

(28th February)

## Combinatorics

There are six lectures that need to be well understood for a test that comes in less than 2 weeks (%20) QUIZ.

- Permutations (Today)
- Subsets
- Set Partitions
- Integer Partitions
- Product Spaces (Gray Codes)
- Grade Coach

## Permutations

"Johnson Trotter" permutation enumeration algorithm...notation we speak of  $[1, 2, \dots, n] = [n]$ .

Proposition: The number of permutations of  $n$  objects of length  $k$  is

$$n(n-1)(n-2)\dots(n-k+1) = (n)_k \quad (1)$$

Of course where the total number of permutations is  $n!$ ... And in combinatorics we want to do three things efficiently

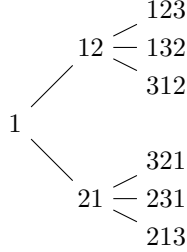
- Enumerate every permutation (each step constant amortized time, "fast")
- Map combinatorial object to an integer (ranking).
- Map a rank to re-build a combinatorial object (de-ranking).

## Lexicographic Ordering

We do the natural enumeration, but each step is not exactly constant, lots of shifting needs to occur hence we lean onto the "Johnson Trotter" style permutation enumeration approach.

## Johnson Trotter

Intuition really is like; *notice the swapping of the 3 simply "zigzagging"....*



## Ranking

Recursive "stripping" definition;  $n$  is the  $n$ th item of the list.  $j$  is insertion position of the stripped  $n$  in  $\pi$ .

$$Rank(\pi) = n \cdot Rank(\pi_j) + \begin{cases} j - 1, & \text{if } Rank(\pi_j) \text{ is odd} \\ n - j, & \text{if } Rank(\pi_j) \text{ is even} \end{cases}$$

### Example; Rank(31254)

We construct down first like so; we stop when we try to compute a Rank of an ordered list. (as Rank is 0 in this case).

$$Rank(31254) = 5 \cdot Rank(3124) + \text{something}, j = 4 \quad (2)$$

$$Rank(3124) = 4 \cdot Rank(312) + \text{something}, j = 4 \quad (3)$$

$$Rank(312) = 3 \cdot Rank(12) + \text{something}, j = 1 \quad (4)$$

Now we build back up and populate all those "somethings", from the bottom using the ranking equation given...

$$Rank(31254) = 5 \cdot Rank(3124) + (5 - 4) = (5 \cdot 8) + 1 = 41, j = 4 \quad (5)$$

$$Rank(3124) = 4 \cdot Rank(312) + (4 - 4) = (4 \cdot 2) + 0 = 8, j = 4 \quad (6)$$

$$Rank(312) = 3 \cdot Rank(12) + (3 - 1) = 0 + 2 = 2, j = 1 \quad (7)$$

### Example; Rank(25134)

Just repeating

$$Rank(25134) = 5 \cdot Rank(2134) + \text{something}, j = 2 \quad (8)$$

$$Rank(2134) = 4 \cdot Rank(213) + \text{something}, j = 4 \quad (9)$$

$$Rank(213) = 3 \cdot Rank(21) + \text{something}, j = 3 \quad (10)$$

$$Rank(21) = 2 \cdot Rank(1) + \text{something}, j = 1 \quad (11)$$

$$Rank(25134) = 5 \cdot Rank(2134) + (2 - 1) = (5 \cdot 23 + 1) = 116, j = 2 \quad (12)$$

$$Rank(2134) = 4 \cdot Rank(213) + (4 - 1) = (4 \cdot 5) + 3 = 23, j = 4 \quad (13)$$

$$Rank(213) = 3 \cdot Rank(21) + (3 - 1) = (3 \cdot 1) + 2 = 5, j = 3 \quad (14)$$

$$Rank(21) = 2 \cdot Rank(1) + (2 - 1) = 0 + 1 = 1, j = 1 \quad (15)$$

**Un-ranking is omitted, we must read the books**