CS761 Artificial Intelligence

First-order Logic

Towards a More Expressive Logic

Logic as a knowledge representation language:

1. It is declarative: Separation between knowledge and control

   E.g., The language describes what an agent senses in the wumpus world, not how to achieve its goals.

2. it is expressive: Ability to handle unknown information or partially-known information.

   E.g., The language can state *"there is a pit in [2,2] or [3,1]"*, or *"if the wumpus is in [1,1], then it is not in [2,2]"*.

3. It is compositional: The meaning of a sentence is a function of the meaning of its parts.

   E.g., The language can compose sentences into longer sentences using "and", "or", "if", "if and only if".

Propositional logic as a knowledge representation language has severe shortcomings:

- Verbosity: Stating simple facts often requires listing a large number of propositions.

  Example. "A breeze is sensed if and only if an adjacent location contains a pit."

  $B_{1,1} \leftrightarrow (P_{1,2} \vee P_{2,1}), \qquad B_{1,2} \leftrightarrow (P_{1,3} \vee P_{2,2} \vee P_{1,1}),$
  $B_{2,1} \leftrightarrow (P_{2,2} \vee P_{3,1} \vee P_{1,1}), \qquad B_{2,2} \leftrightarrow (P_{1,2} \vee P_{2,1} \vee P_{3,2} \vee P_{2,3}), \ldots$

- Low expressiveness: Many facts cannot be expressed by propositional logic in a meaningful way.

  Example.

  - Universal sentence: "For all elements, ...".
    E.g. "Every mammal is breast-feeding"
  - Existential sentence: "There exists an element such that ...".
    E.g. "Some bird does not fly"
  - Reasoning by instantiation:
    E.g. "All mammals are breast-feeding. A dolphin is a mammal. Therefore dolphins are breast-feeding."

- A model of a knowledge base is an interpretation that is consistent with the knowledge base.
- An interpretation of a propositional knowledge base is a truth value function π that maps all propositions to $\{\text{true}, \text{false}\}$.
- In propositional logic, the atomic entities are sentences. But sentences are by far not the basic building blocks of human languages.
- The "atomic building blocks" of a human language are words such as nouns, verbs, adjectives, etc.

  E.g.,
  "If a square has a pit, then all surrounding squares are breezy."

  "Adding one to an even integer gives an odd integer."

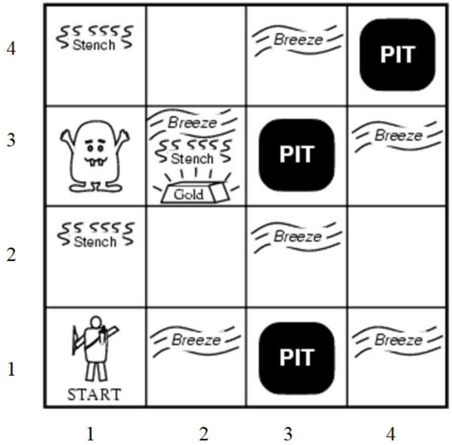
  "Lucy is a sister of Adam's mother."
- We need a knowledge representation language that is expressive enough to represent these atomic building blocks explicitly.

From Natural to Formal Languages

Examples of objects, relations (predicates), and functions:

- Objects: people, houses, numbers, theories, colours, …
- Relations:
  - Unary relations: being red, being big, being smart, being a prime number, being uncertain, …
  - Binary relations: bigger than, older than, divides, is more preferred than, …
  - n-ary relations ($n \geq 3$): mother and father and child relation, …
- Functions: father of, the number that is one more than, addition, the inverse of, …

Question. How would you describe the state of the wumpus world below?



A interpretation should capture:

- Domain: A set of objects.

