Differentiation and Integration

Differentiation

$$\frac{d}{dx}[c] = 0$$

$$\frac{d}{dx}[x^n] = nx^{n-1}$$

$$\frac{d}{dx}[x^4] = 4x^3, \qquad \frac{d}{dx}[x^5] = 5x^4, \qquad \frac{d}{dt}[t^{12}] = 12t^{11}$$

$$\frac{d}{dx}[x^r] = rx^{r-1}$$

$$\frac{d}{dx}[x^\pi] = \pi x^{\pi-1}$$

$$\frac{d}{dx}\left[\frac{1}{x}\right] = \frac{d}{dx}[x^{-1}] = (-1)x^{-1-1} = -x^{-2} = -\frac{1}{x^2}$$

$$\frac{d}{dw}\left[\frac{1}{w^{100}}\right] = \frac{d}{dw}[w^{-100}] = -100w^{-101} = -\frac{100}{w^{101}}$$

$$\frac{d}{dx}[x^{4/5}] = \frac{4}{5}x^{(4/5)-1} = \frac{4}{5}x^{-1/5}$$

$$\frac{d}{dx}[\sqrt[3]{x}] = \frac{d}{dx}[x^{1/3}] = \frac{1}{3}x^{-2/3} = \frac{1}{3\sqrt[3]{x^2}}$$

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}[f(x)]$$

$$\frac{d}{dx}[4x^8] = 4\frac{d}{dx}[x^8] = 4[8x^7] = 32x^7$$

$$\frac{d}{dx}[-x^{12}] = (-1)\frac{d}{dx}[x^{12}] = -12x^{11}$$

$$\frac{d}{dx}\left[\frac{\pi}{x}\right] = \pi\frac{d}{dx}[x^{-1}] = \pi(-x^{-2}) = -\frac{\pi}{x^2}$$

$$\frac{d}{dx}[2x^6 + x^{-9}] = \frac{d}{dx}[2x^6] + \frac{d}{dx}[x^{-9}] = 12x^5 + (-9)x^{-10} = 12x^5 - 9x^{-10}$$

$$\frac{d}{dx}\left[\frac{\sqrt{x} - 2x}{\sqrt{x}}\right] = \frac{d}{dx}[1 - 2\sqrt{x}\,]$$

$$= \frac{d}{dx}[1] - \frac{d}{dx}[2\sqrt{x}\,] = 0 - 2\left(\frac{1}{2\sqrt{x}}\right) = -\frac{1}{\sqrt{x}}$$

Find dy/dx if $y = 3x^8 - 2x^5 + 6x + 1$.

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}[3x^8 - 2x^5 + 6x + 1] \\ &= \frac{d}{dx}[3x^8] - \frac{d}{dx}[2x^5] + \frac{d}{dx}[6x] + \frac{d}{dx}[1] \\ &= 24x^7 - 10x^4 + 6 \blacktriangleleft\end{aligned}$$

If $f(x) = 3x^4 - 2x^3 + x^2 - 4x + 2$, then

$$\begin{aligned}
f'(x) &= 12x^3 - 6x^2 + 2x - 4 \\
f''(x) &= 36x^2 - 12x + 2 \\
f'''(x) &= 72x - 12 \\
f^{(4)}(x) &= 72 \\
f^{(5)}(x) &= 0 \\
&\vdots \\
f^{(n)}(x) &= 0 \quad (n \geq 5)
\end{aligned}$$

$$\frac{d}{dx}[\sin x] = \cos x$$

$$\frac{d}{dx}[\cos x] = -\sin x$$

$$\frac{d}{dx}[\tan x] = \sec^2 x$$

$$\frac{d}{dx}[\sec x] = \sec x \tan x$$

$$\frac{d}{dx}[\cot x] = -\csc^2 x$$

$$\frac{d}{dx}[\csc x] = -\csc x \cot x$$

$$y = \tan^{-1}(x)$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$$y = \tan^{-1}(f(x))$$

$$\frac{dy}{dx} = \frac{1}{1+(f(x))^2} \times f'(x)$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

Integration

CONSTANTS, POWERS, EXPONENTIALS

1. $\int du = u + C$

2. $\int a\,du = a \int du = au + C$

3. $\int u^r\,du = \dfrac{u^{r+1}}{r+1} + C,\ r \neq -1$

4. $\int \dfrac{du}{u} = \ln|u| + C$

5. $\int e^u\,du = e^u + C$

6. $\int b^u\,du = \dfrac{b^u}{\ln b} + C,\ b > 0, b \neq 1$

TRIGONOMETRIC FUNCTIONS

7. $\int \sin u \, du = -\cos u + C$

8. $\int \cos u \, du = \sin u + C$

9. $\int \sec^2 u \, du = \tan u + C$

10. $\int \csc^2 u \, du = -\cot u + C$

11. $\int \sec u \tan u \, du = \sec u + C$

12. $\int \csc u \cot u \, du = -\csc u + C$

13. $\int \tan u \, du = -\ln|\cos u| + C$

14. $\int \cot u \, du = \ln|\sin u| + C$

(1) $$\int \frac{1}{1+x^2} = \tan^{-1}x + c$$

(2) $$\int \frac{1}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)+c$$

(3) $$\int \frac{1}{1+b^2x^2} = \frac{1}{b}\tan^{-1}(bx)+c$$

$\boxed{\text{Example 10}}$

$$\int \frac{1}{2x^2+3}\,dx$$

$b=\sqrt{2} \quad a=\sqrt{3}$

$$\frac{1}{\sqrt{3}} \times \frac{1}{\sqrt{2}} \times \tan^{-1}\left(\frac{\sqrt{2}\,x}{\sqrt{3}}\right)$$

