SHANGHAI UNIVERSITY OF FINANCE AND ECONOMICS

DOCTORAL THESIS

Essays on some mechanisms in the fields of auction, matching and mental competition

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Declaration of Authorship

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- Where I have consulted the published work of others, this is always clearly attributed.
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"Thanks to my solid academic training, today I can write a whole doctoral thesis paper!"

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Abstract

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Essays on some mechanisms in the fields of auction, matching and mental competition

by Yan JU

Mechanisms Design is a very interesting field of economics. In this paper, theories about general mechanisms are presented as well as some kinds of special mechanisms such as matching. With all the needed details provided or given in the literature, the thesis is mostly a reflection of the author's interest and provide insightful arguments and findings in related areas.

Under the interdependent value environment, the paper proposes a sufficient condition for ex post implementation of efficiency in an auction, i.e., allocating the good to the agent who values it most.This condition is less demanding than a sufficient condition that is proposed in (Dasgupta and Maskin, 2000). Then the paper further extend the sufficient condition to the case of multiple goods assignment with interdependent values and call the extended condition Condition ρ. Then we provide a model where Condition ρ is satisfied and we design a mechanism to ex post implement the social efficiency maximizing goal through appropriate money transfer.

For the Chinese college admission mechanism, the paper finds that the allocation results of the ex post equilibrium in the Boston Mechanism, Chinese parallel mechanism and the DA mechanism are the same. However, the ease for the DA mechanism to reach this allocation result can never be reached for the other two mechanisms, which might be a motivation for advocating DA mechanism in the Chinese college admission mechanism.

For a perfect information competitive board game like chess which is a mechanism for deciding win or loss, the last chapter of this paper classify every position into three categories, winning position, losing position, and drawing position, and provides the way to identify them theoretically. It is only theoretical for chess because the work need to be done is astronomical figure. However, the paper manages to use this method to solve an interesting game, which shows the power of the notions.

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Thanks to professor Guoqiang Tian for his teaching of Mechanism Design theory and advice to this doctoral paper, Zhe Yang for the cooperation we have been doing in publishing the paper "existence and generic stability of cooperative equilibria for multi-leader-multi-follower games" which is primarily his idea. Thanks to all the teachers who has helped or taught me in SOE.

Thanks to all the classmates and friends that has been with me!

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List of Abbreviations

DA	**D**eferred **A**cceptance
SOSM	**S**tudent **O**ptimal **S**table **M**atching
TTC	**T**op **T**rading **C**ycle
SD	**S**erial **D**ictatorship

Dedicated To my parents who has been so patient and loving to me since my birth

Chapter 1

An Introduction to Mechanism Design Theory

1.1 Introduction

The starting point of analysis in traditional economics is often the present state of economic institutions and mechanisms, for example, market theory, auctions and resource allocation. In mechanism design theory, an alternative framework and perspective is used. We view the present state of affairs just as a possibility and has an intention to design a better mechanism with respect to a social goal or criterion. Mechanism design is kind of engineering side of economics. To do that, a framework must be provided. Now the framework has become a theory with a large body of literatures. That theory is mechanism design theory. The mechanism design theory was established by leonid Hurwicz, later expanded by Roger B.Myerson, Eric S. Maskin, Guoqiang Tian and many other scholars of the field.

The most important component of Mechanism design theory is implementation theory. (Liang, 2010) has done a survey of this field . The present paper is not another attempt to summarize the implementation theory of mechanism design, but uses it as a central tool for formalizing ideas in this paper. This paper aims to analyze some economic mechanisms for situations where information is not fully known to the social planner. In the process, the author creates some new concepts which the author thought is necessary. However, a little background of the mechanism design theory is needed. This is what the next section is about.

1.2 A survey of the background

A formal study of the informational requirements and informational optimality of resource allocation processes was initiated by Hurwicz (1960). In line with the prevailing tradition, interest in this area was focused on the design of Pareto-satisfactory (non-wasteful) and privacy-preserving mechanisms, i.e., mechanisms that result in Pareto efficient allocations and use informationally decentralized decision making processes. Allocative efficiency and informational efficiency are two highly desired properties for an economic system to have. The importance of Pareto optimality is attributed to what may be regarded as a minimal welfare property. Pareto optimality requires resources be allocated efficiently. If an allocation is not efficient, there is a waste in allocating resources and thus at least one agent is better off without making others worse off under given resources. Informational efficiency requires an economic system have the minimal informational cost of operation. The informational requirements depend upon two basic components: the class and types of economic environments over which a mechanism is supposed to operate and the particular outcomes that a mechanism is required to realize.

For informational decentralized systems, ("On informationally decentralized systems") proved a very important theorem: For the neoclassical pri- vate goods economies, there is no mechanism < M, h > that implements Pareto efficient and individually rational allocations in dominant strategy. Consequently, any revelation mechanism < M, h > that yields Pareto efficient and individually rational allocations is not strongly individually incentive compatible. (Truth-telling about their preferences is not Nash Equilibrium).

A mechanism can be viewed as an abstract planning procedure; it consists of a message space in which communication takes place, rules by which the agents form messages, and an outcome function which translates messages into outcomes (allocations of resources). Mecha- nisms are imagined to operate iteratively. Attention, however, may be focused on mechanisms that have stationary or equilibrium messages for each possible economic environment. A mechanism realizes a prespecified welfare criterion (also called performance, social choice rule, or social choice correspondence) if the outcomes given by the outcome function agree with the welfare criterion of the stationary messages.

1.3 General model framework

In this section, we give the framework of analysis which will reappear many times later in the paper with slightly different forms. Our framework is comprised of 5 parts: economic environments, social goals or criteria, mechanism, expected outcomes(often equilbirums of all kinds), and the concept of implementation of social goals. The following subsections will give a detailed discussion and relative notations of these components.

1.3.1 Economic environments

Economic environments consists of economic entities and their features as well as the state of some relevant things in the world. The entities are of two kinds. One is the principal (or called the social planner), and the other is the agents (or called economic participants). Usually, we have the following notations.

$N = \{1, ..., n\}$: denote the set of the agents.

$e_i \in E_i$: denote the economic feature of agent $i \in N$. It may be preferences, economic status or some other relevant feature.

$E = E_1 \times \cdots \times E_2$: denote the set of profiles of economic features.

The social planner does not know much information of the participants' profile e. These information are decentralized among the agents. If the agents all know the whole profile e, then it is the perfect information case; if every agent i knows his own e_i and knows the distribution of e, then this is the imperfect information case which can be dealt with using Bayesian method; else if every agent i only knows his own e_i, then it is only good to be dealt with using strategyproof mechanisms. The details will be in later chapters.

1.3.2 Social criteria

Given an economic environment e, every agents participate in economic activities, make economic decisions, pay the cost and get the profits. The social planner hope that the results satisfy some criteria. Let us give some notations and talk about it.

A: denote the set of social allocations, or economic results.

$F : E \mapsto A$: denote a social criterion, or social goal, which is a correpondence from the set of environments to the set of results. Given any environment e, there will be a subset of A that satisfy the social criterion, F just denotes this function.

If randomized results on A are acceptable as social results, then we can use ΔA instead of just A.

1.3.3 Mechanism for information collection

The social planner lacks information, so he or she need to design incentive compatible rules of the game to induce everyone reveal their true private information.

If the social planner knows completely the environment, then he or she can simply choose a result that is in the set $F(A)$. However, he or she usually does not know much about these things. That is where mechanism for information collection functions. A mechanism, or a game form, usually contains the following components.

M_i: denote the message space of agent $i \in N$. An agent can only emit message $m_i \in M_i$.

$M = M_1 \times \cdots \times M_n$: denote the space of message profiles. Every message profile $m = (m_1, \ldots, m_n) \in M$.

$h : M \to A$: denote the assignment function for the mechanism, which assigns a result for a given message profile m.

$\Gamma = \langle M, h \rangle$: denote the information collection mechanism, which is just the combination of the space of message profiles and the assignment function that lies on top of it.

Thus, a mechanism prescribes rule of the game. Every agent i chooses a message $m_i \in M_i$ to send, and then the social planner collects all the messages in the message profile m, and finally decides on the allocation result $h(m)$ as the social choice. The Γ must be declared openly to let every agent know, then it is up to every agent i to choose from his or her M_i the m_i to report.

A very important class of mechanisms is the direct revelation mechanism(or simply called direct mechanism) in which M_i is just the possible world state information that agent i has. later we will introduce a very important theorem about this mechanism.

1.3.4 Expected outcomes

When a mechanism $\Gamma = \langle M, h\rangle$ is given, we have a game with rules for the agents to report $m \in M$. The strategy of an agent i is usually denoted $\sigma^i(e_i, \Gamma)$ or $\rho^i(e_i, \Gamma)$. Taking this form is for the reason that different environments e_i may induce i to choose different message for a given mechanism Γ. For a given Γ, the Γ in the above notation can usually be omitted as the default Γ is clear. When the agents send messages to the social planner, they have strategical interaction in choosing which message to send. Now we need to know what will result from the strategical interaction, these are the expected outcomes. Usually the solution concept of equilibriums in games are ideal for this role.

Here, one point concerning the e should be stressed. There is requirement on the preference information which is contained in e_i for every $i \in N$. When people game with each other, the hypothesis for human behavior is very important. A fundamental hypothesis is that human being are self-interested, that is, they will try to maximize some kind of self-utility. With this hypothesis, it is usually implied that human being will not deliberately contribute to the society without considering the returns to himself or herself. Or put it another way, a human being will only concentrate on maximizing self-interest, only a good mechanism can make this self-interesting behavior also beneficial to the society. Let $\succeq_{e_i}$ denote the preference in e_i for agent i. It must be an order relation on the results set A. Put it another way, it must satisfy the following 3 conditions.

- Reflexivity. For any outcome $a \in A$, $a \succeq a$.
- Completeness. For any two outcomes a and b from A, either $a \succeq b$ or $b \succeq a$. That is , any two outcomes are comparable.
- Transitivity. For any three outcomes a, b and c, if both $a \succeq b$ and $b \succeq c$ hold, then $a \succeq c$ hold. This eliminates unwanted circles.

Two other relations $\succ$ and $\sim$ can be generated from a given $\succeq$. For any a and b from the results set A, $a \succ b$ if and only if $a \succeq b$ and $b \not\succeq a$; $a \sim b$ if and only if $a \succeq b$ and $b \succeq a$.

The three relations $\succeq, \succ, \sim$ are also frequently denoted with a single letter R, P, I respectively. Later, we will use these single letter forms for consistency with matching literature in that chapter.

With this kind of preference, every agent i can rank the results in A in such a way that equilibriums are definable.

With this hypothesis, the equilibrium concepts of game theory best suit our needs for the expected outcomes. The equilibriums are the expected outcomes. For different situations, different equilibrium concepts should be taken as the expected outcomes. Dominant strategy equilibrium, Nash equilibrium, Subgame perfect Nash equilibrium, Bayesian Nash equilibrium, Perfect Bayesian equilibrium and so on all have their chances of being the expected outcomes depending on the problem situation. Notations are as follows.

$b(e,\Gamma)$: denote the equilibrium message choices $M* \subset M$ under the economic environment e and mechanism Γ.

$h(b(e,\Gamma))$: denote the expected outcomes of the assignment.

For a given Γ, the Γ in the above notation can usually be omitted as the default Γ is clear.

1.3.5 Implementation of social criteria

Finally, comes the concept of implementation. An important goal of mechanism design is to make individual incentives and social criteria compatible. Roughly speaking, if a social criteria is satisfiable under certain equilibrium solution of a mechanism, then we say that that social criteria is implementable with that mechanism. The idea can be illustrated by the following graph.

$$\begin{tikzcd} & M \arrow[dr, "h(b(e,\Gamma))"] & \\ E \arrow[ur, "b(e,\Gamma)"] \arrow[rr, "F(e)"'] & & A \end{tikzcd}$$

In the above graph, $e \in E$ is the economic environment, A is the possible allocation set. Social criteria F choose what is desirable allocation for the society. The social planner need to design a mechanism to implement the criteria. Under such a mechanism, the strategic interaction of agents should result in an outcome in the equilibrium set $b(e,\Gamma)$ mapped by h, i.e., $h(b(e,\Gamma))$. If $h(b(e,\Gamma))$ is in the set $F(e)$, then the social criteria have been implemented. Formally, we have the following definition

Definition 1. *A mechanism $\Gamma = \langle M, h\rangle$ is said to have implemented the social criteria F in the economic environments space E under the equilibrium b, if for all $e \in E$, we have $h(b(e, \Gamma)) \subset F(e)$.*

Our implementation concept in this paper is weak compared to most in the literature. And what we call social criteria F is usually called social choice, by choosing the wording "criteria" we want to convey the idea that not every point in $F(e)$ must be implemented. However, we still distinguish between two forms of weak implementation.

Here is the weaker implementation concept definition that we call partial implementation.

Definition 2. *A mechanism $\Gamma = \langle M, h\rangle$ is said to have partially implemented the social criteria F in the economic environments space E under the equilibrium b, if for all $e \in E$, we have $h(b(e, \Gamma)) \cap F(e) \neq \emptyset$.*

Below is the strongest implementation concept that we call full implementation.

Definition 3. *A mechanism $\Gamma = \langle M, h\rangle$ is said to have fully implemented the social choice rule F in the economic environments space E under the equilibrium b, if for all $e \in E$, we have $h(b(e, \Gamma)) = F(e)$.*

The above five-component framework covers both the literature review of chapter 2 as well as the messaging mechanisms of auction and matching we will later discussed in-depth in Chapter3 and Chapter4.

For these kind of mesaging mechanism, there is a very famous theorem called revelation principle. I will give here its definition and the proof is given in Appendix A.

Theorem. *revelation principle:*

if a Mechanism $\langle M, h\rangle$ implements the social criteria F in dominant strategy equilibrium. Then there is a direct revelation mechanism which implements F truthfully in dominant strategy equilibrium(truth telling is a dominant strategy equilibrium).

The importance of revelation principle lies in the fact that it narrowes the search space for an implementation mechanism to a special kind of mechanism. This makes the finding of a mechanism implementing some social goal more easily or give a method of proving the nonexistence of such mechanism in general by just proving the unavailability of such mechanism for the revelation games.

1.4 Mechanisms we will discuss

In this section, I would like to talk about the scope of my thesis. The main focus will be around mechanisms for dealing with decentralized information problem. The last chapter is a diversion to some mechanisms that are designed for special purpose and under special requirements.4 chapters follow this introductory one.

Chapter 2 is a review of important results in literatures that influence this paper most, especially those impossibility results. These results establish the impossibility of implementability with most general conditions. And this lead us to put effort in the consideration of more specific environments and conditions. The chapter concludes with an example of this kind of analysis in an environment with more assumptions.

Chapter 3 is a whole chapter devoted to the study of ex post implementation of socially efficient allocation goal through money transfer under the interdependent value environment. Both continuous and discrete cases are considered. A sufficient condition called Condtion ρ for ex post implementation with interdependent values via money transfer is given for the multiple goods assignment problem with interdependent information as an attempt to contribute to the field.

Chapter 4 is devoted to a discussion of matching mechanisms design. Then the focus of attention is turned on an application of the theory to the Chinese college admission mechanism. A few derived propositions show that the score-dictating mechanism is perhaps the best choice among three popular mechanisms.

Chapter 5 is a chapter reflecting upon a special class of mechanisms in which the implementation goal is not preset. It is up to the agents to show their techniques to compete in these mechanism to win . Focus is put upon finite dynamic games with perfect information with chess as a model for the convenience of discussion.

Chapter 2

implementation theory and examples of designing mechanisms

2.1 Introduction

For dominant strategy implementation, we have mentioned the important contributions of Hurwicz, Gibbard, Satterthwaite. In the first section of this chapter, these results will be given in detail. These are important impossibility theorems about which we can not design. Like in physics, we know that we can not produce perpetual motion machine, then we put our efforts on resource consuming machines which are everywhere in use today. Likely, we should understand these proved negative theorems well first, then we can begin our search on mechanisms that can be designed and of some good qualities while avoiding time waste on proved impossible mechanism design.

As we have mentioned in 1, the agents may have perfect information regarding the economic environment e. In this case, Nash Implementation is the solution concept most used. This we will deal with in the next section. If every agent i knows his own e_i and knows the distribution of e, then this is the imperfect information case which can be dealt with using Bayesian method. These are mostly the work of an old generation of economic theorists but serve as good reference points for comparison or provide basic framework for some newer application.

The economic environment e in the following part and most economic literature is mostly the preference profile or utility profile, usually denoted R or u.

2.2 Important impossibility results

(**Gibbard1973**) and (Satterthwaite, 1975) proved a very important theorem:If the assignment results has at least 3 alterna- tives, a social choice function which is strongly individually incentive compatible and defined on a unrestricted domain is dictatorial. (Gibbard, 1973) and (Satterthwaite, 1975) proposed a very important theorem for implementation theory. We will call it Gibbard Satterthwaite Impossibility theorem henceforth. It is stated like this.

Theorem 1. *If the outcome set A has at least 3 alterna- tives, a social choice rule which is strongly individually incentive compatible and defined on a unrestricted preference domain is dictatorial.*

This impossibility result is very important. It lead the research to restricted preference domain, like in matching where the outcome is assumed to be ranked only by a player's self matched object.

Pareto efficiency is often a basic requirement of economic mechanisms. However, in ("On informationally decentralized systems") Hurwics shows that the Pareto efficiency and the truthful revelation is fundamentally inconsistent even for the class of neoclassical economic environments.

Theorem 2. *(Hurwicz Impossibility Theorem, 1972) For the neoclassical pri- vate goods economies, there is no mechanism $< M, h >$ that implements Pareto efficient and individually rational allocations in dominant strategy. Consequently, any revelation mechanism $< M, h >$ that yields Pareto efficient and individually rational allocations is not strongly individually incentive compatible. (Truth-telling about their preferences is not Nash Equilibrium).*

The proof is adapted from Guoqiang Tian 2014.

Proof. By the Revelation Principle, we only need to show that any revelation mecha- nism cannot implement Pareto efficient and individually rational allocations truthfully in dominant equilibrium for a particular pure exchange economy.

Then, it is enough to show that truth-telling is not a Nash equilibrium for any revelation mechanism that yields Pareto efficient and individually rational allocations for a particular pure exchange economy.

Consider a private goods economy with two agents and two goods. The endowments are

$$w_1 = (2, 0), w_2 = (0, 2).$$

The utilities are

$$u_i(x, y) = \begin{cases} 3x_i + y_i & \text{if } x_i \leq y_i \\ x_i + 3y_i & \text{if } x_i > y_i \end{cases}$$

The above things form the economic environment e.

