivative





Gravitational field of a radially layered spherical shell

The **computation points** P_i or P_a are located inside or outside the solid shell Ω (filled with mass).

Q is referred to as the mass point (**source point**).
The following holds: $dm = \rho \cdot d\Omega$, where $d\Omega$ represents the volume elemen.

Spherical coordinates are used; pole = computation point

r = radius of P_i resp. P_a

P_i : $0 \le r < R_1$

P_a : $R_2 < r < \infty$

Coordinates of the mass point Q: r', ψ **(polar distance)** $, \alpha$

$R_1 \le r' \le R_2,\ 0 \le \psi \le \pi,\ 0 \le \alpha \le 2\pi$

Volume element

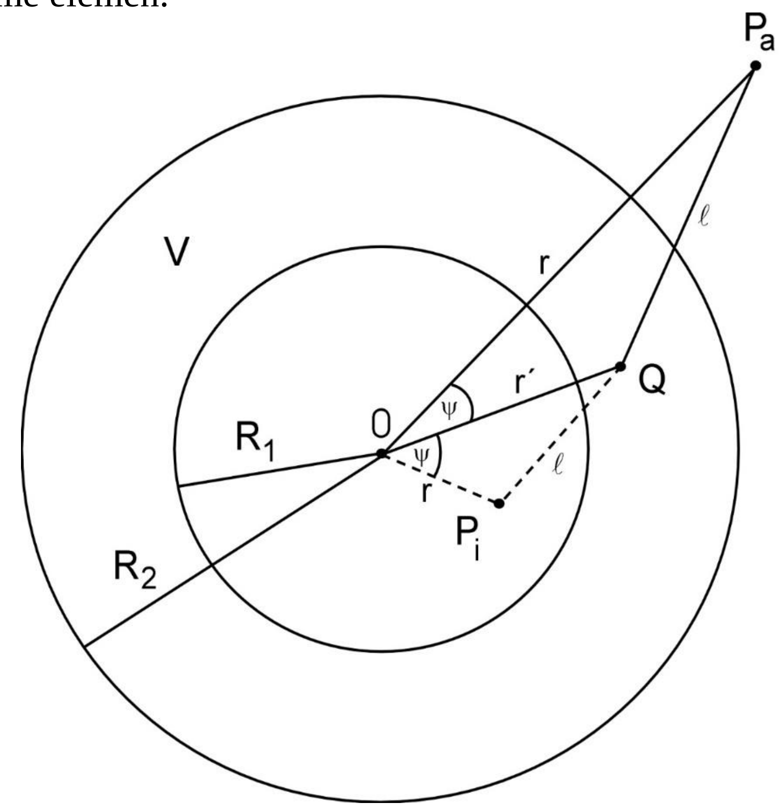
$d\Omega = r'^2 \cdot \sin\psi \cdot dr'\, d\psi\, d\alpha$

Distance between P and Q

$\ell = \sqrt{r^2 + r'^2 - 2rr' \cdot \cos\psi}$

radial mass stratification

$\rho = \rho(r'),\ 0 \le \rho \le \rho_{max} < \infty$ (Density is positive and finite !)



Gravitational field of a radially layered spherical shell

With the radially density function $\rho(r)$

Ansatz: $V(r) = G \cdot \iiint\limits_{\Omega} \frac{dm}{\ell} = G \cdot \iiint\limits_{\Omega} \frac{\rho \cdot d\Omega}{\ell}$

Volume element: $d\Omega = r'^2 \cdot \sin\psi \cdot dr'\, d\psi\, d\alpha$

Assumtion on the density distribution: $\rho = \rho(r')$

Distance between P and Q: $\ell = \sqrt{r^2 + r'^2 - 2rr' \cdot \cos\psi}$

[$G = 6.672 \cdot 10^{-11}\, m^3 s^{-2} kg^{-1}$ Newton’s gravitational constant]

