Waves and Oscillations

Part B

Oscillations

Assuming a suitable solution

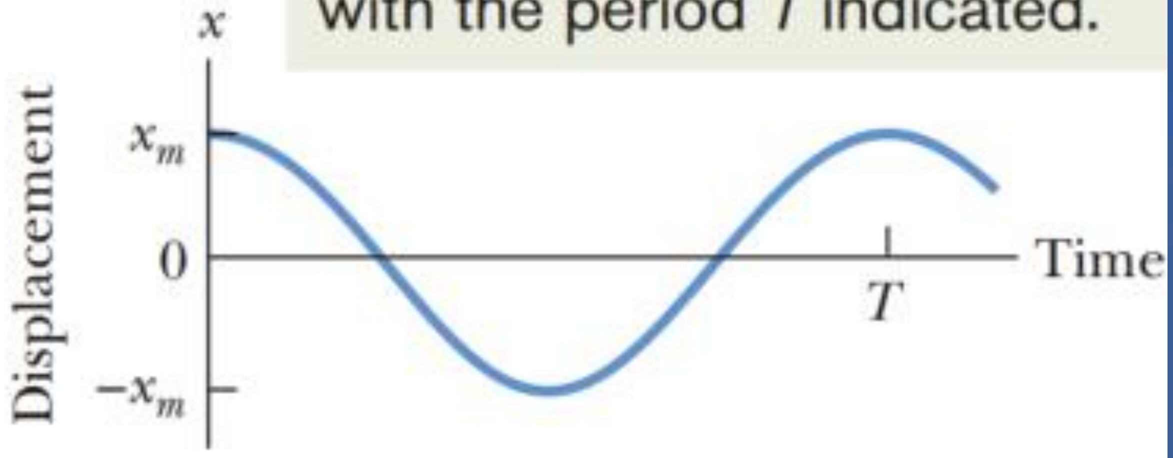
$$\overbrace{x(t)}^{\text{Displacement at time } t} = \underbrace{x_m}_{\text{Amplitude}} \cos\overbrace{(\omega t + \phi)}^{\text{Phase}}$$

Angular frequency (ω)

Time (t)

Phase constant or phase angle (ϕ)

This is a graph of the motion, with the period T indicated.



Position

$$x(t) = x_m \cos(\omega t + \phi)$$

$v(t) = \frac{dx(t)}{dt}$

Velocity

$$v(t) = -\omega x_m \sin(\omega t + \phi)$$

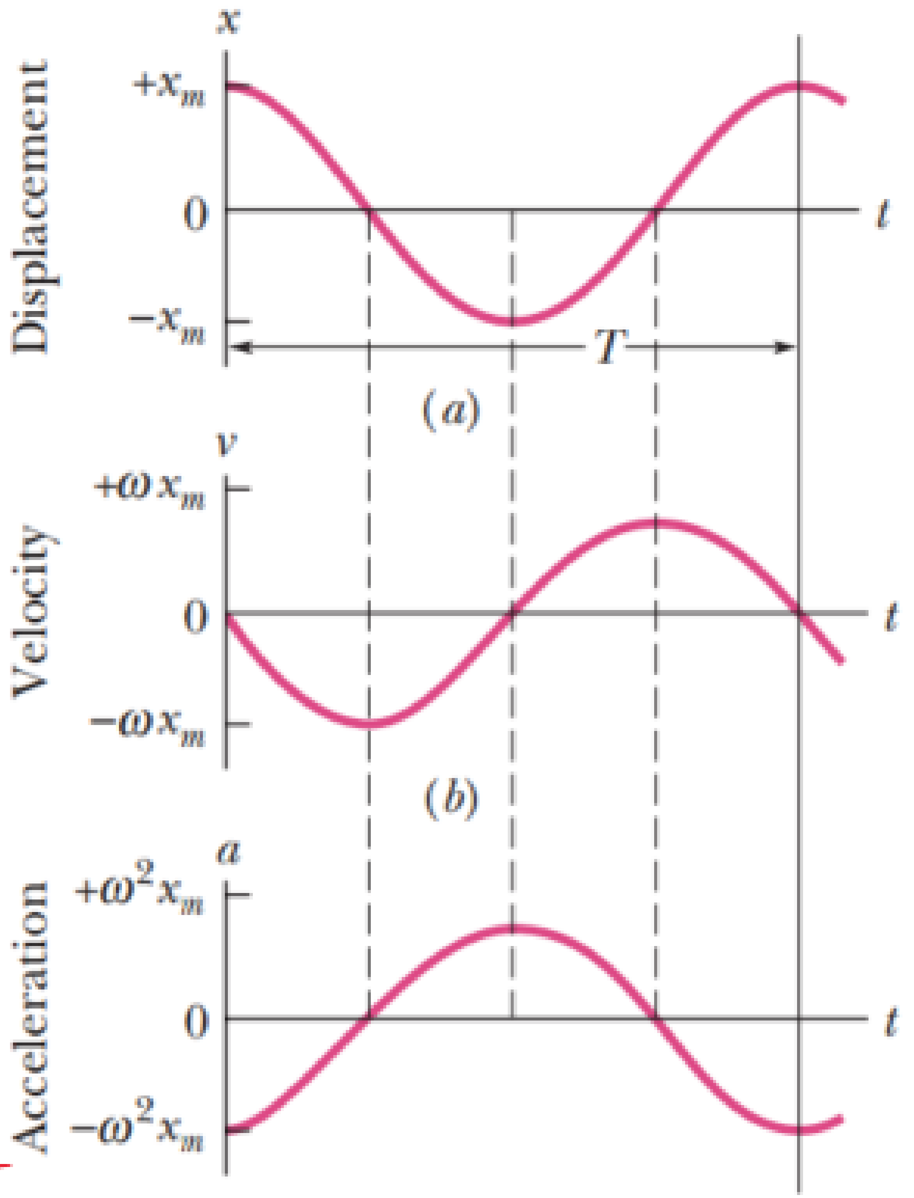
Acceleration

$a(t) = \frac{dv(t)}{dt} = \frac{d^2x(t)}{dt}$

$$a(t) = -\omega^2 x_m \cos(\omega t + \phi)$$

Oscillation signature

$$\boxed{a(t) = -\omega^2 x(t)}$$



Extrer
values
here
mean

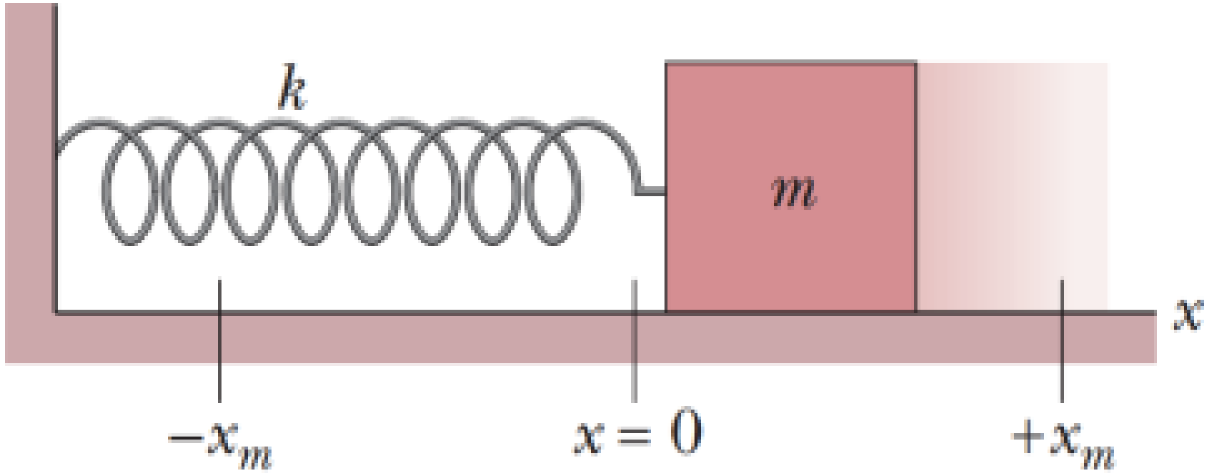
zero
values
here
and ..

extrer
values
here.

