

Solid State Physics Homework

Chapter5 No.1, Due on May 20th 2022, Friday

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Problem1 A&M Chapter31.4

(a)

According to Eq(31.83), one can easily prove that

$$[\mathbf{L}, L_\alpha] = i\hat{\alpha} \times \mathbf{L}, \quad [\mathbf{L}, S_\alpha] = 0, \quad \alpha = x, y, z$$

i.e. for any unit \hat{n}

$$[\mathbf{L}, \hat{n} \cdot (\mathbf{L} + \mathbf{S})] = i\hat{n} \times \mathbf{L}$$

The law is exactly the same for \mathbf{S}

$$[\mathbf{S}, \hat{n} \cdot \mathbf{J}] = i\hat{n} \times \mathbf{S}$$

thus one can add up the two equations

$$[\mathbf{L} + g_0\mathbf{S}, \hat{n} \cdot \mathbf{J}] = i\hat{n} \times (\mathbf{L} + g_0\mathbf{S})$$

which is precisely Eq(31.84).

(b)

According to Eq(31.85), for any unit \hat{n}

$$\hat{n} \cdot \mathbf{J}|0\rangle = 0, \quad \langle 0|\hat{n} \cdot \mathbf{J} = 0$$

multiply Eq(31.84) by $\langle 0|$ and $|0\rangle$

$$i\langle 0|\hat{n} \times (\mathbf{L} + g_0\mathbf{S})|0\rangle = \langle 0|(\mathbf{L} + g_0\mathbf{S}) \cdot 0 + 0 \cdot (\mathbf{L} + g_0\mathbf{S})|0\rangle = 0$$

Because \hat{n} is arbitrary, so

$$\langle 0|(\mathbf{L} + g_0\mathbf{S})|0\rangle = 0$$

which is Eq(31.86).

(c)

when $J = 1/2$, according to Hund's Rule, $L = 0, S = 1/2$, multiply Eq(31.84) by the wave function, one can proof that

$$\langle JLSJ_z | (\mathbf{L} + g_0 \mathbf{S}) | JLSJ'_z \rangle = g(JLS) \langle JLSJ_z | \mathbf{J} | JLSJ'_z \rangle$$

with $g(JLS) = g_0$, which is the special result of Eq(31.34).

Problem2 A&M Chapter31.9
According to Eq(31.89)

$$e^{-\beta F} = \text{Tr}(e^{-\beta \mathcal{H}}) = \text{Tr}[1 - \beta \mathcal{H} + \frac{1}{2}(\beta \mathcal{H})^2 + \dots]$$

when $\beta H \ll 1$, $F \approx F_0$

$$F(H) = F_0 + F' H + \frac{F'' H^2}{2} + \dots$$

$$e^{-\beta F_0} (1 - \beta F' H - \frac{\beta F'' H^2}{2} + \frac{\beta F''' H^3}{6} + \dots) = \sum_{J_z} (1 - \beta \gamma H J_z + \frac{1}{2} (\beta \gamma H J_z)^2 + \dots)$$

one can simply sum from $-J$ to J and get the result

$$e^{-\beta F_0} = 2J + 1, \quad F' = 0$$

$$F'' = -\frac{2}{2J+1} \beta \gamma^2 \frac{J(J+1)(2J+1)}{6} = \frac{J(J+1)(g\mu_B)^2}{3k_B T}$$

so the susceptibility can be write down easily

$$\chi = -\frac{N}{V} F'' = \frac{N}{V} \frac{(g\mu_B)^2}{3} \frac{J(J+1)}{k_B T}, \quad k_B T \gg g\mu_B H$$

which is precisely Eq(31.47).

