Essays on mechanisms for efficient social choice, matching and mental competition

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Declaration of Authorship

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"Economist, unlike Lawyers, believe you have to pay people to tell the truth."

Morton Kamien

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Abstract

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Essays on mechanisms for efficient social choice, matching and mental competition

by Yan JU

Mechanisms Design is a very interesting field of economics. In this paper, theories about general mechanisms are presented as well as some kinds of special mechanisms such as matching. With all the needed details provided or given in the literature, the thesis is mostly a reflection of the author's interest and provide insightful arguments and findings in related areas.

For two particular kinds of interdependent value problems, the VCG mechanism is adapted to achieve ex post implementation of social efficiency result. The uniqueness of tranfer schemes for ex post implementation of continuously differentible social goal with interdependent private signals are shown when such scheme does exist. For the discrete social goals, allocation of one and many goods are considered. Under the interdependent value environment, the paper proposes a sufficient condition for ex post implementation of efficiency in an one-good auction, i.e., allocating the good to the agent who values it most.This condition is less demanding than a sufficient condition that is proposed in (Dasgupta and Maskin, 2000). Then the paper further extend the sufficient condition to the case of multiple goods assignment with interdependent values and call the extended condition Condition ρ. Then we provide a model where Condition ρ is satisfied and we design a mechanism to ex post implement the social efficiency maximizing goal through appropriate money transfer.

For the Chinese college admission mechanism, the paper finds that the allocation results of the ex post equilibrium in the Boston Mechanism, Chinese parallel mechanism and the DA mechanism are the same under the same score-based priority structures. However, the ease for the DA mechanism to reach this allocation result can never be reached for the other two mechanisms, which might be a motivation for advocating DA mechanism in the Chinese college admission mechanism. Under general priority structures of the colleges, the sets of ex post equilibrium matching results for these mechanisms exibits a nested structure with or without province-oriented reserve policy, which still favors DA.

For a perfect information competitive board game like chess which is a mechanism for deciding win or loss, the last chapter of this paper classify every position into three categories, winning position, losing position, and drawing position, and provides the way to identify them theoretically. It is only theoretical for chess because the work

need to be done is astronomical figure. However, the paper manages to use this method to solve an interesting game, which shows the power of the notions.

Acknowledgements

Contents

List of Abbreviations

VCG   Vickery Clark Groves

DA   Deferred Acceptance

SOSM Student Optimal Stable Matching

TTC   Top Trading Cycle

SD   Serial Dictatorship

CCA   Chinese College Admission

IA   Instant Acceptance

Dedicated To my parents who has been so patient and loving to me since my birth

Chapter 1

An Introduction to Mechanism Design Theory

1.1 Introduction

The starting point of analysis in traditional economics is often the present state of economic institutions and mechanisms, for example, market theory, auctions and resource allocation. In mechanism design theory, an alternative framework and perspective is used. We view the present state of affairs just as a possibility and has an intention to design a better mechanism with respect to a social goal or criterion. Mechanism design is kind of engineering side of economics. To do that, a framework must be provided. Now the framework has become a theory with a large body of literatures. That theory is mechanism design theory. The mechanism design theory was established by leonid Hurwicz, later expanded by Roger B.Myerson, Eric S. Maskin, Guoqiang Tian and many other scholars of the field.

The most important component of Mechanism design theory is implementation theory. (Liang, 2010) has done a survey of this field . The present paper is not another attempt to summarize the implementation theory of mechanism design, but uses it as a central tool for formalizing ideas in this paper. This paper aims to analyze some economic mechanisms for situations where information is not fully known to the social planner. In the process, the author creates some new concepts which the author thought is necessary. However, a little background of the mechanism design theory is needed. This is what the next section is about.

1.2   A survey of the background

A formal study of the informational requirements and informational optimality of resource allocation processes was initiated by Hurwicz (1960). In line with the prevailing tradition, interest in this area was focused on the design of Pareto-satisfactory (non-wasteful) and privacy-preserving mechanisms, i.e., mechanisms that result in Pareto efficient allocations and use informationally decentralized decision making processes. Allocative efficiency and informational efficiency are two highly desired properties for an economic system to have. The importance of Pareto optimality is attributed to what may be regarded as a minimal welfare property. Pareto optimality requires resources be allocated efficiently. If an allocation is not efficient, there is a waste in allocating resources and thus at least one agent is better off without making others worse off under given resources. Informational efficiency requires an economic system have the minimal informational cost of operation. The informational requirements depend upon two basic components: the class and types of economic environments over which a mechanism is supposed to operate and the particular outcomes that a mechanism is required to realize.

For informational decentralized systems, ([“On informationally decentralized systems”](#)) proved a very important theorem: For the neoclassical pri- vate goods economies, there is no mechanism < M, h > that implements Pareto efficient and individually rational allocations in dominant strategy. Consequently, any revelation mechanism < M, h > that yields Pareto efficient and individually rational allocations is not strongly individually incentive compatible. (Truth-telling about their preferences is not Nash Equilibrium).

A mechanism can be viewed as an abstract planning procedure; it consists of a message space in which communication takes place, rules by which the agents form messages, and an outcome function which translates messages into outcomes (allocations of resources). Mecha- nisms are imagined to operate iteratively. Attention, however, may be focused on mechanisms that have stationary or equilibrium messages for each possible economic environment. A mechanism realizes a prespecified welfare criterion (also called performance, social choice rule, or social choice correspondence) if the outcomes given by the outcome function agree with the welfare criterion of the

stationary messages.

1.3 General model framework

In this section, we give the framework of analysis which will reappear many times later in the paper with slightly different forms. Our framework is comprised of 5 parts: economic environments, social goals or criteria, mechanism, expected outcomes(often equilbirums of all kinds), and the concept of implementation of social goals. The following subsections will give a detailed discussion and relative notations of these components.

1.3.1 Economic environments

Economic environments consists of economic entities and their features as well as the state of some relevant things in the world. The entities are of two kinds. One is the principal (or called the social planner), and the other is the agents (or called economic participants). Usually, we have the following notations.

$N = \{1, ..., n\}$: denote the set of the agents.

$e_i \in E_i$: denote the economic feature of agent $i \in N$. It may be preferences, economic status or some other relevant feature.

$E = E_1 \times \cdots \times E_2$: denote the set of profiles of economic features.

The social planner does not know much information of the participants' profile e. These information are decentralized among the agents. If the agents all know the whole profile e, then it is the perfect information case; if every agent i knows his own e_i and knows the distribution of e, then this is the imperfect information case which can be dealt with using Bayesian method; else if every agent i only knows his own e_i, then it is only good to be dealt with using strategyproof mechanisms or ex post mechanisms. The details will be in later chapters.

1.3.2 Social criteria

Given an economic environment e, every agents participate in economic activities, make economic decisions, pay the cost and get the profits. The social planner hope that the results satisfy some criteria. Let us give some notations and talk about it.

A: denote the set of social allocations, or economic results.

$F : E \mapsto A$: denote a social criterion, or social goal, which is a correpondence from the set of environments to the set of results. Given any environment e, there will be a subset of A that satisfy the social criterion, F just denotes this function.

If randomized results on A are acceptable as social results, then we can use ΔA instead of just A.

1.3.3   Mechanisms for information collection

The social planner lacks information, so he or she need to design incentive compatible rules of the game to induce everyone reveal their true private information.

If the social planner knows completely the environment, then he or she can simply choose a result that is in the set $F(A)$. However, he or she usually does not know much about these things. That is where mechanism for information collection functions. A mechanism, or a game form, usually contains the following components.

M_i: denote the message space of agent $i \in N$. An agent can only emit message $m_i \in M_i$.

$M = M_1 \times \cdots \times M_n$: denote the space of message profiles. Every message profile $m = (m_1, \ldots, m_n) \in M$.

$h : M \to A$: denote the assignment function for the mechanism, which assigns a result for a given message profile m.

$\Gamma = \langle M, h \rangle$: denote the information collection mechanism, which is just the combination of the space of message profiles and the assignment function that lies on top of it.

Thus, a mechanism prescribes rule of the game. Every agent i chooses a message $m_i \in M_i$ to send, and then the social planner collects all the messages in the message profile m, and finally decides on the allocation result $h(m)$ as the social choice. The Γ must be declared openly to let every agent know, then it is up to every agent i to choose from his or her M_i the m_i to report.

A very important class of mechanisms is the direct revelation mechanism(or simply called direct mechanism) in which M_i is just the possible world state information that agent i has. later we will introduce a very important theorem about this mechanism.

1.3.4 Expected outcomes

When a mechanism $\Gamma = \langle M, h \rangle$ is given, we have a game with rules for the agents to report $m \in M$. The strategy of an agent i is usually denoted $\sigma^i(e_i, \Gamma)$ or $\rho^i(e_i, \Gamma)$. Taking this form is for the reason that different environments e_i may induce i to choose different message for a given mechanism Γ. For a given Γ, the Γ in the above notation can usually be omitted as the default Γ is clear. When the agents send messages to the social planner, they have strategical interaction in choosing which message to send. Now we need to know what will result from the strategical interaction, these are the expected outcomes. Usually the solution concept of equilibriums in games are ideal for this role.

Here, one point concerning the e should be stressed. There is requirement on the preference information which is contained in e_i for every $i \in N$. When people game with each other, the hypothesis for human behavior is very important. A fundamental hypothesis is that human being are self-interested, that is, they will try to maximize some kind of self-utility. With this hypothesis, it is usually implied that human being will not deliberately contribute to the society without considering the returns to himself or herself. Or put it another way, a human being will only concentrate on maximizing self-interest, only a good mechanism can make this self-interesting behavior also beneficial to the society. Let $\succeq_{e_i}$ denote the preference in e_i for agent i. It must be an order relation on the results set A. Put it another way, it must satisfy the following 3 conditions.

- Reflexivity. For any outcome $a \in A$, $a \succeq a$.

- Completeness. For any two outcomes a and b from A, either $a \succeq b$ or $b \succeq a$. That is , any two outcomes are comparable.

- Transitivity. For any three outcomes a, b and c, if both $a \succeq b$ and $b \succeq c$ hold, then $a \succeq c$ hold. This eliminates unwanted circles.

Two other relations $\succ$ and $\sim$ can be generated from a given $\succeq$. For any a and b from the results set A, $a \succ b$ if and only if $a \succeq b$ and $b \not\succeq a$; $a \sim b$ if and only if $a \succeq b$ and $b \succeq a$.

The three relations $\succeq, \succ, \sim$ are also frequently denoted with a single letter R, P, I respectively. Later, we will use these single letter forms for consistency with matching literature in that chapter.

With this kind of preference, every agent i can rank the results in A in such a way that equilibriums are definable.

With this hypothesis, the equilibrium concepts of game theory best suit our needs for the expected outcomes. The equilibriums are the expected outcomes. For different situations, different equilibrium concepts should be taken as the expected outcomes. Dominant strategy equilibrium, Nash equilibrium, Subgame perfect Nash equilibrium, Bayesian Nash equilibrium, Perfect Bayesian equilibrium and so on all have their chances of being the expected outcomes depending on the problem situation. Notations are as follows.

$b(e, \Gamma)$: denote the equilibrium message choices $M* \subset M$ under the economic environment e and mechanism Γ.

$h(b(e, \Gamma))$: denote the expected outcomes of the assignment.

For a given Γ, the Γ in the above notation can usually be omitted as the default Γ is clear.

1.3.5   Implementation of social criteria

Finally, comes the concept of implementation. An important goal of mechanism design is to make individual incentives and social criteria compatible. Roughly speaking, if a social criteria is satisfiable under certain equilibrium solution of a mechanism, then we say that that social criteria is implementable with that mechanism. The idea can be illustrated by the following graph.

$$\begin{array}{ccc} & M & \\ b(e,\Gamma) \nearrow & & \searrow h(b(e,\Gamma)) \\ E & \xrightarrow{\quad F(e) \quad} & A \end{array}$$
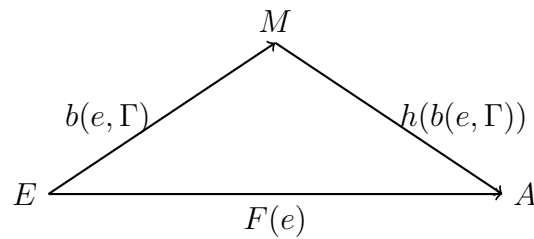
In the above graph, $e \in E$ is the economic environment, A is the possible allocation set. Social criteria F choose what is desirable allocation for the society. The social planner need to design a mechanism to implement the criteria. Under such a mechanism, the strategic interaction of agents should result in an outcome in the equilibrium set $b(e, \Gamma)$ mapped by h, i.e., $h(b(e, \Gamma))$. If $h(b(e, \Gamma))$ is in the set $F(e)$, then the social criteria have been implemented. Formally, we have the following definition

Definition 1. *A mechanism $\Gamma = \langle M, h \rangle$ is said to have implemented the social criteria F in the economic environments space E under the equilibrium b, if for all $e \in E$, we have $h(b(e, \Gamma)) \subset F(e)$.*

Our implementation concept in this paper is weak compared to most in the literature. And what we call social criteria F is usually called social choice, by choosing the wording "criteria" we want to convey the idea that not every point in $F(e)$ must be implemented. However, we still distinguish between two forms of weak implementation.

Here is the weaker implementation concept definition that we call partial implementation.

Definition 2. *A mechanism $\Gamma = \langle M, h \rangle$ is said to have partially implemented the social criteria F in the economic environments space E under the equilibrium b, if for all $e \in E$, we have $h(b(e, \Gamma)) \cap F(e) \neq \emptyset$.*

Below is the strongest implementation concept that we call full implementation.

Definition 3. *A mechanism $\Gamma = \langle M, h \rangle$ is said to have fully implemented the social choice rule F in the economic environments space E under the equilibrium b, if for all $e \in E$, we have $h(b(e, \Gamma)) = F(e)$.*

The above five-component framework covers both the literature review of chapter 2 as well as the messaging mechanisms of auction and matching we will later discussed in-depth in Chapter3 and Chapter4.

For these kind of mesaging mechanism, there is a very famous theorem called revelation principle. I will give here its definition and the proof is given in Appendix A.

Theorem. *revelation principle:*

if a Mechanism $\langle M, h \rangle$ implements the social criteria F in dominant strategy equilibrium. Then there is a direct revelation mechanism which implements F truthfully in dominant strategy equilibrium(truth telling is a dominant strategy equilibrium).

The importance of revelation principle lies in the fact that it narrowes the search space for an implementation mechanism to a special kind of mechanism. This makes the finding of a mechanism implementing some social goal more easily or give a method of proving the nonexistence of such mechanism in general by just proving the unavailability of such mechanism for the revelation games.

1.4   Known impossibility results

A few important impossibility results from the literature of mechanism design are listed in this section. We have mentioned the important contributions of Hurwicz, Gibbard, Satterthwaite. Their contributions contain important impossibility theorems about which social goal we can not design mechanisms to implement. Like in physics, we know that we can not produce perpetual motion machine, then we put our efforts on resource consuming machines which are everywhere in use today. Likely, we should understand these proved negative theorems well first, then we can begin our search on mechanisms that can be designed and of some good qualities while avoiding time waste on mechanism design that has been proved impossible.

The economic environment e in the following part and most economic literature is mostly the preference profile or utility profile, usually denoted R or u.

Theorem. *(Gibbard-Satterthwaite) If the outcome set A has at least 3 alternatives, a social choice rule which is strongly individually incentive compatible and defined on a unrestricted preference domain is dictatorial.*

(Gibbard, 1973) and (Satterthwaite, 1975) first proposed this important theorem for implementation theory. It is generally known as Gibbard Satterthwaite Impossibility theorem. This impossibility result is very important. It has led our research to restricted preference domain, like in matching where the outcome is assumed to be ranked only by a player's self matched object. For its proof, the original paper is recommended for reference.

Pareto efficiency is often a basic requirement of economic mechanisms. However, in ("On informationally decentralized systems") Hurwics shows that the Pareto efficiency and the truthful revelation is fundamentally inconsistent even for the class of neoclassical economic environments.

Theorem. *(Hurwicz Impossibility Theorem) For the neoclassical pri- vate goods economies, there is no mechanism $< M, h >$ that implements Pareto efficient and individually rational allocations in dominant strategy. Consequently, any revelation mechanism $< M, h >$ that yields Pareto efficient and individually rational allocations is not strongly individually incentive*

compatible. (Truth-telling about their preferences is not a dominant strategy equilibrium).

The proof can be found in Hurwicz's original paper.

After introducing these impossibility results, we would like to see what can be implemented with mechanism design through two examples.

1.5   An example of mechanism design

The previous introduction of general model is abstract, and the impossibility results are depressing. To show the usefulness of mechanism design, the following subsection will give an example of mechanism design in more concrete situation. This exemplifies the usefulness of mechanism design. It can be safely skipped without hindering reading of later chapters if you are familiar with the framework of mechanism design.

1.5.1   Assets Inheritance problem

Example 1. *An old man has lots of houses and 2 sons. When he is about to die, he want to give the houses to his sons, but he is not familiar with the market prices for these houses. He does not want to make the sons share any house, so he would like to give every house as a whole to some son. Both sons know the prices of these houses well, and both sons like more values from the inherited houses. How can he achieve his goal of dividing the houses as fairly[1] as possible, and if it can not be perfect fair, he prefers to give his younger son more.*

For the example situation, can a mechanism be designed so that the old man's goal be implemented? With the notations in the previous part, the possible prices of the assets constitute E, the possible division of assets is A. For a true $p \in E$, an allocation $\{a_1, a_2\} \in A$ can give the value $a_1(p)$ for the elder son and $a_2(p)$ for the younger son. The old man's family goal is to $F(p) = (a_1^*, a_2^*)$ which is the solution to $\min_{a_1,a_2} a_2(p) - a_1(p) \quad s.t. \ a_2(p) - a_1(p) > 0$.

