Solid State Physics Homework

Chapter3 No.1, Due on Apr 1st 2022, Friday

Sui Yuan 2000011379

Problem1 A&M Chapter12.2(a)

One can rewrite the energy

$$\varepsilon(\mathbf{k}) = \frac{\hbar^2}{2}\langle \mathbf{k}|\mathbf{M}^{-1}|\mathbf{k}\rangle \tag{1}$$

by taking the minimum energy to be zero and the corresponding wave vector at origin, which doesn't make changes. For calculating cyclotron effective mass, one can get

$$\sum_{a,b=x,y} \mathbf{M}^{-1}{}_{ab}k_a k_b + 2k_z \sum_{c=x,y} \mathbf{M}^{-1}{}_{cz}k_c = \frac{2\varepsilon}{\hbar^2} - \mathbf{M}^{-1}{}_{zz}k_z^2 \tag{2}$$

where ε and k_z should be constant. Make a translation transformation to reset the origin in the k_z plane, then one can simplify (2) as

$$\mathbf{M}^{-1}{}_{xx}{k_x'}^2 + \mathbf{M}^{-1}{}_{yy}{k_y'}^2 + 2\mathbf{M}^{-1}{}_{xy}k_x' k_y' = \frac{2\varepsilon}{\hbar^2} - Ck_z^2 \tag{3}$$

Next, make a rotation transformation

$$A{k_x''}^2 + B{k_y''}^2 = \frac{2\varepsilon}{\hbar^2} - Ck_z^2 \tag{4}$$

Therefore

$$A(\varepsilon, k_z) = \pi\frac{2\varepsilon/\hbar^2 - Ck_z^2}{\sqrt{AB}}, \quad m^* = \frac{\hbar^2}{2\pi}\frac{\partial A(\varepsilon, k_z)}{\partial \varepsilon} = \frac{1}{\sqrt{AB}} \tag{5}$$

A, B can be simply written down by combining (3)(4). Using the linear algebra formula, one can prove

$$m^* = \frac{1}{\sqrt{AB}} = (\frac{|\mathbf{M}|}{\mathbf{M_{zz}}})^{\frac{1}{2}} \tag{6}$$

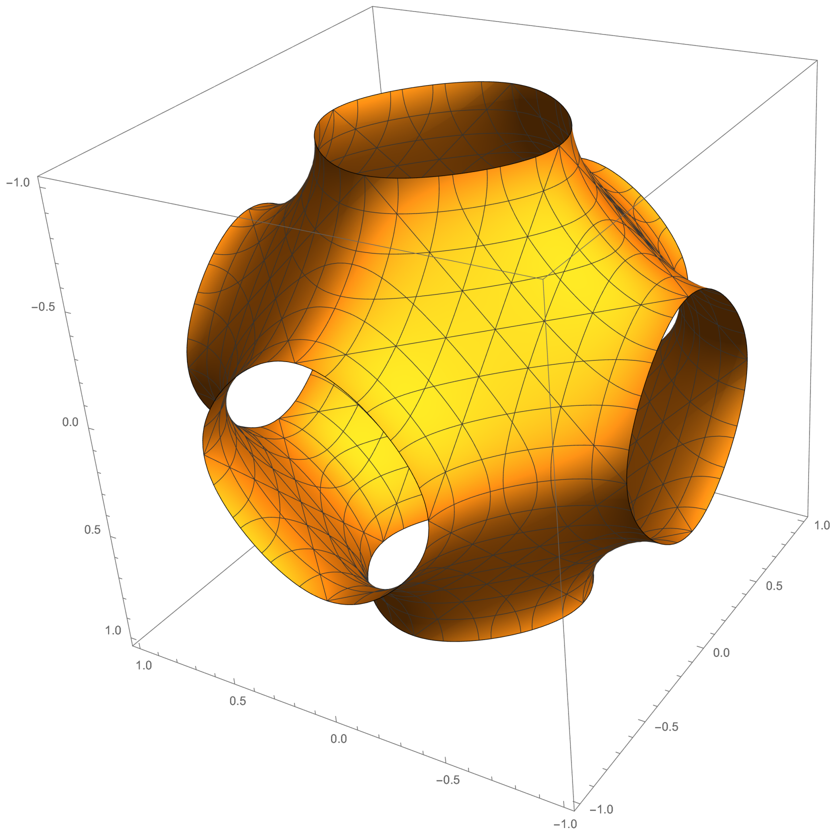
Problem2 Cyclotron effective mass

The tight-binding energy band of the simple cubic lattice is

$$\varepsilon(\mathbf{k}) = \varepsilon_s - J_0 - 2J_1(\cos k_x a + \cos k_y a + \cos k_z a) \tag{7}$$