For the pure exchange economy, it has the following allocation results set

$$A = \{((x_1, y_1), (x_2, y_2)) | x_1 + x_2 = 2 \; and \; y_1 + y_2 = 2\}$$

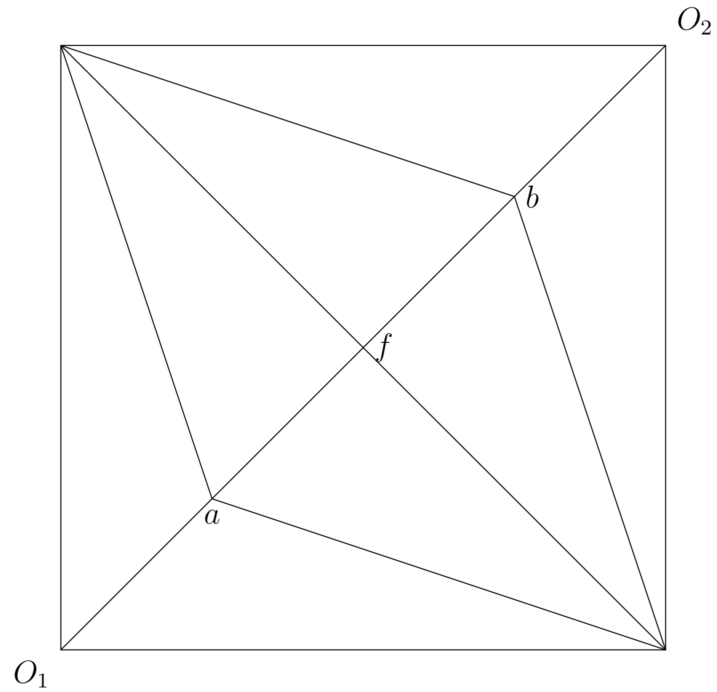
Let U be the set of all neoclassical utility functions, i.e. they are continuous and quasi- concave, which agent i can report to the designer. Thus, the true utility function $u_i \in U$.

Then,

$$h : U \times U \to A$$

Suppose that the true utility function profile u_i were indeed a Nash Equilibrium, it would then satisfy

$$u_i(h(u_i, u_{-i})) \geq u_i(h(u_i', u_{-i}))$$



Let us denote the individual rational allocations by $IR(e)$, the Pareto efficient allocations by $P(e)$, the true report allocation $d = h(u_i, u_{-i})$.

First thing to note is that the Pareto efficient allocation results is all on the 45 degree line. For any other point, move on to the 45 degree line along the shortest path(the path that is orthogonal to it) is a Pareto improvement. $P(e) = O_1O_2$. Individual rational means that the final allocation must be at least as good as the endowment, so $P(e) \cap IR(e) = \overline{ab}$.

Suppose $d \in P(e) \cap IR(e)$, that is, $d \in \overline{ab}$.

If agent 1 misreports his utility function :

$$u_1(x, y) = x_1 + y_1$$

Then, with the misreported u_1 and thus the misreported e', the new set of individually rational and Pareto efficient allocations is given by $P(e') \cap IR(e') = \overline{fb}$

If agent 2 misreports his utility function :

$$u_2(x, y) = x_2 + y_2$$

Then, with the misreported u_2 and thus the misreported e'', the new set of individually rational and Pareto efficient allocations is given by $P(e'') \cap IR(e'') = \overline{af}$

If the d is on $\overline{af}$, agent 1 will choose to misreport a $u_1(x, y) = x_1 + ky_1$ where $0 < k < 1$. For any d that is on $\overline{fb}$, agent 2 will choose to misreport a $u_2(x, y) = x_2 + ky_2$ where $0 < k < 1$.

Thus, no mechanism that yields Pareto efficient and individually rational allocations is incentive compatible for both sides.

□

2.3 Nash implementation

(Maskin, 1999) proposed a monotonicity concept that is later called Maskin monotonicity, which is a necessary condition for Nash implementation. This condition along with no veto power constitutes a simple set of sufficient conditions for full Nash Implementation.

Now, let us see the definintion of Maskin monotonicity.

Definition 4. *A social choice rule $f : \mathscr{R} \to A$ satisfies Maskin monotonicity provided that*

$\forall a \in A, \forall R\ R' \in \mathscr{R},$, if $a \in f(R)$ and [$\forall i \in \{1, \ldots, n\} \forall b \in A,\ aR_ib \Rightarrow aR'_ib$], then $a \in f(R')$.

Quoting (Maskin, 1999), "In words, monotonicity requires that if alternative a is f optimal with respect to some profile of preferences and the profile is then altered so that, in each individual's ordering, a does not fall below any alternative that it was not below before, then a remains f optimal with respect to the new profile." The f optimal in the above quotation means that the alternative is chosen by the social choice rule f.

To illustrate the concept, (Maskin, 1999) provided some examples of mechanisms satisfying Maskin Monotonicity. As a first example, he considered the Pareto optimal correspondence f^{PO}. If a is (weakly) Pareto optimal with respect to R, then for all b, $\forall i\ aR_ib$. Now if we replace R by R' such that, for all i, $aR_ib \Rightarrow aR'_ib$, we conclude that for all b, $\forall i\ aR'_ib$. Hence, a is also (weakly) Pareto optimal with respect to R', establishing the monotonicity of f^{PO}.

The Condorcet correspondence f^{CON} is also Maskin monotonic. If a is a majority winner for a strict profile (a profile consisting of strict orderings) R, then, for any other alternative b, the number of individuals preferring a to b is no less than the number preferring b to a. Formally, $|\{i|aR_ib\}| \geq |\{i|bR_ia\}|$ where the $|$s deliminating a set stands for the number of elements in the set. Now if R' is a profile such that, for all i, $aR_ib \Rightarrow aR'_ib$, then the left-hand side of the inequality cannot fall when we replace R by R'. Furthermore, if the right-hand side of the inequality rises, then a contradiction happens since for strict relation R R' $|\{i|aR_ib\}| + |\{i|bR_ia\}| = |\{i|aR'_ib\}| + |\{i|bR'_ia\}| = n$. Therefore we conclude that the inequality continues to hold when R' replaces R, and so a is still a majority winner with respect to the profile R', establishing the monotonicity of f^{CON}.

The following is an important theorem proposed by (Maskin, 1999).

Theorem 3. *If $f : \mathscr{R} \to A$ is an SCR that is fully implementable in Nash equilibrium, then it is Maskin monotonic.*

Proof. Suppose f is fullly implementable in Nash equilibrium by the game form $h : M_1 \times \cdots \times M_n \to A$. Then for an arbitrary profile $R \in \mathscr{R}$, and any $a \in f(R)$, because of full implementation, there exists a Nash equilibrium m of g with respect to R such that $h(m) = a$. Now consider a profile R' that satisfy the condition $\forall i \in \{1, \dots, n\} \forall b \in A$ $aR_ib \Rightarrow aR'_ib$

If m is not a Nash equilibrium with respect to R', then there exists i and m'_i such that $h(m'_i, m_{-i})P(R'_i)h(m)$. By using the contrapositive of the above condition of R', we get that there exists i and m'_i such that $h(m'_i, m_{-i})P(R_i)h(m)$, this contradicts that m is a Nash equilibrium. Therefore m is a Nash equilibrium with respect to R', by full implementation, $h(m) \in f(R')$, i.e., $a \in f(R')$. Maskin monotonicity of f is proved.

□

(Maskin, 1999) proposes a No Veto Power(NVP) concept, together with Maskin monotonicity will be sufficient to guarantee full Nash implementability. Here is its definition.

Definition 5. *An social choice rule SCR $f : \mathscr{R} \to A$ satisfies NVP if, $\forall R \in \mathscr{R}, \forall a \in A$, and $\forall i \in \{1, \dots, n\}$, $(\forall j \neq i$ and $\forall b \in A$, $aR_jb)$ $\Rightarrow a \in f(R)$.*

Now the famous theorem of Maskin:

Theorem 4. *If $n \geq 3$ and $f : \mathscr{R}$ is a n-person SCR satisfying Maskin monotonicity and No Veto Power, then it is implementable in Nash equilibrium.*

The proof of it is illustrative of how to prove implementability. It is constructive as you can see. This proof method has been in (Moor and Repullo, 1990). (Maskin, 1999) adopted the same approach. Here it is adapted. Only pure strategy is considered for implementability.

Proof. □

Now we have a sufficient condition. However, it is not a necessary condition. Two examples here.

Example 1. *A constant social choice rule . A social choice rule SCR is called a constant social choice rule if there is a* $C \subset A$ *such that* $\forall R \in \mathscr{R}, f(R) = C$*. For a constant social choice function, a Nash implementation mechanism can be given by letting people report their preference, and then choose the constant social choice* C *regardless of what they report. Obviously, what ever the report is, there is not profitable deviation. Therefore, every report profile constitutes a Nash equilibrium whose result is the* C*, i.e.,* $h(m) = f(R) = C$*.*

Example 2. *A dictatorial choice rule. A social choice rule SCR is called a dictatorial choice rule if* $\exists i \in \{1, \ldots, n\}$*such that* $\forall R \in \mathscr{R}$ $f(R) = top(R_i)$*(here* $top(R_i)$ *means the highest ranked* $a \in A$ *according to* R_i*). For a dictatorial social choice function, a Nash implementation mechanism can be given by letting people report their preference, and then choose the dictator's top ranked choice according to his or her reported preference regardless of what others report. Obviously, what ever the others report, if the dictator report his or her true preference, the result is a Nash equilibrium profile. Therefore, every report profile constitutes a Nash equilibrium whose result is the* $top(R_i)$*, i.e.,* $h(m) = f(R) = top(R_i)$*.*

The questions are then: what happens in the grey area between these necessary and sufficient conditions of Nash implementation; and what happens in the case of two agents?

Actually, (Moor and Repullo, 1990) has answered these questions. (Danilov, 1992) provides another essentially monotone condition that is necessary and sufficient. (Yamato, 1992) extended the (Danilov, 1992) conditions for Nash implementation to weak preferences over an arbitrary set of alternatives. Anyway, these necessary and sufficient condition was not easy to identify, and (Maskin, 1999) proposed the easier to identify sufficient condition and a separate necessary condition that are well-known as Maskin monotonicity(Actually, the paper as a working paper is widely known since 1977 and all the later papers had been written under the influence of it). (Maskin, 1999) has been widely known since 1977 as working paper, and all these necessary and sufficient condition has been greatly influenced by Maskin's work. In this retrospective chapter, we would like to have a look at these conditions.

2.3.1 Condition μ

First, let us take a look at the necessary and sufficient condition in (Moor and Repullo, 1990) which is the most general form of condition . It is called condition μ which contains three parts.

Definition 6. *Condition μ: There is a set $B \subset A$, and $\forall i \in I, R \in \mathscr{R}, a \in f(R)$, there is a set $C_i(a,R) \subset B$, with $a \in M_i(C_i(a,R),R)$ such that $\forall R^* \in \mathscr{R}$, the following (i), (ii) and (iii) are satisfied.*

(i) if $a \in \cap_{i \in I} M_i(C_i(a,R),R^*)$, then $a \in f(R^*)$.
(ii)if $c \in M_i(C_i(a,R),R^*) \cap [\cap_{j \neq i} M_j(B,R^*)]$, then $c \in f(R^*)$.
(iii)if $d \in \cap_{i \in I} M_i(B,R^*)$, then $d \in f(R*)$.

In the above definition, $I = \{1, \ldots, n\}$, and the M_i has the following meaning. For any $i \in I, R \in \mathscr{R}, C \subset A$, $M_i(C,R)$ denote the set of maximal elements in C for agent i under preference R_i.

Now, the theorem in (Moor and Repullo, 1990) is cited here.

Theorem 5. *Suppose there are three or more agents. Then a choice rule f is Nash implementable if and only if it satisfies Condition μ.*

Proof. Necessity. There must be a range for the Nash implementing mechanism $\Gamma = \langle S, g \rangle$. Let it be the B in condition μ.

$$B \equiv \{a \in A | a = g(s) \ for \ some \ s \in S\}$$

According to full Nash implementation, $\forall R \in \mathscr{R}, a \in f(R)$, there is a Nash equilibrium profile s implementing the a, denote it by $s(a,R)$. $\forall i \in I$, we choose

$$C_i(a,R) \equiv \{c \in A | c = g(s_i', s_{-i}(a,R)) \ for \ some \ s_i' \in S_i\}$$

After finding the B and C_i, $\forall R^* \in \mathscr{R}$, condition μ (i)(ii)(iii) are directly implied by the full Nash implementation.

if $a \in \cap_{i \in I} M_i(C_i(a, R), R^*)$, then $s(a, R)$ is a Nash equilibrium of the mechanism under R^*. Since the mechanism fully implement the social choice rule f, $a \in f(R^*)$.

if $c \in M_i(C_i(a, R), R^*) \cap [\cap_{j \neq i} M_j(B, R^*)]$, then $c = g(s'_i, s_{-i}(a, R))$ and is the best outcome among outcomes with the form $g(\hat{s}_i, s_{-i}(a, R))\, where\, \hat{s}_i \in S_i$ under R^*, and c is the best outcome for all $j \neq i$ in B under R^*. Therefore $(s'_i, s_{-i}(a, R))$ is a Nash equilibrium of the mechanism under R^*. Since the mechanism fully implement the social choice rule f, $c \in f(R^*)$.

if $d \in \cap_{i \in I} M_i(B, R^*)$, then d is the best outcome for all $i \in I$ under R^* and therefore the strategy profile $s(d, R*)$ producing d is a Nash equilbrium of the mechanism under R^*. Since the mechanism fully implement the social choice rule f, $d \in f(R^*)$. Thus, necessity is proved.

Sufficiency. For any f satisfying the condition μ, it is enough to construct a mechanism Γ which Nash implements the social choice rule f fully. The construction method of such a mechanism is adapted from the appendix of (Moor and Repullo, 1990).

For each agent $i \in I$, the message or strategy space is

$$S_i \equiv \mathscr{R} \times A \times Z^+$$

and the outcome function $g : S \to A$ is defined as follows.

Case 1. If $\forall i \in I, s_i = (R, a, n)$ such that $R \in \mathscr{R}, a \in A, n \in Z^+$, then $g(s) = a$.

Case 2. If $\exists i \in I, \forall j \neq i\ s_j = (R, a, n)\ and\ s_i = (R', a', n')$ where $(R, a, n) \neq (R', a', n')$, then

$$g(s) = \begin{cases} a' & \text{if } a' \in C_i(a, R) \\ a & \text{otherwise} \end{cases}$$

Case 3. For all the other situations, $g(s) = a_i$, where a_i is the agent with the largest reported n's choice of a (ties are broken by random selection).

Thus, the mechanism Γ is fully defined. We need to prove $\forall R in \mathscr{R} f(R) = NE(\Gamma, R)$. For any $R \in \mathscr{R}$, the following two facts are showed to complete the proof.

$f(R) \subset NE(\Gamma, R)$. $\forall a \in f(R)$, we consider the strategy profile s satisfying that $\forall i \in I, s_i = (R, a, 1)$. It is a Nash equilibrium, because if one deviates, Case 2 stipulates that other than a only an $a' \in C_i(a,R)$ can be chosen which is worse than a according to condition $\mu(a \in M_i(C_i(a,R),R)$).

$NE(\Gamma,R) \subset f(R)$. There are three equilibrium cases.

Case1: $\forall i \in I, s_i = (R',a,n)$, then $g(s) = a \in \cap_{i\in I} M_i(C_i(a,R'),R)$ because deviation can lead to any element in $C_i(a,R')$ and Nash equilibrium should guarantee no profitable deviation. By condition μ (i), $g(s) \in f(R)$.

Case2: $\exists i \in I, \forall j \neq i \ s_j = (R',a,n)$ *and* $s_i = (R'',a',n')$ where $(R',a,n) \neq (R'',a',n')$, then

$$g(s) = \begin{cases} a' & \text{if } a' \in C_i(a,R') \\ a & \text{otherwise} \end{cases}$$

and $g(s) \in\in M_i(C_i(a,R'),R) \cap [\cap_{j\neq i} M_j(B,R)]$ because i's deviation can lead to any element in $C_i(a,R')$ while $\forall j \neq i$ j's deviation can lead to any element in B by anouncing a big enough n. By condition μ (ii), $g(s) \in f(R)$.

Case3: For all the other situations other than the previous two cases, anyone can pronouce a big enough n to get any element in B. Nash equilibrium means that the chosen $g(s) = a \in \cap_{i\in I} M_i(B,R)$. By condition μ (iii), $g(s) \in f(R)$.

□

A explanation of the condition μ is needed here. It is not as complex as one may feel at a quick glance. In fact, condition μ (i) is an equivalent of Maskin monotonicity. To see this, let $C_i(a,R) = L_i(a,R)$, then Maskin's monotonicity $\Rightarrow$ condition μ (i). The other direction, condition μ (i) $\Rightarrow$ Maskin's monotonicity , is shown as follows. Choose any $R, R' \in \mathscr{R}$ and $a \in f(R)$ such that $L_i(a,R) \subset L_i(a,R')$ for all $i \in I$. From Condition μ we know that for each $i \in I$ and $R \in \mathscr{R}$ there exists a set $C_i(a,R)$ such that $a \in M_i(Ci(a,R),R)$ implying that $C_i(a,R) \subset L_i(a,R)$. Hence $C_i(a,R) \subset L_i(a,R')$, which is equivalent to $a \in \cap_{i\in I} M_i(C_i(a,R),R')$. Therefore by condition μ (i) we conclude that $a \in f(R')$. f is thus shown to be monotonic.

Condition μ (ii) (iii) are implied by No Veto Power. Let $B = A$, and we can see this easily.

To see the usage of such a necessary and sufficient condition, a proof of a proposition is provided.

First, introduce a concept called neutral social choice rule.

Definition 7. *Formally, a choice rule f is said to be neutral if for all permutations* $\pi : A \to A$ *and* $R \in \mathscr{R}$ *we have an* $R^{\pi} \in \mathscr{R}$ *such that* $f(R^{\pi}) = \pi \circ f(R)$ *and* $\forall i \in I : aR_i^{\pi}b \iff \pi^{-1}(a)R_i\pi^{-1}(b)$.

That is, a neutral social choice function does not care about what the allocation really is, it only cares about what the agents has ranked these allocations and then decides. Apparently, a constant social choice rule is not neutral. However, a dictatorial social choice rule is neutral as you can check according to the definition.

Proposition 1. *Suppose there are three or more agents. Then a choice rule f is Nash implementable if it is monotonic and neutral.*

Proof. Now we will prove it by condition μ and theorem 5.

Monotonicity is equivalent to condition μ (i). Now we need to show monotonicity and neutral imply condition μ (ii)(iii).

For the implication of condition μ (ii). $\forall R, R' \in \mathscr{R}, a \in f(R), i \in I$, if $c \in M_i(C_i(a, R), R') \cap [\cap_{j \neq i} M_j(B, R')]$, we need to show $c \in f(R')$. Choose $R'' \in \mathscr{R}$ such that it is the same as R except that outcomes a and c are switched. By neutrality, $a \in f(R) \Rightarrow c \in f(R'')$. By monotonicity, $c \in f(R'') \Rightarrow c \in f(R')$ as required.