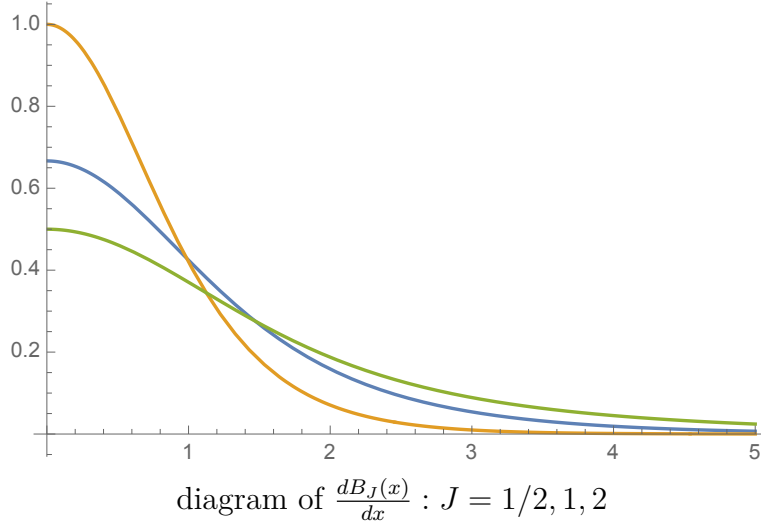
Problem3 susceptibility $\chi(T, H)$

According to LF(5.10)(lecture formula), the susceptibility at any temperature is

$$\chi = \frac{\partial M}{\partial H} = \frac{N}{V} \beta \gamma^2 J^2 \frac{dB_J(x)}{dx}$$

Derivative of Brillouin function is

$$\frac{dB_J(x)}{dx} = \frac{1}{4J^2} \coth^2 \frac{x}{2J} - (1 + \frac{1}{2J})^2 \coth^2(1 + \frac{1}{2J})x + 1 + \frac{1}{J}$$



and it is analytic generically. So there is no phase transition in this system.

Problem4 Quantitative proof of the question of identity

first we consider the whole system. Without taking interaction into consideration, the whole system consists of N equivalent atoms (ions).

First, we take different indexes between single atom(ion) and the whole system.

$$\varepsilon_i : \varepsilon_0, \varepsilon_1, \varepsilon_2, \dots \quad E_n : E_0, E_1, E_2, \dots$$

$$E_n^{(N)} = \sum_j^N \varepsilon_i^{(j)}$$

then we write down the the Helmholtz free energy of the whole system, using the independence of the atoms(ions)

$$\begin{aligned} e^{-\beta F} &= \sum_n e^{-\beta E_n^{(N)}} \\ &= (e^{-\beta \varepsilon_0} + e^{-\beta \varepsilon_1} + e^{-\beta \varepsilon_2} + \dots) \sum_n e^{-\beta E_n^{(N-1)}} \\ &= \sum_i e^{-\beta \varepsilon_i} \sum_n e^{-\beta E_n^{(N-1)}} \\ &= \sum_{i_1} e^{-\beta \varepsilon_{i_1}} \sum_{i_2} e^{-\beta \varepsilon_{i_2}} \dots \sum_{i_N} e^{-\beta \varepsilon_{i_N}} \\ &= \sum_{\alpha=1}^N e^{-\beta f_\alpha} \end{aligned}$$

where $f_\alpha = \sum_{i_\alpha} e^{-\beta \varepsilon_{i_\alpha}}$ is the Helmholtz free energy of the α -th atom(ion). Thus an important conclusion was reached

$$F^{(N)} = \sum_{\alpha=1}^N f_\alpha$$

So one can find the the susceptibility of the whole system and the sum of atoms(ions) is equivalent

$$\chi_E(T) = -\frac{1}{V} \frac{\partial^2 F}{\partial H^2} = -\frac{1}{V} \sum_{\alpha} \frac{\partial^2 f_\alpha}{\partial H^2} = -\frac{N}{V} \frac{\partial^2 f}{\partial H^2} = \chi_\varepsilon(T)$$

i.e. the two methods are consistent.