A possible mechanism is to let the older son to devide the assets or houses into 2 groups, then let the younger son chooses first, and

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[1]The fairness here means that to make the less valued group of assets as valuable as possible.

the older son gets what is left. This is a simple but effective way of implementing the choice rule. With similar notations to the previous part, let $M_1 = A$, and for any $m_1 = (x, y)$, $M_2((x, y)) = \{(x, y), (y, x)\}$. Therefore the strategy space is $M = \Sigma_{m_1:M_1} M_2(m_1)$[2]. The allocation result function is $h(m_1, m_2) = m_2$. Then the mechanism is just $\Gamma = (M, h)$. It can be checked that in equilibrium $b^*(E, \Gamma) = (m_1^*, m_2^*)$, $h(b^*(E, \Gamma)) = F(p)$ where p is the true p known by both son.

Another implementing mechanism constitutes a revelation game. In this revelation mechanism, let both sons report the p of the houses, and then divide the houses as fairly as possible according to what the older son has reported(he divides the houses to let the least valued group of houses as valuable as possible). The social planner picks the more valuable group of houses according to the younger son's reported p and give it to him, and gives the remaining group of houses to the older son.

1.6   Mechanisms we will discuss

In this section, I would like to talk about the scope of my thesis. The main focus will be around mechanisms for dealing with decentralized information problem. The last chapter is a diversion to some mechanisms that are designed for special purpose and under special requirements. 3 chapters follow this introductory one.

Chapter 2 is a whole chapter devoted to the study of ex post implementation of socially efficient allocation goal through money transfer under the interdependent value environment. Both continuous and discrete cases are considered. First, a form of VCG mechanism is proposed in the social choice of minimizing total pollution under a special interdependent value environment. The scope of use for this form of VCG mechanism is investigated and provided. The uniqueness of tranfer schemes for ex post implementation of continuously differentible social goal with interdependent private signals are shown when such scheme does exist. The later theoretical material in this chapter employ the condition-proposing and constructive-proving method frequently found in mechanism design literature.

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[2]This notation is from dependent type theory. It essentially means that for the $(m_1, m_2) \in M$, m_1 is from the set M_1 and m_2 is from the set $M_2(m_1)$ which is dependent on m_1. Cf. (Voevodsky, 2013)

A sufficient condition called Condtion ρ for ex post implementation with interdependent values via money transfer is given for the multiple goods assignment problem as an attempt to contribute to the field.

Chapter 3 is devoted to a discussion of matching mechanisms design. The attention is focused on an application of the theory to the Chinese college admission mechanism. We have two findings. First if all colleges have same priority criterion , the ex post equilibrium matching result of DA, IA and parallel mechanisms is the unique optimal stable matching result for the students. Second under general priority structures of the colleges, the sets of ex post equilibrium matching results for these mechanisms exibits a nested structure with or without reserves or quotas. These derived propositions show that the DA mechanism is perhaps the best choice among these three popular mechanisms in practice.

Chapter 4 is a chapter reflecting upon a special class of mechanisms in which the implementation goal is to let the side with more skills etc to win. It is up to the agents to show their techniques to compete in these mechanism to win . Focus is put upon finite dynamic games with perfect information with chess as a model for the convenience of discussion.

Chapter 2

Expost implementation with interdependent values

2.1 Introduction

We try to characterize truthful implementation in ex post equilibrium in this Chapter. Ex post equilibrium is a good equilibrium in implementation theory for its distribution free property. A large body of literature has been devoted to many aspects of ex post implementation.

As is pointed out in (McLean and Postlewaite, 2014), it is often the case that truthful revelation is not ex post incentive compatible, that is, for a given agent, there are some profiles of the other agents' types for which the agent may be better off by misreporting his type than by truthfully revealing it. However, with the help of money transfer, the authors in (Dasgupta and Maskin, 2000) devised an complicated yet ingenious auction mechanism to Nash implement a class of allocation problems in interdependent value context. It is obvious that money transfer is one key for implementation. VCG mechanisms and its extensions are successful mechanisms for dominant strategy implementation in private value context. For interdependent value framework, however Here, for a certain but not rare kind of setting, we propose a simple Generalized Vickery mechanism which can easily make the buyers reveal their private information truthfully through the help of money transfer. Of course, the Generalized Vickery mechanism is not balanced scheme. Nevertheless, the ability of the mechanism to make people tell truthfully about their private information is a sign of its power.

We would like to show the subtlety of those ingenious mechanisms in implementing social goal within our paper's model framework, and how their implemented social goals have satisfied the sufficient and necessary condition.

In private value settings, ex post equilibrium is equivalent to dominant strategy equilibrium. Thus the well-known VCG mechanism is the ideal mechanism for ex post implementation in private value case. What we care most in this chapter is the interdependent value setting. We also assume quasilinear utility for the agents, which is a common assumption. In (Bergemann and Morris, 2008), the authors focus on identifying conditions for full implementation of a social choice set in ex post equilibrium. They stressed that a conceptual advantage of ex post equilibrium is its robustness to the informational assumptions about the environment. (Dasgupta and Maskin, 2000) proposed an indirect bidding mechanism which give ex post partial implementation of the socially efficient outcome. Another important contribution of (Dasgupta and Maskin, 2000) is that a mechanism for allocating multiple heterogeneous goods is provided. (Perry and Reny, 2002) devised another clever mechanism for this case. Their methods consists of a collection of second-price auctions between each pair of bidders conducted over at most two rounds of bidding. Unlike them, in this paper we do not consider full implementation, and for partial implementation we focus on direct revelation mechanism. we use the direct mechanism and manage to cover some more situations that are not implementable using the indirect bidding mechanism in (Dasgupta and Maskin, 2000). We restrict out attention to the conditions garanteeing that truthful revelation of one's own private signal constitutes an ex post equilibrium in some carefully designed direct revelation mechanism. That is, we focus on incentive compatibility of truthful information revelation in direct mechanism. Partial implementation is emploed in this paper and henceforth just called implementation. In such direct mechanisms, a well devised payment rule is of key importance. (Ausubel, 1999) provides a payment scheme in its generalized Vickery Auction mechanism, which under some conditions achieves the task of implementing social efficiency in ex post equilibrium. (Jehiel and Moldovanu, 2001) considered efficient, Bayes-Nash incentive compatible mechanisms in a social choice setting that allows for informational and

allocative externalities(i.e. interdependence exists). They then apply the results to the study of multi-object auctions. We also consider a multi-goods case, but different from their settings in that, each buyer wants one good at most, which are better known as assignment problem in matching theory literature(Roth and Sotomayor, 1990). We are also quite different from the matching literature in that we are considering implementation with interdependent values for this problem.

(Chung and Ely, 2006) is a work about ex post implementation. However, we provide different formulations which is easier to grasp and applied to different problems.

we first investigate the continuous cases of interdependent value, including allocation of continuous resources and efficient provision of continuous public goods. For example, in a room of dancers, a volume must be chosen for the music so that it can provide the best social efficiency. Each person has a valuation function of the form

$$u_i = v_i(volume) + \alpha_i \sum_{j \neq i} v_j(volume)$$

Here, the α_i is the altruist coefficient of agent i , which are known by social planner, only each $v_i()$ is private information. This is a typical interdependent value situation.

Simpleness often means less mistakes. Also, a simple mechanism is easy to supervise. For an outsider, it is not easy to judge who should win the goods in a very complicated mechanism , so a corrupted social planner might manipulate the result. That is one motivation for us to construct a direct revelation mechanism.

2.2 Key concepts and notations

First, we give some description of the notations used in this chapter. It's similar to the general framework described in the first chapter. In the meantime of depicting the notations here, we give out the settings for this chapter in more detail. The economic environment consists of A: the set of choice possibilities.

$Z = Z_1 \times \cdots \times Z_n$:the outcome space. For this chapter, implementation with money transfer means that the outcome z has the form

$(a, t_1, \cdots, t_n)$ where a is from the set of choice possibilities A, t_i is the payment agent i has to make.

$U = U_1 \times \cdots \times U_n$: the set of all admissible utility functions $u = \{u_i(\cdot, \cdot)\}_{i=1}^n$ whose domain is $Z \times S$. In this paper, quasilinearity is assumed, that is , $u_i((a, t), s)$ takes the form $v_i(a, s) - t_i$. $v = \{v_i(\cdot, \cdot)\}_{i=1}^n$ is in a space $V = R^+ \times R^+$.

$S = S_1 \times \cdots \times S_n$: the set of all admissible signals $s = (s_1, \cdots, s_n) \in S$ that determine types of parametric utility functions $u_i(\cdot, s)$, and so it is called the space of signals or called the state of the world. In many papers, notation θ is used in stead of s. We adopt the notation s of (Dasgupta and Maskin, 2000) in this paper. Here, apparently both independent value and interdependent value models are both incorporated in this framework.

E: a set of environments(states of the economy) $e = (\{u_i(\cdot, \cdot)\}_{i=1}^n, s)$. In this paper, $e = (\{v_i(\cdot, \cdot)\}_{i=1}^n, s)$ since v_i can identify u_i. Moreover, as will be discussed in the information structure part, the v can be elicit out easily by the social planner, we simply let $e = s = (s_1, \cdots, s_n)$, which is the decentralized information part of the model that need mechanism design to tackle.

The information structure of the paper is specified by

(1)$s_i$is privately observed by agent i.

(2)$v = \{v_i(\cdot, \cdot)\}_{i=1}^n$ is common knowledge among the agents.To elicit the common knowledge part of their information, the social planner can adopt methods similar to those (Moor and Repullo, 1990) propose for Nash implementation of social choice rules. If very agent reports the same, then the result is believed to be the truth. Otherwise, some kind of punishment is given to everyone.

Given economic environments, each agent participates economic activities, makes deci- sions, receives benefits and pays costs on economic activities.The designer wants to reach some desired goal that is considered to be socially optimal by some criterion. Let

$F : E \to\to Z$: the social goal or called social choice correspondence in which $F(e)$ is the set of socially desired outcomes at a certain state of the economy under some criterion of social optimality. For this paper, the social goal is denoted

$$F(s) = \{(a^*, t_1, \cdots, t_n) | a^* \text{ is a solution to } \max_a \sum_{i=1}^n v_i(a, s)\}$$

We call this the social goal of efficiency in the paper. Since money transfer is not the concerned part. we can use transfers freely to implement the goal.

A mechanism consists of a message space and an outcome function.

M_i: the message space of agent i.

$M = M_1 \times \cdots \times M_n$: the message space in which communications take place.

$m_i \in M_i$: a message reported by agent i.

$m = (m_1, \cdots, m_n) \in M$: a profile of Messages.

$h : M \to Z$: outcome functions that translate messages into an outcome.

$\Gamma = \langle M, h \rangle$: a mechanism.

A very important class of mechanisms is the direct revelation mechanism in which M_i is just the possible world state information that agent i has. In this paper, the direct revelation mechanism has a message space $M_i = \{(v, s_i) | v \in V, s \in S\}$. The most important case of m is where all reported v is the same, and for this case the outcome function can be written as $h(m) = (a(v, s), t(v, s))$.

Let $b(e, \Gamma)$ be the set of equilibrium messaging strategies that describes the self-interested behavior of individuals. For instance, Nash equilibrium $N(e, \Gamma)$ is the most frequently adopted equilibrium concept.

A Mechanism $\langle M, h \rangle$ is said to implement a social choice correspondence F in equilibrium strategy $b(e, \Gamma)$ on Environment space E if for every $e \in E$, $h(b(e, \Gamma)) \in F(e)$. Incentive compatibility is another way of saying implementation. A Mechanism is said to be incentive-compatible with a social choice correspondence F on E if it implements F in some kind of equilibrium on E.

2.3 General conditions for implementation with money transfer

Obviously, for a direct revelation mechanism to implement the social efficiency, every agent must tell truth.

Theorem 1. *The social goal of maximizing $\Sigma v_i(a, s)$ can be partially implemented in ex post equilibrium with money transfer if and only if there is*

a transfer scheme $t(s)$(taken as a tax from agent to the social planner), such that for the chosen $a(s)$ maximizing $\Sigma v_i(a, s)$ the following formula holds for all i , s and s_i'

$$v_i(a(s), s) - t(s) \geq v_i(a(s_i', s_{-i}), s) - t(s_i', s_{-i})$$

Proof. Just use the $t(s)$ as a taxation and according to the definition of ex post equilibrium, tell truth about one's private signal is an ex post equilibrium.

$\square$

The following single crossing condition is a necessary condition of ex post implementation. It is roughly saying that the value sum of your current type pretending another type(who have different value when truth telling) and that type pretending your current type is less than the value sum of you truthfully report your current type and that other type.

Proposition 1 (single crossing condition). *A social goal of efficiency G can be ex post implememted on E by some mechanism only if $\forall s, \forall v, \forall i$, when $v_i(a(v, s), s) \neq v_i(a(v, s'), s')$ where $s = (s_i, s_{-i})$ and $s' = (s_i', s_{-i})$ (i.e., they only differ in agent's private signal), then*

$$v_i(a(s'), s) - v_i(a(s), s) < v_i(a(s'), s') - v_i(a(s), s')$$

Proof. By the necessary and sufficient condition **??**, there must be a $t(\cdot)$ such that

$$v_i(a(s), s) - t(s) \geq v_i(a(s'), s) - t(s')$$

and

$$v_i(a(s'), s') - t(s') \geq v_i(a(s), s') - t(s)$$

Add the above two inequality together and exchange some terms from both sides, then you get the single crossing condition.

$\square$

2.4 Continuous social goals

2.4.1 Ex post implementation via VCG

For two special forms of interdependent value problems, the VCG(Vickery-Clark-Groves) mechanisms can be used to elicit players' truthful reporting of private signal or value in ex post equilibrium. In (Vickery, 1961),Vickery first proposed the famous second-price sealed bid auction mechanism, which is also called Vickery auction due to his contribution. Later in (Clarke, 1971), (Groves, 1973), taxation scheme in other public choice problems are proposed. The taxation scheme shares similar structures and are later known as VCG mechanisms. However, the VCG mechanism is recognized as a way of eliciting truthtelling in dominant strategy under independent value environment. In the auction theory monograph (Krishna, 2010), an detailed description of VCG can be found. In this part, we consider a case that VCG mechanism or adapted VCG mechanism can be used to truthfully implement efficient public choice in ex post equilibrium.

Value interdependency is through social goal

Let us discuss with some examples

Example 2. *Consider there are two chess players living L miles apart from each other along a road in a city. They drive their cars to meet each other every sunday to play chess face to face, and then go back home. Suppose that they can choose anywhere on the road to meet. Let $c_i(\cdot)$ be the gasoline cost function for i's one time travelled distance. The city governer tries to assign a meeting place minimizing the pollution of their car causing to the city, so he wants to chose a meeting spot to minimize $c_i(l_i) + c_j(l_j)$. The two chess players all care about the pollution but they also care about their oil expenditure, player i feels the disutility $d_i = c_i(l_i) + \phi_i(c_i(l_i) + c_j(l_j))$, where $\phi_i(\cdot)$ is the polution effect function for i, which is monotonically increasing. What is the transfer scheme that can make the two players telling truth about their private c_i an ex post equilibrium.*

Though it is artificially made, it is indeed interesting to see that there is a tax scheme to accomplish truthful revelation in ex post equilibrium. That taxation method is to tax agent $t_i = \hat{c}_j(l_j^*)$ and tax agent $t_j = \hat{c}_i(l_i^*)$, where $\hat{c_i}(\cdot), c_j(\hat{cdot})$ are the reported gasoline

cost functions and l_i^*, l_j^* is the social choice, which is a solution to $\min \sum \hat{c}_i(l_i) + \hat{c}_j(l_j)$.

The total disutility for i is $t_i + d_i = c_i(l_i) + \hat{c}_j(l_j) + \phi_i(c_i(l_i) + c_j(l_j))$. If the other player j has told truth about $c_j(\cdot)$, then $t_i + d_i = c_i(l_i) + c_j(l_j) + \phi_i(c_i(l_i) + c_j(l_j))$. Player i wants the total disutility as small as possible, and therefore hopes to get a pair of l_i, l_j to minimize $c_i(l_i) + c_j(l_j)$. However, we know that the social planner chooses l_i^*, l_j^* to minimize $\hat{c}_i(l_i) + \hat{c}_j(l_j)$. Given the other player reported truthfully, i.e., $\hat{c}_j = c_j$, player i can let the social planner choose l_i, l_j to minimize $c_i(l_i) + c_j(l_j)$ by just telling truth (let $\hat{c}_i = c_i$).

when extending the two player game of chess to four player game like mahjong, let the four people report their gasoline cost function, the governor choose their play site, it is also easy to let them tell truth in ex post equilibrium with a well designed taxation. As you can verify, one possible taxation scheme is

$$t_i = \sum_{j \neq i} \hat{c}_j(l_j^*) - \min_{l_j, j \neq i} \sum_{j \neq i} \hat{c}_j(l_j)$$

where $(l_1^*, l_2^*, l_3^*, l_4^*)$ is a solution to $\min_{l_k, k \in \{1,2,3,4\}} \sum_{k \in \{1,2,3,4\}} c_k(l_k)$.