For the implication of condition μ (iii). $\forall R' \in \mathscr{R}$, if $d \in \cap_{i \in I} M_i(B, R')$, we need to show $d \in f(R')$. Choose $h \in f(R')$. If $h = d$, then the proof is done. If $h \neq d$, choose $\hat{R} \in \mathscr{R}$ such that it is the same as R' except that outcomes d and h are switched. By neutrality, $h \in f(R') \Rightarrow d \in f(\hat{R})$. By monotonicity, $d \in f(\hat{R}) \Rightarrow d \in f(R')$ as required. □

2.3.2 Essentially monotonic

(Danilov, 1992) proposed an essentially monotone concept and (Yamato, 1992) extended its application conditions and called it strongly monotonic. We will call it essentially monotonic in this paper.

Some definitions are needed. First, the notion of essential element for a participant in a particular set of outcomes.

Definition 8. *For any* $i \in I$ *and* $X \subset A$, *an alternative* $x \in X$ *is essential for* i *in set* X *if* $\exists R \in \mathscr{R}$ *such that* $x \in f(R)$ *and* $L_i(x, R) \subset X$.

The set of all essential elements for i in X under a social choice rule f is denoted as $Ess_i(X, f)$.

We can now define the notion of essential monotonicity.

Definition 9. *A social choice rule f is essentially monotonic if* $\forall R, R' \in \mathscr{R}, a \in f(R)$, $Ess_i(L_i(a,R), f) \subset L_i(a,R') for all i \in I$ *implies* $a \in f(R')$.

A useful rewrite of this essentially monotonic condition is provided in (Yamato, 1992). That is, a social choice rule satisfies essential monotonicity if : $\forall R, R' \in \mathscr{R}, a \in f(R), a \notin f(R')$, there exists $i \in I, b \in A$ such that (i) aR_ib and bP_ia; (ii) $\exists \hat{R} \in \mathscr{R}$, $b \in f(\hat{R})$ and $L_i(b, \hat{R}) \in L_i(b, R)$. This rewrite has done a contrapositive to the definition and insert in the definition of essential elements Ess.

Remark. *Some properties of essential elements and essential monotonicity are listed here.*

For any $B \subset C \subset A$, $Ess_i(B, f) \subset Ess_i(C, f) \subset Ess_i(C, f) \subset Ess_i(A, f) = Im(f)$. *Essential monotonicity implies Maskin monotonicity, because* $Ess_i(L_i(a,R), f) \subset L_i(a,R)$.

(Yamato, 1992) proposed a requirement of Condition D on the preference domain $\mathscr{R}$.

Definition 10. *The preference domain $\mathscr{R}$ satisfies Condition D if* $\forall a \in A, r \in R, i \in I, b \in L_i(a,R)$, *there exists* $R' \in \mathscr{R}$ *such that* $L_i(a,R) = L_i(b,R')$ *and* $\forall j \neq i L_j(b,R') = A$

Condition D can be thought as a requirement that the preference domain $\mathscr{R}$ should contain sufficiently many preferences. For example, unrestricted domain therefore easily satisfies condition D.

The following necessity theorem of Nash implementation is from (Yamato, 1992).

Theorem 6. *If the preference domain $\mathscr{R}$ satisfies Condition D, and a social choice rule $f : \mathscr{R} \to A$ is fully Nash implementable, then f is essentially monotonic.*

Proof. Suppose mechanism $\Gamma = (S, g)$ Nash implements a social choice rule f, $R, R' \in \mathscr{R}$. Due to full implementation, $f(R) = g(NE(R))$ and $f(R') = g(NE(R'))$. We then use the contrapositive rewrite of the essential monotonicity definition mentioned above to show that f is essentially monotonic. For any $a \in f(R)$ and $a \notin f(R')$, there is a $s \in NE(R)$ such that $a = g(s)$; there is a $i \in I, b = g(s'_i, s_{-i})$ such that bP'_ia. Obviously $b \in L_i(a,R)$ by Nash equilibrium concept, so the (i) of the rewrite of essential monotonicity is fulfilled. From condition D, $\exists \tilde{R} \in \mathscr{R}$ such that $L_i(b, \tilde{R}) = L_i(a, R)$ and $\forall j \neq i, L_j(b, \tilde{R}) = A$. Since $g(S_i, s_{-i}) \subset L_i(a, R)$ and $\forall j \neq i, g(S_j, s_{-j}) \subset A$, therefore $g(S_i, s_{-i}) \subset L_i(b, \tilde{R})$ and $\forall j \neq i, g(S_j, s_{-j}) \subset L_j(b, \tilde{R})$. $b \in NE(\tilde{R}) = f(\tilde{R})$. □

The sufficiency theorem of Nash implementation for more than 3 participants are proposed and proved in both (Danilov, 1992) and (Yamato, 1992). The following is the theorem.

Theorem 7. *If there are more than 3 participants, then an essentially monotonic social choice rule is fully Nash implementable.*

An application of this essential monotonicity concept is provided in (Kara and Sonmez, 1996). There they proved that

2.4 Some examples of mechanisms

The previous results are formal and mathematical. The following will give some examples of mechanism design.

2.4.1 Assets Inheritance problem

Example 3. *An old man has lots of houses and three sons. When he is about to die, he want to give the houses to his sons, but he is not familiar with the market prices for these houses. He does not want to make the sons share any house, so he would like to give every house as a whole to some son. All sons know the prices of these houses well. all sons like more values from the inherited houses. How can he achieve his goal of dividing the houses as fairly as possible, but if it can not be perfect fair, he prefers to give his younger sons more.*

The fairness here means that to make the least valued group of assets as close to the average as possible.

For the example situation, can a mechanism be designed so that the old man's goal be implemented? The best mechanism should be let the oldest son to devide the assets or houses into 3 groups, then let the youngest son chooses first, and the middle son chooses second, and the oldest son gets what is left. This is the simple but effective way of implementing the choice rule. Another form of the mechanism that is basically the same thing is that let the three sons report the values of the assets . and then devide the assets as fairly as possible according to what the oldest son has reported. The social planner picks the most valuable group of assets according to the youngest son's report and give it to him, then picks the more valuable group from the remaining assets according to the middle son's value report, and finally gives the last group of goods to the oldest son.

2.4.2 An example of Auction Design

In the last part of the chapter, we would like to design a mechanism for selling goods with uncertain values. Here a distribution assumption is used. Auctions are one of the most successful areas where the development of economics theory and practice help each other. I will try to investigate this area from a mechanism design perspective in this section. The material is put here as an example of what can be implemented for some special market.

Here we focus on revenue maximization for the seller of an indivisible experience good under a situation in which buyers are all ex-ante identical. The only information they all have is an identical value distribution of the experience good. In our setting, the seller can freely give any buyer access to get his true private value, which process we call experiencing the good. If the seller is unable to charge experience fee, she may maximize her revenue by excluding just one buyer from experiencing, which is in conflict with the social efficiency. If the seller is able to charge experience fee, she will maximize her revenue through setting the price at a level such that every potential buyer will buy the experience, resulting in the maximal trading surplus which is fully extracted by the seller.

Of course, the equilibrium of implementation in such a mechanism is ex ante, the following chapter will be concerned with ex post implementation.

Now the details are as follows. A seller(she) has a single indivisible good, and wants to sell the good to a group of N buyers indexed by $i = 1, 2, ..., N$(buyers and bidders are used interchangeably in this paper, and are assumed to be male). All parties are risk neutral. The value of retaining the good to the seller is publicly known as $\underline{v}(\geq 0)$ in this paper. Bidder i's valuation of the object is v_i, and v_i has independent identical distribution $F(x)$ with support $[\underline{v}, \overline{v}]$, where $\overline{v} > \underline{v}$,

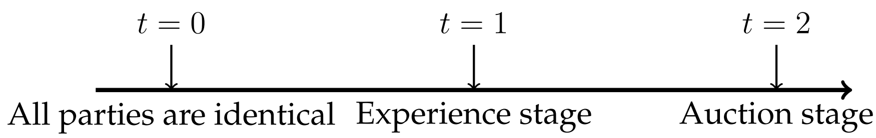


FIGURE 2.1: Timing of experience good auction

and $\overline{v} = +\infty$ is allowed. In the beginning, no one can directly observe the v_i. Only the distribution is common knowledge among all the parties. However, if the seller gives a chance of experiencing the good to a buyer, the buyer can learn of his v_i after experiencing.

The seller designs a sales mechanism to sell the good. The sales mechanism consists of two stages, the experience stage, and the auction stage. After the experience stage, the seller will allow those who have experienced the good to participate the auction. By Revenue Equivalence Principle, we analyze the case that the seller chooses the second-price sealed-bid auction in the auction stage without loss of generality. In the auction stage, if no bidder bids a value exceeding a reserve price , then the good is sold to the unexperienced buyer at the price μ(the expectation of v) when there is an unexperienced buyer. Here we assume that the buyer will choose to buy the good if he is indifferent between buying and not buying the good. The timeline is shown in the Figure 1.

Setting Reserve price to extract trade surplus

In the situation that the seller cannot charge fee for experiencing the good, the seller will set a reserve price in the auction stage to maximize the revenue in a second-price sealed-bid auction.

Lemma 1. *The hazard rate function associated with the distribution F is defined as $\lambda(x) = f(x)/(1 - F(x))$. The optimal reserve price r satisfies $r - 1/\lambda(r) = v^*$. The v^* denotes the seller's reserve revenue/valuation.*

Proof. Let $R(r, m)$ denote the revenue of the seller when the reserve price is r and the number of buyers experiencing the good is m.

Obviously, r should be higher than v^* to have an effect of enhancing the price. Then the expected value of seller revenue can be calculated as the sum of three parts, the revenue from the auction when the second highest bid is above the reserve price r, the revenue from the auction when the highest bid is above the reserve price but the second highest price is below the reserve price r, and the reserve revenue/valuationv^* when the highest bid is below the reserve price r.

$$R(r, m) = \int_r^{\overline{v}} x \mathrm{d}G_{(2)}^m(x) + rmF^{m-1}(r)[1 - F(r)] + v^* F^m(r) \tag{2.1}$$

where $G_{(2)}^m(x) = F^m(x) + mF^{m-1}(x)[1 - F(x)]$.

Differentiating this with respect to r and simplify, we obtain

$$-rmF^{m-1}f(r) + mF^{m-1}(r)(1 - F(r)) + v^* mF^{m-1}(r)f(r)$$

The first order condition is

$$r^* - \frac{1 - F(r^*)}{f(r^*)} = v^*$$

□

Remark. *Under regularity conditions, the first order condition is also sufficient for optimality.*

If the seller gives a buyer the chance to experience the good, no buyer will refuse since experiencing the good means some chance of gain. Thus the number of buyers who can experience the good is at her control. Intuitively, she wants to obtain a high revenue from the auction by letting more buyers to experience as long as there is still an unexperienced buyer, for the seller can sell the good to him at price μ when no bidder bids over the reserve price. This is summarized in the following proposition.

Proposition 2. *As the number m of buyers experiencing the good increases, the expected value of seller revenue increases as long as $m \leq n - 1$.*

Proof. When the reserve revenue v^* is obtained by selling the good to a unexperienced buyer, $v^* = \mu$. Then by equation 3 in the proof of Lemma 1,

$$R(r, m) = \int_r^{\overline{v}} x \mathrm{d}G_{(2)}^m(x) + rF^{m-1}(r)[1 - F(r)] + \mu F^m(r) \tag{2.2}$$

where $G_{(2)}^m(x) = F^m(x) + mF^{m-1}(x)[1 - F(x)]$.

When $m \leq N - 1$, the best reserve price is $r = r^*$ which is the solution to $r^* - 1/\lambda(r^*) = \mu$. The revenue is $R(r^*, m)$. By the first

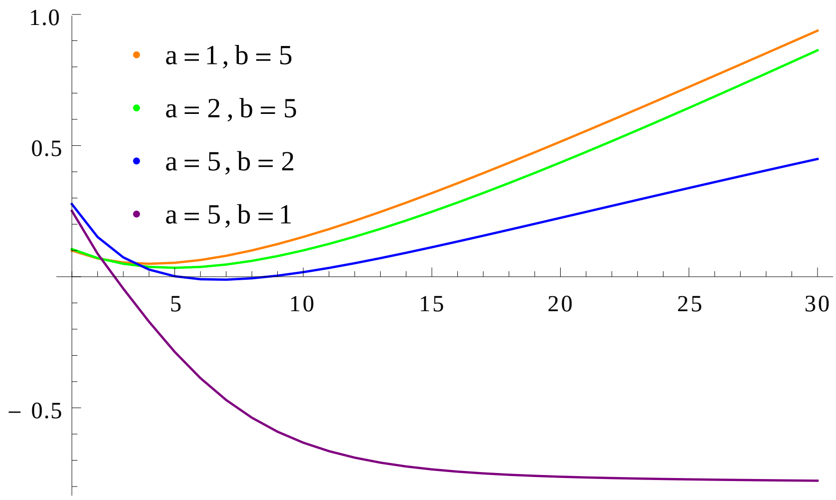


FIGURE 2.2: Difference in revenue: $R(r^*, N-1) - R(\hat{r}, N)$

order stochastic dominance, it is straightforward to show $R(r^*, m) > R(r^*, m')$ for any $N-1 \geq m > m'$. □

However, letting N potential buyers to all experience the good may not be better than just letting $N-1$ buyers to experience. Because the former may let go the safe option of selling the good at μ. Indeed, it depends on the distribution F(x). A careful research of us has led to the following conclusion.

Theorem 8. *The seller does not always want all buyers to experience the good under the setting here.*

Proof. When every buyer has experienced the good, then by lemma 1, the reserve price should now be set to $\hat{r}$, which is the solution to $\hat{r} - 1/\lambda(\hat{r}) = \underline{v}$, since the reserve valuation is $\underline{v}$ and the revenue is $R(\hat{r}, N)$ using equation 3. Generally speaking, $R(\hat{r}, N) < R(r^*, N-1)$ for many distributions and the potential buyers' number N. □

Figure 2 shows the values of $R(r^*, N-1) - R(\hat{r}, N)$ for different values of parameters in Beta distribution(with the horizontal axis denoting N).

Remark. *Specially for the uniform distribution, we have, when* $N \leq 7$, $R(\hat{r}, N) < R(r^*, N-1)$.

Charging Experience fee to extract trade surplus

We then consider the case where the seller can charge experience fee. For the seller, at the experience-stage, how to optimally set price of exprience? If a certain number of buyers are permitted to experience the good, each of those buyers is supposed to get a nonnegative ex ante expected payoff from participating the subsequent second-price auction. Suppose that buyers will buy the experience when they are indifferent between remaining uninformed and buying the experience to be informed. Then the seller should set the price equal to a buyer's ex ante expected payoff of participating the auction to extract all the expected trade surplus. Intuitively, the ex ante expected payoff of experiencing buyers depends on the number of auction participants.

Next, we will offer the formula for the experience fee in the following lemma.

Lemma 2. *The experience fee $P(m)$ can be formulated as*

$$P(m) = \begin{cases} \int_{\mu}^{\overline{v}} \int_{\mu}^{v} F^{m-1}(t)\mathrm{d}t f(v)\mathrm{d}v, & \textit{if} \quad m \leq N-1 \\ \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{v} F^{N-1}(t)\mathrm{d}t f(v)\mathrm{d}v, & \textit{if} \quad m = N \end{cases} \tag{2.3}$$

Proof. When there are $m(\leq n-1)$ buyers knowing one's own value information, the value of the information is

$$\begin{aligned} P(m) &= \int_{\mu}^{\overline{v}} (\int_{\mu}^{v} (v-t)\mathrm{d}F^{m-1}(t) + (v-\mu)F^{m-1}(\mu))f(v)\mathrm{d}v \\ &= \int_{\mu}^{\overline{v}} \int_{\mu}^{v} F^{m-1}(t)\mathrm{d}t f(v)\mathrm{d}v \end{aligned}$$

$P(N)$ can be formulated as,

$$\begin{aligned} P(N) &= \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{v} (v-t)\mathrm{d}F^{N-1}(t)\mathrm{d}t f(v)\mathrm{d}v \\ &= \int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{v} F^{N-1}(t)\mathrm{d}t f(v)\mathrm{d}v \end{aligned}$$

□

Lemma 3. *As the number of buyers knowing one's own value information increases, the expected value of knowing one's own value information decreases.*

Notice that for many distributions and potential buyers' number n, $P(N) < P(N-1)$. In such cases, the seller set price at $P(N)$, and the buyers' domininant stratey is to buy the experience. Since every buyer knows the true value in the auction and expected gain of every buyer is zero now, the seller can extract full surplus through setting appropriate experience fee. the social surplus is maximized and fully extracted by the seller.

The above reasoning leads to the following conclusion

Theorem 9. *If the seller can charge fee for letting one buyer know his true value, then the seller will set the price* $\int_{\underline{v}}^{\overline{v}} \int_{\underline{v}}^{v} F^{n-1}(t)\mathrm{d}t f(v)\mathrm{d}v$ *to fully extract all the surplus of trade, and the revenue she get is*

$$E(V_{(1)}^N) = \int_{\underline{v}}^{\overline{v}} x \mathrm{d}F^N(x) \tag{2.4}$$

where $V_{(1)}^N$ *denotes the highest bid.*

This is the best possible result for the seller, maximizing the trade surplus and minimizing all the buyers' share.

The example is supposed to shed light on a seller's optimal sales mechamism design when buyers have identical value distribution about the experience good, but do not know the private value before the experience. The seller controls the number of buyers to experience.

When the seller is not able to charge experience fee, her optimal sales mechanism might be to exclude a buyer from experiencing the good, which harms social efficiency. When the seller is allowed to charge fee, the result is efficient. However, the defect is that all the trade surplus is extracted by the seller.

Chapter 3

Expost implementation in economies with interdependent values

3.1 Introduction

We try to characterize truthful implementation in ex post equilibrium in this Chapter. Due to "Wilson criterion", ex post equilibrium is a good equilibrium in implementation theory for its distribution free property. A large body of literature has been devoted to many aspects of ex post implementation.

As is pointed out in (McLean and Postlewaite, 2014), it is often the case that truthful revelation is not ex post incentive compatible, that is, for a given agent, there are some profiles of the other agents' types for which the agent may be better off by misreporting his type than by truthfully revealing it. However, with the help of money transfer, the authors in (Dasgupta and Maskin, 2000) devised an complicated yet ingenious auction mechanism to Nash implement a rather general kind of allocation problems in interdependent value context. It is obvious that money transfer is one key for implementation. In this chapter, we provide a sufficient and necessary condition for truthful implementation with money transfer. VCG mechanisms in private value context and all its generalized forms in interdependent value context have all successfully find a suitable money transfer scheme for specific economic environments. We would like to show the subtlety of those ingenious mechanisms in implementing social goal within our paper's model framework, and how their implemented social goals have satisfied the sufficient and necessary condition. Here, for a certain but not rare kind of setting, we propose a simple Generalized Vickery mechanism which can easily make the buyers reveal their private information truthfully through the help of money transfer. Of course, the Generalized Vickery mechanism is not balanced scheme. Nevertheless, the ability of the mechanism to make people tell truthfully about their private information is a sign of its power. For the common knowledge part of agents, we make the agents tell truth in Nash equilibrium using insights got from the paper (Moor and Repullo, 1990).