Solid State Physics Homework

Chapter5 No.2, Due on May 27th 2022, Friday

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Problem1 A&M Chapter32.5

(a)

According to Eq(32.30) and Eq(32.31), one can write down the Hamiltonian in $|\mathbf{R}\rangle$ and $|\mathbf{R}'\rangle$ directly

$$\mathcal{H} = \begin{pmatrix} \mathcal{E} & -t \\ -t & \mathcal{E} \end{pmatrix}$$

and for Schrodinger equation

$$\mathcal{H}\psi = E\psi$$

the solutions can be write down directly

$$\psi_{\pm} = \frac{1}{\sqrt{2}}(|\mathbf{R}\rangle \pm |\mathbf{R}'\rangle), \quad E_{\pm} = \mathcal{E} \mp t \quad (\text{shown})$$

which are precisely Eq(32.32) and Eq(32.33). Note that $|\mathbf{R}\rangle$ and $|\mathbf{R}'\rangle$ are orthonormalized.

(b)

Using Hartree approximation, one can write down the ground-state wave function

$$\Phi_{ie} = \psi_+^{(1)}\psi_+^{(2)} = \frac{1}{2}(|\mathbf{R}\rangle|\mathbf{R}\rangle + |\mathbf{R}'\rangle|\mathbf{R}'\rangle + |\mathbf{R}\rangle|\mathbf{R}'\rangle + |\mathbf{R}'\rangle|\mathbf{R}\rangle)$$

Obviously, the possibility of two electrons being on the same site is equal to that are not. So the possibility is $\frac{1}{2}$ (shown). In this case, the system Hamiltonian is

$$\mathcal{H}' = \mathcal{H}^{(1)} + \mathcal{H}^{(2)} + \mathcal{U}$$

where \mathcal{U} is the electron-interaction-potential operator. In the improved independent electron estimate, it can be written as

$$\mathcal{U}\phi = \begin{cases} U\phi, & \phi = |\mathbf{R}\rangle|\mathbf{R}'\rangle, |\mathbf{R}'\rangle|\mathbf{R}\rangle \\ 0, & \phi = |\mathbf{R}\rangle|\mathbf{R}\rangle, |\mathbf{R}'\rangle|\mathbf{R}'\rangle \end{cases}$$

then the ground-state energy should be

$$E_{ie} = E_+^{(1)} + E_+^{(2)} + \langle \Phi_{ie} | \mathcal{U} | \Phi_{ie} \rangle = 2(\mathcal{E} - t) + \frac{1}{2}U \quad (\text{shown})$$

which is precisely Eq(32.34).

(c)

Rewrite the ground-state wave function with Eq(32.35)

$$\Phi_{ie} = \frac{1}{\sqrt{2}}\Phi_0 + \frac{1}{2}(\Phi_1 + \Phi_2) \quad (\text{shown})$$

which is precisely Eq(32.36). So one can write down the system Hamiltonian in (Φ_0, Φ_1, Φ_2) , using Eq(32.29)-(32.31) and Eq(32.35)

$$\begin{aligned} H_{00} &= \langle \Phi_0 | \mathcal{H}' | \Phi_0 \rangle = 2\mathcal{E} \\ H_{10} &= \langle \Phi_1 | \mathcal{H}' | \Phi_0 \rangle = -\sqrt{2}t \\ H_{11} &= \langle \Phi_1 | \mathcal{H}' | \Phi_1 \rangle = 2\mathcal{E} + U \\ H_{21} &= \langle \Phi_2 | \mathcal{H}' | \Phi_1 \rangle = 0 \end{aligned}$$

Others are all the same. So one can find the matrix form is precisely Eq(32.38)

$$\mathcal{H}' = \begin{pmatrix} 2\mathcal{E} & -\sqrt{2}t & -\sqrt{2}t \\ -\sqrt{2}t & 2\mathcal{E} + U & 0 \\ -\sqrt{2}t & 0 & 2\mathcal{E} + U \end{pmatrix} \quad (\text{shown})$$

(d)

One can easily write down the eigenvalues of the above matrix

$$\lambda = 2\mathcal{E} + U, \quad 2\mathcal{E} + \frac{1}{2}U \pm \sqrt{4t^2 + \frac{U^2}{4}}$$

Obviously the exact ground state energy is

$$E = 2\mathcal{E} + \frac{1}{2}U - \sqrt{4t^2 + \frac{U^2}{4}} \quad (\text{shown})$$

for fixed \mathcal{E} and t , one can plot the energy in next page.

