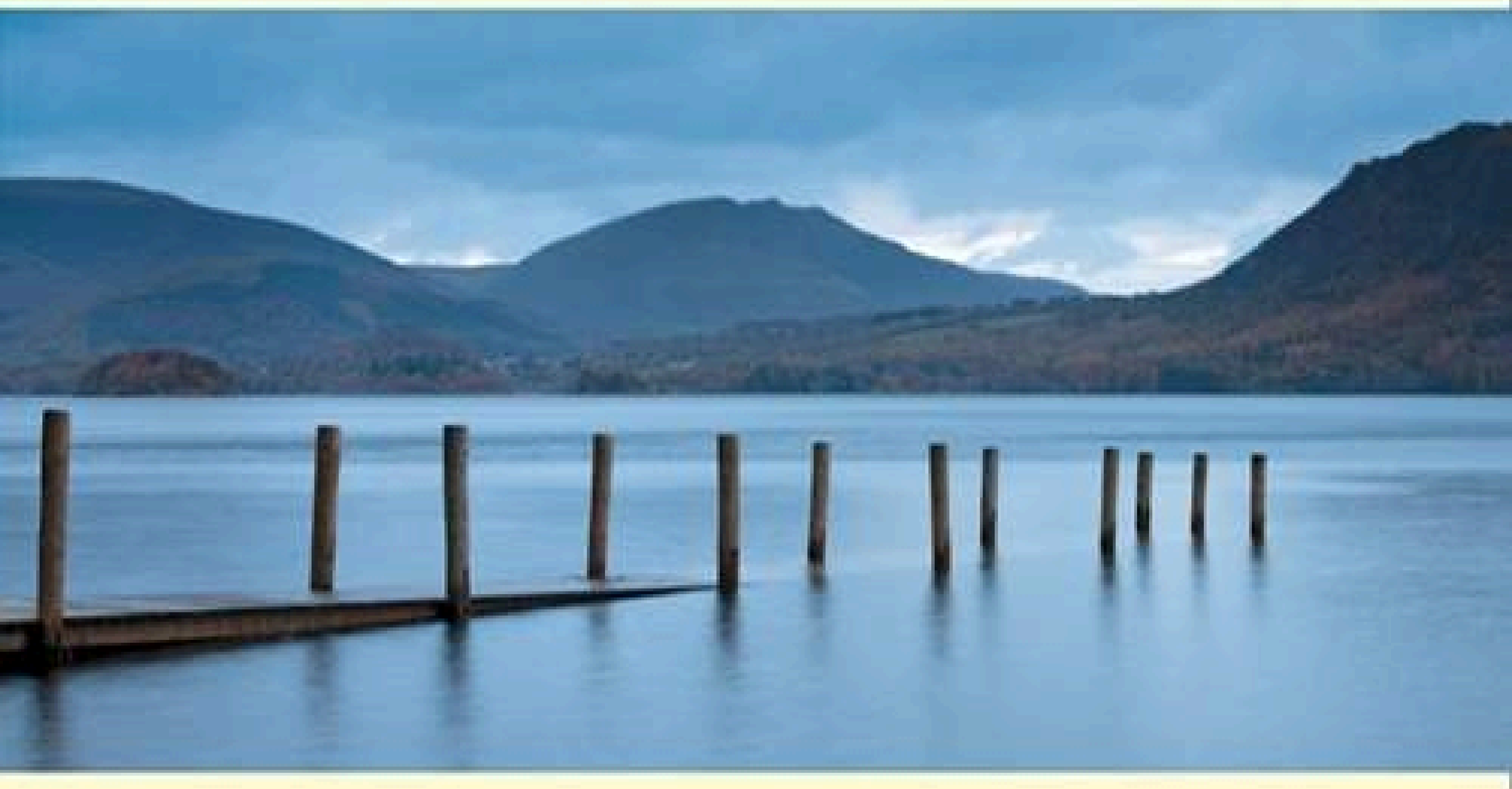
CALCULUS

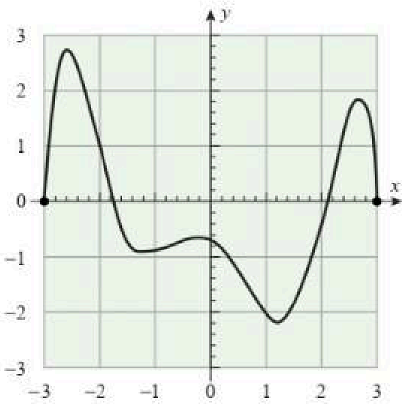
ANTON BIVENS DAVIS

EARLY TRANSCENDENTALS 10TH EDITION



EXERCISE SET 0.1 Graphing Utility

1. Use the accompanying graph to answer the following questions, making reasonable approximations where needed.
   (a) For what values of x is $y = 1$?
   (b) For what values of x is $y = 3$?
   (c) For what values of y is $x = 3$?
   (d) For what values of x is $y \leq 0$?
   (e) What are the maximum and minimum values of y and for what values of x do they occur?



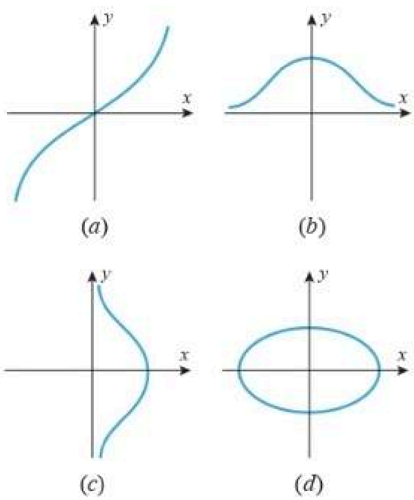
◀ Figure Ex-1

2. Use the accompanying table to answer the questions posed in Exercise 1.

x	−2	−1	0	2	3	4	5	6
y	5	1	−2	7	−1	1	0	9

▲ Table Ex-2

3. In each part of the accompanying figure, determine whether the graph defines y as a function of x.

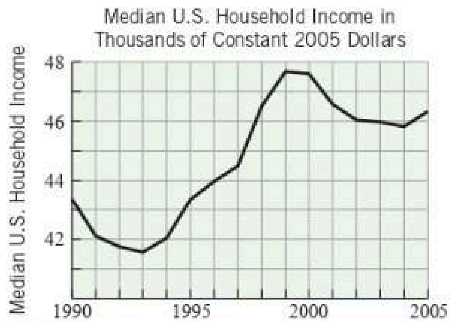


▲ Figure Ex-3

4. In each part, compare the natural domains of f and g.
   (a) $f(x) = \dfrac{x^2 + x}{x + 1}$; $g(x) = x$
   (b) $f(x) = \dfrac{x\sqrt{x} + \sqrt{x}}{x + 1}$; $g(x) = \sqrt{x}$

FOCUS ON CONCEPTS

5. The accompanying graph shows the median income in U.S. households (adjusted for inflation) between 1990 and 2005. Use the graph to answer the following questions, making reasonable approximations where needed.
   (a) When was the median income at its maximum value, and what was the median income when that occurred?
   (b) When was the median income at its minimum value, and what was the median income when that occurred?
   (c) The median income was declining during the 2-year period between 2000 and 2002. Was it declining more rapidly during the first year or the second year of that period? Explain your reasoning.



Source: U.S. Census Bureau, August 2006.

▲ Figure Ex-5

6. Use the median income graph in Exercise 5 to answer the following questions, making reasonable approximations where needed.
   (a) What was the average yearly growth of median income between 1993 and 1999?
   (b) The median income was increasing during the 6-year period between 1993 and 1999. Was it increasing more rapidly during the first 3 years or the last 3 years of that period? Explain your reasoning.
   (c) Consider the statement: "After years of decline, median income this year was finally higher than that of last year." In what years would this statement have been correct?

7. Find $f(0)$, $f(2)$, $f(-2)$, $f(3)$, $f(\sqrt{2})$, and $f(3t)$.

(a) $f(x) = 3x^2 - 2$ (b) $f(x) = \begin{cases} \dfrac{1}{x}, & x > 3 \\ 2x, & x \leq 3 \end{cases}$

8. Find $g(3)$, $g(-1)$, $g(\pi)$, $g(-1.1)$, and $g(t^2 - 1)$.

(a) $g(x) = \dfrac{x+1}{x-1}$ (b) $g(x) = \begin{cases} \sqrt{x+1}, & x \geq 1 \\ 3, & x < 1 \end{cases}$

9–10 Find the natural domain and determine the range of each function. If you have a graphing utility, use it to confirm that your result is consistent with the graph produced by your graphing utility. [*Note:* Set your graphing utility in radian mode when graphing trigonometric functions.]

9. (a) $f(x) = \dfrac{1}{x-3}$ (b) $F(x) = \dfrac{x}{|x|}$

(c) $g(x) = \sqrt{x^2 - 3}$ (d) $G(x) = \sqrt{x^2 - 2x + 5}$

(e) $h(x) = \dfrac{1}{1 - \sin x}$ (f) $H(x) = \sqrt{\dfrac{x^2 - 4}{x - 2}}$

10. (a) $f(x) = \sqrt{3 - x}$ (b) $F(x) = \sqrt{4 - x^2}$

(c) $g(x) = 3 + \sqrt{x}$ (d) $G(x) = x^3 + 2$

(e) $h(x) = 3 \sin x$ (f) $H(x) = (\sin \sqrt{x})^{-2}$

FOCUS ON CONCEPTS

11. (a) If you had a device that could record the Earth's population continuously, would you expect the graph of population versus time to be a continuous (unbroken) curve? Explain what might cause breaks in the curve.

(b) Suppose that a hospital patient receives an injection of an antibiotic every 8 hours and that between injections the concentration C of the antibiotic in the bloodstream decreases as the antibiotic is absorbed by the tissues. What might the graph of C versus the elapsed time t look like?

12. (a) If you had a device that could record the temperature of a room continuously over a 24-hour period, would you expect the graph of temperature versus time to be a continuous (unbroken) curve? Explain your reasoning.

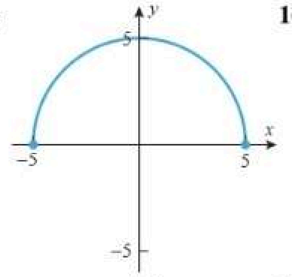
(b) If you had a computer that could track the number of boxes of cereal on the shelf of a market continuously over a 1-week period, would you expect the graph of the number of boxes on the shelf versus time to be a continuous (unbroken) curve? Explain your reasoning.

13. A boat is bobbing up and down on some gentle waves. Suddenly it gets hit by a large wave and sinks. Sketch a rough graph of the height of the boat above the ocean floor as a function of time.

14. A cup of hot coffee sits on a table. You pour in some cool milk and let it sit for an hour. Sketch a rough graph of the temperature of the coffee as a function of time.

15–18 As seen in Example 3, the equation $x^2 + y^2 = 25$ does not define y as a function of x. Each graph in these exercises is a portion of the circle $x^2 + y^2 = 25$. In each case, determine whether the graph defines y as a function of x, and if so, give a formula for y in terms of x.

15.



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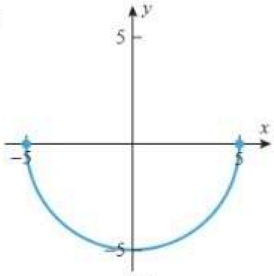

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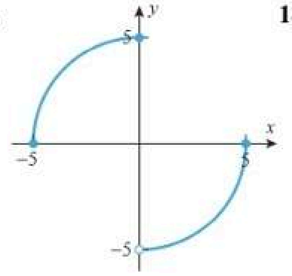

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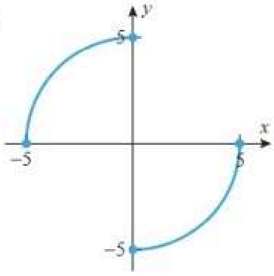


19–22 True–False Determine whether the statement is true or false. Explain your answer.

19. A curve that crosses the x-axis at two different points cannot be the graph of a function.

20. The natural domain of a real-valued function defined by a formula consists of all those real numbers for which the formula yields a real value.

21. The range of the absolute value function is all positive real numbers.

22. If $g(x) = 1/\sqrt{f(x)}$, then the domain of g consists of all those real numbers x for which $f(x) \neq 0$.

23. Use the equation $y = x^2 - 6x + 8$ to answer the following questions.

(a) For what values of x is $y = 0$?

(b) For what values of x is $y = -10$?

(c) For what values of x is $y \geq 0$?

(d) Does y have a minimum value? A maximum value? If so, find them.

24. Use the equation $y = 1 + \sqrt{x}$ to answer the following questions.

(a) For what values of x is $y = 4$?

(b) For what values of x is $y = 0$?

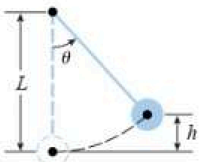
(c) For what values of x is $y \geq 6$?

(cont.)

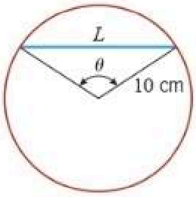
(d) Does y have a minimum value? A maximum value? If so, find them.

25. As shown in the accompanying figure, a pendulum of constant length L makes an angle θ with its vertical position. Express the height h as a function of the angle θ.

26. Express the length L of a chord of a circle with radius 10 cm as a function of the central angle θ (see the accompanying figure).



▲ **Figure Ex-25**



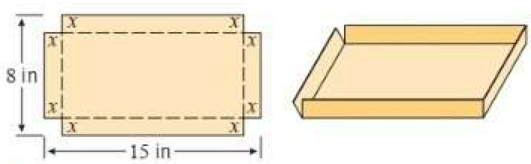
▲ **Figure Ex-26**

27–28 Express the function in piecewise form without using absolute values. [*Suggestion:* It may help to generate the graph of the function.] ■

27. (a) $f(x) = |x| + 3x + 1$ (b) $g(x) = |x| + |x - 1|$

28. (a) $f(x) = 3 + |2x - 5|$ (b) $g(x) = 3|x - 2| - |x + 1|$

29. As shown in the accompanying figure, an open box is to be constructed from a rectangular sheet of metal, 8 in by 15 in, by cutting out squares with sides of length x from each corner and bending up the sides.
(a) Express the volume V as a function of x.
(b) Find the domain of V.
(c) Plot the graph of the function V obtained in part (a) and estimate the range of this function.
(d) In words, describe how the volume V varies with x, and discuss how one might construct boxes of maximum volume.

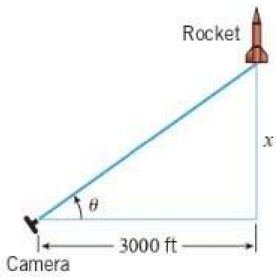


▲ **Figure Ex-29**

30. Repeat Exercise 29 assuming the box is constructed in the same fashion from a 6-inch-square sheet of metal.

31. A construction company has adjoined a 1000 ft^2 rectangular enclosure to its office building. Three sides of the enclosure are fenced in. The side of the building adjacent to the enclosure is 100 ft long and a portion of this side is used as the fourth side of the enclosure. Let x and y be the dimensions of the enclosure, where x is measured parallel to the building, and let L be the length of fencing required for those dimensions.
(a) Find a formula for L in terms of x and y.
(b) Find a formula that expresses L as a function of x alone.
(c) What is the domain of the function in part (b)?
(d) Plot the function in part (b) and estimate the dimensions of the enclosure that minimize the amount of fencing required.

32. As shown in the accompanying figure, a camera is mounted at a point 3000 ft from the base of a rocket launching pad. The rocket rises vertically when launched, and the camera's elevation angle is continually adjusted to follow the bottom of the rocket.
(a) Express the height x as a function of the elevation angle θ.
(b) Find the domain of the function in part (a).
(c) Plot the graph of the function in part (a) and use it to estimate the height of the rocket when the elevation angle is $\pi/4 \approx 0.7854$ radian. Compare this estimate to the exact height.



◀ **Figure Ex-32**

33. A soup company wants to manufacture a can in the shape of a right circular cylinder that will hold 500 cm^3 of liquid. The material for the top and bottom costs 0.02 cent/cm^2, and the material for the sides costs 0.01 cent/cm^2.
(a) Estimate the radius r and the height h of the can that costs the least to manufacture. [*Suggestion:* Express the cost C in terms of r.]
(b) Suppose that the tops and bottoms of radius r are punched out from square sheets with sides of length $2r$ and the scraps are waste. If you allow for the cost of the waste, would you expect the can of least cost to be taller or shorter than the one in part (a)? Explain.
(c) Estimate the radius, height, and cost of the can in part (b), and determine whether your conjecture was correct.

34. The designer of a sports facility wants to put a quarter-mile (1320 ft) running track around a football field, oriented as in the accompanying figure on the next page. The football field is 360 ft long (including the end zones) and 160 ft wide. The track consists of two straightaways and two semicircles, with the straightaways extending at least the length of the football field.
(a) Show that it is possible to construct a quarter-mile track around the football field. [*Suggestion:* Find the shortest track that can be constructed around the field.]
(b) Let L be the length of a straightaway (in feet), and let x be the distance (in feet) between a sideline of the football field and a straightaway. Make a graph of L versus x.

(cont.)

5. $y=-2(x+1)^2-3$
6. $y=\frac{1}{2}(x-3)^2+2$
7. $y=x^2+6x$
8. $y=\frac{1}{2}(x^2-2x+3)$
9. $y=3-\sqrt{x+1}$
10. $y=1+\sqrt{x-4}$
11. $y=\frac{1}{2}\sqrt{x}+1$
12. $y=-\sqrt{3x}$
13. $y=\dfrac{1}{x-3}$
14. $y=\dfrac{1}{1-x}$
15. $y=2-\dfrac{1}{x+1}$
16. $y=\dfrac{x-1}{x}$
17. $y=|x+2|-2$
18. $y=1-|x-3|$
19. $y=|2x-1|+1$
20. $y=\sqrt{x^2-4x+4}$
21. $y=1-2\sqrt[3]{x}$
22. $y=\sqrt[3]{x-2}-3$
23. $y=2+\sqrt[3]{x+1}$
24. $y+\sqrt[3]{x-2}=0$

25. (a) Sketch the graph of $y=x+|x|$ by adding the corresponding y-coordinates on the graphs of $y=x$ and $y=|x|$.
(b) Express the equation $y=x+|x|$ in piecewise form with no absolute values, and confirm that the graph you obtained in part (a) is consistent with this equation.

26. Sketch the graph of $y=x+(1/x)$ by adding corresponding y-coordinates on the graphs of $y=x$ and $y=1/x$. Use a graphing utility to confirm that your sketch is correct.

27–28 Find formulas for $f+g$, $f-g$, fg, and f/g, and state the domains of the functions.

27. $f(x)=2\sqrt{x-1},\ g(x)=\sqrt{x-1}$

28. $f(x)=\dfrac{x}{1+x^2},\ g(x)=\dfrac{1}{x}$

29. Let $f(x)=\sqrt{x}$ and $g(x)=x^3+1$. Find
(a) $f(g(2))$ (b) $g(f(4))$ (c) $f(f(16))$
(d) $g(g(0))$ (e) $f(2+h)$ (f) $g(3+h)$.

30. Let $g(x)=\sqrt{x}$. Find
(a) $g(5s+2)$ (b) $g(\sqrt{x}+2)$ (c) $3g(5x)$
(d) $\dfrac{1}{g(x)}$ (e) $g(g(x))$ (f) $(g(x))^2-g(x^2)$
(g) $g(1/\sqrt{x}\,)$ (h) $g((x-1)^2)$ (i) $g(x+h)$.

31–34 Find formulas for $f\circ g$ and $g\circ f$, and state the domains of the compositions.

31. $f(x)=x^2,\ g(x)=\sqrt{1-x}$

32. $f(x)=\sqrt{x-3},\ g(x)=\sqrt{x^2+3}$

33. $f(x)=\dfrac{1+x}{1-x},\ g(x)=\dfrac{x}{1-x}$

34. $f(x)=\dfrac{x}{1+x^2},\ g(x)=\dfrac{1}{x}$

35–36 Find a formula for $f\circ g\circ h$.

35. $f(x)=x^2+1,\ g(x)=\dfrac{1}{x},\ h(x)=x^3$

36. $f(x)=\dfrac{1}{1+x},\ g(x)=\sqrt[3]{x},\ h(x)=\dfrac{1}{x^3}$

37–42 Express f as a composition of two functions; that is, find g and h such that $f=g\circ h$. [*Note:* Each exercise has more than one solution.]

37. (a) $f(x)=\sqrt{x+2}$ (b) $f(x)=|x^2-3x+5|$
38. (a) $f(x)=x^2+1$ (b) $f(x)=\dfrac{1}{x-3}$
39. (a) $f(x)=\sin^2 x$ (b) $f(x)=\dfrac{3}{5+\cos x}$
40. (a) $f(x)=3\sin(x^2)$ (b) $f(x)=3\sin^2 x+4\sin x$
41. (a) $f(x)=\left(1+\sin(x^2)\right)^3$ (b) $f(x)=\sqrt{1-\sqrt[3]{x}}$
42. (a) $f(x)=\dfrac{1}{1-x^2}$ (b) $f(x)=|5+2x|$

43–46 True–False Determine whether the statement is true or false. Explain your answer.

43. The domain of $f+g$ is the intersection of the domains of f and g.

44. The domain of $f\circ g$ consists of all values of x in the domain of g for which $g(x)\neq 0$.

45. The graph of an even function is symmetric about the y-axis.

46. The graph of $y=f(x+2)+3$ is obtained by translating the graph of $y=f(x)$ right 2 units and up 3 units.

FOCUS ON CONCEPTS

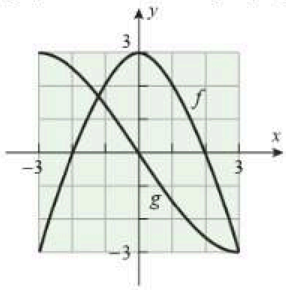
47. Use the data in the accompanying table to make a plot of $y=f(g(x))$.

x	−3	−2	−1	0	1	2	3
$f(x)$	−4	−3	−2	−1	0	1	2
$g(x)$	−1	0	1	2	3	−2	−3

▲ Table Ex-47

48. Find the domain of $g\circ f$ for the functions f and g in Exercise 47.

49. Sketch the graph of $y=f(g(x))$ for the functions graphed in the accompanying figure.



◀ Figure Ex-49

50. Sketch the graph of $y=g(f(x))$ for the functions graphed in Exercise 49.

51. Use the graphs of f and g in Exercise 49 to estimate the solutions of the equations $f(g(x))=0$ and $g(f(x))=0$.

52. Use the table given in Exercise 47 to solve the equations $f(g(x))=0$ and $g(f(x))=0$.

53–56 Find

$$\frac{f(x+h)-f(x)}{h} \quad \text{and} \quad \frac{f(w)-f(x)}{w-x}$$

Simplify as much as possible. ■

53. $f(x)=3x^2-5$ **54.** $f(x)=x^2+6x$

55. $f(x)=1/x$ **56.** $f(x)=1/x^2$

57. Classify the functions whose values are given in the accompanying table as even, odd, or neither.

x	−3	−2	−1	0	1	2	3
$f(x)$	5	3	2	3	1	−3	5
$g(x)$	4	1	−2	0	2	−1	−4
$h(x)$	2	−5	8	−2	8	−5	2

▲ **Table Ex-57**

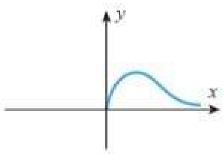
58. Complete the accompanying table so that the graph of $y=f(x)$ is symmetric about
(a) the y-axis (b) the origin.

x	−3	−2	−1	0	1	2	3
$f(x)$	1		−1	0		−5	

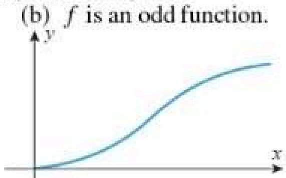
▲ **Table Ex-58**

59. The accompanying figure shows a portion of a graph. Complete the graph so that the entire graph is symmetric about
(a) the x-axis (b) the y-axis (c) the origin.

60. The accompanying figure shows a portion of the graph of a function f. Complete the graph assuming that
(a) f is an even function (b) f is an odd function.



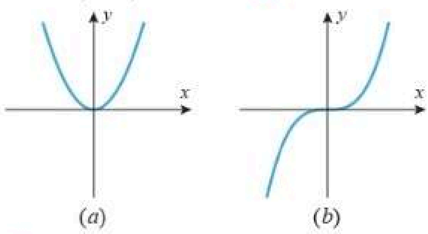
▲ **Figure Ex-59**



▲ **Figure Ex-60**

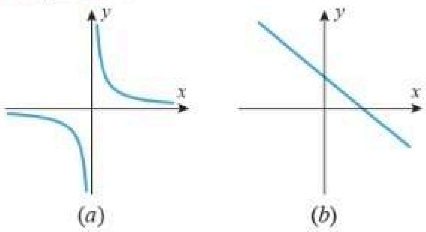
61–62 Classify the functions graphed in the accompanying figures as even, odd, or neither. ■

61.



▲ **Figure Ex-61**

62.



▲ **Figure Ex-62**

63. In each part, classify the function as even, odd, or neither.
(a) $f(x)=x^2$ (b) $f(x)=x^3$
(c) $f(x)=|x|$ (d) $f(x)=x+1$
(e) $f(x)=\dfrac{x^5-x}{1+x^2}$ (f) $f(x)=2$

64. Suppose that the function f has domain all real numbers. Determine whether each function can be classified as even or odd. Explain.
(a) $g(x)=\dfrac{f(x)+f(-x)}{2}$ (b) $h(x)=\dfrac{f(x)-f(-x)}{2}$

65. Suppose that the function f has domain all real numbers. Show that f can be written as the sum of an even function and an odd function. [*Hint:* See Exercise 64.]

66–67 Use Theorem 0.2.3 to determine whether the graph has symmetries about the x-axis, the y-axis, or the origin. ■

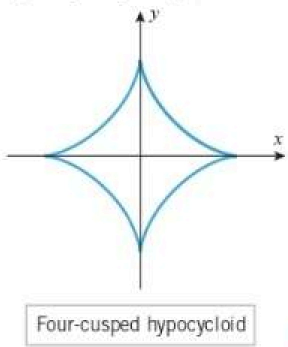
66. (a) $x=5y^2+9$ (b) $x^2-2y^2=3$
(c) $xy=5$

67. (a) $x^4=2y^3+y$ (b) $y=\dfrac{x}{3+x^2}$
(c) $y^2=|x|-5$

68–69 (i) Use a graphing utility to graph the equation in the first quadrant. [*Note:* To do this you will have to solve the equation for y in terms of x.] (ii) Use symmetry to make a hand-drawn sketch of the entire graph. (iii) Confirm your work by generating the graph of the equation in the remaining three quadrants. ■

68. $9x^2+4y^2=36$ **69.** $4x^2+16y^2=16$

70. The graph of the equation $x^{2/3}+y^{2/3}=1$, which is shown in the accompanying figure, is called a ***four-cusped hypocycloid***.
(a) Use Theorem 0.2.3 to confirm that this graph is symmetric about the x-axis, the y-axis, and the origin.
(b) Find a function f whose graph in the first quadrant coincides with the four-cusped hypocycloid, and use a graphing utility to confirm your work.
(c) Repeat part (b) for the remaining three quadrants.



