Mixing Samples to Address Weak Overlap in Causal Inference

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Suehyun Kim

Department of Statistics, Seoul National University

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Collaborators

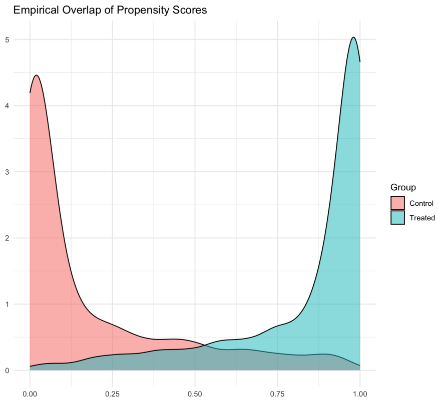
Kwonsang Lee
Seoul National University
kwonsanglee@snu.ac.kr

Jaehyuk (Jay) Jang
Seoul National University
bbq12340@snu.ac.kr

Overview

Overview

Handling weak overlap in weighting methods



- **Overlap assumption**:
  $0 < e(x) = \mathbb{P}(Z = 1 \mid X = x) < 1$

- Weak overlap is problematic in weighting methods due to units with extreme weights such as $1/e \simeq \infty$ or $1/(1-e) \simeq \infty$.

Remedies so far

There have been three main approaches to handle weak overlap in the literature:

1. **Trimming/truncating units with extreme weights**
   - Loss of sample size, sensitivity to the choice of cutoff
2. **Targeting an alternative causal estimand**
   - Overlap weights and ATO (Average Treatment of the Overlap Population)
   - Lack of interpretability
3. **Balancing weights**
   - Entropy Balancing, Covariate Balancing Propensity Score, etc.
   - Optimization may be infeasible under weak overlap

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   - Optimization may be infeasible under weak overlap

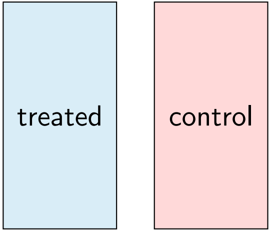
Our idea

We propose the **mixing framework**, which helps overcome the limitations of the above approaches by creating a synthetic sample of mixed treated and control units.

Main idea: Simple mixing strategy

Our strategy aims to intentionally increase overlap by mixing treated and control units.

Original dataset



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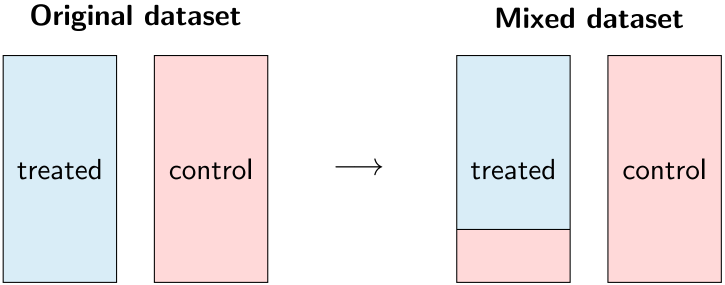

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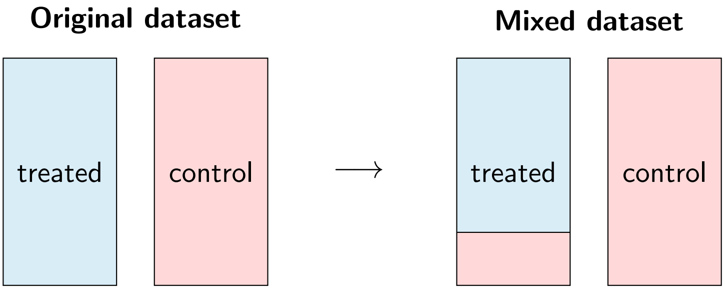
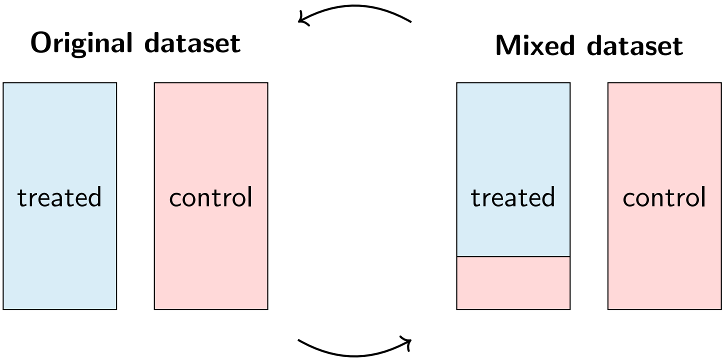