$$\frac{\sqrt{6}}{6}\tan^{-1}\left(\frac{\sqrt{6}\,x}{3}\right) + C$$

HYPERBOLIC FUNCTIONS

15. $\displaystyle\int \sinh u \, du = \cosh u + C$

16. $\displaystyle\int \cosh u \, du = \sinh u + C$

17. $\displaystyle\int \operatorname{sech}^2 u \, du = \tanh u + C$

18. $\displaystyle\int \operatorname{csch}^2 u \, du = -\coth u + C$

19. $\displaystyle\int \operatorname{sech} u \tanh u \, du = -\operatorname{sech} u + C$

20. $\displaystyle\int \operatorname{csch} u \coth u \, du = -\operatorname{csch} u + C$

ALGEBRAIC FUNCTIONS ($a > 0$)

21. $$\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1} \frac{u}{a} + C \qquad (|u| < a)$$

22. $$\int \frac{du}{a^2 + u^2} = \frac{1}{a} \tan^{-1} \frac{u}{a} + C$$

23. $$\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a} \sec^{-1} \left|\frac{u}{a}\right| + C \qquad (0 < a < |u|)$$

24. $$\int \frac{du}{\sqrt{a^2 + u^2}} = \ln(u + \sqrt{u^2 + a^2}) + C$$

25. $$\int \frac{du}{\sqrt{u^2 - a^2}} = \ln\left|u + \sqrt{u^2 - a^2}\right| + C \qquad (0 < a < |u|)$$

26. $$\int \frac{du}{a^2 - u^2} = \frac{1}{2a} \ln\left|\frac{a + u}{a - u}\right| + C$$

27. $$\int \frac{du}{u\sqrt{a^2 - u^2}} = -\frac{1}{a} \ln\left|\frac{a + \sqrt{a^2 - u^2}}{u}\right| + C \qquad (0 < |u| < a)$$

28. $$\int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a} \ln\left|\frac{a + \sqrt{a^2 + u^2}}{u}\right| + C$$

$$\int \sin^n(x)\,dx = -\frac{1}{n}\sin^{n-1}(x)\cos(x) + \frac{n-1}{n}\int \sin^{n-2}(x)\,dx$$

$$\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx$$

$$\int \tan^n(x)\,dx = \frac{1}{n-1}\tan^{n-1}(x) - \int \tan^{n-2}(x)\,dx$$

$$\int \csc^n(x)\,dx = -\frac{1}{n-1}\csc^{n-2}(x)\cot(x) + \frac{n-2}{n-1}\int \csc^{n-2}(x)\,dx$$

$$\int \sec^n(x)\,dx = \frac{1}{n-1}\tan(x)\sec^{n-2}(x) + \frac{n-2}{n-1}\int \sec^{n-2}(x)\,dx$$

$$\int \cot^n(x)\,dx = \frac{-1}{n-1}\cot^{n-1}(x) - \int \cot^{n-2}(x)\,dx$$

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x\,dx.$$

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x\,dx,$$

$$du = (n-1)\cos^{n-2} x\,(-\sin x\,dx) \qquad \text{and} \qquad v = \sin x.$$

$$\begin{aligned}
\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1) \int \sin^2 x \cos^{n-2} x\,dx \\
&= \cos^{n-1} x \sin x + (n-1) \int (1 - \cos^2 x) \cos^{n-2} x\,dx, \\
&= \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\,dx - (n-1) \int \cos^n x\,dx.
\end{aligned}$$

$$n \int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1) \int \cos^{n-2} x\,dx.$$

$$
\begin{aligned}
\int \sin^n x \, dx &= \int \sin^{n-1} x \sin x \, dx \\
&= \sin^{n-1} x \, (-\cos x) - \int (-\cos x) \left((n-1) \sin^{n-2} x \cos x \right) \, dx \\
&= \int (n-1) \sin^{n-2} x \cos^2 x \, dx - \sin^{n-1} x \cos x \\
&= \int (n-1) \sin^{n-2} x \left(1 - \sin^2 x \right) \, dx - \sin^{n-1} x \cos x \\
&= (n-1) \int \sin^{n-2} x \, dx - (n-1) \int \sin^n x \, dx - \sin^{n-1} x \cos x \\
n \int \sin^n x \, dx &= (n-1) \int \sin^{n-2} x \, dx - \sin^{n-1} x \cos x \\
\int \sin^n x \, dx &= \frac{n-1}{n} \int \sin^{n-2} x \, dx - \frac{\sin^{n-1} x \cos x}{n}
\end{aligned}
$$