For calculating cyclotron effective mass, the route in the k_z plane is determined by

$$\cos k_x a + \cos k_y a = const \tag{8}$$



Contour surface of energy in First Brillouin Zone: $\varepsilon = \varepsilon_s - J_0$. The unit coordinate is π/a

The area enclosed by the route(8) is $A(\varepsilon, k_z)$(which is in the figure), and the effective mass is

$$m^* = \frac{\hbar^2}{2\pi}\frac{\partial A(\varepsilon, k_z)}{\partial \varepsilon} \tag{9}$$

Problem3 A&M Chapter14.1(b)

According to the definition of the reciprocal lattice vector, the pattern density in ΔA is

$$\sigma = (\frac{L}{2\pi})^2 \tag{10}$$

Where L is the length of the cubic. So the number of energy states in the area is

$$N = 2\sigma\Delta A = 2(\frac{L}{2\pi})^2\frac{2\pi eH}{\hbar c} = \frac{2e}{hc}HL^2 \tag{11}$$

Which is concisely equal to the degeneracy. Note the former factor 2 is for spin degeneracy.

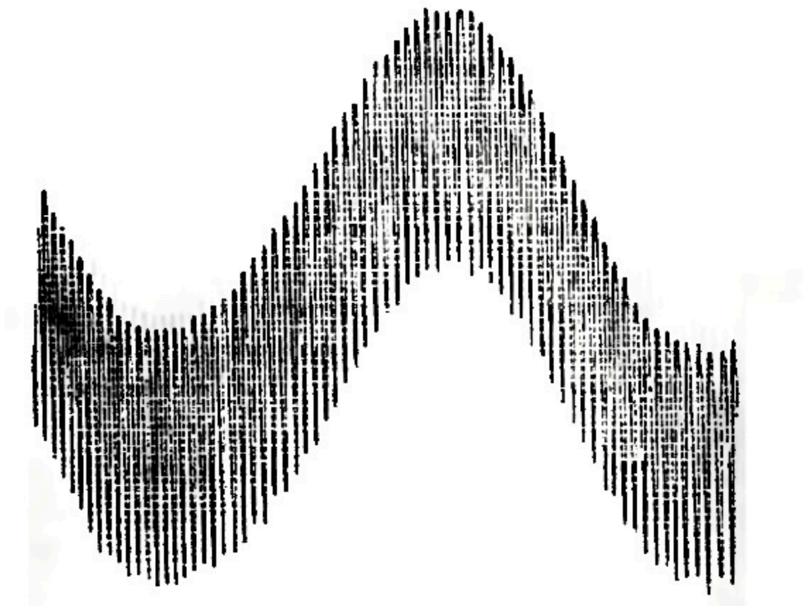
Problem4 A&M Chapter14.2

From the fundamental relation

$$\Delta(\frac{1}{H}) = \frac{2\pi e}{\hbar c}\frac{1}{A_e} \tag{12}$$

one can get the relation between the area of extremal orbits and its oscillation cycle

$$A_e \Delta T_A = const \tag{13}$$



De Haas-van Alphen oscillations in silver

For the oscillation cycle in the figure, one can find

$$0.5T_l \approx 30T_s \tag{14}$$

So the ratio of the area of two extremal orbits is

$$\frac{A_s}{A_l} \approx 60 \tag{15}$$

Solid State Physics Homework

Chapter3 No.8, Due on May 13th 2022, Friday

Sui Yuan 2000011379

Problem1 A&M Chapter13.2

According to (13.24) and (13.25), the conductivity tensor is a sum of contributions from each band

$$\sigma = \sum_n e^2 \int \frac{d\mathbf{k}}{4\pi^3} \tau_n(\varepsilon_n(\mathbf{k}))\mathbf{v}_n(\mathbf{k})\mathbf{v}_n(\mathbf{k}) \left(-\frac{\partial f}{\partial \varepsilon}\right)_{\varepsilon=\varepsilon_n(\mathbf{k})} \tag{1}$$

one can simplify it by taking the anisotropy and irrelevant bands of crystals of cubic symmetry into consideration