In private value settings, ex post equilibrium is equivalent to dominant strategy equilibrium. Thus the well-known VCG mechanism is the ideal mechanism for ex post implementation in private value case. What we care most in this paper is the interdependent value setting. We also assume quasilinear utility for the agents, which is a common assumption. In (Bergemann and Morris, 2008), the authors focus on identifying conditions for full implementation of a social choice set in ex post equilibrium. They stressed that a conceptual advantage of ex post equilibrium is its robustness to the informational assumptions about the environment. (Dasgupta and Maskin, 2000) proposed an indirect bidding mechanism which give ex post partial implementation of the socially efficient outcome. An important contribution of (Dasgupta and Maskin, 2000) is that a mechanism for allocating multiple heterogeneous goods is provided. (Perry and Reny, 2002) devised another clever mechanism for this case. Their methods consists of a collection of second-price auctions between each pair of bidders conducted over at most two rounds of bidding. Unlike them, in this paper we do not consider full implementation, and for partial implementation we focus on direct revelation mechanism. we use the direct mechanism and get to cover some more situations that are not implementable using the indirect bidding mechanism in (Dasgupta and Maskin, 2000). We restrict out attention to the conditions garanteeing that truthful revelation of one's own private signal constitutes an ex post equilibrium in some carefully designed direct revelation mechanism. That is, we focus on incentive compatibility of truthful information revelation in direct mechanism. Partial implementation is enough in this paper. In such direct mechanisms, a well devised payment rule is of key importance. (Ausubel, 1999) provides a payment scheme in its generalized Vickery Auction mechanism, which under some conditions achieves the task of implementing social efficiency in ex post equilibrium. (Jehiel and Moldovanu, 2001) takes into consideration discreet social choice possibilities.

In this paper, we would like to also investigate the continuous cases of interdependent value, including allocation of continuous resources and efficient provision of continuous public goods. For example, in a room of dancers, a volume must be chosen for the music so that it can provide the best social efficiency. Each person has a valuation function of the form

$$u_i = v_i(volume) + \alpha_i \sum_{j \neq i} v_j(volume)$$

Here, the α_i is the altruist coefficient of agent i. We will see how to implement social efficiency in this case later in the paper. (Chung and Ely, 2006) is a work dedicated to similar problems. We are going to mention it when we draw on some of their results later.

Our intuition is mainly taken from (Dasgupta and Maskin, 2000) Vickery Mechanism and auction theory. Simpleness often means less mistakes. Also, a simple mechanism is easy to supervise. For an outsider, it is not easy to judge who should win the goods in a very complicated mechanism , so a corrupted social planner might manipulate the result. That is one motivation for us to construct a direct revelation

In interdependent value settings, a winner often suffers from winners' curse. In this paper, we first introduce the notion and model of interdependent value goods, then for some kind of setting, we find a unique expost Implementation of the socially efficient results. Finally, we find an application of the model in the mineral rights assignment problem by aggregating signals in the designed market.

3.2 Key concepts and notations

First, we give some description of the notations used in this chapter. It's similar to the general framework described in the first chapter. In the meantime of depicting the notations here, we give out the settings for this chapter in more detail. The economic environment consists of A: the set of choice possibilities.

$Z = Z_1 \times \cdots \times Z_n$:the outcome space. For this chapter, implementation with money transfer means that the outcome z has the form $(a, t_1, \cdots, t_n)$ where a is from the set of choice possibilities A, t_i is the payment agent i has to make.

$U = U_1 \times \cdots \times U_n$: the set of all admissible utility functions $u = \{u_i(\cdot, \cdot)\}_{i=1}^n$ whose domain is $Z \times S$. In this paper, quasilinearity is assumed, that is , $u_i((a, t), s)$ takes the form $v_i(a, s) - t_i$. $v = \{v_i(\cdot, \cdot)\}_{i=1}^n$ is in a space $V = R^+ \times R^+$.

$S = S_1 \times \cdots \times S_n$: the set of all admissible signals $s = (s_1, \cdots, s_n) \in S$ that determine types of parametric utility functions $u_i(\cdot, s)$, and so it is called the space of signals or called the state of the world. In many papers, notation θ is used in stead of s. We adopt the notation s of (Dasgupta and Maskin, 2000) in this paper. Here, apparently both independent value and interdependent value models are both incorporated in this framework.

E: a set of environments(states of the economy) $e = (\{u_i(\cdot, \cdot)\}_{i=1}^n, s)$. In this paper, $e = (\{v_i(\cdot, \cdot)\}_{i=1}^n, s)$ since v_i can identify u_i. Moreover, as will be discussed in the information structure part, the v can be elicit out easily by the social planner, we simply let $e = s = (s_1, \cdots, s_n)$, which is the decentralized information part of the model that need mechanism design to tackle.

The information structure of the paper is specified by

(1)$s_i$is privately observed by agent i.

(2)$v = \{v_i(\cdot, \cdot)\}_{i=1}^n$ is common knowledge among the agents.To elicit the common knowledge part of their information, the social planner can adopt methods similar to those (Moor and Repullo, 1990) propose for Nash implementation of social choice rules. If very agent reports the same, then the result is believed to be the truth. Otherwise, some kind of punishment is given to everyone.

Given economic environments, each agent participates economic activities, makes deci- sions, receives benefits and pays costs on economic activities.The designer wants to reach some desired goal that is considered to be socially optimal by some criterion. Let

$F : E \rightarrow\rightarrow Z$: the social goal or called social choice correspondence in which $F(e)$ is the set of socially desired outcomes at a certain state of the economy under some criterion of social optimality.

For this paper, the social goal is denoted

$$F(s) = \{(a^*, t_1, \cdots, t_n) | a^* \; is \; a \; solution \; to \; \max_a \sum_{i=1}^{n} v_i(a, s)\}$$

We call this the social goal of efficiency in the paper. Since money transfer is not the concerned part. we can use transfers freely to implement the goal.

A mechanism consists of a message space and an outcome function.

M_i: the message space of agent i.

$M = M_1 \times \cdots \times M_n$: the message space in which communications take place.

$m_i \in M_i$: a message reported by agent i.

$m = (m_1, \cdots, m_n) \in M$: a profile of Messages.

$h : M \to Z$: outcome functions that translate messages into an outcome.

$\Gamma = \langle M, h \rangle$: a mechanism.

A very important class of mechanisms is the direct revelation mechanism in which M_i is just the possible world state information that agent i has. In this paper, the direct revelation mechanism has a message space $M_i = \{(v, s_i) | v \in V, s \in S\}$. The most important case of m is where all reported v is the same, and for this case the outcome function can be written as $h(m) = (a(v, s), t(v, s))$.

Let $b(e, \Gamma)$ be the set of equilibrium messaging strategies that describes the self-interested behavior of individuals. For instance, Nash equilibrium $N(e, \Gamma)$ is the most frequently adopted equilibrium concept.

A Mechanism $\langle M, h \rangle$ is said to implement a social choice correspondence F in equilibrium strategy $b(e, \Gamma)$ on Environment space E if for every $e \in E$, $h(b(e, \Gamma)) \in F(e)$. Incentive compatibility is another way of saying implementation. A Mechanism is said to be incentive-compatible with a social choice correspondence F on E if it implements F in some kind of equilibrium on E.

3.3 The sufficient and necessary condition for implementation with money transfer

Obviously, for a direct revelation mechanism to implement the social efficiency, every agent must tell truth.

We assume outside choice is a_0, and $v_i(a_0, s) = v_i(a_0, s_{-i})$, that is, the agent i's value of the outside choice does not depend on s_i.

Theorem 10. *The social goal of maximizing $\Sigma v_i(a, s)$ can be partially implemented in ex post equilibrium with money transfer if and only if there is a transfer scheme $t(s)$(taken as a tax from agent to the social planner), such that for the chosen $a(s)$ maximizing $\Sigma v_i(a, s)$ the following formula holds for all i , s and s'_i*

$$v_i(a(s), s) - t(s) \geq v_i(a(s'_i, s_{-i}), s) - t(s'_i, s_{-i})$$

Proof. Just use the $t(s)$ as a taxation and according to the definition of ex post equilibrium, tell truth about one's private signal is an ex post equilibrium.

□

In order to facilitate the proof, we give the following fundamental theorem of this paper. It is roughly saying that the value sum of you pretending another type(who have different value when truth telling) and that type pretending you is less than the value sum of you and that type truthfully report your types.

Proposition 3 (single crossing condition)**.** *A social goal of efficiency G can be ex post implememted on E by some mechanism only if $\forall s, \forall v, \forall i$, when $v_i(a(v, s), s) \neq v_i(a(v, s'), s')$ where $s = (s_i, s_{-i})$ and $s' = (s'_i, s_{-i})$ (i.e., they only differ in agent's private signal), then*

$$v_i(a(s'), s) - v_i(a(s), s) < v_i(a(s'), s') - v_i(a(s), s')$$

Proof. By the necessary and sufficient condition10, there must be a $t(\cdot)$ such that

$$v_i(a(s), s) - t(s) \geq v_i(a(s'), s) - t(s')$$

and

$$v_i(a(s'), s') - t(s') \geq v_i(a(s), s') - t(s)$$

Add the above two inequality together and exchange some terms from both sides, then you get the single crossing condition.

□

3.4 Continuous cases

Let us begin with some examples

Example 4. *Consider there are two chess players living L miles apart from each other along a road in a city. They drive their cars to meet each other every sunday to play chess face to face, and then go back home. Suppose that they can choose anywhere on the road to meet. The city governer tries to assign a meeting place minimizing the pollution of their car causing to the city, so he wants to chose a meeting spot to minimize* $c_i l_i + c_j(L - l_i)$*. They all care about the pollution but they also care about their oil expenditure, player i wants to minimize* $v_i = c_i l_i + \alpha(c_i l_i + c_j(L - l_i))$*. What is the transfer scheme that can make the two players telling truth about their private* c_i *an ex post equilibrium.*

when extending the two player game of chess to four player game like bridge, and the four people meet each other, it is more complicated.

Example 5. n *firms at* n *corners of a regular n polygon area transporting their produced goods to a city for sale. The city planner tries to build the city in a place such that it can minimize the pollution caused by transportation of the goods, and the firms cares about its own transportation cost as well as the total pollution(the reason why a firm care about pollution may be that the goods are vegetables and pollution can hurt the production). The oil consumption and thus pollution caused by a given amount of goods transportation for each firm is a private value . Each day the goods transported to the city is also private information for each firm. What is the taxation scheme that can make each firm report their true private information so that the city planner can choose the best place to build the city in order to reduce transporation pollution. Here the meaning of the taxation is not to reduce pollution directly, but to lead the firms to tell truth about their private information such that the planner can choose a pollution minimizing choice for building the city.*

Another economically more interesting example is as following

Example 6. *A country has N oligopoly firms producing the same goods ,say oil. They face an exogenous demand function. Competition among them hurt aggregate profits of the firms. The country want to devise a taxation scheme such that every firm reveal their true production cost so that the country can plan the production quantity for each firms to maximize total profit of the country. This is the private value case that can be implemented using classical Clark mechanism.*

Now change the scenario to another case where the firms' CEOs know the private cost informations of their own firms. They are all shareholders, and all have a large percent of their own firm's stock and a different amount of share on other firms, and these shares are common knowledge. Now the valuation of each firm's CEO on the country's production assignment plan are dependent upon all the cost information and each firm's final production quantity as specified by the country.

Now the taxation scheme is a true challenge. In this paper, we try to classify these interdependent value problems into two category, the ones that can be implemented with money transfer(suitable taxation)in ex post equilibrium, and the ones that can not. For those that can be implemented with money transfer, we give the taxation scheme that can be used to implement the socially efficient choice.

One interesting finding is that the money transfer scheme is unique up to a shifting constant if the functions involved are continuously differentiable.

Theorem 11. *Assuming all the partial derivatives exists and are continuous, then solve the following differential equations*

$$\frac{\partial t_i}{\partial s_i} = \frac{\partial v_i(a(s_i, s_{-i}), s_i, s_{-i})}{\partial a} \frac{\partial a}{\partial s_i}$$

$i = 1, \cdots, n$

gives us a transfer scheme $t(\cdot)$. A sufficient and necessary condition for implementation with money transfer in such continuous case is that, for all i,s by shifting up or down the $t(\cdot, s_{-i})$ to let it be tangent with $v_i(a(\cdot, s_{-i}), s_i, s_{-i})$ at s_i, the curve $t_i(\cdot, s_{-i})$ is below $v_i(a(\cdot, s_{-i}), s_i, s_{-i})$, namely, $v_i(a(\cdot, s_{-i}), s_i, s_{-i}) - t(\cdot, s_{-i})$ is maximized at s_i.

Proof. If truth telling is a ex post equilbrium, it must be the case that s_i is the solution to

$$\max_{s_i'}\{v_i(a(s_i', s_{-i}), s_i, s_{-i}) - t(s_i', s_{-i})\}$$

solving it, we get the conclusions in the theorem. □

The theorem is simple, but it contains much information. One thing is that not every continuously differentiable social goal $a(s)$ is implementable in ex post equilibrium with money transfer. It must satisfy the conditions in the theorem to be implemented with money transfer. The second thing is that even if it can be implemented with money transfer, the scheme for implementing it is unique in essence. They only differ by a constant(the solution to the differential equations are some specific function plus a something of the form $f(s_{-i})$ that is not changing with s_{-i}).

Now let us consider the oil producers example. We should give a mathematical structure to it in order to show how the process of mechanism design is done in this case. Market demand function is $q(p)$, inverse demand function is $p(q)$. The cost function facing firm i is $c(q_i, s_i)$, the parameter s_i is privately known. The share of firm j that the leader of firm i holds is m_{ij}, and the leader i cares the overall profit $v_i(q, s) = \sum_{j=1}^{n} m_{ij}(q_j p(q) - c(q_j, s_j))$. The country's goal is

$$\max_{\{q_1,\cdots,q_n\}} \sum_{j=1}^{n}(q_j p(q) - c(q_j, s_j))$$

The solution to this maximization problem must satisfy the first order conditions

$$q_j \frac{\partial p(q)}{\partial q} + p(q) = \frac{\partial c(q_j, s_j)}{\partial q_j}$$
$$j = 1, \cdots, n$$

For many function forms, the above conditions are also sufficient, and we calculate out $q_1(s), \cdots, q_n(s)$ from the above equations. Next, by using the conditions in the above theorem, we get

$$\frac{\partial t}{\partial s_i} = \sum_{j=1}^{n} m_{ij}\{q_j \frac{\partial p(q)}{q}(\frac{\partial q_1}{s_i} + \cdots + \frac{\partial q_n}{s_i}) - \frac{\partial c(q_j, s_j)}{\partial q_j}\frac{\partial q_j}{s_i}\}$$

Now that the transfer scheme is calculated, we only need to verify whether it satisfies the requirement that s_i maximizes

$$\sum_{j=1}^{n} m_{ij}\{q_j(\cdot, s_{-i})p(q(\cdot, s_{-i})) - c(q_j(\cdot, s_{-i}), s_j)\} - t_i(\cdot, s_{-i})$$

For this continuously differentiable situation, the private value case of the problem always has the unique solution, the uniqueness has been shown by the above more general result. And the unique mechanism is the well-known VCG mechanism. This has been shown by Laffont and Maskin(Econometrica, 1980). Tian

And for private value case, ex post implementation is indeed dominant strategy implementation. A specialness of the private value situation is that the calculated VCG money transfer mechanism can always satisfy the additional requirement for implementability. We will show why now.

3.5 Discrete social goal

The discrete social choice possibility set A does not allow differential analysis like we do above in the continuous social choice possibility set case. However, the allocation of discrete social resources, or the determination of whether a project should be carried are important social choice problem we have to face. We are also interested at whether we can implement such social choices with money transfer. We start from allocation of one good.

3.5.1 Allocation of one good

Imagine the following scenario : there is a group of agents who are familiar with each other, and the social planner has one good to allocate to one of them. The socially efficient outcome is to give it to the one who values it the most. Suppose each agent can only detects one of the qualities of the good, and the total value of the good for any agent is only determined by all the qualities of the good and the importance of each quality to the agent himself. This is a situation completely covered by our model framework in the previous section. The social choice possibility set $A = \{(1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)\}$. Every body only cares about getting the object, and when one does not get it his or her utility gain is 0 from the allocation. Now what the social planner want is a mechanism to implement the socially efficient outcome.

(Dasgupta and Maskin, 2000) has partly solved the problem and give us an auction mechanism to implement a rather large economic environment set E in this one good allocation environment. However, as the authors pointed out in the footnote 26 of their paper, some utility forms can not be implemented by their auction mechanism. In this section, through combining the insights of the two authors with the power of revelation mechanism, we try and find a direct revelation mechanism which enlarge the E on which the social goal of efficiency G is implementble, for example, the economic environment in footnote 26 of their paper. In Theorem 10, we have shown what is the sufficient and necessary condition for implementation with money transfer. In this simple case, we give some simple assumptions parallel to the sufficient and necessary condition so that we can implement the social efficiency.

In this one good allocation problem,let $v_i(s) = v_i(\epsilon_i, s)$ where ϵ_i is the allocation of the good to i. Then we have the following sufficient condition for ex post implementation of efficient result in an auction.

Proposition 4. *The efficient allocation of one good can be implemented as an ex post equilibrium if*

$$\forall i, \forall s_{-i}, \exists s_i^* \in S_i, such\ that$$

$$(a) v_i(s_i^*, s_{-i}) = max_{j \neq i} v_j(s_i^*, s_{-i});$$

$$(b) When\ v_i(s_i, s_{-i}) > v_i(s_i^*, s_{-i}),$$

$$\forall j \neq i, v_i(s_i, s_{-i}) - v_i(s_i^*, s_{-i}) > v_j(s_i, s_{-i}) - v_j(s_i^*, s_{-i});$$

$$When\ v_i(s_i, s_{-i}) < v_i(s_i^*, s_{-i}),$$

$$\forall j \neq i, v_i(s_i, s_{-i}) - v_i(s_i^*, s_{-i}) < v_j(s_i, s_{-i}) - v_j(s_i^*, s_{-i});$$

Proof. The proof is through constructing the needed mechanism and then examining how the mechanism works in all possible situations. We only need to find an appropriate mechanism to implement it. Now we give it as follows and then prove it can indeed implement the social goal of efficiency.