Example 3. *$n(n \geq 3)$ vegetable firms at n corners of a regular n polygon area transporting their produced goods to a city for sale. The city planner tries to build the city in a place such that it can minimize the pollution caused by transportation of the goods, and the firms cares about its own transportation cost as well as the total pollution[1]. The oil consumption and thus pollution caused by a given amount of goods transportation for each firm is a private value . Each day the goods will be transported to the city once. What is the taxation scheme that can make each firm report their true private information so that the city planner can choose the best place to build the city in order to reduce transporation pollution. Here the meaning of the taxation is not to reduce pollution directly, but to lead the firms to tell truth about their private information such that the planner can choose a pollution minimizing choice for building the city.*

This is a more realistic economic problem than the previous chess and mahjong problems, and the taxation scheme to let the vegetable

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[1]the reason why a firm care about pollution may be that the goods are vegetables and pollution can hurt the production

firms to report the true $c_i()$ is

$$t_i = \sum_{j \neq i} \hat{c}_j(l_j^*) - \min_{l_j, j \neq i} \sum_{j \neq i} \hat{c}_j(l_j)$$

where $(l_1^*, \cdots, l_n^*)$ is a solution to $\min_{l_k, k \in \{1, \dots, n\}} \sum_{k \in \{1, \dots, n\}} c_k(l_k)$

Remark. *A feature of VCG taxation scheme is retained in the above problems. The scheme taxes player i the aggravation of gasoline consumption that the participation of him or her causes to the other players.*

Value interdependency is through a known altruistic inclination

In the beginning of this chapter, we have given a dancers example. Let us revisit it. In a room of dancers, a volume must be chosen for the music so that it can provide the best social efficiency. Each person has a valuation function of the form

$$u_i = v_i(volume) + \alpha \sum_{j \neq i} v_j(volume)$$

Here, the α is the altruist coefficient of all the agents , which is known by the social planner, only each $v_i()$ is private information. This problem is an interdependent value problem. Suppose the social planner aims to choose a music volume to maximize $\sum_{i \in \{1, \cdots, n\}} u_i(volume)$.

An adapted VCG mechanism can be devised to elicit every dancer's true report of $v_i()$. After receiving the reports $\hat{v}_1, \cdots, \hat{v}_n$, the social planner find a solution $volume^*$ to $\max_{volume} \sum_{i \in \{1, \cdots, n\}} \hat{v}_i(volume)$ because $\sum_{i \in \{1, \cdots, n\}} \hat{u}_i(volume) = (1 + (n-1)\alpha) \sum_{i \in \{1, \cdots, n\}} \hat{v}_i(volume)$. The taxation is

$$t_i = (1 - \alpha)\big(\max_{volume} \sum_{j \neq i} \hat{v}_j(volume) - \sum_{j \neq i} \hat{v}_j(volume^*)\big)$$

The total welfare for i is $u_i - t_i = v_i(volume^*) + \alpha \sum_{j \neq i} v_j(volume^*) + (1 - \alpha) \sum_{j \neq i} \hat{v}_j(volume^*) - (1 - \alpha) \max_{volume} \sum_{j \neq i} \hat{v}_j(volume)$. If the other player j has told truth about $v_j(\cdot)$, then $u_i - t_i = \sum_{i \in \{1, \cdots, n\}} v_i(volume^*) - (1 - \alpha) \max_{volume} \sum_{j \neq i} v_j(volume)$, the term after $-$ is independent with $\hat{v}_i$. However, we know that the social planner chooses $volume^*$ to maximize $\sum_{i \in \{1, \cdots, n\}} \hat{v}_i(volume)$. Given the other player reported truthfully, i.e., $\hat{v}_j = v_j, j \neq i$, player i can let the social planner choose

$volume^*$ to maximize $\sum_{i \in \{1, \cdots, n\}} v_i(volume)$ by just telling truth (let $\hat{v}_i = v_i$).

Remark. *Truthful implementation in ex post equilibrium under this kind of interdependent altruistic utility function is very demanding. It needs the altruist parameter $alpha$ being the same across the agents. Only with this requirement fulfilled, can the problem be implemented by the above adapted VCG mechanism.This same $alpha$ can be interpreted as a general altrustic inclination among a group of similar people.*

2.4.2   Uniqueness of transfer scheme for continuously differentiable cases

First, an economical problem is provided.

Example 4. *A country has N oligopoly firms producing the same goods ,say oil. They face an exogenous demand function. Competition among them hurt aggregate profits of the firms. The country want to devise a taxation scheme such that every firm reveal their true production cost so that the country can plan the production quantity for each firms to maximize total profit of the country. This is the private value case that can be implemented using classical VCG mechanism.*

Now change the scenario to another case where the firms' CEOs know the private cost informations of their own firms. They are all shareholders, and all have a large percent of their own firm's stock and a different amount of share on other firms, and these shares are common knowledge. Now the valuation of each firm's CEO on the country's production assignment plan are dependent upon all the cost information and each firm's final production quantity as specified by the country.

The taxation scheme is a new challenge. We try to face this kind of challenge by designing mechanisms to implement a social goal in ex post equilibrium with money transfer(suitable taxation). For those that can be implemented with money transfer, we give the taxation scheme that can be used to implement the socially efficient choice.

One interesting finding is that the money transfer scheme is unique up to a shifting constant if the functions involved are continuously differentiable as to the private signal.

Lemma 1. *Assuming all the partial derivatives exists and are continuous, then solve the following differential equations*

$$\frac{\partial t_i}{\partial s_i} = \frac{\partial v_i(a(s_i, s_{-i}), s_i, s_{-i})}{\partial a} \frac{\partial a}{\partial s_i}$$

$i = 1, \cdots, n$

gives us a transfer scheme $t(\cdot)$. A sufficient and necessary condition for implementation with money transfer in such continuous case is that, for all i,s by shifting up or down the $t(\cdot, s_{-i})$ to let it be tangent with $v_i(a(\cdot, s_{-i}), s_i, s_{-i})$ at s_i, the curve $t_i(\cdot, s_{-i})$ is below $v_i(a(\cdot, s_{-i}), s_i, s_{-i})$, namely, $v_i(a(\cdot, s_{-i}), s_i, s_{-i}) - t(\cdot, s_{-i})$ is maximized at s_i.

Proof. If truth telling is a ex post equilbrium, it must be the case that s_i is the solution to

$$\max_{s_i'}\{v_i(a(s_i', s_{-i}), s_i, s_{-i}) - t(s_i', s_{-i})\}$$

solving it, we get the conclusions in the theorem. □

Thus for any two t_i, t_i' which can lead to truthful implementation in ex post equilibrium, $\frac{\partial(t_i - t_i')}{\partial s_i} = 0$, and we have the uniqueness of transfer.

From Lemma 1, the uniqueness theorem followes.

Theorem 2. *(Uniqueness) If a mechanism with money transfer exists which truthfully implements the social efficiency in ex post equilibrium, then given any s_{-i}, the transfer tax scheme $tax(x) = t_i^*(x, s_{-i})$ for agent i is essentially the tangent curve of i's profit function family $f(x, s_i) = v_i(a(x, s_{-i}), s_i, s_{-i})$ where $s_i, x \in S_i$, that is, $tax(x)$ has the same slope as $f(x, s_i)$ at s_i. Moreover, any other implementing scheme is an upward or downward shift of it, ie., $tax(x) = t_i^*(x, s_{-i}) + c(s_{-i})$.*

The theorem is simple, but it contains much information. One thing is that not every continuously differentiable social goal $a(s)$ is implementable in ex post equilibrium with money transfer. It must satisfy the conditions in the theorem to be implemented with money transfer, that is, there must be an upper envelope curve for a family of function curves. The second thing is that even if it can be implemented with money transfer, the scheme for implementing it is unique in essence. They only differ by a constant[2]. The model here

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[2]the solution to the differential equations are some specific function plus something of the form $c(s_{-i})$ that is not changing with s_i

does not include the previous VCG mechanism case as c_i or v_i are functions there, not a s_i. The idea to convey is that i's transfer difference of any two mechanisms truthfully implementing an efficient social goal must be a function which is not changing with the reported value of i.

Now let us consider the oil producers example again. We should give a mathematical structure to it in order to show how the process of mechanism design is done in this case. Market demand function is $q(p)$, inverse demand function is $p(q)$. The cost function facing firm i is $c(q_i, s_i)$, the parameter s_i is privately known. The share of firm j that the leader of firm i holds is m_{ij}, and the leader i cares the overall profit $v_i(q, s) = \sum_{j=1}^{n} m_{ij}(q_j p(q) - c(q_j, s_j))$. The country's goal is

$$\max_{\{q_1, \cdots, q_n\}} \sum_{j=1}^{n} (q_j p(q) - c(q_j, s_j))$$

The solution to this maximization problem must satisfy the first order conditions

$$q_j \frac{\partial p(q)}{\partial q} + p(q) = \frac{\partial c(q_j, s_j)}{\partial q_j}$$

$$j = 1, \cdots, n$$

For many function forms, the above conditions are also sufficient, and we calculate out $q_1(s), \cdots, q_n(s)$ from the above equations. Next, by using the conditions in the above theorem, we get

$$\frac{\partial t}{\partial s_i} = \sum_{j=1}^{n} m_{ij} \{ q_j \frac{\partial p(q)}{q} (\frac{\partial q_1}{s_i} + \cdots + \frac{\partial q_n}{s_i}) - \frac{\partial c(q_j, s_j)}{\partial q_j} \frac{\partial q_j}{s_i} \}$$

Now that the transfer scheme is calculated, we only need to verify whether it satisfies the requirement that for all i, s, s_i maximizes

$$\sum_{j=1}^{n} m_{ij} \{ q_j(\cdot, s_{-i}) p(q(\cdot, s_{-i})) - c(q_j(\cdot, s_{-i}), s_j) \} - t_i(\cdot, s_{-i})$$

When the requirement holds, then t is the money transfer scheme.

To have an intuitionistic idea about the above discussion, we will give a numerical illustration. Let $n = 2, p(q) = 5-q, m_{11} = 2/3, m_{12} = 1/3, c(q, s_i) = qs_i$ for $i = 1, 2$. Suppose that the agent 2 has reported the real $s_2 = 1$. Now what is $tax(x)$ for agent 1 which can make him or her tell truth?
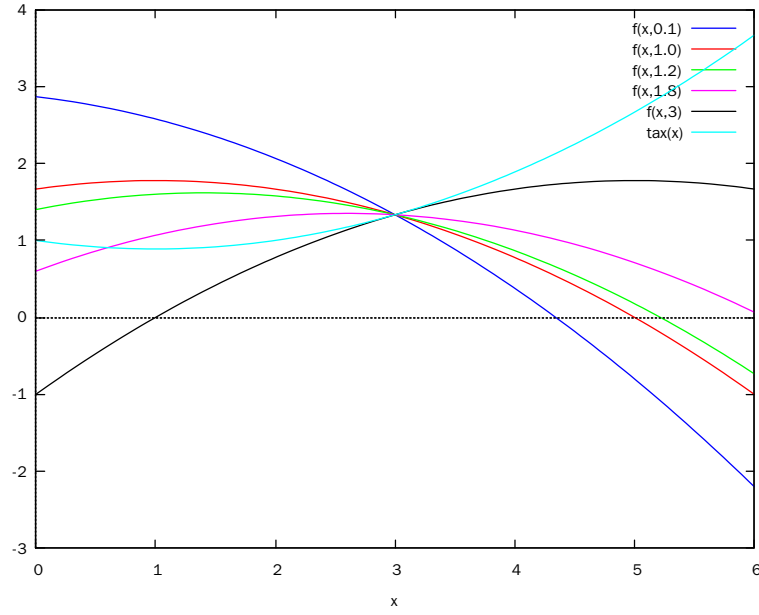
FIGURE 2.1: tax scheme $tax(x)$ etc.

A possible tax scheme for player 1 is $tax(x) = x^2/9 - 2x/9 + 1$, which can be deduced from the just sketched method.[3]

In the graph, we depict the tax curve $tax(x)$ together with the family of $f(x, s_1)$s that is parallel to it[4] at five xs $(0.1, 1.0, 1.2, 1.8, 3)$. If $S_i = (0, 3]$, then the $tax(x) = x^2/9 - 2x/9 + 1$ is individual rational for the agent 1. We can also see that only $tax(x)$ is convex while all the family of $f(x, s_1)$s with s_1 changing from 0 to 3 are concave. Therefore it is a tax scheme which can make the agent 1 tell truth as shifting any curve $f(x, s_1)$ downward such that it is tangent to $tax(x)$ at s_1 will make the curve $f(x, s_1)$ lie under $tax(x)$. As for the agent 2, the taxation is similarly calculated given any s_1, m_{21}, m_{22} and we omit it here.

For this continuously differentiable situation, the private value case of the problem always has the unique solution, the uniqueness has been shown by the above more general result. And the unique mechanism is the well-known VCG mechanism. This has been shown by Laffont and Maskin(Econometrica, 1980).

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[3]We use the software Maxima to calculate the functions and draw the graph. A detailed guide for doing such jobs is obtained from (Hammock and Mixon, 2013). See the Appendix B for the code of producing the graph.

[4]Tangent after an appropriate downward shift.

2.5   Discrete social goal

The discrete social choice possibility set A does not allow differential analysis like we do above in the continuous social choice possibility set case. However, the allocation of discrete social resources, or the determination of whether a project should be carried are important social choice problem we have to face. We are also interested at whether we can implement such social choices with money transfer. We start from allocation of one good.

2.5.1   Allocation of one good

Imagine the following scenario : there is a group of agents who are familiar with each other, and the social planner has one good to allocate to one of them. The socially efficient outcome is to give it to the one who values it the most. Suppose each agent can only detects one of the qualities of the good, and the total value of the good for any agent is only determined by all the qualities of the good and the importance of each quality to the agent himself. This is a situation completely covered by our model framework in the previous section. The social choice possibility set $A = \{(1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)\}$. Every body only cares about getting the object, and when one does not get it his or her utility gain is 0 from the allocation. Now what the social planner want is a mechanism to implement the socially efficient outcome.

(Dasgupta and Maskin, 2000) has partly solved the problem and give us an auction mechanism to implement a rather large economic environment set E in this one good allocation environment. However, as the authors pointed out in the footnote 26 of their paper, some utility forms can not be implemented by their auction mechanism. In this section, through combining the insights of the two authors with the power of revelation mechanism, we try and find a direct revelation mechanism which enlarge the E on which the social goal of efficiency G is implementble, for example, the economic environment in footnote 26 of their paper. In Theorem 1, we have shown what is the sufficient and necessary condition for implementation with money transfer. In this simple case, we give some simple assumptions parallel to the sufficient and necessary condition so that we can implement the social efficiency.

In this one good allocation problem,let $v_i(s) = v_i(\epsilon_i, s)$ where ϵ_i is the allocation of the good to i. Then we have the following sufficient condition for ex post implementation of efficient result in an auction.

Proposition 2. *The efficient allocation of one good can be implemented as an ex post equilibrium if*

$$\forall i, \forall s_{-i}, \exists s_i^* \in S_i, such\ that$$

$$(a) v_i(s_i^*, s_{-i}) = max_{j \neq i} v_j(s_i^*, s_{-i});$$

$$(b) When\ v_i(s_i, s_{-i}) > v_i(s_i^*, s_{-i}),$$

$$\forall j \neq i, v_i(s_i, s_{-i}) - v_i(s_i^*, s_{-i}) > v_j(s_i, s_{-i}) - v_j(s_i^*, s_{-i});$$

$$When\ v_i(s_i, s_{-i}) < v_i(s_i^*, s_{-i}),$$

$$\forall j \neq i, v_i(s_i, s_{-i}) - v_i(s_i^*, s_{-i}) < v_j(s_i, s_{-i}) - v_j(s_i^*, s_{-i});$$

Proof. The proof is through constructing the needed mechanism and then examining how the mechanism works in all possible situations. We only need to find an appropriate mechanism to implement it. Now we give it as follows and then prove it can indeed implement the social goal of efficiency.

Step 1. Every agent reports a (v, s_i) from the set $V \times S_i$;

Step 2. If the v reported by every agent has some disagreement, then the result is that no body gets the good.

Step 3. If the v reported by every agent is the same, then solve the maximization problem

$$\max_{i \in \{1, \cdots, n\}} v_i(s)$$

and give the good to the solution agent. The winner pays $v_i(s_i^*, s_{-i})$ which s_i^* is the solution to the following minimization problem

$$\min_{s_i^* \in S_i} v_i(s_i^*, s_{-i}) \quad s.t. \quad v_i(s_i^*, s_{-i}) \geqslant \max_{j \neq i, j \in \{1, \cdots, n\}} v_j(s_i^*, s_{-i}) \qquad (2.1)$$

Explanation of this mechanism being able to induce truthful revelation of every agent's private signal is not difficult with the help of case analysis for all possibilities.

Devide the actually received signal s_i^* into three cases:

(i)When $v_i(s_i, s_{-i}) > v_i(s_i^*, s_{-i})$, given others truthfully revealing their private signals s_{-i}, the agent i wishes to win the auction. Truthful revelation of i's private signal is enough to insure the win, since

the s_i^* is enough to guarantee a tied first according to assumption (a), further by the first part of assumption (b), the true s_i gives agent i the clear first. Therefore truthful revelation of the signal s_i in this case is incentive compatible.

(ii)When $v_i(s_i, s_{-i}) < v_i(s_i^*, s_{-i})$, given others truthfully revealing their private signals s_{-i}, the agent i's best choice is to lose the auction. Truthful revelation of i's private signal is sure to make him lose the auction, since the s_i^* only gives him a tied first according to assumption (a), further by the second part of assumption (b), the true s_i will not make him the first. Therefore truthful revelation of the signal s_i in this case is also incentive compatible.

(iii)When $v_i(s_i, s_{-i}) = v_i(s_i^*, s_{-i})$, given others truthfully revealing their private signals s_{-i}, winning or losing the auction has the same net value of 0. Therefore truthful revelation of the signal s_i in this case is incentive compatible trivially.

$\square$

The improvement of our mechanism on (Dasgupta and Maskin, 2000) is that they have four assumptions for implementation, but we only need two assumptions which are essentially parallel to their first three assumptions and less demanding.

Their three assumptions are listed below.