$$= ma = m(-\omega^2 x) = -(m\omega^2)x. \quad F = -kx,$$

quating the forms

$$k = m\omega^2$$



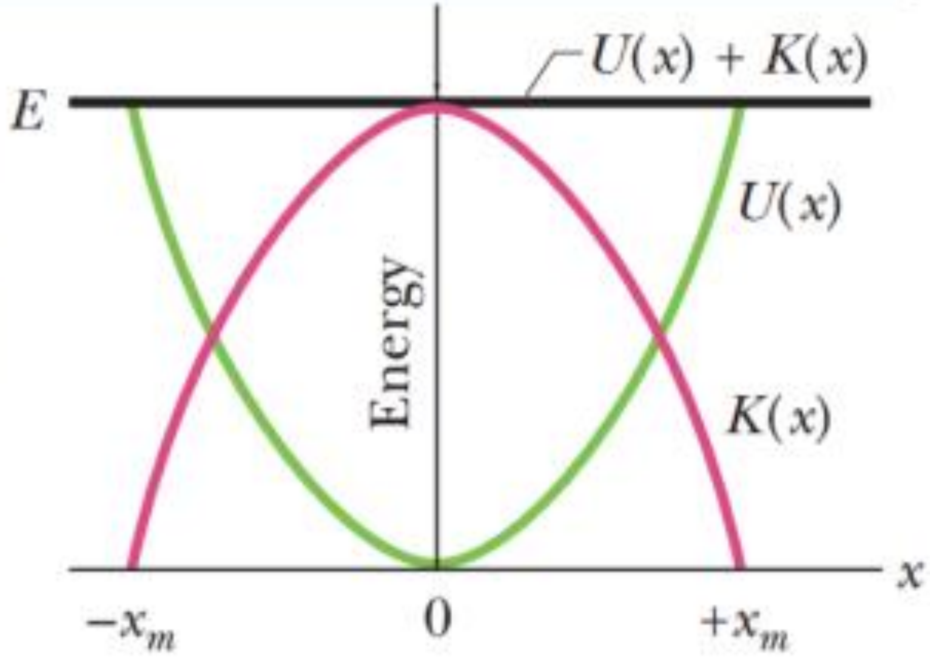
$$= \sqrt{\frac{k}{m}} \quad \text{(angular frequency).}$$

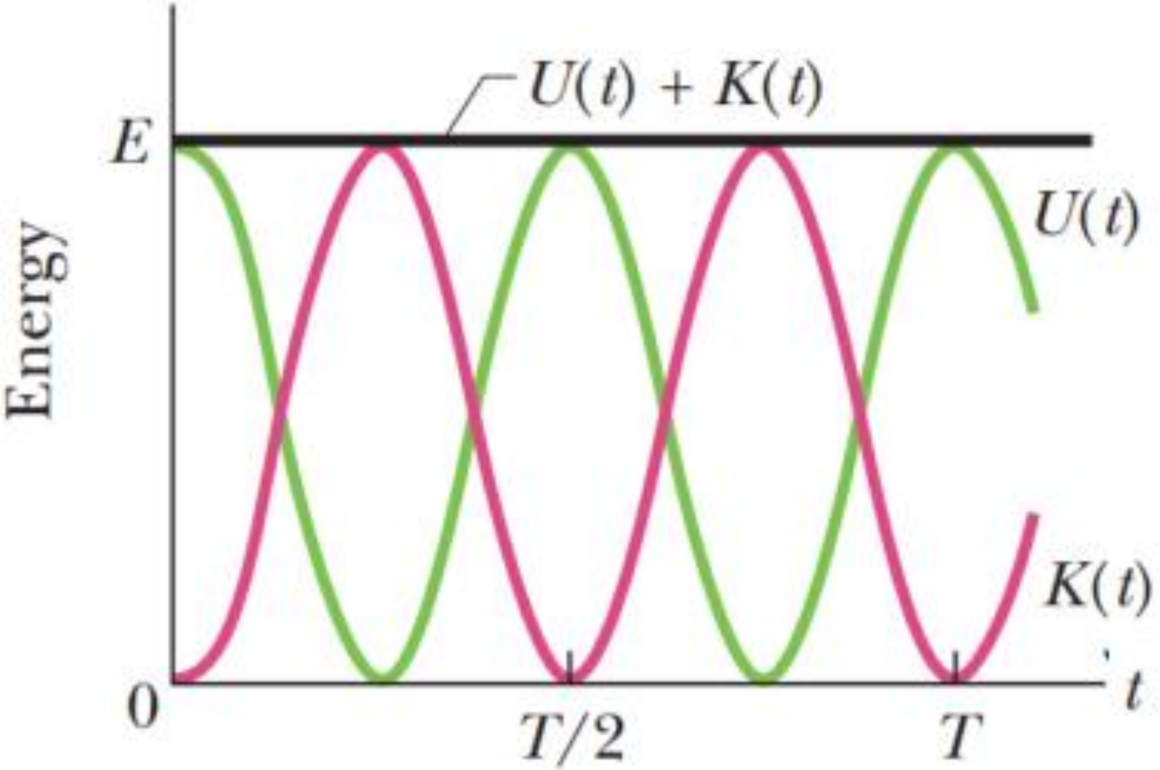
Natural oscillation frequency

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{(period).}$$

$$U(t) = \frac{1}{2}kx^2 = \frac{1}{2}kx_m^2 \cos^2(\omega t + \phi).$$

$$K(t) = \frac{1}{2}mv^2 = \frac{1}{2}kx_m^2 \sin^2(\omega t + \phi).$$

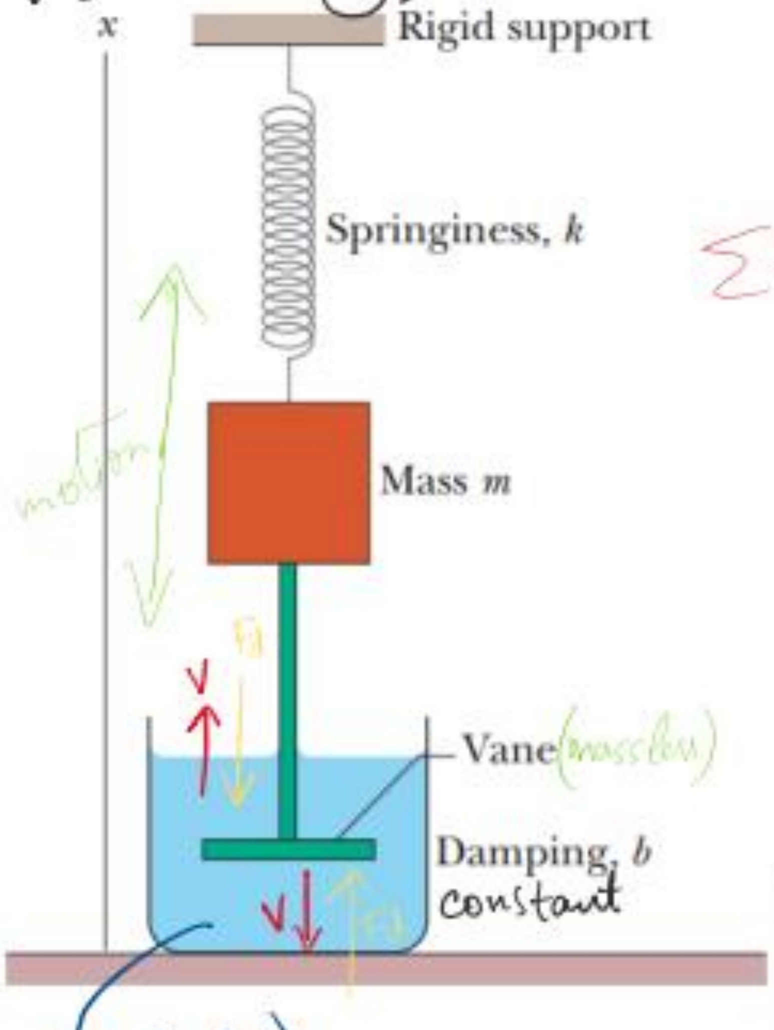




$$E = U + K = \frac{1}{2}kx_m^2.$$

Total energy remains constant

DAMPED OSCILLATIONS



damping force

$$F_d = -bv$$

velocity

damping constant (depends on fluid and vane properties)

damping force always opposes the velocity

$$\Sigma F = \underbrace{-bv}_{F_d} \underbrace{- kx}_{F_s} = ma.$$

using Newton's second law

rearranging

$$m \frac{d^2x}{dt^2} + b \frac{dx}{dt} + kx = 0.$$

Solution to this differential equation

[Because solving this differential equation is not part of our course.]

