2018-17119 박수현 컴퓨터비전 hw1 Theory

Note : My answers to the questions are highlighted in yellow.

3.1 Composing Filters [10 pts]

Consider the following three filters $\mathcal{G}$, $\mathcal{E}$ and $\mathcal{M}$. $\mathcal{G}$ is a Gaussian smoothing kernel, $\mathcal{E}$ is one of the linear kernels used by the Sobel edge detector and $\mathcal{M}$ is a median filter. Is applying $\mathcal{G}$ to an image followed by $\mathcal{E}$ equivalent to applying $\mathcal{E}$ to an image followed by $\mathcal{G}$? How about if $\mathcal{M}$ is used in place of $\mathcal{G}$? In both cases, explain your answer.

Hint: Think about the properties of convolution.

i) Applying $\mathcal{G}$ to an image followed by $\mathcal{E}$ is equivalent to applying $\mathcal{E}$ to an image followed by $\mathcal{G}$.

$\because$ $\mathcal{G}$ and $\mathcal{E}$ are both linear filters, and thus can both be applied to an image by convolution.

Since convolution satisfies commutativity, the order of filters to convolve doesn't affect the final result.

e.g., Given image I, the convolution with filters $\mathcal{G}$ and $\mathcal{E}$ can be expressed as follows, using properties of convolution:

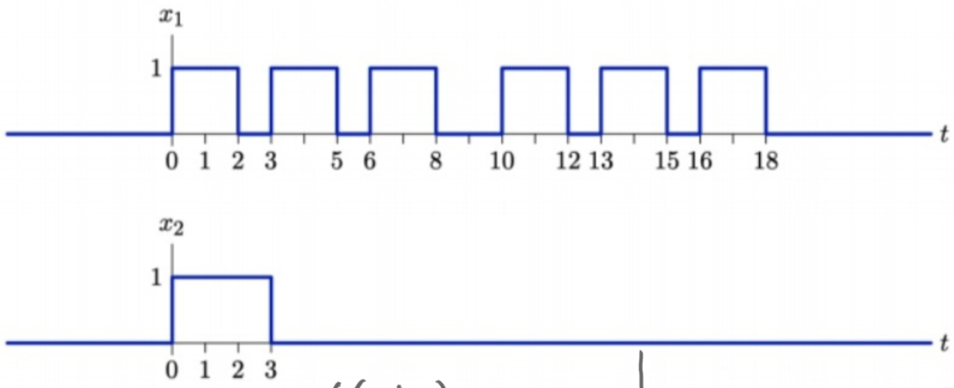
$$
\begin{aligned}
(I * \mathcal{G}) * \mathcal{E} &= I * (\mathcal{G} * \mathcal{E}) && \because \text{associativity} \\
&= I * (\mathcal{E} * \mathcal{G}) && \because \text{commutativity} \\
&= (I * \mathcal{E}) * \mathcal{G} && \because \text{associativity}
\end{aligned}
$$

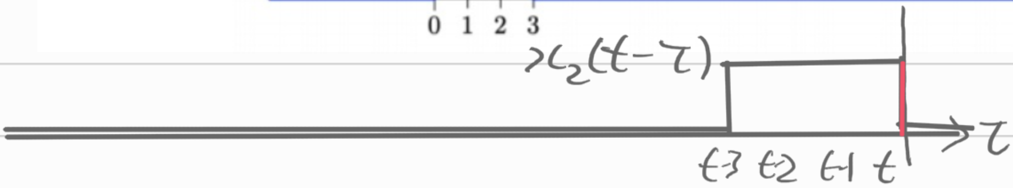
ii) Applying $\mathcal{M}$ to an image followed by $\mathcal{E}$ is <u>not</u> equivalent to applying $\mathcal{E}$ to an image followed by $\mathcal{M}$.

$\because$ $\mathcal{M}$ is a non-linear filter and cannot be implemented with convolution. $\mathcal{E}$, on the other hand, still makes use of convolution. Since commutativity of convolution can't take effect at this particular procedure, the order of using $\mathcal{M}$ can change the final result.

3.2 Convolution [20 pts]

Signals $x_1(t)$ and $x_2(t)$ are shown in the plots below, and are zero outside the indicated intervals. Draw the result of convolving $x_1(t)$ with $x_2(t)$. Make sure that the important break-points are clear. $= y(t)$





$t :=$ distance from the $\tau=0$ axis to the leading edge

$0 \le t < 2$: $y(t) = \int_0^t x_1(t)\, x_2(t-\tau)\, d\tau = \int_0^t 1 \cdot 1\, d\tau = t$

$2 \le t < 3$: $y(t) = \int_0^2 1 \cdot 1\, d\tau + \int_2^t 0 \cdot 1\, d\tau = 2$