1. Insertion of the volume element and the radial mass distribution

$$V(r) = G \cdot \iiint\limits_{\Omega} \frac{\rho \cdot d\Omega}{\ell} = G \cdot \iiint\limits_{\Omega} \frac{\rho(r') \cdot r'^2 \cdot \sin\psi \cdot dr'\, d\psi\, d\alpha}{\ell}$$

2. Insertion of the formula for the distance between P and Q

$$V(r) = G \cdot \iiint\limits_{\Omega} \frac{\rho(r') \cdot r'^2 \cdot \sin\psi \cdot dr'\, d\psi\, d\alpha}{\ell} = G \cdot \iiint\limits_{\Omega} \frac{\rho(r') \cdot r'^2 \cdot \sin\psi \cdot dr'\, d\psi\, d\alpha}{\sqrt{r^2 + r'^2 - 2rr' \cdot \cos\psi}}$$

$$\boxed{V(r) = G \cdot \int_{\alpha=0}^{2\pi} \int_{\psi=0}^{\pi} \int_{r'=R_1}^{R_2} \frac{\rho(r') \cdot r'^2 dr' \cdot \sin\psi\, d\psi\, d\alpha}{\sqrt{r^2 + r'^2 - 2rr' \cdot \cos\psi}}}$$

Gravitational field of a radially layered spherical shell

3. Consideration of the partial integrals

   a) Integration by $d\alpha$

   $$\int_0^{2\pi} d\alpha = 2\pi$$

   b) Integration by $d\psi$ (Substitution $\cos\psi = t; \quad -\sin\psi \, d\psi = dt$)

$$\int_{\psi=0}^{\pi} \frac{\sin\psi \; d\psi}{\sqrt{r^2 + r'^2 - 2rr' \cdot \cos\psi}} = \int_{1}^{-1} \frac{-dt}{\sqrt{r^2 + r'^2 - 2rr' \cdot t}} = \left. \frac{\sqrt{r^2 + r'^2 - 2rr' \cdot \cos\psi}}{rr'} \right|_0^{\pi}$$

$$= \frac{1}{rr'} \cdot \left(\sqrt{(r + r')^2} - \sqrt{(r - r')^2} \right) = \frac{1}{rr'} \cdot \left(|r + r'| - |r - r'| \right) = \frac{1}{rr'} \cdot \left(r + r' - |r - r'| \right)$$

4. General formula for the gravitational potential
   (depends only on r, with radial density variation)

$$\boxed{V(r) = 2\pi G \cdot \int_{R_1}^{R_2} \rho(r') \cdot \frac{r'}{r} \cdot \left(r + r' - |r - r'| \right) dr'}$$

Conclusion:

- You should be familiar with the **Newton integral** for representing the gravitational potential.

$$V = G \cdot \iiint_{Erde} \frac{\rho \cdot d\Omega}{\ell}$$

- You can express the potential effect of a spherical shell in spherical polar coordinates