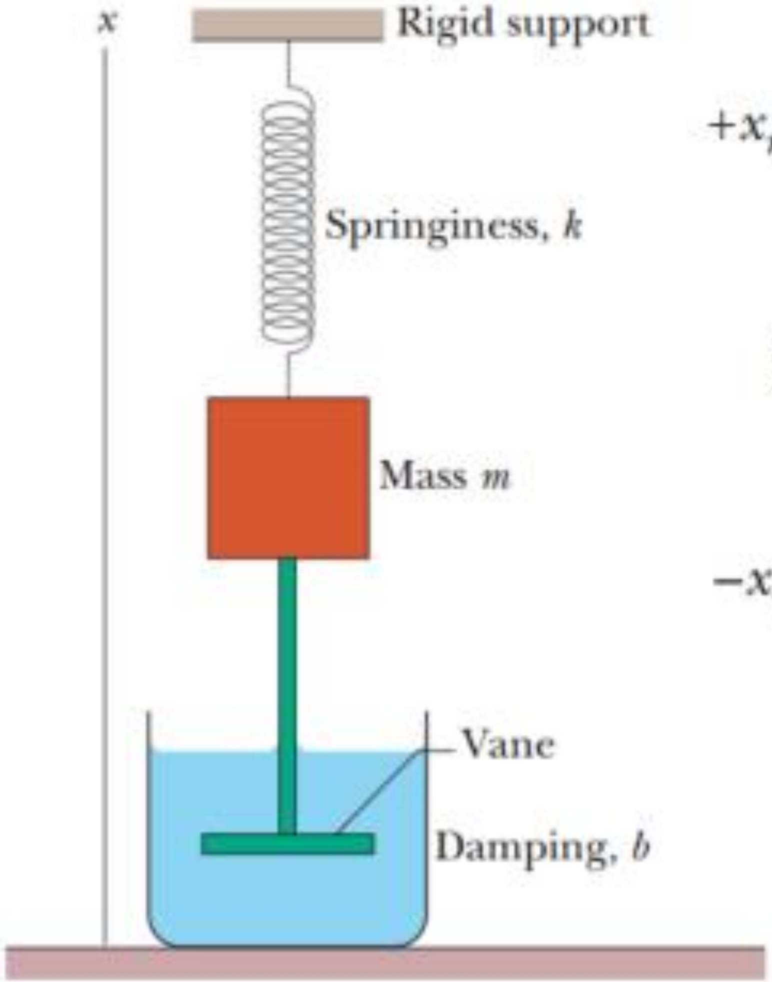
$$x(t) = x_m e^{-bt/2m} \cos(\omega' t + \phi),$$