$3 \le t < 5$: $y(t) = \int_{t-3}^2 1 \cdot 1\, d\tau + \int_3^t 1 \cdot 1\, d\tau = 2 - \cancel{t} + \cancel{3} + \cancel{t} - \cancel{3} = 2$

$5 \le t < 6$: $y(t) = \int_3^5 1 \cdot 1\, d\tau = 2$

$6 \le t < 8$: $y(t) = \int_{t-3}^5 1 \cdot 1\, d\tau + \int_6^t 1 \cdot 1\, d\tau = 5 - t + 3 + t - 6 = 2$

$8 \le t < 9$: $y(t) = \int_6^8 1 \cdot 1\, d\tau = 2$

$9 \le t < 10$: $y(t) = \int_{t-3}^8 1 \cdot 1\, d\tau = 8 - t + 3 = 11 - t$

$10 \le t < 11$: $y(t) = \int_{t-3}^8 1 \cdot 1\, d\tau + \int_{10}^t 1 \cdot 1\, d\tau = 8 - t + 3 + t - 10 = 1$

$11 \le t < 12$: $y(t) = \int_{10}^t 1 \cdot 1\, d\tau = t - 10$

$12 \le t < 13$: $y(t) = 2$

$13 \le t < 15$: $y(t) = \int_{t-3}^{12} 1 \cdot 1\, d\tau + \int_{13}^t 1 \cdot 1\, d\tau = 12 - t + 3 + t - 13 = 2$

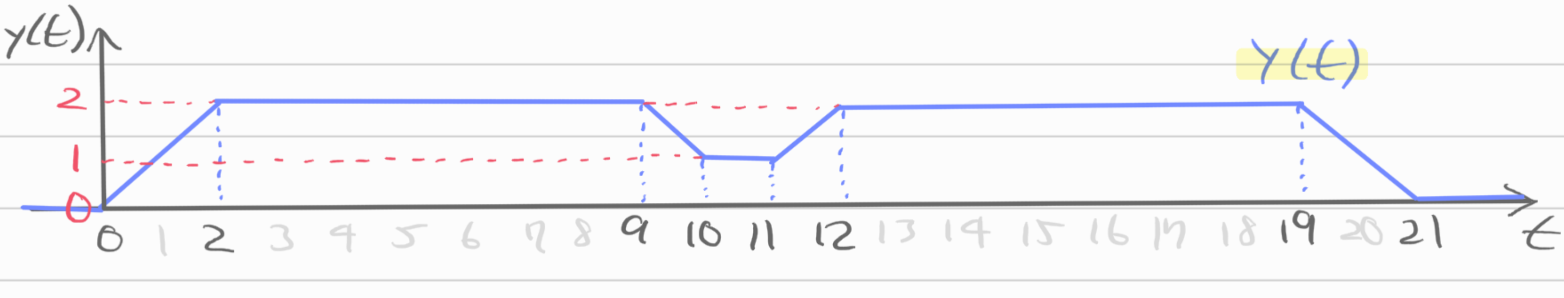
$15 \le t < 16$: $y(t) = 2$

$16 \le t < 18$: $y(t) = \int_{t-3}^{15} 1 \cdot 1\, d\tau + \int_{16}^t 1 \cdot 1\, d\tau = 15 - t + 3 + t - 16 = 2$

$18 \le t < 19$: $y(t) = 2$

$19 \le t < 21$: $y(t) = \int_{t-3}^{18} 1 \cdot 1\, d\tau = 18 - t + 3 = 21 - t$

$t < 0$, $t \ge 21$: $y(t) = 0$



3.3 Decomposing a Steerable Filter [10 pts]

In the continuous domain, a two dimensional Gaussian kernel $\mathcal{G}$ with standard deviation σ is given by $G(x,y) = \frac{1}{2\pi\sigma^2}\exp\left(-\frac{x^2+y^2}{2\sigma^2}\right)$. Show that convolution with $\mathcal{G}$ is equivalent to convolving with $\mathcal{G}_x$ followed by $\mathcal{G}_y$, where $\mathcal{G}_x$ and $\mathcal{G}_y$ are 1-dimensional Gaussian kernels in the x and y coordinate respectively, with standard deviation σ. From a computational efficiency perspective, explain which is better, convolving with $\mathcal{G}$ in a single step, or the two step $\mathcal{G}_x$-and-$\mathcal{G}_y$ approach.

