Criticality and Quenched Disorder: Harris Criterion Versus Rare Regions

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We employ scaling arguments and optimal fluctuation theory to establish a general relation between quantum Griffiths singularities and the Harris criterion for quantum phase transitions in disordered systems. If a clean critical point violates the Harris criterion, it is destabilized by weak disorder. At the same time, the Griffiths dynamical exponent z' diverges upon approaching the transition, suggesting unconventional critical behavior. In contrast, if the Harris criterion is fulfilled, power-law Griffiths singularities can coexist with clean critical behavior, but z' saturates at a finite value. We present applications of our theory to a variety of systems including quantum spin chains, classical reaction-diffusion systems and metallic magnets, and we discuss modifications for transitions above the upper critical dimension. Based on these results we propose a unified classification of phase transitions in disordered systems.

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The effects of quenched disorder on continuous phase transitions have been the subject of intense theoretical and experimental interest, with applications ranging from condensed matter and atomic physics to chemistry and biology. Over the course of this research, two different frameworks for classifying disorder effects have emerged, one based on the behavior of the average disorder strength under coarse graining, and the other focused on the properties of rare large disorder fluctuations.

The traditional approach is based on analyzing the stability of clean critical points against weak disorder via the Harris criterion [1]. If the clean correlation length exponent ν fulfills the inequality $d\nu > 2$ (d is the space dimensionality), weak disorder decreases under coarse graining and becomes unimportant on large length scales. The critical behavior of the disordered system is thus identical to that of the corresponding clean one. If $d\nu < 2$, weak disorder is relevant; i.e., it increases under coarse graining, and the transition must change. Motrunich *et al.* [2] generalized this idea and classified critical points according to the fate of the (average) disorder strength under coarse graining: If it vanishes on large length scales, the clean critical behavior is unchanged. If it reaches a nonzero finite value, the critical behavior remains conventional, but the critical exponents differ from the clean ones. Finally, if the disorder strength diverges under coarse graining, the transition is governed by an exotic infinite-randomness critical point.

In recent years, it has become clear that the average disorder strength is insufficient to characterize disordered critical points. Rare large disorder fluctuations and the corresponding spatial regions play an important role. Such rare regions can be locally in one phase while the bulk system is in the other. Griffiths showed that they cause nonanalyticities, now known as the Griffiths singularities, in the free energy in an entire parameter region around the transition [3]. The strength of these singularities can be classified [4] by comparing the rare region dimensionality d_{RR} with the lower critical dimension d_c^- of the transition [5]. If $d_{RR} < d_c^-$, individual rare regions cannot undergo the transition independently. The resulting weak essential Griffiths singularities are likely unobservable in experiment. If $d_{RR} > d_c^-$, individual rare regions do order independently; this destroys the global phase transition by smearing. In the marginal case, $d_{RR} = d_c^-$, rare regions cannot order, but their slow dynamics leads to enhanced *quantum* Griffiths singularities characterized by power laws with a nonuniversal dynamical exponent z'.

These two classification schemes have been used successfully to analyze a plethora of classical, quantum, and nonequilibrium transitions (see, e.g., Refs. [6]). However, they look at different aspects of the disorder problem, which may lead to seemingly incompatible predictions, e.g., if Harris's inequality $d\nu > 2$ is fulfilled while the rare regions produce strong power-law Griffiths singularities.

In this Letter, we use scaling and optimal fluctuation theory to establish a general relation between quantum Griffiths singularities and the Harris criterion: If disorder is introduced into a system that fulfills Harris's inequality $d\nu > 2$, power-law Griffiths singularities coexist with clean critical behavior, and the dynamical exponent z' governing the Griffiths singularities saturates at a finite, disorder-dependent value at the transition. In contrast, if Harris's inequality is violated, z' diverges upon approaching the transition, suggesting strong-randomness or infinite-randomness critical behavior. In the remainder of this Letter, we sketch the derivation of the results, and we discuss several examples in quantum spin chains [7], classical reaction-diffusion systems [8], random quantum Ising models [9,10], and metallic magnets [11]. We also consider modifications for transitions above the upper critical dimension where hyperscaling is violated, and we present computer simulations illustrating our theory. Finally, we use these results to propose a refined classification of phase transitions in disordered systems.

Our first example is the Ashkin-Teller model [7] which consists of two coupled transverse-field Ising chains,

$$H = -\sum_{\alpha=1}^{2}\sum_{i}(J_i\sigma^z_{\alpha,i}\sigma^z_{\alpha,i+1} + h_i\sigma^x_{\alpha,i}) - \sum_i \epsilon(J_i\sigma^z_{1,i}\sigma^z_{1,i+1}\sigma^z_{2,i}\sigma^z_{2,i+1} + h_i\sigma^x_{1,i}\sigma^x_{2,i}), \quad (1)$$

where σ^x, σ^z are Pauli matrices, and J_i and h_i denote the interactions and transverse fields. In the clean case, $J_i \equiv J$, $h_i \equiv h$, the system undergoes a quantum phase transition from a paramagnetic phase to a ferromagnetic (Baxter) phase at $h = J$ for all ϵ between $-1/\sqrt{2}$ and 1. The critical exponents vary continuously with ϵ. In particular, the correlation length exponent ν is below 2 for $\epsilon > -1/2$ but above 2 for $\epsilon < -1/2$. Upon introducing weak (random mass) disorder, the local distance from criticality $r_i = \ln(h_i/J_i)$ becomes a random variable. It is governed by a probability distribution $W(r_i)$ which we take to be a binary distribution, $W(r_i) = p\delta(r_i - r_h) + (1-p)\delta(r_i - r_l)$ with $r_h > r_l$, for simplicity.