$$\sigma_{\mu\nu} = e^2 \int \frac{d\mathbf{k}}{4\pi^3} \tau(\varepsilon_{\mathbf{k}})\mathbf{v}_\mu(\mathbf{k})\mathbf{v}_\nu(\mathbf{k}) \left(-\frac{\partial f}{\partial \varepsilon_{\mathbf{k}}}\right) \tag{2}$$

$$\sigma_{\mu\nu} = \sigma\delta_{\mu\nu} \tag{3}$$

i.e. the conductivity can be described as a vector. Divide the momentum space into equal energy areas

$$d\mathbf{k} = dSdk_\perp = dS\frac{d\varepsilon_{\mathbf{k}}}{|\nabla_{\mathbf{k}}\varepsilon_{\mathbf{k}}|} = \frac{dS}{\hbar|\mathbf{v}(\mathbf{k})|}d\varepsilon_{\mathbf{k}} \tag{4}$$

when $T = 0$, use the approximation

$$\left(-\frac{\partial f}{\partial \varepsilon_{\mathbf{k}}}\right) = \delta(\varepsilon_{\mathbf{k}} - \varepsilon_F) \tag{5}$$

then one can get the result below

$$\begin{aligned} \sigma_{\mu\nu} &= \frac{e^2}{4\pi^3\hbar}\int \tau(\varepsilon_F)\frac{\mathbf{v}_\mu(\mathbf{k})\mathbf{v}_\nu(\mathbf{k})}{|\mathbf{v}(\mathbf{k})|}dS_F, \\ \sigma &= \frac{1}{3}(\sigma_{xx}+\sigma_{yy}+\sigma_{zz}) \\ &= \frac{e^2}{12\pi^3\hbar}\int \tau(\varepsilon_F)|\mathbf{v}(\mathbf{k})|dS_F \\ &= \frac{e^2}{12\pi^3\hbar}\tau(\varepsilon_F)\bar{v}S_F \end{aligned} \tag{6}$$

which is same as (13.71) and (13.72). And it is easy to write down the result in free electron limit

$$\sigma = \frac{ne^2\tau(\varepsilon_F)}{m^*} \tag{7}$$

and it is precisely the Drude result, by replacing the electron mass with effective mass, and the relaxation-time is qualified by the Fermi surface.

Problem2 A&M Chapter13.5

For a metal subject simultaneously to a nonzero electric field and nonzero thermal gradient, the heat production per unit volume is given by

$$\frac{dq}{dt} = \frac{du}{dt} - \mu\frac{dn}{dt} = -\nabla \cdot \mathbf{j}^{\epsilon} + \mathbf{E} \cdot \mathbf{j} + \mu\nabla \cdot \mathbf{j}^{n} \tag{8}$$

where $\mathbf{j} = -e\mathbf{j}^n$ is the charge current density, $\mathbf{j}^\epsilon$ is the energy current density. substitute (13.40) into (8), one can get

$$\frac{dq}{dt} = -\nabla \cdot \mathbf{j}^q + \varepsilon \cdot \mathbf{j} \tag{9}$$

where $\varepsilon = \mathbf{E} + \frac{1}{e}\nabla\mu$ is the field associated with the electrochemical potential. And it is same as (13.83). Use (13.45) and the relations in A&M and take the symmetry into consideration, one can get

$$\mathbf{j} = L^{11}\varepsilon - L^{12}\nabla T, \quad \mathbf{j}^q = L^{21}\varepsilon - L^{22}\nabla T \tag{10}$$

$$\frac{1}{\rho} = L^{11}, \quad K = L^{22} - \frac{L^{21}L^{12}}{L^{11}}, \quad Q = \frac{L^{12}}{L^{11}} \tag{11}$$

substitute (10) and (11) into (9), one can get

$$\begin{aligned}\frac{dq}{dt} =& -\nabla \cdot \mathbf{j}^q + \varepsilon \cdot \mathbf{j} \\ =&\rho\mathbf{j}^2 + \nabla K \cdot \nabla T - T\nabla Q \cdot \mathbf{j} \\ =&\rho\mathbf{j}^2 + \frac{dK}{dT}(\nabla T)^2 - T\frac{dQ}{dT}\nabla T \cdot \mathbf{j}\end{aligned}$$

which is precisely (13.84).