The picture is: For the two electrons system, when U/t is small, **the effect of electron interaction on the eigenstates of the system is negligible, so the ground-state energy should only get a fix of the expected value of the interaction potential**, which is same with the improved independent electron approximation; when U/t is large, **the interaction effect changes**

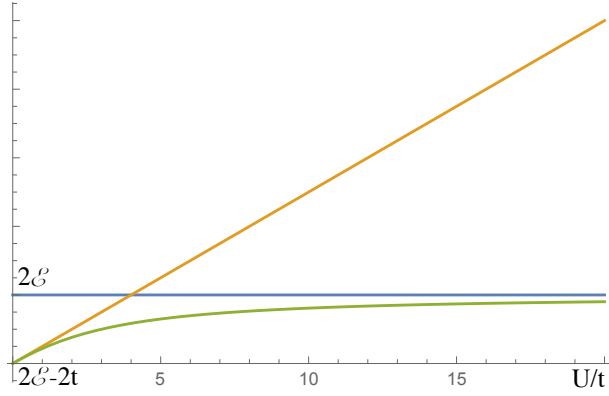


Figure1: Ground-state energy with variation of U/t
 (Green: Exact ground-state energy, Blue: Heitler-London approximation,
 Yellow: Improved independent electron approximation)

the eigenstates completely, so the electrons must be at different states, which is precisely Heitler-London approximation. When $U = 2t$, one can find $E_{ex} < E_{ie} < E_{HL}$ in the figure.(shown)

(e)

From the matrix, one can write down the eigenstate of minimum eigenvalue

$$\Phi = \frac{1}{\sqrt{2}}\Phi_0 + \left(\sqrt{1 + \left(\frac{U}{4t}\right)^2} - \frac{U}{4t} \right) \frac{1}{2}(\Phi_1 + \Phi_2) \quad (\text{shown})$$

which is precisely Eq(32.41). Denote $x = U/4t$, the possibility of two electrons being at same site is

$$P(x) = 1 - \frac{1}{2}(1 + x^2 - x\sqrt{1 + x^2})^{-1} \quad (\text{shown})$$

The function figure is shown below

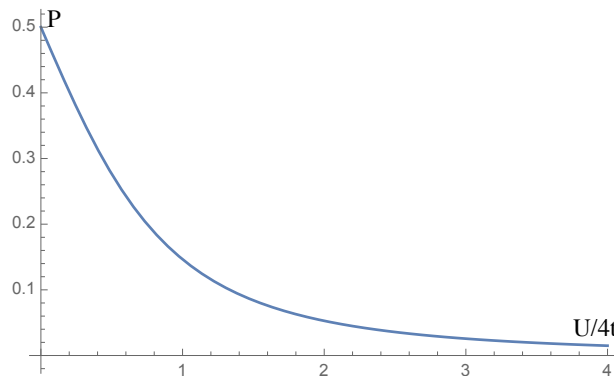


Figure2: possibility of electrons being at same site with variation of $U/4t$

From the figure one can find $P \rightarrow 0.5$ when U/t is small, corresponding to the improved independent electron approximation; and $P \rightarrow 0$ when U/t is large, corresponding to Heitler-London approximation.(shown)

Problem2 Hartree-Fock theory of electron gas

(a)