$$
\begin{aligned}
\int \tan^n x\,dx &= \int \tan^{n-2} x \tan^2 x\,dx = \int \tan^{n-2} x\,(\sec^2 x - 1)\,dx \\
&= \int \tan^{n-2} x \sec^2 x\,dx - \int \tan^{n-2} x\,dx \\
&= \int \tan^{n-2} x\,d(\tan x) - \int \tan^{n-2} x\,dx \\
&= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx.
\end{aligned}
$$

$$\int \operatorname{cosec}^n x\,dx = \int \operatorname{cosec}^{n-2} x \operatorname{cosec}^2 x\,dx$$

Integrating by parts,

$$\int \operatorname{cosec}^n x\,dx = \operatorname{cosec}^{n-2} x\,(-\cot x) - \int (n-2)\operatorname{cosec}^{n-3} x\,(-\operatorname{cosec} x \cot x)(-\cot x)\,dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\int \operatorname{cosec}^{n-2} x\,(\operatorname{cosec}^2 x - 1)\,dx$$

$$= -\cot x \operatorname{cosec}^{n-2} x - (n-2)\left(\int \operatorname{cosec}^n x - \int \operatorname{cosec}^{n-2} x\,dx\right)$$

$$[1+(n-2)]\int \operatorname{cosec}^n x\,dx = -\cot x \operatorname{cosec}^{n-2} x + (n-2)\int \operatorname{cosec}^{n-2} x\,dx$$

$$\int \operatorname{cosec}^n x\,dx = \frac{-\cot x \operatorname{cosec}^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \operatorname{cosec}^{n-2} x\,dx$$

Reduction formulae for $\int sec^n x.dx$

Let $I_n = \int sec^n x.dx$

$$I_n = \int sec^{n-2}x.sec^2 x dx$$

$$\int uv' = uv - \int u'v$$

$u = sec^{n-2}x$, $v' = sec^2 x$

$u' = (n-2)sec^{n-3}x.secx.tanx$, $v = tanx$

Integration by parts

$$I_n = sec^{n-2}x(tanx) - \int (n-2)\, sec^{n-3}x.(secx.tanx).tanx.dx$$

$$= sec^{n-2}x(tanx) - (n-2)\int sec^{n-2}x\,.tan^2 x dx$$

$$= sec^{n-2}x(tanx) - (n-2)\int sec^{n-2}x\,.(sec^2 x - 1)dx$$

$$= sec^{n-2}x(tanx) - (n-2)\int sec^n x\, dx + (n-2)\int sec^{n-2}x\,.dx$$

$$I_n = sec^{n-2}x(tanx) - (n-2)I_n + (n-2)I_{n-2}$$

$$I_n + (n-2)I_n = sec^{n-2}x(tanx) + (n-2)I_{n-2}$$

$$(n-1)I_n = sec^{n-2}x(tanx) + (n-2)I_{n-2}$$

$$\boldsymbol{I_n = \frac{sec^{n-2}x(tanx)}{n-1} + \frac{(n-2)I_{n-2}}{n-1}}$$

EXPRESSION IN THE INTEGRAND	SUBSTITUTION
$\sqrt{a^2 - x^2}$	$x = a \sin\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta$

In summary, we have shown that the substitution $u = \tan(x/2)$ can be implemented in a rational function of $\sin x$ and $\cos x$ by letting

$$\sin x = \frac{2u}{1+u^2}, \quad \cos x = \frac{1-u^2}{1+u^2}, \quad dx = \frac{2}{1+u^2}\,du \tag{5}$$

$$\int \frac{\sec x\,dx}{1} \cdot \frac{(\sec x + \tan x)}{(\sec x + \tan x)}$$

$u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)\,dx$

$$\int \frac{[\sec^2 x + \sec x \tan x]}{\sec x + \tan x}\,dx$$

$$\int \frac{du}{u} = \int \frac{1}{u}\,du = \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C$$

2:10 / 2:15

Trigonometric Identities

Calculus and Analytical Geometry

Trigonometry

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta}$$

$$\cos^2\theta + \sin^2\theta \equiv 1, \qquad 1 + \tan^2\theta \equiv \sec^2\theta, \qquad \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$$

$$\sin(A \pm B) \equiv \sin A\cos B \pm \cos A\sin B$$

$$\cos(A \pm B) \equiv \cos A\cos B \mp \sin A\sin B$$

$$\tan(A \pm B) \equiv \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B}$$

$$\sin 2A \equiv 2\sin A\cos A$$

$$\cos 2A \equiv \cos^2 A - \sin^2 A \equiv 2\cos^2 A - 1 \equiv 1 - 2\sin^2 A$$

$$\tan 2A \equiv \frac{2\tan A}{1 - \tan^2 A}$$