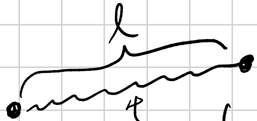
PBD with implicit HMM

Start with no tendon



What is the PBD constraint $C(x)$ that gives the proper HMM force?

In PBD, the potential and force for a constraint are given by:

$$V = \frac{1}{2} \alpha^{-1} C(x)^2 \quad \text{and} \quad f = -\frac{dV}{dx} . \tag{1}$$

We want the force to be equal to the muscle force. The muscle force, ignoring pennation:

$$f_0 \underbrace{\left(a f^L(\tilde{\ell}) f^V(\tilde{v}) + f^P(\tilde{\ell}) + \beta \tilde{v} \right)}_{f^M} . \tag{2}$$

Note this is a scalar equation, but the PBD force is 3D.
The direction of this force is given by $\frac{\Delta x}{\ell}$ because

$$\ell = \|\Delta x\| , \qquad \frac{d\,\|\Delta x\|}{d\Delta x} = \frac{\Delta x}{\|\Delta x\|} . \tag{3}$$

Combining (1) and (2), with the magnitude of the muscle force f^M and the direction $\frac{\Delta x}{\ell}$,

$$
\begin{aligned}
f_0 \, f^M \frac{\Delta x}{\ell} &= -\frac{dV}{dx} \\
&= -\frac{dV}{d\ell} \frac{d\ell}{dx} \\
&= -\frac{d}{d\ell}\left(\frac{1}{2} \alpha^{-1} C^2\right) \frac{\Delta x}{\ell} \\
&= -\alpha^{-1} C \frac{dC}{d\ell} \frac{\Delta x}{\ell} .
\end{aligned}
\tag{4}
$$

The directions match between the RHS and LHS as expected. The magnitudes must also match:

$$
f_0 \, f^M = -\alpha^{-1} C \frac{dC}{d\ell} . \tag{5}
$$

Let $\alpha = \frac{1}{f_0}$, so $f^M = -C \, \nabla_\ell C$.

The constraint function is a fuction of ℓ and v.

Find a network that minimizes

$$
\left\| C(\ell, v) \, \nabla_\ell C(\ell, v) + f^M(\ell, v) \right\|_2^2 . \tag{6}
$$

We need to modify this to use $\hat{l}$ and $\tilde{v}$ because we want to use the same network for all muscles, no matter what the length is. Going back to (4), we have:

$$
\begin{aligned}
f_0 f^M \frac{\Delta x}{l} &= \frac{-dV}{dx} \\
&= -\frac{dV}{d\tilde{l}} \frac{d\tilde{l}}{dl} \frac{dl}{dx} \qquad (7) \\
&= -\frac{d}{d\tilde{l}}\left(\frac{1}{2}\alpha^{-1} C^2\right) \frac{1}{l_{opt}} \frac{\Delta x}{l} \\
&= -\alpha^{-1} C \nabla_{\tilde{l}} C \frac{1}{l_{opt}} \frac{\Delta x}{l}
\end{aligned}
$$

Let $\alpha = \frac{1}{f_0 l_{opt}}$, so $f^M = -C \nabla_{\tilde{l}} C$.

Find a network that minimizes

$$\| C(\tilde{l}, \tilde{v}) \nabla_{\tilde{l}} C(\tilde{l}, \tilde{v}) + f^M(\tilde{l}, \tilde{v}) \|_2^2 .$$