F

ABSOLUTE VALUE

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F.1 DEFINITION The ***absolute value*** or ***magnitude*** of a real number a is denoted by $|a|$ and is defined by

$$|a| = \begin{cases} a & \text{if} \quad a \geq 0 \\ -a & \text{if} \quad a < 0 \end{cases}$$

▶ Example 1

$$|5| = 5 \qquad \left|-\tfrac{4}{7}\right| = -\left(-\tfrac{4}{7}\right) = \tfrac{4}{7} \qquad |0| = 0$$

Since $5 > 0$ Since $-\frac{4}{7} < 0$ Since $0 \geq 0$ ◀

Note that the effect of taking the absolute value of a number is to strip away the minus sign if the number is negative and to leave the number unchanged if it is nonnegative.

▶ Example 2 Solve $|x - 3| = 4$.

Solution. Depending on whether $x - 3$ is positive or negative, the equation $|x - 3| = 4$ can be written as

$$x - 3 = 4 \quad \text{or} \quad x - 3 = -4$$

Solving these two equations gives $x = 7$ and $x = -1$. ◀

▶ Example 3 Solve $|3x - 2| = |5x + 4|$.

Solution. Because two numbers with the same absolute value are either equal or differ in sign, the given equation will be satisfied if either

$$3x - 2 = 5x + 4 \quad \text{or} \quad 3x - 2 = -(5x + 4)$$

Solving the first equation yields $x = -3$ and solving the second yields $x = -\frac{1}{4}$; thus, the given equation has the solutions $x = -3$ and $x = -\frac{1}{4}$. ◀

RELATIONSHIP BETWEEN SQUARE ROOTS AND ABSOLUTE VALUES

Recall from algebra that a number is called a ***square root*** of a if its square is a. Recall also that every positive real number has two square roots, one positive and one negative; the positive square root is denoted by $\sqrt{a}$ and the negative square root by $-\sqrt{a}$. For example, the positive square root of 9 is $\sqrt{9} = 3$, and the negative square root of 9 is $-\sqrt{9} = -3$.

REMARK Readers who may have been taught to write $\sqrt{9}$ as ± 3 should stop doing so, since it is incorrect.

It is a common error to replace $\sqrt{a^2}$ by a. Although this is correct when a is nonnegative, it is false for negative a. For example, if $a = -4$, then

$$\sqrt{a^2} = \sqrt{(-4)^2} = \sqrt{16} = 4 \neq a$$

A result that is correct for all a is given in the following theorem.

F.2 THEOREM *For any real number a,*

$$\sqrt{a^2} = |a|$$

PROOF Since $a^2 = (+a)^2 = (-a)^2$, the numbers $+a$ and $-a$ are square roots of a^2. If $a \geq 0$, then $+a$ is the nonnegative square root of a^2, and if $a < 0$, then $-a$ is the nonnegative square root of a^2. Since $\sqrt{a^2}$ denotes the nonnegative square root of a^2, it follows that

$$\begin{aligned} \sqrt{a^2} &= +a \quad \text{if} \quad a \geq 0 \\ \sqrt{a^2} &= -a \quad \text{if} \quad a < 0 \end{aligned}$$

That is, $\sqrt{a^2} = |a|$. ■

PROPERTIES OF ABSOLUTE VALUE

F.3 THEOREM *If a and b are real numbers, then*

(*a*) $|-a| = |a|$ A number and its negative have the same absolute value.

(*b*) $|ab| = |a||b|$ The absolute value of a product is the product of the absolute values.

(*c*) $|a/b| = |a|/|b|$ The absolute value of a ratio is the ratio of the absolute values.

We will prove parts (a) and (b) only.

PROOF (*a*) From Theorem F.2,

$$|-a| = \sqrt{(-a)^2} = \sqrt{a^2} = |a|$$

PROOF (*b*) From Theorem F.2 and a basic property of square roots,

$$|ab| = \sqrt{(ab)^2} = \sqrt{a^2b^2} = \sqrt{a^2}\sqrt{b^2} = |a||b| \quad ■$$

In part (c) of Theorem F.3 we did not explicitly state that $b \neq 0$, but this must be so since division by zero is not allowed. Whenever divisions occur in this text, it will be assumed that the denominator is not zero, even if we do not mention it explicitly.