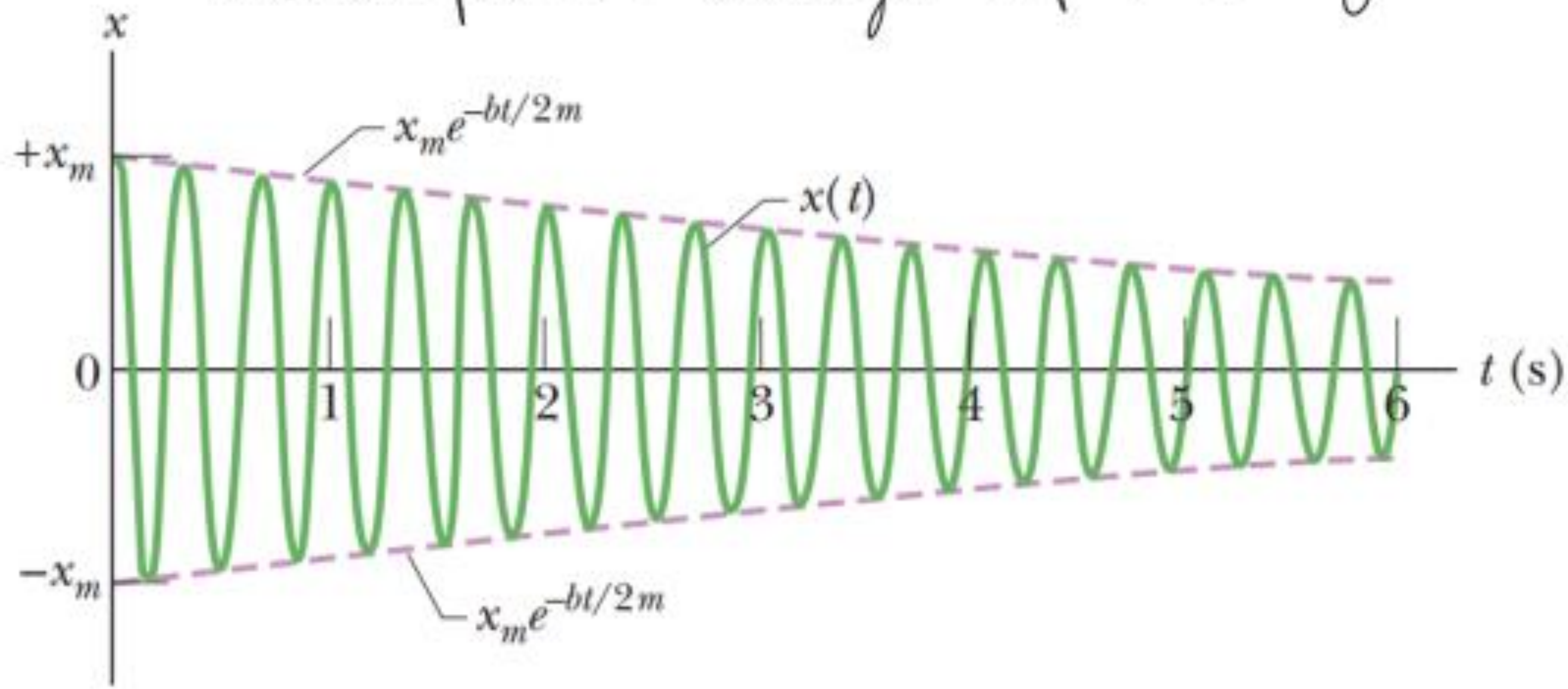
function of time

exponential factor

Angular frequency of damped oscillations

The amplitude decays exponentially

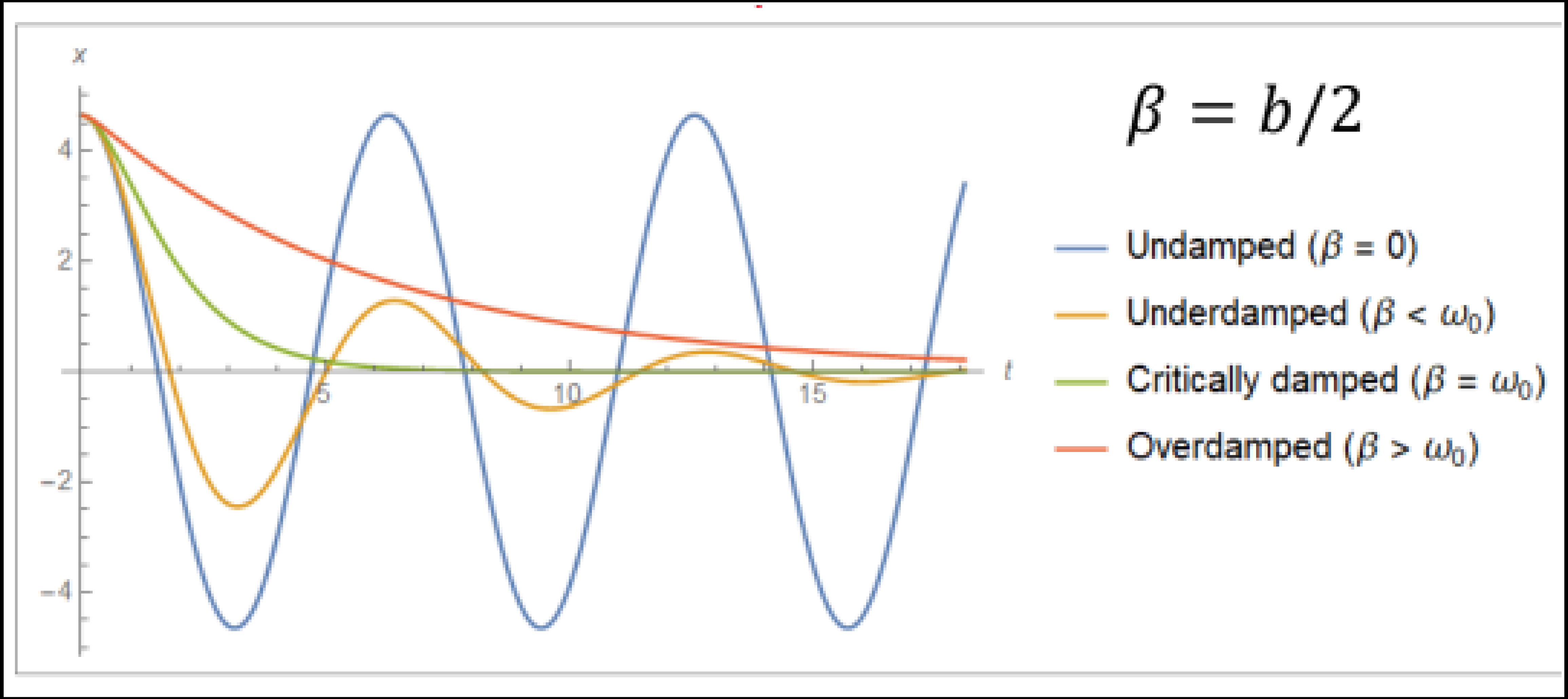




Solution to this differential equation

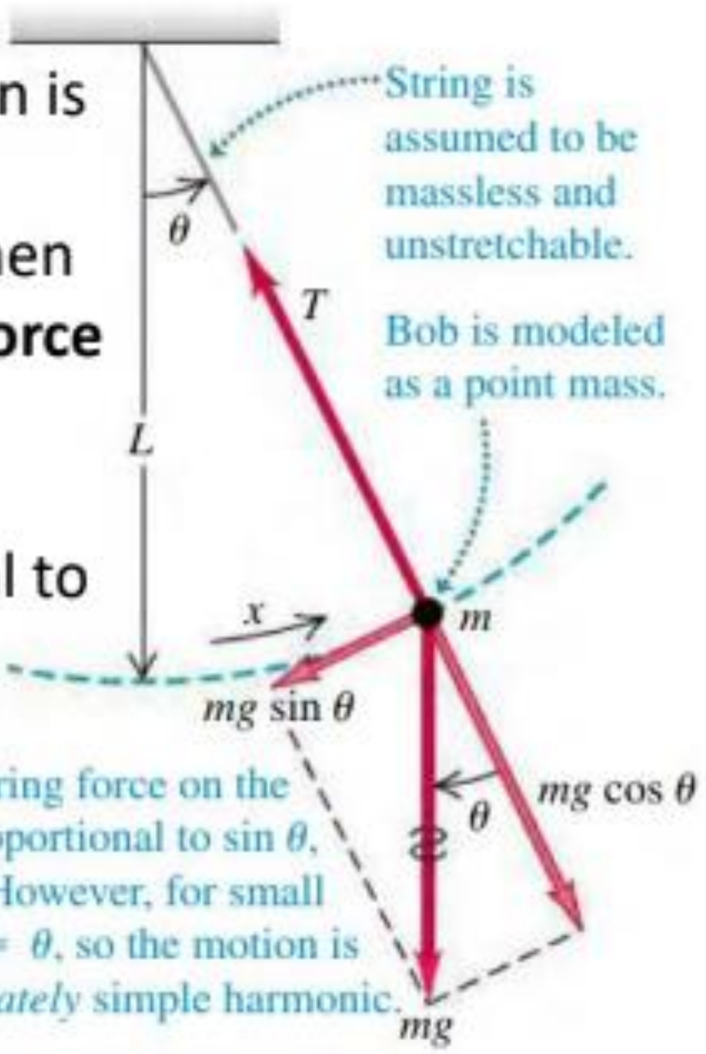
$$x(t) = x_m e^{-bt/2m} \cos(\omega' t + \phi),$$

Amplitude of Damped oscillations



x
4
2
−2
−4
t
5
10
15
$\beta = b/2$
Undamped (β = 0)
Underdamped ($\beta < \omega_0$)
Critically damped ($\beta = \omega_0$)
Overdamped ($\beta > \omega_0$)

If the motion is simple harmonic then **Restoring Force** must be directly proportional to x (or θ)



$$F_\theta = -mg\sin\theta$$

if angle θ is *smal*

$$F_\theta = -mg\theta$$

$$= -mg\frac{x}{L} = -\frac{mg}{L}x$$

Angular frequency of simple pendulum, small amplitude

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/L}{m}} = \sqrt{\frac{g}{L}}$$

Pendulum mass (cancels)

Acc due

Pe le

Period of simple pendulum, small amplitude

$$T = \frac{2\pi}{\omega} = \frac{1}{f} = 2\pi\sqrt{\frac{L}{g}}$$

Angular frequency

Frequency

Pendu length

Acc due

Useful Formulae

$$x = x_m \cos(\omega t + \phi) \quad \text{(displacement)}$$

$$v = -\omega x_m \sin(\omega t + \phi) \quad \text{(velocity)}$$

$$a = -\omega^2 x_m \cos(\omega t + \phi) \quad \text{(acceleration)}$$

$$\omega = \frac{2\pi}{T} = 2\pi f \quad \text{(angular frequency)}.$$

For any oscillatory object

$$\omega = \sqrt{\frac{k}{m}} \quad \text{(angular frequency)}$$

$$T = 2\pi\sqrt{\frac{m}{k}} \quad \text{(period)}.$$

Spring mass system

$$E(t) \approx \tfrac{1}{2}kx_m^2 e^{-bt/m}.$$

$$K = \tfrac{1}{2}mv^2$$

$$U = \tfrac{1}{2}kx^2$$

for spring

For damped oscillations

Waves

Wave function

(A mathematical function that can describe the path/trajectory of oscillations and waves)

Simplest wave model [in SHM (Transverse wave)]

