第八次作业

(1)

Consider a potential

$$V = 0 \quad \text{for } r > R, \qquad V = V_0 = \text{constant} \quad \text{for } r < R,$$

where V_0 may be positive or negative. Using the method of partial waves, show that for $|V_0| \ll E = \hbar^2 k^2/2m$ and $kR \ll 1$ the differential cross section is isotropic and that the total cross section is given by

$$\sigma_{\text{tot}} = \left(\frac{16\pi}{9}\right)\frac{m^2 V_0^2 R^6}{\hbar^4}.$$

Suppose the energy is raised slightly. Show that the angular distribution can then be written as

$$\frac{d\sigma}{d\Omega} = A + B\cos\theta.$$

Obtain an approximate expression for B/A.

(2)

Consider the scattering of a particle by an impenetrable sphere

$$V(r) = \begin{cases} 0 & \text{for } r > a \\ \infty & \text{for } r < a. \end{cases}$$

a. Derive an expression for the s-wave ($l = 0$) phase shift. (You need not know the detailed properties of the spherical Bessel functions to be able to do this simple problem!)

b. What is the total cross section σ $[\sigma = \int (d\sigma/d\Omega)\, d\Omega]$ in the extreme low-energy limit $k \to 0$? Compare your answer with the geometric cross section πa^2. You may assume without proof:

$$\frac{d\sigma}{d\Omega} = |f(\theta)|^2,$$

$$f(\theta) = \left(\frac{1}{k}\right)\sum_{l=0}^{\infty}(2l+1)e^{i\delta_l}\sin\delta_l P_l(\cos\theta).$$

(3)

The Lippmann-Schwinger formalism can also be applied to a *one*-dimensional transmission-reflection problem with a finite-range potential, $V(x) \neq 0$ for $0 < |x| < a$ only.

ι. Suppose we have an incident wave coming from the left: $\langle x|\phi\rangle = e^{ikx}/\sqrt{2\pi}$. How must we handle the singular $1/(E - H_0)$ operator if we are to have a transmitted wave only for $x > a$ and a reflected wave and the original wave for $x < -a$? Is the $E \rightarrow E + i\varepsilon$ prescription still correct? Obtain an expression for the appropriate Green's function and write an integral equation for $\langle x|\psi^{(+)}\rangle$.

(4)

Consider scattering by a repulsive δ-shell potential:

$$\left(\frac{2m}{\hbar^2}\right)V(r) = \gamma\delta(r - R), \quad (\gamma > 0).$$

a. Set up an equation that determines the s-wave phase shift δ_0 as a function of k $(E = \hbar^2k^2/2m)$.

b. Assume now that γ is very large,

$$\gamma \gg \frac{1}{R}, k.$$

Show that if $\tan kR$ is *not* close to zero, the s-wave phase shift resembles the hard-sphere result discussed in the text. Show also that for $\tan kR$ close to (but not exactly equal to) zero, resonance behavior is possible; that is, $\cot\delta_0$ goes through zero from the positive side as k increases.

第九次作业

(1)

Consider scattering by a repulsive δ-shell potential:

$$\left(\frac{2m}{\hbar^2}\right)V(r) = \gamma\delta(r-R), \quad (\gamma > 0).$$

Assume now that γ is very large,

$$\gamma \gg \frac{1}{R}, k.$$

Determine approximately the positions of the resonances keeping terms of order $1/\gamma$; compare them with the bound-state energies for a particle confined *inside* a spherical wall of the same radius,

$$V = 0, \quad r < R; \qquad V = \infty, \quad r > R.$$

(2)

The Lippmann-Schwinger formalism can also be applied to a *one-dimensional* transmission-reflection problem.

Consider the special case of an attractive δ-function potential

$$V = -\left(\frac{\gamma\hbar^2}{2m}\right)\delta(x) \quad (\gamma > 0).$$

Solve the integral equation to obtain the transmission and reflection amplitudes.

The one-dimensional δ-function potential with $\gamma > 0$ admits one (and only one) bound state for any value of γ. Show that the transmission and reflection amplitudes you computed have bound-state poles at the expected positions when k is regarded as a complex variable.