◀ **Figure Ex-70**

71. The equation $y=|f(x)|$ can be written as

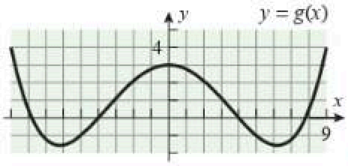
$$y=\begin{cases} f(x), & f(x)\ge 0 \\ -f(x), & f(x)<0 \end{cases}$$

which shows that the graph of $y=|f(x)|$ can be obtained from the graph of $y=f(x)$ by retaining the portion that lies

EXERCISE SET 1.1 Graphing Utility C CAS

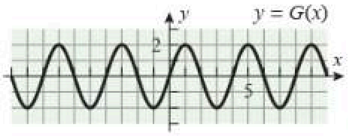
1–10 In these exercises, make reasonable assumptions about the graph of the indicated function outside of the region depicted. ■

1. For the function g graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} g(x)$ (b) $\lim_{x\to 0^+} g(x)$
(c) $\lim_{x\to 0} g(x)$ (d) $g(0)$.



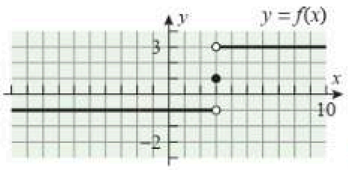
◀ **Figure Ex-1**

2. For the function G graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} G(x)$ (b) $\lim_{x\to 0^+} G(x)$
(c) $\lim_{x\to 0} G(x)$ (d) $G(0)$.



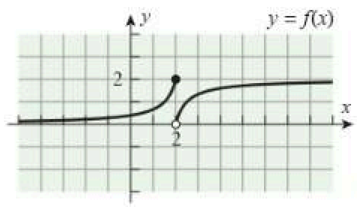
◀ **Figure Ex-2**

3. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to 3^-} f(x)$ (b) $\lim_{x\to 3^+} f(x)$
(c) $\lim_{x\to 3} f(x)$ (d) $f(3)$.



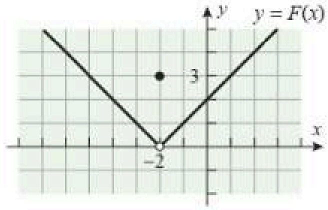
◀ **Figure Ex-3**

4. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to 2^-} f(x)$ (b) $\lim_{x\to 2^+} f(x)$
(c) $\lim_{x\to 2} f(x)$ (d) $f(2)$.



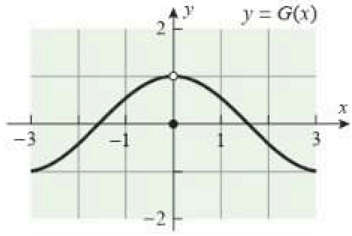
◀ **Figure Ex-4**

5. For the function F graphed in the accompanying figure, find
(a) $\lim_{x\to -2^-} F(x)$ (b) $\lim_{x\to -2^+} F(x)$
(c) $\lim_{x\to -2} F(x)$ (d) $F(-2)$.



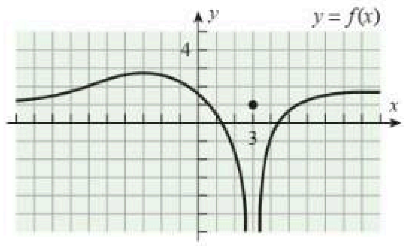
◀ **Figure Ex-5**

6. For the function G graphed in the accompanying figure, find
(a) $\lim_{x\to 0^-} G(x)$ (b) $\lim_{x\to 0^+} G(x)$
(c) $\lim_{x\to 0} G(x)$ (d) $G(0)$.



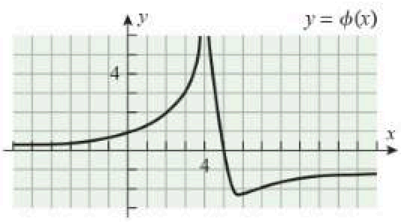
◀ **Figure Ex-6**

7. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to 3^-} f(x)$ (b) $\lim_{x\to 3^+} f(x)$
(c) $\lim_{x\to 3} f(x)$ (d) $f(3)$.



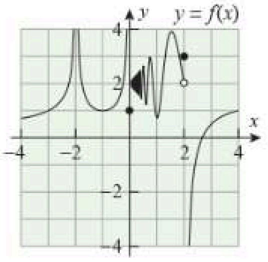
◀ **Figure Ex-7**

8. For the function ϕ graphed in the accompanying figure, find
(a) $\lim_{x\to 4^-} \phi(x)$ (b) $\lim_{x\to 4^+} \phi(x)$
(c) $\lim_{x\to 4} \phi(x)$ (d) $\phi(4)$.



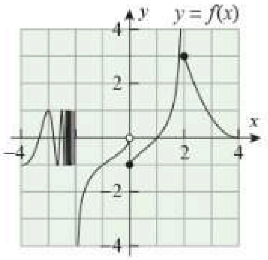
◀ **Figure Ex-8**

9. For the function f graphed in the accompanying figure on the next page, find
(a) $\lim_{x\to -2} f(x)$ (b) $\lim_{x\to 0^-} f(x)$
(c) $\lim_{x\to 0^+} f(x)$ (d) $\lim_{x\to 2^-} f(x)$
(e) $\lim_{x\to 2^+} f(x)$
(f) the vertical asymptotes of the graph of f.



◀ Figure Ex-9

10. For the function f graphed in the accompanying figure, find
(a) $\lim_{x\to-2^-} f(x)$ (b) $\lim_{x\to-2^+} f(x)$ (c) $\lim_{x\to 0^-} f(x)$
(d) $\lim_{x\to 0^+} f(x)$ (e) $\lim_{x\to 2^-} f(x)$ (f) $\lim_{x\to 2^+} f(x)$
(g) the vertical asymptotes of the graph of f.



◀ Figure Ex-10

11–12 (i) Complete the table and make a guess about the limit indicated. (ii) Confirm your conclusions about the limit by graphing a function over an appropriate interval. [*Note:* For the inverse trigonometric function, be sure to put your calculating and graphing utilities in radian mode.]

11. $f(x) = \dfrac{e^x - 1}{x}$; $\lim_{x\to 0} f(x)$

x	−0.01	−0.001	−0.0001	0.0001	0.001	0.01
$f(x)$						

▲ Table Ex-11

12. $f(x) = \dfrac{\sin^{-1} 2x}{x}$; $\lim_{x\to 0} f(x)$

x	−0.1	−0.01	−0.001	0.001	0.01	0.1
$f(x)$						

▲ Table Ex-12

13–16 (i) Make a guess at the limit (if it exists) by evaluating the function at the specified x-values. (ii) Confirm your conclusions about the limit by graphing the function over an appropriate interval. (iii) If you have a CAS, then use it to find the limit. [*Note:* For the trigonometric functions, be sure to put your calculating and graphing utilities in radian mode.]

13. (a) $\lim_{x\to 1} \dfrac{x-1}{x^3-1}$; $x = 2, 1.5, 1.1, 1.01, 1.001, 0, 0.5, 0.9, 0.99, 0.999$
(b) $\lim_{x\to 1^+} \dfrac{x+1}{x^3-1}$; $x = 2, 1.5, 1.1, 1.01, 1.001, 1.0001$
(c) $\lim_{x\to 1^-} \dfrac{x+1}{x^3-1}$; $x = 0, 0.5, 0.9, 0.99, 0.999, 0.9999$

14. (a) $\lim_{x\to 0} \dfrac{\sqrt{x+1}-1}{x}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$
(b) $\lim_{x\to 0^+} \dfrac{\sqrt{x+1}+1}{x}$; $x = 0.25, 0.1, 0.001, 0.0001$
(c) $\lim_{x\to 0^-} \dfrac{\sqrt{x+1}+1}{x}$; $x = -0.25, -0.1, -0.001, -0.0001$

15. (a) $\lim_{x\to 0} \dfrac{\sin 3x}{x}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$
(b) $\lim_{x\to -1} \dfrac{\cos x}{x+1}$; $x = 0, -0.5, -0.9, -0.99, -0.999, -1.5, -1.1, -1.01, -1.001$

16. (a) $\lim_{x\to -1} \dfrac{\tan(x+1)}{x+1}$; $x = 0, -0.5, -0.9, -0.99, -0.999, -1.5, -1.1, -1.01, -1.001$
(b) $\lim_{x\to 0} \dfrac{\sin(5x)}{\sin(2x)}$; $x = \pm 0.25, \pm 0.1, \pm 0.001, \pm 0.0001$

17–20 True–False Determine whether the statement is true or false. Explain your answer.

17. If $f(a) = L$, then $\lim_{x\to a} f(x) = L$.

18. If $\lim_{x\to a} f(x)$ exists, then so do $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$.

19. If $\lim_{x\to a^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist, then so does $\lim_{x\to a} f(x)$.

20. If $\lim_{x\to a^+} f(x) = +\infty$, then $f(a)$ is undefined.

21–26 Sketch a possible graph for a function f with the specified properties. (Many different solutions are possible.)

21. (i) the domain of f is $[-1, 1]$
(ii) $f(-1) = f(0) = f(1) = 0$
(iii) $\lim_{x\to -1^+} f(x) = \lim_{x\to 0} f(x) = \lim_{x\to 1^-} f(x) = 1$

22. (i) the domain of f is $[-2, 1]$
(ii) $f(-2) = f(0) = f(1) = 0$
(iii) $\lim_{x\to -2^+} f(x) = 2$, $\lim_{x\to 0} f(x) = 0$, and $\lim_{x\to 1^-} f(x) = 1$

23. (i) the domain of f is $(-\infty, 0]$
(ii) $f(-2) = f(0) = 1$
(iii) $\lim_{x\to -2} f(x) = +\infty$

24. (i) the domain of f is $(0, +\infty)$
(ii) $f(1) = 0$
(iii) the y-axis is a vertical asymptote for the graph of f
(iv) $f(x) < 0$ if $0 < x < 1$

✔QUICK CHECK EXERCISES 1.2 *(See page 88 for answers.)*

1. In each part, find the limit by inspection.
(a) $\lim_{x\to 8} 7 =$ ______ (b) $\lim_{y\to 3^+} 12y =$ ______
(c) $\lim_{x\to 0^-} \dfrac{x}{|x|} =$ ______ (d) $\lim_{w\to 5} \dfrac{w}{|w|} =$ ______
(e) $\lim_{z\to 1^-} \dfrac{1}{1-z} =$ ______

2. Given that $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = 2$, find the limits.
(a) $\lim_{x\to a} [3f(x) + 2g(x)] =$ ______
(b) $\lim_{x\to a} \dfrac{2f(x)+1}{1-f(x)g(x)} =$ ______
(c) $\lim_{x\to a} \dfrac{\sqrt{f(x)+3}}{g(x)} =$ ______

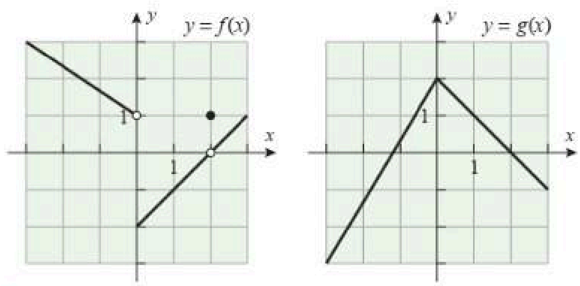
3. Find the limits.
(a) $\lim_{x\to -1} (x^3 + x^2 + x)^{101} =$ ______
(b) $\lim_{x\to 2^-} \dfrac{(x-1)(x-2)}{x+1} =$ ______
(c) $\lim_{x\to -1^+} \dfrac{(x-1)(x-2)}{x+1} =$ ______
(d) $\lim_{x\to 4} \dfrac{x^2-16}{x-4} =$ ______

4. Let
$$f(x) = \begin{cases} x+1, & x \le 1 \\ x-1, & x > 1 \end{cases}$$
Find the limits that exist.
(a) $\lim_{x\to 1^-} f(x) =$ ______
(b) $\lim_{x\to 1^+} f(x) =$ ______
(c) $\lim_{x\to 1} f(x) =$ ______

EXERCISE SET 1.2

1. Given that
$$\lim_{x\to a} f(x) = 2, \quad \lim_{x\to a} g(x) = -4, \quad \lim_{x\to a} h(x) = 0$$
find the limits.
(a) $\lim_{x\to a} [f(x) + 2g(x)]$
(b) $\lim_{x\to a} [h(x) - 3g(x) + 1]$
(c) $\lim_{x\to a} [f(x)g(x)]$ (d) $\lim_{x\to a} [g(x)]^2$
(e) $\lim_{x\to a} \sqrt[3]{6+f(x)}$ (f) $\lim_{x\to a} \dfrac{2}{g(x)}$

2. Use the graphs of f and g in the accompanying figure to find the limits that exist. If the limit does not exist, explain why.
(a) $\lim_{x\to 2} [f(x) + g(x)]$ (b) $\lim_{x\to 0} [f(x) + g(x)]$
(c) $\lim_{x\to 0^+} [f(x) + g(x)]$ (d) $\lim_{x\to 0^-} [f(x) + g(x)]$
(e) $\lim_{x\to 2} \dfrac{f(x)}{1+g(x)}$ (f) $\lim_{x\to 2} \dfrac{1+g(x)}{f(x)}$
(g) $\lim_{x\to 0^+} \sqrt{f(x)}$ (h) $\lim_{x\to 0^-} \sqrt{f(x)}$



▲ **Figure Ex-2**

3–30 Find the limits. ■

3. $\lim_{x\to 2} x(x-1)(x+1)$
4. $\lim_{x\to 3} x^3 - 3x^2 + 9x$
5. $\lim_{x\to 3} \dfrac{x^2-2x}{x+1}$
6. $\lim_{x\to 0} \dfrac{6x-9}{x^3-12x+3}$
7. $\lim_{x\to 1^+} \dfrac{x^4-1}{x-1}$
8. $\lim_{t\to -2} \dfrac{t^3+8}{t+2}$
9. $\lim_{x\to -1} \dfrac{x^2+6x+5}{x^2-3x-4}$
10. $\lim_{x\to 2} \dfrac{x^2-4x+4}{x^2+x-6}$
11. $\lim_{x\to -1} \dfrac{2x^2+x-1}{x+1}$
12. $\lim_{x\to 1} \dfrac{3x^2-x-2}{2x^2+x-3}$
13. $\lim_{t\to 2} \dfrac{t^3+3t^2-12t+4}{t^3-4t}$
14. $\lim_{t\to 1} \dfrac{t^3+t^2-5t+3}{t^3-3t+2}$
15. $\lim_{x\to 3^+} \dfrac{x}{x-3}$
16. $\lim_{x\to 3^-} \dfrac{x}{x-3}$
17. $\lim_{x\to 3} \dfrac{x}{x-3}$
18. $\lim_{x\to 2^+} \dfrac{x}{x^2-4}$
19. $\lim_{x\to 2^-} \dfrac{x}{x^2-4}$
20. $\lim_{x\to 2} \dfrac{x}{x^2-4}$
21. $\lim_{y\to 6^+} \dfrac{y+6}{y^2-36}$
22. $\lim_{y\to 6^-} \dfrac{y+6}{y^2-36}$
23. $\lim_{y\to 6} \dfrac{y+6}{y^2-36}$
24. $\lim_{x\to 4^+} \dfrac{3-x}{x^2-2x-8}$
25. $\lim_{x\to 4^-} \dfrac{3-x}{x^2-2x-8}$
26. $\lim_{x\to 4} \dfrac{3-x}{x^2-2x-8}$
27. $\lim_{x\to 2^+} \dfrac{1}{|2-x|}$
28. $\lim_{x\to 3^-} \dfrac{1}{|x-3|}$
29. $\lim_{x\to 9} \dfrac{x-9}{\sqrt{x}-3}$
30. $\lim_{y\to 4} \dfrac{4-y}{2-\sqrt{y}}$

31. Let
$$f(x) = \begin{cases} x-1, & x \le 3 \\ 3x-7, & x > 3 \end{cases}$$

(cont.)

Find

(a) $\lim_{x\to 3^-} f(x)$ (b) $\lim_{x\to 3^+} f(x)$ (c) $\lim_{x\to 3} f(x)$.

32. Let

$$g(t) = \begin{cases} t-2, & t<0 \\ t^2, & 0 \le t \le 2 \\ 2t, & t>2 \end{cases}$$

Find

(a) $\lim_{t\to 0} g(t)$ (b) $\lim_{t\to 1} g(t)$ (c) $\lim_{t\to 2} g(t)$.

33–36 True–False Determine whether the statement is true or false. Explain your answer. ■

33. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist, then so does $\lim_{x\to a}[f(x)+g(x)]$.

34. If $\lim_{x\to a} g(x) = 0$ and $\lim_{x\to a} f(x)$ exists, then $\lim_{x\to a}[f(x)/g(x)]$ does not exist.

35. If $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist and are equal, then $\lim_{x\to a}[f(x)/g(x)] = 1$.

36. If $f(x)$ is a rational function and $x = a$ is in the domain of f, then $\lim_{x\to a} f(x) = f(a)$.

37–38 First rationalize the numerator and then find the limit. ■

37. $\lim_{x\to 0} \dfrac{\sqrt{x+4}-2}{x}$ **38.** $\lim_{x\to 0} \dfrac{\sqrt{x^2+4}-2}{x}$

39. Let

$$f(x) = \frac{x^3-1}{x-1}$$

(a) Find $\lim_{x\to 1} f(x)$.
(b) Sketch the graph of $y = f(x)$.

40. Let

$$f(x) = \begin{cases} \dfrac{x^2-9}{x+3}, & x \neq -3 \\ k, & x = -3 \end{cases}$$

(a) Find k so that $f(-3) = \lim_{x\to -3} f(x)$.
(b) With k assigned the value $\lim_{x\to -3} f(x)$, show that $f(x)$ can be expressed as a polynomial.

FOCUS ON CONCEPTS

41. (a) Explain why the following calculation is incorrect.

$$\lim_{x\to 0^+}\left(\frac{1}{x}-\frac{1}{x^2}\right) = \lim_{x\to 0^+}\frac{1}{x} - \lim_{x\to 0^+}\frac{1}{x^2} = +\infty - (+\infty) = 0$$

(b) Show that $\lim_{x\to 0^+}\left(\dfrac{1}{x}-\dfrac{1}{x^2}\right) = -\infty$.

42. (a) Explain why the following argument is incorrect.

$$\lim_{x\to 0}\left(\frac{1}{x}-\frac{2}{x^2+2x}\right) = \lim_{x\to 0}\frac{1}{x}\left(1-\frac{2}{x+2}\right) = \infty\cdot 0 = 0$$

(b) Show that $\lim_{x\to 0}\left(\dfrac{1}{x}-\dfrac{2}{x^2+2x}\right) = \dfrac{1}{2}$.

43. Find all values of a such that

$$\lim_{x\to 1}\left(\frac{1}{x-1}-\frac{a}{x^2-1}\right)$$

exists and is finite.

44. (a) Explain informally why

$$\lim_{x\to 0^-}\left(\frac{1}{x}+\frac{1}{x^2}\right) = +\infty$$

(b) Verify the limit in part (a) algebraically.

45. Let $p(x)$ and $q(x)$ be polynomials, with $q(x_0) = 0$. Discuss the behavior of the graph of $y = p(x)/q(x)$ in the vicinity of $x = x_0$. Give examples to support your conclusions.

46. Suppose that f and g are two functions such that $\lim_{x\to a} f(x)$ exists but $\lim_{x\to a}[f(x)+g(x)]$ does not exist. Use Theorem 1.2.2. to prove that $\lim_{x\to a} g(x)$ does not exist.

47. Suppose that f and g are two functions such that both $\lim_{x\to a} f(x)$ and $\lim_{x\to a}[f(x)+g(x)]$ exist. Use Theorem 1.2.2 to prove that $\lim_{x\to a} g(x)$ exists.

48. Suppose that f and g are two functions such that

$$\lim_{x\to a} g(x) = 0 \quad\text{and}\quad \lim_{x\to a}\frac{f(x)}{g(x)}$$

exists. Use Theorem 1.2.2 to prove that $\lim_{x\to a} f(x) = 0$.

49. Writing According to Newton's Law of Universal Gravitation, the gravitational force of attraction between two masses is inversely proportional to the square of the distance between them. What results of this section are useful in describing the gravitational force of attraction between the masses as they get closer and closer together?

50. Writing Suppose that f and g are two functions that are equal except at a finite number of points and that a denotes a real number. Explain informally why both

$$\lim_{x\to a} f(x) \quad\text{and}\quad \lim_{x\to a} g(x)$$

exist and are equal, or why both limits fail to exist. Write a short paragraph that explains the relationship of this result to the use of "algebraic simplification" in the evaluation of a limit.

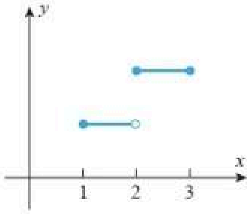
✔ QUICK CHECK ANSWERS 1.2

1. (a) 7 (b) 36 (c) −1 (d) 1 (e) $+\infty$ **2.** (a) 7 (b) −3 (c) 1 **3.** (a) −1 (b) 0 (c) $+\infty$ (d) 8
4. (a) 2 (b) 0 (c) does not exist

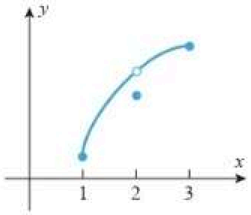
EXERCISE SET 1.5 Graphing Utility

1–4 Let f be the function whose graph is shown. On which of the following intervals, if any, is f continuous?
(a) $[1, 3]$ (b) $(1, 3)$ (c) $[1, 2]$
(d) $(1, 2)$ (e) $[2, 3]$ (f) $(2, 3)$
For each interval on which f is not continuous, indicate which conditions for the continuity of f do not hold. ■

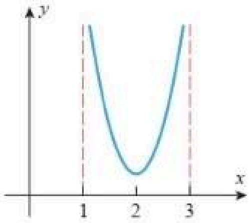
1.



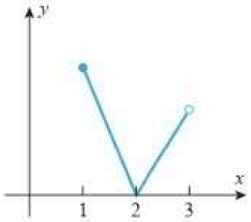
2.



3.



4.



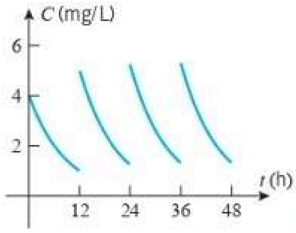
5. Consider the functions
$$f(x) = \begin{cases} 1, & x \neq 4 \\ -1, & x = 4 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 4x - 10, & x \neq 4 \\ -6, & x = 4 \end{cases}$$
In each part, is the given function continuous at $x = 4$?
(a) $f(x)$ (b) $g(x)$ (c) $-g(x)$ (d) $|f(x)|$
(e) $f(x)g(x)$ (f) $g(f(x))$ (g) $g(x) - 6f(x)$

6. Consider the functions
$$f(x) = \begin{cases} 1, & 0 \leq x \\ 0, & x < 0 \end{cases} \quad \text{and} \quad g(x) = \begin{cases} 0, & 0 \leq x \\ 1, & x < 0 \end{cases}$$
In each part, is the given function continuous at $x = 0$?
(a) $f(x)$ (b) $g(x)$ (c) $f(-x)$ (d) $|g(x)|$
(e) $f(x)g(x)$ (f) $g(f(x))$ (g) $f(x) + g(x)$

FOCUS ON CONCEPTS

7. In each part sketch the graph of a function f that satisfies the stated conditions.
(a) f is continuous everywhere except at $x = 3$, at which point it is continuous from the right.
(b) f has a two-sided limit at $x = 3$, but it is not continuous at $x = 3$.
(c) f is not continuous at $x = 3$, but if its value at $x = 3$ is changed from $f(3) = 1$ to $f(3) = 0$, it becomes continuous at $x = 3$.
(d) f is continuous on the interval $[0, 3)$ and is defined on the closed interval $[0, 3]$; but f is not continuous on the interval $[0, 3]$.

8. The accompanying figure models the concentration C of medication in the bloodstream of a patient over a 48-hour period of time. Discuss the significance of the discontinuities in the graph.



◄ **Figure Ex-8**

9. A student parking lot at a university charges \$2.00 for the first half hour (or any part) and \$1.00 for each subsequent half hour (or any part) up to a daily maximum of \$10.00.
(a) Sketch a graph of cost as a function of the time parked.
(b) Discuss the significance of the discontinuities in the graph to a student who parks there.

10. In each part determine whether the function is continuous or not, and explain your reasoning.
(a) The Earth's population as a function of time.
(b) Your exact height as a function of time.
(c) The cost of a taxi ride in your city as a function of the distance traveled.
(d) The volume of a melting ice cube as a function of time.

11–22 Find values of x, if any, at which f is not continuous. ■

11. $f(x) = 5x^4 - 3x + 7$ **12.** $f(x) = \sqrt[3]{x - 8}$

13. $f(x) = \dfrac{x + 2}{x^2 + 4}$ **14.** $f(x) = \dfrac{x + 2}{x^2 - 4}$

15. $f(x) = \dfrac{x}{2x^2 + x}$ **16.** $f(x) = \dfrac{2x + 1}{4x^2 + 4x + 5}$

17. $f(x) = \dfrac{3}{x} + \dfrac{x - 1}{x^2 - 1}$ **18.** $f(x) = \dfrac{5}{x} + \dfrac{2x}{x + 4}$

19. $f(x) = \dfrac{x^2 + 6x + 9}{|x| + 3}$ **20.** $f(x) = \left|4 - \dfrac{8}{x^4 + x}\right|$

21. $f(x) = \begin{cases} 2x + 3, & x \leq 4 \\ 7 + \dfrac{16}{x}, & x > 4 \end{cases}$

22. $f(x) = \begin{cases} \dfrac{3}{x - 1}, & x \neq 1 \\ 3, & x = 1 \end{cases}$

23–28 True–False Determine whether the statement is true or false. Explain your answer. ■

23. If $f(x)$ is continuous at $x = c$, then so is $|f(x)|$.

24. If $|f(x)|$ is continuous at $x = c$, then so is $f(x)$.

25. If f and g are discontinuous at $x = c$, then so is $f + g$.

26. If f and g are discontinuous at $x = c$, then so is fg.

27. If $\sqrt{f(x)}$ is continuous at $x = c$, then so is $f(x)$.
28. If $f(x)$ is continuous at $x = c$, then so is $\sqrt{f(x)}$.

29–30 Find a value of the constant k, if possible, that will make the function continuous everywhere. ■

29. (a) $f(x) = \begin{cases} 7x - 2, & x \leq 1 \\ kx^2, & x > 1 \end{cases}$
    (b) $f(x) = \begin{cases} kx^2, & x \leq 2 \\ 2x + k, & x > 2 \end{cases}$

30. (a) $f(x) = \begin{cases} 9 - x^2, & x \geq -3 \\ k/x^2, & x < -3 \end{cases}$
    (b) $f(x) = \begin{cases} 9 - x^2, & x \geq 0 \\ k/x^2, & x < 0 \end{cases}$

31. Find values of the constants k and m, if possible, that will make the function f continuous everywhere.
$$f(x) = \begin{cases} x^2 + 5, & x > 2 \\ m(x+1) + k, & -1 < x \leq 2 \\ 2x^3 + x + 7, & x \leq -1 \end{cases}$$

32. On which of the following intervals is
$$f(x) = \frac{1}{\sqrt{x-2}}$$
continuous?
(a) $[2, +\infty)$ (b) $(-\infty, +\infty)$ (c) $(2, +\infty)$ (d) $[1, 2)$

33–36 A function f is said to have a ***removable discontinuity*** at $x = c$ if $\lim_{x \to c} f(x)$ exists but f is not continuous at $x = c$, either because f is not defined at c or because the definition for $f(c)$ differs from the value of the limit. This terminology will be needed in these exercises. ■

33. (a) Sketch the graph of a function with a removable discontinuity at $x = c$ for which $f(c)$ is undefined.
    (b) Sketch the graph of a function with a removable discontinuity at $x = c$ for which $f(c)$ is defined.