For electron gas

$$N = N_{\uparrow} + N_{\downarrow} = \frac{k_{\uparrow}^3}{6\pi^2}V + \frac{k_{\downarrow}^3}{6\pi^2}V = \frac{k_F^3}{3\pi^2}V$$

by doting $\alpha = |N_{\uparrow} - N_{\downarrow}|/(N_{\uparrow} + N_{\downarrow})$, one can proof

$$f[N_{\uparrow}] + f[N_{\downarrow}] = f\left[\frac{1+\alpha}{2}N\right] + f\left[\frac{1-\alpha}{2}N\right]$$

so the total energy is

$$\begin{aligned} E(N, \alpha) &= E_{\uparrow} + E_{\downarrow} \\ &= \left[\frac{3}{5}a_0^2(k_{\uparrow}^2N_{\uparrow} + k_{\downarrow}^2N_{\downarrow}) - \frac{3}{2\pi}a_0(k_{\uparrow}N_{\uparrow} + k_{\downarrow}N_{\downarrow}) \right] \text{Ry} \\ &= N \left[\frac{3}{5}(k_F a_0)^2 g^{\frac{5}{3}}(\alpha) - \frac{3}{2\pi}(k_F a_0) g^{\frac{4}{3}}(\alpha) \right] \text{Ry} \end{aligned} \quad (\text{shown})$$

where we donote

$$g^x(\alpha) = \frac{1}{2}[(1+\alpha)^x + (1-\alpha)^x]$$

(b)

The behaviour of $E(N, \alpha)$ with variation of α is equivalent to

$$e(\alpha, \beta) = \beta g^{\frac{5}{3}}(\alpha) - g^{\frac{4}{3}}(\alpha)$$

where $\beta = \frac{2\pi}{5}k_F a_0$ is variable. It seems that different β will lead to different behaviours. However, we can use Maclaurin's series

$$g^{\frac{5}{3}}(\alpha) = 1 + \sum_{n=1}^{\infty} C_n \alpha^{2n}, \quad g^{\frac{4}{3}}(\alpha) = 1 + \sum_{n=1}^{\infty} D_n \alpha^{2n}$$

and one can proof numerically

$$\frac{D_1}{C_1} < \frac{D_2}{C_2} < \frac{D_3}{C_3} < \dots, \quad \frac{D_2}{C_2} = 1$$

i.e. D_n/C_n is monotonically increasing and $D_2 = C_2$. Thus

$$e(\alpha, \beta) = (\beta - 1) + C_1(\beta - \frac{D_1}{C_1})\alpha^2 + C_2(\beta - 1)\alpha^4 + C_3(\beta - \frac{D_3}{C_3})\alpha^6 + \dots$$

so we can discuss the behaviour on different cases.

1. $\beta \leq \frac{D_1}{C_1}$ and $\frac{\partial e}{\partial \alpha}$ is always negative on $(0, 1)$. In this case $e_{min} = e|_{\alpha=1}$.

2. β is sufficiently large and $\frac{\partial e}{\partial \alpha}$ is always positive on $(0, 1)$. In this case $e_{min} = e|_{\alpha=0}$.

3. $\beta > \frac{D_1}{C_1}$ but $\frac{\partial e}{\partial \alpha}$ have a zero point on $(0, 1)$. In this case, there must be a natural number N that makes

$$\frac{D_N}{C_N} \leq \beta < \frac{D_{N+1}}{C_{N+1}}$$

and we can rewrite the function (denote the zero point is α_0)

$$e(\alpha) = e_1(\alpha) - e_2(\alpha), \quad e_1(\alpha_0) = e_2(\alpha_0)$$

$$e_1(\alpha) = E_0 + E_1\alpha^2 + \dots + E_N\alpha^{2N}$$

$$e_2(\alpha) = E_{N+1}\alpha^{2N+2} + E_{N+2}\alpha^{2N+4} + \dots$$

all the coefficient is non-negative. Obviously, when $\alpha > \alpha_0$, $e_2(\alpha)$ must grow faster than $e_1(\alpha)$. So $\frac{\partial e}{\partial \alpha}$ is always positive on $(0, \alpha)$ and negative on $(\alpha, 1)$. In this case $e_{min} = \min\{e|_{\alpha=0}, e|_{\alpha=1}\}$

Thus we can find that the minimal energy corresponds to the case with either $\alpha = 0$ or $\alpha = 1$. Q.E.D.

