

## 第一次作业

- (1) 若  $[[\hat{B}, \hat{A}], \hat{A}] = 0$ , 求  $e^{-\lambda \hat{A}} \hat{B} e^{\lambda \hat{A}} = ?$
- (2) 证明若  $[\hat{B}, \hat{A}] \neq 0$ ,  $[\hat{B}, \hat{C}] = 0$ ,  $[\hat{A}, \hat{C}] = 0$ ,  $\hat{C}$  为厄密算符,  $|\psi\rangle$  为  $\hat{C}$  的非简并本征态, 则  $\langle \psi | [\hat{B}, \hat{A}] | \psi \rangle = 0$ 。
- (3) 证明  $\text{Tr}(|\psi\rangle\langle\phi|) = \langle\phi|\psi\rangle$ 。
- (4) 若某量子态在实空间的波函数为  $\psi(\mathbf{r}) = \left(\frac{\lambda}{\pi}\right)^{3/2} \exp(-\lambda r^2)$ , 求它在动量空间的波函数  $\psi(\mathbf{p}) = ?$

## 第二次作业

- (1) 利用 $[\hat{\mathbf{r}}, -i\hat{\mathbf{K}} \cdot d\mathbf{r}']|\mathbf{r}'\rangle = d\mathbf{r}'|\mathbf{r}' + d\mathbf{r}'\rangle$ , 证明 $[\hat{r}_\alpha, \hat{K}_\beta] = i\delta_{\alpha\beta}$

提示：在 $\beta$ 方向上取 $d\mathbf{r}'$ 趋于零的极限。

- (2) 利用 $-\frac{i}{\hbar}\hat{\mathbf{P}} \cdot \Delta\mathbf{r}'|\psi\rangle = -\int d^3r' \Delta\mathbf{r}' \cdot (\nabla_{\mathbf{r}'}\psi)|\mathbf{r}'\rangle$ ,

证明 $\langle\mathbf{r}|\hat{\mathbf{P}}|\mathbf{r}'\rangle = -i\hbar\nabla_{\mathbf{r}}\delta(\mathbf{r} - \mathbf{r}')$

提示：方程两边左侧乘 $\langle\mathbf{r}|$ ，对方程右边再采用分部积分

- (3) 设 $|g(E)|^2\rho(E) = Be^{-(E-E_0)^2/\Delta^2}$ ，利用公式

$$C(t) = \exp\left(\frac{-iE_0t}{\hbar}\right) \int dE |g(E)|^2 \rho(E) \exp\left[\frac{-i(E-E_0)t}{\hbar}\right],$$

计算相关振幅  $C(t)$ ，解释关于时间和能量的测不准原理，

并指出它与普通的测不准原理的区别。

- (4)

Let  $x(t)$  be the coordinate operator for a free particle in one dimension in the Heisenberg picture. Evaluate

$$[x(t), x(0)].$$

# 第一周高等量子力学作业

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## 1

令

$$\begin{aligned}\hat{B}_0 &= \hat{B} \\ \hat{B}_1 &= [\hat{B}_0, \hat{A}] \\ \hat{B}_2 &= [\hat{B}_1, \hat{A}] = [[\hat{B}_0, \hat{A}], \hat{A}] \\ &\dots\end{aligned}$$

则对函数

$$f(\lambda) = e^{-\lambda\hat{A}}\hat{B}e^{\lambda\hat{A}}$$

有

$$\begin{aligned}\frac{d}{d\lambda}f(\lambda) &= e^{-\lambda\hat{A}}[\hat{B}_0, \hat{A}]e^{\lambda\hat{A}} = e^{-\lambda\hat{A}}\hat{B}_1e^{\lambda\hat{A}} \\ \frac{d^2}{d\lambda^2}f(\lambda) &= e^{-\lambda\hat{A}}[\hat{B}_1, \hat{A}]e^{\lambda\hat{A}} = e^{-\lambda\hat{A}}\hat{B}_2e^{\lambda\hat{A}} \\ \frac{d^3}{d\lambda^3}f(\lambda) &= e^{-\lambda\hat{A}}[\hat{B}_2, \hat{A}]e^{\lambda\hat{A}} = e^{-\lambda\hat{A}}\hat{B}_3e^{\lambda\hat{A}} \\ &\dots\end{aligned}$$

由  $\hat{B}_2 = 0$  可知  $f(\lambda)$  的二阶及更高阶导数全为 0。即函数此时是线性变化。于是

$$f(\lambda) = f(0) + \frac{df}{d\lambda}\lambda = \hat{B} + \lambda[\hat{B}, \hat{A}]$$

## 2

由  $[\hat{B}, \hat{C}] = 0$  可知

$$\hat{C}\hat{B}|\psi\rangle = \hat{B}\hat{C}|\psi\rangle = \lambda\hat{B}|\psi\rangle$$

其中  $\lambda$  是对应本征值。因  $|\psi\rangle$  非简并，可得

$$\hat{B}|\psi\rangle = b|\psi\rangle, \quad \hat{A}|\psi\rangle = a|\psi\rangle$$

$a, b$  均为复数。故

$$\langle \psi | [\hat{B}, \hat{A}] | \psi \rangle = \langle \psi | [b, a] | \psi \rangle = 0$$