(i)$\frac{\partial v_i}{\partial s_i} > 0$;

(ii) $\forall i, j \neq i, \frac{\partial v_i}{\partial s_i}(s_1, ..., s_n) > \frac{\partial v_j}{\partial s_i}(s_1, ..., s_n),$

at any point where $v_i(s_1, ..., s_n) = v_j(s_1, ..., s_n) = max_k v_k(s_1, ..., s_n)$;

(iii)$\forall s_{-i} \in S_{-i}, \exists s_i' \in S_i, such\ that\ v_i(s_i', s_{-i}) > max_{j \neq i} v_j(s_i', s_{-i}).$

Therefore, we can implement social goal of efficiency on all the economic environments E that their auction mechanism can implement and can implement beyond their scope. As a demonstration of the power of our direct revelation mechanism as shown in the proof, we now implement the efficiency goal for (Dasgupta and Maskin, 2000) footnote 26.

Example 5 (Footnote 26 of the paper Efficient Auctions)**.**

$$v_1(s_1, s_2) = s_1 - 2s_2 + 5$$

$$v_2(s_1, s_2) = s_2 - \frac{1}{2}s_1 + 5$$

The v clearly satisfy Assumption 1 and 2, therefore it is implementable using our direct revalation mechanism described above. By using the minimization value in Formula 2.5.1, we get the pay that the winner has to make, which is a constant 5. It is easy to verify that truthful report is a Nash Equilibrium.

For the above example, the reason why the auction in (Dasgupta and Maskin, 2000) cannot implement social goal of efficiency, is perhaps that their contingent biding function loses the information revelation role in this case, since $b_1(v_2) = 15 - 2v_2$, $b_2(v_1) = \frac{15-v_1}{2}$.

The next section will be with multiple goods, and there we will provide a sufficient condition for ex post implementation. When that condition is applied to the one object case(1 is a special case of n), we will have a further relaxed condition than the one we proposed in this part.

2.5.2 Assignment of multiple goods

For allocation of multiple goods under interdependent value environment, not much has been done to our knowledge. (Che, Kim, and Kojima, 2015) involve assigning indivisible objects to individuals without monetary transfer under interdependent values setting. Now in this subsection we extend the previous result to the allocation of multiple goods with money transfer under interdependent value environment. The problem we'll consider is actually the assignment model with a special kind of interdependency. [5]

Suppose that there are m agents and n goods such that $m > n$. Each agent can get at most one object. Agent i's valuations of the n goods is given by $(v_{i1}(s), v_{i2}(s), ..., v_{in}(s))$ respectively, where $s = (s_1, s_2, ..., s_m)$ is a vector of signals held privately by agent 1,agent 2,..., agent m respectively. Allocation result is a $m \times n$ matrix X where $\sum_{i \in \{1,...,m\}} x_{ij} \leq 1$ for all j and $\sum_{j \in \{1,...,n\}} x_{ij} \leq 1$ for all i. The social goal is to allocate the goods to maximize the social efficiency $\sum_{ij} x_{ij} v_{ij}(s)$.

This problem is generally not ex post implementable. However, we find under the following condition, we can design a mechanism to implement the socially efficient goal in ex post equilibrium.

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[5]For the private value assignment problem, the knowledge of which may help the understanding of this section, see Chapter 8 (Roth and Sotomayor, 1990)for reference.

Condition ρ:

$\forall i, \forall s_{-i}, \exists s_i^*$,for all $k \in \{1, \cdots, n\}$, such that :

from the maximizing solution xs to $\sum_{ij} x_{ij} v_{ij}(s)$, one can designate an allocation scheme, such that

if $v_{ik}(s_i) < v_{ik}(s_i^*)$, agent i will have no probability to be allocated the good k.

if $v_{ik}(s_i) > v_{ik}(s_i^*)$, agent i will have a probability to be allocated the certain good k and charged $v_{ik}(s_i^*)$. However, the probability does not depend on s_i.

Proof. The proof is through constructing the needed mechanism and then examining how the mechanism works in all possible situations.

We first need to find an appropriate mechanism. Now we give it as follows and then prove it can indeed implement the social goal of efficiency.

Step 1. Every agent reports a (v, s_i) from the set $V \times S_i$.[6]

Step 2. If the v reported by every agent has some disagreement, then the result is that no body gets any good.

Step 3. If the v reported by every agent is the same, then to allocate the goods, choose the social value maximizing allocation scheme designated in Condition ρ which satisfy the additional two assumptions in it.

Explanation of this mechanism being able to induce truthful revelation of every agent's private signal is not difficult with the help of case analysis for all possibilities.

For any k, devide the actually received signal s_i into three cases:

(i)When $v_{ik}(s_i, s_{-i}) > v_{ik}(s_i^*, s_{-i})$, given others truthfully revealing their private signals s_{-i}, the agent i wishes to be assigned good k to gain $v_{ik}(s_i, s_{-i}) - v_{ik}(s_i^*, s_{-i})$ surplus. Truthful revelation of i's private signal gives i a probability of the assignment of k to i, since s_i will not influence this probability, truthful revelation of the signal s_i in this case is incentive compatible.[7]

(ii)When $v_{ik}(s_i, s_{-i}) < v_{ik}(s_i^*, s_{-i})$, given others truthfully revealing their private signals s_{-i}, the agent i wishes not to be allocated

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[6]Here v is the matrix $\begin{pmatrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{m1} & \cdots & v_{mn} \end{pmatrix}$, where v_{ij} is the value function of agent i getting good j.

[7]Not influencing probability is important, because if not, agent i would have incentive to manipulate the probability by report a false s_i.

the good and being charged $v_{ik}(s_i, s_{-i}) < v_{ik}(s_i^*, s_{-i})$. Truthful revelation of i's private signal s_i is enough to insure that he is not allocated good k by Condition ρ. truthful revelation of the signal s_i in this case is also incentive compatible.

(iii)When $v_{ik}(s_i, s_{-i}) = v_{ik}(s_i^*, s_{-i})$, given others truthfully revealing their private signals s_{-i}, winning or losing the assignment of good k has the same net value of 0. Therefore truthful revelation of the signal s_i in this case is incentive compatible trivially.

Therefore, truthful revelation is incentive compatible overall for this mechanism.

$\square$

Condition ρ does not seem to be easy to satisfy. Nevertheless, in the following we give a more concrete v which satisfy Condition ρ to show that it is indeed satisfiable in some situations so that we can ex post implement the socially efficient goal.

Proposition 3. *The social efficient goal of* $\max_x x_{ij} v_{ij}(s)$ *can be ex post implemented if: For all i and j,*

$$v_{ij} = b_{ij} + o_i(s_i) + \sum_{l \neq i} r_l(s_l)$$

where $b_{ij} > 0$ is the base value, $s_i \in [0, +\infty)$, $\forall i, o_i(0) = r_i(0) = 0$
Assumptions are listed below.
(i)For all i, $\frac{\partial o_i}{\partial s_i} > 0$, $\frac{\partial r_i}{\partial s_i} > 0$;
(ii) $\forall i, \frac{\partial o_i}{\partial s_i} > \frac{\partial r_i}{\partial s_i}$;

The intuition of these assumptions implying Condition ρ is that the own effect of a signal s_i is larger than its effect to the rest of the agents. And therefore there exists a threshold s_i^* as required by Condition ρ for agent i to get into the group of good winners. As for the fixedness of the good winning probability by agent i, it's in part due to the fact that as s_i increases, the increasing rate of v_{ik} is at least as large as any other v_{ab}, where ab stands for arbitrary agent and good combination not equal to ik. The formal proof can be seen in Appendix C.

2.6   Conclusion

We have focused on designing mechanisms implementing social goals truthfully in ex post equilibrium. The one caveat is that there may be collusions among the players. The way to deter this is to form laws against such behaviors globally or to negotiate on contracts against such behaviour locally. Why not just form laws against lying and save all the efforts of designing incentive compatible mechanisms ? The answer is that lying by a person is much harder to detect than collusion by a groups of people, and we can hardly penalize lying with law due to lackness of concrete evidence.

The whole chapter builds on the assumption of quasilinearity of agents' value function. Under such preference, money can be utilized as a tool to help elicit true information about the world that are scattered among the agents. When the mechanism designer cares more on efficiency than fairness, our mechanism is ideal. Since the redistributional effect of money transfer has no influence on the aggregate social value.

Chapter 3

Matching

3.1 Introduction of the basic concepts

We consider two-sided matching in this chapter. There was a monograph dedicated to it, see (Roth and Sotomayor, 1990). However, we will focus on application of the theory to Chinese college admission mechanism, and use some notions that have recently developed. In this section, some of the most important theoretical concepts and results are introduced.

In two sided matching, the two distinguishable sides of matching are assumed to be men and women, students and schools, workers and employers, tenants and houses, etc. Here, students and colleges are chosen as the economic background. For this specific topic, there is (Abdulkadiroglu and Sonmez, 2003) which give a description of the literatures in this field till that time.

Now A description of the model and relevant notations that will be used in this chapter. There is a finite set of students and a finite set of colleges denoted by $S = \{1, ..., n\}$ and $C = \{1, ..., m\}$, respectively. For any $j \in C$, college j has q_j quota, which is college j's seats for students. Students wish to enter at most one college and have an option not to enter any college at all. This outside option is formally represented by a null college, denoted by 0. This null college has unlimited quota, ie, $q_0 = +\infty$.

A matching(sometimes also called assignment or allocation) is a mapping $\mu : S \to C \cup \{0\}$ such that for any $j \in C$, $|\mu^{-1}(j)| \leq q_j$. Denote by $\mu(0)$ the set of colleges with seats that is not assigned to any student at matching μ. The null college is always included in this set, as its seat supply is unlimited. Hence:

$$\mu(0) = \{j \in C : |\mu^{-1}(j)| < q_j\} \cup \{0\}$$

We assume that students have preferences on colleges. Denote by R_i student i's preference on the set of colleges $C \cup \{0\}$. The corresponding strict preference and indifference relations are denoted by P_i and I_i, respectively. The meaning of the notation is as follows: if $c, c' \in C \cup \{0\}$ and cR_ic', then student $i \in S$ weakly prefers college c to college c'. Change "weakly prefers" to "strictly prefer" we get the meaning for the strict P_i. Change "weakly prefers" to "are indifferent" we get the meaning for the indifferent I_i. Preferences are assumed to be rational in the sense that for all $i \in S$, R_i is complete, reflexive and transitive. A preference profile is a list $R = (R_1, ..., R_n)$ of the students' preferences. All the discussions here and after are based on the strict preference assumption. That is , if cI_ic', then $c = c'$.[1]

Here, a priority structure π is assumed for the colleges, that is, for each college $j \in C$, there is an exogenously given strict priority-order π_j. Formally, $\pi_j : S \to \{1, \cdots, n\}$ is a bijection where the highest-ranked student is the student $i \in S$ with $\pi_j(i) = 1$, the second highest ranked student $i' \in S$ has $\pi_j(i') = 2$, and so on. When π_j is the same for all $j \in C$, we get the serial dictator matching mechanism. A priority structure is a list $\pi = (\pi_1, ..., \pi_m)$ of the colleges' priority-order. For the concept of pairwise stable, this priority structure is enough. [2]

A matching μ is pairwise stable(or called priority respecting as in (Andersson and Svensson, 2014)) if there is no student $i \in S$ who strictly prefers some college j to $\mu(i)$ and $\mu^{-1}(j)$ contains some other student $i' \in S$ who has lower priority for college j than student i(i.e., $\pi_j(i') > \pi_j(i)$), and furthermore, all students weakly prefer their assigned college seats to any unassigned seats in $\mu(0)$(seats from the same college are obviously equal). We'll simply use stable to mean pairwise stable later in the chapter.

Formally ,the definition of stableness is as follows.

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[1] Because the students have strict preference, a college can only be indifferent to itself in any student's preference. No two things are exactly the same, and indifference are usually the result of inadequate information, so here we assume aspects of the university are known to the extent that the colleges can be ranked by a student.

[2] For group stable or core, further definition of college preference on groups of students should be provided. And for substitutable college preferences which is an implied assumption of this chapter, the two concepts coincide. See Proposition 6.4 in (Roth and Sotomayor, 1990) for more details.

Definition. *(stableness) A matching μ is stable for a given preference profile R and a given priority structure π, if :*

(i) for all $i, i' \in S$, $\mu(i')P_i\mu(i)$ only if $\pi_{\mu(i')}(i') < \pi_{\mu(i')}(i)$;

(ii) for all $i \in S$, $\mu(i)R_ij$ if $j \in \mu(0)$, where $\mu(0) = \{j \in C : |\mu^{-1}(j)| < q_j\} \cup \{0\}$.

Condition (i) is called fairness condition, with the meaning that there is no justified envy[3]. Condition (ii) is the combination of individual rationality and non-wastefulness. Rationality means that the current assignment of college for a student must be weakly better than the outside option of null college 0. Non-wastefulness means that the current assignment of college for a student must be weakly better than any unoccupied college seat. For more detailed discussion of these concepts , see also (Balinsky and Sonmez, 1999).

The next important concept is Pareto efficiency for students or simply efficiency. A matching is pareto efficient if no other matching can make at least one student get a strictly more preferred college while no student get a strictly less preferred college.

Formally, we have the following definition.

Definition 4. *A matching μ is efficient for a given preference profile R, if for all μ' such that there exists $i \in S$ satisfying $\mu'(i)P_i\mu(i)$, then there must be some other $i' in S$ satisfying $\mu(i')P_{i'}\mu'(i')$*

It is clear that efficiency for students does not take into account of college priority while stableness does need it. Stableness is priority respecting and includes a flavor of fairness in it. Now that we have some idea about matching, we begin to investigate mechanism that can lead to stable and efficient matching. An important difference between matching mechanism design and other mechanism design such as auction is that efficiency is usually not the only requirement, stableness or priority respecting is at least of equal importance. We will illustrate this later in examples.

Why do we care about priority? In the previous chapter, we only care about efficiency. which means to maximize the sum of all participants' utilities. However, there is a well known saying that the whole is more than the parts put together. Priority should belong to the more-than part of the whole. suppose that only students entering

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[3]justified envy means that you envy somebody else's college and according to your priority in that school, you should be admitted prior to that person

their most preferred college will study hard, and any student would feel the same level of happiness when they enter their most preferred college but those with high scores will contribute more to the society, so high scored students should have priority. However just by the limited maximizing sum efficiency criterion we would send arbitrary student to their most preferred college, while from the society as a whole giving the high scored students high priority is a better choice.[4]

When it comes to mechanism design, the usual problem of unilaterally misreporting one's private information(here is one's preference) emerges. In the literature, a mechanism that is immune to such problem is often called strategyproof. Viewing the reporting of one's preference under a mechanism as a game among the students, strategyproofness means that it is a dominant strategy to report one's true preference.

There is also a related concept called group-strategyproof. What is the connotation of this concept? It means that no group can get a pareto improvement by unilaterally changing the reports of students in this group under the mechanism. Viewing the reporting of one's preference under a mechanism as a game among the students, group-strategyproofness means that it is a core equilibrium for everyone to report the private true preference.

3.2   Popular matching mechanisms

There are many matching mechanisms. Three kinds are most studied. They are the Deferred Acceptance Mechanism(DA), the Top Trading Cycle Mechanism(TTC), and the serial dictatorship mechanism(SD). These three mechanism are all related to the later analyzed Chinese student-college matching mechanism, especial the DA mechanism. Why are they most popular? One reason is that all of these mechanisms are strategyproof. Therefore the reported preference profile are true preferences for rational agents.

Among these, DA mechanism is what will be of key importance in the later modelling and discussion of Chinese college admission

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[4]I never believe this myself, so it is only a possible reason coined to show that priority may matter to the social welfare; other possibility of it may be fairness itself is an component of social happiness.

mechanisms. Thus an indepth investigation is provided in the following subsection.

3.2.1 Deferred Acceptance Mechanism

In the seminal work (Gale and Shapley, 1962), DA mechanism is first proposed. This mechanism is the first and probably the most studied mechanism in modern matching theory.

In the setting of student-college matching, it is also called student-optimal stable mechanism (SOSM) for it always finds the stable matching that is most favorable to each student. Its outcome can be calculated via the following Deferred Acceptance (DA) algorithm for a given problem:

Step 1: Each student applies to his or her favorite school. For each college j, up to q_j applicants who have the highest college j priority are tentatively assigned to college j. The remaining applicants are rejected.

Step k ($k \geq 2$): Each student rejected from a college at step k -1 applies to her next favorite college. For each college j, up to q_j students who have the highest college j priority among the new applicants and those tentatively on hold from an earlier step, are tentatively assigned to college j. The remaining applicants are rejected.

The algorithm terminates when no student applies to any new college in some step. That is, every student is either tentatively placed to a college or has been rejected by every college that is better than null college in his or her preference list.

We list important properties concerning the DA mechanism here.

Under strict priority(that is, no two colleges have the same utility level for a student.), the following properties hold. (Gale and Shapley, 1962) first proposed the two theorems and proved them. The proofs for the theorems are short and elegant, and therefore provided in Appendix D.

Theorem. *(Gale-Shapley) The matching given by DA mechanism is stable.*

Theorem. *(Gale-Shapley) Every student is at least as well off under the assignment given by the DA mechanism as he would be under any other stable assignment.*

These two theorems can be combined into one concise statement. The SOSM matching produced by the DA mechanism is the optimal

stable matching. An alternative way to express this is that if you want to find a matching that is stable, the SOSM matching selected by the DA mechanism is the most efficient matching(constrained efficiency).

Theorem. *The DA mechanism is strategy-proof for the students.*

The proof of it is complex and need more notions, please see (Roth, 1982) or (Roth and Sotomayor, 1990)for proof. However, the DA mechanism is not pareto efficient. This fact can be shown by an example.

Example 6. *There are two colleges A and B, each with 1 seat for students. A null college 0 with unlimited seats and accepts who ever applies for it. There are 3 students 1,2,3. The preferences of the students are as follows.*

1	2	3
A	B	A
B	A	B
0	0	0

The two colleges' base priority structure is the same as in the following table

A	B
2	1
3	3
1	2

The DA mechanism is run round by round as in the table below.

