第十次作业

（1）　利用定义式，证明玻色子的产生和湮灭算符满足

$$[\hat{a}_i, \hat{a}_j^+] = \delta_{ij},\ \ [\hat{a}_i, \hat{a}_j] = 0,\ \ \left[\hat{a}_i^+, \hat{a}_j^+\right] = 0$$

（2）利用玻色子产生和湮灭算符的对易关系以及单体波函数$\psi_i(\xi)$的完备性，证明场算符满足 $\left[\hat{\Phi}(\xi), \hat{\Phi}^+(\xi')\right] = \delta(\xi - \xi')$。

（3）证明关系式$\hat{\Phi}^+(\xi)|0\rangle = |\xi\rangle$，即场算符作用在真空态上得到的是广义坐标算符的单体本征态。（提示：利用

$$\hat{a}_i^+|0\rangle = |\psi_i\rangle \text{以及} \langle\xi'|\xi\rangle = \delta(\xi - \xi')\text{。）}$$

（4）证明费米子产生和湮灭算符满足对反易关系

$$\{\hat{a}_i, \hat{a}_j^+\} = \delta_{ij},\ \ \ \{\hat{a}_i, \hat{a}_j\} = 0,\ \ \{\hat{a}_i^+, \hat{a}_j^+\} = 0$$

第十一周高等量子力学作业

隋源　2000011379

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1

当 $i \neq j$ 时

$$\begin{aligned}
\hat{a}_i\hat{a}_j^\dagger &= \sum_{\{N\}}\sum_{\{N'\}}\sqrt{N_i(N_j'+1)}|N_1...N_i-1...\rangle\langle N_1...N_i...|N_1'...N_j'+1...\rangle\langle N_1'...N_j'...| \\
&= \sum_{\{N\}}\sqrt{N_iN_j}|N_1...N_i-1...\rangle\langle N_1...N_j-1...|
\end{aligned}$$

$$\begin{aligned}
\hat{a}_j^\dagger\hat{a}_i &= \sum_{\{N\}}\sum_{\{N'\}}\sqrt{N_i(N_j'+1)}|N_1'...N_j'+1...\rangle\langle N_1'...N_j'...|N_1...N_i-1...\rangle\langle N_1...N_i...| \\
&= \sum_{\{N\}}\sqrt{N_i(N_j+1)}|N_1...N_i-1...N_j+1...\rangle\langle N_1...N_i...N_j...| \\
&= \hat{a}_i\hat{a}_j^\dagger
\end{aligned}$$

这里默认 $i<j, i>j$ 的情形完全类似。若 $i=j$，则有

$$\begin{aligned}
\hat{a}_i\hat{a}_i^\dagger &= \sum_{\{N\}}\sum_{\{N'\}}\sqrt{N_i(N_i'+1)}|N_1...N_i-1...\rangle\langle N_1...N_i...|N_1'...N_i'+1...\rangle\langle N_1'...N_i'...| \\
&= \sum_{\{N\}}N_i|N_1...N_i-1...\rangle\langle N_1...N_i-1...|
\end{aligned}$$

$$\begin{aligned}
\hat{a}_i^\dagger\hat{a}_i &= \sum_{\{N\}}\sum_{\{N'\}}\sqrt{N_i(N_i'+1)}|N_1'...N_i'+1...\rangle\langle N_1'...N_i'...|N_1...N_i-1...\rangle\langle N_1...N_i...| \\
&= \sum_{\{N\}}N_i|N_1...N_i...\rangle\langle N_1...N_i...| \\
&= \hat{a}_i\hat{a}_i^\dagger - 1
\end{aligned}$$

据此可得

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$$

而当 $i \neq j$ 时

$$\hat{a}_i\hat{a}_j = \sum_{\{N\}} \sqrt{N_iN_j}|N_1...N_i-1...N_j-1...\rangle\langle N_1...N_i...N_j| = \hat{a}_j\hat{a}_i$$

$$\hat{a}_i^\dagger\hat{a}_j^\dagger = \sum_{\{N\}} \sqrt{(N_i+1)(N_j+1)}|N_1...N_i+1...N_j+1...\rangle\langle N_1...N_i...N_j| = \hat{a}_j^\dagger\hat{a}_i^\dagger$$

于是有 ($i = j$ 时显然成立)

$$[\hat{a}_i, \hat{a}_j] = 0, \quad [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0$$