34. (a) The terminology *removable discontinuity* is appropriate because a removable discontinuity of a function f at $x = c$ can be "removed" by redefining the value of f appropriately at $x = c$. What value for $f(c)$ removes the discontinuity?
    (b) Show that the following functions have removable discontinuities at $x = 1$, and sketch their graphs.
$$f(x) = \frac{x^2 - 1}{x - 1} \quad \text{and} \quad g(x) = \begin{cases} 1, & x > 1 \\ 0, & x = 1 \\ 1, & x < 1 \end{cases}$$
    (c) What values should be assigned to $f(1)$ and $g(1)$ to remove the discontinuities?

35–36 Find the values of x (if any) at which f is not continuous, and determine whether each such value is a removable discontinuity. ■

35. (a) $f(x) = \dfrac{|x|}{x}$ (b) $f(x) = \dfrac{x^2 + 3x}{x + 3}$
    (c) $f(x) = \dfrac{x - 2}{|x| - 2}$

36. (a) $f(x) = \dfrac{x^2 - 4}{x^3 - 8}$ (b) $f(x) = \begin{cases} 2x - 3, & x \leq 2 \\ x^2, & x > 2 \end{cases}$
    (c) $f(x) = \begin{cases} 3x^2 + 5, & x \neq 1 \\ 6, & x = 1 \end{cases}$

37. (a) Use a graphing utility to generate the graph of the function $f(x) = (x + 3)/(2x^2 + 5x - 3)$, and then use the graph to make a conjecture about the number and locations of all discontinuities.
    (b) Check your conjecture by factoring the denominator.

38. (a) Use a graphing utility to generate the graph of the function $f(x) = x/(x^3 - x + 2)$, and then use the graph to make a conjecture about the number and locations of all discontinuities.
    (b) Use the Intermediate-Value Theorem to approximate the locations of all discontinuities to two decimal places.

39. Prove that $f(x) = x^{3/5}$ is continuous everywhere, carefully justifying each step.

40. Prove that $f(x) = 1/\sqrt{x^4 + 7x^2 + 1}$ is continuous everywhere, carefully justifying each step.

41. Prove:
    (a) part (*a*) of Theorem 1.5.3
    (b) part (*b*) of Theorem 1.5.3
    (c) part (*c*) of Theorem 1.5.3.

42. Prove part (*b*) of Theorem 1.5.4.

43. (a) Use Theorem 1.5.5 to prove that if f is continuous at $x = c$, then $\lim_{h \to 0} f(c + h) = f(c)$.
    (b) Prove that if $\lim_{h \to 0} f(c + h) = f(c)$, then f is continuous at $x = c$. [*Hint:* What does this limit tell you about the continuity of $g(h) = f(c + h)$?]
    (c) Conclude from parts (a) and (b) that f is continuous at $x = c$ if and only if $\lim_{h \to 0} f(c + h) = f(c)$.

44. Prove: If f and g are continuous on $[a, b]$, and $f(a) > g(a)$, $f(b) < g(b)$, then there is at least one solution of the equation $f(x) = g(x)$ in (a, b). [*Hint:* Consider $f(x) - g(x)$.]

FOCUS ON CONCEPTS

45. Give an example of a function f that is defined on a closed interval, and whose values at the endpoints have opposite signs, but for which the equation $f(x) = 0$ has no solution in the interval.

46. Let f be the function whose graph is shown in Exercise 2. For each interval, determine (i) whether the hypothesis of the Intermediate-Value Theorem is satisfied, and (ii) whether the conclusion of the Intermediate-Value Theorem is satisfied.
    (a) $[1, 2]$ (b) $[2, 3]$ (c) $[1, 3]$

47. Show that the equation $x^3 + x^2 - 2x = 1$ has at least one solution in the interval $[-1, 1]$.

EXERCISE SET 2.3 Graphing Utility

1–8 Find dy/dx.

1. $y = 4x^7$ **2.** $y = -3x^{12}$

3. $y = 3x^8 + 2x + 1$ **4.** $y = \frac{1}{2}(x^4 + 7)$

5. $y = \pi^3$ **6.** $y = \sqrt{2}x + (1/\sqrt{2})$

7. $y = -\frac{1}{3}(x^7 + 2x - 9)$ **8.** $y = \dfrac{x^2+1}{5}$

9–16 Find $f'(x)$.

9. $f(x) = x^{-3} + \dfrac{1}{x^7}$ **10.** $f(x) = \sqrt{x} + \dfrac{1}{x}$

11. $f(x) = -3x^{-8} + 2\sqrt{x}$ **12.** $f(x) = 7x^{-6} - 5\sqrt{x}$

13. $f(x) = x^e + \dfrac{1}{x^{\sqrt{10}}}$ **14.** $f(x) = \sqrt[3]{\dfrac{8}{x}}$

15. $f(x) = (3x^2 + 1)^2$

16. $f(x) = ax^3 + bx^2 + cx + d$ $(a, b, c, d$ constant)

17–18 Find $y'(1)$.

17. $y = 5x^2 - 3x + 1$ **18.** $y = \dfrac{x^{3/2}+2}{x}$

19–20 Find dx/dt.

19. $x = t^2 - t$ **20.** $x = \dfrac{t^2+1}{3t}$

21–24 Find $dy/dx|_{x=1}$.

21. $y = 1 + x + x^2 + x^3 + x^4 + x^5$

22. $y = \dfrac{1 + x + x^2 + x^3 + x^4 + x^5 + x^6}{x^3}$

23. $y = (1 - x)(1 + x)(1 + x^2)(1 + x^4)$

24. $y = x^{24} + 2x^{12} + 3x^8 + 4x^6$

25–26 Approximate $f'(1)$ by considering the difference quotient

$$\frac{f(1+h) - f(1)}{h}$$

for values of h near 0, and then find the exact value of $f'(1)$ by differentiating.

25. $f(x) = x^3 - 3x + 1$ **26.** $f(x) = \dfrac{1}{x^2}$

27–28 Use a graphing utility to estimate the value of $f'(1)$ by zooming in on the graph of f, and then compare your estimate to the exact value obtained by differentiating.

27. $f(x) = \dfrac{x^2+1}{x}$ **28.** $f(x) = \dfrac{x + 2x^{3/2}}{\sqrt{x}}$

29–32 Find the indicated derivative.

29. $\dfrac{d}{dt}[16t^2]$ **30.** $\dfrac{dC}{dr}$, where $C = 2\pi r$

31. $V'(r)$, where $V = \pi r^3$ **32.** $\dfrac{d}{d\alpha}[2\alpha^{-1} + \alpha]$

33–36 True–False Determine whether the statement is true or false. Explain your answer.

33. If f and g are differentiable at $x = 2$, then

$$\left.\frac{d}{dx}[f(x) - 8g(x)]\right|_{x=2} = f'(2) - 8g'(2)$$

34. If $f(x)$ is a cubic polynomial, then $f'(x)$ is a quadratic polynomial.

35. If $f'(2) = 5$, then

$$\left.\frac{d}{dx}[4f(x) + x^3]\right|_{x=2} = \left.\frac{d}{dx}[4f(x) + 8]\right|_{x=2} = 4f'(2) = 20$$

36. If $f(x) = x^2(x^4 - x)$, then

$$f''(x) = \frac{d}{dx}[x^2] \cdot \frac{d}{dx}[x^4 - x] = 2x(4x^3 - 1)$$

37. A spherical balloon is being inflated.
(a) Find a general formula for the instantaneous rate of change of the volume V with respect to the radius r, given that $V = \frac{4}{3}\pi r^3$.
(b) Find the rate of change of V with respect to r at the instant when the radius is $r = 5$.

38. Find $\dfrac{d}{d\lambda}\left[\dfrac{\lambda\lambda_0 + \lambda^6}{2 - \lambda_0}\right]$ (λ_0 is constant).

39. Find an equation of the tangent line to the graph of $y = f(x)$ at $x = -3$ if $f(-3) = 2$ and $f'(-3) = 5$.

40. Find an equation of the tangent line to the graph of $y = f(x)$ at $x = 2$ if $f(2) = -2$ and $f'(2) = -1$.

41–42 Find d^2y/dx^2.

41. (a) $y = 7x^3 - 5x^2 + x$ (b) $y = 12x^2 - 2x + 3$
(c) $y = \dfrac{x+1}{x}$ (d) $y = (5x^2 - 3)(7x^3 + x)$

42. (a) $y = 4x^7 - 5x^3 + 2x$ (b) $y = 3x + 2$
(c) $y = \dfrac{3x-2}{5x}$ (d) $y = (x^3 - 5)(2x + 3)$

43–44 Find y'''.

43. (a) $y = x^{-5} + x^5$ (b) $y = 1/x$
(c) $y = ax^3 + bx + c$ $(a, b, c$ constant)

44. (a) $y = 5x^2 - 4x + 7$ (b) $y = 3x^{-2} + 4x^{-1} + x$
(c) $y = ax^4 + bx^2 + c$ $(a, b, c$ constant)

45. Find
(a) $f'''(2)$, where $f(x) = 3x^2 - 2$
(b) $\left.\dfrac{d^2y}{dx^2}\right|_{x=1}$, where $y = 6x^5 - 4x^2$
(c) $\left.\dfrac{d^4}{dx^4}[x^{-3}]\right|_{x=1}$.

46. Find
(a) $y'''(0)$, where $y = 4x^4 + 2x^3 + 3$
(b) $\left.\dfrac{d^4y}{dx^4}\right|_{x=1}$, where $y = \dfrac{6}{x^4}$.

47. Show that $y = x^3 + 3x + 1$ satisfies $y''' + xy'' - 2y' = 0$.

48. Show that if $x \neq 0$, then $y = 1/x$ satisfies the equation $x^3y'' + x^2y' - xy = 0$.

49–50 Use a graphing utility to make rough estimates of the locations of all horizontal tangent lines, and then find their exact locations by differentiating.

49. $y = \frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x$ **50.** $y = \dfrac{x^2 + 9}{x}$

FOCUS ON CONCEPTS

51. Find a function $y = ax^2 + bx + c$ whose graph has an x-intercept of 1, a y-intercept of -2, and a tangent line with a slope of -1 at the y-intercept.

52. Find k if the curve $y = x^2 + k$ is tangent to the line $y = 2x$.

53. Find the x-coordinate of the point on the graph of $y = x^2$ where the tangent line is parallel to the secant line that cuts the curve at $x = -1$ and $x = 2$.

54. Find the x-coordinate of the point on the graph of $y = \sqrt{x}$ where the tangent line is parallel to the secant line that cuts the curve at $x = 1$ and $x = 4$.

55. Find the coordinates of all points on the graph of $y = 1 - x^2$ at which the tangent line passes through the point $(2, 0)$.

56. Show that any two tangent lines to the parabola $y = ax^2$, $a \neq 0$, intersect at a point that is on the vertical line halfway between the points of tangency.

57. Suppose that L is the tangent line at $x = x_0$ to the graph of the cubic equation $y = ax^3 + bx$. Find the x-coordinate of the point where L intersects the graph a second time.

58. Show that the segment of the tangent line to the graph of $y = 1/x$ that is cut off by the coordinate axes is bisected by the point of tangency.

59. Show that the triangle that is formed by any tangent line to the graph of $y = 1/x$, $x > 0$, and the coordinate axes has an area of 2 square units.

60. Find conditions on a, b, c, and d so that the graph of the polynomial $f(x) = ax^3 + bx^2 + cx + d$ has
(a) exactly two horizontal tangents
(b) exactly one horizontal tangent
(c) no horizontal tangents.

61. Newton's Law of Universal Gravitation states that the magnitude F of the force exerted by a point with mass M on a point with mass m is
$$F = \frac{GmM}{r^2}$$
where G is a constant and r is the distance between the bodies. Assuming that the points are moving, find a formula for the instantaneous rate of change of F with respect to r.

62. In the temperature range between 0°C and 700°C the resistance R [in ohms (Ω)] of a certain platinum resistance thermometer is given by
$$R = 10 + 0.04124T - 1.779 \times 10^{-5}T^2$$
where T is the temperature in degrees Celsius. Where in the interval from 0°C to 700°C is the resistance of the thermometer most sensitive and least sensitive to temperature changes? [*Hint:* Consider the size of dR/dT in the interval $0 \leq T \leq 700$.]

63–64 Use a graphing utility to make rough estimates of the intervals on which $f'(x) > 0$, and then find those intervals exactly by differentiating.

63. $f(x) = x - \dfrac{1}{x}$ **64.** $f(x) = x^3 - 3x$

65–68 You are asked in these exercises to determine whether a piecewise-defined function f is differentiable at a value $x = x_0$, where f is defined by different formulas on different sides of x_0. You may use without proof the following result, which is a consequence of the Mean-Value Theorem (discussed in Section 4.8). ***Theorem.*** *Let f be continuous at x_0 and suppose that $\lim_{x\to x_0} f'(x)$ exists. Then f is differentiable at x_0, and $f'(x_0) = \lim_{x\to x_0} f'(x)$.*

65. Show that
$$f(x) = \begin{cases} x^2 + x + 1, & x \leq 1 \\ 3x, & x > 1 \end{cases}$$
is continuous at $x = 1$. Determine whether f is differentiable at $x = 1$. If so, find the value of the derivative there. Sketch the graph of f.

66. Let
$$f(x) = \begin{cases} x^2 - 16x, & x < 9 \\ \sqrt{x}, & x \geq 9 \end{cases}$$
Is f continuous at $x = 9$? Determine whether f is differentiable at $x = 9$. If so, find the value of the derivative there.

67. Let
$$f(x) = \begin{cases} x^2, & x \leq 1 \\ \sqrt{x}, & x > 1 \end{cases}$$
Determine whether f is differentiable at $x = 1$. If so, find the value of the derivative there.

68. Let
$$f(x) = \begin{cases} x^3 + \frac{1}{16}, & x < \frac{1}{2} \\ \frac{3}{4}x^2, & x \geq \frac{1}{2} \end{cases}$$
Determine whether f is differentiable at $x = \frac{1}{2}$. If so, find the value of the derivative there.

69. Find all points where f fails to be differentiable. Justify your answer.
(a) $f(x) = |3x - 2|$ (b) $f(x) = |x^2 - 4|$

EXERCISE SET 2.4 Graphing Utility

1–4 Compute the derivative of the given function $f(x)$ by (a) multiplying and then differentiating and (b) using the product rule. Verify that (a) and (b) yield the same result.

1. $f(x) = (x+1)(2x-1)$
2. $f(x) = (3x^2-1)(x^2+2)$
3. $f(x) = (x^2+1)(x^2-1)$
4. $f(x) = (x+1)(x^2-x+1)$

5–20 Find $f'(x)$.

5. $f(x) = (3x^2+6)\left(2x-\frac{1}{4}\right)$
6. $f(x) = (2-x-3x^3)(7+x^5)$
7. $f(x) = (x^3+7x^2-8)(2x^{-3}+x^{-4})$
8. $f(x) = \left(\dfrac{1}{x}+\dfrac{1}{x^2}\right)(3x^3+27)$
9. $f(x) = (x-2)(x^2+2x+4)$
10. $f(x) = (x^2+x)(x^2-x)$
11. $f(x) = \dfrac{3x+4}{x^2+1}$
12. $f(x) = \dfrac{x-2}{x^4+x+1}$
13. $f(x) = \dfrac{x^2}{3x-4}$
14. $f(x) = \dfrac{2x^2+5}{3x-4}$
15. $f(x) = \dfrac{(2\sqrt{x}+1)(x-1)}{x+3}$
16. $f(x) = (2\sqrt{x}+1)\left(\dfrac{2-x}{x^2+3x}\right)$
17. $f(x) = (2x+1)\left(1+\dfrac{1}{x}\right)(x^{-3}+7)$
18. $f(x) = x^{-5}(x^2+2x)(4-3x)(2x^9+1)$
19. $f(x) = (x^7+2x-3)^3$
20. $f(x) = (x^2+1)^4$

21–24 Find $dy/dx|_{x=1}$.

21. $y = \dfrac{2x-1}{x+3}$
22. $y = \dfrac{4x+1}{x^2-5}$
23. $y = \left(\dfrac{3x+2}{x}\right)(x^{-5}+1)$
24. $y = (2x^7-x^2)\left(\dfrac{x-1}{x+1}\right)$

25–26 Use a graphing utility to estimate the value of $f'(1)$ by zooming in on the graph of f, and then compare your estimate to the exact value obtained by differentiating.

25. $f(x) = \dfrac{x}{x^2+1}$
26. $f(x) = \dfrac{x^2-1}{x^2+1}$

27. Find $g'(4)$ given that $f(4)=3$ and $f'(4)=-5$.
 (a) $g(x) = \sqrt{x}f(x)$ (b) $g(x) = \dfrac{f(x)}{x}$

28. Find $g'(3)$ given that $f(3)=-2$ and $f'(3)=4$.
 (a) $g(x) = 3x^2-5f(x)$ (b) $g(x) = \dfrac{2x+1}{f(x)}$

29. In parts (a)–(d), $F(x)$ is expressed in terms of $f(x)$ and $g(x)$. Find $F'(2)$ given that $f(2)=-1$, $f'(2)=4$, $g(2)=1$, and $g'(2)=-5$.
 (a) $F(x) = 5f(x)+2g(x)$ (b) $F(x) = f(x)-3g(x)$
 (c) $F(x) = f(x)g(x)$ (d) $F(x) = f(x)/g(x)$

30. Find $F'(\pi)$ given that $f(\pi)=10$, $f'(\pi)=-1$, $g(\pi)=-3$, and $g'(\pi)=2$.
 (a) $F(x) = 6f(x)-5g(x)$ (b) $F(x) = x(f(x)+g(x))$
 (c) $F(x) = 2f(x)g(x)$ (d) $F(x) = \dfrac{f(x)}{4+g(x)}$

31–36 Find all values of x at which the tangent line to the given curve satisfies the stated property.

31. $y = \dfrac{x^2-1}{x+2}$; horizontal
32. $y = \dfrac{x^2+1}{x-1}$; horizontal
33. $y = \dfrac{x^2+1}{x+1}$; parallel to the line $y=x$
34. $y = \dfrac{x+3}{x+2}$; perpendicular to the line $y=x$
35. $y = \dfrac{1}{x+4}$; passes through the origin
36. $y = \dfrac{2x+5}{x+2}$; y-intercept 2

FOCUS ON CONCEPTS

37. (a) What should it mean to say that two curves intersect at right angles?
 (b) Show that the curves $y=1/x$ and $y=1/(2-x)$ intersect at right angles.
38. Find all values of a such that the curves $y=a/(x-1)$ and $y=x^2-2x+1$ intersect at right angles.
39. Find a general formula for $F''(x)$ if $F(x)=xf(x)$ and f and f' are differentiable at x.
40. Suppose that the function f is differentiable everywhere and $F(x)=xf(x)$.
 (a) Express $F'''(x)$ in terms of x and derivatives of f.
 (b) For $n \geq 2$, conjecture a formula for $F^{(n)}(x)$.

41. A manufacturer of athletic footwear finds that the sales of their ZipStride brand running shoes is a function $f(p)$ of the selling price p (in dollars) for a pair of shoes. Suppose that $f(120)=9000$ pairs of shoes and $f'(120)=-60$ pairs of shoes per dollar. The revenue that the manufacturer will receive for selling $f(p)$ pairs of shoes at p dollars per pair is $R(p)=p\cdot f(p)$. Find $R'(120)$. What impact would a small increase in price have on the manufacturer's revenue?
42. Solve the problem in Exercise 41 under the assumption that $f(120)=9000$ and $f'(120)=-80$.
43. Use the quotient rule (Theorem 2.4.2) to derive the formula for the derivative of $f(x)=x^{-n}$, where n is a positive integer.

✔QUICK CHECK EXERCISES 2.5

(See page 174 for answers.)

1. Find dy/dx.
(a) $y = \sin x$ (b) $y = \cos x$
(c) $y = \tan x$ (d) $y = \sec x$

2. Find $f'(x)$ and $f'(\pi/3)$ if $f(x) = \sin x \cos x$.

3. Use a derivative to evaluate each limit.
(a) $\lim\limits_{h\to 0} \dfrac{\sin\left(\frac{\pi}{2}+h\right)-1}{h}$ (b) $\lim\limits_{h\to 0} \dfrac{\csc(x+h)-\csc x}{h}$

EXERCISE SET 2.5

Graphing Utility

1–18 Find $f'(x)$.

1. $f(x) = 4\cos x + 2\sin x$
2. $f(x) = \dfrac{5}{x^2} + \sin x$
3. $f(x) = -4x^2 \cos x$
4. $f(x) = 2\sin^2 x$
5. $f(x) = \dfrac{5-\cos x}{5+\sin x}$
6. $f(x) = \dfrac{\sin x}{x^2 + \sin x}$
7. $f(x) = \sec x - \sqrt{2}\tan x$
8. $f(x) = (x^2+1)\sec x$
9. $f(x) = 4\csc x - \cot x$
10. $f(x) = \cos x - x\csc x$
11. $f(x) = \sec x \tan x$
12. $f(x) = \csc x \cot x$
13. $f(x) = \dfrac{\cot x}{1+\csc x}$
14. $f(x) = \dfrac{\sec x}{1+\tan x}$
15. $f(x) = \sin^2 x + \cos^2 x$
16. $f(x) = \sec^2 x - \tan^2 x$
17. $f(x) = \dfrac{\sin x \sec x}{1 + x\tan x}$
18. $f(x) = \dfrac{(x^2+1)\cot x}{3 - \cos x \csc x}$

19–24 Find d^2y/dx^2.

19. $y = x\cos x$
20. $y = \csc x$
21. $y = x\sin x - 3\cos x$
22. $y = x^2\cos x + 4\sin x$
23. $y = \sin x \cos x$
24. $y = \tan x$

25. Find the equation of the line tangent to the graph of $\tan x$ at
(a) $x = 0$ (b) $x = \pi/4$ (c) $x = -\pi/4$.

26. Find the equation of the line tangent to the graph of $\sin x$ at
(a) $x = 0$ (b) $x = \pi$ (c) $x = \pi/4$.

27. (a) Show that $y = x\sin x$ is a solution to $y'' + y = 2\cos x$.
(b) Show that $y = x\sin x$ is a solution of the equation $y^{(4)} + y'' = -2\cos x$.

28. (a) Show that $y = \cos x$ and $y = \sin x$ are solutions of the equation $y'' + y = 0$.
(b) Show that $y = A\sin x + B\cos x$ is a solution of the equation $y'' + y = 0$ for all constants A and B.

29. Find all values in the interval $[-2\pi, 2\pi]$ at which the graph of f has a horizontal tangent line.
(a) $f(x) = \sin x$ (b) $f(x) = x + \cos x$
(c) $f(x) = \tan x$ (d) $f(x) = \sec x$

30. (a) Use a graphing utility to make rough estimates of the values in the interval $[0, 2\pi]$ at which the graph of $y = \sin x \cos x$ has a horizontal tangent line.
(b) Find the exact locations of the points where the graph has a horizontal tangent line.

31. A 10 ft ladder leans against a wall at an angle θ with the horizontal, as shown in the accompanying figure. The top of the ladder is x feet above the ground. If the bottom of the ladder is pushed toward the wall, find the rate at which x changes with respect to θ when $\theta = 60°$. Express the answer in units of feet/degree.

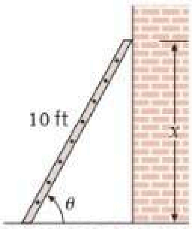


Figure Ex-31

32. An airplane is flying on a horizontal path at a height of 3800 ft, as shown in the accompanying figure. At what rate is the distance s between the airplane and the fixed point P changing with respect to θ when $\theta = 30°$? Express the answer in units of feet/degree.

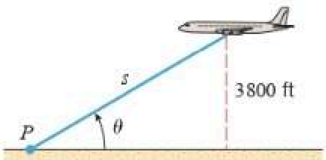


Figure Ex-32

33. A searchlight is trained on the side of a tall building. As the light rotates, the spot it illuminates moves up and down the side of the building. That is, the distance D between ground level and the illuminated spot on the side of the building is a function of the angle θ formed by the light beam and the horizontal (see the accompanying figure). If the searchlight is located 50 m from the building, find the rate at which D is changing with respect to θ when $\theta = 45°$. Express your answer in units of meters/degree.

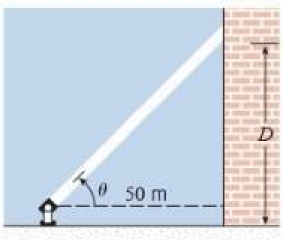


Figure Ex-33

FOCUS ON CONCEPTS

5. Given the following table of values, find the indicated derivatives in parts (a) and (b).

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
3	5	−2	5	7
5	3	−1	12	4

(a) $F'(3)$, where $F(x) = f(g(x))$
(b) $G'(3)$, where $G(x) = g(f(x))$

6. Given the following table of values, find the indicated derivatives in parts (a) and (b).

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
−1	2	3	2	−3
2	0	4	1	−5

(a) $F'(-1)$, where $F(x) = f(g(x))$
(b) $G'(-1)$, where $G(x) = g(f(x))$

7–26 Find $f'(x)$.

7. $f(x) = (x^3 + 2x)^{37}$
8. $f(x) = (3x^2 + 2x - 1)^6$
9. $f(x) = \left(x^3 - \dfrac{7}{x}\right)^{-2}$
10. $f(x) = \dfrac{1}{(x^5 - x + 1)^9}$
11. $f(x) = \dfrac{4}{(3x^2 - 2x + 1)^3}$
12. $f(x) = \sqrt{x^3 - 2x + 5}$
13. $f(x) = \sqrt{4 + \sqrt{3x}}$
14. $f(x) = \sqrt[3]{12 + \sqrt{x}}$
15. $f(x) = \sin\left(\dfrac{1}{x^2}\right)$
16. $f(x) = \tan\sqrt{x}$
17. $f(x) = 4\cos^5 x$
18. $f(x) = 4x + 5\sin^4 x$
19. $f(x) = \cos^2(3\sqrt{x})$
20. $f(x) = \tan^4(x^3)$
21. $f(x) = 2\sec^2(x^7)$
22. $f(x) = \cos^3\left(\dfrac{x}{x+1}\right)$
23. $f(x) = \sqrt{\cos(5x)}$
24. $f(x) = \sqrt{3x - \sin^2(4x)}$
25. $f(x) = [x + \csc(x^3 + 3)]^{-3}$
26. $f(x) = [x^4 - \sec(4x^2 - 2)]^{-4}$

27–40 Find dy/dx.

27. $y = x^3\sin^2(5x)$
28. $y = \sqrt{x}\tan^3(\sqrt{x})$
29. $y = x^5\sec(1/x)$
30. $y = \dfrac{\sin x}{\sec(3x+1)}$
31. $y = \cos(\cos x)$
32. $y = \sin(\tan 3x)$
33. $y = \cos^3(\sin 2x)$
34. $y = \dfrac{1 + \csc(x^2)}{1 - \cot(x^2)}$
35. $y = (5x+8)^7\left(1 - \sqrt{x}\right)^6$
36. $y = (x^2 + x)^5\sin^8 x$
37. $y = \left(\dfrac{x-5}{2x+1}\right)^3$
38. $y = \left(\dfrac{1+x^2}{1-x^2}\right)^{17}$
39. $y = \dfrac{(2x+3)^3}{(4x^2-1)^8}$
40. $y = [1 + \sin^3(x^5)]^{12}$

C **41–42** Use a CAS to find dy/dx.

