Solid State Physics Homework

Chapter6 No.1, Due on Jun 10th 2022, Friday

Sui Yuan 2000011379

Problem1 A&M Chapter34.1: Thermodynamics of the Superconducting State

(a)

Along the critical field $H_c(T)$, the Gibbs free energy of the normal and superconducting states must be equal for equilibrium, i.e.

$$G_n(H_c, T) = G_s(H_c, T)$$

so along the critical field there must be $dG_n(H_c, T) = dG_s(H_c, T)$, i.e.

$$S_n dT + \mathfrak{M}_n dH_c = S_s dT + \mathfrak{M}_s dH_c$$

which is precisely Eq(34.38)

$$\frac{dH_c}{dT} = \frac{S_n - S_s}{\mathfrak{M}_s - \mathfrak{M}_n} \qquad \text{(shown)}$$

(b)

For comparing the entropy of the two states at the critical field, we first extend the Gibbs free energy of different states to the whole $H-T$ plane, then we can write down the path integral

$$G(H_c, T) = G(H, T) - \int_H^{H_c} \mathfrak{M} dH$$

note that T is the critical temperature corresponding to H_c, and H is arbitrary. For the fact the superconducting state displays perfect diamagnetism, while the normal state is negligible, we can use the approximation below

$$\mathfrak{M}_s = \frac{HV}{4\pi}, \quad \mathfrak{M}_n = 0$$

substitute, we can find that

$$G_n(H, T) - G_s(H, T) = -\frac{V}{4\pi} \int_H^{H_c} H dH = \frac{V}{8\pi}(H_c^2 - H^2)$$

then we can get the difference between the two states

$$S_n - S_s = \left[\frac{\partial(G_n - G_s)}{\partial T}\right]_H = -\frac{V}{4\pi} H_c \frac{dH_c}{dT} \qquad \text{(shown)}$$

which is precisely Eq(34.39). Note that it is suitable for all H theoretically, and surely for H_c, which is the only actual intersection. So the latent heat is precisely Eq(34.40)

$$Q = T(S_n - S_s) = -TV \frac{H_c}{4\pi} \frac{dH_c}{dT} \qquad \text{(shown)}$$

(c)

The specific heat around $H_c(T)$ is given by entropy

$$(c_p)_n - (c_p)_s = \frac{1}{V}\left(\frac{dQ_n}{dT} - \frac{dQ_s}{dT}\right) = \frac{T}{V}\frac{d(S_n - S_s)}{dT} = -\frac{T}{4\pi}\frac{d}{dT}\left(H_c \frac{dH_c}{dT}\right)$$

substitute $H_c = 0$, the result is precisely Eq(34.41)

$$(c_p)_n - (c_p)_s = -\frac{T}{4\pi}\left(\frac{dH_c}{dT}\right)^2 \qquad \text{(shown)}$$

Problem2 A&M Chapter34.4: The Cooper Problem

(a)

According to Eq(34.46), we can get the integral

$$\int d\mathbf{k}(E - 2\mathcal{E})\chi(\mathbf{k}) = \iint \frac{d\mathbf{k}d\mathbf{k}'}{(2\pi)^3} V(\mathbf{k}, \mathbf{k}')\chi(\mathbf{k}')$$

substitute Eq(34.47)-(34.48)

$$\int_{\mathbf{k}_F}^{\mathbf{k}_F+\delta} \chi(\mathbf{k})d\mathbf{k}(E - 2\mathcal{E}) = -V \int_{\mathbf{k}_F}^{\mathbf{k}_F+\delta} \chi(\mathbf{k}')d\mathbf{k}' \int_{\mathcal{E}_F}^{\mathcal{E}_F+\hbar\omega} N(\mathcal{E}')d\mathcal{E}'$$

it is directly to find a solution, which is precisely Eq(34.50)

$$1 = V \int_{\mathcal{E}_F}^{\mathcal{E}_F+\hbar\omega} \frac{N(\mathcal{E})d\mathcal{E}}{2\mathcal{E} - E} \qquad \text{(shown)}$$

where $N(\mathcal{E})$ is the density of one-electron levels.

(b)

For the arbitrary weak V, we can estimate roughly

$$1 \approx V\frac{N(\mathcal{E}_F)\hbar\omega}{2\mathcal{E}_F - E} \quad \Rightarrow \quad E \approx 2\mathcal{E}_F - VN(\mathcal{E}_F)\hbar\omega \qquad \text{(shown)}$$

so Eq(34.50) should have a solution with $E < 2\mathcal{E}_F$.

(c)

Substitute $N(\mathcal{E}) = (\mathcal{E}_F)$ to Eq(34.50)

$$1 = VN(\mathcal{E}_F)\int_0^{\hbar\omega}\frac{d\mathcal{E}}{2\mathcal{E}+\Delta} = \frac{1}{2}VN(\mathcal{E}_F)\ln\left(1+\frac{2\hbar\omega}{\Delta}\right)$$

get the solution of Δ from above

$$\Delta = 2\hbar\omega\frac{e^{-2N(\mathcal{E}_F)/V}}{1+e^{-2N(\mathcal{E}_F)/V}} \approx 2\hbar\omega e^{-2N(\mathcal{E}_F)/V} \qquad \text{(shown)}$$

which is precisely Eq(34.51)-(34.52).