Problem3 Heisenberg model of system with three spins

The system Hamiltonian in a external magnetic field along z direction is

$$\mathcal{H} = -J \sum_{i < j}^3 \left(S_{iz}S_{jz} + \frac{1}{2}S_{i+}S_{j-} + \frac{1}{2}S_{i-}S_{j+} \right) + g\mu_B H \sum_i^3 S_{iz}$$

denote $x = \frac{g\mu_B H}{J}$, one can find that

$$\mathcal{H}\psi = \begin{cases} -J \left(\frac{3}{4} - \frac{3}{2}x \right) \psi, & \psi = |\uparrow\uparrow\uparrow\rangle \\ -J \left(\frac{3}{4} + \frac{3}{2}x \right) \psi, & \psi = |\downarrow\downarrow\downarrow\rangle \\ -J\mathcal{H}_+\psi, & \psi = |\uparrow\uparrow\downarrow\rangle, |\uparrow\downarrow\uparrow\rangle, |\downarrow\uparrow\uparrow\rangle \\ -J\mathcal{H}_-\psi, & \psi = |\downarrow\downarrow\uparrow\rangle, |\downarrow\uparrow\downarrow\rangle, |\uparrow\downarrow\downarrow\rangle \end{cases}$$

where

$$\mathcal{H}_+ = \frac{1}{2} \begin{pmatrix} -\frac{1}{2} - x & 1 & 1 \\ 1 & -\frac{1}{2} - x & 1 \\ 1 & 1 & -\frac{1}{2} - x \end{pmatrix}, \quad \mathcal{H}_- = \frac{1}{2} \begin{pmatrix} -\frac{1}{2} + x & 1 & 1 \\ 1 & -\frac{1}{2} + x & 1 \\ 1 & 1 & -\frac{1}{2} + x \end{pmatrix}$$

the eigenvalues are

$$\lambda_{\pm} = \frac{3}{4} \mp \frac{1}{2}x(\text{non-degenerate}), \quad -\frac{3}{4} \mp \frac{1}{2}x(\text{double-degenerate})$$

if x is small (the external magnetic field is weak), then the ground states are

(a)

$$J > 0, \quad E = -\frac{3}{4}J - \frac{3}{2}g\mu_B H, \quad \psi = |\downarrow\downarrow\downarrow\rangle \quad (\text{shown})$$

(b)

$$J < 0, \quad E = \frac{3}{4}J - \frac{1}{2}g\mu_B H, \quad (\text{shown})$$

$$\psi = \begin{cases} \frac{1}{\sqrt{3}} \left(e^{\pi i/6} |\downarrow\downarrow\uparrow\rangle + e^{5\pi i/6} |\uparrow\downarrow\downarrow\rangle + e^{-3\pi i/6} |\downarrow\uparrow\downarrow\rangle \right) \\ \frac{1}{\sqrt{3}} \left(e^{-\pi i/6} |\downarrow\downarrow\uparrow\rangle + e^{-5\pi i/6} |\uparrow\downarrow\downarrow\rangle + e^{3\pi i/6} |\downarrow\uparrow\downarrow\rangle \right) \end{cases}$$

if we define

$$\eta = |n_{\uparrow} - n_{\downarrow}| / (n_{\uparrow} + n_{\downarrow})$$

as the polarization ratio of spins, then

(a)

$$J < 0, \quad \eta = 1 \quad (\text{shown})$$

(b)

$$J > 0, \quad \eta = \frac{1}{3} \quad (\text{shown})$$

The direction of the spin and the external magnetic field are always opposite.