	1	2	3
round1		B	A
round2	B		A
round3	B	A	
round4	B	A	0

Because 3 is the only student not tentatively accepted after 3 rounds, and 3 has been rejected by both A and B, the DA algorithm with quota terminates with student 1 entering B, 2 entering A, and 3 entering null college 0.

Now consider the following alternative preference profile.

1	2	3
A	B	0
B	A	A
0	0	B

With this profile, the DA mechanism is run as below.

	1	2	3
round1	A	B	0

Just 1 round, and the DA process is terminated.

Remark. *If a mechanism is able to produce every possible matching results, then not pareto efficient obviously means that the mechanism is not group-strategyproof, since reporting the profile that can result in a pareto improvement assignment is a profitable group-deviation.*

3.3 Analysis of Chinese college admission mechanisms

As an application of the matching theory, we will focus on the Chinese college admission mechanisms in this section.

The Chinese college admission is a centralized matching mechanism with an entrance exam taken national wide as a means of student-priority deciding tool for the colleges. Students from high school in China need to take an entrance exam to decide the student priority for colleges, and the priority structure is also influenced by the application form that the students write to inform authority their own preferences. The college entrance exam, or Gaokao in Chinese Pinyin, forms the foundation of the priority determination process for the current admission system. However,the stage of priority determination through examimations is not the concern of us. There is a complication in the real system, the colleges are classified into batches, where each batch of colleges share the same enrollment scoreline, that is, the minimum score that a student must have in order to be considered eligible for colleges of that batch. However, Shanghai has first abandoned the classification of colleges into batches with different enrollment scoreline from 2016, and many

other provinces have anounced their plan to reduce batches or simply abandon batches since then. This complication is not considered in our analysis.

We are interested in mechanisms about how to match the college and students once the priority has been determined. Starting from the early 1950s, the mechanisms adopted for the stage of matching has gone through series of reforms, and what kind of mechanism should be adopted is currently still in debate. There are millions of high school seniors compete for seats at various colleges and universities in China each year. The matching of students to these colleges and universities has profound implications for the education and labor market outcomes for these students. The fairness and efficiency of this matching process, is of great importance to the society.

Many Chinese scholars have written articles discussing problems related to Chinese college admission mechanisms. (Zhong, Cheng, and He, 2004) has investigated the effects of three preference report timings on the efficiency of the final matching, i.e. reporting preference before the exam, after the exam but not knowing the exact scores, and after the scores are known. This study has focused on the strategical interaction of students in the games, not on how to induce the truthful reporting with a suitable matching mechanism later in the match stage. As to the strategical importance of preference reporting, Haifeng Nie has written many articles. Both (Nie, 2007a) and (Nie, 2007b) has stressed the point that a good score may not be as useful as a good strategy for reporting the preference of the university. This somewhat strange phenomenon has motivated us in studying the issue of reducing manipulation and inducing better and fairer results.

The Chinese matching mechanisms has also recently caught attention of overseas scholars from the economic field of matching. (Chen and Kesten, 2017) has done much to analyze theoretically the sutleties of the different mechanisms adopted, and has gone even further to design experiments for studying features of these mechanisms, see (Chen and Kesten, working paper). Though much has been investigated by them, we still find something interesting in addition to what they find.

3.3.1 A simple characterization of the popular matching mechanisms in CCA

The mechanisms are hard to analyze with all the real application-rejection procedures mixed in the definition, so we will deal with the equivalent direct mechanisms of the mechanisms actually in use.

When studying ex post equilibrium results for existing mechanisms which are not direct revelation mechanism, we will convert it to aa equivalent direct revelation mechanism with the same matching results for expositional convenience[5] Also to aid analysis and exposition, a notion of successful manipulation which means a best report choice given others' reports is used in later analysis of the strategical interaction of students in the CCA matching mechanisms. [6]

Now, let's introduce three important class of mechanisms: DA mechanism, IA[7] mechanism and parallel mechanisms. DA mechanism is what we have investigated in the previous section, but it has never been truely adopted in China due to the fact that the length of reported preference list is too small a number compared to the hundreds of colleges. The Immediate Acceptance (IA) mechanism had been the main mechanism used in Chinese student assignments both at the high school and college level (Nie, 2007b). However, this mechanism has a serious limitation: "a good score in the college entrance exam is often worth less than a good preference reporting strategy" (Nie, 2007a). In our paper's wording, a successful manipulation is often more important than your score level. This problem arises from the special priority structure in the mechanism. Priority of a student in a college is based on the reported preference list firstly and based on your scores secondly, a dictionary order that put too much emphasis on preference reporting. Given in (Nie, 2007b), and also cited in (Chen and Kesten, 2017), one parent has the following explanation of the problem.

"My child has been among the best students in his school and school district. He achieved a score of 632 in the college entrance

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[5]We put the mechanisms in the same revelation mechanism class. and this make it easier to do comparison.

[6] It is essentially the best response concept in Nash equilibrium but sounds more fluent when used in description of the strategical interactions among the students.

[7]Immediate Acceptance mechanism, also known abroad as the Boston Mechanism for its adoption in Boston district, uses the same accept-reject procedure as DA, only the acceptance is not tentative but final.

exam last year. Unfortunately, he was not accepted by his first choice. After his first choice rejected him, his second and third choices were already full. My child had no choice but to repeat his senior year."

To alleviate this problem of high-scoring students not being accepted by any universities, the parallel mechanism was proposed. In the parallel mechanism, students select several "parallel" colleges within each choice-band. The priority structure is changed. A detailed description from (Chen and Kesten, 2017) is cited here for your full understanding of the mechanism.

"a student's first choice-band may contain a set of three colleges, A, B, and C while her second choice-band may contain another set of three colleges, D, E, and F (in decreasing desirability within each band). Once students submit their choices, colleges process the student applications, using a mechanism where students gain priority for colleges they have listed in their first band over other students who have listed the same college in the second band. Assignments for parallel colleges listed in the same band are considered temporary until all choices of that band have been considered. Thus, this mechanism lies between the IA mechanism, where every choice is final, and the DA mechanism, where every choice is temporary until all seats are filled."

In 2003, Hunan fisrt implemented the parallel mechanism, allowing 3 parallel colleges in the first choice-band(or called group), 5 in the second choice-band, 5 in the third choice-band, 5 in the fourth choice-band, and so on, see (Chen and Kesten, 2017). The parallel mechanism is soon to be widely perceived as having improved allocation outcomes for students. Citing a parent from a newpaper report [8],

"My child really wanted to go to Tsinghua University. However, in order not to take any risks, we unwillingly listed a less prestigious university as her first choice. Had Beijing allowed parallel colleges in the first choice band, we could at least give Tsinghua a try."

In the paper (Chen and Kesten, 2017), they have formulated the IA mechanism, parallel mechanisms, and DA mechanism as a parametric family of application-rejection mechanisms. In this chapter,

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[8]Li Li. "Ten More Provinces Switch to Parallel College Admissions Mechanism This Year." Beijing Evening News, June 8, 2009.

they are all converted to their equivalent direct mechanisms. The following are the definitions.

Definition. *direct revelation DA mechanism:*

Each student reports his or her preference list, each college reports its priority criteria(usually scores plus some tie-breaking rules),then after collecting all these preference lists and priority criteria, the social planner runs the DA procedure to produce a final matching result.

Definition. *direct revelation IA mechanism:*

Each students reports his or her preference list, each college reports its priority criteria(usually scores plus some tie-breaking rules),then after collecting all these preference lists and priority criteria, the social planner adjusts the priority rule such that the reported priority criterion of a college is made only a tie-breaking rule, and the rank of a college in a student's reported preference list is the main priority criterion. Finally, the social planner runs the DA procedure to produce a final matching result.

The parallel mechanisms are a wide group of mechanisms. Therefore we need to devise a group of equivalent direct mechanism for them all. However, parallel mechanisms only differ in the choice-band sizes, so different list of choice-band sizes characterizes a different parallel mechanism.

Definition. *characteristic list of a parallel mechanism:*

An infinite list of integers greater than zero $(e_1, e_2, \cdots)$ is the characteristic list of a parallel mechanism if the $e_1, e_2, \cdots$ define the choice-band sizes for the 1st ranked, 2nd ranked, $\cdots$ choice-bands.

Henceforth, a parallel mechanism with a characteristic list $(e_1, e_2, \cdots)$ will just be called mechanism $(e_1, e_2, \cdots)$.

Definition. *direct revelation parallel mechanism $(e_1, e_2, \cdots)$:*

Each student reports his or her preference list, each college reports its priority criteria(usually scores plus some tie-breaking rules). After collecting all these preference lists and priority criteria, the social planner uses the preference list to produce a list of college choice-bands according to the characteristic list $(e_1, e_2, \cdots)$ such that the 1st ranked choice-band includes the top e_1 colleges in the preference list, the 2nd ranked choice-band includes the top e_2 of the remaining colleges in the preference list, and so on. Then the social planner adjusts the priority rule such that the reported priority

criteria of a college is made only a tie-breaking rule, and the rank of the choice-band which a college belongs to in a student's reported preference list is the main priority criterion. Finally, the social planner runs the DA procedure to produce a final matching result.

These three direct mechanisms are henceforth just called DA, IA and parallel mechanisms. From the definition of parallel mechanisms, we can see that DA and IA mechanisms are two special forms of parallel mechanisms with characteristic list $(+\infty, 0, 0, \cdots)$ and $(1, 1, 1, \cdots)$. Therefore they can also be called mechanism $(+\infty, 0, 0, \cdots)$ and mechanism $(1, 1, 1, \cdots)$.

The IA mechanism and parallel mechanism are qualitatively different from DA. Their priority structure are closely related with the reported preference while the priority structure of DA has no connection with the reported preference. The underlying idea for the IA mechanism or parallel mechanisms is "Colleges should like those who like them". This is exactly where manipulations creep in.

Formally, under the setting of student-college matching, we have the following definition.

Definition 5. *A mechanism of matching is non-manipulable, if given any preference R, and priority structure π, $\mu = h(R, \pi(R))$ and for any $i \in S$, denote $h(R'_i, R_{-i}, \pi(R'_i, R_{-i}))$ by μ', then we have $\mu(i) R_i \mu'(i)$.*

We have the following claim. IA and parallel mechanisms are manipulable. DA is non-manipulable. Its proof can been found in (Chen and Kesten, 2017).

Remark. *The efficiency and stableness concept is apparently not very useful when the reported R is not true. What we should be concerned is the status of the matching result in the perspective of true R. So we have investigated the manipulability first.*

Because manipulation exists, investigating the matching result to the reported preference is of little meaning for the true social welfare. Therefore matching theory about the property of matching results to the reported preference is not applied in investigating Chinese college admission mechanisms. Without strategyproofness, we have to find another equilibrium solution concept other than dominant strategy equilibrium to study the likely matching results.

In this section we would like to view the mechanisms from a perspective of ex post Nash equilibrium. Here, the ex post should not be taken literally, it does not mean that evaluating the action after some particular time. Actually, it means that assuming that every action and information of the other players have been known, the player still insists on his choice due to its optimality with respect to the others' actual action or choice. In fact, it may happen that the player will never know all the actions of the other players and therefore there is not a time point after which all information is revealed. The important thing is that the action or manipulation can be called successful when all information and actions are known. Formally, we give the following definition in the college admission setting.

Definition 6. *A student i has a successful report if let R denote the true preference, R' the reported preference, $\mu' = h(R')$, then for any R''_i, let $\mu'' = h(R''_i, R'_{-i})$, we have $\mu'(i)R_i\mu''(i)$*

We extract this concept from the definition of ex post Nash equilibrium. It is in fact a component of ex post Nash Equilibrium. An ex post Nash equilibrium is just where all agents have successful reports simutaneously.

In this chapter, We consider deterministic game forms and only consider pure strategies. Our "ex post" concept requires that the strategies to be considered must be pure because you cannot report this preference and that preference at the same time when every thing is known[9].

From the social point of view, the ex post equilibrium with every player successfully reports is in a sense the fairest result that a mechanism can have. In an ex post equilibrium, no body has done worse because of lack of information[10].

We will show that in the Chinese style college admission, the IA mechanism, the parallel mechanisms and the DA mechanism have the same matching result in their expost equilibrium. However, it is not easy to achieve the expost equilibrium for the IA mechanism and the parellel mechanism because they need lots of information to achieve it but the informtion is not easily acquired. On the contrary, the DA's ex post equilibrium can be achieved just by every player

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[9]Only scholars who study quantum games may have disagreement. However,we do not consider quantum games here.

[10]Disparity of information acquisition is a very important source of unfairness

reporting the true preference, and it is indeed a dominant strategy equilibrium, stronger than merely an ex post equilibrium.

Now a definition that is useful in formally expressing the ideas in the previous paragraphs.

Definition 7. *under a preference reporting direct mechanism $h(.)$, students report some preference profile R, then the matching result is $\mu = h(R)$. If all students have successful reports in this case, then the matching result μ is an ex post equilibrium outcome, and called achievable with perfect manipulation under the mechanism $h(.)$.*

There is a difficulty in analysing the true welfare property of a matching outcome achievable with perfect manipulation under a mechanism. The difficulty is that even if you know that in the ex post equilibrium the matching result is successfully manipulated by everyone, you do not know if it satisfies a social goal $f(R)$, because you do not know the true preference R in the economic environment because of manipulation of preference reporting!

In (Chen and Kesten, 2017), they started analysis of parallel mechanisms with the symmetric parameter setting, i.e., $e = e_1 = e_2 = \cdots$ before studying the general case where $e_1, e_2, \cdots$ might be unequal parameters. In the general case, they proved a nested manipulability structure for parallel mechanisms, and then they paid attention to stability comparisons between parallel mechanisms with an additive decomposition relationship. However, for ex post equilibrium outcomes, they only give the Proposition 4 in the symmetric parameter setting. As a new contribution, we find a nested structure of the ex post equilibrium outcomes in parallel mechanisms, and have proved that the set of ex post equilibrium outcomes is only related to e_1 but not influenced by $e_2, e_3, \cdots$.

Before diving into the case of general priority structure, we first focus our attention on a very important kind of priority structure, the score-based priority structure and discuss what kind of matching results can be achieved with perfect manipulation under such uniformly score-based priority. This interest comes from reality. In CCA, the priority criteria is basically the scores with some similar tie-breaking rules across colleges, and the examination paper is usually the same nationwide[11] .

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[11]Only Beijing, Shanghai, and a few provinces choose to devise an exam paper of their own

DA mechanism just use the score high-low comparison to determine priority of students for every college. The more score you get, the high priority you get.

As to the adjusted priority structure in IA and parallel mechanisms, it is described as following. To determine a student i's priority in college j, parallel mechanism fisrt group the colleges according to i's reported preference list. For example, in Tibetan's parallel mechanism, every group has 10 colleges, then you group the first 10 together and label every college in this Group 1, group the second 10 together and label every college in this Group 2, and so on. A student's priority in a college is determined in a dictionary order. First, compare the group number of the college for the student. For instance, if Beijing University is in the first 10 college group for a tibetan student, then this student has a higher priority in Beijing University than tthose who has not put Beijing University in the first 10 college group. If two students put the college in the same Group n, then comes the second procedure of comparison. The student with higher score(or high original priority) has higher priority. IA mechanism is just a parallel mechanism with every group size equal to 1.

In fact, the important thing is not that the priority structure is score-based nationwide, but that it is the same across colleges. We have the following important theorem.

Theorem 3. *If every college has the same base priority for the students, then the matching achievable with perfect manipulation,that is, the ex post equilibrium matching result, under the IA and parallel mechanism is the same with the one under the DA mechanism, which is unique , stable and constrained efficient.*

To get this result, we first propose a property of the DA mechanism which is convenient for discussion of its matching result.

Definition 8. *free-style DA:*

Actions includes:

1.some students not tentatively holding by a college apply to their head college in their preference lists, and erase these colleges from the lists.

2.some colleges with more than quota number of students applying rejects some students that are not in their top quota priority.

Each round, action 1 happens or action 2 happens or action 1 and 2 happens together.

The termination condition for the free-style DA:

When there is no college holding more than quota students, and no student is rejected by any college, the free-style DA ends.

The DA mechanism described before is a free-style DA such that each round all students not tentatively holding apply and all colleges with more than quota number of students reject all that are not in the top quota priority. The standard DA mechanism is the most efficient in operation among the free style DA mechanisms, but when the focus is on the final matching result, they are all the same DA mechanism. The following is a proposition for this.

Proposition 4. *The final matching result of free-style DA mechanism is influenced by neither the number and order of applications of the students in each step nor the number and order of the rejection of colleges, as long as there is some student who applies for a college in the preference list order or a college rejects some student not in the top quota priority.*

Proof. During the proof of the optimal stability of ordinary DA mechanism as shown in Appendix D, no specific order of application is assumed, therefore the order does not influence the final result's property of optimal stability. However, there is only one such result, hence the final matching result is the same.   □

Since the order of application does not influence the final result, we can rearrange the DA application orders as we see convenient. In fact, the DA mechanism produces the same matching result as SD(Serial Dictatorship) mechanism when the base priority structure is the same. This can be seen easily by rearrange the DA application order by priority. The 1st prioritied student applies to his or her favorite college and is accepted "finally" because no later application can be from a student with higher priority. The 2nd prioritied student applies to his or her favorite college in the colleges with unfilled seats because he cannot compete with the previous applied students who have higher priority, and is accepted "finally". The 3rd, the 4th,... ,the nth, applies in the same manner. Therefore, in the following discussion, we take the SD mechanism as equal to the DA mechanism under the environment of same priority structure. And

we use terms of the DA matching result and the SD matching result interchangeably since they are the same in the same priority setting.

With this idea in mind, now an informal proof of theorem 3 is given below. The proof is intuitionistic . The perfect manipulation of serial dictatorship mechanism is just non-manipulation, that is, stating the true preference. For the parallel mechanism, we would prove that for everyone to report as his or her favorite the college he or she is matched to in the SD mechanism forms an ex post equilibrium. We only need to show that every student has got a successful manipulation by doing so. Using induction on the students' priority list(Note that we assumes strict priorities in this chapter).