(3) 证明波算符满足关系式 $\hat{H}\hat{U}(0,\pm\infty) = \hat{U}(0,\pm\infty)\hat{H}_0$

(4) 证明散射矩阵满足关系式 $\hat{S}|E,l,m\rangle = e^{2i\delta_l}|E,l,m\rangle$

第九周高等量子力学作业

隋源　2000011379

Nov 8th 2022

1

按照分波法，$r > R$ 时波函数可展开为球面波

$$\psi = \sum_{l=0} i^l (2l+1) \left[j_l(kr) + \frac{a_l}{2} h_l(kr) \right] P_l(\cos\theta)$$

式中 $a_l = \exp(2i\delta_l) - 1$ 代表相移因子，显然当无散射时 $a_l = 0$。对径向方程积分可以得到相移和势能 $V(r)$ 的关系

$$\frac{a_l}{2i} = -\frac{2mk}{\hbar^2} \int_0^\infty V(r) j_l(kr) A_l(r) r^2 \mathrm{d}r$$

其中 $A_l(r)$ 代表径向波函数，$r > R$ 时满足 $A_l(r) = j_l(kr) + \frac{a_l}{2} h_l(kr)$。若使用低能情形的一级玻恩近似，上式简化为

$$\delta_l = -\frac{2mk^{2l+1}}{[(2l+1)!!\hbar]^2} \int_0^R V(r) r^{2l+2} \mathrm{d}r$$

代入 $V(r) = V_0$ 得到 l 分波的相移（零阶近似）

$$\delta_0 = -\frac{2mV_0 kR^3}{3\hbar^2}, \quad \delta_1 = -\frac{2mV_0 k^3 R^5}{45\hbar^2}, \quad \ldots$$

由此可见低能情形可用 l 从小到大进行近似展开。若取 s 波和 p 波，散射截面应为

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{k^2} |e^{i\delta_0} \sin\delta_0 + 3e^{i\delta_1} \sin\delta_1 \cos\theta|^2 \approx \frac{1}{k^2} [\sin^2\delta_0 + 6\cos(\delta_1 - \delta_0) \sin\delta_1 \sin\delta_0 \cos\theta] \qquad (\star)$$

可见满足 $A + B\cos\theta$ 形式。比值的零阶近似为

$$\frac{B}{A} \approx \frac{6\delta_1}{\delta_0} = \frac{2}{5} k^2 R^2 \qquad (\star)$$

若只取 s 波，显然各向同性，总散射截面的零阶近似为

$$\sigma_{tot} \approx 4\pi \left(\frac{\delta_0}{k} \right)^2 = \frac{16\pi m^2 V_0^2 R^6}{9\hbar^4} \qquad (\star)$$

2

$r > R$ 时径向波函数为

$$A_l(r) = j_l(kr) + \frac{a_l}{2}h_l(kr) = e^{i\delta_l}[\cos\delta_l j_l(kr) - \sin\delta_l n_l(kr)]$$

由刚性球的边界条件得到相移

$$A_l(a) = 0 \quad \Rightarrow \quad \tan\delta_l = \frac{j_l(ka)}{n_l(ka)}$$

对 s 波有严格解

$$\tan\delta_0 = -\tan(ka) \quad \Rightarrow \quad \delta_0 = -ka \tag{$\star$}$$

低能近似下有

$$\tan\delta_l \approx -\frac{(2l-1)!!}{(2l+1)!!}(kr)^{2l+1}$$

显然零阶近似取 s 波即可。因此

$$\sigma_{tot} \approx 4\pi\left(\frac{\delta_0}{k}\right)^2 = 4\pi a^2 \tag{$\star$}$$

即计算结果是几何截面的 4 倍。

3

一维 Lippman-Schwinger 方程

$$\psi^{(+)}(x) = \phi(x) + \frac{2m}{\hbar^2}\int \mathrm{d}x' G(x,x')V(x')\psi^{(+)}(x') \tag{$\star$}$$

此时格林函数为

$$G(x,x') = \frac{\hbar^2}{2m}\langle x|(E-H_0)^{-1}|x'\rangle = \frac{1}{2\pi}\int \mathrm{d}q \frac{e^{iq(x-x')}}{k^2-q^2}$$