2

根据定义

$$\begin{aligned}
[\widehat{\Phi}(\xi), \widehat{\Phi}^\dagger(\xi')] &= \sum_{ij} \psi_i(\xi)\psi_j^*(\xi')[\hat{a}_i, \hat{a}_j^\dagger] \\
&= \sum_{ij} \psi_i(\xi)\psi_j^*(\xi')\delta_{ij} \\
&= \sum_{i} \psi_i(\xi)\psi_i^*(\xi') \\
&= \sum_{i} \langle\xi|\psi_i\rangle\langle\psi_i|\xi'\rangle \\
&= \langle\xi|\xi'\rangle \\
&= \delta(\xi-\xi')
\end{aligned}$$

3

由于场算符不依赖于单体基函数

$$\widehat{\Phi}(\xi) = \sum_i \psi_i(\xi)\hat{a}_i = \sum_j \phi_j(\xi)\hat{b}_j$$

可以得到选取不同的单体基函数时湮灭算符之间的变换关系

$$\hat{b}_j = \sum_i \langle\phi_j|\psi_i\rangle\hat{a}_i$$

因而有

$$\begin{aligned}
\widehat{\Phi}^\dagger(\xi)|0\rangle &= \sum_i \psi_i^*(\xi)\hat{a}_i^\dagger|0\rangle \\
&= \sum_i \psi_i^*(\xi)|\psi_i\rangle \\
&= \sum_i |\psi_i\rangle\langle\psi_i|\xi\rangle \\
&= |\xi\rangle
\end{aligned}$$

4

当 $i \neq j$ 时

$$\begin{aligned}\hat{a}_i\hat{a}_j^\dagger &= \sum_{\{N\}}\sum_{\{N'\}} P_iP_j|N_1...N_i=0...\rangle\langle N_1...N_i=1...|N_1'...N_j'=1...\rangle\langle N_1'...N_j'=0...| \\ &= \sum_{\{N\}} P_i(N_j=1)P_j(N_i=1)|N_1...N_i=0...N_j=1...\rangle\langle N_1...N_i=1...N_j=0...|\end{aligned}$$

$$\begin{aligned}\hat{a}_j^\dagger\hat{a}_i &= \sum_{\{N\}}\sum_{\{N'\}} P_iP_j|N_1'...N_j'=1...\rangle\langle N_1'...N_j'=0...|N_1...N_i=0...\rangle\langle N_1...N_i=1...| \\ &= \sum_{\{N\}} P_i(N_j=0)P_j(N_i=0)|N_1...N_i=0...N_j=1...\rangle\langle N_1...N_i=1...N_j=0...|\end{aligned}$$

这里默认 $i<j, i>j$ 的情形完全类似。其中

$$P_i = (-)^{\sum_{l=1}^{i-1} N_l}$$

显然有

$$P_i(N_j=1)P_j(N_i=1) = -P_i(N_j=0)P_j(N_i=0)$$

此时有

$$\hat{a}_i\hat{a}_j^\dagger + \hat{a}_j^\dagger\hat{a}_i = 0$$

若 $i=j$，则有

$$\begin{aligned}\hat{a}_i\hat{a}_i^\dagger + \hat{a}_i^\dagger\hat{a}_i &= \sum_{\{N\}} P_i^2(|N_1...N_i=0...\rangle\langle N_1...N_i=0...| + |N_1...N_i=1...\rangle\langle N_1...N_i=1...|) \\ &= \sum_{\{N\}} |N_1...N_i...\rangle\langle N_1...N_i...| \\ &= 1\end{aligned}$$

据此可得

$$\{\hat{a}_i, \hat{a}_j^\dagger\} = \delta_{ij}$$

而当 $i \neq j$ 时

$$\hat{a}_i\hat{a}_j = \sum_{\{N\}} P_i(N_j=1)P_j(N_i=0)|...0...0...\rangle\langle ...1...1...| = -\hat{a}_j\hat{a}_i$$

$$\hat{a}_i^\dagger\hat{a}_j^\dagger = \sum_{\{N\}} P_i(N_j=1)P_j(N_i=0)|...1...1...\rangle\langle ...0...0...| = -\hat{a}_j^\dagger\hat{a}_i^\dagger$$

于是有 ($i=j$ 时显然成立)

$$\{\hat{a}_i, \hat{a}_j\} = 0, \quad \{\hat{a}_i^\dagger, \hat{a}_j^\dagger\} = 0$$