Convolving $f(x,y)$ with $G(x,y)$,

$\therefore f(x,y) * G(x,y) = G(x,y) * f(x,y)$ $\because$ commutativity of convolution

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{2\pi\sigma^2} \exp\left(-\frac{x^2+y^2}{2\sigma^2}\right) f(x-\tau_1, y-\tau_2)\, d\tau_1\, d\tau_2$$

$$= \frac{1}{2\pi\sigma^2} \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x-\tau_1, y-\tau_2)\cdot\exp\left(-\frac{x^2}{2\sigma^2}\right) d\tau_1\right] \exp\left(-\frac{y^2}{2\sigma^2}\right) d\tau_2$$

$$= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(x-\tau_1, y-\tau_2)\, \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{x^2}{2\sigma^2}\right) d\tau_1\right] \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{y^2}{2\sigma^2}\right) d\tau_2$$

$$= \int_{-\infty}^{\infty} \left[f(x,y) * G_x(x)\right] \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{y^2}{2\sigma^2}\right) d\tau_2$$

$$= \left[f(x,y) * G_x(x)\right] * G_y(y)$$

$$= f(x,y) * G_x(x) * G_y(y)$$

If the 2D Gaussian kernel is of size $n \times n$,

- during full convolution, computation is done for each n^2 values seperately, yielding complexity of $O(n^2)$
- during the seperable convolution, computation can be performed on rows and columns, which is in total $O(2n)$.

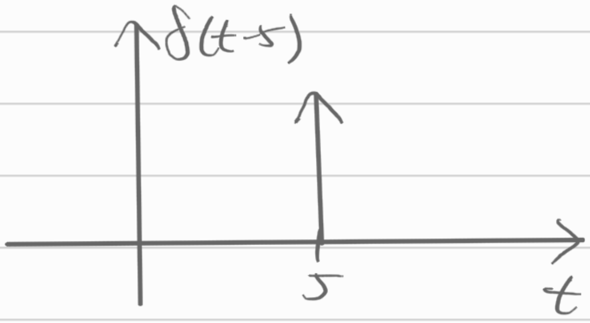
$\therefore$ From a computational efficiency perspective, convolving in a two step approach is much better.

3.4 Fourier Transform [20 pts]

a) Find the Fourier transform of the signal $\delta(t-5)$ and draw the magnitude and phase as a function of frequency, including both positive and negative frequencies.

b) Show that if $x_3(t) = ax_1(t) + bx_2(t)$, then $X_3(w) = aX_1(w) + bX_2(w)$.

a)



$\int_{-\infty}^{\infty} \delta(t-5)\,dt = 1$

Dirac Delta function!

$f(t) = \delta(t-5) \Rightarrow F(w) = \int_{-\infty}^{\infty} \delta(t-5) \exp(-iwt)\,dt$
$= \exp(-5iw)$

$\because$ the Sifting Property of the Dirac Delta function:

for any function $g(x)$ continuous at t_0, $\int_{-\infty}^{\infty} g(t)\,\delta(t-t_0)\,dt = g(t_0)$

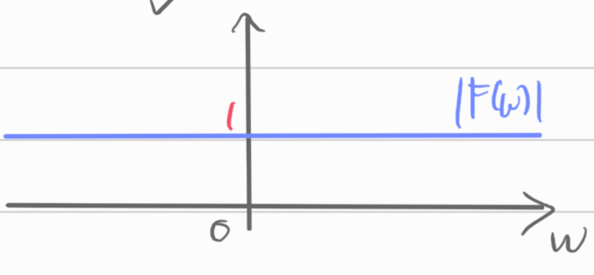
$F(w) = \exp(-5iw)$
$= \cos(-5w) + i\sin(-5w)$ $\quad\because$ Euler's formula
$= \cos(-5w) - i\sin(-5w)$

magnitude : $|F(w)| = \sqrt{\cos(-5w)^2 + \sin(-5w)^2} = 1$

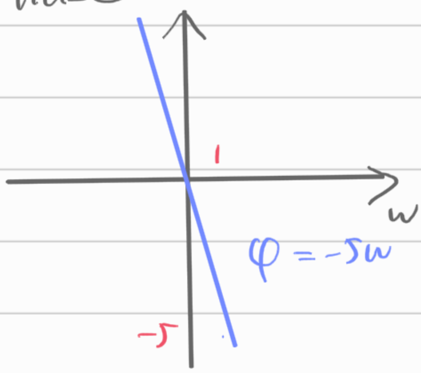
phase : $\varphi = \tan^{-1}\left[\frac{\sin(-5w)}{\cos(-5w)}\right] = \tan^{-1}[\tan(-5w)] = -5w$

$\therefore$ $F(w) = \exp(-5iw)$

→ magnitude



→ phase



b) $x_3(t) = ax_1(t) + bx_2(t)$

$X_3(w) = \int_{-\infty}^{\infty} (ax_1(t) + bx_2(t)) \exp(-iwt)\,dt$
$= a\int_{-\infty}^{\infty} x_1(t)\exp(-iwt)\,dt + b\int_{-\infty}^{\infty} x_2(t)\exp(-iwt)\,dt$
$= aX_1(w) + bX_2(w)$

$\therefore$ $ax_1(t) + bx_2(t) \xrightarrow{F.T} aX_1(w) + bX_2(w)$,

i.e., the Fourier Transform is linear.