其中 $E = \hbar^2k^2/2m, H_0 = \hbar^2q^2/2m$。仍然用 $E \to E + i\varepsilon$ 方法，$x > x'$ 时积分取绕 k 的逆时针围道，$x < x'$ 时积分取绕 $-k$ 的顺时针围道。计算得到格林函数

$$G(x,x') = \frac{1}{2\pi}(+2\pi i)\frac{e^{+ik(x-x')}}{-2k} = \frac{1}{2ik}e^{+ik(x-x')}, \quad x > x' \tag{$\star$}$$

$$G(x,x') = \frac{1}{2\pi}(-2\pi i)\frac{e^{-ik(x-x')}}{+2k} = \frac{1}{2ik}e^{-ik(x-x')}, \quad x < x' \tag{$\star$}$$

将格林函数 $G(x,x')$ 和 $\phi(x) = e^{ikx}/\sqrt{2\pi}$ 代回一维 Lippman-Schwinger 方程即是所求的积分方程。

4

由相移和势能的关系可以写出 s 波径向波函数 $r=R$ 时的值

$$A_0(R)=j_0(kR)-\frac{2mk}{\hbar^2}ih_0(kR)\int_0^\infty V(r)j_l(kr)A_l(r)r^2\mathrm{d}r$$

代入 $V(r)=\hbar^2\gamma\delta(r-R)/2m$ 得到

$$A_0(kR)=\frac{j_0(kR)}{1+\gamma Rj_0(kR)e^{ikR}}$$

由此得到 s 波相移

$$\tan\delta_0=\frac{\Im(A_0)}{\Re(A_0)}=-\frac{\sin^2(kR)}{k/\gamma+\sin(kR)\cos(kR)} \qquad (\star)$$

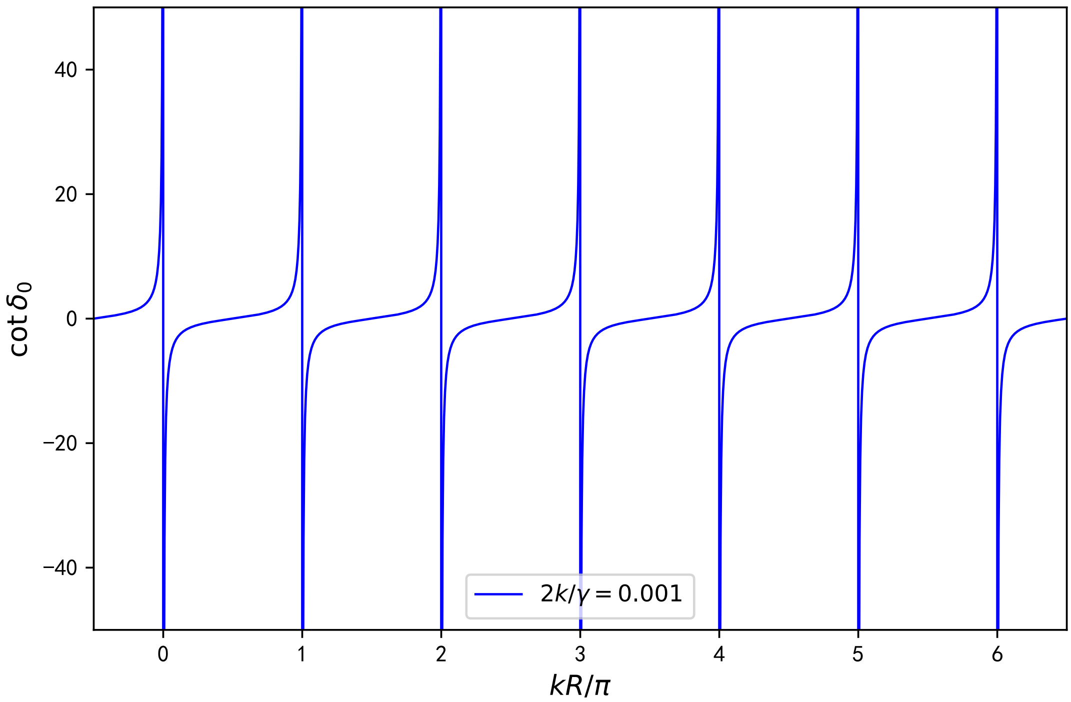
当 γ 很大，但 $\tan(kR)$ 又不接近零时，s 波相移近似为

$$\tan\delta_0\approx-\tan(kR)\quad\Rightarrow\quad\delta_0\approx-kR \qquad (\star)$$