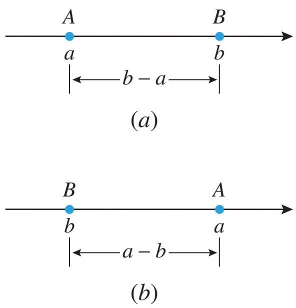
The result in part (b) of Theorem F.3 can be extended to three or more factors. More precisely, for any n real numbers, $a_1, a_2, \ldots, a_n$, it follows that

$$|a_1 a_2 \cdots a_n| = |a_1||a_2| \cdots |a_n| \tag{1}$$

In the special case where $a_1, a_2, \ldots, a_n$ have the same value, a, it follows from (1) that

$$|a^n| = |a|^n \tag{2}$$

GEOMETRIC INTERPRETATION OF ABSOLUTE VALUE



▲ **Figure F.1**

The notion of absolute value arises naturally in distance problems. For example, suppose that A and B are points on a coordinate line that have coordinates a and b, respectively. Depending on the relative positions of the points, the distance d between them will be $b - a$ or $a - b$ (Figure F.1). In either case, the distance can be written as $d = |b - a|$, so we have the following result.

> **F.4 THEOREM** ***(Distance Formula)*** *If A and B are points on a coordinate line with coordinates a and b, respectively, then the distance d between A and B is $d = |b - a|$.*

This theorem provides useful geometric interpretations of some common mathematical expressions:

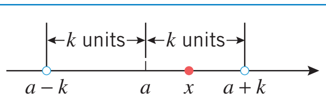
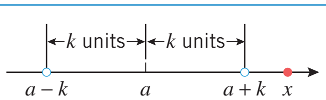
EXPRESSION	GEOMETRIC INTERPRETATION ON A COORDINATE LINE
$\|x - a\|$	The distance between x and a
$\|x + a\|$	The distance between x and $-a$ (since $\|x + a\| = \|x - (-a)\|$)
$\|x\|$	The distance between x and the origin (since $\|x\| = \|x - 0\|$)

INEQUALITIES WITH ABSOLUTE VALUES

Inequalities of the form $|x - a| < k$ and $|x - a| > k$ arise so often that we have summarized the key facts about them in Table F.1.

The statements in Table F.1 remain true if $<$ is replaced by $\leq$ and $>$ by $\geq$, and if the open dots are replaced by closed dots in the illustrations.

Table F.1

INEQUALITY $(k > 0)$	GEOMETRIC INTERPRETATION	FIGURE	ALTERNATIVE FORMS OF THE INEQUALITY
$\|x - a\| < k$	x is within k units of a.		$-k < x - a < k$ $a - k < x < a + k$
$\|x - a\| > k$	x is more than k units away from a.		$x - a < -k$ or $x - a > k$ $x < a - k$ or $x > a + k$

▶ **Example 4** Solve

(a) $|x - 3| < 4$ (b) $|x + 4| \geq 2$ (c) $\dfrac{1}{|2x - 3|} > 5$

Solution (a). The inequality $|x - 3| < 4$ can be rewritten as

$$-4 < x - 3 < 4$$

Adding 3 throughout yields

$$-1 < x < 7$$

▲ **Figure F.2**

which can be written in interval notation as $(-1, 7)$. Observe that this solution set consists of all x that are within 4 units of 3 on a number line (Figure F.2), which is consistent with Table F.1.

Solution (b). The inequality $|x+4| \geq 2$ will be satisfied if

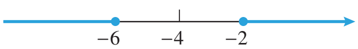
$$x + 4 \leq -2 \quad \text{or} \quad x + 4 \geq 2$$

Solving for x in the two cases yields

$$x \leq -6 \quad \text{or} \quad x \geq -2$$

which can be expressed in interval notation as

$$(-\infty, -6] \cup [-2, +\infty)$$



▲ **Figure F.3**

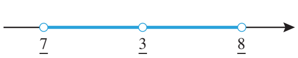
Observe that the solution set consists of all x that are at least 2 units away from -4 on a number line (Figure F.3), which is consistent with Table F.1 and the remark that accompanies it.

