Solid State Physics

Homework Ch6-No.1, Due on Jun 14th, Tuesday

1. Problem No.1 in Chapter 34 of "Solid State Physics" by Ashcroft/Mermin, page 753: "Thermodynamics of the superconducting state".

2. Problem No.4 in Chapter 34 of "Solid State Physics" by Ashcroft/Mermin, page 755: "The Cooper problem".

3. (a) Use the anticommutation relations of electron annihilation and creation operators $\{c_{\mathbf{k},s}, c^\dagger_{\mathbf{k}',s'}\} = c_{\mathbf{k},s}c^\dagger_{\mathbf{k}',s'} + c^\dagger_{\mathbf{k}',s'}c_{\mathbf{k},s} = \delta_{\mathbf{k}\mathbf{k}'}\delta_{ss'}$ and $\{c_{\mathbf{k},s}, c_{\mathbf{k}',s'}\} = \{c^\dagger_{\mathbf{k},s}, c^\dagger_{\mathbf{k}',s'}\} = 0$ to show that the s-wave interaction between electrons

$$H_{\text{int}} = -\frac{\lambda}{2\Omega}\sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q};s,s'} c^\dagger_{\mathbf{k}_1-\mathbf{q},s}c^\dagger_{\mathbf{k}_2+\mathbf{q},s'}c_{\mathbf{k}_2,s'}c_{\mathbf{k}_1,s} \tag{1}$$

is nonzero only for $s \neq s'$. In other words, show that the contribution from $s = s'$ is zero.
(b) Show that when keeping only $\mathbf{k}_1 = -\mathbf{k}_2$, the formula reads

$$H_{\text{int}} = -\frac{\lambda}{\Omega}\sum_{\mathbf{k},\mathbf{k}'} c^\dagger_{\mathbf{k}',\uparrow}c^\dagger_{-\mathbf{k}',\downarrow}c_{-\mathbf{k},\downarrow}c_{\mathbf{k},\uparrow}. \tag{2}$$

4. (This is optional, but I strongly encourage you to work out this problem) The self-consistent equation of the s-wave superconducting order can be obtained through BCS theory and reads

$$\Delta(T) = \frac{\lambda}{\Omega}\sum_{\mathbf{k}} u^*_{\mathbf{k}}v_{\mathbf{k}}\big[1 - 2f(E_{\mathbf{k}})\big], \tag{3}$$

where Ω is the total volume. The coefficients $u_{\mathbf{k}}$ and $v_{\mathbf{k}}$ are obtained by solving BdG equation, satisfying $|u_{\mathbf{k}}|^2 + |v_{\mathbf{k}}|^2 = 1$ and $|u_{\mathbf{k}}|^2 - |v_{\mathbf{k}}|^2 = \xi_{\mathbf{k}}/E_{\mathbf{k}}$, with $\xi_{\mathbf{k}} = \hbar^2\mathbf{k}^2/2m - \mu_F$ and $E_{\mathbf{k}} = (\xi^2_{\mathbf{k}} + |\Delta|^2)^{1/2}$, and $f(E_{\mathbf{k}}) = 1/(e^{\beta E_{\mathbf{k}}} + 1)$.
(a) Show that the above formula leads to the following BCS gap equation

$$1 = \frac{\lambda}{(2\pi)^3}\int d^3\mathbf{k}\frac{1}{2E_{\mathbf{k}}}\tanh(\frac{1}{2k_BT}E_{\mathbf{k}}). \tag{4}$$

(b) For $T = T_c$, use the integration by parts to show that

$$1 = \frac{1}{2}N(0)\lambda\left[(\tanh x\ln x)\Big|_0^{\hbar\omega_D/(2k_BT_c)} - \int_0^\infty \frac{\ln x}{\cosh^2 x}dx\right], \tag{5}$$

where $N(0)$ is density of states at Fermi surface, i.e. at $\xi_{\mathbf{k}} = 0$. The first term is obtained by restricting the integral range to be $-\hbar\omega_D \leq \xi_{\mathbf{k}} \leq \hbar\omega_D$.
(c) Apply the result that $\int_0^\infty \frac{\ln x}{\cosh^2 x}dx = \ln(\pi/4) - C$, with $C = \lim_{n\to 0}(1 + 1/2 + ... + 1/n - \log(n))$ and show that The critical temperature T_c can be obtained by taking that $\Delta(T_c) = 0$. Show that

$$k_BT_c = \frac{2e^C}{\pi}\hbar\omega_D\exp[-\frac{2}{N(0)\lambda}] \approx 1.134\hbar\omega_D\exp[-\frac{2}{N(0)\lambda}]. \tag{6}$$

(d) At $T = 0$, $\Delta(T) = \Delta_0$. Show for $\hbar\omega_D \gg \Delta_0$ that

$$\Delta_0 \approx 2\hbar\omega_D\exp[-\frac{2}{N(0)\lambda}]. \tag{7}$$

With the above results one can find that $2\Delta_0/(k_BT_C) = 2\pi e^C \simeq 3.528$. This is a universal relation in the BCS theory, valid for the weak coupling regime.