- Target population

- Stronger overlap
- More stable estimation within the mixed sample

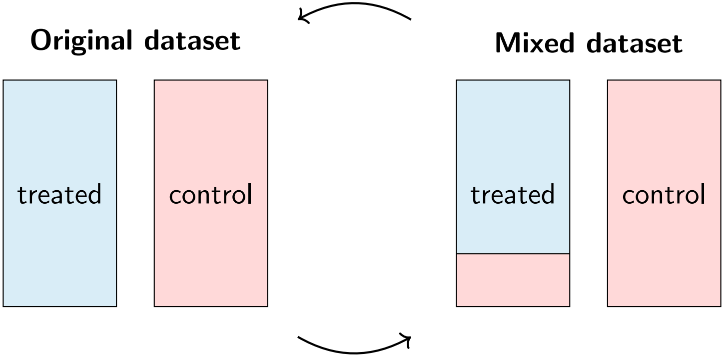
Main idea: Simple mixing strategy



Main idea: Simple mixing strategy

① Mixed distribution

Mixed PS, Mixed IPW and their properties



Main idea: Simple mixing strategy

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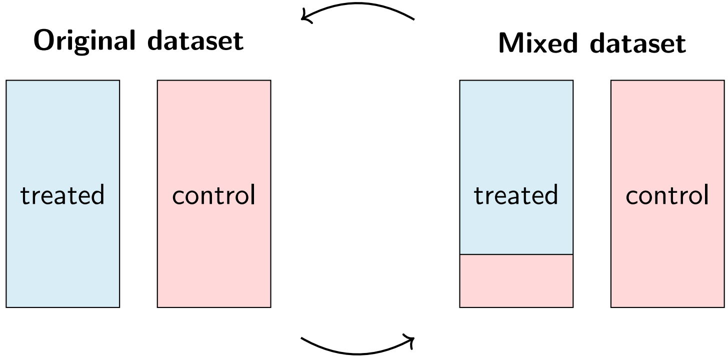


② Mixing implementation

(i) M-estimation (ii) Resampling algorithm

Overview

Notation & Setup

Under Rubin's Potential Outcome Framework, our aim is to apply mixing to weighting estimators of the Average Treatment Effect on the Treated (ATT).

Assumptions

1. **Unconfoundedness**: $(Y(1), Y(0)) \perp\!\!\!\perp Z \mid X$
2. **Overlap**: $0 < e(x) = \mathbb{P}(Z = 1 \mid X = x) < 1$ for all x

- $(Y(0), Y(1))$: Potential outcomes
- Y: Observed outcome
- X: Observed covariates
- Z: Binary treatment indicator
- $\tau = E\left[Y(1) - Y(0) \mid Z = 1\right]$: Target estimand (ATT)
- $f_{Y,X}$: Joint density of (Y, X)
- $f_{Y,X|Z=z}$: Joint density of (Y, X) given $Z = z$

Mixed distribution

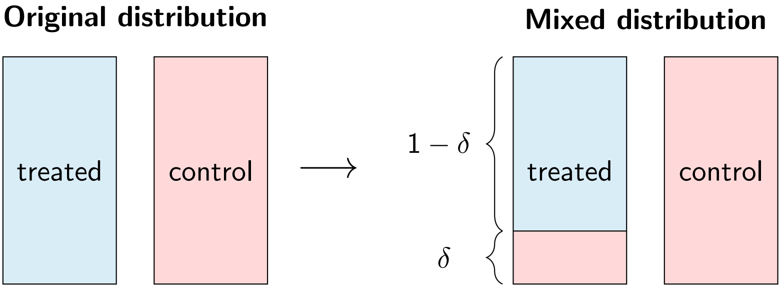
Definition (Mixed distribution)