Gravitational field of a radially layered spherical shell

This results in the potential at the point of application P

a) P in the outer space of the sphere R_2

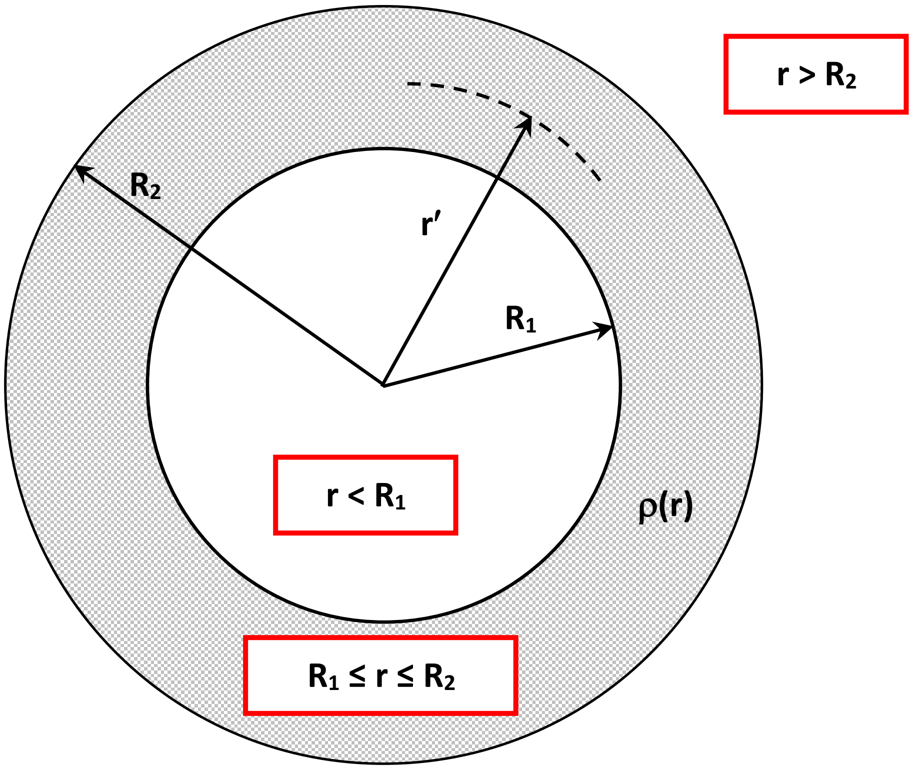
$$r > R_2 \geq r' \geq R_1 \quad \Leftrightarrow \quad |r - r'| = r - r'$$

$$\boxed{\begin{aligned} V(r) &= 2\pi G \cdot \int_{R_1}^{R_2} \rho(r') \cdot \frac{r'}{r} \cdot (r + r' - (r - r'))\, dr' \\ &= \frac{4\pi G}{r} \cdot \int_{R_1}^{R_2} \rho(r') \cdot r'^2 \, dr' \end{aligned}}$$

b) P in the interior of the sphere (cavity) R_1

$$r < R_1 \leq r' \leq R_2 \quad \Leftrightarrow \quad |r - r'| = r' - r$$

$$\boxed{\begin{aligned} \mathrm{V}(\mathrm{r}) &= 2\pi \mathrm{G} \cdot \int_{\mathrm{R}_1}^{\mathrm{R}_2} \rho(\mathrm{r}') \cdot \frac{\mathrm{r}'}{\mathrm{r}} \cdot (\mathrm{r} + \mathrm{r}' - (\mathrm{r}' - \mathrm{r}))\, \mathrm{dr}' \\ &= 4\pi \mathrm{G} \cdot \int_{\mathrm{R}_1}^{\mathrm{R}_2} \rho(\mathrm{r}') \cdot \mathrm{r}' \, \mathrm{dr}' = \mathrm{const} \end{aligned}}$$



Gravitational field of a radially layered spherical shell

Derivation of the total mass of the spherical shell

$$M_\Omega = \iiint\limits_\Omega dm = \iiint\limits_\Omega \rho \cdot d\Omega = \int_0^{2\pi}\int_0^{\pi}\int_{R_1}^{R_2} \rho(r') \cdot r'^2 \cdot \sin\psi\, dr'\, d\psi\, d\alpha = 2\pi \cdot \int_0^{\pi}\int_{R_1}^{R_2} \rho(r') \cdot r'^2 \cdot \sin\psi \cdot d\psi\, dr'$$

$$= 2\pi \cdot \int_{R_1}^{R_2} \underbrace{\left(-\cos\psi\right)\Big|_0^{\pi}}_{=+2} \cdot \rho(r') \cdot r'^2 \cdot dr' = 4\pi \cdot \int_{R_1}^{R_2} \rho(r') \cdot r'^2 \cdot dr'$$

Gravitational field of a radially layered spherical shell

Conclusions

The following applies to points in the outer space (outside) of the spherical shell:

$$\boxed{V(r) = \frac{GM_\Omega}{r}}, \; r \geq R_2$$

This expression is formally identical to the potential of a point mass M_Ω at the origin O or with the potential of a homogeneous sphere of total mass M_Ω.
Consequently, the density function cannot be unambiguously determined from the potential in the outer space (**Inverse problem of gravimetry**).