为了判断共振行为，改写

$$\cot\delta_0=\frac{\sin(2kR)+2k/\gamma}{\cos(2kR)-1}$$

作图如下



图中可以看出共振点位于

$$kR\approx n\pi,\quad n\in\mathbb{Z} \qquad (\star)$$

即 $\tan(kR)$ 接近零时会有共振行为。

第十周高等量子力学作业

隋源　2000011379

Nov 15th 2022

1

由上次作业的结果

$$\cot\delta_0 = -\frac{k/\gamma + \sin kR\cos kR}{\sin^2 kR}$$

由于零点在 $kR = n\pi$ 附近，因此当分子等于零时有

$$-\frac{k}{\gamma} = \sin kR\cos kR = \frac{1}{2}\sin(2kR) = \frac{1}{2}\sin(2kR - 2n\pi) \approx kR - n\pi$$

因此得到一阶近似下的零点

$$k = \frac{n\pi}{R}\left(1 - \frac{1}{\gamma R}\right)$$

对应的自由粒子能量

$$E_n = \frac{\hbar^2 k^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mR^2}\left(1 - \frac{2}{\gamma R}\right)$$

如果忽略修正 $1 - 2/\gamma R$，这正好是无穷深球势阱的能量本征值。

证明如下：$l = 0$ 时无穷深球势阱的径向方程为

$$-\frac{\hbar^2}{2mr^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}u}{\mathrm{d}r}\right) = Eu$$

对应的本征函数和能量本征值为

$$j_0(r) = \frac{\sin kr}{kr}, \quad E = \frac{\hbar^2 k^2}{2m}$$

其中 k 满足 $j_0(kR) = 0$ 即 $kR = \pi, 2\pi, 3\pi, ...$，因此有

$$E_n = \frac{n^2\pi^2\hbar^2}{2mR^2}, \quad n = 1, 2, 3...$$

和上面的共振态能量零阶近似一致，证毕。

2

由上次作业的结果

$$\psi(x)=\phi(x)+\frac{2m}{\hbar^2}\int \mathrm{d}x' G(x,x')V(x')\psi(x')$$

其中

$$\phi(x)=\frac{1}{\sqrt{2\pi}}e^{ikx},\quad V(x)=-\frac{\hbar^2\gamma}{2m}\delta(x),\quad G(x,x')=\frac{1}{2ik}e^{ik|x-x'|}$$

代入可以得到

$$\psi(x)=\phi(x)-\gamma\frac{1}{2ik}e^{ik|x|}\psi(0)$$

$$\psi(0)=\phi(0)-\frac{\gamma}{2ik}\psi(0)$$

联立上两式子可得

$$\psi(x)=\frac{1}{\sqrt{2\pi}}\left(e^{ikx}-\frac{\gamma}{2ik+\gamma}e^{ik|x|}\right)$$

显然 $x<0$ 时为入射和反射波，$x>0$ 时为透射波，反射率和透射率为

$$R(k)=-\frac{\gamma}{2ik+\gamma},\quad T(k)=\frac{2ik}{2ik+\gamma}$$

对应的极点为 $k=i\gamma/2$，自由粒子能量为

$$E=\frac{\hbar^2k^2}{2m}=-\frac{\hbar^2\gamma^2}{8m}$$

正好是 $V(x)$ 对应的基态能量。

证明如下：薛定谔方程满足

$$\psi(x)''+\gamma\delta(x)\psi(x)=-\frac{2mE}{\hbar^2}\psi(x)$$

根据无穷远处边界条件和 0 处的连续条件得到波函数形式

$$\psi(x)=Ae^{-ik|x|},\quad \hbar k=\sqrt{2mE}$$

对方程积分得到 0 处一阶导的连续条件

$$\psi'_{0+}-\psi'_{0-}+\gamma\psi(0)=0$$

代入波函数得到 $k=i\gamma/2$，正好是极点，证毕。

3

根据 Moller 波算符的定义，有

$$|\psi_{\boldsymbol{p}\nu}^{(\pm)}\rangle = \hat{U}(0, \pm\infty)|\boldsymbol{p}\nu\rangle$$