We define the distribution of (Y^*, Z^*, X^*) as the distribution with the conditional joint densities of (Y^*, X^*) given $Z^* = 1, 0$, respectively,

$$
\begin{aligned}
f_{Y^*,X^*|Z^*=1} &= (1-\delta) f_{Y,X|Z=1} + \delta f_{Y,X|Z=0} \\
f_{Y^*,X^*|Z^*=0} &= f_{Y,X|Z=0}
\end{aligned}
$$

for a fixed constant $0 < \delta < 1$ and Z^* to satisfy $\mathbb{P}(Z^* = 1) = \mathbb{P}(Z = 1) =: \pi$. We refer to the mixed distribution with a constant δ as the **simple mixed distribution**.

Mixed distribution



Mixed propensity score

Lemma 1 (Mixed propensity score and its robustness)

Let $e^*(x) = \mathbb{P}(Z^* = 1 \,|\, X^* = x)$ *be the propensity score of the mixed distribution. Then,*

$$\frac{e^*}{1-e^*}(x) = (1-\delta)\frac{e}{1-e}(x) + \delta\frac{\pi}{1-\pi} \quad \textit{for all } x.$$

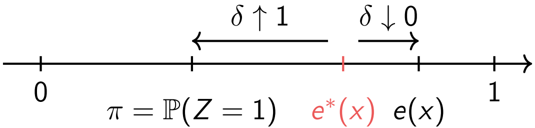


Figure 1: Behavior of $e^*(x)$ with respect to δ

Mixed IPW (MIPW) estimator

Theorem 1 (MIPW estimator and its consistency)

Using the mixed propensity score

$$\frac{e}{1-e}(x) = \frac{\frac{e^*}{1-e^*}(x) - \delta\frac{\pi}{1-\pi}}{1-\delta},$$

we define the **Mixed IPW (MIPW)** *estimator as follows:*

$$\hat{\tau} := \frac{\sum_i Z_i Y_i}{\sum_i Z_i} - \frac{\sum_i \left(\frac{e^*}{1-e^*}(X_i^*) - \delta\frac{\pi}{1-\pi}\right)(1-Z_i^*)Y_i^*}{\sum_i \left(\frac{e^*}{1-e^*}(X_i^*) - \delta\frac{\pi}{1-\pi}\right)(1-Z_i^*)}$$

Under the strong ignorability assumptions, $\hat{\tau}$ is a consistent estimator of τ.

Mixing implementation 1: M-estimation

Proposition 1 (Asymptotic normality based on observed samples)

Under the strong ignorability assumptions, $\hat{\theta} = \mathit{Solve}_\theta \left[\sum_i \psi^{**}(\theta; Y_i, X_i, Z_i) = 0\right]$ *is an M-estimator of* $\theta = (\beta, \pi, E[Y(1) \mid Z = 1], E[Y(0) \mid Z = 1])$*, where, for* $0 < \delta < 1$*,*

$$\psi^{**}(\theta; Y, X, Z) = \begin{pmatrix} \left\{ \frac{1-\delta}{e^*(X;\beta)} Z + \left(\frac{\delta\pi}{(1-\pi)e^*(X;\beta)} - \frac{1}{1-e^*(X;\beta)} \right) (1-Z) \right\} \nabla_\beta e^*(X;\beta) \\ Z - \pi \\ ZY - ZE[Y(1) \mid Z = 1] \\ \frac{e(X;\beta)}{1-e(X;\beta)}(1-Z)Y - \frac{e(X;\beta)}{1-e(X;\beta)}(1-Z)E[Y(0) \mid Z = 1] \end{pmatrix}.$$

$$\begin{tikzcd}
\{(Y_i, Z_i, X_i)\}_{i=1}^n \arrow[dr, dashed] & & \\
\{(Y_i, Z_i, X_i, Y_i^*, Z_i^*, X_i^*)\}_{i=1}^n \arrow[r, "\psi^*"'] & \hat{\tau} \arrow[r, "n \to \infty"] & \tau
\end{tikzcd}$$