41. $y = [x\sin 2x + \tan^4(x^7)]^5$
42. $y = \tan^4\left(2 + \dfrac{(7-x)\sqrt{3x^2+5}}{x^3 + \sin x}\right)$

43–50 Find an equation for the tangent line to the graph at the specified value of x.

43. $y = x\cos 3x,\ x = \pi$
44. $y = \sin(1 + x^3),\ x = -3$
45. $y = \sec^3\left(\dfrac{\pi}{2} - x\right),\ x = -\dfrac{\pi}{2}$
46. $y = \left(x - \dfrac{1}{x}\right)^3,\ x = 2$
47. $y = \tan(4x^2),\ x = \sqrt{\pi}$
48. $y = 3\cot^4 x,\ x = \dfrac{\pi}{4}$
49. $y = x^2\sqrt{5 - x^2},\ x = 1$
50. $y = \dfrac{x}{\sqrt{1-x^2}},\ x = 0$

51–54 Find d^2y/dx^2.

51. $y = x\cos(5x) - \sin^2 x$
52. $y = \sin(3x^2)$
53. $y = \dfrac{1+x}{1-x}$
54. $y = x\tan\left(\dfrac{1}{x}\right)$

55–58 Find the indicated derivative.

55. $y = \cot^3(\pi - \theta)$; find $\dfrac{dy}{d\theta}$.
56. $\lambda = \left(\dfrac{au+b}{cu+d}\right)^6$; find $\dfrac{d\lambda}{du}$ $(a, b, c, d$ constants).
57. $\dfrac{d}{d\omega}[a\cos^2\pi\omega + b\sin^2\pi\omega]$ $(a, b$ constants)
58. $x = \csc^2\left(\dfrac{\pi}{3} - y\right)$; find $\dfrac{dx}{dy}$.

59. (a) Use a graphing utility to obtain the graph of the function $f(x) = x\sqrt{4 - x^2}$.
(b) Use the graph in part (a) to make a rough sketch of the graph of f'.
(c) Find $f'(x)$, and then check your work in part (b) by using the graphing utility to obtain the graph of f'.
(d) Find the equation of the tangent line to the graph of f at $x = 1$, and graph f and the tangent line together.

60. (a) Use a graphing utility to obtain the graph of the function $f(x) = \sin x^2\cos x$ over the interval $[-\pi/2, \pi/2]$.
(b) Use the graph in part (a) to make a rough sketch of the graph of f' over the interval.
(c) Find $f'(x)$, and then check your work in part (b) by using the graphing utility to obtain the graph of f' over the interval.
(d) Find the equation of the tangent line to the graph of f at $x = 1$, and graph f and the tangent line together over the interval.

DIFFERENTIABILITY OF FUNCTIONS DEFINED IMPLICITLY

When differentiating implicitly, it is assumed that y represents a differentiable function of x. If this is not so, then the resulting calculations may be nonsense. For example, if we differentiate the equation

$$x^2 + y^2 + 1 = 0 \tag{13}$$

we obtain

$$2x + 2y\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{y}$$

However, this derivative is meaningless because there are no real values of x and y that satisfy (13) (why?); and hence (13) does not define any real functions implicitly.

The nonsensical conclusion of these computations conveys the importance of knowing whether an equation in x and y that is to be differentiated implicitly actually defines some differentiable function of x implicitly. Unfortunately, this can be a difficult problem, so we will leave the discussion of such matters for more advanced courses in analysis.

✔ QUICK CHECK EXERCISES 3.1 (See page 192 for answers.)

1. The equation $xy + 2y = 1$ defines implicitly the function $y =$ ______.
2. Use implicit differentiation to find dy/dx for $x^2 - y^3 = xy$.
3. The slope of the tangent line to the graph of $x + y + xy = 3$ at $(1, 1)$ is ______.
4. Use implicit differentiation to find d^2y/dx^2 for $\sin y = x$.

EXERCISE SET 3.1 C CAS

1–2
(a) Find dy/dx by differentiating implicitly.
(b) Solve the equation for y as a function of x, and find dy/dx from that equation.
(c) Confirm that the two results are consistent by expressing the derivative in part (a) as a function of x alone. ■

1. $x + xy - 2x^3 = 2$
2. $\sqrt{y} - \sin x = 2$

3–12 Find dy/dx by implicit differentiation. ■

3. $x^2 + y^2 = 100$
4. $x^3 + y^3 = 3xy^2$
5. $x^2y + 3xy^3 - x = 3$
6. $x^3y^2 - 5x^2y + x = 1$
7. $\dfrac{1}{\sqrt{x}} + \dfrac{1}{\sqrt{y}} = 1$
8. $x^2 = \dfrac{x+y}{x-y}$
9. $\sin(x^2y^2) = x$
10. $\cos(xy^2) = y$
11. $\tan^3(xy^2 + y) = x$
12. $\dfrac{xy^3}{1 + \sec y} = 1 + y^4$

13–18 Find d^2y/dx^2 by implicit differentiation. ■

13. $2x^2 - 3y^2 = 4$
14. $x^3 + y^3 = 1$
15. $x^3y^3 - 4 = 0$
16. $xy + y^2 = 2$
17. $y + \sin y = x$
18. $x \cos y = y$

19–20 Find the slope of the tangent line to the curve at the given points in two ways: first by solving for y in terms of x and differentiating and then by implicit differentiation. ■

19. $x^2 + y^2 = 1$; $(1/2, \sqrt{3}/2)$, $(1/2, -\sqrt{3}/2)$
20. $y^2 - x + 1 = 0$; $(10, 3)$, $(10, -3)$

21–24 True–False Determine whether the statement is true or false. Explain your answer. ■

21. If an equation in x and y defines a function $y = f(x)$ implicitly, then the graph of the equation and the graph of f are identical.
22. The function
$$f(x) = \begin{cases} \sqrt{1-x^2}, & 0 < x \le 1 \\ -\sqrt{1-x^2}, & -1 \le x \le 0 \end{cases}$$
is defined implicitly by the equation $x^2 + y^2 = 1$.
23. The function $|x|$ is not defined implicitly by the equation $(x + y)(x - y) = 0$.
24. If y is defined implicitly as a function of x by the equation $x^2 + y^2 = 1$, then $dy/dx = -x/y$.

25–28 Use implicit differentiation to find the slope of the tangent line to the curve at the specified point, and check that your answer is consistent with the accompanying graph on the next page. ■

25. $x^4 + y^4 = 16$; $(1, \sqrt[4]{15})$ [*Lamé's special quartic*]
26. $y^3 + yx^2 + x^2 - 3y^2 = 0$; $(0, 3)$ [*trisectrix*]
27. $2(x^2 + y^2)^2 = 25(x^2 - y^2)$; $(3, 1)$ [*lemniscate*]
28. $x^{2/3} + y^{2/3} = 4$; $(-1, 3\sqrt{3})$ [*four-cusped hypocycloid*]

(b) Given that $dy/dt = 8$, find dx/dt when $(x, y) = \left(\frac{1}{3}, -\frac{\sqrt{5}}{9}\right)$.

4. Equation: $x^2 + y^2 = 2x + 4y$.

(a) Given that $dx/dt = -5$, find dy/dt when $(x, y) = (3, 1)$.

(b) Given that $dy/dt = 6$, find dx/dt when $(x, y) = (1 + \sqrt{2}, 2 + \sqrt{3})$.

FOCUS ON CONCEPTS

5. Let A be the area of a square whose sides have length x, and assume that x varies with the time t.

(a) Draw a picture of the square with the labels A and x placed appropriately.

(b) Write an equation that relates A and x.

(c) Use the equation in part (b) to find an equation that relates dA/dt and dx/dt.

(d) At a certain instant the sides are 3 ft long and increasing at a rate of 2 ft/min. How fast is the area increasing at that instant?

6. In parts (a)–(d), let A be the area of a circle of radius r, and assume that r increases with the time t.

(a) Draw a picture of the circle with the labels A and r placed appropriately.

(b) Write an equation that relates A and r.

(c) Use the equation in part (b) to find an equation that relates dA/dt and dr/dt.

(d) At a certain instant the radius is 5 cm and increasing at the rate of 2 cm/s. How fast is the area increasing at that instant?

7. Let V be the volume of a cylinder having height h and radius r, and assume that h and r vary with time.

(a) How are dV/dt, dh/dt, and dr/dt related?

(b) At a certain instant, the height is 6 in and increasing at 1 in/s, while the radius is 10 in and decreasing at 1 in/s. How fast is the volume changing at that instant? Is the volume increasing or decreasing at that instant?

8. Let l be the length of a diagonal of a rectangle whose sides have lengths x and y, and assume that x and y vary with time.

(a) How are dl/dt, dx/dt, and dy/dt related?

(b) If x increases at a constant rate of $\frac{1}{2}$ ft/s and y decreases at a constant rate of $\frac{1}{4}$ ft/s, how fast is the size of the diagonal changing when $x = 3$ ft and $y = 4$ ft? Is the diagonal increasing or decreasing at that instant?

9. Let θ (in radians) be an acute angle in a right triangle, and let x and y, respectively, be the lengths of the sides adjacent to and opposite θ. Suppose also that x and y vary with time.

(a) How are $d\theta/dt$, dx/dt, and dy/dt related?

(b) At a certain instant, $x = 2$ units and is increasing at 1 unit/s, while $y = 2$ units and is decreasing at $\frac{1}{4}$ unit/s. How fast is θ changing at that instant? Is θ increasing or decreasing at that instant?

10. Suppose that $z = x^3y^2$, where both x and y are changing with time. At a certain instant when $x = 1$ and $y = 2$, x is decreasing at the rate of 2 units/s, and y is increasing at the rate of 3 units/s. How fast is z changing at this instant? Is z increasing or decreasing?

11. The minute hand of a certain clock is 4 in long. Starting from the moment when the hand is pointing straight up, how fast is the area of the sector that is swept out by the hand increasing at any instant during the next revolution of the hand?

12. A stone dropped into a still pond sends out a circular ripple whose radius increases at a constant rate of 3 ft/s. How rapidly is the area enclosed by the ripple increasing at the end of 10 s?

13. Oil spilled from a ruptured tanker spreads in a circle whose area increases at a constant rate of 6 mi^2/h. How fast is the radius of the spill increasing when the area is 9 mi^2?

14. A spherical balloon is inflated so that its volume is increasing at the rate of 3 ft^3/min. How fast is the diameter of the balloon increasing when the radius is 1 ft?

15. A spherical balloon is to be deflated so that its radius decreases at a constant rate of 15 cm/min. At what rate must air be removed when the radius is 9 cm?

16. A 17 ft ladder is leaning against a wall. If the bottom of the ladder is pulled along the ground away from the wall at a constant rate of 5 ft/s, how fast will the top of the ladder be moving down the wall when it is 8 ft above the ground?

17. A 13 ft ladder is leaning against a wall. If the top of the ladder slips down the wall at a rate of 2 ft/s, how fast will the foot be moving away from the wall when the top is 5 ft above the ground?

18. A 10 ft plank is leaning against a wall. If at a certain instant the bottom of the plank is 2 ft from the wall and is being pushed toward the wall at the rate of 6 in/s, how fast is the acute angle that the plank makes with the ground increasing?

19. A softball diamond is a square whose sides are 60 ft long. Suppose that a player running from first to second base has a speed of 25 ft/s at the instant when she is 10 ft from second base. At what rate is the player's distance from home plate changing at that instant?

20. A rocket, rising vertically, is tracked by a radar station that is on the ground 5 mi from the launchpad. How fast is the rocket rising when it is 4 mi high and its distance from the radar station is increasing at a rate of 2000 mi/h?

21. For the camera and rocket shown in Figure 3.4.5, at what rate is the camera-to-rocket distance changing when the rocket is 4000 ft up and rising vertically at 880 ft/s?

Example 6 Find $\lim_{x\to 0}(1+\sin x)^{1/x}$.

Solution. As discussed above, we begin by introducing a dependent variable

$$y = (1+\sin x)^{1/x}$$

and taking the natural logarithm of both sides:

$$\ln y = \ln(1+\sin x)^{1/x} = \frac{1}{x}\ln(1+\sin x) = \frac{\ln(1+\sin x)}{x}$$

Thus,

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1+\sin x)}{x}$$

which is an indeterminate form of type $0/0$, so by L'Hôpital's rule

$$\lim_{x\to 0}\ln y = \lim_{x\to 0}\frac{\ln(1+\sin x)}{x} = \lim_{x\to 0}\frac{(\cos x)/(1+\sin x)}{1} = 1$$

Since we have shown that $\ln y \to 1$ as $x \to 0$, the continuity of the exponential function implies that $e^{\ln y} \to e^1$ as $x \to 0$, and this implies that $y \to e$ as $x \to 0$. Thus,

$$\lim_{x\to 0}(1+\sin x)^{1/x} = e \blacktriangleleft$$

✔ QUICK CHECK EXERCISES 3.6

(See page 228 for answers.)

1. In each part, does L'Hôpital's rule apply to the given limit?
   (a) $\lim_{x\to 1}\frac{2x-2}{x^3+x-2}$ (b) $\lim_{x\to 0}\frac{\cos x}{x}$
   (c) $\lim_{x\to 0}\frac{e^{2x}-1}{\tan x}$

2. Evaluate each of the limits in Quick Check Exercise 1.

3. Using L'Hôpital's rule, $\lim_{x\to +\infty}\frac{e^x}{500x^2} =$ ________.

EXERCISE SET 3.6 Graphing Utility C CAS

1–2 Evaluate the given limit without using L'Hôpital's rule, and then check that your answer is correct using L'Hôpital's rule. ■

1. (a) $\lim_{x\to 2}\frac{x^2-4}{x^2+2x-8}$ (b) $\lim_{x\to +\infty}\frac{2x-5}{3x+7}$

2. (a) $\lim_{x\to 0}\frac{\sin x}{\tan x}$ (b) $\lim_{x\to 1}\frac{x^2-1}{x^3-1}$

3–6 True–False Determine whether the statement is true or false. Explain your answer. ■

3. L'Hôpital's rule does not apply to $\lim_{x\to -\infty}\frac{\ln x}{x}$.

4. For any polynomial $p(x)$, $\lim_{x\to +\infty}\frac{p(x)}{e^x} = 0$.

5. If n is chosen sufficiently large, then $\lim_{x\to +\infty}\frac{(\ln x)^n}{x} = +\infty$.

6. $\lim_{x\to 0^+}(\sin x)^{1/x} = 0$

7–45 Find the limits. ■

7. $\lim_{x\to 0}\frac{e^x-1}{\sin x}$

8. $\lim_{x\to 0}\frac{\sin 2x}{\sin 5x}$

9. $\lim_{\theta\to 0}\frac{\tan\theta}{\theta}$

10. $\lim_{t\to 0}\frac{te^t}{1-e^t}$

11. $\lim_{x\to \pi^+}\frac{\sin x}{x-\pi}$

12. $\lim_{x\to 0^+}\frac{\sin x}{x^2}$

13. $\lim_{x\to +\infty}\frac{\ln x}{x}$

14. $\lim_{x\to +\infty}\frac{e^{3x}}{x^2}$

15. $\lim_{x\to 0^+}\frac{\cot x}{\ln x}$

16. $\lim_{x\to 0^+}\frac{1-\ln x}{e^{1/x}}$

17. $\lim_{x\to +\infty}\frac{x^{100}}{e^x}$

18. $\lim_{x\to 0^+}\frac{\ln(\sin x)}{\ln(\tan x)}$

19. $\lim_{x\to 0}\frac{\sin^{-1}2x}{x}$

20. $\lim_{x\to 0}\frac{x-\tan^{-1}x}{x^3}$

21. $\lim_{x\to +\infty} xe^{-x}$

22. $\lim_{x\to \pi^-}(x-\pi)\tan\frac{1}{2}x$

23. $\lim_{x\to +\infty} x\sin\frac{\pi}{x}$

24. $\lim_{x\to 0^+}\tan x \ln x$

25. $\lim_{x\to \pi/2^-}\sec 3x\cos 5x$

26. $\lim_{x\to \pi}(x-\pi)\cot x$

27. $\lim_{x\to +\infty}(1-3/x)^x$

28. $\lim_{x\to 0}(1+2x)^{-3/x}$

29. $\lim_{x\to 0}(e^x + x)^{1/x}$
30. $\lim_{x\to+\infty}(1 + a/x)^{bx}$

31. $\lim_{x\to 1}(2 - x)^{\tan[(\pi/2)x]}$
32. $\lim_{x\to+\infty}[\cos(2/x)]^{x^2}$

33. $\lim_{x\to 0}(\csc x - 1/x)$
34. $\lim_{x\to 0}\left(\dfrac{1}{x^2} - \dfrac{\cos 3x}{x^2}\right)$

35. $\lim_{x\to+\infty}(\sqrt{x^2 + x} - x)$
36. $\lim_{x\to 0}\left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right)$

37. $\lim_{x\to+\infty}[x - \ln(x^2 + 1)]$
38. $\lim_{x\to+\infty}[\ln x - \ln(1 + x)]$

39. $\lim_{x\to 0^+} x^{\sin x}$
40. $\lim_{x\to 0^+}(e^{2x} - 1)^x$

41. $\lim_{x\to 0^+}\left[-\dfrac{1}{\ln x}\right]^x$
42. $\lim_{x\to+\infty} x^{1/x}$

43. $\lim_{x\to+\infty}(\ln x)^{1/x}$
44. $\lim_{x\to 0^+}(-\ln x)^x$

45. $\lim_{x\to\pi/2^-}(\tan x)^{(\pi/2)-x}$

46. Show that for any positive integer n

(a) $\lim_{x\to+\infty}\dfrac{\ln x}{x^n} = 0$ (b) $\lim_{x\to+\infty}\dfrac{x^n}{\ln x} = +\infty.$

FOCUS ON CONCEPTS

47. (a) Find the error in the following calculation:

$$\lim_{x\to 1}\frac{x^3 - x^2 + x - 1}{x^3 - x^2} = \lim_{x\to 1}\frac{3x^2 - 2x + 1}{3x^2 - 2x} = \lim_{x\to 1}\frac{6x - 2}{6x - 2} = 1$$

(b) Find the correct limit.

48. (a) Find the error in the following calculation:

$$\lim_{x\to 2}\frac{e^{3x^2-12x+12}}{x^4 - 16} = \lim_{x\to 2}\frac{(6x - 12)e^{3x^2-12x+12}}{4x^3} = 0$$

(b) Find the correct limit.

49–52 Make a conjecture about the limit by graphing the function involved with a graphing utility; then check your conjecture using L'Hôpital's rule. ■

49. $\lim_{x\to+\infty}\dfrac{\ln(\ln x)}{\sqrt{x}}$
50. $\lim_{x\to 0^+} x^x$

51. $\lim_{x\to 0^+}(\sin x)^{3/\ln x}$
52. $\lim_{x\to(\pi/2)^-}\dfrac{4\tan x}{1 + \sec x}$

53–56 Make a conjecture about the equations of horizontal asymptotes, if any, by graphing the equation with a graphing utility; then check your answer using L'Hôpital's rule. ■

53. $y = \ln x - e^x$
54. $y = x - \ln(1 + 2e^x)$

55. $y = (\ln x)^{1/x}$
56. $y = \left(\dfrac{x + 1}{x + 2}\right)^x$

57. Limits of the type

$$0/\infty,\quad \infty/0,\quad 0^\infty,\quad \infty\cdot\infty,\quad +\infty + (+\infty),$$
$$+\infty - (-\infty),\quad -\infty + (-\infty),\quad -\infty - (+\infty)$$

are *not* indeterminate forms. Find the following limits by inspection.

(a) $\lim_{x\to 0^+}\dfrac{x}{\ln x}$ (b) $\lim_{x\to+\infty}\dfrac{x^3}{e^{-x}}$

(c) $\lim_{x\to(\pi/2)^-}(\cos x)^{\tan x}$ (d) $\lim_{x\to 0^+}(\ln x)\cot x$

(e) $\lim_{x\to 0^+}\left(\dfrac{1}{x} - \ln x\right)$ (f) $\lim_{x\to-\infty}(x + x^3)$

58. There is a myth that circulates among beginning calculus students which states that all indeterminate forms of types 0^0, ∞^0, and 1^∞ have value 1 because "anything to the zero power is 1" and "1 to any power is 1." The fallacy is that 0^0, ∞^0, and 1^∞ are not powers of numbers, but rather descriptions of limits. The following examples, which were suggested by Prof. Jack Staib of Drexel University, show that such indeterminate forms can have any positive real value:

(a) $\lim_{x\to 0^+}[x^{(\ln a)/(1+\ln x)}] = a$ (form 0^0)

(b) $\lim_{x\to+\infty}[x^{(\ln a)/(1+\ln x)}] = a$ (form ∞^0)

(c) $\lim_{x\to 0}[(x + 1)^{(\ln a)/x}] = a$ (form 1^∞).

Verify these results.

59–62 Verify that L'Hôpital's rule is of no help in finding the limit; then find the limit, if it exists, by some other method. ■

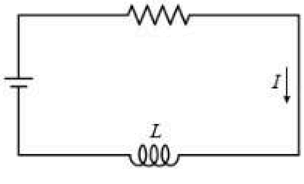
59. $\lim_{x\to+\infty}\dfrac{x + \sin 2x}{x}$
60. $\lim_{x\to+\infty}\dfrac{2x - \sin x}{3x + \sin x}$

61. $\lim_{x\to+\infty}\dfrac{x(2 + \sin 2x)}{x + 1}$
62. $\lim_{x\to+\infty}\dfrac{x(2 + \sin x)}{x^2 + 1}$

63. The accompanying schematic diagram represents an electrical circuit consisting of an electromotive force that produces a voltage V, a resistor with resistance R, and an inductor with inductance L. It is shown in electrical circuit theory that if the voltage is first applied at time $t = 0$, then the current I flowing through the circuit at time t is given by

$$I = \frac{V}{R}(1 - e^{-Rt/L})$$

What is the effect on the current at a fixed time t if the resistance approaches 0 (i.e., $R\to 0^+$)?



◀ **Figure Ex-63**

64. (a) Show that $\lim_{x\to\pi/2}(\pi/2 - x)\tan x = 1$.

(b) Show that

$$\lim_{x\to\pi/2}\left(\frac{1}{\pi/2 - x} - \tan x\right) = 0$$

(c) It follows from part (b) that the approximation

$$\tan x \approx \frac{1}{\pi/2 - x}$$

10.

INTERVAL	SIGN OF $f'(x)$	SIGN OF $f''(x)$
$x < 1$	$+$	$+$
$1 < x < 3$	$+$	$-$
$3 < x$	$+$	$+$

11–14 True–False Assume that f is differentiable everywhere. Determine whether the statement is true or false. Explain your answer.

11. If f is decreasing on $[0, 2]$, then $f(0) > f(1) > f(2)$.

12. If $f'(1) > 0$, then f is increasing on $[0, 2]$.

13. If f is increasing on $[0, 2]$, then $f'(1) > 0$.

14. If f' is increasing on $[0, 1]$ and f' is decreasing on $[1, 2]$, then f has an inflection point at $x = 1$.

15–32 Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points.

15. $f(x) = x^2 - 3x + 8$

16. $f(x) = 5 - 4x - x^2$

17. $f(x) = (2x + 1)^3$

18. $f(x) = 5 + 12x - x^3$

19. $f(x) = 3x^4 - 4x^3$

20. $f(x) = x^4 - 5x^3 + 9x^2$

21. $f(x) = \dfrac{x - 2}{(x^2 - x + 1)^2}$

22. $f(x) = \dfrac{x}{x^2 + 2}$

23. $f(x) = \sqrt[3]{x^2 + x + 1}$

24. $f(x) = x^{4/3} - x^{1/3}$

25. $f(x) = (x^{2/3} - 1)^2$

26. $f(x) = x^{2/3} - x$

27. $f(x) = e^{-x^2/2}$

28. $f(x) = xe^{x^2}$

29. $f(x) = \ln\sqrt{x^2 + 4}$

30. $f(x) = x^3 \ln x$

31. $f(x) = \tan^{-1}(x^2 - 1)$

32. $f(x) = \sin^{-1} x^{2/3}$

33–38 Analyze the trigonometric function f over the specified interval, stating where f is increasing, decreasing, concave up, and concave down, and stating the x-coordinates of all inflection points. Confirm that your results are consistent with the graph of f generated with a graphing utility.

33. $f(x) = \sin x - \cos x;\ [-\pi, \pi]$

34. $f(x) = \sec x \tan x;\ (-\pi/2, \pi/2)$

35. $f(x) = 1 - \tan(x/2);\ (-\pi, \pi)$

36. $f(x) = 2x + \cot x;\ (0, \pi)$

37. $f(x) = (\sin x + \cos x)^2;\ [-\pi, \pi]$

38. $f(x) = \sin^2 2x;\ [0, \pi]$

FOCUS ON CONCEPTS

39. In parts (a)–(c), sketch a continuous curve $y = f(x)$ with the stated properties.
(a) $f(2) = 4,\ f'(2) = 0,\ f''(x) > 0$ for all x
(b) $f(2) = 4,\ f'(2) = 0,\ f''(x) < 0$ for $x < 2,\ f''(x) > 0$ for $x > 2$
(c) $f(2) = 4,\ f''(x) < 0$ for $x \neq 2$ and $\lim_{x\to 2^+} f'(x) = +\infty,\ \lim_{x\to 2^-} f'(x) = -\infty$

40. In each part sketch a continuous curve $y = f(x)$ with the stated properties.
(a) $f(2) = 4,\ f'(2) = 0,\ f''(x) < 0$ for all x
(b) $f(2) = 4,\ f'(2) = 0,\ f''(x) > 0$ for $x < 2,\ f''(x) < 0$ for $x > 2$
(c) $f(2) = 4,\ f''(x) > 0$ for $x \neq 2$ and $\lim_{x\to 2^+} f'(x) = -\infty,\ \lim_{x\to 2^-} f'(x) = +\infty$

41–46 If f is increasing on an interval $[0, b)$, then it follows from Definition 4.1.1 that $f(0) < f(x)$ for each x in the interval $(0, b)$. Use this result in these exercises.

41. Show that $\sqrt[3]{1 + x} < 1 + \frac{1}{3}x$ if $x > 0$, and confirm the inequality with a graphing utility. [*Hint:* Show that the function $f(x) = 1 + \frac{1}{3}x - \sqrt[3]{1 + x}$ is increasing on $[0, +\infty)$.]

42. Show that $x < \tan x$ if $0 < x < \pi/2$, and confirm the inequality with a graphing utility. [*Hint:* Show that the function $f(x) = \tan x - x$ is increasing on $[0, \pi/2)$.]

43. Use a graphing utility to make a conjecture about the relative sizes of x and $\sin x$ for $x \geq 0$, and prove your conjecture.

44. Use a graphing utility to make a conjecture about the relative sizes of $1 - x^2/2$ and $\cos x$ for $x \geq 0$, and prove your conjecture. [*Hint:* Use the result of Exercise 43.]

45. (a) Show that $\ln(x + 1) \leq x$ if $x \geq 0$.
(b) Show that $\ln(x + 1) \geq x - \frac{1}{2}x^2$ if $x \geq 0$.
(c) Confirm the inequalities in parts (a) and (b) with a graphing utility.

46. (a) Show that $e^x \geq 1 + x$ if $x \geq 0$.
(b) Show that $e^x \geq 1 + x + \frac{1}{2}x^2$ if $x \geq 0$.
(c) Confirm the inequalities in parts (a) and (b) with a graphing utility.