  E.g., $\{(1,1),(1,2),\ldots,(4,3),(4,4)\}$ representing squares in a 4×4 environment

- Relations:
  - Unary relations:
    E.g., $Breeze = \{(1,2),(1,4),(2,3),(3,2),(3,4),(4,3)\}$
    $Pit = \{(1,3),(3,3),(4,4)\}$
    $Stench = \{(2,1),(4,1),(3,2)\}$
    $WumpusAt = \{(3,1)\}$, $AgentAt = \{(1,1)\}$
    $Glitter = \{(3,2)\}$
  - n-ary relations (where $n \geq 2$):
    E.g., $Adjacent = \{((1,1),(1,2)),((1,1),(2,1)),((1,2),(2,2)),\ldots\}$

- Functions:

  E.g., $left(1,2) = (1,1),\ldots$

  $right(2,3) = (2,4),\ldots$

Semantics of First-order Logic

Definition [Semantics]

A first-order interpretation is a tuple

$$(D, R_1, R_2, \ldots, R_k, f_1, \ldots, f_\ell)$$

where

- D is a set of elements, and is called the domain.
- Each R_i where $1 \leq i \leq k$ is a relation (or predicates), of certain arity r_i, defined on D, i.e., R_i is a subset of the Cartesian product D^{r_i}.
- Each f_i where $1 \leq i \leq \ell$ is a function (of certain arity s_i) defined on D, i.e., $f_i \colon D^{s_i} \to D$.

E.g. A first-order interpretation of the Wumpus world could be defined as

- Domain: $D = \{(1,1), (1,2), \ldots, (4,4)\}$
- Relations: $AgentAt, Breeze, Pit, WumpusAt, Stench, Glitter, Adjacent$
- Functions: $left, right, up, down$

It describe a state of the current world.

Syntax of First-order Logic

We need to define sentences that could be interpreted using the type of interpretations above.

Definition [Alphabet]

The alphabet of a first order language contains the following symbols

1. $v_0, v_1, v_2, \ldots,$ (variables)
2. $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$ (connectives)
3. $\forall, \exists$ (quantifiers)
4. $=$ (equality symbol)
5. (,) (parentheses)
6. $R_1, R_2, \ldots, R_k$ (relational (or predicate) symbols) and arity function $r_R : \mathbb{N} \rightarrow \mathbb{N}$
7. $f_1, f_2, \ldots, f_\ell$ (functional symbols) and arity function $r_f : \mathbb{N} \rightarrow \mathbb{N}$

A 0-ary function symbol is called a constant symbol.

Definition [Signature]

The set $S = \{R_1, R_2, \ldots, R_k, f_1, f_2, \ldots, f_\ell\}$ of relational symbols and functional symbols in the alphabet of a first order language is called its signature.

E.g. Signature of the Wumpus world could contain $AgentAt, WumpusAt, Pit, Breeze, Stench, Glitters, Adjacent, left, \ldots$

Definition [Terms]

Let S be a signature. The set of terms T are defined as follows:

- Every variable is a term in T
- Every constant symbol is a term in T
- If $t_0, t_1, t_2, \ldots, t_{r-1}$ are terms and f is a function symbol in S with arity r, then $f(t_0, t_1, \ldots, t_{r-1})$ is a term in T.

A ground term is a term that does not contain any variable symbol (i.e. functions only applied to constant symbols).

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Examples. Take S as the signature of the wumpus world. Then the following are terms:

- $(1, 0)$ (this is ground)
- $left(up(x, y))$ (this is not ground)
- $down(right(down(3, 2)))$ (this is ground)

Definition [first-order formulas]

Let S be a signature. The (first-order) formulas are strings of symbols over the alphabet of S, inductively defined as follows:

1. If t_0, t_1 are terms, then $t_0 = t_1$ is an formula.
2. If $t_0, \dots, t_{n-1}$ are terms, and R is an n-ary relation symbol in S, then
$$R(t_0, \dots, t_{n-1}) \text{ is an formula}$$
3. If φ is an formula, then $\neg\varphi$ is an formula
4. If φ_0, φ_1 are formulas, then $(\varphi_0 \vee \varphi_1), (\varphi_0 \wedge \varphi_1), (\varphi_0 \rightarrow \varphi_1), (\varphi_0 \leftrightarrow \varphi_1)$ are formulas
5. If φ is an formula and x is a variable, then $\forall x : \varphi$ and $\exists x : \varphi$ are both formulas.

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The formulas in (1) and (2) are called atomic.