For points inside the spherical shell, the gravitational potential does not depend on r and is therefore constant (in the cavity $r < R_1$)

Gravitational field of a radially layered spherical shell

Implications

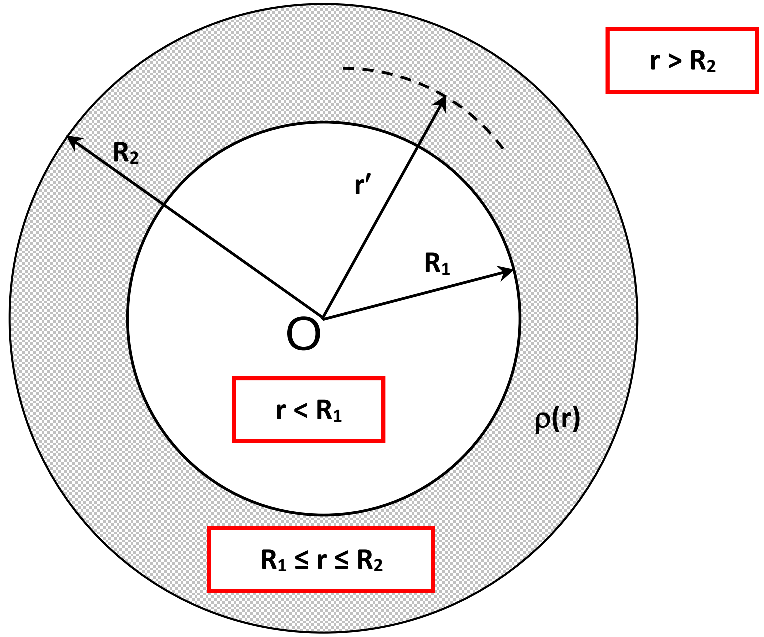
c) The following applies to points inside the masses: $R_1 \le r \le R_2$

$$\boxed{V(r) = \frac{4\pi G}{r} \cdot \int_{R_1}^{r} \rho(r') \cdot r'^2 \, dr' + 4\pi G \cdot \int_{r}^{R_2} \rho(r') \cdot r' \, dr'}$$

resp.

$$\boxed{V(r) = \frac{G \cdot m(r)}{r} + 4\pi G \cdot \int_{r}^{R_2} \rho(r') \cdot r' \, dr'}$$

with $\quad m(r) = 4\pi \int_{R_1}^{r} \rho(r') \cdot r'^2 \, dr'$



This formula is formally valid for the entire definition range of r $(0 \le r < \infty)$;
$m(r)$ represents the mass of a sphere around O through the centre of gravity.

Gravitational field of a radially layered spherical shell

Special cases

(1) **Solid sphere**: $R_1 \to 0; \quad \rho = \rho(r')$

(2) **Homogeneous spherical shell**: $\rho = const.$

a) $r > R_2$

$$V = \frac{4\pi G}{r} \cdot \int_{R_1}^{R_2} \rho(r') \cdot r'^2 \, dr' = \frac{4\pi G\rho}{r} \cdot \frac{1}{3} r'^3 \Big|_{R_1}^{R_2} = \boxed{\frac{4\pi G\rho}{3r} \cdot \left(R_2^3 - R_1^3\right)} \sim \frac{1}{r}$$

b) $r < R_1$

$$V = 4\pi G \cdot \int_{R_1}^{R_2} \rho(r') \cdot r' \, dr' = 4\pi G\rho \cdot \frac{1}{2} r'^2 \Big|_{R_1}^{R_2} = \boxed{2\pi G\rho \cdot \left(R_2^2 - R_1^2\right)} = const.$$