First, the student i with $\pi(i) = 1$, anounce his or her true favorite college as his or her favorite, has successfully manipulated. He can obviously get his or her true favorite this way. Moreover, this top student can only achieve a successful manipulation by reporting this way.

Second, for an arbitrary student i. Suppose every student i' such that $\pi(i') < \pi(i)$, i.e., higher in the priority list has reported as his or her favorite the college that the SD mechanism assigns when he or she has reported truthfully. Now the student i can only expect the SD resultas the best, and can indeed get it if reporting it as his or her favorite. Therefore reporting as his or her favorite the college that the SD mechanism assigns is a successful manipulation for him.

Having showed that the SD matching result is achievable with perfect manipulation, next we prove the uniqueness of the achievable matching result with perfect manipulation.

The proof of unique. Take any other matching result ,we would prove that it is not acheivable with perfect manipulation. Because the SD matching result is pareto-efficient, and the preference is strict, in any other matching result there must be some students who are worse off than in the SD matching result. There must be a student with highest priority among them, then we would like to show that this student has not successfully manipulated. Using case analysis:

First, if this student is the top prioritied student, then he or she definitely has not successfully manipulated because if he or she report whatever college as his or her first choice, then he or she will get it.

Second, if this student is not the top scored student, but all the

higher prioritied students have successfully manipulated, then they will get the SD matching result, so this student can get the SD matching result too. If he or she gets a worse college, he or she definitely has not successfully manipulated.

Except for the informal proof, later we will give another proof using the following theorem.

For general priority structures, as to matching results that can be achieved with perfect manipulation under the IA mechanism, *Theorem 1* in (Ergin and Sönmez, 2006) is of vital importance[12]; as to matching results that can be achieved with perfect manipulation under the parallel mechanism, (Chen and Kesten, 2017) provided many deep insights.

We find the following result which is a new contribution.

Theorem 4. *For two parallel mechanisms $(e_1, e_2, ...)$ and $(e'_1, e'_2, ...)$, if $e_1 < e'_1$, then any ex post equilibrium matching result in mechanisms $(e_1, e_2, ...)$ can be implemented in ex post equilibrium with a properly chosen preference report in mechanism $(e'_1, e'_2, ...)$.*

The concept of underdemanded school has been crucial in (Tang and Yu, 2014). In the proof of the above proposition, a similar concept of underdesiredness is needed and defined below.

Definition 9. *A college c is underdesired at a matching μ if no student i strictly prefers c to $\mu(i)$ relative to the student's true preference.*

Note that there is a difference between the underdesired concept here and the underdemanded concept in (Tang and Yu, 2014). We say "relative to the student's true preference" while they mean "relative to the reported preference".

Lemma 2. *For the DA mechanism, a preference reporting profile $(R_1, ..., R_n)$ gets the matching result μ. For any student i, if he or she changes the preference report to just $\{\mu(i), ...\}$, i.e., put $\mu(i)$ at the first place, the resulting matching result under the new reported preference is μ'. It must be the case that $\mu'(i) = \mu(i)$.*

Proof. If we can show that deleting the report of the college before $\mu(i)$ in R_i does not change i's final matching result, then by repeated

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[12]They called them Nash equilibrium results which we do not think accurate as we have discussed the difference between Nash equilibrium and ex post equilibrium.

deleting the college before $\mu(i)$, we will move $\mu(i)$ to the first place while maintaining its final matching result unchanged.

Suppose R'_i is the preference which results from deleting the college c before $\mu(i)$ in R_i . Arrange the DA so that the student i does not start the report of preference until the DA with all other students reporting has finalized. Then i enters and report his preference list one by one. For $\mu(i)$ to be the final matching for i, all the college reports before it will fail[13]. There are two cases:

First, if applying to the c before $\mu(i)$ results in downright rejection, then deleting the application of it or not does not change the final matching of i.

Second, if applying to the c before $\mu(i)$ results in a rejection circle, then: if the μi is not involved in the circle, the final matching to μi is not changed;if the $\mu(i)$ is a college in the circle, then i can get in $\mu(i)$ after the circle has rotated means that before the circle rotates, it will initiate it and when it rotates back, make $\mu(i)$ reject the student that without i would be tentatively accepted. □

Now is the formal proof of Theorem 4.

Proof. Suppose R is an ex post equilibrium preference report in mechanism $(e_1, e_2, ...)$, and produces the matching result μ. At equilibrium, if a college c has admitted k students who have put it in any of the choice-band $e_2, ...,$ c is an underdesired college. Also, c must have more than k seats unfilled after running DA for the truncated preference lists where we only retain the first e_1 items in every student's preference list. Modify the preference report of R to R' as follows. If a student i is matched to his or her first choice-band e_1, then $R'_i = R_i$. Otherwise, insert $\mu(i)$ to the end of choice-band e_1.

First we claim that in mechanism $(e'_1, e'_2, ...)$ this R' produce the same matching result μ. First, run DA for the truncated preference lists where we only retain the first e_1 items in every student's preference list. This produces the same matching results for the students in the set $\{i|\mu(i)$ is in the first e_1 colleges of $R\}$ since the e_1 items are in the e'_1 items of the $(e'_1, e'_2, ...)$ due to $e_1 < e'_1$. For any other student i, since they all insert $\mu(i)$ to the end of choice-band e_1, they are just matched to $\mu(i)$ too since $\mu(i)$ have enough seats unfilled.

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[13]Downright rejection or first accepted then a rejection cycle is induced so that finally being rejected.

Second we claim that R' is an ex post equilibrium in mechanism $(e_1', e_2', ...)$. Note that with the reported preference R' each student is matched to a college in their first choice-band e_1' in mechanism $(e_1', e_2', ...)$, thus the running of the mechanism is equal to the DA. Suppose a student i can become matched to a $\mu'(i)$ that is strictly preferred to $\mu(i)$ by i relative to the student's true preference. Then by Lemma 2, by moving $\mu(i)$ to the first of reported preference list, i still gets $\mu'(i)$. Now consider i report $\mu'(i)$ as 1st place under mechanism $(e_1, e_2, ...)$, the preference reports of other students there can be viewed as essentially shorter than under mechanism $(e_1', e_2', ...)$, so i can get a matching result at least as good as $\mu'(i)$ under mechanism $(e_1, e_2, ...)$ by *Theorem 5.34* in (Roth and Sotomayor, 1990)[14]. This is contradictory to R being an ex post equilibrium under mechanism $(e_1, e_2, ...)$.

$\square$

An alternative proof of theorem 3 is as follows.

Proof. Theorem 1 in (Ergin and Sönmez, 2006) tells that the ex post(or Nash) equilibrium results for IA(or Boston) mechanism are the same with stable matchings. The only ex post equilibrium matching result in DA with same priority is just the optimal stable matching. Theorem 4 says that the ex post equilibrium results set is a subset of DA, but a superset of IA, and therefore contains only the unique stable matching in this priority structure. $\square$

For two parallel mechanism with the same first choice-band size, we have the following theorem.

Theorem 5. *For two parallel mechanisms $(e_1, e_2, ...)$ and $(e_1', e_2', ...)$, if $e_1 = e_1'$, then the set of all matching results from ex post equilibriums in mechanisms $(e_1, e_2, ...)$ is the same with the set of all matching results from ex post equilibriums in mechanisms $(e_1', e_2', ...)$.*

The proof is similar but slightly different.

Proof. We only need to show that if R is an ex post equilibrium of mechanism $(e_1, e_2, ...)$ and produces the matching result μ, then we can find a R' such that R' under mechanism $(e_1', e_2', ...)$ produces the

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[14] here it means that i is matched to $\mu'(i)$ since $\mu'(i)$ is already the first in the reported list.

same matching result μ and reporting R' is an ex post equilibrium under mechanism $(e'_1, e'_2, ...)$.

At equilibrium R, if a college c has admitted k students who have put it in any of the choice-band $e_2, ...,$ c is an underdesired college. Also, c must have more than k seats unfilled after running DA for the truncated preference lists where we only retain the first e_1 items in every student's preference list.

Modify the preference report of R to R' as follows. If a student i is matched to his or her first choice-band e_1, then $R'_i = R_i$. Otherwise, insert $\mu(i)$ to the end of choice-band e_1.

First we claim that in mechanism $(e'_1, e'_2, ...)$ this R' produce the same matching result μ. First, run DA for the truncated preference lists where we only retain the first e_1 items in every student's preference list. This produces the same matching results for the students in the set $\{i | \mu(i) \text{ is in the first } e_1 \text{ colleges of } R\}$. For any other student i, since they all insert $\mu(i)$ to the end of choice-band e_1, they are just matched to $\mu(i)$ too since $\mu(i)$ have enough seats unfilled.

Second we claim that R' is an ex post equilibrium in mechanism $(e'_1, e'_2, ...)$. Suppose through reporting R''_i a student i can become matched to a $\mu'(i)$ that is strictly preferred to $\mu(i)$ by i relative to the student's true preference. This $\mu'(i)$ must not be underdesired college. By Lemma 2, by moving $\mu(i)$ to the first of reported preference list, i still gets $\mu'(i)$. Now consider i reports $\mu'(i)$ as 1st place under mechanism $(e_1, e_2, ...)$, due to $e_1 = e'_1$, the preference reports R_i of other students can be viewed as essentially the same as under mechanism $(e'_1, e'_2, ...)$(2nd,3rd and ... choice-bands preference do not matter due to low priority), so i can get the same matching result $\mu'(i)$ under mechanism $(e_1, e_2, ...)$. This is contradictory to R being an ex post equilibrium under mechanism $(e_1, e_2, ...)$. $\qquad\square$

The above two theorems together is equivalent to the following one.

Theorem 6. *(Nested theorem)*

For two parallel mechanisms $(e_1, e_2, ...)$ and $(e'_1, e'_2, ...)$, if $e_1 \le e'_1$, then any ex post equilibrium matching result in mechanisms $(e_1, e_2, ...)$ can be implemented in ex post equilibrium with a properly chosen preference report in mechanism $(e'_1, e'_2, ...)$.

We call it nested theorem because it shows the nested structure of the sets of ex post equilibrium outcomes in parallel mechanisms.

Now let us go to the next subsection where we put some more details on.

3.3.2   Student-college matching with affirmative actions

In this subsection, we would like to consider extending the previous model to cover affirmative actions. When there are bonus-score given to some student, if the bonus-score is accepted in every college, then we can just take his or her total score as original-score + bonus-score. The rest is the same as the serial dictatorship case analyzed in the previous subsection. However, if there is difference between the accepted bonus-score among colleges, then the priority structure goes into the general priority model.

In the following parts, we will focus on affirmative action of quota and reserve. Quota is a prevalent affirmative action policy in school choice limits the number of admitted majority students to give minority students higher chances to attend their desired schools. To circumvent the inefficiency caused by majority quotas, (Hafalir, Yenmez, and Yildirim, 2013) offered a different interpretation of the affirmative action policies based on minority reserves. With minority reserves, schools give higher priority to minority students up to the point that the minorities fill the reserves. we would like to adopt the minority reserve definition in (Hafalir, Yenmez, and Yildirim, 2013) with a little extension. we will give every group(including the majority group) a reserve that gives a student of this group priority over other groups whenever the reserve number has not been reached. This model can cover the CCA province reserves policy widely in use. Using this definition of reserve, we give the following example to compare DA and IA.

Example 7. *There are two colleges A and B, each with two seats for students. There are two students groups a and b. Students $1, 2, 3$ are in group a and students $4, 5$ are in group b. The preferences of the students are as follows.*

1	2	3	4	5
A	A	A	B	B
B	B	B	A	A

As can be seen, the students in group a all prefer college A while the students in group b all prefer college B.

The two colleges' base priority structure is the same as in the following table

A	B
1	1
2	2
3	3
4	4
5	5

The quota structure of A and B are both 1 seat for group a, 1 seat for group b. Now the DA mechanism is run round by round as in the table below.

	1	2	3	4	5
round1	A	A		B	B
round2	A	A	B	B	
round3	A		B	B	A
round4	A	B		B	A

Because 3 is the only student not tentatively accepted after 4 rounds, and 3 has been rejected by both A and B, the DA algorithm with quota terminates.

Now consider the IA mechanism. If the quota priority is first compared and considered, then the result is the same as DA. This fact can be easily verified and we omit it here. If the reported preference order is first considered, as the name of Instant Accept is indicating, then the result is as the table below.

	1	2	3	4	5
round1	A	A		B	B

From the above example, we see that the matching result of DA mechanism with quota is not pareto efficient for the students. In fact, the comparison of the tables' matching results shows that IA mechanism's result is a pareto improvement. Student 2 and 5 gets strictly better result in IA mechanism than in DA mechanism. Why the efficiency result of the score-based DA is lost. The reason is that the efficiency in previous subsection comes from the serial dictatorship nature of DA with score as the same priority among colleges. Now

with quota included, the priority structure is changed and that property is lost in the process. However, the resulted matching for DA is still priority-respecting(as (Andersson and Svensson, 2014) call "stable"), and constrained efficient(pareto efficient among the priority-respecting matchings) . Meanwhile, there is not a way for the student to manipulate to get a better result in DA with quota.

Despite the truth-reporting result of IA weakly pareto dominates the DA result, we should notice that the truthful reporting strategy as in the example does not constitutes an ex post equilibrium. The student 3 has not manipulated successfully, if he or she has reported college B as favorite, 3 will not ended with matching no college. What are the relationships of ex post equilibrium matching results among DA with reserve or quota, IA with reserve or quota and parallel mechanism with reserve or quota in general? That is one of the important issues we will deal with in the following parts.

We start the investigation from DA with quota. For the DA mechanism with majority quota, we have the following proposition.

Proposition 5. *A DA mechanism with majority quota is priority-respecting, constrained efficient and non-manipulable.*

Proof. A DA mechanism with majority quota can be converted into an equivalent DA as follows.

Break each college into two colleges, one is the majority college with quota seats, the other is the minority college with remaining seats. The majority college with quota seats inherit the same priority structure, and the minority college with the remaining seats only accept minority students and preserve the priority of them. The preference list for minority is adjusted such that each original college is divided into two college with the minority college put before the majority college. The preference list for majority is adjusted such that each original college is divided into two college with the majority college put before the minority college. Then run DA with these divided colleges and adjusted preferences, one get exactly the same result as running DA with majority quota as described in (Kojima, 2012).

From the above perspective, a DA mechanism with quota is just a DA mechanism essentially, and retains all the feature of a DA mechanism. Therefore it is priority-respecting, constrained efficient and non-manipulable. □

For the DA mechanism with minority reserve, the same assertion holds.

Proposition. *(Hafalir-Yenmez-Yildirim) A DA mechanism with minority reserve is priority-respecting, constrained efficient(optimal stable) and non-manipulable.*

This is equivalent to the proposition 1 in (Hafalir, Yenmez, and Yildirim, 2013). In their proof, they also show that an equivalent way to implement the deferred acceptance algorithm with minority reserves is first to create a new matching problem with no af- firmative action and then to apply the original deferred acceptance algorithm to this market.

A DA mechanism with minority reserve can be converted into an equivalent DA as follows.

Break each college into two colleges, one is the majority college with quota seats, the other is the minority college with remaining seats. The majority college with quota seats inherit the same priority structure, and the minority college with the remaining seats raise the priority of minority students to the front of the majority students while keeping the priority relationship in the group unchanged. The preference list for minority is adjusted such that each original college is divided into two college with the minority college put before the majority college. The preference list for majority is adjusted such that each original college is divided into two college with the majority college put before the minority college. Then run DA with these divided colleges and adjusted preferences, one get exactly the same result as running DA with minority reserve as described in (Hafalir, Yenmez, and Yildirim, 2013).

Quota and reserve are quite distinct measures. Quota is to restrict a group such that it won't occupy more than a number q seats. Reserve is to help a group such that as long as the r seats for a college are not occupied fully by the group, then a member of the group only need to compete with members in the same group to enter the college.

From the previous description of quota and reserve, we see that quota does not help any group directly while reserve does help the first r number of the protected students in each college directly because some of them may not be admitted to their college without the

reserve help. Majority quota q^M puts a natural reserve r^m on minority because no majority will get into the reserve, and besides that, it is more demanding since even if r^m is not full, majority can not be filled.

As is pointed out in (Hafalir, Yenmez, and Yildirim, 2013), the Pareto domination relationship in Theorem 1 of (Hafalir, Yenmez, and Yildirim, 2013) holds if and only if it holds for the student-optimal stable matchings under the corresponding policies. To see this, note that for each affirmative action policy, the student-optimal stable matching Pareto dominates any other stable matching. However, we think the equivalent one is more concise, and give it here as a theorem.

Theorem. *(Hafalir-Yenmez-Yildirim)*

the student-proposing de- ferred acceptance algorithm with minority reserves Pareto dominates the algorithm with majority quotas when $q^M + r^m = q$.

Here we would like to provide another informal proof of it, which is much shorter and may provide some insight.

Proof. Consider the equivalent DAs for the DA with majority quota q^M, and DA with minority reserve r^m.

By proposition 4, we can run the equivalent DA for the DA with minority reserve first to get its final result. Then let the minority colleges ejecting all majority students and continue running the equivalent DA for the DA with majority quota q^M until a termination reach.

The result of this free style DA is the same as the equivalent DA for DA with majority quota q^M, only it postpone the rejection of some majority students in the minority colleges until a matching result of the equivalent DA for the DA with minority reserve is reached.

The continuing of DA is bad for students as they are heading to colleges lower in the preference list. Therefore the student-proposing de- ferred acceptance algorithm with minority reserves Pareto dominates the algorithm with majority quotas when $q^M + r^m = q$.