Solution (c). Observe first that $x = \frac{3}{2}$ results in a division by zero, so this value of x cannot be in the solution set. Putting this aside for the moment, we will begin by taking reciprocals on both sides and reversing the sense of the inequality in accordance with Theorem E.1(e) of Appendix E; then we will use Theorem F.3 to rewrite the inequality $1/|2x-3| > 5$ in a more familiar form:

$$\begin{aligned}
|2x - 3| &< \tfrac{1}{5} \\
|2|\,\left|x - \tfrac{3}{2}\right| &< \tfrac{1}{5} && \boxed{\text{Theorem F.3}(b)} \\
\left|x - \tfrac{3}{2}\right| &< \tfrac{1}{10} && \boxed{\text{We multiplied both sides by } 1/|2| = 1/2.} \\
-\tfrac{1}{10} < x - \tfrac{3}{2} &< \tfrac{1}{10} && \boxed{\text{Table F.1}} \\
\tfrac{7}{5} < x &< \tfrac{8}{5} && \boxed{\text{We added } 3/2 \text{ throughout.}}
\end{aligned}$$

As noted earlier, we must eliminate $x = \frac{3}{2}$ to avoid a division by zero, so the solution set is

$$\tfrac{7}{5} < x < \tfrac{3}{2} \quad \text{or} \quad \tfrac{3}{2} < x < \tfrac{8}{5}$$



▲ **Figure F.4**

which can be expressed in interval notation as $\left(\frac{7}{5}, \frac{3}{2}\right) \cup \left(\frac{3}{2}, \frac{8}{5}\right)$. (See Figure F.4.) ◀

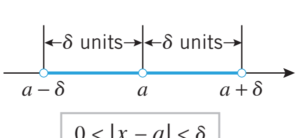
AN INEQUALITY FROM CALCULUS

One of the most important inequalities in calculus is

$$0 < |x - a| < \delta \tag{3}$$

where δ (Greek "delta") is a positive real number. This is equivalent to the two inequalities

$$0 < |x - a| \quad \text{and} \quad |x - a| < \delta$$



▲ **Figure F.5**

the first of which is satisfied by all x except $x = a$, and the second of which is satisfied by all x that are within δ units of a on a coordinate line. Combining these two restrictions, we conclude that the solution set of (3) consists of all x in the interval $(a - \delta, a + \delta)$ except $x = a$ (Figure F.5). Stated another way, the solution set of (3) is

$$(a - \delta, a) \cup (a, a + \delta) \tag{4}$$

THE TRIANGLE INEQUALITY

It is *not* generally true that $|a + b| = |a| + |b|$. For example, if $a = 1$ and $b = -1$, then $|a + b| = 0$, whereas $|a| + |b| = 2$. It is true, however, that *the absolute value of a sum* *is always less than or equal to the sum of the absolute values*. This is the content of the following useful theorem, called the ***triangle inequality***.

The name "triangle inequality" arises from a geometric interpretation of the inequality that can be made when a and b are complex numbers. A more detailed explanation is outside the scope of this text.

F.5 THEOREM (*Triangle Inequality*) *If a and b are any real numbers, then*

$$|a+b| \le |a| + |b| \tag{5}$$

PROOF Observe first that a satisfies the inequality

$$-|a| \le a \le |a|$$

because either $a = |a|$ or $a = -|a|$, depending on the sign of a. The corresponding inequality for b is

$$-|b| \le b \le |b|$$

Adding the two inequalities we obtain

$$-(|a|+|b|) \le a + b \le (|a|+|b|) \tag{6}$$

Let us now consider the cases $a+b \ge 0$ and $a+b < 0$ separately. In the first case, $a+b = |a+b|$, so the right-hand inequality in (6) yields the triangle inequality (5). In the second case, $a+b = -|a+b|$, so the left-hand inequality in (6) can be written as

$$-(|a|+|b|) \le -|a+b|$$

which yields the triangle inequality (5) on multiplying by -1. ■

EXERCISE SET F

1. Compute $|x|$ if
   (a) $x = 7$ (b) $x = -\sqrt{2}$
   (c) $x = k^2$ (d) $x = -k^2$.
2. Rewrite $\sqrt{(x-6)^2}$ without using a square root or absolute value sign.