Consider a large spatial region of linear size L_{RR} containing $N \sim L^d_{RR}$ sites. (We formulate the theory in d dimensions; in our example $d = 1$). The effective distance from criticality r of this region is given by the average of its local r_i. It has a binomial probability distribution

$$P(r, L_{RR}) = \sum_{n=0}^{N}\binom{N}{n}p^n(1-p)^{N-n}\delta[r - r_{RR}(N, n)] \quad (2)$$

with $r_{RR}(N, n) = r_l + (n/N)(r_h - r_l)$. For large regions of roughly average composition, this binomial can be approximated by a Gaussian

$$P_G(r, L_{RR}) \sim \exp[-(1/2b^2)L^d_{RR}(r - r_{av})^2], \quad (3)$$

where $r_{av} = pr_h + (1-p)r_l$ is the average distance from criticality and $b^2 = p(1-p)(r_h - r_l)^2$ measures the strength of the disorder. Regions with $r < 0$ are locally ferromagnetic even if the bulk system is still paramagnetic, $r_{av} > 0$. They thus constitute the rare regions responsible for quantum Griffiths singularities.

The low-energy spectrum of a single, locally ordered, rare region is equivalent to that of two coupled two-level systems. The energy gap ϵ can be easily estimated in perturbation theory [12], yielding

$$\epsilon(L_{RR}) = \epsilon_0 \exp[-aL^d_{RR}] \quad (4)$$

with $\epsilon_0 \approx h$. According to finite-size scaling (FSS) [13], the coefficient a, which has the dimension of an inverse volume, behaves as $a = a'(-r)^{d\nu}$, with r being the distance of the rare region from criticality [14]. Here, ν represents the *clean* correlation length exponent unless the rare region is in the narrow asymptotic critical region.

We now consider a system in the paramagnetic phase, $r_{av} > 0$, but close to the phase transition. The rare-region density of states can be estimated by integrating over all locally ordered regions [15],

$$\rho(\epsilon) \sim \int_{-\infty}^{0} dr \int_{0}^{\infty} dL_{RR} P(r, L_{RR})\delta[\epsilon - \epsilon(L_{RR})]. \quad (5)$$

Using the Gaussian approximation, Eq. (3), for the joint distribution $P(r, L_{RR})$, this expression can be easily evaluated. The L_{RR} integral can be carried out exactly while the remaining integral over r can be performed in a saddle-point approximation in the limit $\epsilon \to 0$. The resulting saddle-point value is

$$r_{sp} = r_{av}\, d\nu/(d\nu - 2). \quad (6)$$

Two cases must be distinguished:

(i) If $d\nu < 2$, r_{sp} is negative and thus within the integration interval $(-\infty, 0)$. Inserting r_{sp} into the integral (5) gives a power-law Griffiths singularity,

$$\rho(\epsilon) \sim \epsilon^{\lambda-1} = \epsilon^{d/z'-1}, \quad (7)$$

in the density of states. Griffiths singularities in various other quantities can be calculated from Eq. (7). The nonuniversal Griffiths exponent λ varies as

$$\lambda \sim b^{-2} r_{av}^{2-d\nu} \quad (8)$$

with the global distance from criticality, implying that the Griffiths dynamical exponent z' diverges as $z' \sim b^2 r_{av}^{d\nu-2}$ upon approaching the transition.

(ii) In the opposite case, $d\nu \geq 2$, the maximum of the exponent in the integrand of (5) is attained for $r \to -\infty$. The density of states is thus dominated by contributions from the far tail of the probability distribution $P(r, L_{RR})$. Thus, the Gaussian approximation, Eq. (3), is not justified. Instead, one needs to work with the tail of the original distribution. For our binomial distribution (2), the far tail consists of regions in which all sites have $r_i = r_l$. For these regions, Eq. (2) simplifies to $P(r, L_{RR}) \sim \exp(-\tilde{p}L^d_{RR})\delta(r - r_l)$ with $\tilde{p} = -\ln(1-p)$. As such compact rare regions remain in the ferromagnetic phase when the bulk reaches criticality, the coefficient a in Eq. (4) takes some *finite nonzero* value a_c at the bulk transition point [16]. Combining $P(r, L_{RR})$ and $\epsilon(L_{RR})$, we again find a power-law density of states as in Eq. (7). However, the Griffiths exponent λ does not vanish at the global transition point but takes the nonzero value

$$\lambda_c = \tilde{p}/a_c. \tag{9}$$

This implies that the dynamical exponent z' does not diverge upon approaching the transition. Its maximum value $z'_c = da_c/\tilde{p}$ vanishes for zero disorder and increases with increasing disorder strength.