47–48 Use a graphing utility to generate the graphs of f' and f'' over the stated interval; then use those graphs to estimate the x-coordinates of the inflection points of f, the intervals on which f is concave up or down, and the intervals on which f is increasing or decreasing. Check your estimates by graphing f.

47. $f(x) = x^4 - 24x^2 + 12x,\quad -5 \leq x \leq 5$

48. $f(x) = \dfrac{1}{1 + x^2},\quad -5 \leq x \leq 5$

C **49–50** Use a CAS to find f'' and to approximate the x-coordinates of the inflection points to six decimal places. Confirm that your answer is consistent with the graph of f.

49. $f(x) = \dfrac{10x - 3}{3x^2 - 5x + 8}$

50. $f(x) = \dfrac{x^3 - 8x + 7}{\sqrt{x^2 + 1}}$

51. Use Definition 4.1.1 to prove that $f(x) = x^2$ is increasing on $[0, +\infty)$.

52. Use Definition 4.1.1 to prove that $f(x) = 1/x$ is decreasing on $(0, +\infty)$.

✔ QUICK CHECK EXERCISES 4.2 (See page 254 for answers.)

1. A function f has a relative maximum at x_0 if there is an open interval containing x_0 on which $f(x)$ is ______ $f(x_0)$ for every x in the interval.

2. Suppose that f is defined everywhere and $x = 2, 3, 5, 7$ are critical points for f. If $f'(x)$ is positive on the intervals $(-\infty, 2)$ and $(5, 7)$, and if $f'(x)$ is negative on the intervals $(2, 3)$, $(3, 5)$, and $(7, +\infty)$, then f has relative maxima at $x =$ ______ and f has relative minima at $x =$ ______.

3. Suppose that f is defined everywhere and $x = -2$ and $x = 1$ are critical points for f. If $f''(x) = 2x + 1$, then f has a relative ______ at $x = -2$ and f has a relative ______ at $x = 1$.

4. Let $f(x) = (x^2 - 4)^2$. Then $f'(x) = 4x(x^2 - 4)$ and $f''(x) = 4(3x^2 - 4)$. Identify the locations of the (a) relative maxima, (b) relative minima, and (c) inflection points on the graph of f.

EXERCISE SET 4.2 Graphing Utility C CAS

FOCUS ON CONCEPTS

1. In each part, sketch the graph of a continuous function f with the stated properties.
   (a) f is concave up on the interval $(-\infty, +\infty)$ and has exactly one relative extremum.
   (b) f is concave up on the interval $(-\infty, +\infty)$ and has no relative extrema.
   (c) The function f has exactly two relative extrema on the interval $(-\infty, +\infty)$, and $f(x) \to +\infty$ as $x \to +\infty$.
   (d) The function f has exactly two relative extrema on the interval $(-\infty, +\infty)$, and $f(x) \to -\infty$ as $x \to +\infty$.

2. In each part, sketch the graph of a continuous function f with the stated properties.
   (a) f has exactly one relative extremum on $(-\infty, +\infty)$, and $f(x) \to 0$ as $x \to +\infty$ and as $x \to -\infty$.
   (b) f has exactly two relative extrema on $(-\infty, +\infty)$, and $f(x) \to 0$ as $x \to +\infty$ and as $x \to -\infty$.
   (c) f has exactly one inflection point and one relative extremum on $(-\infty, +\infty)$.
   (d) f has infinitely many relative extrema, and $f(x) \to 0$ as $x \to +\infty$ and as $x \to -\infty$.

3. (a) Use both the first and second derivative tests to show that $f(x) = 3x^2 - 6x + 1$ has a relative minimum at $x = 1$.
   (b) Use both the first and second derivative tests to show that $f(x) = x^3 - 3x + 3$ has a relative minimum at $x = 1$ and a relative maximum at $x = -1$.

4. (a) Use both the first and second derivative tests to show that $f(x) = \sin^2 x$ has a relative minimum at $x = 0$.
   (b) Use both the first and second derivative tests to show that $g(x) = \tan^2 x$ has a relative minimum at $x = 0$.
   (c) Give an informal verbal argument to explain without calculus why the functions in parts (a) and (b) have relative minima at $x = 0$.

5. (a) Show that both of the functions $f(x) = (x - 1)^4$ and $g(x) = x^3 - 3x^2 + 3x - 2$ have stationary points at $x = 1$.
   (b) What does the second derivative test tell you about the nature of these stationary points?
   (c) What does the first derivative test tell you about the nature of these stationary points?

6. (a) Show that $f(x) = 1 - x^5$ and $g(x) = 3x^4 - 8x^3$ both have stationary points at $x = 0$.
   (b) What does the second derivative test tell you about the nature of these stationary points?
   (c) What does the first derivative test tell you about the nature of these stationary points?

7–14 Locate the critical points and identify which critical points are stationary points. ■

7. $f(x) = 4x^4 - 16x^2 + 17$
8. $f(x) = 3x^4 + 12x$
9. $f(x) = \dfrac{x + 1}{x^2 + 3}$
10. $f(x) = \dfrac{x^2}{x^3 + 8}$
11. $f(x) = \sqrt[3]{x^2 - 25}$
12. $f(x) = x^2(x - 1)^{2/3}$
13. $f(x) = |\sin x|$
14. $f(x) = \sin|x|$

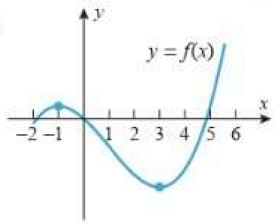
15–18 True–False Assume that f is continuous everywhere. Determine whether the statement is true or false. Explain your answer. ■

15. If f has a relative maximum at $x = 1$, then $f(1) \geq f(2)$.
16. If f has a relative maximum at $x = 1$, then $x = 1$ is a critical point for f.
17. If $f''(x) > 0$, then f has a relative minimum at $x = 1$.
18. If $p(x)$ is a polynomial such that $p'(x)$ has a simple root at $x = 1$, then p has a relative extremum at $x = 1$.

FOCUS ON CONCEPTS

19–20 The graph of a function $f(x)$ is given. Sketch graphs of $y = f'(x)$ and $y = f''(x)$. ■

19.



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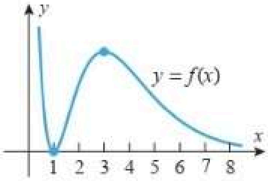
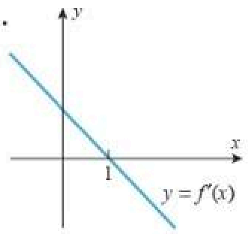


21–24 Use the graph of f' shown in the figure to estimate all values of x at which f has (a) relative minima, (b) relative maxima, and (c) inflection points. (d) Draw a rough sketch of the graph of a function f with the given derivative.

21.



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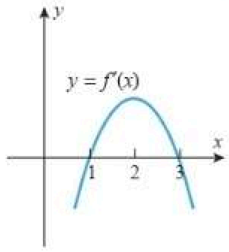

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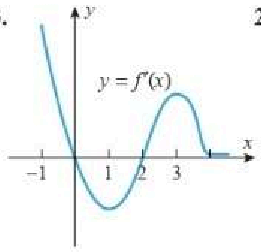

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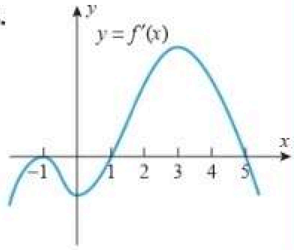


25–32 Use the given derivative to find all critical points of f, and at each critical point determine whether a relative maximum, relative minimum, or neither occurs. Assume in each case that f is continuous everywhere.

25. $f'(x) = x^2(x^3 - 5)$ **26.** $f'(x) = 4x^3 - 9x$

27. $f'(x) = \dfrac{2 - 3x}{\sqrt[3]{x + 2}}$ **28.** $f'(x) = \dfrac{x^2 - 7}{\sqrt[3]{x^2 + 4}}$

29. $f'(x) = xe^{1-x^2}$ **30.** $f'(x) = x^4(e^x - 3)$

31. $f'(x) = \ln\left(\dfrac{2}{1 + x^2}\right)$ **32.** $f'(x) = e^{2x} - 5e^x + 6$

33–36 Find the relative extrema using both first and second derivative tests.

33. $f(x) = 1 + 8x - 3x^2$ **34.** $f(x) = x^4 - 12x^3$

35. $f(x) = \sin 2x,\quad 0 < x < \pi$ **36.** $f(x) = (x - 3)e^x$

37–50 Use any method to find the relative extrema of the function f.

37. $f(x) = x^4 - 4x^3 + 4x^2$ **38.** $f(x) = x(x - 4)^3$

39. $f(x) = x^3(x + 1)^2$ **40.** $f(x) = x^2(x + 1)^3$

41. $f(x) = 2x + 3x^{2/3}$ **42.** $f(x) = 2x + 3x^{1/3}$

43. $f(x) = \dfrac{x + 3}{x - 2}$ **44.** $f(x) = \dfrac{x^2}{x^4 + 16}$

45. $f(x) = \ln(2 + x^2)$ **46.** $f(x) = \ln|2 + x^3|$

47. $f(x) = e^{2x} - e^x$ **48.** $f(x) = (xe^x)^2$

49. $f(x) = |3x - x^2|$ **50.** $f(x) = |1 + \sqrt[3]{x}|$

51–60 Give a graph of the polynomial and label the coordinates of the intercepts, stationary points, and inflection points. Check your work with a graphing utility.

51. $p(x) = x^2 - 3x - 4$ **52.** $p(x) = 1 + 8x - x^2$

53. $p(x) = 2x^3 - 3x^2 - 36x + 5$

54. $p(x) = 2 - x + 2x^2 - x^3$

55. $p(x) = (x + 1)^2(2x - x^2)$

56. $p(x) = x^4 - 6x^2 + 5$

57. $p(x) = x^4 - 2x^3 + 2x - 1$ **58.** $p(x) = 4x^3 - 9x^4$

59. $p(x) = x(x^2 - 1)^2$ **60.** $p(x) = x(x^2 - 1)^3$

61. In each part: (i) Make a conjecture about the behavior of the graph in the vicinity of its x-intercepts. (ii) Make a rough sketch of the graph based on your conjecture and the limits of the polynomial as $x \to +\infty$ and as $x \to -\infty$. (iii) Compare your sketch to the graph generated with a graphing utility.
(a) $y = x(x - 1)(x + 1)$ (b) $y = x^2(x - 1)^2(x + 1)^2$
(c) $y = x^2(x - 1)^2(x + 1)^3$ (d) $y = x(x - 1)^5(x + 1)^4$

62. Sketch the graph of $y = (x - a)^m(x - b)^n$ for the stated values of m and n, assuming that $a < b$ (six graphs in total).
(a) $m = 1,\ n = 1, 2, 3$ (b) $m = 2,\ n = 2, 3$
(c) $m = 3,\ n = 3$

63–66 Find the relative extrema in the interval $0 < x < 2\pi$, and confirm that your results are consistent with the graph of f generated with a graphing utility.

63. $f(x) = |\sin 2x|$ **64.** $f(x) = \sqrt{3}x + 2\sin x$

65. $f(x) = \cos^2 x$ **66.** $f(x) = \dfrac{\sin x}{2 - \cos x}$

67–70 Use a graphing utility to make a conjecture about the relative extrema of f, and then check your conjecture using either the first or second derivative test.

67. $f(x) = x\ln x$ **68.** $f(x) = \dfrac{2}{e^x + e^{-x}}$

69. $f(x) = x^2e^{-2x}$ **70.** $f(x) = 10\ln x - x$

71–72 Use a graphing utility to generate the graphs of f' and f'' over the stated interval, and then use those graphs to estimate the x-coordinates of the relative extrema of f. Check that your estimates are consistent with the graph of f.

71. $f(x) = x^4 - 24x^2 + 12x,\quad -5 \le x \le 5$

72. $f(x) = \sin \frac{1}{2}x \cos x,\quad -\pi/2 \le x \le \pi/2$

C **73–76** Use a CAS to graph f' and f'', and then use those graphs to estimate the x-coordinates of the relative extrema of f. Check that your estimates are consistent with the graph of f.

73. $f(x) = \dfrac{10x^3 - 3}{3x^2 - 5x + 8}$ **74.** $f(x) = \dfrac{\tan^{-1}(x^2 - x)}{x^2 + 4}$

FOCUS ON CONCEPTS

41. Let
$$f(x) = \begin{cases} 3x^2, & x \le 1 \\ ax + b, & x > 1 \end{cases}$$
Find the values of a and b so that f will be differentiable at $x = 1$.

42. (a) Let
$$f(x) = \begin{cases} x^2, & x \le 0 \\ x^2 + 1, & x > 0 \end{cases}$$
Show that
$$\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} f'(x)$$
but that $f'(0)$ does not exist.

(b) Let
$$f(x) = \begin{cases} x^2, & x \le 0 \\ x^3, & x > 0 \end{cases}$$
Show that $f'(0)$ exists but $f''(0)$ does not.

43. Use the Mean-Value Theorem to prove the following result: The graph of a function f has a point of vertical tangency at $(x_0, f(x_0))$ if f is continuous at x_0 and $f'(x)$ approaches either $+\infty$ or $-\infty$ as $x \to x_0^+$ and as $x \to x_0^-$.

44. Writing Suppose that $p(x)$ is a nonconstant polynomial with zeros at $x = a$ and $x = b$. Explain how both the Extreme-Value Theorem (4.4.2) and Rolle's Theorem can be used to show that p has a critical point between a and b.

45. Writing Find and describe a physical situation that illustrates the Mean-Value Theorem.

✔ QUICK CHECK ANSWERS 4.8

1. (a) $[0, 1]$ (b) $c = \frac{1}{2}$ **2.** $[-3, 3]$; $c = -2, 0, 2$ **3.** (a) $b = 2$ (b) $c = 1$ **4.** (a) 1.5 (b) 0.8 **5.** $f(x) = x^2 + 4$

CHAPTER 4 REVIEW EXERCISES

Graphing Utility C CAS

1. (a) If $x_1 < x_2$, what relationship must hold between $f(x_1)$ and $f(x_2)$ if f is increasing on an interval containing x_1 and x_2? Decreasing? Constant?
(b) What condition on f' ensures that f is increasing on an interval $[a, b]$? Decreasing? Constant?

2. (a) What condition on f' ensures that f is concave up on an open interval? Concave down?
(b) What condition on f'' ensures that f is concave up on an open interval? Concave down?
(c) In words, what is an inflection point of f?

3–10 Find: (a) the intervals on which f is increasing, (b) the intervals on which f is decreasing, (c) the open intervals on which f is concave up, (d) the open intervals on which f is concave down, and (e) the x-coordinates of all inflection points. ■

3. $f(x) = x^2 - 5x + 6$

4. $f(x) = x^4 - 8x^2 + 16$

5. $f(x) = \dfrac{x^2}{x^2 + 2}$

6. $f(x) = \sqrt[3]{x + 2}$

7. $f(x) = x^{1/3}(x + 4)$

8. $f(x) = x^{4/3} - x^{1/3}$

9. $f(x) = 1/e^{x^2}$

10. $f(x) = \tan^{-1} x^2$

11–14 Analyze the trigonometric function f over the specified interval, stating where f is increasing, decreasing, concave up, and concave down, and stating the x-coordinates of all inflection points. Confirm that your results are consistent with the graph of f generated with a graphing utility. ■

11. $f(x) = \cos x$; $[0, 2\pi]$

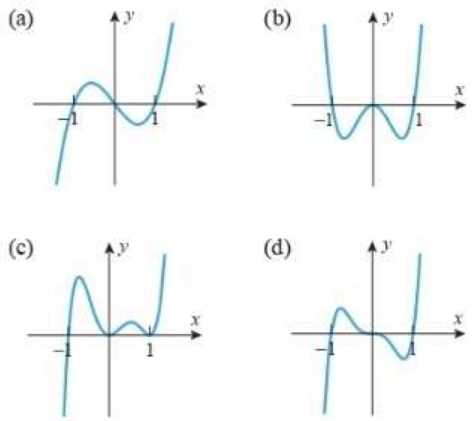
12. $f(x) = \tan x$; $(-\pi/2, \pi/2)$

13. $f(x) = \sin x \cos x$; $[0, \pi]$

14. $f(x) = \cos^2 x - 2 \sin x$; $[0, 2\pi]$

15. In each part, sketch a continuous curve $y = f(x)$ with the stated properties.
(a) $f(2) = 4$, $f'(2) = 1$, $f''(x) < 0$ for $x < 2$, $f''(x) > 0$ for $x > 2$
(b) $f(2) = 4$, $f''(x) > 0$ for $x < 2$, $f''(x) < 0$ for $x > 2$, $\lim_{x \to 2^-} f'(x) = +\infty$, $\lim_{x \to 2^+} f'(x) = +\infty$
(c) $f(2) = 4$, $f''(x) < 0$ for $x \ne 2$, $\lim_{x \to 2^-} f'(x) = 1$, $\lim_{x \to 2^+} f'(x) = -1$

16. In parts (a)–(d), the graph of a polynomial with degree at most 6 is given. Find equations for polynomials that produce graphs with these shapes, and check your answers with a graphing utility.



EXERCISE SET 5.5

1–4 Find the value of

(a) $\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$ (b) $\max \Delta x_k$.

1. $f(x) = x + 1$; $a = 0, b = 4$; $n = 3$;
$\Delta x_1 = 1, \Delta x_2 = 1, \Delta x_3 = 2$;
$x_1^* = \frac{1}{3}, x_2^* = \frac{3}{2}, x_3^* = 3$

2. $f(x) = \cos x$; $a = 0, b = 2\pi$; $n = 4$;
$\Delta x_1 = \pi/2, \Delta x_2 = 3\pi/4, \Delta x_3 = \pi/2, \Delta x_4 = \pi/4$;
$x_1^* = \pi/4, x_2^* = \pi, x_3^* = 3\pi/2, x_4^* = 7\pi/4$

3. $f(x) = 4 - x^2$; $a = -3, b = 4$; $n = 4$;
$\Delta x_1 = 1, \Delta x_2 = 2, \Delta x_3 = 1, \Delta x_4 = 3$;
$x_1^* = -\frac{5}{2}, x_2^* = -1, x_3^* = \frac{1}{4}, x_4^* = 3$

4. $f(x) = x^3$; $a = -3, b = 3$; $n = 4$;
$\Delta x_1 = 2, \Delta x_2 = 1, \Delta x_3 = 1, \Delta x_4 = 2$;
$x_1^* = -2, x_2^* = 0, x_3^* = 0, x_4^* = 2$

5–8 Use the given values of a and b to express the following limits as integrals. (Do not evaluate the integrals.)

5. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (x_k^*)^2 \Delta x_k$; $a = -1, b = 2$

6. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (x_k^*)^3 \Delta x_k$; $a = 1, b = 2$

7. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 4x_k^*(1 - 3x_k^*)\Delta x_k$; $a = -3, b = 3$

8. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (\sin^2 x_k^*)\Delta x_k$; $a = 0, b = \pi/2$

9–10 Use Definition 5.5.1 to express the integrals as limits of Riemann sums. (Do not evaluate the integrals.)

9. (a) $\int_1^2 2x\,dx$ (b) $\int_0^1 \frac{x}{x+1}\,dx$

10. (a) $\int_1^2 \sqrt{x}\,dx$ (b) $\int_{-\pi/2}^{\pi/2} (1 + \cos x)\,dx$

FOCUS ON CONCEPTS

11. Explain informally why Theorem 5.5.4(*a*) follows from Definition 5.5.1.

12. Explain informally why Theorem 5.5.6(*a*) follows from Definition 5.5.1.

13–16 Sketch the region whose signed area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry, where needed.

13. (a) $\int_0^3 x\,dx$ (b) $\int_{-2}^{-1} x\,dx$
(c) $\int_{-1}^{4} x\,dx$ (d) $\int_{-5}^{5} x\,dx$

14. (a) $\int_0^2 \left(1 - \frac{1}{2}x\right) dx$ (b) $\int_{-1}^{1} \left(1 - \frac{1}{2}x\right) dx$
(c) $\int_2^3 \left(1 - \frac{1}{2}x\right) dx$ (d) $\int_0^3 \left(1 - \frac{1}{2}x\right) dx$

15. (a) $\int_0^5 2\,dx$ (b) $\int_0^{\pi} \cos x\,dx$
(c) $\int_{-1}^{2} |2x - 3|\,dx$ (d) $\int_{-1}^{1} \sqrt{1 - x^2}\,dx$

16. (a) $\int_{-10}^{-5} 6\,dx$ (b) $\int_{-\pi/3}^{\pi/3} \sin x\,dx$
(c) $\int_0^3 |x - 2|\,dx$ (d) $\int_0^2 \sqrt{4 - x^2}\,dx$

17. In each part, evaluate the integral, given that

$$f(x) = \begin{cases} |x - 2|, & x \ge 0 \\ x + 2, & x < 0 \end{cases}$$

(a) $\int_{-2}^{0} f(x)\,dx$ (b) $\int_{-2}^{2} f(x)\,dx$
(c) $\int_0^6 f(x)\,dx$ (d) $\int_{-4}^{6} f(x)\,dx$

18. In each part, evaluate the integral, given that

$$f(x) = \begin{cases} 2x, & x \le 1 \\ 2, & x > 1 \end{cases}$$

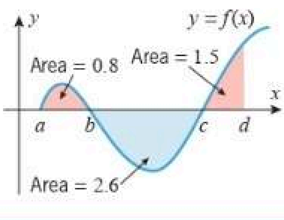
(a) $\int_0^1 f(x)\,dx$ (b) $\int_{-1}^{1} f(x)\,dx$
(c) $\int_1^{10} f(x)\,dx$ (d) $\int_{1/2}^{5} f(x)\,dx$

FOCUS ON CONCEPTS

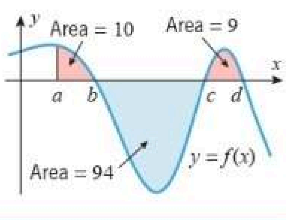
19–20 Use the areas shown in the figure to find

(a) $\int_a^b f(x)\,dx$ (b) $\int_b^c f(x)\,dx$
(c) $\int_a^c f(x)\,dx$ (d) $\int_a^d f(x)\,dx$.

19.



20.



21. Find $\int_{-1}^{2} [f(x) + 2g(x)]\,dx$ if

$$\int_{-1}^{2} f(x)\,dx = 5 \quad \text{and} \quad \int_{-1}^{2} g(x)\,dx = -3$$

22. Find $\int_{1}^{4} [3f(x) - g(x)]\,dx$ if

$$\int_{1}^{4} f(x)\,dx = 2 \quad \text{and} \quad \int_{1}^{4} g(x)\,dx = 10$$

23. Find $\int_{1}^{5} f(x)\,dx$ if

$$\int_{0}^{1} f(x)\,dx = -2 \quad \text{and} \quad \int_{0}^{5} f(x)\,dx = 1$$

24. Find $\int_{3}^{-2} f(x)\,dx$ if

$$\int_{-2}^{1} f(x)\,dx = 2 \quad \text{and} \quad \int_{1}^{3} f(x)\,dx = -6$$

25–28 Use Theorem 5.5.4 and appropriate formulas from geometry to evaluate the integrals. ■

25. $\int_{-1}^{3} (4 - 5x)\,dx$ **26.** $\int_{-2}^{2} (1 - 3|x|)\,dx$

27. $\int_{0}^{1} (x + 2\sqrt{1 - x^2})\,dx$ **28.** $\int_{-3}^{0} (2 + \sqrt{9 - x^2})\,dx$

29–32 True–False Determine whether the statement is true or false. Explain your answer. ■

29. If $f(x)$ is integrable on $[a, b]$, then $f(x)$ is continuous on $[a, b]$.

30. It is the case that

$$0 < \int_{-1}^{1} \frac{\cos x}{\sqrt{1 + x^2}}\,dx$$

31. If the integral of $f(x)$ over the interval $[a, b]$ is negative, then $f(x) \leq 0$ for $a \leq x \leq b$.

32. The function

$$f(x) = \begin{cases} 0, & x \leq 0 \\ x^2, & x > 0 \end{cases}$$

is integrable over every closed interval $[a, b]$.

33–34 Use Theorem 5.5.6 to determine whether the value of the integral is positive or negative. ■

33. (a) $\int_{2}^{3} \frac{\sqrt{x}}{1 - x}\,dx$ (b) $\int_{0}^{4} \frac{x^2}{3 - \cos x}\,dx$

34. (a) $\int_{-3}^{-1} \frac{x^4}{\sqrt{3 - x}}\,dx$ (b) $\int_{-2}^{2} \frac{x^3 - 9}{|x| + 1}\,dx$

35. Prove that if f is continuous and if $m \leq f(x) \leq M$ on $[a, b]$, then

$$m(b - a) \leq \int_{a}^{b} f(x)\,dx \leq M(b - a)$$

36. Find the maximum and minimum values of $\sqrt{x^3 + 2}$ for $0 \leq x \leq 3$. Use these values, and the inequalities in Exercise 35, to find bounds on the value of the integral

$$\int_{0}^{3} \sqrt{x^3 + 2}\,dx$$

37–38 Evaluate the integrals by completing the square and applying appropriate formulas from geometry. ■

37. $\int_{0}^{10} \sqrt{10x - x^2}\,dx$ **38.** $\int_{0}^{3} \sqrt{6x - x^2}\,dx$

39–40 Evaluate the limit by expressing it as a definite integral over the interval $[a, b]$ and applying appropriate formulas from geometry. ■

39. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (3x_k^* + 1)\Delta x_k;\ a = 0, b = 1$

40. $\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{4 - (x_k^*)^2}\,\Delta x_k;\ a = -2, b = 2$

FOCUS ON CONCEPTS

41. Let $f(x) = C$ be a constant function.

(a) Use a formula from geometry to show that

$$\int_{a}^{b} f(x)\,dx = C(b - a)$$

(b) Show that any Riemann sum for $f(x)$ over $[a, b]$ evaluates to $C(b - a)$. Use Definition 5.5.1 to show that

$$\int_{a}^{b} f(x)\,dx = C(b - a)$$

42. Define a function f on $[0, 1]$ by

$$f(x) = \begin{cases} 1, & 0 < x \leq 1 \\ 0, & x = 0 \end{cases}$$

Use Definition 5.5.1 to show that

$$\int_{0}^{1} f(x)\,dx = 1$$

43. It can be shown that every interval contains both rational and irrational numbers. Accepting this to be so, do you believe that the function

$$f(x) = \begin{cases} 1 & \text{if} \quad x \text{ is rational} \\ 0 & \text{if} \quad x \text{ is irrational} \end{cases}$$

is integrable on a closed interval $[a, b]$? Explain your reasoning.

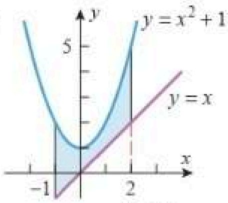
✔ QUICK CHECK EXERCISES 6.1 (See page 421 for answers.)

1. An integral expression for the area of the region between the curves $y = 20 - 3x^2$ and $y = e^x$ and bounded on the sides by $x = 0$ and $x = 2$ is ________.
2. An integral expression for the area of the parallelogram bounded by $y = 2x + 8$, $y = 2x - 3$, $x = -1$, and $x = 5$ is ________. The value of this integral is ________.
3. (a) The points of intersection for the circle $x^2 + y^2 = 4$ and the line $y = x + 2$ are ________ and ________.
   (b) Expressed as a definite integral with respect to x, ________ gives the area of the region inside the circle $x^2 + y^2 = 4$ and above the line $y = x + 2$.
   (c) Expressed as a definite integral with respect to y, ________ gives the area of the region described in part (b).
4. The area of the region enclosed by the curves $y = x^2$ and $y = \sqrt[3]{x}$ is ________.