Solid State Physics Homework

Chapter5 No.3, Due on Jun 3rd 2022, Friday

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Problem1 A&M Chapter33.5

(a)

According to Eq(33.71), the anisotropic Heisenberg spin Hamiltonian within external magnetic field should be

$$\mathcal{H} = -\frac{1}{2} \sum_{\mathbf{R}, \mathbf{R}'} J_z(\mathbf{R} - \mathbf{R}') \mathbf{S}_z(\mathbf{R}) \mathbf{S}_z(\mathbf{R}') - g\mu_B H \sum_{\mathbf{R}} \mathbf{S}_z(\mathbf{R}) \\ - \frac{1}{2} \sum_{\mathbf{R}, \mathbf{R}'} J(\mathbf{R} - \mathbf{R}') \mathbf{S}_-(\mathbf{R}') \mathbf{S}_+(\mathbf{R})$$

From Eq(33.5) and (33.23), one can write down the ground state and one-spin-wave states

$$|0\rangle \prod_{\mathbf{R}} |S\rangle_{\mathbf{R}}, \quad |\mathbf{k}\rangle = \frac{1}{\sqrt{N}} \sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} |\mathbf{R}\rangle$$

Substitute

$$\mathcal{H}|0\rangle = -\frac{1}{2} S^2 \sum_{\mathbf{R}, \mathbf{R}'} J_z(\mathbf{R} - \mathbf{R}') |0\rangle - N g\mu_B H S |0\rangle = E_0 |0\rangle \quad (\text{shown})$$

$$\mathcal{H}|\mathbf{k}\rangle = E_0 |\mathbf{k}\rangle + S \sum_{\mathbf{R}} J_z(\mathbf{R}) |\mathbf{k}\rangle + g\mu_B H |\mathbf{k}\rangle - S \sum_{\mathbf{R}} J(\mathbf{R}) e^{i\mathbf{k}\cdot\mathbf{R}} \\ = E_k |\mathbf{k}\rangle + S \sum_{\mathbf{R}} [J_z(\mathbf{R}) - J(\mathbf{R})] |\mathbf{k}\rangle \quad (\text{shown})$$

So the ground state and one-spin-wave states remain eigenstates of the Hamiltonian, but the spin wave excitation energies are raised by

$$\Delta E = S \sum_{\mathbf{R}} [J_z(\mathbf{R}) - J(\mathbf{R})]$$

which is precisely Eq(33.72).

(b)

According to Eq(33.32), when $H = 0$, the excitation energy of the state $|\mathbf{k}\rangle$ is

$$\mathcal{E}(\mathbf{k}) = \Delta E + 2S \sum_{\mathbf{R}} J(\mathbf{R}) \sin^2 \left(\frac{1}{2} \mathbf{k} \cdot \mathbf{R} \right)$$

it is important that the lowest excitation energy is $\Delta E > 0$. So when $T \rightarrow 0$, we can only consider the two lowest states

$$\begin{aligned} M(T) &= M(0) \left[1 - \frac{1}{NS} \frac{e^{-\beta \Delta E}}{1 + e^{-\beta \Delta E}} \right] \\ &\approx M(0) \left[1 - \frac{1}{NS} e^{-\frac{\Delta E}{k_B T}} \right] \end{aligned} \quad (\text{shown})$$

i.e. the low-temperature spontaneous magnetization deviates from saturation only exponentially in $-1/T$.

(c)

According to Eq(33.34), when $q_0 \rightarrow 0$, the integral

$$\begin{aligned} &\int_0^{q_0} \int_0^{q_0} dq_x dq_y \{ \exp[S \sum_{\mathbf{R}} J(\mathbf{R}) (\mathbf{q} \cdot \mathbf{R})^2 / 2] - 1 \}^{-1} \\ &\approx \int_0^{q_0} \int_0^{q_0} dq_x dq_y [k_x q_x^2 + k_y q_y^2]^{-1} \\ &\sim \int_0^{q_0} \frac{dq}{q} \rightarrow \infty \end{aligned} \quad (\text{shown})$$

i.e. the integral in two dimensions diverges at small q . So there can be no spontaneous magnetization in two dimensions.