$$y(x,t) = y_m \sin(kx - \omega t)$$

Wave function

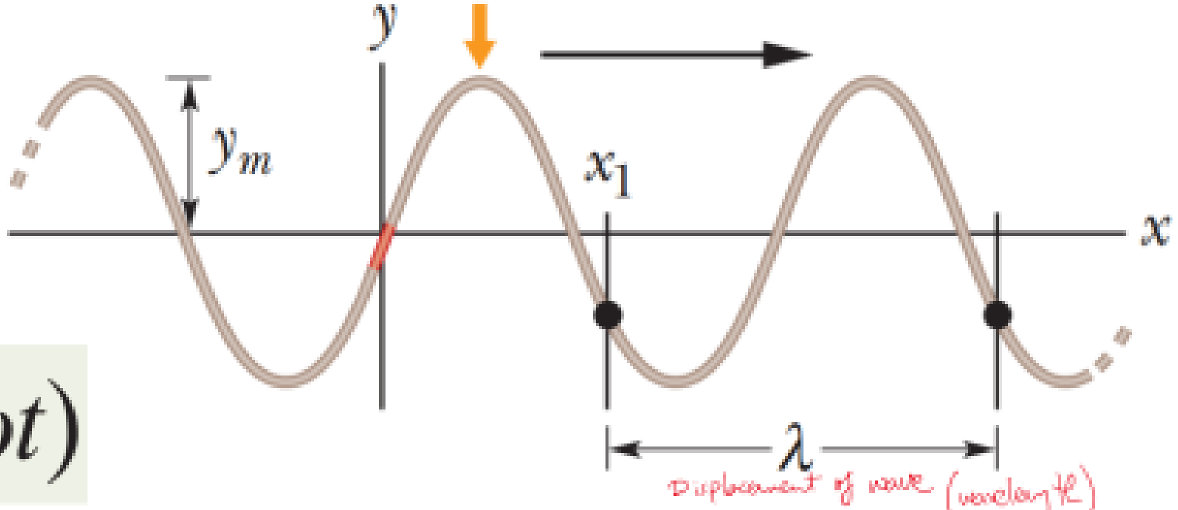
λ —— 2π

1 —— $\frac{2\pi}{\lambda}$

x —— $\frac{2\pi}{\lambda}x$

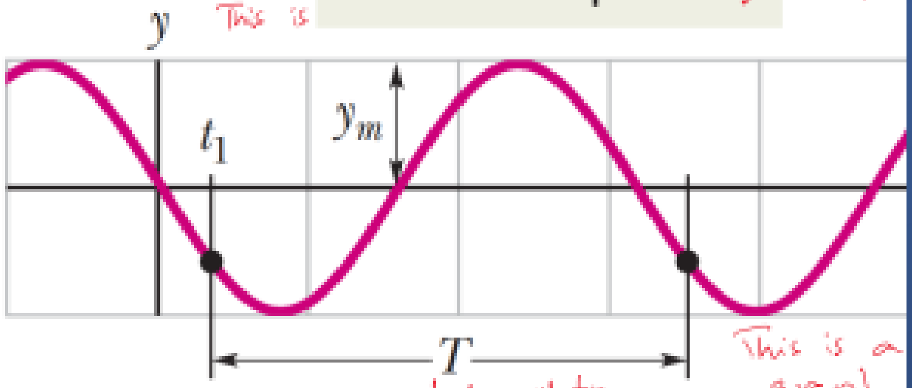
$$k = \frac{2\pi}{\lambda} \quad \text{(angular wave number).}$$

This is snapshot, other picture is not.



Displacement of wave (wavelength)

This is not a snapshot, other pict

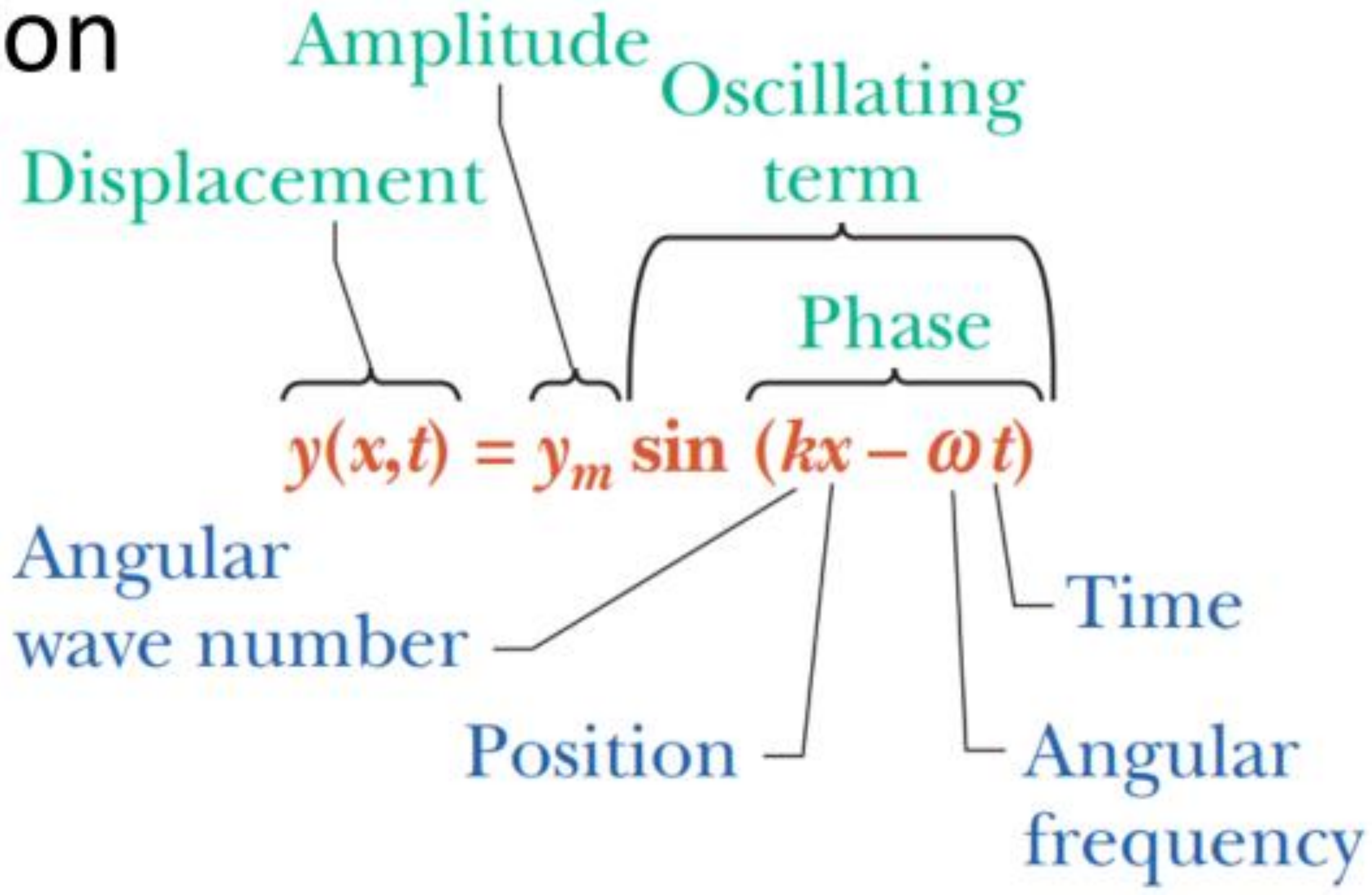


Period of oscillation (Time period)

This is a graph

compare it with $\omega = \frac{2\pi}{T}$

Wave function



Wave function

To take snapshot,

$$y(x,0) = y_m \sin kx.$$

fix a point in time $t=0$

To plot a graph,

fix a point in space $x=0$

$$y(0,t) = y_m \sin(-\omega t)$$
$$= -y_m \sin \omega t$$

The generalized wave function

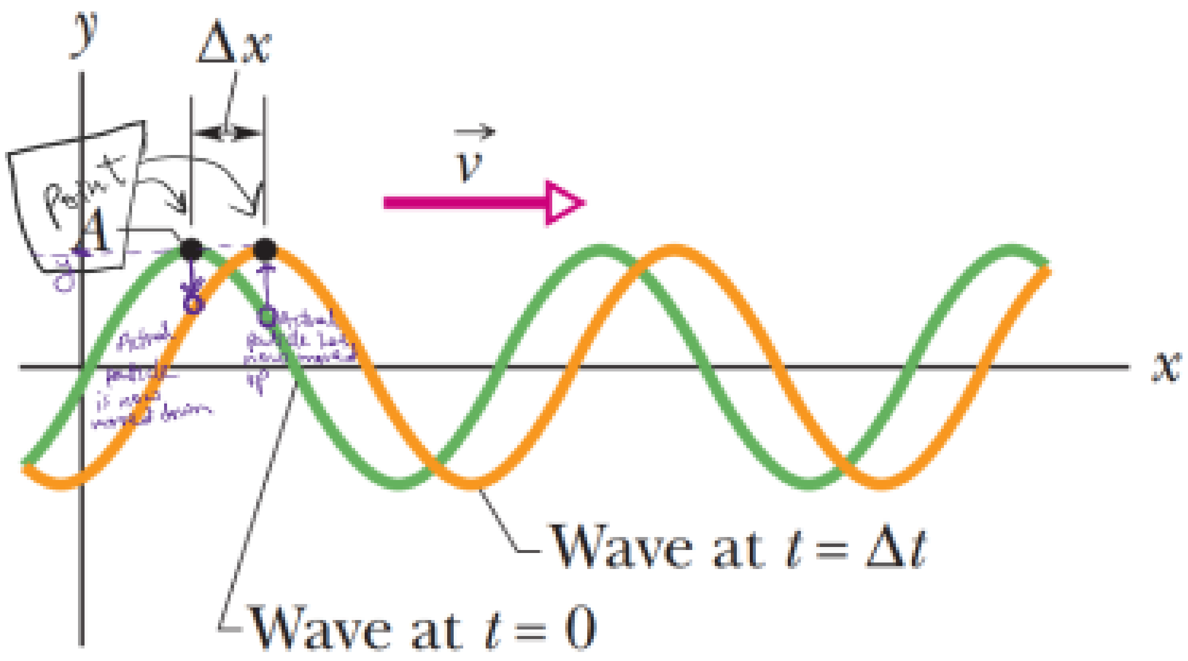
$$\boxed{y = y_m \sin(kx \pm \omega t + \phi).}$$

will add information for second wave

Wave Velocity

point A retains its displacement

$$kx - \omega t = \text{a constant.}$$



$$k\frac{dx}{dt} - \omega = 0$$

$$\frac{dx}{dt} = v = \frac{\omega}{k}.$$

$$\boxed{v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f} \quad \text{(wave speed)}$$

The **frequency** f of a wave is defined as $1/T$ and is related to the angular frequency ω by; $\boldsymbol{\omega = 2\pi f}$

We define the **period** of oscillation T of a wave to be the time any string element takes to move through one full oscillation.