For crude estimate in Problem3, Chapter1, the coefficient is(use (1.7))

$$A_c = \frac{ne\rho\tau}{m}\frac{d\varepsilon}{dT} = \frac{1}{e}\frac{d\varepsilon}{dT} = \frac{3k_B}{2e} \tag{12}$$

and for free quantum-mechanical electrons, the coefficient is(use (2.94))

$$Q = -\frac{\pi^2}{6}\frac{k_B^2 T}{e\epsilon_F}, \quad A_q = \frac{\pi^2}{6}\frac{k_B^2 T}{e\epsilon_F} \tag{13}$$

So the ratio is about

$$\frac{A_q}{A_c} = \frac{2\pi^2}{3}\frac{k_B T}{\epsilon_F} \tag{14}$$

At room temperature, and let $\epsilon_F = 5\text{eV}$, one can find the numerical value of this ratio to be on the order of 10^{-2}.

Problem3 A&M Chapter16.3

(a)

According to the Boltzmann equation and the relaxation-time approximation

$$e\hbar^{-1}(\mathbf{E} + \mathbf{v}(\mathbf{k}) \times \mathbf{B}) \cdot \nabla_{\mathbf{k}} g(\mathbf{k}) = \frac{g(\mathbf{k}) - g^0(\mathbf{k})}{\tau(\mathbf{k})} \tag{15}$$

because the energy band is $\varepsilon(\mathbf{k}) = \frac{\hbar^2 k^2}{2m^*}$, the distribution must be qualified by symmetry

$$g(\mathbf{k}) = g^0(\varepsilon) + \mathbf{a}(\varepsilon) \cdot \mathbf{k} \tag{16}$$

Substitute (16) and the energy band into (15), one can get

$$\hbar\left(\frac{\partial g^0}{\partial \varepsilon}\right)\mathbf{E} \cdot \mathbf{k} + (\mathbf{k} \times \mathbf{B}) \cdot \mathbf{a}(\varepsilon) = \frac{m^* \mathbf{a}(\varepsilon)}{e\tau(\mathbf{k})} \cdot \mathbf{k} \tag{17}$$

(b)

one can simplify (15)

$$\frac{e\hbar}{m^*}\tau(\mathbf{k})\left(\frac{\partial g^0}{\partial \varepsilon}\right)\mathbf{E} = \frac{e\tau(\mathbf{k})}{m^*}\mathbf{a}(\varepsilon) \times \mathbf{B} + \mathbf{a}(\varepsilon) = \mathbf{T}\mathbf{a}(\varepsilon) \tag{18}$$

while $\mathbf{T}$ is a tensor. So the conductivity is

$$\sigma = e^2 \int \frac{d\mathbf{k}}{4\pi^3}\tau(\varepsilon)\left(-\frac{\partial g^0}{\partial \varepsilon}\right)[\mathbf{v}_\mu \mathbf{v}_\nu]\mathbf{T}^{-1} \tag{19}$$

and it is simple to proof the result is equal to (13.69) and (13.70) at free electron band.

Problem4 A&M Chapter16.4

(a)

According to the Boltzmann equation and the impurity scattering approximation

$$e\hbar^{-1}\mathbf{E} \cdot \nabla_{\mathbf{k}} g(\mathbf{k}) = -\int \frac{d\mathbf{k}}{(2\pi)^3} W_{\mathbf{k},\mathbf{k}'}[g(\mathbf{k}) - g(\mathbf{k}')] \tag{20}$$

and substitute the distribution form

$$g(\mathbf{k}) = f(\varepsilon) + \delta g(\mathbf{k}) \tag{21}$$

into (20), one can find

$$e\left(-\frac{\partial f}{\partial \varepsilon}\right)\mathbf{v}(\mathbf{k})\cdot\mathbf{E}=\int\frac{d\mathbf{k}}{(2\pi)^3}W_{\mathbf{k},\mathbf{k}'}[\delta g(\mathbf{k})-\delta g(\mathbf{k}')] \tag{22}$$

Define

$$\mathbf{v}(\mathbf{k})=\int\frac{d\mathbf{k}}{(2\pi)^3}W_{\mathbf{k},\mathbf{k}'}[u(\mathbf{k})-u(\mathbf{k}')] \tag{23}$$

$$g(\mathbf{k})=e\left(-\frac{\partial f}{\partial \varepsilon}\right)u(\mathbf{k})\cdot\mathbf{E} \tag{24}$$

then the conductivity should be

$$\sigma=e^2\int\frac{d\mathbf{k}}{4\pi^3}\left(-\frac{\partial f}{\partial \varepsilon}\right)\mathbf{v}(\mathbf{k})\mathbf{u}(\mathbf{k}) \tag{25}$$

which are same as (16.37) and (16.38).