Mixing implementation 1: M-estimation

Proposition 2 (Asymptotic normality based on observed samples)

Under the strong ignorability assumptions, $\hat{\theta} = \textit{Solve}_{\theta}\left[\sum_i \psi^{**}(\theta; Y_i, X_i, Z_i) = 0\right]$ *is an M-estimator of* $\theta = (\beta, \pi, E[Y(1) \mid Z = 1], E[Y(0) \mid Z = 1])$, *where, for* $0 < \delta < 1$,

$$\psi^{**}(\theta; Y, X, Z) = \begin{pmatrix} \left\{ \frac{1-\delta}{e^*(X;\beta)} Z + \left(\frac{\delta\pi}{(1-\pi)e^*(X;\beta)} - \frac{1}{1-e^*(X;\beta)} \right)(1-Z) \right\} \nabla_{\beta} e^*(X;\beta) \\ Z - \pi \\ ZY - ZE[Y(1) \mid Z = 1] \\ \frac{e(X;\beta)}{1-e(X;\beta)}(1-Z)Y - \frac{e(X;\beta)}{1-e(X;\beta)}(1-Z)E[Y(0) \mid Z = 1] \end{pmatrix}.$$

$$\begin{tikzcd} \{(Y_i, Z_i, X_i)\}_{i=1}^{n} \arrow[dr, "\psi^{**}"] & & \\ \{(Y_i, Z_i, X_i, Y_i^*, Z_i^*, X_i^*)\}_{i=1}^{n} \arrow[r, "\psi^*"'] & \hat{\tau} \arrow[r, "n \to \infty"] & \tau \end{tikzcd}$$

Mixing implementation 2: Resampling algorithm

Another way to implement mixing is to use a **resampling algorithm** that directly estimates $\hat{f}^*_{Y,X|Z=z}$ from the observed dataset.

Observed dataset

$$\{(Y_i, Z_i, X_i)\}_{i=1}^n \sim \hat{f}_{Y,X|Z=z}$$

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$$\left(\{(Y_i^*, Z_i^*, X_i^*)\}_{i=1}^n\right)_{m=1}$$

$$\vdots$$

Observed dataset

$$\{(Y_i, Z_i, X_i)\}_{i=1}^n \sim \hat{f}_{Y,X|Z=z} \Longrightarrow \hat{f}^*_{Y,X|Z=z}$$

Resample M times

$$\vdots$$

$$\left(\{(Y_i^*, Z_i^*, X_i^*)\}_{i=1}^n\right)_{m=M}$$

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Observed dataset

$$\{(Y_i, Z_i, X_i)\}_{i=1}^n \sim \hat{f}_{Y,X|Z=z} \Longrightarrow \hat{f}^*_{Y,X|Z=z}$$

Resample M times

$$\left(\{(Y_i^*, Z_i^*, X_i^*)\}_{i=1}^n\right)_{m=1}$$

$$\vdots$$

$$\vdots$$

$$\left(\{(Y_i^*, Z_i^*, X_i^*)\}_{i=1}^n\right)_{m=M}$$

$$\xrightarrow{\text{Aggregate}} \hat{\tau}$$

Mixing implementation 2: Resampling algorithm

The resampling algorithm allows for extensions to various weighting schemes, such as Entropy Balancing or Covariate Balancing Propensity Score.

Proposition 3 (Extension to balancing weights)

Suppose W^* is a **balancing weight** *for X^*, satisfying*

$$E[X^* \mid Z^* = 1] = E[W^* X^* \mid Z^* = 0].$$

Then,

$$W := \frac{W^* - \delta \frac{\pi}{1-\pi}}{1 - \delta}$$

is a balancing weight for X.