The **wavelength** of a wave is the distance (parallel to the direction of the wave's travel) between repetitions of the shape of the wave (or *wave shape*).

The **amplitude** y_m of a wave is the magnitude of the maximum displacement of the elements from their equilibrium positions as the wave passes through them.

The **phase** of the wave is the *argument* $kx - \omega t$ of the sine. As the wave sweeps through a string element at a particular position x, the phase changes linearly with time t. This means that the sine also changes, oscillating between +1 and -1.

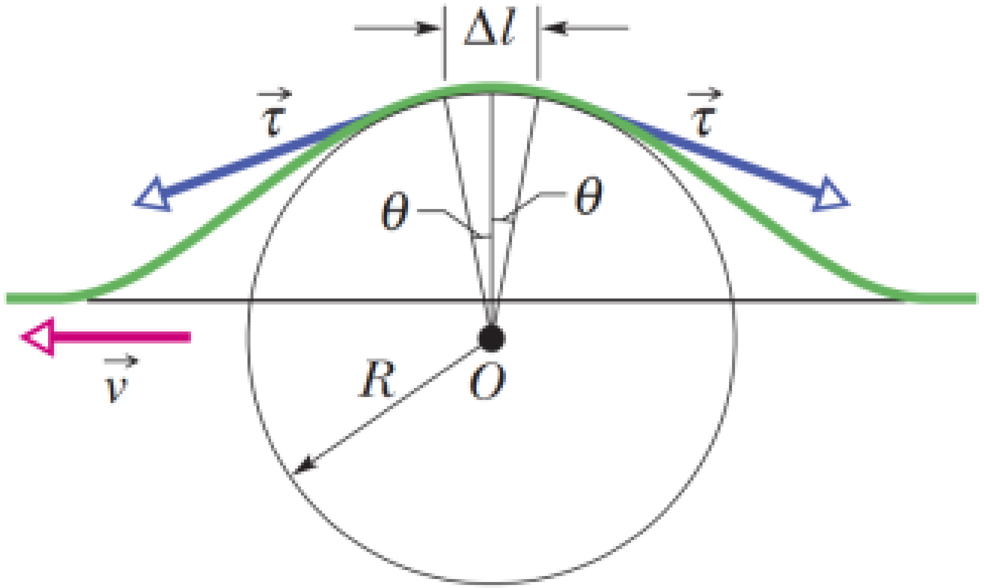
We call k the **angular wave number** of the wave; its SI unit is the radian per meter, or the inverse meter.

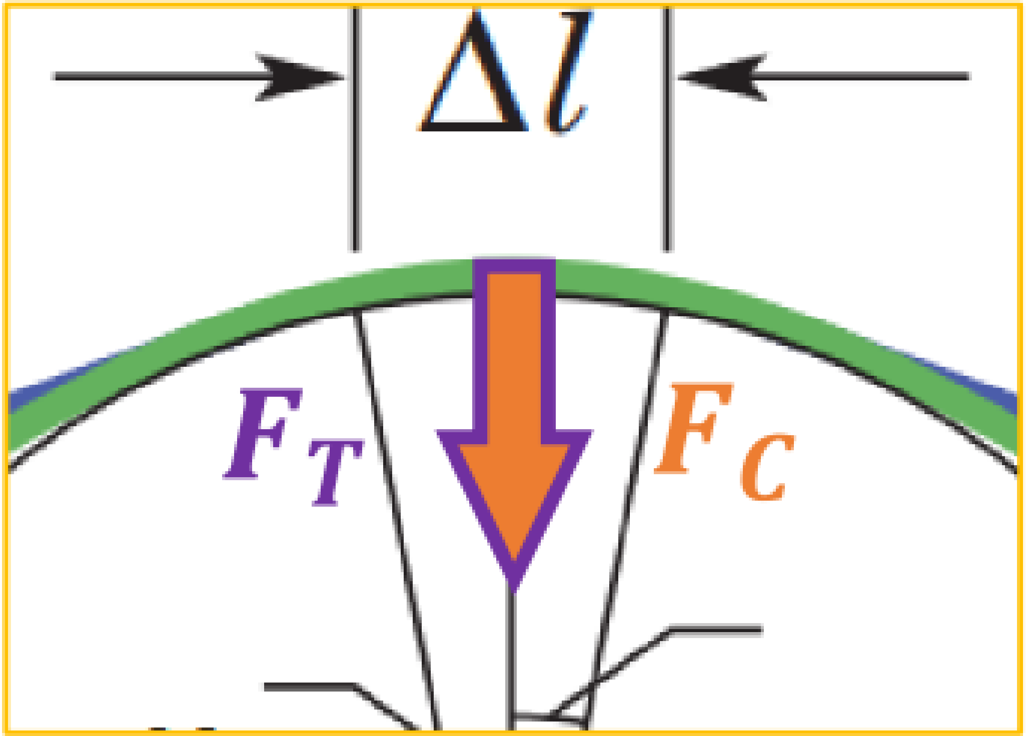
A phase constant φ in the wave function:

$\mathbf{y = y_m \sin(kx - \omega t + \varphi)}$**.**

The value of φ can be chosen so that the function gives some other displacement and slope at $x = 0$ when $t = 0$.

Wave in a string





$$F = 2(\tau \sin\theta) \approx \tau(2\theta) = \tau \frac{\Delta l}{R}$$

$$\frac{\tau \Delta l}{R} = (\mu \Delta l) \frac{v^2}{R}$$

$$\boxed{v = \sqrt{\frac{\tau}{\mu}}}$$

μ (dimension ML^{-1})
τ (dimension MLT^{-2})