(b)

it is easy to prove

$$\begin{aligned}
\{\alpha,\gamma\}=&e^2\int\frac{d\mathbf{k}}{4\pi^3}\left(-\frac{\partial f}{\partial \varepsilon}\right)\alpha(\mathbf{k})\int\frac{d\mathbf{k}'}{(2\pi)^3}W_{\mathbf{k},\mathbf{k}'}[\gamma(\mathbf{k})-\gamma(\mathbf{k}')]\\
=&e^2\int\int\frac{d\mathbf{k}}{4\pi^3}\frac{d\mathbf{k}'}{(2\pi)^3}\left(-\frac{\partial f}{\partial \varepsilon}\right)W_{\mathbf{k},\mathbf{k}'}[\alpha(\mathbf{k})\gamma(\mathbf{k})-\alpha(\mathbf{k})\gamma(\mathbf{k}')]\\
=&e^2\int\int\frac{d\mathbf{k}}{4\pi^3}\frac{d\mathbf{k}'}{(2\pi)^3}\left(-\frac{\partial f}{\partial \varepsilon}\right)W_{\mathbf{k},\mathbf{k}'}[\alpha(\mathbf{k})\gamma(\mathbf{k})-\alpha(\mathbf{k}')\gamma(\mathbf{k})]\\
=&\{\gamma,\alpha\}
\end{aligned}$$

(c)

Obviously$\{\alpha,\alpha\}\geqslant 0$.Then

$$\begin{aligned}
\{\alpha+\lambda\gamma,\alpha+\lambda\gamma\}=&\{\alpha,\alpha\}+2\lambda\{\alpha,\gamma\}+\lambda^2\{\gamma,\gamma\}\\
=&\{\gamma,\gamma\}\left(\lambda+\frac{\{\alpha,\gamma\}}{\{\gamma,\gamma\}}\right)^2+\{\alpha,\alpha\}-\frac{\{\alpha,\gamma\}^2}{\{\gamma,\gamma\}}\\
\geqslant&0
\end{aligned}$$

let $\lambda=-\{\alpha,\gamma\}/\{\gamma,\gamma\}$,then one can get the result

$$\{\alpha,\alpha\}\geqslant\frac{\{\alpha,\gamma\}^2}{\{\gamma,\gamma\}} \tag{26}$$

(d)

Let $\alpha = u_x$ in (26), then

$$\begin{aligned}\sigma_{xx} =& \{u_x, u_x\} \\ \geqslant& \frac{\{u_x, \gamma\}^2}{\{\gamma, \gamma\}} \\ =& \frac{(v_x, \gamma)^2}{\{\gamma, \gamma\}} \\ =& \frac{e^2[\int \frac{d\mathbf{k}}{4\pi^3}\left(-\frac{\partial f}{\partial \varepsilon}\right) v_x(\mathbf{k})\gamma(\mathbf{k})]^2}{\int \frac{d\mathbf{k}}{4\pi^3}\left(-\frac{\partial f}{\partial \varepsilon}\right)\gamma(\mathbf{k})\int \frac{d\mathbf{k}'}{(2\pi)^3} W_{\mathbf{k},\mathbf{k}'}[\gamma(\mathbf{k}) - \gamma(\mathbf{k}')]}\end{aligned}$$

(e)

One can define

$$\sigma = \{u, u\}_W = \{u, u_1\}_{W1} = \{u, u_2\}_{W2} \tag{27}$$

$$\sigma_1 = \{u_1, u_1\}_{W1}, \quad \sigma_2 = \{u_2, u_2\}_{W2} \tag{28}$$

so

$$\sigma^2 = \{u, u_1\}_{W1}\{u, u_2\}_{W2} \leqslant \sqrt{\sigma_1\sigma_2\{u, u\}_{W1}\{u, u\}_{W2}} \leqslant \sqrt{\sigma_1\sigma_2}\sigma/2 \tag{29}$$

$$2\sigma = \{u, u_1\}_{W1} + \{u, u_2\}_{W2} \geqslant (\sigma_1 + \sigma_2)/2 \tag{30}$$

using (29) and (30) one can derive

$$\frac{1}{\sigma} \geqslant \frac{1}{\sigma_1} + \frac{1}{\sigma_2} \tag{31}$$