EXERCISE SET 6.1 Graphing Utility C CAS

1–4 Find the area of the shaded region. ■

1.



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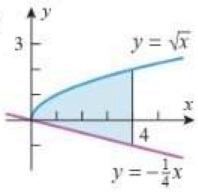

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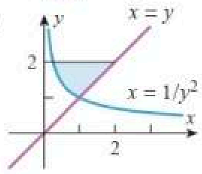

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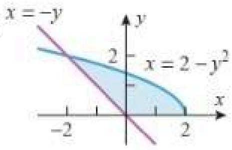
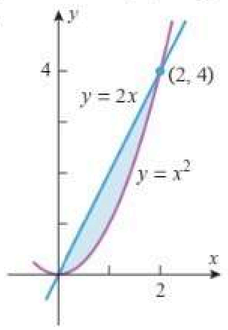
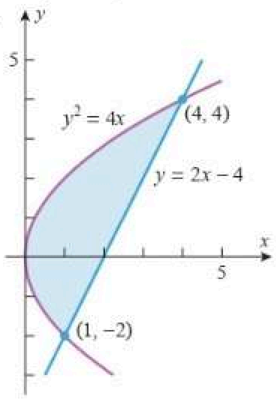


5–6 Find the area of the shaded region by (a) integrating with respect to x and (b) integrating with respect to y. ■

5.



6.



7–18 Sketch the region enclosed by the curves and find its area. ■

7. $y = x^2,\ y = \sqrt{x},\ x = \frac{1}{4},\ x = 1$

8. $y = x^3 - 4x,\ y = 0,\ x = 0,\ x = 2$

9. $y = \cos 2x,\ y = 0,\ x = \pi/4,\ x = \pi/2$

10. $y = \sec^2 x,\ y = 2,\ x = -\pi/4,\ x = \pi/4$

11. $x = \sin y,\ x = 0,\ y = \pi/4,\ y = 3\pi/4$

12. $x^2 = y,\ x = y - 2$

13. $y = e^x,\ y = e^{2x},\ x = 0,\ x = \ln 2$

14. $x = 1/y,\ x = 0,\ y = 1,\ y = e$

15. $y = \dfrac{2}{1 + x^2},\ y = |x|$ **16.** $y = \dfrac{1}{\sqrt{1 - x^2}},\ y = 2$

17. $y = 2 + |x - 1|,\ y = -\frac{1}{5}x + 7$

18. $y = x,\ y = 4x,\ y = -x + 2$

19–26 Use a graphing utility, where helpful, to find the area of the region enclosed by the curves. ■

19. $y = x^3 - 4x^2 + 3x,\ y = 0$

20. $y = x^3 - 2x^2,\ y = 2x^2 - 3x$

21. $y = \sin x,\ y = \cos x,\ x = 0,\ x = 2\pi$

22. $y = x^3 - 4x,\ y = 0$ **23.** $x = y^3 - y,\ x = 0$

24. $x = y^3 - 4y^2 + 3y,\ x = y^2 - y$

25. $y = xe^{x^2},\ y = 2|x|$

26. $y = \dfrac{1}{x\sqrt{1 - (\ln x)^2}},\ y = \dfrac{3}{x}$

27–30 True–False Determine whether the statement is true or false. Explain your answer. [In each exercise, assume that f and g are distinct continuous functions on $[a, b]$ and that A denotes the area of the region bounded by the graphs of $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$.] ■

27. If f and g differ by a positive constant c, then $A = c(b - a)$.

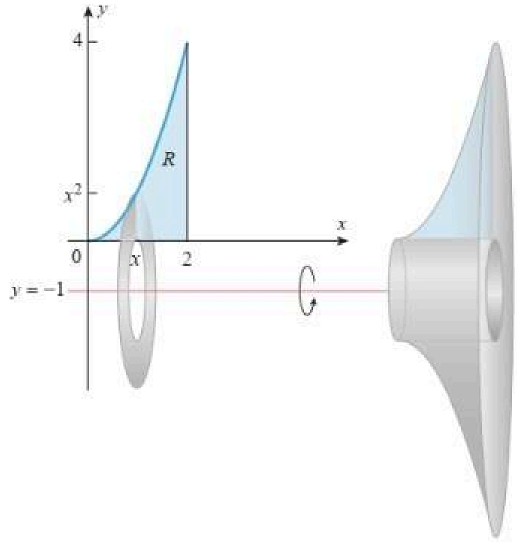
28. If
$$\int_a^b [f(x) - g(x)]\,dx = -3$$
then $A = 3$.

29. If
$$\int_a^b [f(x) - g(x)]\,dx = 0$$
then the graphs of $y = f(x)$ and $y = g(x)$ cross at least once on $[a, b]$.

30. If
$$A = \left|\int_a^b [f(x) - g(x)]\,dx\right|$$
then the graphs of $y = f(x)$ and $y = g(x)$ don't cross on $[a, b]$.

it follows by (3) that the volume of the solid is

$$V = \int_0^2 A(x)\,dx = \int_0^2 \pi\left(x^4 + 2x^2\right)dx = \pi\left[\frac{1}{5}x^5 + \frac{2}{3}x^3\right]_0^2 = \frac{176\pi}{15}$$ ◀



▶ **Figure 6.2.17**

✔ QUICK CHECK EXERCISES 6.2 *(See page 431 for answers.)*

1. A solid S extends along the x-axis from $x = 1$ to $x = 3$. For x between 1 and 3, the cross-sectional area of S perpendicular to the x-axis is $3x^2$. An integral expression for the volume of S is ______. The value of this integral is ______.

2. A solid S is generated by revolving the region between the x-axis and the curve $y = \sqrt{\sin x}$ $(0 \le x \le \pi)$ about the x-axis.

(a) For x between 0 and π, the cross-sectional area of S perpendicular to the x-axis at x is $A(x) =$ ______.

(b) An integral expression for the volume of S is ______.

(c) The value of the integral in part (b) is ______.

3. A solid S is generated by revolving the region enclosed by the line $y = 2x + 1$ and the curve $y = x^2 + 1$ about the x-axis.

(a) For x between ______ and ______, the cross-sectional area of S perpendicular to the x-axis at x is $A(x) =$ ______.

(b) An integral expression for the volume of S is ______.

4. A solid S is generated by revolving the region enclosed by the line $y = x + 1$ and the curve $y = x^2 + 1$ about the y-axis.

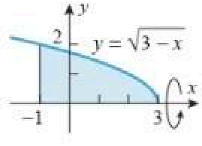
(a) For y between ______ and ______, the cross-sectional area of S perpendicular to the y-axis at y is $A(y) =$ ______.

(b) An integral expression for the volume of S is ______.

EXERCISE SET 6.2 Ⓒ CAS

1–8 Find the volume of the solid that results when the shaded region is revolved about the indicated axis. ■

1.



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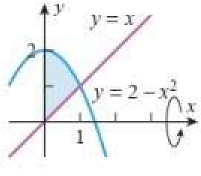

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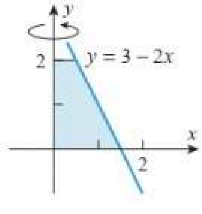

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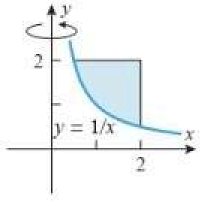
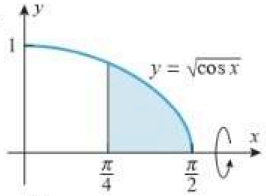
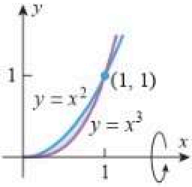
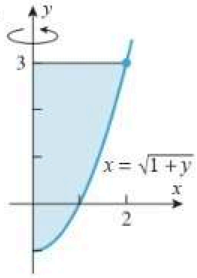
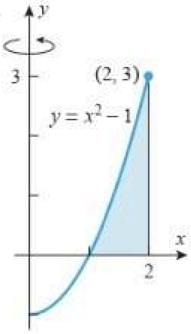


5. 

6. 

7. 

8. 

9. Find the volume of the solid whose base is the region bounded between the curve $y = x^2$ and the x-axis from $x = 0$ to $x = 2$ and whose cross sections taken perpendicular to the x-axis are squares.

10. Find the volume of the solid whose base is the region bounded between the curve $y = \sec x$ and the x-axis from $x = \pi/4$ to $x = \pi/3$ and whose cross sections taken perpendicular to the x-axis are squares.

11–18 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the x-axis.

11. $y = \sqrt{25 - x^2},\ y = 3$

12. $y = 9 - x^2,\ y = 0$

13. $x = \sqrt{y},\ x = y/4$

14. $y = \sin x,\ y = \cos x,\ x = 0,\ x = \pi/4$
[*Hint:* Use the identity $\cos 2x = \cos^2 x - \sin^2 x$.]

15. $y = e^x,\ y = 0,\ x = 0,\ x = \ln 3$

16. $y = e^{-2x},\ y = 0,\ x = 0,\ x = 1$

17. $y = \dfrac{1}{\sqrt{4 + x^2}},\ x = -2,\ x = 2,\ y = 0$

18. $y = \dfrac{e^{3x}}{\sqrt{1 + e^{6x}}},\ x = 0,\ x = 1,\ y = 0$

19. Find the volume of the solid whose base is the region bounded between the curve $y = x^3$ and the y-axis from $y = 0$ to $y = 1$ and whose cross sections taken perpendicular to the y-axis are squares.

20. Find the volume of the solid whose base is the region enclosed between the curve $x = 1 - y^2$ and the y-axis and whose cross sections taken perpendicular to the y-axis are squares.

21–26 Find the volume of the solid that results when the region enclosed by the given curves is revolved about the y-axis.

21. $x = \csc y,\ y = \pi/4,\ y = 3\pi/4,\ x = 0$

22. $y = x^2,\ x = y^2$

23. $x = y^2,\ x = y + 2$

24. $x = 1 - y^2,\ x = 2 + y^2,\ y = -1,\ y = 1$

25. $y = \ln x,\ x = 0,\ y = 0,\ y = 1$

26. $y = \sqrt{\dfrac{1 - x^2}{x^2}}\quad (x > 0),\ x = 0,\ y = 0,\ y = 2$

27–30 True–False Determine whether the statement is true or false. Explain your answer. [In these exercises, assume that a solid S of volume V is bounded by two parallel planes perpendicular to the x-axis at $x = a$ and $x = b$ and that for each x in $[a, b]$, $A(x)$ denotes the cross-sectional area of S perpendicular to the x-axis.]

27. If each cross section of S perpendicular to the x-axis is a square, then S is a rectangular parallelepiped (i.e., is box shaped).

28. If each cross section of S is a disk or a washer, then S is a solid of revolution.

29. If x is in centimeters (cm), then $A(x)$ must be a quadratic function of x, since units of $A(x)$ will be square centimeters (cm^2).

30. The average value of $A(x)$ on the interval $[a, b]$ is given by $V/(b - a)$.

31. Find the volume of the solid that results when the region above the x-axis and below the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > 0, b > 0)$$

is revolved about the x-axis.

32. Let V be the volume of the solid that results when the region enclosed by $y = 1/x$, $y = 0$, $x = 2$, and $x = b$ $(0 < b < 2)$ is revolved about the x-axis. Find the value of b for which $V = 3$.

33. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x + 1}$, $y = \sqrt{2x}$, and $y = 0$ is revolved about the x-axis. [*Hint:* Split the solid into two parts.]

34. Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 6 - x$, and $y = 0$ is revolved about the x-axis. [*Hint:* Split the solid into two parts.]

FOCUS ON CONCEPTS

35. Suppose that f is a continuous function on $[a, b]$, and let R be the region between the curve $y = f(x)$ and the line $y = k$ from $x = a$ to $x = b$. Using the method of disks, derive with explanation a formula for the volume of a solid generated by revolving R about the line $y = k$. State and explain additional assumptions, if any, that you need about f for your formula.

36. Suppose that v and w are continuous functions on $[c, d]$, and let R be the region between the curves $x = v(y)$ and $x = w(y)$ from $y = c$ to $y = d$. Using the method of washers, derive with explanation a formula for the volume of a solid generated by revolving R about the line

24. $\displaystyle\int \frac{du}{\sqrt{a^2+u^2}} = \ln(u+\sqrt{u^2+a^2})+C$

25. $\displaystyle\int \frac{du}{\sqrt{u^2-a^2}} = \ln\left|u+\sqrt{u^2-a^2}\right|+C \qquad (0<a<|u|)$

26. $\displaystyle\int \frac{du}{a^2-u^2} = \frac{1}{2a}\ln\left|\frac{a+u}{a-u}\right|+C$

27. $\displaystyle\int \frac{du}{u\sqrt{a^2-u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2-u^2}}{u}\right|+C \qquad (0<|u|<a)$

28. $\displaystyle\int \frac{du}{u\sqrt{a^2+u^2}} = -\frac{1}{a}\ln\left|\frac{a+\sqrt{a^2+u^2}}{u}\right|+C$

REMARK Formula 25 is a generalization of a result in Theorem 6.9.6. Readers who did not cover Section 6.9 can ignore Formulas 24–28 for now, since we will develop other methods for obtaining them in this chapter.

✔ QUICK CHECK EXERCISES 7.1 *(See page 491 for answers.)*

1. Use algebraic manipulation and (if necessary) u-substitution to integrate the function.

(a) $\displaystyle\int \frac{x+1}{x}\,dx =$ ________

(b) $\displaystyle\int \frac{x+2}{x+1}\,dx =$ ________

(c) $\displaystyle\int \frac{2x+1}{x^2+1}\,dx =$ ________

(d) $\displaystyle\int xe^{3\ln x}\,dx =$ ________

2. Use trigonometric identities and (if necessary) u-substitution to integrate the function.

(a) $\displaystyle\int \frac{1}{\csc x}\,dx =$ ________

(b) $\displaystyle\int \frac{1}{\cos^2 x}\,dx =$ ________

(c) $\displaystyle\int (\cot^2 x+1)\,dx =$ ________

(d) $\displaystyle\int \frac{1}{\sec x+\tan x}\,dx =$ ________

3. Integrate the function.

(a) $\displaystyle\int \sqrt{x-1}\,dx =$ ________

(b) $\displaystyle\int e^{2x+1}\,dx =$ ________

(c) $\displaystyle\int (\sin^3 x\cos x+\sin x\cos^3 x)\,dx =$ ________

(d) $\displaystyle\int \frac{1}{(e^x+e^{-x})^2}\,dx =$ ________

EXERCISE SET 7.1

1–30 Evaluate the integrals by making appropriate u-substitutions and applying the formulas reviewed in this section. ■

1. $\displaystyle\int (4-2x)^3\,dx$

2. $\displaystyle\int 3\sqrt{4+2x}\,dx$

3. $\displaystyle\int x\sec^2(x^2)\,dx$

4. $\displaystyle\int 4x\tan(x^2)\,dx$

5. $\displaystyle\int \frac{\sin 3x}{2+\cos 3x}\,dx$

6. $\displaystyle\int \frac{1}{9+4x^2}\,dx$

7. $\displaystyle\int e^x\sinh(e^x)\,dx$

8. $\displaystyle\int \frac{\sec(\ln x)\tan(\ln x)}{x}\,dx$

9. $\displaystyle\int e^{\tan x}\sec^2 x\,dx$

10. $\displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx$

11. $\displaystyle\int \cos^5 5x\sin 5x\,dx$

12. $\displaystyle\int \frac{\cos x}{\sin x\sqrt{\sin^2 x+1}}\,dx$

13. $\displaystyle\int \frac{e^x}{\sqrt{4+e^{2x}}}\,dx$

14. $\displaystyle\int \frac{e^{\tan^{-1}x}}{1+x^2}\,dx$

15. $\displaystyle\int \frac{e^{\sqrt{x-1}}}{\sqrt{x-1}}\,dx$

16. $\displaystyle\int (x+1)\cot(x^2+2x)\,dx$

17. $\displaystyle\int \frac{\cosh\sqrt{x}}{\sqrt{x}}\,dx$

18. $\displaystyle\int \frac{dx}{x(\ln x)^2}$

19. $\displaystyle\int \frac{dx}{\sqrt{x}\,3^{\sqrt{x}}}$

20. $\displaystyle\int \sec(\sin\theta)\tan(\sin\theta)\cos\theta\,d\theta$

21. $\displaystyle\int \frac{\operatorname{csch}^2(2/x)}{x^2}\,dx$ **22.** $\displaystyle\int \frac{dx}{\sqrt{x^2-4}}$

23. $\displaystyle\int \frac{e^{-x}}{4-e^{-2x}}\,dx$ **24.** $\displaystyle\int \frac{\cos(\ln x)}{x}\,dx$

25. $\displaystyle\int \frac{e^x}{\sqrt{1-e^{2x}}}\,dx$ **26.** $\displaystyle\int \frac{\sinh(x^{-1/2})}{x^{3/2}}\,dx$

27. $\displaystyle\int \frac{x}{\csc(x^2)}\,dx$ **28.** $\displaystyle\int \frac{e^x}{\sqrt{4-e^{2x}}}\,dx$

29. $\displaystyle\int x4^{-x^2}\,dx$ **30.** $\displaystyle\int 2^{\pi x}\,dx$

FOCUS ON CONCEPTS

31. (a) Evaluate the integral $\int \sin x\cos x\,dx$ using the substitution $u = \sin x$.
(b) Evaluate the integral $\int \sin x\cos x\,dx$ using the identity $\sin 2x = 2\sin x\cos x$.
(c) Explain why your answers to parts (a) and (b) are consistent.

32. (a) Derive the identity

$$\frac{\operatorname{sech}^2 x}{1+\tanh^2 x} = \operatorname{sech} 2x$$

(b) Use the result in part (a) to evaluate $\int \operatorname{sech} x\,dx$.
(c) Derive the identity

$$\operatorname{sech} x = \frac{2e^x}{e^{2x}+1}$$

(d) Use the result in part (c) to evaluate $\int \operatorname{sech} x\,dx$.
(e) Explain why your answers to parts (b) and (d) are consistent.

33. (a) Derive the identity

$$\frac{\sec^2 x}{\tan x} = \frac{1}{\sin x\cos x}$$

(b) Use the identity $\sin 2x = 2\sin x\cos x$ along with the result in part (a) to evaluate $\int \csc x\,dx$.
(c) Use the identity $\cos x = \sin[(\pi/2)-x]$ along with your answer to part (a) to evaluate $\int \sec x\,dx$.

✔ QUICK CHECK ANSWERS 7.1

1. (a) $x+\ln|x|+C$ (b) $x+\ln|x+1|+C$ (c) $\ln(x^2+1)+\tan^{-1}x+C$ (d) $\dfrac{x^5}{5}+C$ **2.** (a) $-\cos x+C$ (b) $\tan x+C$ (c) $-\cot x+C$ (d) $\ln(1+\sin x)+C$ **3.** (a) $\frac{2}{3}(x-1)^{3/2}+C$ (b) $\frac{1}{2}e^{2x+1}+C$ (c) $\frac{1}{2}\sin^2 x+C$ (d) $\frac{1}{4}\tanh x+C$

7.2 INTEGRATION BY PARTS

In this section we will discuss an integration technique that is essentially an antiderivative formulation of the formula for differentiating a product of two functions.

THE PRODUCT RULE AND INTEGRATION BY PARTS

Our primary goal in this section is to develop a general method for attacking integrals of the form

$$\int f(x)g(x)\,dx$$

As a first step, let $G(x)$ be *any* antiderivative of $g(x)$. In this case $G'(x) = g(x)$, so the product rule for differentiating $f(x)G(x)$ can be expressed as

$$\frac{d}{dx}[f(x)G(x)] = f(x)G'(x) + f'(x)G(x) = f(x)g(x) + f'(x)G(x) \tag{1}$$

This implies that $f(x)G(x)$ is an antiderivative of the function on the right side of (1), so we can express (1) in integral form as

$$\int [f(x)g(x) + f'(x)G(x)]\,dx = f(x)G(x)$$

✔QUICK CHECK EXERCISES 7.2 (See page 500 for answers.)

1. (a) If $G'(x) = g(x)$, then
$$\int f(x)g(x)\,dx = f(x)G(x) - \underline{\qquad}$$
(b) If $u = f(x)$ and $v = G(x)$, then the formula in part (a) can be written in the form $\int u\,dv = \underline{\qquad}$.

2. Find an appropriate choice of u and dv for integration by parts of each integral. Do not evaluate the integral.
(a) $\int x\ln x\,dx$; $u = \underline{\qquad}$, $dv = \underline{\qquad}$
(b) $\int (x-2)\sin x\,dx$; $u = \underline{\qquad}$, $dv = \underline{\qquad}$
(c) $\int \sin^{-1} x\,dx$; $u = \underline{\qquad}$, $dv = \underline{\qquad}$
(d) $\int \dfrac{x}{\sqrt{x-1}}dx$; $u = \underline{\qquad}$, $dv = \underline{\qquad}$

3. Use integration by parts to evaluate the integral.
(a) $\int xe^{2x}\,dx$ (b) $\int \ln(x-1)\,dx$
(c) $\int_0^{\pi/6} x\sin 3x\,dx$

4. Use a reduction formula to evaluate $\int \sin^3 x\,dx$.

EXERCISE SET 7.2

1–38 Evaluate the integral. ■

1. $\int xe^{-2x}\,dx$
2. $\int xe^{3x}\,dx$
3. $\int x^2e^{x}\,dx$
4. $\int x^2e^{-2x}\,dx$
5. $\int x\sin 3x\,dx$
6. $\int x\cos 2x\,dx$
7. $\int x^2\cos x\,dx$
8. $\int x^2\sin x\,dx$
9. $\int x\ln x\,dx$
10. $\int \sqrt{x}\ln x\,dx$
11. $\int (\ln x)^2\,dx$
12. $\int \dfrac{\ln x}{\sqrt{x}}\,dx$
13. $\int \ln(3x-2)\,dx$
14. $\int \ln(x^2+4)\,dx$
15. $\int \sin^{-1} x\,dx$
16. $\int \cos^{-1}(2x)\,dx$
17. $\int \tan^{-1}(3x)\,dx$
18. $\int x\tan^{-1} x\,dx$
19. $\int e^{x}\sin x\,dx$
20. $\int e^{3x}\cos 2x\,dx$
21. $\int \sin(\ln x)\,dx$
22. $\int \cos(\ln x)\,dx$
23. $\int x\sec^2 x\,dx$
24. $\int x\tan^2 x\,dx$
25. $\int x^3e^{x^2}\,dx$
26. $\int \dfrac{xe^{x}}{(x+1)^2}\,dx$
27. $\int_0^2 xe^{2x}\,dx$
28. $\int_0^1 xe^{-5x}\,dx$
29. $\int_1^e x^2\ln x\,dx$
30. $\int_{\sqrt{e}}^{e} \dfrac{\ln x}{x^2}\,dx$
31. $\int_{-1}^1 \ln(x+2)\,dx$
32. $\int_0^{\sqrt{3}/2} \sin^{-1} x\,dx$
33. $\int_2^4 \sec^{-1}\sqrt{\theta}\,d\theta$
34. $\int_1^2 x\sec^{-1} x\,dx$
35. $\int_0^{\pi} x\sin 2x\,dx$
36. $\int_0^{\pi} (x + x\cos x)\,dx$
37. $\int_1^3 \sqrt{x}\tan^{-1}\sqrt{x}\,dx$
38. $\int_0^2 \ln(x^2+1)\,dx$

39–42 True–False Determine whether the statement is true or false. Explain your answer. ■

39. The main goal in integration by parts is to choose u and dv to obtain a new integral that is easier to evaluate than the original.

40. Applying the LIATE strategy to evaluate $\int x^3\ln x\,dx$, we should choose $u = x^3$ and $dv = \ln x\,dx$.

41. To evaluate $\int \ln e^x\,dx$ using integration by parts, choose $dv = e^x\,dx$.

42. Tabular integration by parts is useful for integrals of the form $\int p(x)f(x)\,dx$, where $p(x)$ is a polynomial and $f(x)$ can be repeatedly integrated.

43–44 Evaluate the integral by making a u-substitution and then integrating by parts. ■

43. $\int e^{\sqrt{x}}\,dx$
44. $\int \cos\sqrt{x}\,dx$

45. Prove that tabular integration by parts gives the correct answer for
$$\int p(x)f(x)\,dx$$
where $p(x)$ is any *quadratic polynomial* and $f(x)$ is any function that can be repeatedly integrated.

46. The computations of any integral evaluated by repeated integration by parts can be organized using tabular integration by parts. Use this organization to evaluate $\int e^x\cos x\,dx$ in

two ways: first by repeated differentiation of $\cos x$ (compare Example 5), and then by repeated differentiation of e^x.

47–52 Evaluate the integral using tabular integration by parts. ■

47. $\int (3x^2 - x + 2)e^{-x}\,dx$ **48.** $\int (x^2 + x + 1)\sin x\,dx$

49. $\int 4x^4 \sin 2x\,dx$ **50.** $\int x^3\sqrt{2x+1}\,dx$

51. $\int e^{ax}\sin bx\,dx$ **52.** $\int e^{-3\theta}\sin 5\theta\,d\theta$

53. Consider the integral $\int \sin x \cos x\,dx$.
(a) Evaluate the integral two ways: first using integration by parts, and then using the substitution $u = \sin x$.
(b) Show that the results of part (a) are equivalent.
(c) Which of the two methods do you prefer? Discuss the reasons for your preference.

54. Evaluate the integral

$$\int_0^1 \frac{x^3}{\sqrt{x^2+1}}\,dx$$

using
(a) integration by parts
(b) the substitution $u = \sqrt{x^2+1}$.

55. (a) Find the area of the region enclosed by $y = \ln x$, the line $x = e$, and the x-axis.
(b) Find the volume of the solid generated when the region in part (a) is revolved about the x-axis.

56. Find the area of the region between $y = x\sin x$ and $y = x$ for $0 \le x \le \pi/2$.

57. Find the volume of the solid generated when the region between $y = \sin x$ and $y = 0$ for $0 \le x \le \pi$ is revolved about the y-axis.

58. Find the volume of the solid generated when the region enclosed between $y = \cos x$ and $y = 0$ for $0 \le x \le \pi/2$ is revolved about the y-axis.

59. A particle moving along the x-axis has velocity function $v(t) = t^3 \sin t$. How far does the particle travel from time $t = 0$ to $t = \pi$?

60. The study of sawtooth waves in electrical engineering leads to integrals of the form

$$\int_{-\pi/\omega}^{\pi/\omega} t\sin(k\omega t)\,dt$$

where k is an integer and ω is a nonzero constant. Evaluate the integral.

61. Use reduction formula (9) to evaluate
(a) $\int \sin^4 x\,dx$ (b) $\int_0^{\pi/2} \sin^5 x\,dx$.

62. Use reduction formula (10) to evaluate
(a) $\int \cos^5 x\,dx$ (b) $\int_0^{\pi/2} \cos^6 x\,dx$.

63. Derive reduction formula (9).

64. In each part, use integration by parts or other methods to derive the reduction formula.

(a) $\int \sec^n x\,dx = \dfrac{\sec^{n-2} x\tan x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2} x\,dx$

(b) $\int \tan^n x\,dx = \dfrac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx$

(c) $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$

65–66 Use the reduction formulas in Exercise 64 to evaluate the integrals. ■

65. (a) $\int \tan^4 x\,dx$ (b) $\int \sec^4 x\,dx$ (c) $\int x^3 e^x\,dx$

66. (a) $\int x^2 e^{3x}\,dx$ (b) $\int_0^1 xe^{-\sqrt{x}}\,dx$
[*Hint:* First make a substitution.]