### 3

对有限维希尔伯特空间，取一完备基矢  $|n\rangle$ ,  $n = 0, 1, 2, \dots$

$$\sum_n |n\rangle \langle n| = 1$$

则

$$\langle \psi | \phi \rangle = \sum_n \langle \psi | n \rangle \langle n | \phi \rangle = \sum_n \langle n | \phi \rangle \langle \psi | n \rangle = \sum_n (|\phi\rangle \langle \psi|)_{nn} = \text{Tr}(|\phi\rangle \langle \psi|)$$

### 4

动量空间波函数

$$\begin{aligned} \varphi(\mathbf{p}) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(\mathbf{r}) e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar} d^3r \\ &= \frac{1}{(2\pi\hbar)^{3/2}} \left(\frac{\lambda}{\pi}\right)^{3/2} \int e^{-\lambda r^2 - i\mathbf{p}\cdot\mathbf{r}/\hbar} d^3r \\ &= \frac{1}{(2\pi\hbar)^{3/2}} e^{-p^2/4\lambda\hbar^2} \end{aligned}$$

其中利用了积分公式

$$\int e^{-\lambda x^2} e^{-ikx} dx = \sqrt{\pi/\lambda} e^{-k^2/4\lambda}$$

# 第二周高等量子力学作业

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## 1

对平移算符  $\hat{D}(dr_\beta) = 1 - i\hat{K}_\beta dr_\beta$ , 当  $\alpha$  与  $\beta$  相异时

$$[\hat{r}_\alpha, \hat{D}(dr_\beta)]|\mathbf{r}\rangle = 0$$

即

$$[\hat{r}_\alpha, \hat{K}_\beta] = 0$$

当  $\alpha$  与  $\beta$  相同时, 由

$$[\hat{r}_\alpha, -i\hat{K}_\beta dr_\beta]|\mathbf{r}\rangle = dr_\beta|\mathbf{r} + d\mathbf{r}_\beta\rangle$$

可知当  $dr_\beta \rightarrow 0$  时

$$[\hat{r}_\alpha, \hat{K}_\beta] = i$$

综上有

$$[\hat{r}_\alpha, \hat{K}_\beta] = i\delta_{\alpha\beta}$$

## 2

根据条件可知

$$\hat{\mathbf{P}}|\psi\rangle = -i\hbar \int dr' (\nabla_{r'}\psi)|\mathbf{r}'\rangle$$

方程两边左乘  $\langle\mathbf{r}|$ , 方程左侧变为

$$\langle\mathbf{r}|\hat{\mathbf{P}}|\psi\rangle = \int dr' \langle\mathbf{r}|\hat{\mathbf{P}}|\mathbf{r}'\rangle \langle\mathbf{r}'|\psi\rangle = \int dr' \langle\mathbf{r}|\hat{\mathbf{P}}|\mathbf{r}'\rangle \psi(\mathbf{r}')$$

方程右侧分部积分, 变为

$$-i\hbar \int dr' (\nabla_{r'}\psi) \langle\mathbf{r}|\mathbf{r}'\rangle = -i\hbar \int dr' (\nabla_{r'}\psi) \delta(\mathbf{r} - \mathbf{r}') = i\hbar \int dr' (\nabla_{r'}\delta(\mathbf{r} - \mathbf{r}')) \psi(\mathbf{r}')$$

对比左右两侧可得

$$\langle\mathbf{r}|\hat{\mathbf{P}}|\mathbf{r}'\rangle = -i\hbar \nabla_r \delta(\mathbf{r} - \mathbf{r}')$$

### 3

相关振幅

$$\begin{aligned}
C(t) &= B \exp(-iE_0 t/\hbar) \int dE \exp[-(E - E_0)^2/\Delta^2 - i(E - E_0)t/\hbar] \\
&= B \exp(-iE_0 t/\hbar - \Delta^2 t^2/4\hbar^2) \int dE \exp[-(E - E_0 + i\Delta^2 t/2\hbar)^2/\Delta^2] \\
&= \Delta B \exp(-iE_0 t/\hbar - \Delta^2 t^2/4\hbar^2) \int dx \exp(-x^2) \\
&= \sqrt{\pi} \Delta B \exp \left[ -\frac{\Delta^2}{4\hbar^2} (t + 2iE_0 \hbar/\Delta^2)^2 - \frac{E_0^2}{\Delta^2} - i\frac{\pi}{4} \right]
\end{aligned}$$

由此可见相关振幅的大小主要取决于  $\Delta t$  和  $\hbar$  之间的比较关系。故有

$$\Delta E \Delta t \approx \hbar$$

时间-能量不确定关系和普通的不确定关系的区别在于，后者是由非对易性导致的两个力学量的不确定性，而前者是描述由能态密度导致的系统随时间演化的行为。

### 4

对于自由粒子

$$H = \frac{p^2}{2m}, \quad [p, H] = 0$$

即动量是守恒量。根据 Heisenberg 方程

$$\frac{d}{dt} x(t) = \frac{1}{i\hbar} [x(t), H] = \frac{1}{i\hbar} e^{iHt/\hbar} \left[ x(t), \frac{p^2}{2m} \right] e^{-iHt/\hbar} = \frac{p}{m}$$

则

$$\begin{aligned}
x(t) &= x(0) + \frac{p}{m} t \\
[x(t), x(0)] &= \frac{t}{m} [p, x] = -i\hbar t/m
\end{aligned}$$