Problem3 Annihilation and Creation Operators

(a)

The anticommutation relations

$$\begin{aligned}\{c_{\mathbf{k},s}, c^\dagger_{\mathbf{k}',s'}\} =& \delta_{\mathbf{k}\mathbf{k}'}\delta_{ss'}\\ \{c_{\mathbf{k},s}, c_{\mathbf{k}',s'}\} =& 0\\ \{c^\dagger_{\mathbf{k},s}, c^\dagger_{\mathbf{k}',s'}\} =& 0\end{aligned}$$

For $s = s'$, the summation should be

$$N = \sum_{\mathbf{k}_1,\mathbf{k}_2,\mathbf{q};s,s'} c^\dagger_{\mathbf{k}_1-\mathbf{q},s}c^\dagger_{\mathbf{k}_2+\mathbf{q},s'}c_{\mathbf{k}_2,s'}c_{\mathbf{k}_1,s} = \sum_{\mathbf{k}_1,\mathbf{k}_2}\left(\sum_{\mathbf{q}} c^\dagger_{\mathbf{k}_1-\mathbf{q}}c^\dagger_{\mathbf{k}_2+\mathbf{q}}\right)c_{\mathbf{k}_2}c_{\mathbf{k}_1}$$

for $\mathbf{k}_1, \mathbf{k}_2, \mathbf{q}$ is in 1st Brillouin zone, we can simplify the summation

$$\begin{aligned}N =& \sum_{\mathbf{k}_1<\mathbf{k}_2}\left(\sum_{\mathbf{q}} c^\dagger_{\mathbf{k}_1-\mathbf{q}}c^\dagger_{\mathbf{k}_2+\mathbf{q}}\right)c_{\mathbf{k}_2}c_{\mathbf{k}_1} + \sum_{\mathbf{k}_1>\mathbf{k}_2}\left(\sum_{\mathbf{q}} c^\dagger_{\mathbf{k}_1-\mathbf{q}}c^\dagger_{\mathbf{k}_2+\mathbf{q}}\right)c_{\mathbf{k}_2}c_{\mathbf{k}_1}\\ =& \sum_{\mathbf{k}_1<\mathbf{k}_2}\left(\sum_{\mathbf{q}_1+\mathbf{q}_2=\mathbf{k}_1-\mathbf{k}_2} c^\dagger_{\mathbf{k}_1-\mathbf{q}_1}c^\dagger_{\mathbf{k}_2+\mathbf{q}_1} + c^\dagger_{\mathbf{k}_1-\mathbf{q}_2}c^\dagger_{\mathbf{k}_2+\mathbf{q}_2} + \sum_{rest\mathbf{q}} c^\dagger_{\mathbf{k}_1-\mathbf{q}}c^\dagger_{\mathbf{k}_2+\mathbf{q}}\right)c_{\mathbf{k}_2}c_{\mathbf{k}_1} + \sum_{\mathbf{k}_1>\mathbf{k}_2}\\ =& \sum_{\mathbf{k}_1<\mathbf{k}_2}\left(\sum_{rest\mathbf{q}} c^\dagger_{\mathbf{k}_1-\mathbf{q}}c^\dagger_{\mathbf{k}_2+\mathbf{q}}\right)c_{\mathbf{k}_2}c_{\mathbf{k}_1} + \sum_{\mathbf{k}_1>\mathbf{k}_2}\end{aligned}$$

for symmetry and the anticommutation relations, we can change the summation order

$$N = \sum_{\mathbf{k}_1<\mathbf{k}_2} \left(\sum_{rest\mathbf{q}} c^\dagger_{\mathbf{k}_1-\mathbf{q}} c^\dagger_{\mathbf{k}_2+\mathbf{q}} \right) c_{\mathbf{k}_2} c_{\mathbf{k}_1} + \sum_{\mathbf{k}_1>\mathbf{k}_2} \left(\sum_{rest\mathbf{q}} c^\dagger_{\mathbf{k}_2-\mathbf{q}} c^\dagger_{\mathbf{k}_1+\mathbf{q}} \right) c_{\mathbf{k}_1} c_{\mathbf{k}_2} = 0$$

so the contribution from $s = s'$ is zero.(shown)

(b)

From (a), we only consider $s \neq s'$, the formula reads

$$\begin{aligned} H_{int} =& -\frac{\lambda}{2\Omega} \sum_{\mathbf{k},\mathbf{q}} \left(c^\dagger_{\mathbf{k}-\mathbf{q},\uparrow} c^\dagger_{-\mathbf{k}+\mathbf{q},\downarrow} c_{-\mathbf{k},\downarrow} c_{\mathbf{k},\uparrow} + c^\dagger_{\mathbf{k}-\mathbf{q},\downarrow} c^\dagger_{-\mathbf{k}+\mathbf{q},\uparrow} c_{-\mathbf{k},\uparrow} c_{\mathbf{k},\downarrow} \right) \\ =& -\frac{\lambda}{\Omega} \sum_{\mathbf{k},\mathbf{k}'} c^\dagger_{\mathbf{k}',\uparrow} c^\dagger_{-\mathbf{k}',\downarrow} c_{-\mathbf{k},\downarrow} c_{\mathbf{k},\uparrow} \end{aligned} \qquad \text{(shown)}$$

note to use the anticommutation relations and rewrite the summation order.