67. Let f be a function whose second derivative is continuous on $[-1, 1]$. Show that

$$\int_{-1}^1 xf''(x)\,dx = f'(1) + f'(-1) - f(1) + f(-1)$$

FOCUS ON CONCEPTS

68. (a) In the integral $\int x\cos x\,dx$, let

$$u = x, \quad dv = \cos x\,dx,$$
$$du = dx, \quad v = \sin x + C_1$$

Show that the constant C_1 cancels out, thus giving the same solution obtained by omitting C_1.
(b) Show that in general

$$uv - \int v\,du = u(v + C_1) - \int (v + C_1)\,du$$

thereby justifying the omission of the constant of integration when calculating v in integration by parts.

69. Evaluate $\int \ln(x+1)\,dx$ using integration by parts. Simplify the computation of $\int v\,du$ by introducing a constant of integration $C_1 = 1$ when going from dv to v.

70. Evaluate $\int \ln(3x-2)\,dx$ using integration by parts. Simplify the computation of $\int v\,du$ by introducing a constant of integration $C_1 = -\frac{2}{3}$ when going from dv to v. Compare your solution with your answer to Exercise 13.

71. Evaluate $\int x\tan^{-1} x\,dx$ using integration by parts. Simplify the computation of $\int v\,du$ by introducing a constant of integration $C_1 = \frac{1}{2}$ when going from dv to v.

72. What equation results if integration by parts is applied to the integral

$$\int \frac{1}{x\ln x}\,dx$$

with the choices

$$u = \frac{1}{\ln x} \quad \text{and} \quad dv = \frac{1}{x}\,dx?$$

In what sense is this equation true? In what sense is it false?

✔ QUICK CHECK EXERCISES 7.4 *(See page 514 for answers.)*

1. For each expression, give a trigonometric substitution that will eliminate the radical.
(a) $\sqrt{a^2-x^2}$ ______ (b) $\sqrt{a^2+x^2}$ ______
(c) $\sqrt{x^2-a^2}$ ______

2. If $x = 2\sec\theta$ and $0 < \theta < \pi/2$, then
(a) $\sin\theta =$ ______ (b) $\cos\theta =$ ______
(c) $\tan\theta =$ ______.

3. In each part, state the trigonometric substitution that you would try first to evaluate the integral. Do not evaluate the integral.
(a) $\int \sqrt{9+x^2}\,dx$ ______
(b) $\int \sqrt{9-x^2}\,dx$ ______
(c) $\int \sqrt{1-9x^2}\,dx$ ______
(d) $\int \sqrt{x^2-9}\,dx$ ______
(e) $\int \sqrt{9+3x^2}\,dx$ ______
(f) $\int \sqrt{1+(9x)^2}\,dx$ ______

4. In each part, determine the substitution u.
(a) $\int \frac{1}{x^2-2x+10}\,dx = \int \frac{1}{u^2+3^2}\,du$;
$u =$ ______
(b) $\int \sqrt{x^2-6x+8}\,dx = \int \sqrt{u^2-1}\,du$;
$u =$ ______
(c) $\int \sqrt{12-4x-x^2}\,dx = \int \sqrt{4^2-u^2}\,du$;
$u =$ ______

EXERCISE SET 7.4 [C] CAS

1–26 Evaluate the integral. ■

1. $\int \sqrt{4-x^2}\,dx$
2. $\int \sqrt{1-4x^2}\,dx$
3. $\int \frac{x^2}{\sqrt{16-x^2}}\,dx$
4. $\int \frac{dx}{x^2\sqrt{9-x^2}}$
5. $\int \frac{dx}{(4+x^2)^2}$
6. $\int \frac{x^2}{\sqrt{5+x^2}}\,dx$
7. $\int \frac{\sqrt{x^2-9}}{x}\,dx$
8. $\int \frac{dx}{x^2\sqrt{x^2-16}}$
9. $\int \frac{3x^3}{\sqrt{1-x^2}}\,dx$
10. $\int x^3\sqrt{5-x^2}\,dx$
11. $\int \frac{dx}{x^2\sqrt{9x^2-4}}$
12. $\int \frac{\sqrt{1+t^2}}{t}\,dt$
13. $\int \frac{dx}{(1-x^2)^{3/2}}$
14. $\int \frac{dx}{x^2\sqrt{x^2+25}}$
15. $\int \frac{dx}{\sqrt{x^2-9}}$
16. $\int \frac{dx}{1+2x^2+x^4}$
17. $\int \frac{dx}{(4x^2-9)^{3/2}}$
18. $\int \frac{3x^3}{\sqrt{x^2-25}}\,dx$
19. $\int e^x\sqrt{1-e^{2x}}\,dx$
20. $\int \frac{\cos\theta}{\sqrt{2-\sin^2\theta}}\,d\theta$
21. $\int_0^1 5x^3\sqrt{1-x^2}\,dx$
22. $\int_0^{1/2} \frac{dx}{(1-x^2)^2}$
23. $\int_{\sqrt{2}}^2 \frac{dx}{x^2\sqrt{x^2-1}}$
24. $\int_{\sqrt{2}}^2 \frac{\sqrt{2x^2-4}}{x}\,dx$
25. $\int_1^3 \frac{dx}{x^4\sqrt{x^2+3}}$
26. $\int_0^3 \frac{x^3}{(3+x^2)^{5/2}}\,dx$

27–30 True–False Determine whether the statement is true or false. Explain your answer. ■

27. An integrand involving a radical of the form $\sqrt{a^2-x^2}$ suggests the substitution $x = a\sin\theta$.

28. The trigonometric substitution $x = a\sin\theta$ is made with the restriction $0 \le \theta \le \pi$.

29. An integrand involving a radical of the form $\sqrt{x^2-a^2}$ suggests the substitution $x = a\cos\theta$.

30. The area enclosed by the ellipse $x^2+4y^2 = 1$ is $\pi/2$.

FOCUS ON CONCEPTS

31. The integral
$$\int \frac{x}{x^2+4}\,dx$$
can be evaluated either by a trigonometric substitution or by the substitution $u = x^2+4$. Do it both ways and show that the results are equivalent.

32. The integral
$$\int \frac{x^2}{x^2+4}\,dx$$
can be evaluated either by a trigonometric substitution or by algebraically rewriting the numerator of the integrand as $(x^2+4)-4$. Do it both ways and show that the results are equivalent.

33. Find the arc length of the curve $y = \ln x$ from $x = 1$ to $x = 2$.

34. Find the arc length of the curve $y = x^2$ from $x = 0$ to $x = 1$.

35. Find the area of the surface generated when the curve in Exercise 34 is revolved about the x-axis.

36. Find the volume of the solid generated when the region enclosed by $x = y(1 - y^2)^{1/4}$, $y = 0$, $y = 1$, and $x = 0$ is revolved about the y-axis.

37–48 Evaluate the integral.

37. $\displaystyle\int \frac{dx}{x^2 - 4x + 5}$

38. $\displaystyle\int \frac{dx}{\sqrt{2x - x^2}}$

39. $\displaystyle\int \frac{dx}{\sqrt{3 + 2x - x^2}}$

40. $\displaystyle\int \frac{dx}{16x^2 + 16x + 5}$

41. $\displaystyle\int \frac{dx}{\sqrt{x^2 - 6x + 10}}$

42. $\displaystyle\int \frac{x}{x^2 + 2x + 2}\,dx$

43. $\displaystyle\int \sqrt{3 - 2x - x^2}\,dx$

44. $\displaystyle\int \frac{e^x}{\sqrt{1 + e^x + e^{2x}}}\,dx$

45. $\displaystyle\int \frac{dx}{2x^2 + 4x + 7}$

46. $\displaystyle\int \frac{2x + 3}{4x^2 + 4x + 5}\,dx$

47. $\displaystyle\int_1^2 \frac{dx}{\sqrt{4x - x^2}}$

48. $\displaystyle\int_0^4 \sqrt{x(4 - x)}\,dx$

C **49–50** There is a good chance that your CAS will not be able to evaluate these integrals as stated. If this is so, make a substitution that converts the integral into one that your CAS can evaluate.

49. $\displaystyle\int \cos x \sin x \sqrt{1 - \sin^4 x}\,dx$

50. $\displaystyle\int (x \cos x + \sin x)\sqrt{1 + x^2 \sin^2 x}\,dx$

51. (a) Use the ***hyperbolic substitution*** $x = 3 \sinh u$, the identity $\cosh^2 u - \sinh^2 u = 1$, and Theorem 6.9.4 to evaluate

$$\int \frac{dx}{\sqrt{x^2 + 9}}$$

(b) Evaluate the integral in part (a) using a trigonometric substitution and show that the result agrees with that obtained in part (a).

52. Use the hyperbolic substitutionn $x = \cosh u$, the identity $\sinh^2 u = \frac{1}{2}(\cosh 2u - 1)$, and the results referenced in Exercise 51 to evaluate

$$\int \sqrt{x^2 - 1}\,dx, \quad x \ge 1$$

53. Writing The trigonometric substitution $x = a \sin\theta$, $-\pi/2 \le \theta \le \pi/2$, is suggested for an integral whose integrand involves $\sqrt{a^2 - x^2}$. Discuss the implications of restricting θ to $\pi/2 \le \theta \le 3\pi/2$, and explain why the restriction $-\pi/2 \le \theta \le \pi/2$ should be preferred.

54. Writing The trigonometric substitution $x = a \cos\theta$ could also be used for an integral whose integrand involves $\sqrt{a^2 - x^2}$. Determine an appropriate restriction for θ with the substitution $x = a \cos\theta$, and discuss how to apply this substitution in appropriate integrals. Illustrate your discussion by evaluating the integral in Example 1 using a substitution of this type.

✔ QUICK CHECK ANSWERS 7.4

1. (a) $x = a\sin\theta$ (b) $x = a\tan\theta$ (c) $x = a\sec\theta$ **2.** (a) $\dfrac{\sqrt{x^2 - 4}}{x}$ (b) $\dfrac{2}{x}$ (c) $\dfrac{\sqrt{x^2 - 4}}{2}$ **3.** (a) $x = 3\tan\theta$ (b) $x = 3\sin\theta$ (c) $x = \frac{1}{3}\sin\theta$ (d) $x = 3\sec\theta$ (e) $x = \sqrt{3}\tan\theta$ (f) $x = \frac{1}{9}\tan\theta$ **4.** (a) $x - 1$ (b) $x - 3$ (c) $x + 2$

7.5 INTEGRATING RATIONAL FUNCTIONS BY PARTIAL FRACTIONS

Recall that a rational function is a ratio of two polynomials. In this section we will give a general method for integrating rational functions that is based on the idea of decomposing a rational function into a sum of simple rational functions that can be integrated by the methods studied in earlier sections.

PARTIAL FRACTIONS

In algebra, one learns to combine two or more fractions into a single fraction by finding a common denominator. For example,

$$\frac{2}{x-4} + \frac{3}{x+1} = \frac{2(x+1) + 3(x-4)}{(x-4)(x+1)} = \frac{5x - 10}{x^2 - 3x - 4} \tag{1}$$

The second integral on the right now involves a proper rational function and can thus be evaluated by a partial fraction decomposition. Using the result of Example 1 we obtain

$$\int \frac{3x^4+3x^3-5x^2+x-1}{x^2+x-2}\,dx = x^3+x+\frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|+C \blacktriangleleft$$

CONCLUDING REMARKS

There are some cases in which the method of partial fractions is inappropriate. For example, it would be inefficient to use partial fractions to perform the integration

$$\int \frac{3x^2+2}{x^3+2x-8}\,dx = \ln|x^3+2x-8|+C$$

since the substitution $u = x^3+2x-8$ is more direct. Similarly, the integration

$$\int \frac{2x-1}{x^2+1}\,dx = \int \frac{2x}{x^2+1}\,dx - \int \frac{dx}{x^2+1} = \ln(x^2+1) - \tan^{-1}x + C$$

requires only a little algebra since the integrand is already in partial fraction form.

✔QUICK CHECK EXERCISES 7.5 *(See page 523 for answers.)*

1. A partial fraction is a rational function of the form ________ or of the form ________.
2. (a) What is a proper rational function?
   (b) What condition must the degree of the numerator and the degree of the denominator of a rational function satisfy for the method of partial fractions to be applicable directly?
   (c) If the condition in part (b) is not satisfied, what must you do if you want to use partial fractions?
3. Suppose that the function $f(x) = P(x)/Q(x)$ is a proper rational function.
   (a) For each factor of $Q(x)$ of the form $(ax+b)^m$, the partial fraction decomposition of f contains the following sum of m partial fractions: ________
   (b) For each factor of $Q(x)$ of the form $(ax^2+bx+c)^m$, where ax^2+bx+c is an irreducible quadratic, the partial fraction decomposition of f contains the following sum of m partial fractions: ________
4. Complete the partial fraction decomposition.
   (a) $\dfrac{-3}{(x+1)(2x-1)} = \dfrac{A}{x+1} - \dfrac{2}{2x-1}$
   (b) $\dfrac{2x^2-3x}{(x^2+1)(3x+2)} = \dfrac{B}{3x+2} - \dfrac{1}{x^2+1}$
5. Evaluate the integral.
   (a) $\displaystyle\int \frac{3}{(x+1)(1-2x)}\,dx$ (b) $\displaystyle\int \frac{2x^2-3x}{(x^2+1)(3x+2)}\,dx$

EXERCISE SET 7.5 [C] CAS

1–8 Write out the form of the partial fraction decomposition. (Do not find the numerical values of the coefficients.) ■

1. $\dfrac{3x-1}{(x-3)(x+4)}$
2. $\dfrac{5}{x(x^2-4)}$
3. $\dfrac{2x-3}{x^3-x^2}$
4. $\dfrac{x^2}{(x+2)^3}$
5. $\dfrac{1-x^2}{x^3(x^2+2)}$
6. $\dfrac{3x}{(x-1)(x^2+6)}$
7. $\dfrac{4x^3-x}{(x^2+5)^2}$
8. $\dfrac{1-3x^4}{(x-2)(x^2+1)^2}$

9–34 Evaluate the integral. ■

9. $\displaystyle\int \frac{dx}{x^2-3x-4}$
10. $\displaystyle\int \frac{dx}{x^2-6x-7}$
11. $\displaystyle\int \frac{11x+17}{2x^2+7x-4}\,dx$
12. $\displaystyle\int \frac{5x-5}{3x^2-8x-3}\,dx$
13. $\displaystyle\int \frac{2x^2-9x-9}{x^3-9x}\,dx$
14. $\displaystyle\int \frac{dx}{x(x^2-1)}$
15. $\displaystyle\int \frac{x^2-8}{x+3}\,dx$
16. $\displaystyle\int \frac{x^2+1}{x-1}\,dx$
17. $\displaystyle\int \frac{3x^2-10}{x^2-4x+4}\,dx$
18. $\displaystyle\int \frac{x^2}{x^2-3x+2}\,dx$
19. $\displaystyle\int \frac{2x-3}{x^2-3x-10}\,dx$
20. $\displaystyle\int \frac{3x+1}{3x^2+2x-1}\,dx$
21. $\displaystyle\int \frac{x^5+x^2+2}{x^3-x}\,dx$
22. $\displaystyle\int \frac{x^5-4x^3+1}{x^3-4x}\,dx$
23. $\displaystyle\int \frac{2x^2+3}{x(x-1)^2}\,dx$
24. $\displaystyle\int \frac{3x^2-x+1}{x^3-x^2}\,dx$

25. $\displaystyle\int \frac{2x^2 - 10x + 4}{(x+1)(x-3)^2}\,dx$

26. $\displaystyle\int \frac{2x^2 - 2x - 1}{x^3 - x^2}\,dx$

27. $\displaystyle\int \frac{x^2}{(x+1)^3}\,dx$

28. $\displaystyle\int \frac{2x^2 + 3x + 3}{(x+1)^3}\,dx$

29. $\displaystyle\int \frac{2x^2 - 1}{(4x-1)(x^2+1)}\,dx$

30. $\displaystyle\int \frac{dx}{x^3 + 2x}$

31. $\displaystyle\int \frac{x^3 + 3x^2 + x + 9}{(x^2+1)(x^2+3)}\,dx$

32. $\displaystyle\int \frac{x^3 + x^2 + x + 2}{(x^2+1)(x^2+2)}\,dx$

33. $\displaystyle\int \frac{x^3 - 2x^2 + 2x - 2}{x^2 + 1}\,dx$

34. $\displaystyle\int \frac{x^4 + 6x^3 + 10x^2 + x}{x^2 + 6x + 10}\,dx$

35–38 True–False Determine whether the statement is true or false. Explain your answer.

35. The technique of partial fractions is used for integrals whose integrands are ratios of polynomials.

36. The integrand in
$$\int \frac{3x^4 + 5}{(x^2+1)^2}\,dx$$
is a proper rational function.

37. The partial fraction decomposition of
$$\frac{2x+3}{x^2} \quad \text{is} \quad \frac{2}{x} + \frac{3}{x^2}$$

38. If $f(x) = P(x)/(x+5)^3$ is a proper rational function, then the partial fraction decomposition of $f(x)$ has terms with constant numerators and denominators $(x+5)$, $(x+5)^2$, and $(x+5)^3$.

39–42 Evaluate the integral by making a substitution that converts the integrand to a rational function.

39. $\displaystyle\int \frac{\cos\theta}{\sin^2\theta + 4\sin\theta - 5}\,d\theta$

40. $\displaystyle\int \frac{e^t}{e^{2t} - 4}\,dt$

41. $\displaystyle\int \frac{e^{3x}}{e^{2x} + 4}\,dx$

42. $\displaystyle\int \frac{5 + 2\ln x}{x(1 + \ln x)^2}\,dx$

43. Find the volume of the solid generated when the region enclosed by $y = x^2/(9 - x^2)$, $y = 0$, $x = 0$, and $x = 2$ is revolved about the x-axis.

44. Find the area of the region under the curve $y = 1/(1 + e^x)$, over the interval $[-\ln 5, \ln 5]$. [*Hint:* Make a substitution that converts the integrand to a rational function.]

C **45–46** Use a CAS to evaluate the integral in two ways: (i) integrate directly; (ii) use the CAS to find the partial fraction decomposition and integrate the decomposition. Integrate by hand to check the results.

45. $\displaystyle\int \frac{x^2 + 1}{(x^2 + 2x + 3)^2}\,dx$

46. $\displaystyle\int \frac{x^5 + x^4 + 4x^3 + 4x^2 + 4x + 4}{(x^2+2)^3}\,dx$

C **47–48** Integrate by hand and check your answers using a CAS.

47. $\displaystyle\int \frac{dx}{x^4 - 3x^3 - 7x^2 + 27x - 18}$

48. $\displaystyle\int \frac{dx}{16x^3 - 4x^2 + 4x - 1}$

FOCUS ON CONCEPTS

49. Show that
$$\int_0^1 \frac{x}{x^4 + 1}\,dx = \frac{\pi}{8}$$

50. Use partial fractions to derive the integration formula
$$\int \frac{1}{a^2 - x^2}\,dx = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C$$

51. Suppose that $ax^2 + bx + c$ is a quadratic polynomial and that the integration
$$\int \frac{1}{ax^2 + bx + c}\,dx$$
produces a function with no inverse tangent terms. What does this tell you about the roots of the polynomial?

52. Suppose that $ax^2 + bx + c$ is a quadratic polynomial and that the integration
$$\int \frac{1}{ax^2 + bx + c}\,dx$$
produces a function with neither logarithmic nor inverse tangent terms. What does this tell you about the roots of the polynomial?

53. Does there exist a quadratic polynomial $ax^2 + bx + c$ such that the integration
$$\int \frac{x}{ax^2 + bx + c}\,dx$$
produces a function with no logarithmic terms? If so, give an example; if not, explain why no such polynomial can exist.

54. **Writing** Suppose that $P(x)$ is a cubic polynomial. State the general form of the partial fraction decomposition for
$$f(x) = \frac{P(x)}{(x+5)^4}$$
and state the implications of this decomposition for evaluating the integral $\int f(x)\,dx$.

55. **Writing** Consider the functions
$$f(x) = \frac{1}{x^2 - 4} \quad \text{and} \quad g(x) = \frac{x}{x^2 - 4}$$
Each of the integrals $\int f(x)\,dx$ and $\int g(x)\,dx$ can be evaluated using partial fractions and using at least one other integration technique. Demonstrate two different techniques for evaluating each of these integrals, and then discuss the considerations that would determine which technique you would use.

51. $\int \frac{x}{\sqrt{5+4x-x^2}}\,dx$ **52.** $\int \frac{x}{x^2+6x+13}\,dx$

C **53–64** (a) Make an appropriate u-substitution of the form $u = x^{1/n}$ or $u = (x+a)^{1/n}$, and then evaluate the integral. (b) If you have a CAS, use it to evaluate the integral, and then confirm that the result is equivalent to the one that you found in part (a).

53. $\int x\sqrt{x-2}\,dx$ **54.** $\int \frac{x}{\sqrt{x+1}}\,dx$

55. $\int x^5\sqrt{x^3+1}\,dx$ **56.** $\int \frac{1}{x\sqrt{x^3-1}}\,dx$

57. $\int \frac{dx}{x-\sqrt[3]{x}}$ **58.** $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}}$

59. $\int \frac{dx}{x(1-x^{1/4})}$ **60.** $\int \frac{\sqrt{x}}{x+1}\,dx$

61. $\int \frac{dx}{x^{1/2}-x^{1/3}}$ **62.** $\int \frac{1+\sqrt{x}}{1-\sqrt{x}}\,dx$

63. $\int \frac{x^3}{\sqrt{1+x^2}}\,dx$ **64.** $\int \frac{x}{(x+3)^{1/5}}\,dx$

C **65–70** (a) Make u-substitution (5) to convert the integrand to a rational function of u, and then evaluate the integral. (b) If you have a CAS, use it to evaluate the integral (no substitution), and then confirm that the result is equivalent to that in part (a).

65. $\int \frac{dx}{1+\sin x+\cos x}$ **66.** $\int \frac{dx}{2+\sin x}$

67. $\int \frac{d\theta}{1-\cos\theta}$ **68.** $\int \frac{dx}{4\sin x-3\cos x}$

69. $\int \frac{dx}{\sin x+\tan x}$ **70.** $\int \frac{\sin x}{\sin x+\tan x}\,dx$

71–72 Use any method to solve for x.

71. $\int_2^x \frac{1}{t(4-t)}\,dt = 0.5,\ 2 < x < 4$

72. $\int_1^x \frac{1}{t\sqrt{2t-1}}\,dt = 1,\ x > \frac{1}{2}$

73–76 Use any method to find the area of the region enclosed by the curves.

73. $y = \sqrt{25-x^2},\ y = 0,\ x = 0,\ x = 4$

74. $y = \sqrt{9x^2-4},\ y = 0,\ x = 2$

75. $y = \frac{1}{25-16x^2},\ y = 0,\ x = 0,\ x = 1$

76. $y = \sqrt{x}\ln x,\ y = 0,\ x = 4$

77–80 Use any method to find the volume of the solid generated when the region enclosed by the curves is revolved about the y-axis.

77. $y = \cos x,\ y = 0,\ x = 0,\ x = \pi/2$

78. $y = \sqrt{x-4},\ y = 0,\ x = 8$

79. $y = e^{-x},\ y = 0,\ x = 0,\ x = 3$

80. $y = \ln x,\ y = 0,\ x = 5$

81–82 Use any method to find the arc length of the curve.

81. $y = 2x^2,\ 0 \le x \le 2$ **82.** $y = 3\ln x,\ 1 \le x \le 3$

83–84 Use any method to find the area of the surface generated by revolving the curve about the x-axis.

83. $y = \sin x,\ 0 \le x \le \pi$ **84.** $y = 1/x,\ 1 \le x \le 4$

C **85–86** Information is given about the motion of a particle moving along a coordinate line.
(a) Use a CAS to find the position function of the particle for $t \ge 0$.
(b) Graph the position versus time curve.

85. $v(t) = 20\cos^6 t\sin^3 t,\ s(0) = 2$

86. $a(t) = e^{-t}\sin 2t\sin 4t,\ v(0) = 0,\ s(0) = 10$

FOCUS ON CONCEPTS

87. (a) Use the substitution $u = \tan(x/2)$ to show that

$$\int \sec x\,dx = \ln\left|\frac{1+\tan(x/2)}{1-\tan(x/2)}\right| + C$$

and confirm that this is consistent with Formula (22) of Section 7.3.

(b) Use the result in part (a) to show that

$$\int \sec x\,dx = \ln\left|\tan\left(\frac{\pi}{4}+\frac{x}{2}\right)\right| + C$$

88. Use the substitution $u = \tan(x/2)$ to show that

$$\int \csc x\,dx = \frac{1}{2}\ln\left[\frac{1-\cos x}{1+\cos x}\right] + C$$

and confirm that this is consistent with the result in Exercise 65(a) of Section 7.3.

89. Find a substitution that can be used to integrate rational functions of $\sinh x$ and $\cosh x$ and use your substitution to evaluate

$$\int \frac{dx}{2\cosh x+\sinh x}$$

without expressing the integrand in terms of e^x and e^{-x}.

C **90–93** Some integrals that can be evaluated by hand cannot be evaluated by all computer algebra systems. Evaluate the integral by hand, and determine if it can be evaluated on your CAS.

90. $\int \frac{x^3}{\sqrt{1-x^8}}\,dx$

91. $\int (\cos^{32}x\sin^{30}x - \cos^{30}x\sin^{32}x)\,dx$

92. $\int \sqrt{x-\sqrt{x^2-4}}\,dx$ [*Hint:* $\frac{1}{2}(\sqrt{x+2}-\sqrt{x-2})^2 = ?$]

✔ QUICK CHECK EXERCISES 7.8 (See page 557 for answers.)

1. In each part, determine whether the integral is improper, and if so, explain why. Do not evaluate the integrals.

(a) $\int_{\pi/4}^{3\pi/4} \cot x\, dx$ (b) $\int_{\pi/4}^{\pi} \cot x\, dx$

(c) $\int_0^{+\infty} \frac{1}{x^2+1}\, dx$ (d) $\int_1^{+\infty} \frac{1}{x^2-1}\, dx$

2. Express each improper integral in Quick Check Exercise 1 in terms of one or more appropriate limits. Do not evaluate the limits.

3. The improper integral

$$\int_1^{+\infty} x^{-p}\, dx$$

converges to ________ provided ________.

4. Evaluate the integrals that converge.

(a) $\int_0^{+\infty} e^{-x}\, dx$ (b) $\int_0^{+\infty} e^{x}\, dx$

(c) $\int_0^1 \frac{1}{x^3}\, dx$ (d) $\int_0^1 \frac{1}{\sqrt[3]{x^2}}\, dx$

EXERCISE SET 7.8 Graphing Utility C CAS

1. In each part, determine whether the integral is improper, and if so, explain why.

(a) $\int_1^5 \frac{dx}{x-3}$ (b) $\int_1^5 \frac{dx}{x+3}$ (c) $\int_0^1 \ln x\, dx$

(d) $\int_1^{+\infty} e^{-x}\, dx$ (e) $\int_{-\infty}^{+\infty} \frac{dx}{\sqrt[3]{x-1}}$ (f) $\int_0^{\pi/4} \tan x\, dx$

2. In each part, determine all values of p for which the integral is improper.

(a) $\int_0^1 \frac{dx}{x^p}$ (b) $\int_1^2 \frac{dx}{x-p}$ (c) $\int_0^1 e^{-px}\, dx$

3–32 Evaluate the integrals that converge.