5. (This is not homework problem, but you may study it if you are interested)
1D topological superconductor.–The simplest toy model of topological superconductor is the 1D spinless p-wave SC, as proposed by Kitaev (A. Y. Kitaev, Physics-Uspekhi 44, 131 (2001)). The tight-binding Hamiltonian of the model in the second quantization is given below

$$H = -\mu \sum_j c_j^\dagger c_j - \frac{1}{2} \sum_j (t c_j^\dagger c_{j+1} + \Delta c_j c_{j+1} + h.c.), \tag{8}$$

where μ is chemical potential, t is hopping coefficient, and unlike the s-wave pairing which occurs between spin-up and spin-down electrons at the same site, here Δ is the p-wave pairing between *spinless* electrons at two neighboring sites. Thus the p-wave pairing is anti-symmetric when exchanging the two electron (Indeed this is because p-wave pairing corresponds to the channel with orbital angular momentum $l = 1$ which is parity odd). The word *spinless* could mean that we consider only a single spin component of the electons, e.g. fully spin-polarized electrons. Note that the annihilation and creation operators $c_j, c_j^\dagger$ obey anti-commutation relation.

(a) Transforming the above Hamiltonian into momentum space through

$$c_k = \frac{1}{\sqrt{N}} \sum_j c_j e^{-ikx_j}, \quad c_k^\dagger = \frac{1}{\sqrt{N}} \sum_j c_j^\dagger e^{ikx_j},$$

show that the Bogoliubov-de Gennes (BdG) with $\Delta = \Delta_0 e^{i\phi}$ can be written as

$$\begin{aligned} H &= \frac{1}{2} \sum_k \mathcal{C}_k^\dagger \mathcal{H}_k \mathcal{C}_k, \\ \mathcal{H}_k &= (-t\cos k - \mu)\tau_z - \Delta_0 \sin k(\cos\phi\tau_y + \sin\phi\tau_x), \end{aligned} \tag{9}$$

where the operator $\mathcal{C}_k = [c_k, c_{-k}^\dagger]^T$ in the Nambu particle-hole space, and the Pauli matrices $\tau_{x,y,z}$ act on the Nambu space. One can rotate the term $\cos\phi\tau_y + \sin\phi\tau_x \to \tau_y$, or directly take $\phi = 0$ without loss of generality. The Hamiltonian $\mathcal{H}_k$ is very similar to the case for the SSH model or the dimerized 1D lattice (see Chapter II), while the physics is completely different since here it is in the particle-hole space. The topology of the present system can then be studied in the similar way.

(b) Solve the BdG Hamiltonian for the spectra $E_\pm(k)$ and the corresponding Bogoliubov quasi-particle operators $\gamma_{E_\pm}(k)$ and $\gamma_{E_\pm}^\dagger(k)$, which are not independent but satisfy $\gamma_{E_+}(k) = \gamma_{E_-}^\dagger(k)$ due to particle-hole symmetry.

(c) Show that the bulk of the present 1D superconductor is gapped when $\mu \neq \pm t$, and is gapless at $\mu = \pm t$, which corresponds to transition between topological and trivial phases.

(d) The 1D winding number is obtained by

$$N_{1D} = \frac{1}{\pi} \int dk \hat{h}_z \partial_k \hat{h}_y, \tag{10}$$

where $(\hat{h}_y, \hat{h}_z) = (h_y, h_z)/h$, with $h_y = -\Delta_0 \sin k, h_z = -t\cos k - \mu$, and $h = (h_y^2 + h_z^2)^{1/2}$ for $\phi = 0$. Show that the topology is nontrivial for $|\mu| < t$ and compute the topological invariant.

In the topologically nontrivial phase, at each end of this 1D system is located a *Majorana zero bound mode* which is mathematically similar to the boundary states at ends of an SSH chain, but the physics are sharply different. The Majorana zero modes obey *non-Abelian statistics* and can be applied to *topological quantum computation* (You do not need to prove this).