Step 1. Every agent reports a (v, s_i) from the set $V \times S_i$;

Step 2. If the v reported by every agent has some disagreement, then the result is that no body gets the good.

Step 3. If the v reported by every agent is the same, then solve the maximization problem

$$\max_{i\in\{1,\cdots,n\}} v_i(s)$$

and give the good to the solution agent. The winner pays $v_i(s_i^*, s_{-i})$ which s_i^* is the solution to the following minimization problem

$$\min_{s_i^*\in S_i} v_i(s_i^*, s_{-i}) \quad s.t. \quad v_i(s_i^*, s_{-i}) \geqslant \max_{j\neq i, j\in\{1,\cdots,n\}} v_j(s_i^*, s_{-i}) \tag{3.1}$$

Explanation of this mechanism being able to induce truthful revelation of every agent's private signal is not difficult with the help of case analysis for all possibilities.

Devide the actually received signal s_i^* into three cases:

(i)When $v_i(s_i, s_{-i}) > v_i(s_i^*, s_{-i})$, given others truthfully revealing their private signals s_{-i}, the agent i wishes to win the auction. Truthful revelation of i's private signal is enough to insure the win, since the s_i^* is enough to guarantee a tied first according to assumption (a), further by the first part of assumption (b), the true s_i gives agent i the clear first. Therefore truthful revelation of the signal s_i in this case is incentive compatible.

(ii)When $v_i(s_i, s_{-i}) < v_i(s_i^*, s_{-i})$, given others truthfully revealing their private signals s_{-i}, the agent i's best choice is to lose the auction. Truthful revelation of i's private signal is sure to make him lose the auction, since the s_i^* only gives him a tied first according to assumption (a), further by the second part of assumption (b), the true s_i will not make him the first. Therefore truthful revelation of the signal s_i in this case is also incentive compatible.

(iii)When $v_i(s_i, s_{-i}) = v_i(s_i^*, s_{-i})$, given others truthfully revealing their private signals s_{-i}, winning or losing the auction has the same net value of 0. Therefore truthful revelation of the signal s_i in this case is incentive compatible trivially.

□

The improvement of our mechanism on (Dasgupta and Maskin, 2000) is that they have four assumptions for implementation, but we only need two assumptions which are essentially parallel to their first three assumptions and less demanding.

Their three assumptions are listed below.
(i)$\frac{\partial v_i}{\partial s_i} > 0$;
(ii) $\forall i, j \neq i, \frac{\partial v_i}{\partial s_i}(s_1, ..., s_n) > \frac{\partial v_j}{\partial s_i}(s_1, ..., s_n)$;
at any point where $v_i(s_1, ..., s_n) = v_j(s_1, ..., s_n) = max_k v_k(s_1, ..., s_n)$
(iii)$\forall s_{-i} \in S_{-i}, \exists s_i' \in S_i, such\ that\ v_i(s_i', s_{-i}) > max_{j \neq i} v_j(s_i', s_{-i})$;

Therefore, we can implement social goal of efficiency on all the economic environments E that their auction mechanism can implement and can implement beyond their scope. As a demonstration of the power of our direct revelation mechanism as shown in the proof, we now implement the efficiency goal for (Dasgupta and Maskin, 2000) footnote 26.

Example 7 (Footnote 26 of the paper Efficient Auctions)**.**

$$v_1(s_1, s_2) = s_1 - 2s_2 + 5$$

$$v_2(s_1, s_2) = s_2 - \frac{1}{2}s_1 + 5$$

The v clearly satisfy Assumption 1 and 2, therefore it is implementable using our direct revalation mechanism described above. By using the minimization value in Formula 3.5.1, we get the pay that the winner has to make, which is a constant 5. It is easy to verify that truthful report is a Nash Equilibrium.

For the above example, the reason why the auction in (Dasgupta and Maskin, 2000) cannot implement social goal of efficiency, is perhaps that their contingent biding function loses the information revelation role in this case, since $b_1(v_2) = 15 - 2v_2, \quad b_2(v_1) = \frac{15 - v_1}{2}$.

The next section will be with multiple goods, and there we will provide a sufficient condition for ex post implementation. When that condition is applied to the one object case(1 is a special case of n), we will have a further relaxed condition than the one we proposed in this part.

3.5.2 Allocation of multiple goods

For allocation of multiple goods under interdependent value environment, not much has been done to our knowledge. (Che, Kim, and Kojima, 2015) involve assigning indivisible objects to individuals without monetary transfer under interdependent values setting.

Now in this subsection we extend the previous result to the allocation of multiple goods with money transfer under interdependent value environment. The problem we'll consider is actually the assignment model with a special kind of interdependency. [1]

Suppose that there are m agents and n goods such that $m > n$. Each agent can get at most one object. Agent i's valuations of the n goods is given by $(v_{i1}(s), v_{i2}(s), ..., v_{in}(s))$ respectively, where $s = (s_1, s_2, ..., s_m)$ is a vector of signals held privately by agent 1,agent 2,..., agent m respectively. Allocation result is a $m \times n$matrix X where $\sum_{i\in\{1,...,m\}} x_{ij} \leq 1 \ for \ all \ j$ and $\sum_{j\in\{1,...,n\}} x_{ij} \leq 1 \ for \ all \ i$. The social goal is to allocate the goods to maximize the social efficiency $\sum_{ij} x_{ij} v_{ij}(s)$.

This problem is generally not ex post implementable. However, we find under the following condition, we can design a mechanism to implement the socically efficient goal in ex post equilibrium.

Condition ρ:

$\forall i, \forall s_{-i}, \exists s_i^*, forallk \in \{1, \cdots, n\}, such\ that$

From the maximizing solution xs to $\sum_{ij} x_{ij} v_{ij}(s)$, one can designate an allocation scheme, such that

if $v_{ik}(s_i) < v_{ik}(s_i^*)$, agent i will have no probability to be allocated the good k.

if $v_{ik}(s_i) > v_{ik}(s_i^*)$, agent i will have a probability to be allocated the certain good k and charged $v_{ik}(s_i^*)$. However, the probability does not depend on s_i.

Proof. The proof is through constructing the needed mechanism and then examining how the mechanism works in all possible situations.

We first need to find an appropriate mechanism. Now we give it as follows and then prove it can indeed implement the social goal of efficiency.

Step 1. Every agent reports a (v, s_i) from the set $V \times S_i$.[2]

Step 2. If the v reported by every agent has some disagreement, then the result is that no body gets any good.

[1]For the private value assignment problem, the knowledge of which may help the understanding of this section, see Chapter 8 (Roth and Sotomayor, 1990)for reference.

[2]Here v is the matrix $\begin{matrix} v_{11} & \cdots & v_{1n} \\ \vdots & \ddots & \vdots \\ v_{m1} & \cdots & v_{mn} \end{matrix}$, where v_{ij} is the value function of agent i getting good j.

Step 3. If the v reported by every agent is the same, then to allocate the goods, choose the social value maximizing allocation scheme designated in Condition ρ which satisfy the additional two assumptions in it.

Explanation of this mechanism being able to induce truthful revelation of every agent's private signal is not difficult with the help of case analysis for all possibilities.

For any k, devide the actually received signal s_i into three cases:

(i)When $v_{ik}(s_i, s_{-i}) > v_{ik}(s_i^*, s_{-i})$, given others truthfully revealing their private signals s_{-i}, the agent i wishes to be assigned good k to gain $v_{ik}(s_i, s_{-i}) - v_{ik}(s_i^*, s_{-i})$ surplus. Truthful revelation of i's private signal gives i a probability of the assignment of k to i, since s_i will not influence this probability, truthful revelation of the signal s_i in this case is incentive compatible.[3]

(ii)When $v_{ik}(s_i, s_{-i}) < v_{ik}(s_i^*, s_{-i})$, given others truthfully revealing their private signals s_{-i}, the agent i wishes not to be allocated the good and being charged $v_{ik}(s_i, s_{-i}) < v_{ik}(s_i^*, s_{-i})$. Truthful revelation of i's private signal s_i is enough to insure that he is not allocated good k by Condition ρ. truthful revelation of the signal s_i in this case is also incentive compatible.

(iii)When $v_{ik}(s_i, s_{-i}) = v_{ik}(s_i^*, s_{-i})$, given others truthfully revealing their private signals s_{-i}, winning or losing the assignment of good k has the same net value of 0. Therefore truthful revelation of the signal s_i in this case is incentive compatible trivially.

Therefore, truthful revelation is incentive compatible overall for this mechanism.

□

Condition ρ does not seem to be easy to satisfy. Nevertheless, in the following we give a more concrete v which satisfy Condition ρ to show that it is indeed satisfiable in some situations so that we can ex post implement the socially efficient goal.

Proposition 5. *The social efficient goal of* $\max_{i,j} x_{ij} v_{ij}(s)$ *can be ex post implemented if: For all* i *and* j*,*

$$v_{ij} = b_{ij} + o_i(s_i) + \sum_{l \neq i} r_l(s_l)$$

[3] Not influencing probability is important, because if not, agent i would have incentive to manipulate the probability by report a false s_i.

where $b_{ij} > 0$ *is the base value,* $s_i \in [0, +\infty)$, $\forall i, o_i(0) = r_i(0) = 0$

Assumptions are listed below.

(i)For all i, $\frac{\partial o_i}{\partial s_i} > 0$, $\frac{\partial r_i}{\partial s_i} > 0$;

(ii) $\forall i$, $\frac{\partial o_i}{\partial s_i} > \frac{\partial r_i}{\partial s_i}$;

The intuition of these assumptions implying Condition ρ is that the own effect of a signal s_i is larger than its effect to the rest of the agents. And therefore there exists a threshold s_i^* as required by Condition ρ for agent i to get into the group of good winners. As for the fixedness of the good winning probability by agent i, it's in part due to the fact that as s_i increases, the increasing rate of v_{ik} is at least as large as any other v_{ab}, where ab stands for arbitrary agent and good combination not equal to ik. The formal proof can be seen in Appendix B

3.6 Conclusion

The whole chapter builds on the assumption of quasilinearity of agents' value function. Under such preference, money can be utilized as a tool to help elicit true information about the world that are scattered among the agents. When the mechanism designer cares more on efficiency than fairness, our mechanism is ideal. Since the redistributional effect of money transfer has no influence on the aggregate social value.

Chapter 4

Matching

4.1 Introduction of the basic concepts

We only consider two-sided matching in this paper. There was a monograph dedicated to it, see (Roth and Sotomayor, 1990). However, in this section, I would like to summarize some of the most important theoretical concepts and results.

First, we need to provide a setting of discussion. Usually, the two distinctive sides of matching are assumed to be men and women, students and schools, workers and employers, tenants and houses, etc. Here, students and colleges are chosen as the economic background. Actually, any other pairs are OK. Choosing a specific one is to avoid too much abstractness in theory and provide motivation for the discussion. For this specific topic, there is (Abdulkadiroglu and Sonmez, 2003) which give a description of the literatures in this field till that time.

Now let us give the model and relevant notations. There is a finite set of students and a finite set of colleges denoted by $S = \{1, ..., n\}$ and $C = \{1, ..., m\}$, respectively. For any $j \in C$, college j has q_j quota, which is college j's seats for students. Students wish to enter at most one college and have an option not to enter any college at all. This outside option is formally represented by a null college, denoted by 0. This null college has unlimited quota, ie, $q_0 = \infty$.

A matching(sometimes also called assignment or allocation) is a mapping $\mu : S \to C \cup \{0\}$ such that for any $j \in C$, $|\mu^{-1}(j)| \leq q_j$. Denote by $\mu(0)$ the set of colleges with seats that is not assigned to any student at matching μ. The null college is always included in this set, as its seat supply is unlimited. Hence:

$$\mu(0) = \{j \in C : |\mu^{-1}(j)| < q_j\} \cup \{0\}$$

Now for simplicity, we only assume the price of each college is the same and students have preferences on colleges. Later we will add price consideration for colleges. Denote by R_i student i's preference on the set of colleges $C \cup \{0\}$. The corresponding strict preference and indifference relations are denoted by P_i and I_i, respectively. The meaning of the notation is as follows: if $c, c' \in C \cup \{0\}$ and cR_ic', then student $i \in S$ weakly prefers college c to college c'. Change "weakly prefers" to "strictly prefer" we get the meaning for the strict P_i. Change "weakly prefers" to "are indifferent" we get the meaning for the indifferent I_i. Preferences are assumed to be rational in the sense that for all $i \in S$, R_i is complete, reflexive and transitive. A preference profile is a list $R = (R_1, ..., R_n)$ of the students' preferences. All the discussions here and after are based on the strict preference assumption. That is , if cI_ic', then $c = c'$.

Here, a priority structure π is assumed for the colleges, that is, for each college $j \in C$, there is an exogenously given strict priority-order π_j. Formally, $\pi_j : S \to S$ is a bijection where the highest-ranked student is the student $i \in S$ with $\pi_j(i) = 1$, the second highest ranked student $i' \in S$ has $\pi_j(i') = 2$, and so on. When π_j is the same for all $j \in C$, we get the serial dictator matching mechanism. A priority structure is a list $\pi = (\pi_1, ..., \pi_m)$ of the colleges' priority-order.

A matching μ is stable(or called priority respecting as in (Andersson and Svensson, 2014)) if there is no student $i \in S$ who strictly prefers some college j to $\mu(i)$ and $\mu^{-1}(j)$ contains some other student $i' \in S$ who has lower priority for college j than student i(i.e., $\pi_j(i') > \pi_j(i)$), and furthermore, all students weakly prefer their assigned college seats to any unassigned seats in $\mu(0)$(seats from the same college are obviously equal).

Formally ,we have the following definition of stableness.

Definition 11. *A matching μ is stable for a given preference profile R and a given priority structure π, if :*
(i) for all $i, i' \in S$, $\mu(i')P_i\mu(i)$ only if $\pi_{\mu(i')}(i') < \pi_{\mu(i')}(i)$;
(ii)for all $i \in S$, $\mu(i)R_ij$ if $j \in \mu(0)$.

Condition (i) is called fairness condition, with the meaning that there is no justified envy(justified envy means that you envy somebody else's college and according to your priority in that school, you should be admitted prior to that person). Condition (ii) is the combination of individual rationality and non-wastefulness. Rationality means that the current assignment of college for a student must be weakly better than the outside option of null college 0. Non-wastefulness means that the current assignment of college for a student must be weakly better than any unocuppied college seat. For more detailed discussion of these concepts , see also (Balinsky and Sonmez, 1999).

The next important concept is Pareto efficiency or simply efficiency. A matching is pareto efficient if no other matching can make at least one student get a strictly more preferred college while no student get a strictly less preferred college.

Formally, we have the following definition.

Definition 12. *A matching μ is efficient for a given preference profile R, if for all μ' such that there exists $i \in S$ satisfying $\mu'(i) P_i \mu(i)$, then there must be some other $i' in S$ satisfying $\mu(i') P_{i'} \mu'(i')$*

It is clear that efficiency does not take into account of priority while stableness does need it. Stableness is priority respecting and includes a flavor of fairness in it. Now that we have some idea about matching, we begin to investigate mechanism that can lead to stable and efficient matching. An important difference between matching mechanism design and other mechanism design such as auction is that efficiency is usually not the only requirement, stableness or priority respecting is at least of equal importance. We will illustrate this later in examples.

Why do we care about priority? In the previous chapter, we only care about efficiency. which means to maximize the sum of all participants' utilities. However, there is a well known saying that the whole is more than the parts put together. Priority should belong to the more-than part of the whole. suppose that only students entering their most preferred college will study hard, and any students would feel the same happiness when they enter their most preferred college but those with high scores will contribute more to the society. Then just by the narrow efficiency criterion we would send any one to their most preferred college, but from the society as a whole giving the high scored students high priority is a better choice.

When it comes to mechanism design, the usual problem of unilaterally misreporting one's private information(here is one's preference) emerges. In the literature, a mechanism that is immune to such problem is often called strategyproof. Viewing the reporting of one's preference under a mechanism as a game among the students, strategyproofness means that it is a dominant strategy to report one's true preference.

There is also a related concept called group-strategyproof. What is the connotation of this concept? It means that no group can get a pareto improvement by unilaterally changing the reports of students in this group under the mechanism. Viewing the reporting of one's preference under a mechanism as a game among the students, group-strategyproofness means that it is a core equilibrium for everyone to report the private true preference.

4.2 Popular matching mechanisms

There are many matching mechanisms. Three kinds are most studied. They are the Deferred Acceptance Mechanism(DA), the Top Trading Cycle Mechanism(TTC), and the serial dictatorship mechanism(SD). These three mechanism are all related to the later analyzed Chinese student-college matching mechanism, especial the DA mechanism. Why are they most popular? One reason is that all of these mechanisms are strategyproof. Therefore the reported preference profile are true preferences for rational agents. Combined with the fact that the property of the resultant matching result relative to the reported preference profiles are relatively easy to study, these mechanisms are suitable to do research on. According to

Another reason is due to Revelation Principle. If a social result can be implemented in dominant strategy equilibrium in a mechanism, then it can be truthfully implemented in a direct mechanism. For designing new mechanisms, it seems that considering only strategyproof direct mechanism is enough. However, when studying existing mechanisms which are not designed by modern theorists, we have to face other mechanisms. To give expected results for such mechanisms, I created a special term called successful manipulation, which will be provided later in the analysis of Chinese student-college matching mechanisms subsection and as you can see is just the best response concept in Nash equilibrium but which can be better understood literally here.

4.2.1 Deferred Acceptance Mechanism

In the seminal work (Gale and Shapley, 1962), DA mechanism is first proposed. This mechanism is the first and probably the most studied mechanism in modern matching theory.

In the setting of student-college matching, it is also called student-optimal stable mechanism (SOSM) for it always finds the stable matching that is most favorable to each student. Its outcome can be calculated via the following Deferred Acceptance (DA) algorithm for a given problem:

Step 1: Each student applies to his or her favorite school. For each college j, up to q_j applicants who have the highest college j priority are tentatively assigned to college j. The remaining applicants are rejected.

Step k ($k \geq 2$): Each student rejected from a college at step k -1 applies to her next favorite college. For each college j, up to q_j students who have the highest college j priority among the new applicants and those tentatively on hold from an earlier step, are tentatively assigned to college j. The remaining applicants are rejected.

The algorithm terminates when no student applies to any new college in some step. That is, every student is either tentatively placed to a college or has been rejected by every college that is better than null college in his or her preference list.

We list important properties concerning the DA mechanism here.

Under strict preference(that is, no two colleges have the same utility level for a student.), the following properties hold. (Gale and Shapley, 1962) first proposed the two theorems and proved them. The proofs for the theorems are short and elegant, and therefore provided here following the main theorems.

Theorem 12. *The matching given by DA mechanism is stable.*

Proof. We will prove it by proof of contradiction. Suppose not, then there is a student i and a college j such that i strictly prefers j to his final assignment and $|\mu^{-1}(j)| < q_j\ or\ \exists i' \in \mu^{-1}(j)\ satisfying\ \pi_j(i') < \pi_j(i)$. However, according to the student side operation of DA mechanism, j must have been applied before the final assignment college.