Overview

Simulation study

Simulation 1	Simulation 2
M-estimation • IPW vs MIPW • Efficiency gain in terms of both finite- and large-sample perspective	Resampling algorithm • Extension to Entropy Balancing • Performance under model misspecification

- **Data generating process**: $e(X) = \{1 + \exp(-X^T\beta)\}^{-1}, X \sim N_5(0, I)$
  - Overlap level (according to β): Strong / Moderate / Weak
  - Treatment effect: $\tau = 1$ (homogeneous)

Simulation study for implementation 1: M-estimation

- **Performance measures**: Monte-Carlo simulation of (1) standard deviation estimates and (2) Huber-White's robust standard error estimates
- **Benchmark**: ATO estimation via overlap weights (Li et al., 2018)
  - **ATO**: A causal estimand under the subpopulation for which the average treatment effect can be estimated with the smallest variance.
- **True treatment effect**: 1 (homogeneous) $\implies ATO = ATT = 1$

Estimator	IPW	MIPW	OW
Target	ATT	ATT	ATO

Results: IPW vs MIPW vs OW

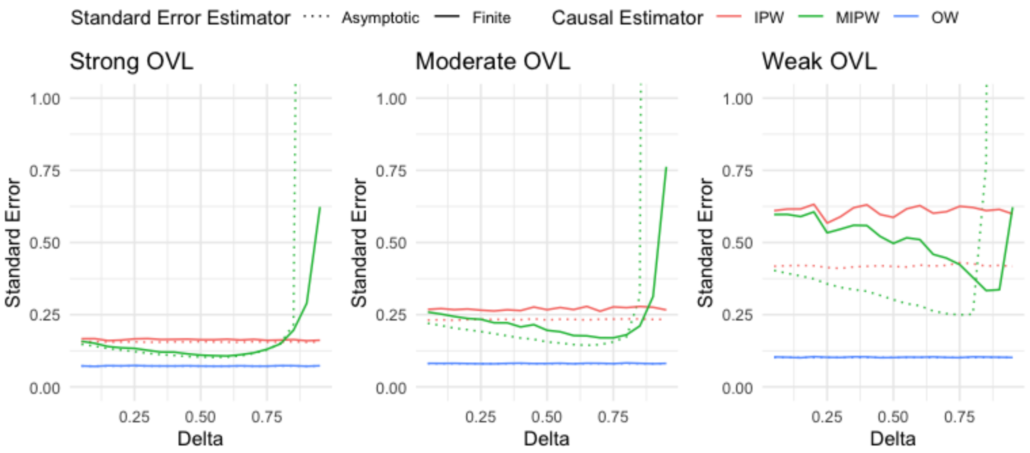

Figure 2: Monte Carlo simulation result: SD estimates (solid) and Huber-White robust SE estimates (dotted) of IPW, MIPW, OW

Simulation study for implementation 2: Resampling algorithm

- **Scenario 1**: The same weak overlap setting from previous study (true treatment effect = 1)
- **Scenario 2**: A modified study from Kang & Schafer (2007) to endow model misspecification but within weak overlap (true treatment effect = 210)
- **Extension to Entropy Balancing (EB)**: Weighting method that estimates $\frac{e}{1-e}(X_i)$ by solving a constrained optimization problem to reduce model dependence (Hainmueller, 2012).