$\square$

Though minority reserve was first proposed explicitly in theory in (Hafalir, Yenmez, and Yildirim, 2013). However, the reserve method was practically in use in China for years. In CCA, a college often reserves a fixed number of students seats for a particular province.

Through the years, students in the provinces with reserves have mostly benefited from such policy. [15]

When considering reserves for provinces, the nested structure of ex post equilibrium results for DA, IA and parallel mechanisms still hold. We have the following theorem.

Theorem 7. *For two parallel mechanisms* $(e_1, e_2, ...)$ *and* $(e'_1, e'_2, ...)$ *with reserves for provinces, if* $e_1 \leq e'_1$, *then any ex post equilibrium matching result in mechanisms* $(e_1, e_2, ...)$ *can be implemented in ex post equilibrium with a properly chosen preference report in mechanism* $(e'_1, e'_2, ...)$.

Proof. We only need to convert the parallel mechanisms with the same reserve into equivalent parallel mechanisms with no reserve, and show that the converted mechanisms produce the same results and their first choice-band size still have the same comparative relationship. After receiving the preferrence report of the students and the priority report of the colleges, break each college into the subcolleges each with the reserved number of seats and raise the priority of the target group of students[16] to the front of the other students while keeping the priority relationship in this group unchanged. The preference list for a student in each reserve group is adjusted such that each original college is replaced by the list of subcolleges with the reserve subcollege for the student being raised to the head. Then multiply the characteristic list of the original parallel mechanism by the number of subcolleges, we get the new characteristic list.

After all the above conversions, run the converted parallel mechanism, we can get a matching result. Combine the subcolleges back into the original college with all the admitted students, we get a matching result for the students and colleges. It is obvious that this matching result is the same with the matching result from original parallel mechanism. For any two converted mechanisms, their first choice-band size still have the same comparative relationship because they are multiplied by the same number.

□

Quota also does not influence the nested structure of ex post equilibrium results for parallel mechanisms with similar reason. The

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style of proof in this section is intuitional and informal, but the ideas are hopefully conveyed. All in all, we find that the nested structure of ex post equilibrium matching results is not influence by the added details of quota or reserve.

3.3.3   Policy suggestions

For student-college matching mechanisms, when no affirmative actions are taken, and the scores in the nationwide college entrance examination is the base priority criterion, then the achievable result under perfect manipulation is the same, which is efficient, stable and unique. However, for the DA or score-based serial dictatorship, the perfect manipulation is easily fulfilled in reality, since the perfect manipulation is just non-manipulation, i.e., reporting the true preference. Therefore, it's very easy to teach how to do a successful manipulation in a score-based serial dictatorship, and even a researcher cannot tell how to do a successful manipulation in any other mechanism such as IA mechanism and parallel mechanism. As a result, a kind of unfairness caused by manipulation occurs. We do not want to judge students by manipulation, but these IA and parallel mechanism allow students' matching results to be influenced by a large extent to such undesired manipulation abilities, especially for the majority of students who are not the top students.

Where there is manipulation, there is corruption. A successful manipulation(reporting of preference) in IA and parallel mechanism needs information about other students' preference reporting. So there might be buying and selling of such information. Even without corruption of relevant institutions, some other unnecessary businesses, like strategy consulting of preference reporting for the college admission mechanisms will arise. These things burden the family of senior high school students unnecessarily both economically and spiritually.

After adding policies of reserves or quotas, the same priority is broken, but the set of possible ex post equilibria outcomes of all the IA and parallel mechanisms are included in the set of possible ex post equilibria outcomes of the DA mechanism. Therefore IA and parallel mechanisms cannot produce any better matching results in ex post

equilibria. Favor is still towards DA as long as ex post equilibria results are concerned.

All in all, we support DA as the best mechanism among DA, IA and parallel mechanisms.

Chapter 4

Mechanism for mental competition

4.1   Introduction

Go, chess, Chinese chess, draughts etc occupy a lot of leisure time
of many people around the world, and all kinds of match and tour-
naments are held everywhere around the world. These game forms
share many things in common as mechanism for determining win,lose,or
draw between two players. They are finite dynamic games with per-
fect information. Many books, articles and manuals are published
about them. However, most centered around how to win with con-
crete moves in certain positions, how to devise traps for the oppo-
nents and how to play certain popular openings. In this chapter, we
analyze these finite games with perfect information through a typ-
ical and popular game,chess. As a mechanism for mental competi-
tion, chess is analyzed from game theoretic point of view and the
propositions and theorems can be applied to general finite games
with perfect information easily.

Let us begin with a comparison of chess with student-college match-
ing mechanism. That mechanism is to collect true preference from
students and allocate student according to the reported preference
and their priority(often determined by score in the entrance exam-
ination). It aims at making the truth telling dominant strategy and
tries to garentee that the game has no manipulation or less. Chess is
different, it determines the result of win, loss, or draw just by how
well two players can manipulate under the rules of the mechanism.
To make it a real mental competition, it is deliberately designed with
a strategy space that seems infinite to human beings or even com-
puters. Therefore the implementation theory employed to analyse

student-college matching mechanism is not suitable here. Incentive is usually not the problem, the players all want to win. The problem is how strong the desire to win is and mental efforts put in. Usually, the expected results are also not as the previous chapter. We do not expect a Nash equilibrium for such game forms. We expect a result of win,loss,or draw based on the extent of successfulness of manipulation from both sides in chess. While in student-college matching we try to design mechanism that avoid the manipulation factor, in chess we encourage manipulation skill of the game and that is where the enjoyment lies.

Zermelo first analyzed the game of chess in 1913 in German. (Schwalbe and Walker, 2001) give a survey of the early studies which are mostly based on Zermelo's seminal article and translated Zermelo's article into English in the appendix of that paper. However, their papers are formal and mathematically oriented whereas our discussion uses mathematics for rigorousness, but is more in an intuitionly simple style.

4.2 Finite games with perfect information: chess

In this model, we do not consider time factor, do not consider the moves limit factor, and agreed draw or resign.

First, the definition of position

Definition 10. *A position q on a chess board, is a certain placement of pieces on the board along with the information that it is who's turn to move.*

Here, the only relevant thing is which piece is placed in which square, and who is to move.

Let us then consider the ending position which should be first analysed in backward induction method. An ending position is a position that the referee must be called to register the result,i.e.,the game is finished according to chess rule.

Let A, B denote the opposing sides. Let q denote the ending position that is B's turn to move. There are only the following cases:

1. A has checkmated B, then A win.

2. A has stalemated B, then draw.

3. Both sides have insufficient material(pieces) to give the opponent checkmate, then draw.

4. Three repetition of the same position, that is, the same position q has been reached twice before, then draw.

Now, in order to analyze the game of chess, we need the concept of a winning position for a side. Similar concept can be found in (Schwalbe and Walker, 2001), but ours here is for all positions, not just positions that is due to move for that side.

Definition 11. *A position q is a winning position for side A if and only if one of the following case is true:*

1. It is an ending position that A win(A has checkmated B).

2. When it is B's turn to move, B can make a legal move, but for every move he can produce according to chess rule, it leads to a position that is a winning position for side A.

3. When it is A's turn to move, A has a legal move that leads to a winning position for A.

Here, we take it for granted that this definition is well defined. Later on , we will prove it.

The definition of a winning position for B is just the above definition with A and B interchanged. With the concept of a winning position, we can define the concept of a losing position.

Definition 12. *A position q is a losing position for A if and only if it is a winning position for B.*

The definition of a losing position for B is just the above definition with A and B interchanged.

Now the only possible other positions is defined as following:

Definition 13. *A position that is neither a winning position nor a losing position for any side is called a drawing position.*

Now there is a lemma for chess.

Lemma 3. *every chess game end in finite moves*

Proof. Suppose not. Then there is a game with infinite moves. According to chess rule, three repetition of a same position makes the game end in draw, therefore there must be infinite positions. However, with limited squares to place limited kinds of pieces for the two sides, the positions of chess is finite. A contradiction. □

Chess as a mechanism to assign win, loss, and draw is not used to elicit preferences from both sides. We assume all the players of chess has the preference $win \succ draw \succ loss$. It is rationality and mental abilities that decides the choice of moves. Now we propose the following important theorem.

Theorem 8. *For two players with perfect rationality and immense mental power, a winning position for a player will end in win of the player, a losing position for a player will end in loss of the player, and a draw position of a player will end in a draw.*

Proof. For two players(you and an opponent) with perfect rationality and immense mental power, classifying a given position into three clear defined category is easy. Then:

In the winning position, when it is your turn to move, you just chooses the move that lead to another winning position that has not been on board before in the game(avoiding cycle) , then since the potential positions are finite, an ending winning position(checkmate) can be reached finally; when it is the opponent's turn to move, any legal move will lead to a winning position for you.

In the losing position , when it is your turn to move, any legal move will lead to a losing position: when it is the opponent's turn to move, the opponent player just chooses the move that lead to another losing position that has not been on board before in the game(avoiding cycle), then since the potential positions are finite, an ending losing position (be checkmated)can be reached finally.

In the drawing position, since no winning position for the side that is to move, and the side is not losing position, so he or she must be able to find a move leading to another drawing position, a cycle is emerging and three repetitions reached (draw acording to the rule) or a drawing ending is reached(insufficient material).   □

All the arguments so far relied on the well-definedness of our winning position concept. Apparently, it is a recursively defined term. First, an edge case, the checkmate, this is easily found according to chess rules. Then, after finding all the checkmate position, we can produce all the position that can reach this position according to the chess rule.

Now,when all the positions that can produce the checkmate position is found. We now need to produce all the position that can

lead to all the position that can produce the checkmate. Here, another step is needed since these position is the opponent's turn to move. We still need to check these positions against another criterion. All the position that can be produced must belong to those that is already found in the winning positions... and so on. The process continued until no new winning positions can be found. After these informal description, I would like to formalize these with mathematical notations below. A proposition regarding a winning position is as follows.

Proposition 6. *The winning position concept is well defined, that is, for every position that can arise on the chessboard according to chess rule, it has a deterministic answer whether it is a winning position or not.*

First, an explanation. Why do we ask the question of well definedness? Russell's Paradox is the reason, not every recursively defined set is consistent. In naive set Theory, the following definition is possible. let $R = \{x | x \notin x\}$, that is R is a set of sets that does not contain itself. It is a definition on the space of all sets. Do we have a deterministic answer whether R is a set in R or not. Unfortunately it cannot be determined, since $R \in R \iff R \notin R$ which is a contradiction.

To prove the recursive definition of a winning position is well defined, we only need to show that every position can be detemined whether it is a winning position or not.

Proof. Let X denote the set of positions that can possibly be reached from the start of the game by finite alternative moves from both sides. We first prove that this set is well defined. Let $f(q)$ stand for the set of a chess position q's next possible positions. For all $A \subset X$. let $f(A) = \cup_{x \in A} f(x)$. Then,

Step 0, $X_0 = x_o$, the starting position.

Step 1, $X_1 = f(X_0) \cup X_0$, the positions that is reachable in two moves.

Step 2, $X_2 = f(X_1) \cup X_1$, the positions that is reachable in three moves.

...

Step n, $X_n = f(X_{n-1}) \cup X_{n-1}$,

$X = lim_{n \to \infty} X_n$. This set is decidable, since $X(\cdot)$ is well defined by chess rule, and possible chess placements is finite(therefore for

all $n \in N$, $|X_n| < M$). With the increasing of n, whenever $X_n = X_{n-1}$, you can see according to the definition the X is found. And since $X_0, X_1...$ is a sequence of sets with bound for the number of elements. This n will occur inevitably. So X is well defined.

For a finite set, you need to be able to identify all the elements satisfying a certain definition, or else that definition is not well defined. Now we devise an algorithm to find all the positions that is a winning position. We search for the winning positions from the ending position.

Step 0, filter out all the checkmate positions from X, this can be done by applying chess rule, put these positions into W_0.

Step 1, $W_1 = \{x | f(x) \cap W_0 \neq \emptyset\}$.

Step 2, $W_2 = \{x | f(x) \neq \emptyset and f(x) \subset W_1\} \cup W_0$.

...

Step 2n-1, $W_{2n-1} = \{x | f(x) \cap W_{2n-2} \neq \emptyset\}$.

Step 2n, $W_{2n} = \{x | f(x) \neq \emptyset and f(x) \subset W_{2n-1}\} \cup W_0$.

Obviously, $W_0 \subset W_2 \subset W_4 \cdots$ and $W_1 \subset W_3 \subset W_5 \cdots$. Since these are increasing sets, and the number of potential position is finite, there must be somewhere that the $\subset$ can be replaced with $=$ for the first time, and in fact after that all the $\subset$ can be replaced with $=$. The set of winning positions is $W = lim_{n \to \infty} W_{2n-1} \cup W_{2n}$.

$\square$

Chess is too complicated for us to tell all the winning positions. A simple finite game with perfect information is provided in the following example to illustrate the use of the winning positions concept.

Example 8. *The game of taking out stones. There is a pile of stones and two players. Each is required to take 1 or 2 stones out of the pile one time. They take turns to do it alternatively. The one player who takes the last stone out is the winner. For a pile of 20 stones, can the first player to move the stones win? How?*

One may feel headless if not having a pattern to do it. The way of solving this problem is just the way of finding the winning positions in the above proof by way of backward induction. In the following we use the notation n_o to represent positions when there are n stones left in the pile while it is the opponent to move ; the notation n_s to represent positions when there are n stones left in the pile while it is self to move .

Step 0, we need to find the "checkmate" positions W_0. There is only one "checkmate" situation in this game: it is the opponent to move, but he or she find that there is no stone . We denote it with 0_o meaning a position of 0 stones with the oppenent turn to move. Therefore $W_0 = 0_o$.

Step 1, the positions that can directly be led into a position in W_0 after self's well chosen move, $W_1 = \{1_s, 2_s\}$.

Step 2, the positions that the opponent is to move but have to face a position in W_0 or every move will result in a position in W_1, $W_2 = \{0_o, 3_o\}$.

Step 3, the positions that can directly be led into a position in W_2 after self's well chosen move, $W_3 = \{1_s, 2_s, 4_s, 5_s\}$.

Step 4, the positions that the opponent is to move but have to face a position in W_0 or every move will result in a position in W_1, $W_4 = \{0_o, 3_o, 6_o\}$.

...

We can therefore get $W_{2n} = \{0_o, \ldots, 3(n-1)_o, 3n_o\}$
and $W_{2n-1} = \{1_s, 2_s, 4_s, 5_s, \ldots, 3n-2_s, 3n-1_s\}$.

In informal words, the winning positions include the positions with a pile of stone with a number of multiples of 3 when it is the opponent to move, and the positions with a pile of stone with a number of nonmultiples of 3 when it is self to move.

The moving strategy is to move to a winning position whenever is possible, and it is possible when the starting position is a winning position. Since 20 is a nonmultiple of 3, the first person is to move will win following the "from winning to winning" strategy.

For chess, typical winning position are collected as finite move checkmate problems in chess books. These kind of books can train a player's ability to find winning positions and identify the move order to transpose to the final checkmate ending. Besides, even if you cannot find out the path of transition, you can remember them after seeing the answer, thus enrich your database of winning positions and the transposing moves. It is widely admitted among chess players that when the amount of such positions are accumulated to some certain degree(different players may have a different threshold), they feel much easier to detect winning positions and the transposing moves. This is an important phenomenon: the brain can nearly automatically extract patterns from remembered winning chess positions.

Appendix A

Revelation principles

Theorem. *revelation principle:*

if a Mechanism $\langle M, h \rangle$ implements the social criteria F in dominant strategy equilibrium. Then there is a direct revelation mechanism which implements F truthfully in dominant strategy equilibrium(truth telling is a dominant strategy equilibrium).

Proof. [1] Since F can be implemented in dominant strategies by $\langle M, h \rangle$, there is a profile of strategies $(\sigma^1, \cdots, \sigma^n) \in (E_1 \mapsto M_1) \times \cdots \times (E_n \mapsto M_n)$ that forms an dominant strategy equilibrium in the game induced by $\langle M, h \rangle$. Thus, for all $e = (e^1, \cdots, e^n) \in E = E_1 \times \cdots \times E_n$, we have

$$h(\sigma^1(e_1), \cdots, \sigma^n(e_n)) \in F(e_1, \cdots, e_n)$$

Furthermore, implementability in dominant strategies means that in the mechanism $\langle M, h \rangle$, for all $i \in N$, $e \in E$, and any strategy profile $\rho = (\rho^1, \cdots, \rho^n) in (E_1 \mapsto M_1) \times \cdots \times (E_n \mapsto M_n)$,

$$h(\rho^1(e_1), \cdots, \sigma^i(e_i), \cdots, \rho^n(e_n)) \succeq_{e_i} h(\rho^1(e_1), \cdots, \rho^i(e_i), \cdots, \rho^n(e_n)) \tag{A.1}$$

Consider the following direct mechanism $(E_1 \times \cdots \times E_n, g)$ where for all $(e_1, \cdots, e_n) \in E_1 \times \cdots \times E_n$,

$$g(e_1, \cdots, e_n) = h(\sigma^1(e_1), \cdots, \sigma^n(e_n)) \in F(e_1, \cdots, e_n)$$

It suffices to show that in the game induced by $(E_1 \times \cdots \times E_n, g)$, it is a dominant strategy for each agent i with type e_i to report e_i. Suppose not. Then there is a profile $e = (e_1, \cdots, e'_i, \cdots, e_n) \in E_1 \times \cdots \times E_i \times$

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[1]The proof is adapted from (Vohra, 2011).