And according to the college side operation of DA mechanism, if $|\mu^{-1}(j)| < q_j \ or \ \exists i' \in \mu^{-1}(j) \ satisfying \ \pi_j(i') < \pi_j(i)$, then i must not be rejected by j which contradicts to the fact that he or she was rejected(implied by the fact that his or her final college is lower in the application order than college j). □

Theorem 13. *Every student is at least as well off under the assignment given by the DA mechanism as he would be under any other stable assignment.*

Proof. Call a college j achievable for a student i in a particular environment if the student can be assigned the college in some stable matching result.

The proof is to show the following proposition. Up to a certain step n, no student has ever been rejected by a college achievable for the student, then no student will be rejected by a college achievable for the student in the step n too.

Since up to step 1, no student has been rejected by any college obivously, we can show that up to any finite step no student will ever be rejected by a college achievable for the student by repeatedly applying the previous proposition. The end step is a finite number, so in the final result no student is ever rejected by a college achievable for the student. And the students apply colleges from the most favorite to the least favorite, we therefore know that the final assignment is most favorite for every student among the stable matchings.

Now we will prove that up to a certain step n, no student has ever been rejected by a college achievable for the student, then no student will be rejected by a college achievable for the student in the step n too. That is eqivalent to say, suppose a student i is rejected by a college j in the step n, then the college is not achievable for the student. Since j reject i, that means that after step n every student assigned to j is higher in priority than i, and the quota of j has been full. If i is assigned to j in some stable matching result, then some i' that is assigned to j in the step n must be assigned to some college that is less preferred than j because up to n no student is rejected by an achievable college. A contradiction occurs here for such a matching result cannot be a stable matching result due to the existence of pair i', j which blocks i, j. Therefore by reduction to absurdity, we show that i cannot be assigned to j in any stable matching result, i.e., j is not achievable to$i$. □

See (Gale and Shapley, 1962) for more details. These two theorems can be combined into one concise statement. The SOSM matching produced by the DA mechanism is the optimal stable matching. An alternative way to express this is that if you want to find a matching that is stable, the SOSM matching selected by the DA mechanism is the only most efficient matching(constrained efficiency).

Theorem 14. *The DA mechanism is strategy-proof.*

See (Roth, 1982) for proof.

However, the DA mechanism is not pareto efficient. This fact can be shown by an example.

Example 8. *There are two colleges A and B, each with 1 seat for students. A null college 0 with unlimited seats and accepts who ever applies for it. There are 3 students 1,2,3. The preferences of the students are as follows.*

1	2	3
A	B	A
B	A	B
0	0	0

The two colleges' base priority structure is the same as in the following table

A	B
2	1
3	3
1	2

The DA mechanism is run round by round as in the table below.

	1	2	3
round1		B	A
round2	B		A
round3	B	A	
round4	B	A	0

Because 3 is the only student not tentatively accepted after 3 rounds, and 3 has been rejected by both A and B, the DA algorithm with quota terminates with student 1 entering B, 2 entering A, and 3 entering null college 0.

Now consider the following alternative preference profile.

1	2	3
A	B	0
B	A	A
0	0	B

With this profile, the DA mechanism is run as below.

	1	2	3
round1	A	B	0

Just 1 round, and the DA process is terminated.

If a mechanism is able to produce every possible results, then not pareto efficient obviously means that the mechanism is not group-strategyproof, since reporting the profile that can result in a pareto improvement assignment is a profitable group-deviation.

4.3 Analysis of Chinese student-college matching mechanisms

As an application of matching theory, we will focus on the Chinese student-college matching mechanisms for the college admission of high school students in this section.

The Chinese college admission is a centralized matching mechanism with an entrance exam taken national wide as a means of student-priority deciding for the colleges. In China, students from high school need to take an entrance exam to decide the priority structure of colleges, and the priority structure is also influenced by the application form that the students write to inform authority their own preferences. Starting from the early 1950s, the mechanisms adopted has gone through series of reforms, and what kind of mechanism should be adopted is currently still in hot debate.

The college entrance exam, or Gaokao in Chinese Pinyin, forms the foundation of the priority determination process for the current admission system. There are millions of high school seniors compete for seats at various colleges and universities in China each year. The matching of students to these colleges and universities has profound implications for the education and labor market outcomes for these students. The fairness and efficiency of this matching process, is of great importance to the society.

Many Chinese scholars have writen articles discussing problems related to Gaokao. (Zhong, Cheng, and He, 2004) has done the efficiency comparison of the three preference reporting mechanisms, i.e. reporting preference before the exam, after the exam but not knowing the exact scores, and after the scores are known. This kind of study apparently focused on the strategical interaction of studnets in the games, not on how to induce the truthful reporting. As to the strategical importance of preference reporting, Haifeng Nie has written many articles. Both (Nie, 2007a) and (Nie, 2007b) has stressed the point that a good score may not be as useful as a good strategy for reporting the preference of the university. This somewhat strange phenomenon has motivated me in studying the issue of reducing manipulation and inducing better and fairer results.

The Chinese matching mechanisms has recently caught attention of scholars from the economic field of matching overseas. (Chen and Kesten, forthcoming) has done much to analyze theoretically the sutleties of the different mechanisms adopted, and has gone even further to design experiments for studying features of these mechanisms, see (Chen and Kesten, working paper).

4.3.1 A simple model of the Chinese student-college matching

The mechanisms are hard to analyze with all the details on, so I have decide on a simple classification of the mechanisms with details removed in this subsection. And I have managed to find a theorem that is enlightening.

First, we introduce two most important mechanisms: sequential mechanism, parallel mechanism. The sequential mechanism, or Boston or Immediate Acceptance (IA) mechanism, had been the only mechanism used in Chinese student assignments both at the high school and college level (Nie, 2007b). However, this mechanism has a serious limitation: "a good score in the college entrance exam is worth less than a good strategy in the ranking of colleges" (Nie, 2007a). In our paper's wording, a successful manipulation is often more important than your score level. The problem arise from the special priority structure in the mechanism. Priority of a student in a college is based on your reported preference list firstly and based on your score secondly, a dictionary order that put too much emphasis on preference reporting. Given in (Nie, 2007b), and also cited in (Chen and Kesten, forthcoming), one parent has the following explanation of the problem.

"My child has been among the best students in his school and school district. He achieved a score of 632 in the college entrance exam last year. Unfortunately, he was not accepted by his first choice. After his first choice rejected him, his second and third choices were already full. My child had no choice but to repeat his senior year."

To alleviate this problem of high-scoring students not being accepted by any universities, the parallel mechanism was proposed. In the parallel mechanism, students select several "parallel" colleges within each choice-band. The priority structure is changed. The priority of a student in a college of the same choice-band is the same. A more detailed description from (Chen and Kesten, forthcoming) is cited here for the importance of fully understanding the mechanism.

"a student's first choice-band may contain a set of three colleges, A, B, and C while her second choice-band may contain another set of three colleges, D, E, and F (in decreasing desirability within each band). Once students submit their choices, colleges process the student applications, using a mechanism where students gain priority for colleges they have listed in their first band over other students who have listed the same college in the second band. Assignments for parallel colleges listed in the same band are considered temporary until all choices of that band have been considered. Thus, this mechanism lies between the IA mechanism, where every choice is final, and the DA mechanism, where every choice is temporary until all seats are filled."

In 2003, Hunan fisrt implemented the parallel mechanism, allowing 3 parallel colleges in the first choice-band(or called group), 5 in the second choice-band, 5 in the third choice-band, 5 in the fourth choice-band, and so on, see (Chen and Kesten, forthcoming). The parallel mechanism is soon to be widely perceived as having improved allocation outcomes for students. Citing a parent from a newpaper report [1],

[1] Li Li. "Ten More Provinces Switch to Parallel College Admissions Mechanism This Year." Beijing Evening News, June 8, 2009.

"My child really wanted to go to Tsinghua University. However, in order not to take any risks, we unwillingly listed a less prestigious university as her first choice. Had Beijing allowed parallel colleges in the first choice band, we could at least give Tsinghua a try."

In the paper (Chen and Kesten, forthcoming), they have formulated the IA mechanism, parallel mechanism, and DA mechanism as a parametric family of application-rejection mechanisms. That is a correct description. However, I would like to describe all the three mechanisms as generalized DA mechanisms with different priority structures. The IA mechanism and parallel mechanism as kinds of DA are very special. The priority structure are closely related with the reported preference. The underlying idea for the colleges may be "I like those who like me". This is where manipulation can creep in.

Formally, under the setting of student-college matching, we have the following definition.

Definition 13. *A mechanism of matching is non-manipulable, if given any preference R, and priority structure π, $\mu = h(R, \pi(R))$ and for any $i \in S$, denote $h(R'_i, R_{-i}, \pi(R'_i, R_{-i}))$ by μ', then we have $\mu(i) R_i \mu'(i)$.*

We have the following proposition.

Proposition 6. *IA and parallel mechanisms are manipulable. DA is non-manipulable.*

Now let us define the concepts of efficient mechanism and stable mechanism in the setting of student-college matching.

Definition 14. *A mechanism is efficient if for any reported preference profile R and priority structure π, the resulting matching result is pareto efficient according to the R.*

Definition 15. *A mechanism is stable if for any reported preference profile R and priority structure π, the resulting matching result is stable according to the R and $\pi(R)$.*

According to these definition, we have the following proposition.

Proposition 7. *IA is efficient. IA , parallel and DA are all stable.*

The proof is relegated to Appendices.

Remark. *The efficiency and stableness concept here is apparently not very useful if the reported R is not true. What we are concerned is the status of the matching result in the perspective of true R.*

Here I would like to view the mechanisms from a new perspective. The concept is from ex post Nash equilibrium. For ex ante Nash equilibrium or Bayesian Nash equilibrium, there is a best reponse concept. A best response is a strategy for your type that maximize the ex ante payoff given the others' probability distribution on all probable types and each type's strategy choice. An ex ante Nash equilibrium or Bayesian equilibrium is a strategy profile in which every player type chooses the best response. For ex post equilibrium, a successful manipulation is similar in meaning to a best response to ex ante Nash equilibrium. It means that given the other students' actual choice(action), the action you have chosen is best to your interest. It deprives the intermediary probability distribution structure. This way we can concentrate on the final assignment or allocation result. In real life situations, we have to use probability to make decisions ex ante, but when we can avoid it in mechanism design, we had better design a sure way to implement a certain social criteria or goal, not relying on probability distribution. The only exception should be entertainment activity like mahjong, poker games, or lottery where no prescribed social goals need to be fulfilled. Here, the ex post should not be taken literally, it does not mean that evaluating the action after some particular time. Actually, it means that assuming that every action of the other players have been known, the player still insists his choice due to its optimality with respect to the others' actual action or choice. In fact, it may happen that the player will never know all the actions of the other players and therefore there is not a time point after which all information is revealed. The important thing is that the action or manipulation can be called successful when all information is known. Formally, we give the following definition in the student-college matching setting.

Definition 16. *A student i has a successful manipulation if let R denote the true preference, R' the reported preference, $\mu' = h(R')$, then for any R''_i, let $\mu'' = h(R''_i, R'_{-i})$, we have $\mu'(i) R_i \mu''(i)$*

Here, I would like to explain this seemingly trivial concept. I think it is even more basic than ex post Nash Equilibrium. In fact, I extract it from the definition of ex post Nash equilibrium and find it more applicable than Nash equilibrium in my every day life. Its definition is in fact half of the Nash Equilibrium.

You may say that it is just the best response function. Almost, best response function is a way of getting a successful manipulation when the others' action choice is known. But calling it a successful manipulation is more appropriate in most cases as this concept does not assume that you know the payoff or the strategies of the others. Nash equilibrium usually assumes that the payoffs are known. That is, it may not be a purposeful response but just happen to be successful and optimal. It is an ex post equilibrium but it maybe the case that nobody knows the payoffs of the game fully. Here, our "ex post" requires that the strategies to be considered must be pure.

In this chapter, I only consider deterministic game forms and only consider pure strategies.

Why this concept deserves a special mention? Well, that is because in most real life situation it is the player who always choose a successful manipulation wins the game.

Example 9. *Consider a*

However, when you have clear social criteria in mind, it is not good to devise a mechanism in which the social criteria are satisfied only when every agents have a successful manipulation. Not everyone is adapted at manipulation.

In this chapter, I find that in the Chinese style college admission, the Boston mechanism, the parallel mechanism and the DA mechanism have the same assignment result in their expost equilibrium. However, it is not easy to achieve the expost equilibrium for the Boston mechanism and the parellel mechanism because they are not direct mechanisms while the DA can be seen as an almost direct mechanism which makes the "truthful revealation" the expost equilibrium strategy and easy to find for every student.

Then we can have an important concept defined now.

Definition 17. *under a preference reporting direct mechanism $h(.)$, and students report some preference profile R, therefore the matching result is $\mu = h(R)$. If all students have successful manipulation in this case, then the matching result μ is called achievable with perfect manipulation under the mechanism $h(.)$.*

In fact, the concept is just a special Nash implementation in an indirect mechanism. The difference is that when in a direct mechanism, you nash implement a result and know whether it satisfies a social goal $f(p)$, but in a indirect mechanism even if you know that is successfully manipulated by everyone, you do not know if it satisfies a social goal $f(p)$, because you do not know the economics environment p!

If you see carefully, you will know that this concept is a kind of ex post nash implementation in the particular case of matching mechanism. For general priority structures, what kind of matching results can be achieved with perfect manipulation under the IA mechanism, see (Ergin and Sönmez, 2006); what kind of matching results can be achieved with perfect manipulation under the parallel mechanism, see (Chen and Kesten, forthcoming). In this chapter, we focus our attention on a very important kind of priority structure, the score-based priority structure and discuss what kind of matching results can be achieved with perfect manipulation under such score-based priority.

DA mechanism just use the score high-low comparison to determine priority of students for every college. The more score you get, the high priority you get.

To determine a student i's priority in college j, parallel mechanism fisrt group the colleges according to i's reported preference list. For example, in Tibetan's parallel mechanism, every group has 10 colleges, then you group the first 10 together and label every college in this Group 1, group the second 10 together and label every college in this Group 2, and so on. A student's priority in a college is determined in a dictionary order. First, compare the group number of the college for the student. For instance, if Beijing University is in the first 10 college group for a tibetan student, then this student has a higher priority in Beijing Unversity than tthose who has not put Beijing University in the first 10 college group. If two students put the college in the same Group n, then comes the second procedure of comparison. The student with higher score(or high original priority) has higher priority. IA mechanism is just a parallel mechanism with every group size equal to 1.

We have the following important theorem.

Theorem 15. *If every college has the same base priority for the students, and only modify the priority structure as is necessary for the IA and parallel mechanism, then the only matching achievable with perfect manipulation* *under the IA and parallel mechanism is the same with the one under the DA mechanism, which is unique , efficient and stable.*

To get this result, we first propose a property of the DA mechanism which is convenient for discussion of its matching result.

Proposition 8. *The final matching result of the DA mechanism is not influenced by the number and order of applications of the students in each step, as long as there is some student in each step who applies for the college that is his or her favorite among the colleges having not rejected him.*

Because of the finiteness of the problem, any mechanism with the above requirement will end, and we can say that the result is the same as the ordinary DA according to this proposition. We call this group of mechanisms as the DA family of mechanisms.

As a direct result of the above propostion and the optimal stableness and non-manipulability, we have the following corollary.

Corollary 1. *The DA family of mechanisms are all optimally stable, and non-manipulable.*

Proof. Now is the proof of the above theorem 15. In fact, the Serial dictatorship mechanism is just a member of the DA family of mechanisms and has the property mentioned in the above corollary. Therefore, in the following discussion, we take the Score-based serial dictatorship mechanism for producing the DA matching result.

Now we give a intuitionistic proof of the important theorem. The perfect manipulation of DA mechanism is just non-manipulation, that is, stating the true preference according to 14. For the parallel mechanism, we would first prove that for everyone to report as his or her favorite the college that the DA matching mechanism assigns is a perfect manipulation.

To prove perfect manipulation, we only need to show that every student has got a successful manipulation by doing so. Using induction on the students' score list(without loss of generality, assuming no same scores).

First, the student with the highest score, anounce his or her true favorite college as his or her favorite, has successfully manipulated. His or her true favorite is obviously the DA matching result. Therefore, this top student has done a successful manipulation by reporting as his or her favorite the college that the DA matching mechanism assigns.

Second, for an arbitrary student i. Suppose every student higher in the score list hasreported as his or her favorite the college that the DA matching mechanism assigns. Then since the DA result is the same as the score-based serial dictatorship result, these students will get what they get in the DA. Now the student i can only expect as the best the DA result, and can indeed get it if reporint it as his or her favorite. Therefore reporting as his or her favorite the college that the DA matching mechanism assigns is a successful manipulation for him.

Having proved that the DA matching result is achievable with perfect manipulation, next we prove the uniqueness of the achievable matching result with perfect manipulation. We use terms of the DA matching result and the score-based serial dictatorship matching result interchangeably since they are the same in our setting. The proof of unique. Take any other matching result ,we would prove that it is not acheivable with perfect manipulation. Because the score-based serial dictatorship matching result is pareto-efficient, and the preference is strict, in any other matching result there must be some students who are worse off thanin the DA matching result. There must be a student with highest score among them, then we would like to show that this student has not successfully manipulated. Still using induction,

first, if this student is the top scored student, then he or she definitely has not successfully manipulated because if he or she report whatever college as his or her first choice, then he or she will get it.

second, if all the higher scored students has successfully manipulated, then they will get the score-based serial dictatorship matching college, so this student can get the score-based serial dictatorship matching result. If he or she gets a worse college, he or she definitely has not successfully manipulated.

Done. □

Now let us go to the next subsection where we put some details on.

4.3.2 Student-college matching with affirmative actions of bonus-score and quota

In this subsection, we would like to consider extending the previous model to cover affirmative actions of bonus-score and quota. When there are bonus-score given to some student, if the bonus-score is accepted in every college, then we can just take his or her total score as original score + bonus-score. The rest is the same as the serial dictatorship case analyzed in the previous subsection. However, if there is difference between the accepted bonus-score among colleges, then the priority structure again goes into the chaotic state that the score-based serial dictatorship mechanism has successfully avoided in our previous simple model.

Next is the more discussed affirmative action of quota, A prevalent affirmative action policy in school choice limits the number of ad- mitted majority students to give minority students higher chances to attend their desired schools. To circumvent the inefficiency caused by majority quo- tas, (Hafalir, Yenmez, and Yildirim, 2013) offered a different interpretation of the affirmative action policies based on minority reserves. With minority reserves, schools give higher priority to minor- ity students up to the point that the minorities fill the reserves. we would like to adopt the minority reserve definition in (Hafalir, Yenmez, and Yildirim, 2013) with a little extension. we will give every group(including the majority group) a reserve (or called quota, whatever) that gives a student of this group priority over other group whenever the reserve number has not been reached. Using this definition of quota, we give the following enlightening example.