Estimator	EB	MEB	OW
Target	ATT	ATT	ATO

Results: EB vs MEB vs OW

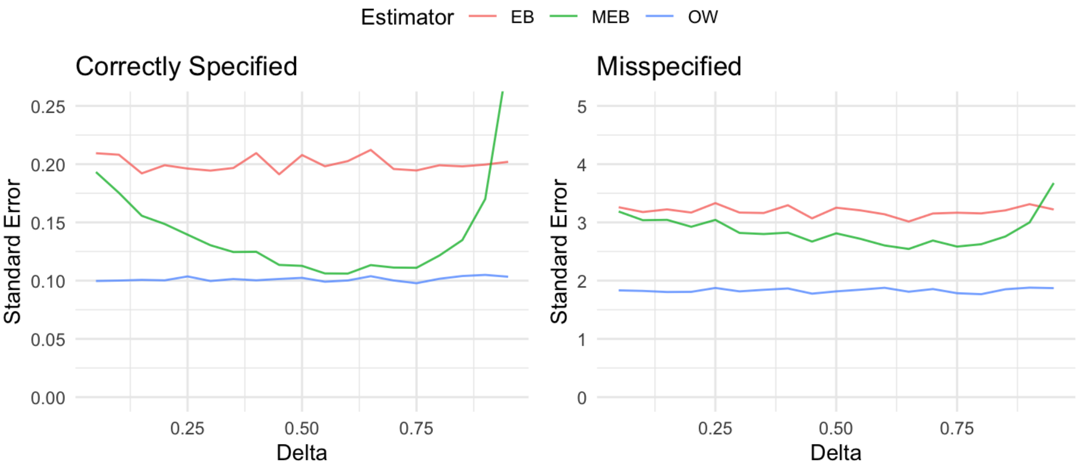

Figure 3: Monte Carlo simulation result: SD estimates of EB, MEB, OW

Results: EB vs MEB vs OW

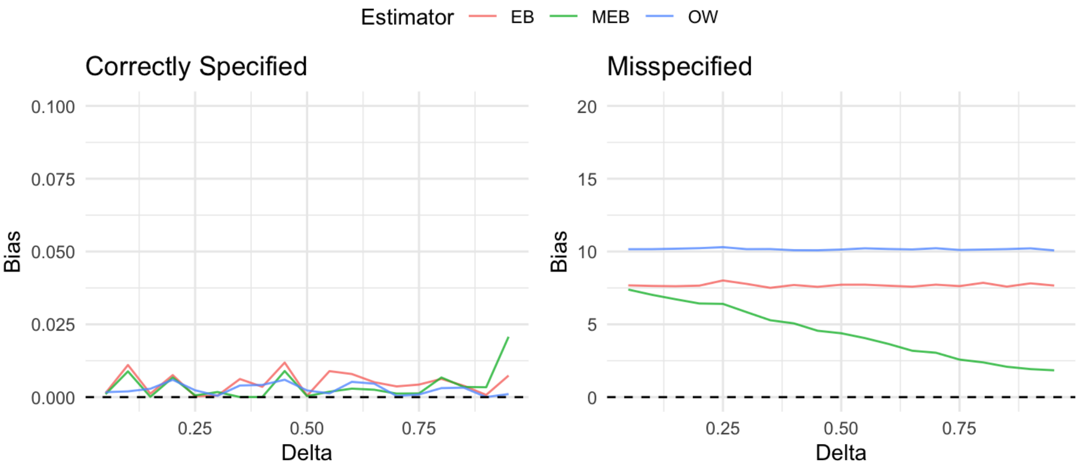

Figure 4: Monte Carlo simulation result: Finite-sample bias of EB, MEB, OW

Overview

Future Work

- **Heterogeneous mixing strategy**: What if we allow δ to vary according to the values of covariates?

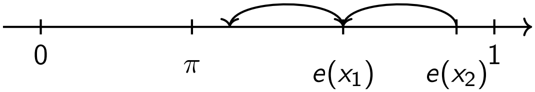

Figure 5: Homogeneous (Simple) Mixing: shrink with same ratio

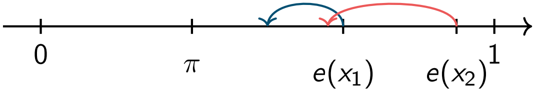

Figure 6: Heterogeneous (Advanced) Mixing: (Blue) shrink less (Red) shrink more

Future Work

- Primary results

Estimator	Bias	SD
IPW	0.1812	0.2656
Simple Mixing	0.1590	0.2440
Heterogeneous Mixing	0.1497	0.2232

Table 1: Advanced mixing strategy

- Other interesting topics remain, including application of mixing to matching methods.