Examples. S is the signature of wumpus world.
The following are formulas

- $\forall x\colon \exists y\colon Adjacent(x, y)$
- $\forall x\colon Wumpus(x) \rightarrow (\forall y\colon Adjacent(x, y) \rightarrow Stench(y))$
- $Wumpus((1, 2)) \vee Wumpus((2, 1))$
- $Wumpus(x) \wedge Adjacent(x, (4, 2))$
- $\forall x\colon Pit(x) \rightarrow (Adjacent(x, y) \wedge Breeze(y))$

Free Variables and Sentences

Definition [Free Variables and Sentences]

- In any formula of the form $\exists x : \varphi$ (or $\forall x : \varphi$), we refer to φ as the scope of $\exists x$ (or $\forall x$).
- A variable x occurring in a formula φ is called a bounded variable if it is within the scope of some $\exists x$ or $\forall x$ (sub-formula of the formula φ).
- If x is not bounded, then it is called a free variable.
- If a formula does not contain any free variable, then it is called a sentence.

E.g. These formulas are not sentences:

- $AgentAt(x)$
- $\forall x\colon Pit(x) \rightarrow (Adjacent(x, y) \wedge Breeze(y))$
- $WumpusAt(x) \wedge \forall x\colon Adjacent(x, (4, 2))$

Satisfaction Relation

Definition [Satisfaction Relation]

Let φ be a sentence and $\mathcal{I}$ be an interpretation with the same signature. We inductively define the satisfaction relation $\models$ in such a way that $\mathcal{I}$ satisfies φ, written $\mathcal{I} \models \varphi$, if

- $\mathcal{I} \models t_0 = t_1$ iff the terms t_0 and t_1 receive the same meaning in $\mathcal{I}$
- $\mathcal{I} \models R(t_0, \ldots, t_{n-1})$ iff the tuple $(t_0, \ldots, t_{n-1})$ belong to the relation R in $\mathcal{I}$
- $\mathcal{I} \models \neg\varphi$ iff not $\mathcal{I} \models \varphi$
- $\mathcal{I} \models (\varphi \wedge \psi)$ iff $\mathcal{I} \models \varphi$ and $\mathcal{I} \models \psi$
- $\mathcal{I} \models (\varphi \vee \psi)$ iff $\mathcal{I} \models \varphi$ or $\mathcal{I} \models \psi$
- $\mathcal{I} \models (\varphi \rightarrow \psi)$ iff $\mathcal{I} \models \varphi$ implies $\mathcal{I} \models \psi$
- $\mathcal{I} \models (\varphi \leftrightarrow \psi)$ iff ($\mathcal{I} \models \varphi$ if and only if $\mathcal{I} \models \psi$)
- $\mathcal{I} \models \forall x : \varphi$ iff for all $a \in D$, $\mathcal{I}[x/a] \models \varphi$ ($\mathcal{I}[x/a]$ is the interpretation $\mathcal{I}$ where we associate the meaning of x to a)
- $\mathcal{I} \models \exists x : \varphi$ iff there is an $a \in D$ such that $\mathcal{I}[x/a] \models \varphi$

Essentially $\mathcal{I} \models \varphi$ if φ is true according to the interpretation $\mathcal{I}$.

Logical Equivalence

Suppose φ_1 and φ_2 are sentences that have the same signature. We say that φ_1 and φ_2 are logically equivalent if for any interpretation, we have

$$\mathcal{I} \models \varphi_1 \text{ if and only if } \mathcal{I} \models \varphi_2.$$

Examples.

- $\neg(\varphi_1 \vee \varphi_2)$ and $\neg\varphi_1 \wedge \neg\varphi_2$
- $\neg(\varphi_1 \wedge \varphi_2)$ and $\neg\varphi_1 \vee \neg\varphi_2$
- $\neg\forall x\colon \varphi(x)$ and $\exists x\colon \neg\varphi(x)$
- $\neg\exists x\colon \varphi(x)$ and $\forall x\colon \neg\varphi(x)$
- $\exists x\colon \varphi_1(x) \vee \varphi_2(x)$ and $\exists x\colon \varphi_1(x) \vee \exists x\colon \varphi_2(x)$
- $\forall x\colon \varphi_1(x) \wedge \varphi_2(x)$ and $\forall x\colon \varphi_1(x) \wedge \forall x\colon \varphi_2(x)$
- $\neg\forall x\colon \varphi_1(x) \rightarrow \varphi_2(x)$ and $\exists x\colon \varphi_1(x) \wedge \neg\varphi_2(x)$

First-order Knowledge Base

Definition.