Example 10. *There are two colleges A and B, each with two seats for students. There are two students groups a and b. Students $1, 2, 3$ are in group a and students $4, 5$ are in group b. The preferences of the students are as follows.*

1	2	3	4	5
A	A	A	B	B
B	B	B	A	A

As can be seen, the students in group a all prefer college A while the students in group b all prefer college B.

The two colleges' base priority structure is the same as in the following table

A	B
1	1
2	2
3	3
4	4
5	5

The quota structure of A and B are both 1 seat for group a, 1 seat for group b. Now the DA mechanism is run round by round as in the table below.

	1	2	3	4	5
round1	A	A		B	B
round2	A	A	B	B	
round3	A		B	B	A
round4	A	B		B	A

Because 3 is the only student not tentatively accepted after 4 rounds, and 3 has been rejected by both A and B, the DA algorithm with quota terminates.

Now consider the IA mechanism. If the quota priority is first compared and considered, then the result is the same as DA. This fact can be easily verified and we omit it here. If the reported preference order is first considered, as the name of Instant Accept is indicating, then the result is as the table below.

	1	2	3	4	5
round1	A	A		B	B

From the above example, we see that the matching result of DA mechanism with quota is not pareto efficient for the students. In fact, the comparison of the tables' matching results shows that IA mechanism's result is a pareto improvement. Student 2 and 5 gets strictly better result in IA mechanism than in DA mechanism. Why the efficiency result of the the previous subsection lost. The reason is that the efficiency in previous subsection comes from the serial dictatorship nature of DA in that special priority structure. Now with quota included, the priority structure is changed and that property is lost in the process. However, the resulted matching is still priority-respecting(as (Andersson and Svensson, 2014) call "stable"), and constrained efficient(pareto efficient among the priority-respecting matchings). Meanwhile, there is not way for the student to manipulate to get a better result for this example.

This is true for the DA mechanism with quota. Because of its importance, we state it as a formal theorem here.

Theorem 16. *A DA mechanism with quota is priority-respecting, constrained efficient and non-manipulable.*

The proof is relegated to the Appendices.

4.3.3 Policy suggestions

For student-college matching mechanisms, when no affirmative actions are taken, and the scores in the nationwide college entrance examination is the base priority criterion, then the achievable result under perfect manipulation is the same, which is efficient, stable and unique. However, only for the DA or score-based serial dictatorship, the perfect manipulation is easily fulfilled in reality, since the perfect manipulation is just non-manipulation, i.e., reporting the true preference. Therefore, it's very easy to teach how to do a successful manipulation in a score-based serial dictatorship, and even a researcher cannot tell how to do a successful manipulation in another mechanism such as IA mechanism and parallel mechanism. As a result, a kind of unfairness caused by manipulation occurs. We do not want to judge students by manipulation, but these IA and parallel mechanism allow students' matching results to be influenced by a large extent to such undesired manipulation abilities, especially for the majority of students who are not the top students.

Where there is manipulation, there is corruption. A successful manipulation(reporting of preference) in IA and parallel mechanism needs information about other students' preference reporting. So there might be buying and selling of such information and even without corruption of relevant institutions, some other unnecessary businesses, like strategy consulting of preference reporting for the college admission mechanisms. These things burden the family of senior high school students unnecessarily both economically and spiritually.

Chapter 5

A popular mechanism for mental competition— finite dynamic games with perfect information

5.1 Introduction

Go, chess, Chinese chess, draughts etc occupy a lot of leisure time of many people around the world, and all kinds of match and tournaments are held everywhere around the world. These game forms share many things in common as mechanism for determining win,lose,or draw between two players. Many books, articles and manuals are published about them. However, most centered around how to win with concrete moves in certain positions, how to devise traps for the opponents and how to play certain popular openings. In this chapter, we analyze these finite games with perfect information through a typical and popular game,chess. As a mechanism for mental competition, chess is analyzed from game theoretic point of view and the propositions and theorems can be applied to general finite games with perfect information easily.

Let us begin with a comparison of chess with student-college matching mechanism. That mechanism is to collect true preference from students and allocate student according to the reported preference and their priority(often determined by score in the entrance examination). It aims at making the truth telling dominant strategy and tries to garentee that the game has no manipulation or less. Chess is different, it determines the result of win, loss, or draw just by how well two players can manipulate under the rules of the mechanism.

To make it a real mental competition, it is deliberately designed with a strategy space that seems infinite to human beings or even computers. Therefore the implementation theory employed to analyse student-college matching mechanism is not suitable here. Incentive is usually not the problem, the players all want to win. The problem is how strong the desire to win is and mental efforts put in. Usually, the expected results are also not as the previous chapter. We do not expect a Nash equilibrium for such game forms. We expect a result of win,loss,or draw based on the extent of successfulness of manipulation from both sides in chess. While in student-college matching we try to design mechanism that avoid the manipulation factor, in chess we encourage manipulation skill of the game and that is where the enjoyment lies.

Zermelo first analyzed the game of chess in 1913 in German. (Schwalbe and Walker, 2001) give a survey of the early studies which are mostly based on Zermelo's seminal article and translated Zermelo's article into English in the appendix of that paper.

5.2 Finite games with perfect information: chess as a simple model

In this model, we do not consider time factor, do not consider the moves limit factor, and agreed draw or resign.

First, the definition of position

Definition 18. *A position q on a chess board, is a certain placement of pieces on the board along with the information that it is who's turn to move.*

Here, the only relevant thing is which piece is placed in which square, and who is to move.

Let us then consider the ending position which should be first analysed in backward induction method. An ending position is a position that the referee must be called to register the result,i.e.,the game is finished according to chess rule.

Let A, B denote the opposing sides. Let q denote the ending position that is B's turn to move. There are only the following cases:

1. A has checkmated B, then A win.
2. A has stalemated B, then draw.

3. Both sides have insufficient material(pieces) to give the opponent checkmate, then draw.

4. Three repetition of the same position, that is, the same position q has been reached twice before, then draw.

Now, in order to analyze the game of chess, we need the concept of a winning position for a side. Similar concept can be found in (Schwalbe and Walker, 2001), but ours here is for all positions, not just positions that is due to move for that side.

Definition 19. *A position q is a winning position for side A if and only if one of the following case is true:*

1. It is an ending position that A win(A has checkmated B).

2. When it is B's turn to move, B can make a legal move, but for every move he can produce according to chess rule, it leads to a position that is a winning position for side A.

3.When it is A's turn to move, A has a legal move that leads to a winning position for A.

Here, we take it for granted that this definition is well defined. Later on , we will prove it.

The definition of a winning position for B is just the above definition with A and B interchanged. With the concept of a winning position, we can define the concept of a losing position.

Definition 20. *A position q is a losing position for A if and only if it is a winning position for B.*

The definition of a losing position for B is just the above definition with A and B interchanged.

Now the only possible other positions is defined as following:

Definition 21. *A position that is neither a winning position nor a losing position for any side is called a drawing position.*

Now there is a lemma for chess.

Lemma 4. *every chess game end in finite moves*

Proof. Suppose not. Then there is a game with infinite moves. According to chess rule, three repetition of a same position makes the game end in draw, therefore there must be infinite positions. However, with limited squares to place limited kinds of pieces for the two sides, the positions of chess is finite. A contradiction. □

Chess as a mechanism to assign win, loss, and draw is not used to elicit preferences from both sides. We assume all the players of chess has the preference $win \succ draw \succ loss$. It is rationality and mental abilities that decides the choice of moves. Now we propose the following important theorem.

Theorem 17. *For two players with perfect rationality and immense mental power, a winning position for a player will end in win of the player, a losing position for a player will end in loss of the player, and a draw position of a player will end in a draw.*

Proof. For two players(you and an opponent) with perfect rationality and immense mental power, classifying a given position into three clear defined category is easy. Then:

In the winning position, when it is your turn to move, you just chooses the move that lead to another winning position that has not been on board before in the game(avoiding cycle) , then since the potential positions are finite, an ending winning position(checkmate) can be reached finally; when it is the opponent's turn to move, any legal move will lead to a winning position for you.

In the losing position , when it is your turn to move, any legal move will lead to a losing position: when it is the opponent's turn to move, the opponent player just chooses the move that lead to another losing position that has not been on board before in the game(avoiding cycle), then since the potential positions are finite, an ending losing position (be checkmated)can be reached finally.

In the drawing position, since no winning position for the side that is to move, and the side is not losing position, so he or she must be able to find a move leading to another drawing position, a cycle is emerging and three repetitions reached (draw acording to the rule) or a drawing ending is reached(insufficient material). □

All the arguments so far relied on the well-definedness of our winning position concept. Apparently, it is a recursively defined term. First, an edge case, the checkmate, this is easily found according to chess rules. Then, after finding all the checkmate position, we can produce all the position that can reach this position according to the chess rule.

Now,when all the positions that can produce the checkmate position is found. We now need to produce all the position that can lead to all the position that can produce the checkmate. Here, another step is needed since these position is the opponent's turn to move. We still need to check these positions against another criterion. All the position that can be produced must belong to those that is already found in the winning positions... and so on. The process continued until no new winning positions can be found. After these informal description, I would like to formalize these with mathematical notations below. A proposition regarding a winning position is as follows.

Proposition 9. *The winning position concept is well defined, that is, for every position that can arise on the chessboard according to chess rule, it has a deterministic answer whether it is a winning position or not.*

First, an explanation. Why do we ask the question of well definedness? Russell's Paradox is the reason, not every recursively defined set is consistent. In naive set Theory, the following definition is possible. let $R = \{x|x \notin x\}$, that is R is a set of sets that does not contain itself. It is a definition on the space of all sets. Do we have a deterministic answer whether R is a set in R or not. Unfortunately it cannot be determined, since $R \in R \iff R \notin R$ which is a contradiction.

To prove the recursive definition of a winning position is well defined, we only need to show that every position can be detemined whether it is a winning position or not.

Proof. Let X denote the set of positions that can possibly be reached from the start of the game by finite alternative moves from both sides. We first prove that this set is well defined. Let $f(q)$ stand for the set of a chess position q's next possible positions. For all $A \subset X$. let $f(A) = \cup_{x \in A} f(x)$. Then,

Step 0, $X_0 = x_o$, the starting position.

Step 1, $X_1 = f(X_0) \cup X_0$, the positions that is reachable in two moves.

Step 2, $X_2 = f(X_1) \cup X_1$, the positions that is reachable in three moves.

...

Step n, $X_n = f(X_{n-1}) \cup X_{n-1}$,

$X = lim_{n \to \infty} X_n$. This set is decidable, since $X(\cdot)$ is well defined by chess rule, and possible chess placements is finite(therefore for all $n \in N$, $|X_n| < M$). With the increasing of n, whenever $X_n = X_{n-1}$, you can see according to the definition the X is found. And since $X_0, X_1...$ is a sequence of sets with bound for the number of elements. This n will occur inevitably. So X is well defined.

For a finite set, you need to be able to identify all the elements satisfying a certain definition, or else that definition is not well defined. Now we devise an algorithm to find all the positions that is a winning position. We search for the winning positions from the ending position.

Step 0, filter out all the checkmate positions from X, this can be done by applying chess rule, put these positions into W_0.

Step 1, $W_1 = \{x|f(x) \cap W_0 \neq \emptyset\}$.

Step 2, $W_2 = \{x|f(x) \neq \emptyset andf(x) \subset W_1\} \cup W_0$.

...

Step 2n-1, $W_{2n-1} = \{x|f(x) \cap W_{2n-2} \neq \emptyset\}$.

Step 2n, $W_{2n} = \{x|f(x) \neq \emptyset andf(x) \subset W_{2n-1}\} \cup W_0$.

Obviously, $W_0 \subset W_2 \subset W_4 \cdots$ and $W_1 \subset W_3 \subset W_5 \cdots$. Since these are increasing sets, and the number of potential position is finite, there must be somewhere that the $\subset$ can be replaced with $=$ for the first time, and in fact after that all the $\subset$ can be replaced with $=$. The set of winning positions is $W = lim_{n\to\infty} W_{2n-1} \cup W_{2n}$.

□

Chess is too complicated for us to tell all the winning positions. A simple finite game with perfect information is provided in the following example to illustrate the use of the winning positions concept.

Example 11. *The game of taking out stones. There is a pile of stones and two players. Each is required to take 1 or 2 stones out of the pile one time. They take turns to do it alternatively. The one player who takes the last stone out is the winner. For a pile of 20 stones, can the first player to move the stones win?How?*

One may feel headless if not having a pattern to do it. The way of solving this problem is just the way of finding the winning positions in the above proof by way of backward induction. In the following we use the notation n_o to represent positions when there are n stones left in the pile while it is the opponent to move ; the notation n_s to represent positions when there are n stones left in the pile while it is self to move .

Step 0, we need to find the "checkmate" positions W_0. There is only one "checkmate" situation in this game: it is the opponent to move, but he or she find that there is no stone . We denote it with 0_o meaning a position of 0 stones with the oppenent turn to move. Therefore $W_0 = 0_o$.

Step 1, the positions that can directly be led into a position in W_0 after self's well chosen move, $W_1 = \{1_s, 2_s\}$.

Step 2, the positions that the opponent is to move but have to face a position in W_0 or every move will result in a position in W_1, $W_2 = \{0_o, 3_o\}$.

Step 3, the positions that can directly be led into a position in W_2 after self's well chosen move, $W_3 = \{1_s, 2_s, 4_s, 5_s\}$.

Step 4, the positions that the opponent is to move but have to face a position in W_0 or every move will result in a position in W_1, $W_4 = \{0_o, 3_o, 6_o\}$.

...

We can therefore get $W_{2n} = \{0_o, \ldots, 3(n-1)_o, 3n_o\}$

and $W_{2n-1} = \{1_s, 2_s, 4_s, 5_s, \ldots, 3n-2_s, 3n-1_s\}$.

In informal words, the winning positions include the positions with a pile of stone with a number of multiples of 3 when it is the opponent to move, and the positions with a pile of stone with a number of nonmultiples of 3 when it is self to move.

The moving strategy is to move to a winning position whenever is possible, and it is possible when the starting position is a winning position. Since 20 is a nonmultiple of 3, the first person is to move will win following the "from winning to winning" strategy.

For chess, typical winning position are collected as finite move checkmate problems in chess books. These kind of books can train a player's ability to find winning positions and identify the move order to transpose to the final checkmate ending. Besides, even if you cannot find out the path of transition, you can remember them after seeing the answer, thus enrich your database of winning positions and the transposing moves. It is widely admitted among chess players that when the amount of such positions are accumulated to some certain degree(different players may have a different threshold), they feel much easier to detect winning positions and the transposing moves. This is an important phenomenon: the brain can nearly automatically extract patterns from remembered winning chess positions.

Appendix A

revelation principles

Theorem. *revelation principle:*

if a Mechanism $\langle M, h\rangle$ implements the social criteria F in dominant strategy equilibrium. Then there is a direct revelation mechanism which implements F truthfully in dominant strategy equilibrium(truth telling is a dominant strategy equilibrium).

Proof. Since F can be implemented in dominant strategies by $\langle M, h\rangle$, there is a profile of strategies $(\sigma^1, \cdots, \sigma^n) \in (E_1 \mapsto M_1) \times \cdots \times (E_n \mapsto M_n)$ that forms an dominant strategy equilibrium in the game induced by $\langle M, h\rangle$. Thus, for all $e = (e^1, \cdots, e^n) \in E = E_1 \times \cdots \times E_n$, we have

$$h(\sigma^1(e_1), \cdots, \sigma^n(e_n)) \in F(e_1, \cdots, e_n)$$

Furthermore, implementability in dominant strategies means that in the mechanism $\langle M, h\rangle$, for all $i \in N$, $e \in E$, and any strategy profile $\rho = (\rho^1, \cdots, \rho^n) in (E_1 \mapsto M_1) \times \cdots \times (E_n \mapsto M_n)$,

$$h(\rho^1(e_1), \cdots, \sigma^i(e_i), \cdots, \rho^n(e_n)) \succeq_{e_i} h(\rho^1(e_1), \cdots, \rho^i(e_i), \cdots, \rho^n(e_n)) \tag{A.1}$$

Consider the following direct mechanism $(E_1 \times \cdots \times E_n, g)$ where for all $(e_1, \cdots, e_n) \in E_1 \times \cdots \times E_n$,

$$g(e_1, \cdots, e_n) = h(\sigma^1(e_1), \cdots, \sigma^n(e_n)) \in F(e_1, \cdots, e_n)$$

It suffices to show that in the game induced by $(E_1 \times \cdots \times E_n, g)$, it is a dominant strategy for each agent i with type e_i to report e_i. Suppose not. Then there is a profile $e = (e_1, \cdots, e'_i, \cdots, e_n) \in E_1 \times \cdots \times E_i \times \cdots \times E_n$ and an agent $i \in N$ and another type e'_i such that

$$g(e_1, \cdots, e'_i, \cdots, e_n) \succ_{e_i} g(e)$$

$$\Longleftrightarrow$$

$$h(\sigma^1(e_1), \cdots, \sigma^i(e_i'), \cdots, \sigma^n(e_n)) \succ_{e_i} h(\sigma^1(e_1), \cdots, \sigma^i(e_i), \cdots, \sigma^n(e_n))$$

Choose $\rho = (\rho^1, \cdots, \rho^i, \cdots, \rho^n) \in (E_1 \mapsto M_1) \times \cdots \times (E_i \mapsto M_i) \times \cdots \times (E_n \mapsto M_n)$ such that $\rho^j(e_j) = \sigma^j(e_j)$ for every $j \in N, j \neq i$ and $\rho^i(e_i) = \sigma^i(e_i')$. Then, the last inequality can be rewritten as

$$h(\rho^1(e_1), \cdots, \rho^i(e_i), \cdots, \rho^n(e_n)) \succ_{e_i} h(\rho^1(e_1), \cdots, \sigma^i(e_i), \cdots, \rho^n(e_n))$$

which contradicts inequality A.1. □

Proposition. *revelation principle in an interdependent value environment:*

if a Mechanism $\langle M, h\rangle$ implements the social choice rule F in ex post equilibrium. Then there is a direct revelation mechanism which implements F truthfully in ex post equilibrium(truth telling is a ex post equilibrium).

