第七次作业

（1）

Let $\phi(\mathbf{p}')$ be the momentum-space wave function for state $|\alpha\rangle$, that is, $\phi(\mathbf{p}') = \langle \mathbf{p}'|\alpha\rangle$. Is the momentum-space wave function for the time-reversed state $\Theta|\alpha\rangle$ given by $\phi(\mathbf{p}')$, $\phi(-\mathbf{p}')$, $\phi^*(\mathbf{p}')$, or $\phi^*(-\mathbf{p}')$? Justify your answer.

（2）

a. What is the time-reversed state corresponding to $\mathscr{D}(R)|j, m\rangle$?
b. Using the properties of time reversal and rotations, prove

$$\mathscr{D}^{(j)*}_{m'm}(R) = (-1)^{m-m'}\mathscr{D}^{(j)}_{-m',-m}(R).$$

提示：时间反演算符与转动算符对易。

（3）

Suppose a spinless particle is bound to a fixed center by a potential $V(\mathbf{x})$ so asymmetrical that no energy level is degenerate. Using time-reversal invariance prove

$$\langle \mathbf{L} \rangle = 0$$

for any energy eigenstate. (This is known as **quenching** of orbital angular momentum.) If the wave function of such a nondegenerate eigenstate is expanded as

$$\sum_l \sum_m F_{lm}(r) Y_l^m(\theta, \phi),$$

what kind of phase restrictions do we obtain on $F_{lm}(r)$?

提示：最后一问利用球谐函数的正交性和在时间反演下的性质。

(4)

The Hamiltonian for a spin 1 system is given by

$$H = AS_z^2 + B\left(S_x^2 - S_y^2\right).$$

Solve this problem *exactly* to find the normalized energy eigenstates and eigenvalues. (A spin-dependent Hamiltonian of this kind actually appears in crystal physics.) Is this Hamiltonian invariant under time reversal? How do the normalized eigenstates you obtained transform under time reversal?

提示：在$|jm\rangle$表示下写出 H 的矩阵，再对角化。

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1

根据时间反演算符的定义

$$\begin{aligned}
\langle \mathbf{p}'|\Theta|\alpha\rangle &= \langle \mathbf{p}'|\Theta \int \mathrm{d}^3 p |\mathbf{p}''\rangle\langle \mathbf{p}''|\alpha\rangle \\
&= \langle \mathbf{p}'| \int \mathrm{d}^3 p |-\mathbf{p}''\rangle\langle \mathbf{p}''|\alpha\rangle^* \\
&= \int \mathrm{d}^3 p \langle \mathbf{p}'|\mathbf{p}''\rangle\langle -\mathbf{p}''|\alpha\rangle^* \\
&= \langle -\mathbf{p}''|\alpha\rangle^*
\end{aligned}$$

因此 $|\alpha\rangle$ 的时间反演态在动量空间波函数为 $\phi^*(-\mathbf{p}')$ 给出。

2

a. 对于转动算符

$$\mathscr{D}(\hat{\boldsymbol{n}}, \phi) = e^{-i\frac{\boldsymbol{J}\cdot\hat{\boldsymbol{n}}}{\hbar}\phi} = 1 + (-iJ_n\phi/\hbar) + \frac{(-iJ_n\phi/\hbar)^2}{2!} + \frac{(-iJ_n\phi/\hbar)^3}{3!} + \dots$$

可以看到所有的项都由 iJ_n 和 J_n^2 组成。根据时间反演算符的定义有

$$\Theta(iJ_n)\Theta^{-1} = iJ_n, \quad \Theta(J_n^2)\Theta^{-1} = J_n^2$$

因而有

$$\Theta\mathscr{D}(R)|j,m\rangle = \mathscr{D}(R)\Theta|j,m\rangle$$

b. 根据时间反演算符和角动量的关系，容易写出

$$J_z\Theta|j,m\rangle = -\Theta J_z|j,m\rangle = -m\Theta|j,m\rangle$$

$$J_\pm\Theta|j,m\rangle = -c_\mp(j,m)\Theta|j,m\mp 1\rangle$$

由此可以看出 Θ 应有如下形式

$$\Theta|j,m\rangle = e^{i\delta}(-)^m|j,-m\rangle$$

通过计算

$$\langle j,-m'|\mathscr{D}\Theta|j,m\rangle = e^{i\delta}(-)^m\langle j,-m'|\mathscr{D}|j,-m\rangle = e^{i\delta}(-)^m\mathscr{D}^{(j)}_{-m',-m}$$

$$\langle j,-m'|\Theta\mathscr{D}|j,m\rangle = \langle j,-m'|\Theta\sum_{m''}|j,m''\rangle\langle j,m''|\mathscr{D}|j,m\rangle = e^{i\delta}(-)^{m'}\mathscr{D}^{(j)*}_{m',m}$$

由此得到

$$\mathscr{D}^{(j)*}_{m',m} = (-)^{m-m'}\mathscr{D}^{(j)}_{-m',-m}$$

3

由于时间反演算符和哈密顿算符互易，对能量本征态有

$$H\Theta|E\rangle = \Theta H|E\rangle = \Theta E|E\rangle = E\Theta|E\rangle$$

又因为能量本征态非兼并，故有

$$|\tilde{E}\rangle = \Theta|E\rangle = e^{i\delta}|E\rangle$$

由此得到

$$\langle E|\mathbf{L}|E\rangle = e^{-i\delta}\langle E|\mathbf{L}|E\rangle e^{i\delta} = \langle\tilde{E}|\mathbf{L}|\tilde{E}\rangle = \langle E|\Theta^{-1}\mathbf{L}\Theta|E\rangle = -\langle E|\mathbf{L}|E\rangle$$

因此

$$\langle\mathbf{L}\rangle = 0$$

能量本征态波函数

$$\psi(\mathbf{x}) = \langle\mathbf{x}|E\rangle = \sum_{l,m}F_{l,m}(r)Y_l^m(\theta,\phi)$$

一方面，时间反演态的波函数可写作

$$\tilde{\psi}(\mathbf{x}) = \langle\mathbf{x}|\tilde{E}\rangle = e^{i\delta}\sum_{l,m}F_{l,m}(r)Y_l^m(\theta,\phi)$$

另一方面

$$\tilde{\psi}(\mathbf{x}) = \psi^*(\mathbf{x}) = \sum_{l,m}F^*_{l,m}(r)(-)^mY_l^{-m}(\theta,\phi) = \sum_{l,m}F^*_{l,-m}(r)(-)^mY_l^m(\theta,\phi)$$

由此得到相位关系

$$F_{l,m}(r) = e^{-i\delta}(-)^mF^*_{l,-m}(r)$$

4

根据

$$\Theta S_i^2 \Theta^{-1} = \Theta S_i \Theta^{-1} \Theta S_i \Theta^{-1} = (-S_i)(-S_i) = S_i^2$$

显然看出哈密顿量是时间反演不变的。在 S_z 本征态下写出哈密度算符的矩阵形式

$$H = \hbar^2 \begin{pmatrix} A & 0 & B \\ 0 & 0 & 0 \\ B & 0 & A \end{pmatrix}$$

容易解得能量本征态

$$|E_\pm\rangle = \frac{1}{\sqrt{2}}(|1,1\rangle \pm |1,-1\rangle), \quad E_\pm = \hbar^2(A \pm B)$$

$$|E_0\rangle = |1,0\rangle, \quad E_0 = 0$$

由此可以直接得到

$$\Theta|E_\pm\rangle = \mp|E_\pm\rangle, \quad \Theta|E_0\rangle = |E_0\rangle$$