A first-order knowledge base (FO-KB) is a set of first-order sentences in some signature.

To define model of a FO-KB, we make the following two assumptions:

1. Closed-world assumption: In a model of the KB, if any atomic sentence is not to be known or shown to be true, then it is false.
2. Domain-closure assumption: In a model of the KB, all elements of the domain appear as ground terms that can be expressed using constants.

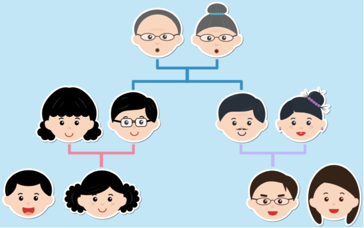
Definition

A model of a FO-KB if a first-order interpretation that satisfies all sentences in the KB and the assumptions above.

First-order Logic Example 1: Kinship

This domain aims to describe kinship ties among family members.
Signature:

- Relations:
  - Unary relation: *Male*
  - Binary relations: *Parent*, *Sibling*, *Brother*, *Sister*, *Child*, *Spouse*, *Grandparent*, *Grandchild*, *Cousin*, *Aunt*, *Uncle*
- Functions: *mother*, *father*



Axioms provide the basic factual information from which useful conclusions can be derived.

Note that axioms often take the form of definitions:

- $\forall m, c\colon mother(c) = m \leftrightarrow (\neg male(m) \land Parent(m, c))$
- $\forall g, c\colon Grandparent(g, c) \leftrightarrow (\exists p\colon Parent(g, p) \land Parent(p, c))$
- $\forall x, y\colon Sibling(x, y) \leftrightarrow (x \neq y \land \exists p\colon Parent(p, x) \land Parent(p, y))$

Theorems are factual information derived from the axioms.

- $\forall x, y\colon Sibling(x, y) \leftrightarrow Sibling(y, x)$
- $\forall x\colon (\exists y\colon mother(y) = x) \rightarrow \neg Male(x)$

First-order Logic Example 2: Numbers

Signature:

- Unary relation: *NatNum* (denoting natural numbers)
- Constant: 0
- Function: s (denoting successors)
- Function: + (denoting addition)

Axioms/Definitions:

- $NatNumber(0)$
- $\forall n\colon NatNum(n) \rightarrow NatNum(s(n))$
- $\forall n\colon 0 \neq s(n)$
- $\forall m, n\colon m \neq n \rightarrow s(m) \neq s(n)$
- $\forall m\colon NatNum(m) \rightarrow +(0, m) = m$
- $\forall m, n\colon NatNum(m) \wedge NatNum(n) \rightarrow +(s(m), n) = s(+(m, n))$

First-order Logic Example 3: Sets

Signature:

- Constant: $\varnothing$
- Unary relation: *Set*
- Binary relations: $\in$, $\subseteq$
- Function: $\cap$, $\cup$,
- Function: adj ($adj(x, s)$ denotes adjoining element x to a set s)

Axioms/Definitions:

- $\forall s\colon Set(s) \leftrightarrow (s = \varnothing) \vee (\exists x, s_2\colon Set(s_2) \wedge s = adj(x, s_2))$
- $\neg\exists x, s\colon adj(x, s) = \varnothing$
- $\forall x, s\colon x \in s \leftrightarrow s = adj(x, s)$
- $\forall s_1, s_2\colon s_1 \subseteq s_2 \leftrightarrow (\forall x\colon x \in s_1 \rightarrow x \in s_2)$
- $\forall x, s_1, s_2\colon x \in (s_1 \cap s_2) \leftrightarrow (x \in s_1 \wedge x \in s_2)$
- $\forall x, s_1, s_2\colon x \in (s_1 \cup s_2) \leftrightarrow (x \in s_1 \vee x \in s_2)$

Summary of The Topic

The following are the main knowledge points covered:

- Shortcomings of propositional logic as a knowledge representation language:
  - Verbosity
  - Low expressiveness
- Semantics of FO Logic: FO interpretation
  - domain
  - relations
  - functions
- Syntax of FO Logic: FO formulas
  - symbols
  - terms
  - formulas
  - existential/universal quantification
- Bounded and free variables in a formula. First-order sentence.
- Satisfaction: $\mathcal{I} \models \varphi$
- FO-KB and their models.