c) $R_1 \le r \le R_2$

$$V = \frac{4\pi G}{r} \cdot \int_{R_1}^{r} \rho(r') \cdot r'^2 \, dr' + 4\pi G \cdot \int_{r}^{R_2} \rho(r') \cdot r' \, dr'$$

$$V = \frac{4\pi G\rho}{3r} \cdot \left(r^3 - R_1^3\right) + 2\pi G\rho \cdot \left(R_2^2 - r^2\right)$$

$$\boxed{V = 2\pi G\rho \cdot \left(R_2^2 - \frac{2}{3r} \cdot R_1^3 - \frac{1}{3} r^2\right)}$$

V is **continuous** everywhere, even for $r = R_1, r = R_2$

This generally applies to any mass distribution with $\rho \le \rho_{max} < \infty$.

Gravitational field of a radially layered spherical shell

Specific cases

(3) **Homogeneous solid sphere** $R_1 \to 0$, $R_2 = R$, $\rho = const.$, M = mass of the solid sphere

Potential in outer space: $r > R$

$$\boxed{V = \frac{4\pi G\rho}{3r} \cdot R^3 = \frac{GM}{r}} \sim \frac{1}{r} \qquad \text{(hyperbola)}$$

Potential in the interior: $r \leq R$

$$\boxed{V = 2\pi G\rho \cdot \left(R^2 - \frac{1}{3} r^2 \right)} \sim r^2 \qquad \text{(parabola)}$$

on the edge ($r = R$) applies with the mass M of the homogeneous solid sphere $\boxed{\mathrm{M} = \frac{4\pi\rho}{3} \cdot \mathrm{R}^3}$

$$V = \frac{4\pi G\rho}{3} \cdot R^2 = \frac{GM}{R}$$

First derivatives of the potential in geocentric polar coordinates

$$V(r) = 2\pi G \cdot \int_{R_1}^{R_2} \rho(r') \cdot \frac{r'}{r} \cdot \left(r + r' - |r - r'|\right) dr'$$

Tangential: $\frac{\partial V}{\partial \alpha} = \frac{\partial V}{\partial \psi} = 0$ (E-W- resp. N-S-direction)

Radial: $\frac{\partial V}{\partial r}$

With the signum function, the norm $|r - r'|$ can be written as:

$|r - r'| = (r - r') \cdot sign(r - r')$ and it follows from that

$$\frac{\partial V}{\partial r} = \frac{\partial\left(2\pi G \cdot \int_{R_1}^{R_2} \rho(r') \cdot \frac{r'}{r} \cdot (r + r' - (r - r') \cdot sign(r - r'))dr'\right)}{\partial r}$$

$$= 2\pi G \cdot \int_{R_1}^{R_2} \rho(r') \cdot r' \cdot \frac{\partial}{\partial r}\left(\frac{(r + r' - (r - r') \cdot sign(r - r'))}{r}\right) dr'$$

$$= \frac{2\pi G}{r^2} \cdot \int_{R_1}^{R_2} \rho(r') \cdot r' \cdot \left[(1 - sign(r - r'))r - r - r' + (r - r') \cdot sign(r - r')\right] dr'$$

$$= -\frac{2\pi G}{r^2} \cdot \int_{R_1}^{R_2} \rho(r') \cdot r'^2 \cdot \left[(1 + sign(r - r'))\right] dr'$$

Gravitational field of a radially layered spherical shell

If

$$r' > r \;\rightarrow\; \left(1 + sign\left(r - r'\right)\right) = 0$$
$$r' < r \;\rightarrow\; \left(1 + sign\left(r - r'\right)\right) = 2$$