Summary

Key Takeaways

Mixing: A simple & practical statistical tool for handling overlap to control extremeness of inverse probability weights without additional assumptions

- **Performance**: Efficiency is enhanced without bias trade-off (even in sufficient overlap)
- **Straightforward interpretation**: No need to shift the target estimand
- **Flexibility**: Applicable to broad range of weighting methods
- **Open to further exploration**: Heterogeneous mixing strategy

Thank you!

Email: `suehyunkim@snu.ac.kr`
Preprint link: https://arxiv.org/abs/2411.10801v3

Supplementary Materials 1

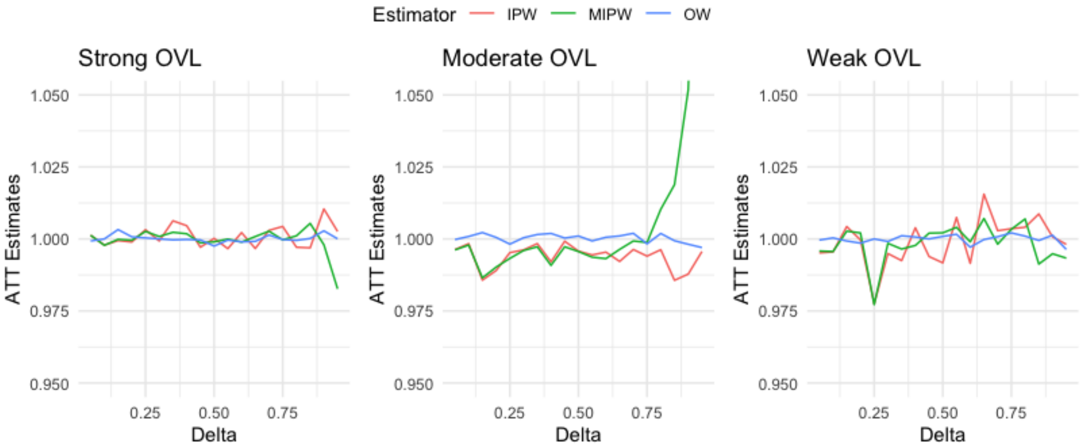

Figure 7: Monte Carlo Simulation Result: ATT Estimates of IPW, MIPW, OW

Supplementary Materials 2

Definition 2 (Mixed Distribution)

Let Z^* be a binary variable with marginal probability, π^*. Define the joint distribution of (Y^*, X^*) as distribution with density,

$$h^* = h^*_{\pi^*,\theta_1,\theta_0} = \pi^* h^*_1 + (1-\pi^*)h^*_0$$

where
$h^*_z = \theta_z h_1 + (1-\theta_z)h_0 \in \mathcal{M} := \{\theta h_1 + (1-\theta)h_0 : 0 \leq \theta = \theta(x) \leq 1, \forall x \in \mathcal{X}\}, z = 0, 1$ are the densities of conditional distribution given $Z^* = z, z = 0, 1$. We will call its distribution, H^* as "the mixed distribution of H".

Supplementary Materials 3

Lemma 2 (Propensity Score of the Mixed Distribution)

Let $e^*(x) = P(Z^* = 1 \mid X^* = x), \forall x \in \mathcal{X}$ *be the propensity score of the mixed distribution. Then,*

$$\frac{e^*}{1-e^*}(x) = \frac{\pi^*}{1-\pi^*} \cdot \frac{\theta_1(x)\frac{e}{1-e}(x) + (1-\theta_1(x))\frac{\pi}{1-\pi}}{\theta_0(x)\frac{e}{1-e}(x) + (1-\theta_0(x))\frac{\pi}{1-\pi}}, \qquad \forall x \in \mathcal{X}$$