3–10 Find all values of x for which the given statement is true. ■

3. $|x-3| = 3 - x$
4. $|x+2| = x+2$
5. $|x^2+9| = x^2+9$
6. $|x^2+5x| = x^2+5x$
7. $|3x^2+2x| = x|3x+2|$
8. $|6-2x| = 2|x-3|$
9. $\sqrt{(x+5)^2} = x+5$
10. $\sqrt{(3x-2)^2} = 2-3x$
11. Verify $\sqrt{a^2} = |a|$ for $a = 7$ and $a = -7$.
12. Verify the inequalities $-|a| \le a \le |a|$ for $a = 2$ and for $a = -5$.
13. Let A and B be points with coordinates a and b. In each part find the distance between A and B.
    (a) $a = 9,\ b = 7$ (b) $a = 2,\ b = 3$
    (c) $a = -8,\ b = 6$ (d) $a = \sqrt{2},\ b = -3$
    (e) $a = -11,\ b = -4$ (f) $a = 0,\ b = -5$
14. Is the equality $\sqrt{a^4} = a^2$ valid for all values of a? Explain.
15. Let A and B be points with coordinates a and b. In each part, use the given information to find b.
    (a) $a = -3$, B is to the left of A, and $|b-a| = 6$.
    (b) $a = -2$, B is to the right of A, and $|b-a| = 9$.
    (c) $a = 5$, $|b-a| = 7$, and $b > 0$.
16. Let E and F be points with coordinates e and f. In each part, determine whether E is to the left or to the right of F on a coordinate line.
    (a) $f - e = 4$ (b) $e - f = 4$
    (c) $f - e = -6$ (d) $e - f = -7$

17–24 Solve for x. ■

17. $|6x-2| = 7$
18. $|3+2x| = 11$
19. $|6x-7| = |3+2x|$
20. $|4x+5| = |8x-3|$
21. $|9x| - 11 = x$
22. $2x - 7 = |x+1|$
23. $\left|\dfrac{x+5}{2-x}\right| = 6$
24. $\left|\dfrac{x-3}{x+4}\right| = 5$

25–36 Solve for x and express the solution in terms of intervals. ■

25. $|x+6| < 3$
26. $|7-x| \le 5$
27. $|2x-3| \le 6$

28. $|3x+1|<4$ **29.** $|x+2|>1$ **30.** $\left|\frac{1}{2}x-1\right|\geq 2$

31. $|5-2x|\geq 4$ **32.** $|7x+1|>3$ **33.** $\dfrac{1}{|x-1|}<2$

34. $\dfrac{1}{|3x+1|}\geq 5$ **35.** $\dfrac{3}{|2x-1|}\geq 4$

36. $\dfrac{2}{|x+3|}<1$

37. For which values of x is $\sqrt{(x^2-5x+6)^2}=x^2-5x+6$?

38. Solve $3\leq|x-2|\leq 7$ for x.

39. Solve $|x-3|^2-4|x-3|=12$ for x. [*Hint:* Begin by letting $u=|x-3|$.]

40. Verify the triangle inequality $|a+b|\leq|a|+|b|$ (Theorem F.5) for
(a) $a=3,\ b=4$ (b) $a=-2,\ b=6$
(c) $a=-7,\ b=-8$ (d) $a=-4,\ b=4$.

41. Prove: $|a-b|\leq|a|+|b|$.

42. Prove: $|a|-|b|\leq|a-b|$.

43. Prove: $\big||a|-|b|\big|\leq|a-b|$. [*Hint:* Use Exercise 42.]