From this the following expressions result:

a) P outside of the spherical shell

$$r > R_2 \geq r' \qquad \frac{\partial V}{\partial r} = -\frac{2\pi G}{r^2} \cdot \int_{R_1}^{R_2} \rho\left(r'\right) \cdot r'^2 \cdot [2]\, dr' = \boxed{-\frac{4\pi G}{r^2} \cdot \int_{R_1}^{R_2} \rho\left(r'\right) \cdot r'^2\, dr' = -\frac{GM_\Omega}{r^2}}$$

b) P in the cavity of the spherical shell

$$r < R_1 \leq r' \qquad \boxed{\frac{\partial V}{\partial r} = -\frac{2\pi G}{r^2} \cdot \int_{R_1}^{R_2} \rho\left(r'\right) \cdot r'^2 \cdot [0]\, dr' = 0}$$

c) P in the mass-filled spherical shell

$$\mathrm{R}_1 \leq \mathrm{r} \leq \mathrm{R}_2 \qquad \frac{\partial V}{\partial r} = -\frac{2\pi G}{r^2} \cdot \int_{R_1}^{r} \rho\left(r'\right) \cdot r'^2 \cdot [2]\, dr'$$

$$\boxed{\frac{\partial V}{\partial r} = -\frac{4\pi G}{r^2} \cdot \int_{R_1}^{r} \rho\left(r'\right) \cdot r'^2\, dr' = -\frac{G \cdot m(r)}{r^2}}$$

I.e. only the masses lying within the sphere of radius r has an influence on $\frac{\partial \mathrm{V}}{\partial \mathrm{r}}$!

$\frac{\partial \mathrm{V}}{\partial \mathrm{r}}$ is continuous in the entire space, even for $\mathrm{r} = \mathrm{R}_1, \mathrm{r} = \mathrm{R}_2$

This generally applies to any mass distribution with $\rho \leq \rho_{\max} < \infty$.

Gravitational field of a radially layered spherical shell

Special cases

(1) **Solid sphere**: $R_1 \to 0$

(2) **homogeneous spherical shell:**: $\rho = const.$

a) $r > R_2$

$$\boxed{\frac{\partial V}{\partial r} = -\frac{4\pi G\rho}{r^2} \cdot \int_{R_1}^{R_2} r'^2\, dr' = -\frac{4\pi G\rho}{3r^2} \cdot \left(R_2^3 - R_1^3\right)}$$

b) $r < R_1$

$$\boxed{\frac{\partial V}{\partial r} = 0}$$

c) $R_1 \le r \le R_2$

$$\boxed{\frac{\partial V}{\partial r} = -\frac{4\pi G\rho}{r^2} \cdot \int_{R_1}^{r} r'^2\, dr' = -\frac{4\pi G\rho}{3r^2} \cdot \left(r^3 - R_1^3\right) = -\frac{4\pi G\rho}{3} \cdot \left(r - \frac{R_1^3}{r^2}\right)}$$

Note: It is not dependent on R_2 !

(3) **homogeneous solid sphere** $R_1 \to 0$, $R_2 = R$, $\rho = const.$, M = mass of the solid sphere

gravitational effect in outer space: $r > R$

$$\boxed{\frac{\partial V}{\partial r} = -\frac{4\pi G\rho}{3r^2} \cdot R^3 = -\frac{GM}{r^2}}$$

gravitational effect in the interior: $r \le R$

$$\boxed{\frac{\partial V}{\partial r} = -\frac{4\pi G\rho}{3} \cdot r}$$

Conclusion:

- You should be familiar with the Newton integral to represent the gravitational potential
- You should be able to express the potential effect of a spherical shell in spherical polar coordinates
- You should have an idea of the qualitative course of the potential and its 1st radial derivative of a spherical shell
  - The following applies to a radial density function:
    - ✓ The potential in the cavity of the spherical shell is constant
    - ✓ The radial derivative in the cavity of the spherical shell is zero
    - ✓ In the outer space of the spherical shell, the potential and its radial derivative behave as for a point mass

Exercise sheet

- Is set on ILIAS
- It can be worked on in pairs

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