Problem2 A&M Chapter33.6

(a)

According to mean field theory, the self-consistent equation for spontaneous magnetization is

$$\lambda M = \frac{J_0 S}{g \mu_B} B_S(g \mu_B S \frac{\lambda M}{k_B T})$$

where

$$\lambda = \frac{V J_0}{N (g \mu_B)^2}, \quad J_0 = \sum_{\mathbf{R}} J(\mathbf{R})$$

denote $x = g\mu_B S \frac{\lambda M}{k_B T}$ and simplified the equation

$$x = \frac{J_0 S^2}{k_B T} B_S(x)$$

for small x

$$B_S(x) \approx \frac{S+1}{3S} x - C_S x^3$$

where C_S is a positive constant. Substitute the approximation by order(T approaches T_c from below)

$$T_c = \frac{S(S+1)}{3k_B} J_0$$

$$x^2 \propto \frac{T_c}{T} - 1$$

so the behaviour of spontaneous magnetization is

$$M \propto xT \propto (T_c - T)^{1/2}, \quad T \rightarrow T_{c-} \quad (\text{shown})$$

(b)

when H, M, x are small, the equation should be

$$x = \frac{J_0 S^2}{k_B T_c} B_S[x(1 + \frac{H}{\lambda M})] \approx x + x \frac{H}{\lambda M} - C'_S x^3$$

so the behaviour of spontaneous magnetization when $T = T_c$ should be

$$M \propto (Mx^2)^{1/3} \propto H^{1/3}, \quad H \rightarrow 0 \quad (\text{shown})$$

Problem3 Spontaneous symmetry-breaking

Calculate the different spontaneous magnetizations only to the first nonzero term

$$\begin{aligned} \langle M \rangle^{(1)} &= g\mu_B \lim_{N \rightarrow \infty} \frac{1}{V} \frac{\sum_{J_z} (1 - \beta g\mu_B H_z J_z) J_z}{\sum_{J_z} J_z} \\ &= -g\mu_B \lim_{N \rightarrow \infty} \frac{1}{V} \frac{2\beta g\mu_B H_z N S(NS+1)(2NS+1)}{6(2NS+1)} \\ &= -g\mu_B \lim_{N \rightarrow \infty} \frac{\beta g\mu_B H_z S(NS+1)}{3v_0} \\ &= -\frac{\beta N (g\mu_B S)^2}{3v_0} H_z \end{aligned} \quad (\text{shown})$$

$$\begin{aligned}
\langle M \rangle^{(2)} &= g\mu_B \lim_{H_z \rightarrow 0} \lim_{N \rightarrow \infty} \frac{-NS}{Nv_0} \\
&= - \frac{g\mu_B S}{v_0}
\end{aligned}
\tag{shown}$$

note that in (2) we only consider the ground state. So the difference is clear, $\langle M \rangle^{(1)}$ is related with H_z and do not exist when $H_z = 0$, while $\langle M \rangle^{(2)}$ is not. For the same Hamiltonian with symmetry, $\langle M \rangle^{(2)}$ is the one that has spontaneous magnetization, corresponding to the spontaneous symmetry-breaking.

Problem4 The mean-field theory: magnetization when $T \rightarrow 0$

According to mean field theory, the self-consistent equation for spontaneous magnetization is

$$M = \frac{J_0 S}{g\mu_B \lambda} B_S(g\mu_B S \frac{\lambda M}{k_B T})$$

when $T \rightarrow 0$, $x = g\mu_B S \frac{\lambda M}{k_B T} \rightarrow \infty$, so

$$B_S(x) \rightarrow 1, \quad M = M_0$$

try to write down the first nonzero term

$$M = M_0 \left[1 - \frac{1}{S} e^{-x/S} \right] = M_0 [1 - S^{-1} \exp(-\beta g\mu_B \lambda M)] \tag{shown}$$

i.e. the magnetization varies by $\exp(-1/T)$, while Eq(33.34) varies by $T^{3/2}$. For $\lim_{T \rightarrow 0} \frac{\exp(-1/T)}{T^{3/2}} = 0$, the mean field theory is not very suitable for the low-temperature condition.