$$\hat{H}|\psi_{\boldsymbol{p}\nu}^{(\pm)}\rangle = E_p|\psi_{\boldsymbol{p}\nu}^{(\pm)}\rangle$$

$$\hat{H}_0|\boldsymbol{p}\nu\rangle = E_p|\boldsymbol{p}\nu\rangle$$

因此

$$\hat{H}\hat{U}(0, \pm\infty)|\boldsymbol{p}\nu\rangle = \hat{H}|\psi_{\boldsymbol{p}\nu}^{(\pm)}\rangle = E_p|\psi_{\boldsymbol{p}\nu}^{(\pm)}\rangle = E_p\hat{U}(0, \pm\infty)|\boldsymbol{p}\nu\rangle = \hat{U}(0, \pm\infty)\hat{H}_0|\boldsymbol{p}\nu\rangle$$

即

$$\hat{H}\hat{U}(0, \pm\infty) = \hat{U}(0, \pm\infty)\hat{H}_0$$

证毕。

4

根据散射矩阵性质

$$\langle\boldsymbol{p}'|\hat{S} - 1|\boldsymbol{p}\rangle = \frac{i}{2\pi\hbar m}f(\boldsymbol{p}', \boldsymbol{p})\delta(E_{p'} - E_p)$$

得到

$$f(\boldsymbol{p}', \boldsymbol{p}) = -2\pi\hbar mi\int\langle\boldsymbol{p}'|\hat{S} - 1|\boldsymbol{p}\rangle \mathrm{d}E_{p'}$$

由于 $\hat{S}$ 是 $|Elm\rangle$ 本征态 (和 H_0 互易且满足旋转不变)，设

$$\hat{S}|Elm\rangle = S_l|Elm\rangle$$

利用

$$\langle\boldsymbol{p}|Elm\rangle = \frac{1}{\sqrt{mp}}\delta(E - E_p)\mathrm{Y}_l^m(\hat{\boldsymbol{p}})$$

代入得到

$$\begin{aligned}
f(\boldsymbol{p}', \boldsymbol{p}) &= -2\pi\hbar mi\sum_{l,l',m,m'}\iiint\langle\boldsymbol{p}'|E'l'm'\rangle\langle E'l'm'|\hat{S} - 1|Elm\rangle\langle Elm|\boldsymbol{p}\rangle \mathrm{d}E_{p'}\mathrm{d}E\mathrm{d}E' \\
&= -2\pi\hbar mi\sum_{l,m}(S_l - 1)\iint\langle\boldsymbol{p}'|Elm\rangle\langle Elm|\boldsymbol{p}\rangle \mathrm{d}E_{p'}\mathrm{d}E \\
&= -2\pi\hbar i\sum_{l,m}(S_l - 1)\frac{1}{p}\mathrm{Y}_l^m(\hat{\boldsymbol{p}}')\mathrm{Y}_l^{m*}(\hat{\boldsymbol{p}})
\end{aligned}$$

由于积分中有条件 $|\boldsymbol{p}'| = |\boldsymbol{p}|$，令二者夹角为 θ，且 $\hat{\boldsymbol{p}} = \hat{\boldsymbol{z}}$，于是

$$\begin{aligned} f(\theta) &= -2\pi\hbar i \sum_l (S_l - 1)\frac{1}{p}\frac{2l+1}{4\pi}\mathrm{P}_l(\cos\theta) \\ &= \sum_l (2l+1)\left(\frac{S_l - 1}{2ik}\right)\mathrm{P}_l(\cos\theta) \end{aligned}$$

根据相移的定义，立即得到

$$S_l = e^{2i\delta_l}$$