Energy Transportation

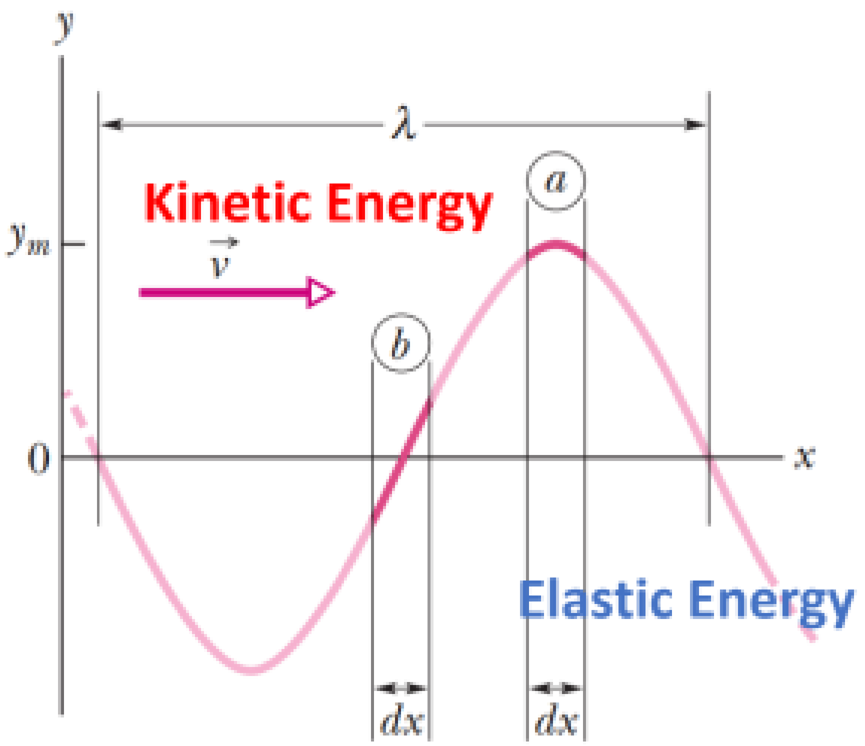
$$dK = \frac{1}{2}\, dm\, u^2$$

Using wave function $\boxed{y(x,t) = y_m \sin(kx - \omega t)}$

$$u = \frac{\partial y}{\partial t} = -\omega y_m \cos(kx - \omega t).$$

$$dK = \tfrac{1}{2}(\mu\, dx)(-\omega y_m)^2 \cos^2(kx - \omega t).$$

$$\frac{dK}{dt} = \tfrac{1}{2}\mu v \omega^2 y_m^2 \cos^2(kx - \omega t).$$

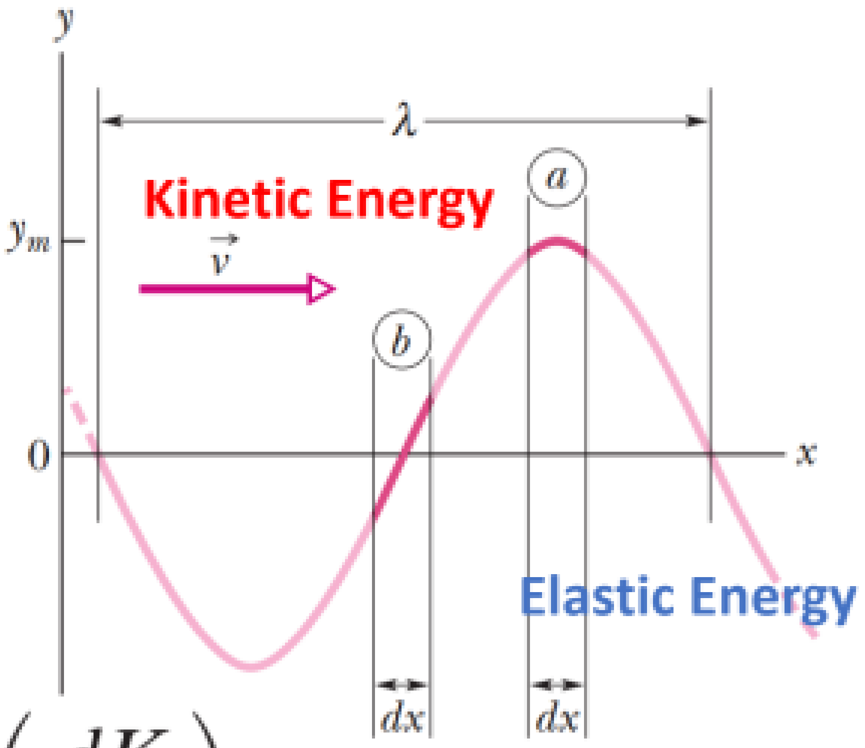


$$\left(\frac{dK}{dt}\right)_{\text{avg}} = \tfrac{1}{4}\mu v \omega^2 y_m^2.$$

Energy Transportation

$$dK = \frac{1}{2}\, dm\, u^2$$

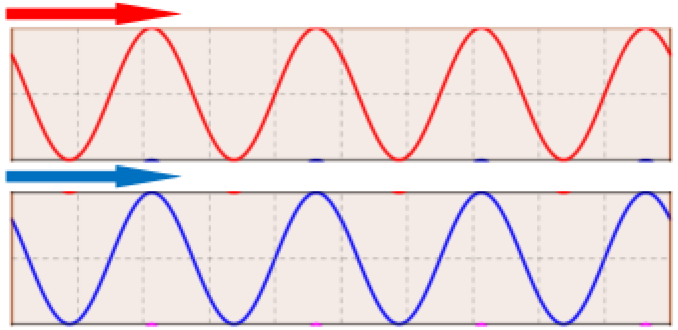
Using wave function $\boxed{y(x,t) = y_m \sin(kx - \omega t)}$



$$\left(\frac{dK}{dt}\right)_{\text{avg}} = \frac{1}{4}\mu v \omega^2 y_m^2.$$

$$P_{\text{avg}} = 2\left(\frac{dK}{dt}\right)_{\text{avg}}$$

$$\boxed{P_{\text{avg}} = \frac{1}{2}\mu v \omega^2 y_m^2 \quad \text{(average power).}}$$



Wave Interference

$$y'(x,t) = 2y_m \sin(kx - \omega t) \qquad (\phi = 0).$$

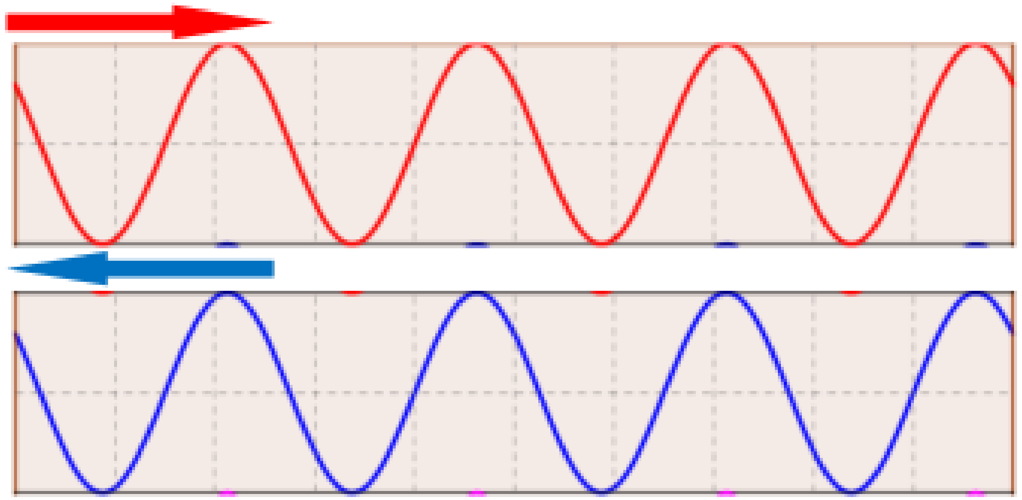
$$y'(x,t) = 0 \qquad (\phi = \pi\,\text{rad}).$$

$$\overbrace{y'(x,t)}^{\text{Displacement}} = \underbrace{[2y_m \cos \tfrac{1}{2}\phi]}_{\substack{\text{Magnitude}\\ \text{gives}\\ \text{amplitude}}} \underbrace{\sin(kx - \omega t + \tfrac{1}{2}\phi)}_{\substack{\text{Oscillating}\\ \text{term}}}$$

Wave Interference

$$y_1(x, t) = y_m \sin(kx - \omega t)$$

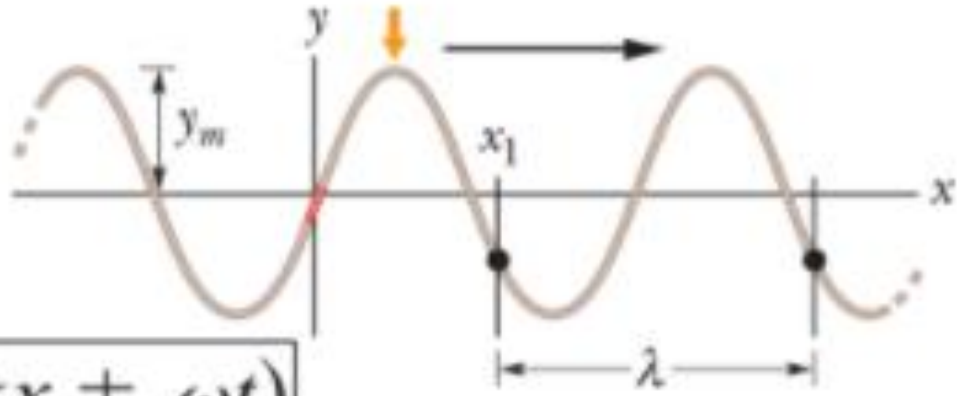
$$y_2(x, t) = y_m \sin(kx + \omega t).$$