$\cdots \times E_n$ and an agent $i \in N$ and another type e_i' such that

$$g(e_1, \cdots, e_i', \cdots, e_n) \succ_{e_i} g(e)$$

$$\Longleftrightarrow$$

$$h(\sigma^1(e_1), \cdots, \sigma^i(e_i'), \cdots, \sigma^n(e_n)) \succ_{e_i} h(\sigma^1(e_1), \cdots, \sigma^i(e_i), \cdots, \sigma^n(e_n))$$

Choose $\rho = (\rho^1, \cdots, \rho^i, \cdots, \rho^n) \in (E_1 \mapsto M_1) \times \cdots \times (E_i \mapsto M_i) \times \cdots \times (E_n \mapsto M_n)$ such that $\rho^j(e_j) = \sigma^j(e_j)$ for every $j \in N, j \neq i$ and $\rho^i(e_i) = \sigma^i(e_i')$. Then, the last inequality can be rewritten as

$$h(\rho^1(e_1), \cdots, \rho^i(e_i), \cdots, \rho^n(e_n)) \succ_{e_i} h(\rho^1(e_1), \cdots, \sigma^i(e_i), \cdots, \rho^n(e_n))$$

which contradicts inequality A.1.   □

Proposition. *revelation principle in an interdependent value environment:*
if a Mechanism $\langle M, h \rangle$ implements the social choice rule F in ex post equilibrium. Then there is a direct revelation mechanism which implements F truthfully in ex post equilibrium(truth telling is an ex post equilibrium).

Proof. Since F can be implemented in ex post equilibrium strategies by $\langle M, h \rangle$, there is a profile of strategies $(\sigma^1, \cdots, \sigma^n) \in (S_1 \mapsto M_1) \times \cdots \times (S_n \mapsto M_n)$ that forms an ex post equilibrium in the game induced by $\langle M, h \rangle$. Thus, for all $(s^1, \cdots, s^n) \in S_1 \times \cdots \times S_n$, we have

$$h(\sigma^1(s_1), \cdots, \sigma^n(s_n)) \in F(s_1, \cdots, s_n)$$

Furthermore, implementability in ex post strategies means that for all $i \in N$, $s \in S$, and $\rho^i : S_i \mapsto M_i$,

$$v_i(h(\sigma^1(s_1), \cdots, \sigma^i(s_i), \cdots, \sigma^n(s_n)), s) \geqslant v_i(h(\sigma^1(s_1), \cdots, \rho^i(s_i), \cdots, \sigma^n(s_n)), s)$$
$$\text{(A.2)}$$

Consider the following direct mechanism $(S_1 \times \cdots \times S_n, g)$ where for all $(s_1, \cdots, s_n) \in S_1 \times \cdots \times S_n$,

$$g(s_1, \cdots, s_n) = h(\sigma^1(s_1), \cdots, \sigma^n(s_n)) \in F(s_1, \cdots, s_n)$$

It suffices to show that in the game induced by $(S_1 \times \cdots \times S_n, g)$, it is ex post incentive compatible for each agent i with type s_i to report s_i. Suppose not. Then there is a profile $(s_1, \cdots, s_n) \in S_1 \times \cdots \times S_n$

and an agent $i \in N$ and a type $q \in S_i$ such that

$$v_i(g(q, s_{-i}), s) > v_i(g(s), s)$$

$$\Longleftrightarrow$$

$$v_i(h(\sigma^1(s_1), \cdots, \sigma^i(q), \cdots, \sigma^n(s_n)), s) > v_i(h(\sigma^1(s_1), \cdots, \sigma^i(s_i), \cdots, \sigma^n(s_n)), s)$$

Choose any $\rho^i : S_i \mapsto M_i$ such that $\rho^i(s_i) = \sigma^i(q)$. Then, the last inequality can be written as

$$v_i(h(\sigma^1(s_1), \cdots, \rho^i(s_i), \cdots, \sigma^n(s_n)), s) > v_i(h(\sigma^1(s_1), \cdots, \sigma^i(s_i), \cdots, \sigma^n(s_n)), s)$$

which contradicts inequality A.2. $\qquad\square$

Appendix B

wxMaxima session code for producing picture 2.1

First, we solve the FOC. of the social planner's maximization problem.

(% i1) solve([-q[1]+5-q[1]-q[2]=s[1], -q[2]+5-q[1]-q[2]=s[2]], [q[1],q[2]]);

$$[[q_1 = -\frac{-s_2 + 2s_1 - 5}{3}, q_2 = \frac{-2s_2 + s_1 + 5}{3}]] \qquad \text{(% o1)}$$

Price p:

(% i2) p: ratsimp(5-(-(-s[2]+2*s[1]-5)/3+(-2*s[2]+s[1]+5)/3));

$$\frac{s_2 + s_1 + 5}{3} \qquad \text{(p)}$$

Second, calculate agent 1 's profit f(x,s_1) given the share 2/3,1/3, and s_2=1.

(% i3) ratsimp(1/3 * (-2+x+5)/3 *((1+x+5)/3 - 1) + 2/3 * (-(-1+2*x-5)/3 *((1+x+5)/3 -s[1])));

$$-\frac{x^2 + (2 - 4s_1)x + 12s_1 - 27}{9} \qquad \text{(% o3)}$$

(% i4) f(x,s) := -(x^2+(2-4*s)*x+12*s-27)/9;

$$\mathrm{f}(x, s) := \frac{-(x^2 + (2 - 4s)x + 12s - 27)}{9} \qquad \text{(% o4)}$$

Third, calculate the partial derivative of f(x,s) to x at s=x, and then from the differential equation $\partial t(x)/\partial x = \partial f(x,s)/\partial x$ hold when s=x, we derive a t(x).

(% i5)   diff(f(x,s),x);

$$\frac{-2x + 4s - 2}{9} \qquad\qquad (\% \text{ o5})$$

(% i6)   partial: (2*x-2)/9;

$$\frac{2x - 2}{9} \qquad\qquad (\text{partial})$$

(% i7)   desolve('diff(t(x),x) =(2*x-2)/9, t(x));

$$\mathrm{t}(x) = \frac{x^2}{9} - \frac{2x}{9} + \mathrm{t}(0) \qquad\qquad (\% \text{ o7})$$

We find from the following that all f(x,s)s go through (3,4/3), and let t(x) goes through it too, we get a tax(x).

(% i8)   ratsimp(f(3,s));

$$\frac{4}{3} \qquad\qquad (\% \text{ o8})$$

(% i9)   tax(x) := x^2/9-(2*x)/9+1;

$$\mathrm{tax}(x) := \frac{x^2}{9} - \frac{2x}{9} + 1 \qquad\qquad (\% \text{ o9})$$

(% i10)   tax(3);

$$\frac{4}{3} \qquad\qquad (\% \text{ o10})$$

Now we can draw the picture.

(% i11)   wxplot2d([f(x,0.1),f(x,1.0),f(x,1.2),f(x,1.8), f(x,3), tax(x)], [x, 0,6],[legend, "f(x,0.1)","f(x,1.0)","f(x,1.2)","f(x,1.8)", "f(x,3)", "tax(x)"]);

Appendix C

Proof of proposition 3

Proposition. *The social efficient goal of* $\max_{i,j} x_{ij} v_{ij}(s)$ *can be ex post implemented if:*

For all i and j,

$$v_{ij}(s) = b_{ij} + o_i(s_i) + \sum_{l \neq i} r_l(s_l)$$

where $b_{ij} > 0$ is the base value, $s_i \in [0, +\infty)$, $\forall i, o_i(0) = r_i(0) = 0$

Further assumptions are listed below.

(i)For all i, $\frac{\partial o_i}{\partial s_i} > 0$, $\frac{\partial r_i}{\partial s_i} > 0$;

(ii) $\forall i$, $\frac{\partial o_i}{\partial s_i} > \frac{\partial r_i}{\partial s_i} > 0$;

Proof. We need to show that the assumptions imply Condition ρ, that is, from the maximizing solution xs to $\sum_{ij} x_{ij} v_{ij}(s)$, one can designate an allocation scheme satisfying if $v_{ik}(s_i) < v_{ik}(s_i^*)$, agent i will have no probability to be allocated the good k.

if $v_{ik}(s_i) > v_{ik}(s_i^*)$, agent i will have a probability to be allocated the certain good k and charged $v_{ik}(s_i^*)$. However, the probability does not depend on s_i.

The wanted allocation scheme under the propostion's environment: Randomly pick an integer solution matrix x of the assignment maximization problem [1]

$$\max_{i,j} x_{ij} v_{ij}(s)$$

s.t.

$$\sum_i x_{ij} \leq 1 \ for \ all \ j$$

$$\sum_j x_{ij} \leq 1 \ for \ all \ i$$

---

[1]By Dantzig's linear programming theory, such a solution exists.

$$x_{ij} \geq 0 \ for \ all \ i, j$$

Then assign the goods according to the matrix x such that if $x_{ij} = 1$, agent i is assigned good j, and i is charged $v_{ij}(s_i^*(s_{-i}), s_{-i})$.

Now we need to check that this scheme satisfy the two requirements in Condition ρ to complete the proof. Let us show two useful properties.

First, we assert that whenever agent i is never allocated the good k in any maximizing solution xs to $\sum_{ij} x_{ij} v_{ij}(s)$ under the reported profile (s_i, s_{-i}), then agent i is never allocated the good k in any maximizing solution x's to $\sum_{ij} x'_{ij} v_{ij}(s'_i, s_{-i})$ under the reported profile (s'_i, s_{-i}) if $s'_i < s_i$. Suppose not, then there is an $\hat{x}$ allocating k to i such that $\sum_{ij} \hat{x}_{ij} v_{ij}(s'_i, s_{-i}) \geq \sum_{ij} x_{ij} v_{ij}(s'_i, s_{-i})$. Notice that s_i influence v_{ij} only through $o_i(s_i)$ for all $j \in \{1, \cdots, n\}$; for all other agent $l \neq i$, s_i influence v_{lj} only through $r_i(s_i)$. Furthermore, $\frac{\partial o_i}{\partial s_i} > \frac{\partial r_i}{\partial s_i} > 0$. Thus $\sum_{ij} \hat{x}_{ij} v_{ij}(s_i, s_{-i}) \geq \sum_{ij} x_{ij} v_{ij}(s_i, s_{-i})$ as the increase from s'_i to s_i add to the left hand side at least as much as the right hand side[2]. So $\hat{x}$ is also a maximizing scheme. Contrary to maximizing solution never allocate k to i.

Second, we assert that whenever agent i is allocated the good k in a integer maximizing solution xs to $\sum_{ij} x_{ij} v_{ij}(s)$ under the reported profile (s_i, s_{-i}), then agent i must also be allocated the good k in some maximizing solution x's to $\sum_{ij} x'_{ij} v_{ij}(s'_i, s_{-i})$ under the reported profile (s'_i, s_{-i}) if $s'_i > s_i$. Suppose not, then there is a x not allocating k to i such that $\sum_{ij} \hat{x}_{ij} v_{ij}(s'_i, s_{-i}) > \sum_{ij} x_{ij} v_{ij}(s'_i, s_{-i})$. Notice that s_i influence v_{ij} only through $o_i(s_i)$ for all $j \in \{1, \cdots, n\}$; for all other agent $l \neq i$, s_i influence v_{lj} only through $r_i(s_i)$. Furthermore, $\frac{\partial o_i}{\partial s_i} > \frac{\partial r_i}{\partial s_i} > 0$. Thus $\sum_{ij} \hat{x}_{ij} v_{ij}(s_i, s_{-i}) > \sum_{ij} x_{ij} v_{ij}(s_i, s_{-i})$ as the decrease from s'_i to s_i reduce the right hand side at least as much as the right hand side. This is contrary to that x maximizes $\sum_{ij} x_{ij} v_{ij}(s)$.

From these two properties, we have for each agent good pair (i, j) a particular $s_i = s_i(j)$ such that when $s_i < s_i(j)$, $x_{ij} = 1$ never hold for any integer maximizing solution xs to $\sum_{ij} x_{ij} v_{ij}(s)$; when $s_i > s_i(j)$, $x_{ij} = 1$ hold for some integer maximizing solution xs to $\sum_{ij} x_{ij} v_{ij}(s)$.[3]

A notational declaration: all the x in the following part is integral.

---

[2]Note that all n goods are allocated in any maximizing scheme

[3]when j is never allocated to i in a integer maximizing solution, let $s_i(j) = 0$; When j is always allocated to i in some integer maximizing solution, let $s_i(j) = +\infty$.

Now we claim that for any i, $s_i(j) = s_i(j')$ for all j, j' if neither is $+\infty$. Suppose not, then as s_i increases from 0 to $+\infty$, after we pass the first $s_i(j), j \in \{1, \cdots, n\}$, we will meet a second $s_i(j'), j' \in \{1, \cdots, n\}$, the set $\{j \in \{1, \cdots, n\} | x_{ij} = 1, x \ maximizes \sum_{ij} x_{ij} v_{ij}(s_i, s_{-i})\}$ increases. Let x' denote the integer solution to $\max_x \sum_{ij} x_{ij} v_{ij}(s_i, s_{-i})$ such that $x_{ij'} = 1$ when $s_i > s_i(j')$. Then for any other assignment x, $x'_{ij} v_{ij}(s_i, s_{-i}) \geq x_{ij} v_{ij}(s_i, s_{-i})$. Consider s_i decreases to $s_i < s_i(j')$. Since s_i influence v_{ij} only through $o_i(s_i)$ for all $j \in \{1, \cdots, n\}$, for all other agent $l \neq i$, s_i influence v_{lj} only through $r_i(s_i)$, and $\frac{\partial o_i}{\partial s_i} > \frac{\partial r_i}{\partial s_i} > 0$, it is still the case that $x'_{ij} v_{ij}(s_i, s_{-i}) \geq x_{ij} v_{ij}(s_i, s_{-i})$ as the decrease from s'_i to s_i reduce the right hand side at least as much as the right hand side. But it's contrary to the description of $s_i(j')$ which says that if $s_i < s_j(j')$, $x_{ij} = 1$ never hold for any integer maximizing solution xs to $\sum_{ij} x_{ij} v_{ij}(s)$.

Let s_i^* be the value that all $s_i(j)$ which is not ∞ equal to. Then the sets $\{j \in \{1, \cdots, n\} | x_{ij} = 1, x \ maximizes \sum_{ij} x_{ij} v_{ij}(s_i, s_{-i})\}$ and $\{x | x \ maximizies \sum_{ij} x_{ij} v_{ij}(s_i, s_{-i})\}$ stay the same for any $s_i > s_i^*$.[4]

We can see that Condition ρ holds, because for any i, k,

if $v_{ik}(s_i) < v_{ik}(s_i^*)$, since s_i influences $v_{ik}(s_i)$ only through $o_i(s_i)$ and $\frac{\partial o_i}{\partial s_i} > 0$, $s_i < s_i*$. According to s_i*'s definition, agent i will have no probability to be allocated the good k.

if $v_{ik}(s_i) > v_{ik}(s_i^*)$, then $s_i > s_i^*$, agent i will have a probability to be allocated the certain good k and charged $v_{ik}(s_i^*)$. However, the probability does not rely on s_i for it is a fixed value for all $s_i > s_i^*$, $|\{x | x_{ij} = 1, x \ maximizes \sum_{ij} x_{ij} v_{ij}(s_i, s_{-i})\}| / |\{x | x \ maximizes \sum_{ij} x_{ij} v_{ij}(s_i, s_{-i})\}|$.

<div align="right">□</div>

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[4]The fact that $\{x | x \ maximizes \sum_{ij} x_{ij} v_{ij}(s_i, s_{-i})\}$ stays the same for any $s_i > s_i^*$ has not been shown explicitly yet. However, it is easy to show its truthfulness by the method of "reduction to absurdity" like the previous paragraphs.

Appendix D

Proofs to theorems in Chapter 3

Theorem. *(Gale-Shapley) The matching given by DA mechanism is stable.*

Proof. We will prove it by reduction to absurdity. Suppose not, then there is a student i and a college j such that i strictly prefers j to his final assignment and $|\mu^{-1}(j)| < q_j$ or $\exists i' \in \mu^{-1}(j)$ $satisfying$ $\pi_j(i') < \pi_j(i)$. However, according to the student side operation of DA mechanism, j must have been applied before the final assignment college. And according to the college side operation of DA mechanism, if $|\mu^{-1}(j)| < q_j$ or $\exists i' \in \mu^{-1}(j)$ $satisfying$ $\pi_j(i') < \pi_j(i)$, then i must not be rejected by j which contradicts to the fact that he or she was rejected(implied by the fact that his or her final college is lower in the application order than college j). $\square$

Theorem. *(Gale-Shapley) Every student is at least as well off under the assignment given by the DA mechanism as he would be under any other stable assignment.*

Proof. Call a college j achievable for a student i in a particular environment if the student can be assigned the college in some stable matching result.

The proof is to show the following proposition. Up to a certain step n, no student has ever been rejected by a college achievable for the student, then no student will be rejected by a college achievable for the student in the step n too.

Since up to step 1, no student has been rejected by any college obivously, we can show that up to any finite step no student will ever be rejected by a college achievable for the student by repeatedly applying the previous proposition. The end step is a finite number, so in the final result no student is ever rejected by a college achievable for the student. And the students apply colleges from the most

favorite to the least favorite, we therefore know that the final assignment is most favorite for every student among the stable matchings.

Now we will prove that up to a certain step n, no student has ever been rejected by a college achievable for the student, then no student will be rejected by a college achievable for the student in the step n too. That "then" part is eqivalent to say, suppose a student i is rejected by a college j in the step n, then the college is not achievable for the student. Since j reject i, that means that after step n every student assigned to j is higher in priority than i, and the quota of j has been full. If i is assigned to j in some stable matching result, then some i' that is assigned to j in the step n must be assigned to some college that is less preferred than j[1]. A contradiction occurs here for such a matching result cannot be a stable matching result due to the existence of pair i', j which blocks i, j. Therefore by reduction to absurdity, we show that i cannot be assigned to j in any stable matching result, i.e., j is not achievable to i. □

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[1] because up to n no student is rejected by an achievable college and i' is rejected by all the preferred colleges, so in a stable matching i' must be matched to some less preferred colleges.

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Resume

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