Proof. Since F can be implemented in ex post strategies by $\langle M, h\rangle$, there is a profile of strategies $(\sigma^1, \cdots, \sigma^n) \in (S_1 \mapsto M_1) \times \cdots \times (S_n \mapsto M_n)$ that forms an ex post equilibrium in the game induced by $\langle M, h\rangle$. Thus, for all $(s^1, \cdots, s^n) \in S_1 \times \cdots \times S_n$, we have

$$h(\sigma^1(s_1), \cdots, \sigma^n(s_n)) \in F(s_1, \cdots, s_n)$$

Furthermore, implementability in ex post strategies means that for all $i \in N$, $s \in S$, and $\rho^i : S_i \mapsto M_i$,

$$v_i(h(\sigma^1(s_1), \cdots, \sigma^i(s_i), \cdots, \sigma^n(s_n)), s) \geqslant v_i(h(\sigma^1(s_1), \cdots, \rho^i(s_i), \cdots, \sigma^n(s_n)), s) \tag{A.2}$$

Consider the following direct mechanism $(S_1 \times \cdots \times S_n, g)$ where for all $(s_1, \cdots, s_n) \in S_1 \times \cdots \times S_n$,

$$g(s_1, \cdots, s_n) = h(\sigma^1(s_1), \cdots, \sigma^n(s_n)) \in F(s_1, \cdots, s_n)$$

It suffices to show that in the game induced by $(S_1 \times \cdots \times S_n, g)$, it is ex post incentive compatible for each agent i with type s_i to report s_i. Suppose not. Then there is a profile $(s_1, \cdots, s_n) \in S_1 \times \cdots \times S_n$ and an agent $i \in N$ and a type $q \in S_i$ such that

$$v_i(g(q, s_{-i}), s) > v_i(g(s), s)$$

$$\Longleftrightarrow$$

$$v_i(h(\sigma^1(s_1), \cdots, \sigma^i(q), \cdots, \sigma^n(s_n)), s) > v_i(h(\sigma^1(s_1), \cdots, \sigma^i(s_i), \cdots, \sigma^n(s_n)), s)$$

Choose any $\rho^i : S_i \mapsto M_i$ such that $\rho^i(s_i) = \sigma^i(q)$. Then, the last inequality can be written as

$$v_i(h(\sigma^1(s_1), \cdots, \rho^i(s_i), \cdots, \sigma^n(s_n)), s) > v_i(h(\sigma^1(s_1), \cdots, \sigma^i(s_i), \cdots, \sigma^n(s_n)), s)$$

which contradicts inequality A.2. $\square$

Appendix B

Proof of proposition 5

Proposition. *The social efficient goal of* $\max_{i,j} x_{ij}v_{ij}(s)$ *can be ex post implemented if:*

For all i *and* j,

$$v_{ij}(s) = b_{ij} + o_i(s_i) + \sum_{l \neq i} r_l(s_l)$$

where $b_{ij} > 0$ *is the base value,* $s_i \in [0, +\infty)$, $\forall i, o_i(0) = r_i(0) = 0$

Further assumptions are listed below.

(i)For all i, $\frac{\partial o_i}{\partial s_i} > 0$, $\frac{\partial r_i}{\partial s_i} > 0$;

(ii) $\forall i$, $\frac{\partial o_i}{\partial s_i} > \frac{\partial r_i}{\partial s_i} > 0$;

Before the proof of this proposition, two lemmas which are useful later is given and proved first.

Lemma. *(u)For the* v, s *described in the main proposition, when* $s = s'$, *a good* k *will be allocated to agent* i *in a maximizing scheme, then when* $s_i > s'_i$ *and* $s_{-i} = s'_{-i}$, *the good* k *will also be allocated to agent* i *in the maximizing schemes.*

Proof. Notice that s_i influence v_{ij} only through $o_i(s_i)$ for all $j \in \{1, \cdots, n\}$;for all other agent $l \neq i$, s_i influence v_{lj} only through $r_i(s_i)$.

Now Suppose when $s = s'$, a good k will be allocated to agent i in the maximizing scheme, but contrary to lemma(u)'s assertion, for some $s_i > s'_i$ and $s_{-i} = s'_{-i}$, no good is allocated to agent i or a good $q \neq k$ is allocated to agent i in a maximizing scheme. Let x' denote the value maximizing allocation scheme when $s = s'$ and k is allocated to agent i, x denote the value maximizing allocation scheme when $s_i > s'_i$, $s_{-i} = s_{-i}$ and a good $q \neq k$ is allocated to agent i. We must have $\sum x_{ij}v_{ij}(s_i, s_{-i}) \geq \sum x'_{ij}v_{ij}(s_i, s_{-i})$, □

Lemma. *(d)For the* v, s *described in the main proposition, when* $s = s'$, *no good will be allocated to agent* i *in a maximizing scheme, then when* $s_i < s'_i$

and $s_{-i} = s'_{-i}$, it is also the case that no good will be allocated to agent i in the maximizing schemes.

Proof. We need to show that the assumptions imply Condition ρ, that is, from the maximizing solution xs to $\sum_{ij} x_{ij}v_{ij}(s)$, one can designate an allocation scheme satisfying if $v_{ik}(s_i) < v_{ik}(s_i^*)$, agent i will have no probability to be allocated the good k.

if $v_{ik}(s_i) > v_{ik}(s_i^*)$, agent i will have a probability to be allocated the certain good k and charged $v_{ik}(s_i^*)$. However, the probability does not depend on s_i.

The wanted allocation scheme under the propostion's environment: Randomly pick an integer solution matrix x of the assignment maximization problem [1]

$$\max_{i,j} x_{ij}v_{ij}(s)$$

s.t.

$$\sum_i x_{ij} \leq 1 \; for \; all \; j$$

$$\sum_j x_{ij} \leq 1 \; for \; all \; i$$

$$x_{ij} \geq 0 \; for \; all \; i, j$$

Then assign the goods according to the matrix x such that if $x_{ij} = 1$, agent i is assigned good j, and i is charged $v_{ij}(s_i^*(s_{-i}), s_{-i})$.

Now we need to check that this scheme satisfy the two requirements in Condition ρ to complete the proof. Let us show two useful properties.

First, we assert that whenever agent i is never allocated the good k in any maximizing solution xs to $\sum_{ij} x_{ij}v_{ij}(s)$ under the reported profile (s_i, s_{-i}), then agent i is never allocated the good k in any maximizing solution x's to $\sum_{ij} x'_{ij}v_{ij}(s'_i, s_{-i})$ under the reported profile (s'_i, s_{-i}) if $s'_i < s_i$. Suppose not, then there is an $\hat{x}$ allocating k to i such that $\sum_{ij} \hat{x}_{ij}v_{ij}(s'_i, s_{-i}) \geq \sum_{ij} x_{ij}v_{ij}(s'_i, s_{-i})$. Notice that s_i influence v_{ij} only through $o_i(s_i)$ for all $j \in \{1, \cdots, n\}$;for all other agent $l \neq i$, s_i influence v_{lj} only through $r_i(s_i)$. Furthermore, $\frac{\partial o_i}{\partial s_i} > \frac{\partial r_i}{\partial s_i} > 0$. Thus $\sum_{ij} \hat{x}_{ij}v_{ij}(s_i, s_{-i}) \geq \sum_{ij} x_{ij}v_{ij}(s_i, s_{-i})$ as the increase from s'_i to s_i add to the left hand side at least as much as the right hand side[2].

[1] By Dantzig's linear programming theory, such a solution exists.
[2] Note that all n goods are allocated in any maximizing scheme

So $\hat{x}$ is also a maximizing scheme. Contrary to maximizing solution never allocate k to i.

Second, we assert that whenever agent i is allocated the good k in a integer maximizing solution xs to $\sum_{ij} x_{ij}v_{ij}(s)$ under the reported profile (s_i, s_{-i}), then agent i must also be allocated the good k in some maximizing solution x's to $\sum_{ij} x'_{ij}v_{ij}(s'_i, s_{-i})$ under the reported profile (s'_i, s_{-i}) if $s'_i > s_i$. Suppose not, then there is a x not allocating k to i such that $\sum_{ij} \hat{x}_{ij}v_{ij}(s'_i, s_{-i}) > \sum_{ij} x_{ij}v_{ij}(s'_i, s_{-i})$. Notice that s_i influence v_{ij} only through $o_i(s_i)$ for all $j \in \{1, \cdots, n\}$;for all other agent $l \neq i$, s_i influence v_{lj} only through $r_i(s_i)$. Furthermore, $\frac{\partial o_i}{\partial s_i} > \frac{\partial r_i}{\partial s_i} > 0$. Thus $\sum_{ij} \hat{x}_{ij}v_{ij}(s_i, s_{-i}) > \sum_{ij} x_{ij}v_{ij}(s_i, s_{-i})$ as the decrease from s'_i to s_i reduce the right hand side at least as much as the right hand side. This is contrary to that x maximizes $\sum_{ij} x_{ij}v_{ij}(s)$.

From these two properties, we have for each agent good pair (i, j) a particular $s_i = s_i(j)$ such that when $s_i < s_i(j)$, $x_{ij} = 1$ never hold for any integer maximizing solution xs to $\sum_{ij} x_{ij}v_{ij}(s)$; when $s_i > s_i(j)$, $x_{ij} = 1$ hold for some integer maximizing solution xs to $\sum_{ij} x_{ij}v_{ij}(s)$.[3]

Now we claim that for any i, $s_i(j) = s_i(j')$ for all j, j' if neither is $+\infty$. Suppose not, then as s_i increases from 0 to $+\infty$, after we pass the first $s_i(j), j \in \{1, \cdots, n\}$, we will meet a second $s_i(j'), j' \in \{1, \cdots, n\}$, the set $\{j \in \{1, \cdots, n\} | x_{ij} = 1 \ for\ some\ integer\ solution\ x\ to\ \max_x \sum_{ij} x_{ij}v_{ij}(s_i, s_{-i})\}$ increases. Let x' denote the integer solution to $\max_x \sum_{ij} x_{ij}v_{ij}(s_i, s_{-i})$ such that $x_{ij'} = 1$ when $s_i > s_i(j')$. Then for any other assignment x, $x'_{ij}v_{ij}(s_i, s_{-i}) \geq x_{ij}v_{ij}(s_i, s_{-i})$. Consider s_i decreases to $s_i < s_i(j')$. Since s_i influence v_{ij} only through $o_i(s_i)$ for all $j \in \{1, \cdots, n\}$,for all other agent $l \neq i$, s_i influence v_{lj} only through $r_i(s_i)$,and $\frac{\partial o_i}{\partial s_i} > \frac{\partial r_i}{\partial s_i} > 0$, it is still the case that $x'_{ij}v_{ij}(s_i, s_{-i}) \geq x_{ij}v_{ij}(s_i, s_{-i})$ as the decrease from s'_i to s_i reduce the right hand side at least as much as the right hand side. But it's contrary to the description of $s_i(j')$ which says that if $s_i < s_j(j')$, $x_{ij} = 1$ never hold for any integer maximizing solution xs to $\sum_{ij} x_{ij}v_{ij}(s)$.

Let s^*_i be the value that all $s_i(j)$ which is not ∞ equal to. Then the sets $\{j \in \{1, \cdots, n\} | x_{ij} = 1 for some integer solution x to \max_x \sum_{ij} x_{ij}v_{ij}(s_i, s_{-i})\}$ and $\{x | x\ is\ an\ integer\ \ solution\ \ to \max_x \sum_{ij} x_{ij}v_{ij}(s_i, s_{-i})\}$ stay the

[3]when j is never allocated to i in a integer maximizing solution, let $s_i(j) = 0$; When j is always allocated to i in some integer maximizing solution, let $s_i(j) = +\infty$.

same for any $s_i > s_i^*$.[4] We can see that Condition ρ holds,because for any i,k,

if $v_{ik}(s_i) < v_{ik}(s_i^*)$, since s_i influences $v_{ik}(s_i)$ only through $o_i(s_i)$ and $\frac{\partial o_i}{\partial s_i} > 0$, $s_i < s_i*$. According to s_i*'s definition, agent i will have no probability to be allocated the good k.

if $v_{ik}(s_i) > v_{ik}(s_i^*)$, then $s_i > s_i^*$, agent i will have a probability to be allocated the certain good k and charged $v_{ik}(s_i^*)$. However, the probability does not depend on s_i since it is $\{x|x_{ij} = 1 and x is an integer solution to \max_x \sum_{ij} x_{ij}v$ which is the same for all $s_i > s_i^*$.

To verify Condition ρ, we have to find s_i^*, for each s_{-i}, k. Case (a):Given s_{-i} , suppose a good k must always be allocated to agent i in the maximizing scheme for any $s_i \in [0, +\infty)$, then the s_i^* is chosen to be $-\epsilon$, where $epsilon$ is a small positive number.

Case (b):Given s_{-i}, suppose no good will be allocated to agent i in the maximizing scheme for any $s_i \in [0, +\infty)$, then the s_i^* is chosen to be $+\infty$, and k can be chosen arbitrarily for it is not relevant in this case; In the above two cases, Condition ρ hold trivially as can be easily verified. The last case is more complex: Given s_{-i}, for some $s_i \in [0, +\infty)$ no good will be allocated to agent i in the maximizing scheme, and for other $s_i \in [0, +\infty)$ some good will be allocated to agent i in the maximizing scheme.

□

[4] $\{x|x \textit{ is an integer solution to} \max_x \sum_{ij} x_{ij}v_{ij}(s_i, s_{-i})\}$ stays the same for any $s_i > s_i^*$ has not been shown explicitly yet. However, it is easy to show its truthfulness by the method of "reduction to absurdity" like the previous paragraphs.

Bibliography

Abdulkadiroglu, Atila and Tayfun Sonmez (2003). “School Choice: A Mechanism Design Approach”. In: *American Economic Review* 93.3, pp. 729–747.

Andersson, Tommy and Lars-Gunnar Svensson (2014). “Non-manipulable House Allocation with rent control”. In: *Econometrica* 82.2, pp. 507–539.

Ausubel, Lawrence (1999). “A Generalized Vickrey Auction”. In: *mimeo*.

Balinsky, Michel and Tayfun Sonmez (1999). “A Tale of Two Mechanisms: Student Placement”. In: *Journal of Economic Theory* 84, pp. 73–94.

Bergemann, Dirk and Stephen Morris (2008). “Ex post implementation.” In: *Games and Economic Behavior* 63.2, pp. 527–566. URL: http://dblp.uni-trier.de/db/journals/geb/geb63.html#BergemannM08;http://dx.doi.org/10.1016/j.geb.2006.10.006;http://www.bibsonomy.org/bibtex/22093f3b8738e2e76053d8156f7c1b11b/dblp.

Che, Yeon-Koo, Jinwoo Kim, and Fuhito Kojima (2015). “Efficient assignment with interdependent values”. In: *Journal of Economic Theory* 158, pp. 54–86.

Chen, Yan and Onur Kesten (forthcoming). “Chinese College Admissions and School Choice Reforms: A Theoretical Analysis”. In: *Journal of Political Economy*.

— (working paper). “Chinese College Admissions and School Choice Reforms: An Experimental Study”. In:

Chung, Kim-Sau and Jeffrey C. Ely (2006). “Ex post incentive compatible mechanism design”. In: *mimeo*.

Danilov, Vladimir (1992). “Implementation Via Nash Equilibria”. In: *Econometrica* 60.1, pp. 43–56.

Dasgupta, Partha and Eric Maskin (2000). “Efficient Auctions”. In: *Quarterly Jounal of Economics* 115.2, pp. 341–388.

Ergin, Haluk and Tayfun Sönmez (2006). "Games of school choice under the Boston mechanism". In: *Journal of Public Economics* 90.1-2, pp. 215–237. URL: http://www.sciencedirect.com/science/article/B6V76-4GGXX7T-1/1/f5de48520ea7ea1ad4152c64009843b0

Gale, D. and L. S. Shapley (1962). "College Admissions and the Stability of Marriage." In: *The American Mathematical Monthly* 69, pp. 9–15. URL: http://dblp.uni-trier.de/db/journals/tamm/tamm120.html#GaleS13;http://dx.doi.org/10.4169/amer.math.monthly.120.05.386;http://www.bibsonomy.org/bibtex/212c189a930c3901f28055c644fb93406/dblp.

Gibbard, Allan (1973). "Manipulation of voting schems: a general result". In: *Econometrica* 41.4, pp. 587–601.

Hafalir, Isa E., M. Bumin Yenmez, and Muhammed A. Yildirim (2013). "Effective affirmative action in school choice". In: *Theoretical Economics*, pp. 325–363.

Hurwicz, Leonid. "On informationally decentralized systems". In:

Jehiel, Philippe and Benny Moldovanu (2001). "Efficient Design with Interdependent Valuations". In: *Econometrica* 69.5. doi: 10.1111/1468-0262.00240, pp. 1237–1259. URL: http://www.blackwell-synergy.com/doi/abs/10.1111/1468-0262.00240;http://www.bibsonomy.org/bibtex/2bcd8d4aa0aa0406e2acbe00e52c9b893/smicha.

Kara, Tarik and Tayfun Sonmez (1996). "Nash Implementation of Matching Rules". In: *Journal of Economic Theory* 68, pp. 425–439.

Liang, Haiyin (2010). "A Survey of Implementation Theory of Mechanism Design". PhD thesis.

Maskin, Eric (1999). "Nash Implementation and Welfare Optimality". In: *Review of Economic Studies* 66, pp. 23–38.

McLean, Richard P. and Andrew Postlewaite (2014). "Implementation with Interdependent Valuations".

Moor, John and Rafeal Repullo (1990). "Nash Implementation: A Full Characterization". In: *Econometrica* 58, pp. 1083–1099.

Nie, Haifeng (2007a). "A game theoretic analysis of college admissions". In: *China Economic Quarterly* 6.3, pp. 899–916.

— (2007b). "Is a High Score Enough? An Analysis of Strategies under the College Admissions Mechanism". In: *Southern Economics* 7, pp. 23–36.

Perry, Motty and Philip J. Reny (2002). "An Efficient Auction". In: *Econometrica* 70.3. doi: 10.1111/1468-0262.00322, pp. 1199–1212. URL: http://www.blackwell-synergy.com/doi/abs/10.1111/1468-0262.00322.

Roth, A.E. and M.A.O. Sotomayor (1990). *Two-sided matching: A study in game-theoretic modeling and analysis*. Cambridge University Press. ISBN: 052139015X.

Roth, Alvin E. (1982). "The Economics of Matching: Stability and Incentives". In: *Mathematics Of Operations Research* 7.4, pp. 617–628.

Satterthwaite, Mark Allen (1975). "Strategy-Proofness and Arrow's Conditions: Existence and Correspondence Theorems for Voting Procedures and Social Welfare Functions". In: *Journal of Economic Theory* 10, pp. 187–217.

Schwalbe, Ulrich and Paul Walker (2001). "Zermelo and the Early Theory of Game Theory". In: *Games and Economic Behavior* 34.1, pp. 123–137.

Yamato, Takehiko (1992). "On Nash implementation of Social Choice Correspondences". In: *Games and Economic Behavior*, pp. 484–492.

Zhong, Xiaohan, Na Cheng, and Yunfan He (2004). "Where Have All the Flowers Gone? A Game-Theoretic Analysis of the National College Entrance Exam Preference Submission Mechanisms". In: *China Economic Quarterly* 3.3, pp. 763–778.