3. $\int_0^{+\infty} e^{-2x}\, dx$

4. $\int_{-1}^{+\infty} \frac{x}{1+x^2}\, dx$

5. $\int_3^{+\infty} \frac{2}{x^2-1}\, dx$

6. $\int_0^{+\infty} xe^{-x^2}\, dx$

7. $\int_e^{+\infty} \frac{1}{x \ln^3 x}\, dx$

8. $\int_2^{+\infty} \frac{1}{x\sqrt{\ln x}}\, dx$

9. $\int_{-\infty}^0 \frac{dx}{(2x-1)^3}$

10. $\int_{-\infty}^3 \frac{dx}{x^2+9}$

11. $\int_{-\infty}^0 e^{3x}\, dx$

12. $\int_{-\infty}^0 \frac{e^x\, dx}{3-2e^x}$

13. $\int_{-\infty}^{+\infty} x\, dx$

14. $\int_{-\infty}^{+\infty} \frac{x}{\sqrt{x^2+2}}\, dx$

15. $\int_{-\infty}^{+\infty} \frac{x}{(x^2+3)^2}\, dx$

16. $\int_{-\infty}^{+\infty} \frac{e^{-t}}{1+e^{-2t}}\, dt$

17. $\int_0^4 \frac{dx}{(x-4)^2}$

18. $\int_0^8 \frac{dx}{\sqrt[3]{x}}$

19. $\int_0^{\pi/2} \tan x\, dx$

20. $\int_0^4 \frac{dx}{\sqrt{4-x}}$

21. $\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

22. $\int_{-3}^1 \frac{x\, dx}{\sqrt{9-x^2}}$

23. $\int_{\pi/3}^{\pi/2} \frac{\sin x}{\sqrt{1-2\cos x}}\, dx$

24. $\int_0^{\pi/4} \frac{\sec^2 x}{1-\tan x}\, dx$

25. $\int_0^3 \frac{dx}{x-2}$

26. $\int_{-2}^2 \frac{dx}{x^2}$

27. $\int_{-1}^8 x^{-1/3}\, dx$

28. $\int_0^1 \frac{dx}{(x-1)^{2/3}}$

29. $\int_0^{+\infty} \frac{1}{x^2}\, dx$

30. $\int_1^{+\infty} \frac{dx}{x\sqrt{x^2-1}}$

31. $\int_0^1 \frac{dx}{\sqrt{x}(x+1)}$

32. $\int_0^{+\infty} \frac{dx}{\sqrt{x}(x+1)}$

33–36 True–False Determine whether the statement is true or false. Explain your answer.

33. $\int_1^{+\infty} x^{-4/3}\, dx$ converges to 3.

34. If f is continuous on $[a, +\infty)$ and $\lim_{x\to+\infty} f(x) = 1$, then $\int_a^{+\infty} f(x)\, dx$ converges.

35. $\int_1^2 \frac{1}{x(x-3)}\, dx$ is an improper integral.

36. $\int_{-1}^1 \frac{1}{x^3}\, dx = 0$

37–40 Make the u-substitution and evaluate the resulting definite integral.

37. $\int_0^{+\infty} \frac{e^{-\sqrt{x}}}{\sqrt{x}}\, dx$; $u = \sqrt{x}$ [*Note:* $u \to +\infty$ as $x \to +\infty$.]

38. $\int_{12}^{+\infty} \frac{dx}{\sqrt{x}(x+4)}$; $u = \sqrt{x}$ [*Note:* $u \to +\infty$ as $x \to +\infty$.]

39. $\int_0^{+\infty} \frac{e^{-x}}{\sqrt{1-e^{-x}}}\, dx$; $u = 1 - e^{-x}$
[*Note:* $u \to 1$ as $x \to +\infty$.]

✔ QUICK CHECK EXERCISES 11.5 (See page 812 for answers.)

1. Let L be the line through $(2, 5)$ and parallel to $\mathbf{v} = \langle 3, -1\rangle$.
   (a) Parametric equations of L are
   $$x = \_\_\_\_\_\_ \quad y = \_\_\_\_\_\_$$
   (b) A vector equation of L is $\langle x, y\rangle = \_\_\_\_\_\_$.

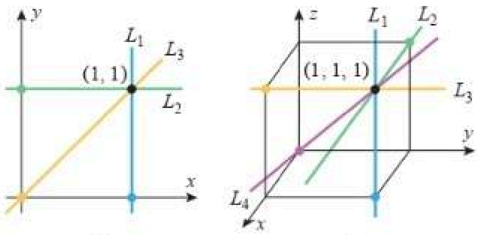
2. Parametric equations for the line through $(5, 3, 7)$ and parallel to the line $x = 3 - t$, $y = 2$, $z = 8 + 4t$ are
   $$x = \_\_\_\_\_\_, \quad y = \_\_\_\_\_\_, \quad z = \_\_\_\_\_\_$$

3. Parametric equations for the line segment joining the points $(3, 0, 11)$ and $(2, 6, 7)$ are
   $$x = \_\_\_\_\_\_, \quad y = \_\_\_\_\_\_, \quad z = \_\_\_\_\_\_ \quad (\_\_\_\_\_\_)$$

4. The line through the points $(-3, 8, -4)$ and $(1, 0, 8)$ intersects the yz-plane at ______.

EXERCISE SET 11.5 Graphing Utility C CAS

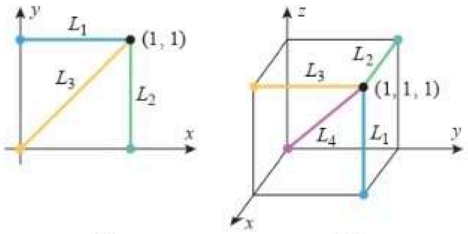
1. (a) Find parametric equations for the lines through the corner of the unit square shown in part (a) of the accompanying figure.
   (b) Find parametric equations for the lines through the corner of the unit cube shown in part (b) of the accompanying figure.



(a) (b)

▲ **Figure Ex-1**

2. (a) Find parametric equations for the line segments in the unit square in part (a) of the accompanying figure.
   (b) Find parametric equations for the line segments in the unit cube shown in part (b) of the accompanying figure.



(a) (b)

▲ **Figure Ex-2**

3–4 Find parametric equations for the line through P_1 and P_2 and also for the line segment joining those points. ■

3. (a) $P_1(3, -2)$, $P_2(5, 1)$ (b) $P_1(5, -2, 1)$, $P_2(2, 4, 2)$
4. (a) $P_1(0, 1)$, $P_2(-3, -4)$
   (b) $P_1(-1, 3, 5)$, $P_2(-1, 3, 2)$

5–6 Find parametric equations for the line whose vector equation is given. ■

5. (a) $\langle x, y\rangle = \langle 2, -3\rangle + t\langle 1, -4\rangle$
   (b) $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \mathbf{k} + t(\mathbf{i} - \mathbf{j} + \mathbf{k})$
6. (a) $x\mathbf{i} + y\mathbf{j} = (3\mathbf{i} - 4\mathbf{j}) + t(2\mathbf{i} + \mathbf{j})$
   (b) $\langle x, y, z\rangle = \langle -1, 0, 2\rangle + t\langle -1, 3, 0\rangle$

7–8 Find a point P on the line and a vector $\mathbf{v}$ parallel to the line by inspection. ■

7. (a) $x\mathbf{i} + y\mathbf{j} = (2\mathbf{i} - \mathbf{j}) + t(4\mathbf{i} - \mathbf{j})$
   (b) $\langle x, y, z\rangle = \langle -1, 2, 4\rangle + t\langle 5, 7, -8\rangle$
8. (a) $\langle x, y\rangle = \langle -1, 5\rangle + t\langle 2, 3\rangle$
   (b) $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (\mathbf{i} + \mathbf{j} - 2\mathbf{k}) + t\mathbf{j}$

9–10 Express the given parametric equations of a line using bracket notation and also using $\mathbf{i}, \mathbf{j}, \mathbf{k}$ notation. ■

9. (a) $x = -3 + t$, $y = 4 + 5t$
   (b) $x = 2 - t$, $y = -3 + 5t$, $z = t$
10. (a) $x = t$, $y = -2 + t$
    (b) $x = 1 + t$, $y = -7 + 3t$, $z = 4 - 5t$

11–14 True–False Determine whether the statement is true or false. Explain your answer. In these exercises L_0 and L_1 are lines in 3-space whose parametric equations are
$$L_0: x = x_0 + a_0t, \quad y = y_0 + b_0t, \quad z = z_0 + c_0t$$
$$L_1: x = x_1 + a_1t, \quad y = y_1 + b_1t, \quad z = z_1 + c_1t$$ ■

11. By definition, if L_1 and L_2 do not intersect, then L_1 and L_2 are parallel.
12. If L_1 and L_2 are parallel, then $\mathbf{v}_0 = \langle a_0, b_0, c_0\rangle$ is a scalar multiple of $\mathbf{v}_1 = \langle a_1, b_1, c_1\rangle$.
13. If L_1 and L_2 intersect at a point (x, y, z), then there exists a single value of t such that
    $$L_0: x = x_0 + a_0t, \quad y = y_0 + b_0t, \quad z = z_0 + c_0t$$
    $$L_1: x = x_1 + a_1t, \quad y = y_1 + b_1t, \quad z = z_1 + c_1t$$
    are satisfied.

14. If L_0 passes through the origin, then the vectors $\langle a_0, b_0, c_0\rangle$ and $\langle x_0, y_0, z_0\rangle$ are parallel.

15–22 Find parametric equations of the line that satisfies the stated conditions. ■

15. The line through $(-5, 2)$ that is parallel to $2\mathbf{i} - 3\mathbf{j}$.
16. The line through $(0, 3)$ that is parallel to the line $x = -5 + t$, $y = 1 - 2t$.
17. The line that is tangent to the circle $x^2 + y^2 = 25$ at the point $(3, -4)$.
18. The line that is tangent to the parabola $y = x^2$ at the point $(-2, 4)$.
19. The line through $(-1, 2, 4)$ that is parallel to $3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$.
20. The line through $(2, -1, 5)$ that is parallel to $\langle -1, 2, 7\rangle$.
21. The line through $(-2, 0, 5)$ that is parallel to the line given by $x = 1 + 2t$, $y = 4 - t$, $z = 6 + 2t$.
22. The line through the origin that is parallel to the line given by $x = t$, $y = -1 + t$, $z = 2$.
23. Where does the line $x = 1 + 3t$, $y = 2 - t$ intersect
    (a) the x-axis (b) the y-axis
    (c) the parabola $y = x^2$?
24. Where does the line $\langle x, y\rangle = \langle 4t, 3t\rangle$ intersect the circle $x^2 + y^2 = 25$?

25–26 Find the intersections of the lines with the xy-plane, the xz-plane, and the yz-plane. ■

25. $x = -2,\ y = 4 + 2t,\ z = -3 + t$
26. $x = -1 + 2t,\ y = 3 + t,\ z = 4 - t$
27. Where does the line $x = 1 + t$, $y = 3 - t$, $z = 2t$ intersect the cylinder $x^2 + y^2 = 16$?
28. Where does the line $x = 2 - t$, $y = 3t$, $z = -1 + 2t$ intersect the plane $2y + 3z = 6$?

29–30 Show that the lines L_1 and L_2 intersect, and find their point of intersection. ■

29. $L_1\colon x = 2 + t,\ y = 2 + 3t,\ z = 3 + t$
    $L_2\colon x = 2 + t,\ y = 3 + 4t,\ z = 4 + 2t$
30. $L_1\colon x + 1 = 4t,\ y - 3 = t,\ z - 1 = 0$
    $L_2\colon x + 13 = 12t,\ y - 1 = 6t,\ z - 2 = 3t$

31–32 Show that the lines L_1 and L_2 are skew. ■

31. $L_1\colon x = 1 + 7t,\ y = 3 + t,\ z = 5 - 3t$
    $L_2\colon x = 4 - t,\ y = 6,\ z = 7 + 2t$
32. $L_1\colon x = 2 + 8t,\ y = 6 - 8t,\ z = 10t$
    $L_2\colon x = 3 + 8t,\ y = 5 - 3t,\ z = 6 + t$

33–34 Determine whether the lines L_1 and L_2 are parallel. ■

33. $L_1\colon x = 3 - 2t,\ y = 4 + t,\ z = 6 - t$
    $L_2\colon x = 5 - 4t,\ y = -2 + 2t,\ z = 7 - 2t$
34. $L_1\colon x = 5 + 3t,\ y = 4 - 2t,\ z = -2 + 3t$
    $L_2\colon x = -1 + 9t,\ y = 5 - 6t,\ z = 3 + 8t$

35–36 Determine whether the points P_1, P_2, and P_3 lie on the same line. ■

35. $P_1(6, 9, 7),\ P_2(9, 2, 0),\ P_3(0, -5, -3)$
36. $P_1(1, 0, 1),\ P_2(3, -4, -3),\ P_3(4, -6, -5)$

37–38 Show that the lines L_1 and L_2 are the same. ■

37. $L_1\colon x = 3 - t,\ y = 1 + 2t$
    $L_2\colon x = -1 + 3t,\ y = 9 - 6t$
38. $L_1\colon x = 1 + 3t,\ y = -2 + t,\ z = 2t$
    $L_2\colon x = 4 - 6t,\ y = -1 - 2t,\ z = 2 - 4t$

FOCUS ON CONCEPTS

39. Sketch the vectors $\mathbf{r}_0 = \langle -1, 2\rangle$ and $\mathbf{v} = \langle 1, 1\rangle$, and then sketch the six vectors $\mathbf{r}_0 \pm \mathbf{v}$, $\mathbf{r}_0 \pm 2\mathbf{v}$, $\mathbf{r}_0 \pm 3\mathbf{v}$. Draw the line $L\colon x = -1 + t$, $y = 2 + t$, and describe the relationship between L and the vectors you sketched. What is the vector equation of L?
40. Sketch the vectors $\mathbf{r}_0 = \langle 0, 2, 1\rangle$ and $\mathbf{v} = \langle 1, 0, 1\rangle$, and then sketch the vectors $\mathbf{r}_0 + \mathbf{v}$, $\mathbf{r}_0 + 2\mathbf{v}$, and $\mathbf{r}_0 + 3\mathbf{v}$. Draw the line $L\colon x = t$, $y = 2$, $z = 1 + t$, and describe the relationship between L and the vectors you sketched. What is the vector equation of L?
41. Sketch the vectors $\mathbf{r}_0 = \langle -2, 0\rangle$ and $\mathbf{r}_1 = \langle 1, 3\rangle$, and then sketch the vectors
$$\tfrac{1}{3}\mathbf{r}_0 + \tfrac{2}{3}\mathbf{r}_1,\quad \tfrac{1}{2}\mathbf{r}_0 + \tfrac{1}{2}\mathbf{r}_1,\quad \tfrac{2}{3}\mathbf{r}_0 + \tfrac{1}{3}\mathbf{r}_1$$
Draw the line segment $(1 - t)\mathbf{r}_0 + t\mathbf{r}_1$ $(0 \le t \le 1)$. If n is a positive integer, what is the position of the point on this line segment corresponding to $t = 1/n$, relative to the points $(-2, 0)$ and $(1, 3)$?
42. Sketch the vectors $\mathbf{r}_0 = \langle 2, 0, 4\rangle$ and $\mathbf{r}_1 = \langle 0, 4, 0\rangle$, and then sketch the vectors
$$\tfrac{1}{4}\mathbf{r}_0 + \tfrac{3}{4}\mathbf{r}_1,\quad \tfrac{1}{2}\mathbf{r}_0 + \tfrac{1}{2}\mathbf{r}_1,\quad \tfrac{3}{4}\mathbf{r}_0 + \tfrac{1}{4}\mathbf{r}_1$$
Draw the line segment $(1 - t)\mathbf{r}_0 + t\mathbf{r}_1$ $(0 \le t \le 1)$. If n is a positive integer, what is the position of the point on this line segment corresponding to $t = 1/n$, relative to the points $(2, 0, 4)$ and $(0, 4, 0)$?

43–44 Describe the line segment represented by the vector equation. ■

43. $\langle x, y\rangle = \langle 1, 0\rangle + t\langle -2, 3\rangle \quad (0 \le t \le 2)$
44. $\langle x, y, z\rangle = \langle -2, 1, 4\rangle + t\langle 3, 0, -1\rangle \quad (0 \le t \le 3)$
45. Find the point on the line segment joining $P_1(3, 6)$ and $P_2(8, -4)$ that is $\frac{2}{5}$ of the way from P_1 to P_2.
46. Find the point on the line segment joining $P_1(1, 4, -3)$ and $P_2(1, 5, -1)$ that is $\frac{2}{3}$ of the way from P_1 to P_2.

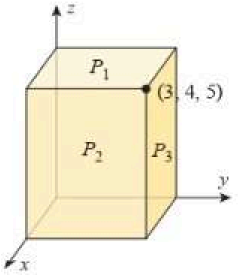
47–48 Use the method in Exercise 32 of Section 11.3 to find the distance from the point P to the line L, and then check your answer using the method in Exercise 30 of Section 11.4. ■

47. $P(-2, 1, 1)$
    $L\colon x = 3 - t,\ y = t,\ z = 1 + 2t$

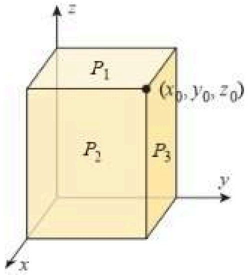
EXERCISE SET 11.6

1. Find equations of the planes P_1, P_2, and P_3 that are parallel to the coordinate planes and pass through the corner $(3, 4, 5)$ of the box shown in the accompanying figure.

2. Find equations of the planes P_1, P_2, and P_3 that are parallel to the coordinate planes and pass through the corner (x_0, y_0, z_0) of the box shown in the accompanying figure.



▲ Figure Ex-1

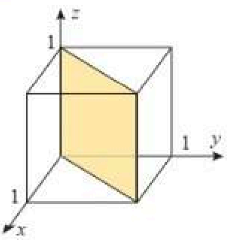


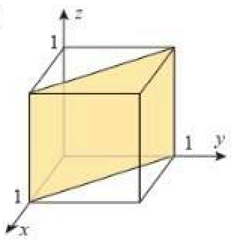
▲ Figure Ex-2

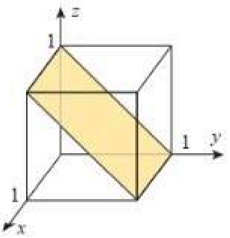
3–6 Find an equation of the plane that passes through the point P and has the vector $\mathbf{n}$ as a normal.

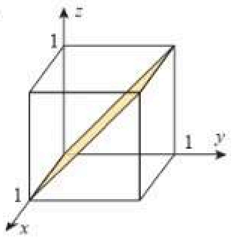
3. $P(2, 6, 1)$; $\mathbf{n} = \langle 1, 4, 2 \rangle$
4. $P(-1, -1, 2)$; $\mathbf{n} = \langle -1, 7, 6 \rangle$
5. $P(1, 0, 0)$; $\mathbf{n} = \langle 0, 0, 1 \rangle$
6. $P(0, 0, 0)$; $\mathbf{n} = \langle 2, -3, -4 \rangle$

7–10 Find an equation of the plane indicated in the figure.

7.


8.


9.


10.


11–12 Find an equation of the plane that passes through the given points.

11. $(-2, 1, 1)$, $(0, 2, 3)$, and $(1, 0, -1)$
12. $(3, 2, 1)$, $(2, 1, -1)$, and $(-1, 3, 2)$

13–14 Determine whether the planes are parallel, perpendicular, or neither.

13. (a) $2x - 8y - 6z - 2 = 0$
$-x + 4y + 3z - 5 = 0$
(b) $3x - 2y + z = 1$
$4x + 5y - 2z = 4$
(c) $x - y + 3z - 2 = 0$
$2x + z = 1$

14. (a) $3x - 2y + z = 4$
$6x - 4y + 3z = 7$
(b) $y = 4x - 2z + 3$
$x = \frac{1}{4}y + \frac{1}{2}z$
(c) $x + 4y + 7z = 3$
$5x - 3y + z = 0$

15–16 Determine whether the line and plane are parallel, perpendicular, or neither.

15. (a) $x = 4 + 2t,\ y = -t,\ z = -1 - 4t;$
$3x + 2y + z - 7 = 0$
(b) $x = t,\ y = 2t,\ z = 3t;$
$x - y + 2z = 5$
(c) $x = -1 + 2t,\ y = 4 + t,\ z = 1 - t;$
$4x + 2y - 2z = 7$

16. (a) $x = 3 - t,\ y = 2 + t,\ z = 1 - 3t;$
$2x + 2y - 5 = 0$
(b) $x = 1 - 2t,\ y = t,\ z = -t;$
$6x - 3y + 3z = 1$
(c) $x = t,\ y = 1 - t,\ z = 2 + t;$
$x + y + z = 1$

17–18 Determine whether the line and plane intersect; if so, find the coordinates of the intersection.

17. (a) $x = t,\ y = t,\ z = t;$
$3x - 2y + z - 5 = 0$
(b) $x = 2 - t,\ y = 3 + t,\ z = t;$
$2x + y + z = 1$

18. (a) $x = 3t,\ y = 5t,\ z = -t;$
$2x - y + z + 1 = 0$
(b) $x = 1 + t,\ y = -1 + 3t,\ z = 2 + 4t;$
$x - y + 4z = 7$

19–20 Find the acute angle of intersection of the planes to the nearest degree.

19. $x = 0$ and $2x - y + z - 4 = 0$
20. $x + 2y - 2z = 5$ and $6x - 3y + 2z = 8$

21–24 True–False Determine whether the statement is true or false. Explain your answer.

21. Every plane has exactly two unit normal vectors.
22. If a plane is parallel to one of the coordinate planes, then its normal vector is parallel to one of the three vectors $\mathbf{i}$, $\mathbf{j}$, or $\mathbf{k}$.
23. If two planes intersect in a line L, then L is parallel to the cross product of the normals to the two planes.

24. If $a^2 + b^2 + c^2 = 1$, then the distance from $P(x_0, y_0, z_0)$ to the plane $ax + by + cz = 0$ is $|\langle a, b, c\rangle \cdot \langle x_0, y_0, z_0\rangle|$.

25–34 Find an equation of the plane that satisfies the stated conditions. ■

25. The plane through the origin that is parallel to the plane $4x - 2y + 7z + 12 = 0$.

26. The plane that contains the line $x = -2 + 3t$, $y = 4 + 2t$, $z = 3 - t$ and is perpendicular to the plane $x - 2y + z = 5$.

27. The plane through the point $(-1, 4, 2)$ that contains the line of intersection of the planes $4x - y + z - 2 = 0$ and $2x + y - 2z - 3 = 0$.

28. The plane through $(-1, 4, -3)$ that is perpendicular to the line $x - 2 = t$, $y + 3 = 2t$, $z = -t$.

29. The plane through $(1, 2, -1)$ that is perpendicular to the line of intersection of the planes $2x + y + z = 2$ and $x + 2y + z = 3$.

30. The plane through the points $P_1(-2, 1, 4)$, $P_2(1, 0, 3)$ that is perpendicular to the plane $4x - y + 3z = 2$.

31. The plane through $(-1, 2, -5)$ that is perpendicular to the planes $2x - y + z = 1$ and $x + y - 2z = 3$.

32. The plane that contains the point $(2, 0, 3)$ and the line $x = -1 + t$, $y = t$, $z = -4 + 2t$.

33. The plane whose points are equidistant from $(2, -1, 1)$ and $(3, 1, 5)$.

34. The plane that contains the line $x = 3t$, $y = 1 + t$, $z = 2t$ and is parallel to the intersection of the planes $y + z = -1$ and $2x - y + z = 0$.

35. Find parametric equations of the line through the point $(5, 0, -2)$ that is parallel to the planes $x - 4y + 2z = 0$ and $2x + 3y - z + 1 = 0$.

36. Let L be the line $x = 3t + 1$, $y = -5t$, $z = t$.
(a) Show that L lies in the plane $2x + y - z = 2$.
(b) Show that L is parallel to the plane $x + y + 2z = 0$. Is the line above, below, or on this plane?

37. Show that the lines

$$\begin{aligned} x &= -2 + t, & y &= 3 + 2t, & z &= 4 - t \\ x &= 3 - t, & y &= 4 - 2t, & z &= t \end{aligned}$$

are parallel and find an equation of the plane they determine.

38. Show that the lines

$$\begin{aligned} L_1\colon x + 1 &= 4t, & y - 3 &= t, & z - 1 &= 0 \\ L_2\colon x + 13 &= 12t, & y - 1 &= 6t, & z - 2 &= 3t \end{aligned}$$

intersect and find an equation of the plane they determine.

FOCUS ON CONCEPTS

39. Do the points $(1, 0, -1)$, $(0, 2, 3)$, $(-2, 1, 1)$, and $(4, 2, 3)$ lie in the same plane? Justify your answer two different ways.

40. Show that if a, b, and c are nonzero, then the plane whose intercepts with the coordinate axes are $x = a$, $y = b$, and $z = c$ is given by the equation

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

41–42 Find parametric equations of the line of intersection of the planes. ■

41. $-2x + 3y + 7z + 2 = 0$
$x + 2y - 3z + 5 = 0$

42. $3x - 5y + 2z = 0$
$z = 0$

43–44 Find the distance between the point and the plane. ■

43. $(1, -2, 3)$; $2x - 2y + z = 4$

44. $(0, 1, 5)$; $3x + 6y - 2z - 5 = 0$

45–46 Find the distance between the given parallel planes. ■

45. $-2x + y + z = 0$
$6x - 3y - 3z - 5 = 0$

46. $x + y + z = 1$
$x + y + z = -1$

47–48 Find the distance between the given skew lines. ■

47. $x = 1 + 7t$, $y = 3 + t$, $z = 5 - 3t$
$x = 4 - t$, $y = 6$, $z = 7 + 2t$

48. $x = 3 - t$, $y = 4 + 4t$, $z = 1 + 2t$
$x = t$, $y = 3$, $z = 2t$

49. Find an equation of the sphere with center $(2, 1, -3)$ that is tangent to the plane $x - 3y + 2z = 4$.

50. Locate the point of intersection of the plane $2x + y - z = 0$ and the line through $(3, 1, 0)$ that is perpendicular to the plane.

51. Show that the line $x = -1 + t$, $y = 3 + 2t$, $z = -t$ and the plane $2x - 2y - 2z + 3 = 0$ are parallel, and find the distance between them.

FOCUS ON CONCEPTS

52. Formulas (1), (2), (3), (5), and (10), which apply to planes in 3-space, have analogs for lines in 2-space.
(a) Draw an analog of Figure 11.6.3 in 2-space to illustrate that the equation of the line that passes through the point $P(x_0, y_0)$ and is perpendicular to the vector $\mathbf{n} = \langle a, b\rangle$ can be expressed as

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0) = 0$$

where $\mathbf{r} = \langle x, y\rangle$ and $\mathbf{r}_0 = \langle x_0, y_0\rangle$.
(b) Show that the vector equation in part (a) can be expressed as

$$a(x - x_0) + b(y - y_0) = 0$$

This is called the ***point-normal form of a line***.
(c) Using the proof of Theorem 11.6.1 as a guide, show that if a and b are not both zero, then the graph of the equation

$$ax + by + c = 0$$

is a line that has $\mathbf{n} = \langle a, b\rangle$ as a normal. *(cont.)*