$$\boxed{y'(x, t) = [2y_m \sin kx] \cos \omega t.}$$

Wave

$$y = y_m \sin(kx \pm \omega t + \phi).$$



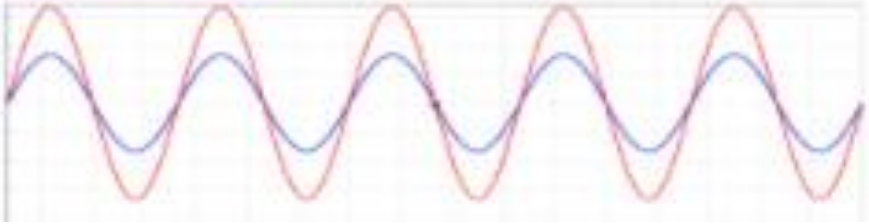
$$y(x,t) = h(kx \pm \omega t)$$

$$P_{\text{avg}} = \tfrac{1}{2}\mu v \omega^2 y_m^2 \quad \text{(average power)}.$$

$$v = \frac{\omega}{k} = \frac{\lambda}{T} = \lambda f$$

$$y'(x,t) = [2y_m \cos \tfrac{1}{2}\phi] \sin(kx - \omega t + \tfrac{1}{2}\phi).$$

$$y'(x,t) = [2y_m \sin kx] \cos \omega t.$$

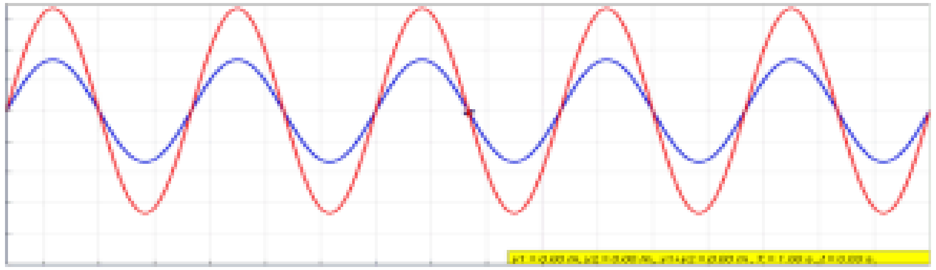


Nodes

where the sinusoidal part of amplitude is minimum

$\sin kx = 0$

Points of Zero Amplitude



$$kx = n\pi, \qquad \text{for } n = 0, 1, 2, \ldots.$$

$\boxed{k = \frac{2\pi}{\lambda}}$

$$x = n\frac{\lambda}{2}, \qquad \text{for } n = 0, 1, 2, 3, \ldots \qquad \text{(nodes)},$$

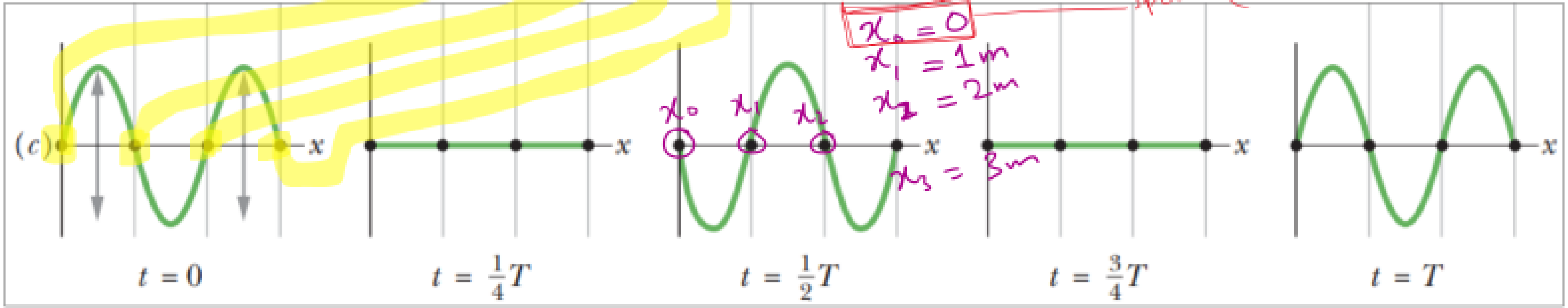
If $\lambda = 2m$ then

$\boxed{x_0 = 0}$ special (minimum) condition

$x_1 = 1m$

$x_2 = 2m$

$x_3 = 3m$



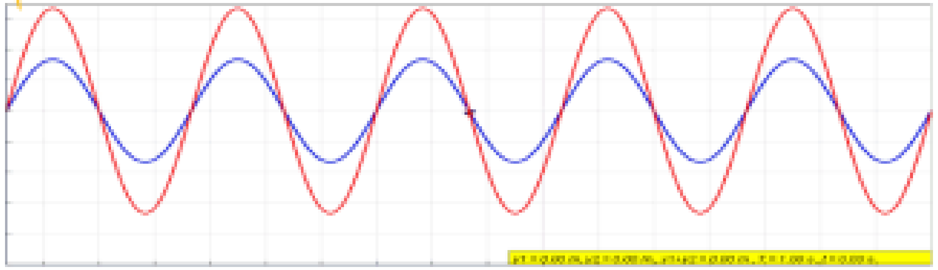
Antinodes

where the sinusoidal part of standing wave amplitude is 1

$\sin kx = 1$

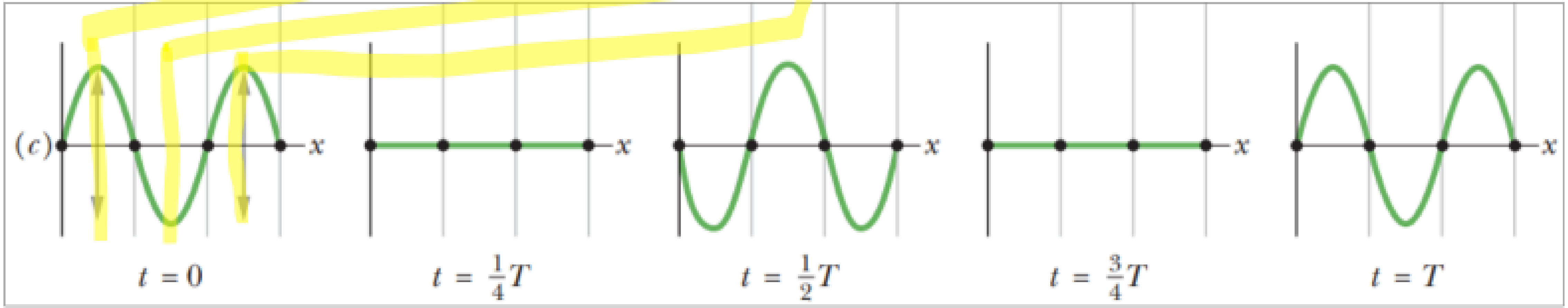
Points of Maximum Amplitude

$$kx = (n + \tfrac{1}{2})\pi, \qquad \text{for } n = 0, 1, 2, \ldots$$



$\boxed{k = \frac{2\pi}{\lambda}}$

$$x = \left(n + \frac{1}{2}\right)\frac{\lambda}{2}, \qquad \text{for } n = 0, 1, 2, \ldots \qquad \text{(antinodes)},$$



Wave Reflections and Harmonics

Conditions to produce Harmonics:

$L = n\left(\frac{\lambda}{2}\right)$

$$\boxed{\lambda = \frac{2L}{n}}, \qquad \text{for } n = 1, 2, 3, \ldots.$$

this gives all the allowed wavelengths inside a section of length L

for a section of length 0.42 m

$\lambda = \frac{2(0.42)}{1} = 0.84\,m$ — wave of wavelength 0.84 m can exist as first harmonic

$\lambda = \frac{2(0.42)}{2} = 0.42\,m$ — wave of wavelength 0.42 m can exist as second harmonic

$\lambda = \frac{2(0.42)}{3} = 0.28\,m$ — wave of wavelength 0.28 m can exist as third harmonic

$$f = \frac{v_{\text{wave velocity}}}{\lambda} = n\frac{v}{2L}, \qquad \text{for } n = 1, 2, 3, \ldots.$$

allowed frequencies inside a section.