We emphasize that our optimal fluctuation theory describes the disorder scaling close to the clean critical (fixed) point. Therefore, it correctly describes the asymptotic critical behavior in the case $d\nu > 2$. In contrast, for $d\nu < 2$, it does not hold in the asymptotic critical region because nontrivial disorder renormalizations beyond the tree-level analysis underlying Eqs. (2) and (3) become important as the disorder strength increases. To explore the limits of our approach, we can use scaling theory, which states that the clean description breaks down when the scaling combination $b^2 r^{d\nu-2}$ exceeds a constant of order one [17]. Up to a numerical factor, the Griffiths dynamical exponent z' in Eq. (8) equals this scaling combination. It thus reaches a value of order one independent of the bare disorder strength before (8) breaks down. The further evolution of z' in the asymptotic critical region is beyond the scope of our method [18].

We have thus established our main result: The same inequality that controls the scaling of the average disorder strength also governs the quantum Griffiths singularities. If the clean correlation length exponent ν fulfills the inequality $d\nu > 2$, the average disorder strength scales to zero under coarse graining. Moreover, the Griffiths dynamical exponent at the transition takes a finite value that vanishes in the limit of zero disorder and increases with increasing disorder strength. This means that, for sufficiently weak disorder, clean critical behavior coexists with subleading power-law quantum Griffiths singularities. In contrast, for $d\nu < 2$, the average disorder strength grows under coarse graining, destabilizing the clean critical behavior. The Griffiths dynamical exponent increases in parallel with the renormalized disorder strength. Even for arbitrarily weak bare disorder, it reaches a value of order one at the crossover from the clean to the disordered critical fixed point. In the asymptotic critical region it either diverges or saturates at a large value. Density matrix renormalization group calculations [19] of short random Ashkin-Teller chains for selected ϵ between -1 and 1 are compatible with these predictions.

We now discuss how general our result is. The optimal fluctuation theory applies as long as three assumptions are fulfilled. (i) The disorder is of random-mass (random-T_c) type with short-range correlations and a bounded probability distribution. (ii) The gap or characteristic energy of a rare region depends exponentially on its volume, putting the system in class B of the rare region classification of Refs. [4,6]. (iii) The transition is of non-mean-field type (i.e., below the upper critical dimension d_c^+) such that conventional FSS can be used for the relation between the rare region size and its distance from criticality, $a = a'(-r)^{d\nu}$. These conditions are fulfilled for a large variety of classical, quantum, and nonequilibrium phase transitions in realistic systems [6].

However, other important (quantum) phase transitions are above d_c^+. How is our theory modified in this case? For $d > d_c^+$, conventional FSS breaks down due to dangerously irrelevant variables. Instead, several systems fulfill a modified FSS [20] (dubbed "q-scaling" [21]) that replaces the scaling combination $rL^{1/\nu}$ with $rL^{q/\nu}$, where $q = d/d_c^+$. Examples include the classical Ising model, directed percolation, and the large-N limits of various $O(N)$ order-parameter field theories. In the following, we assume that q-scaling is fulfilled for $d > d_c^+$.

Repeating the derivation of the rare-region density of states, we find that the only change is in the relation between the rare region size and its distance from criticality in Eq. (4). Here, $a = a'(-r)^{d\nu}$ gets replaced by $a = a'(-r)^{d\nu/q} = a'(-r)^{d_c^+\nu}$. As a result, the average disorder strength and the power-law Griffiths singularities are controlled by different inequalities: The disorder strength increases under coarse graining if $d\nu < 2$ [22], while the dynamical exponent z' diverges if $d_c^+\nu < 2$.

Let us apply these ideas to our second example, the nonequilibrium transition in the contact process [23], which can be mapped to a quantum problem using the Hamiltonian formalism [24]. In this problem, each lattice site is in one of two states: infected or healthy. The time evolution is a Markov process during which infected sites heal at a rate μ or infect their neighbors at a rate κ. If $\mu \gg \kappa$, healing dominates, and the infection eventually dies out completely. For $\kappa \gg \mu$, the infection never dies out, leading to a nonzero steady-state density ρ of infected sites. These two regimes are separated by a nonequilibrium phase transition in the directed percolation [25] universality class. It has an upper critical dimension of $d_c^+ = 4$. The FSS above d_c^+ is of q-scaling type [26].

In $d = 1$, 2, and 3, the clean correlation length exponent violates the inequality $d\nu > 2$. According to the Harris criterion, weak disorder is relevant. Moreover, Eq. (8) predicts the Griffiths dynamical exponent z' to diverge at the transition. In agreement with these predictions, infinite-randomness critical behavior has been found in the disordered contact process for $d = 1$, 2, and 3 [27,28]. For $d > 4$, weak disorder is irrelevant according to the Harris criterion because $d\nu = d/2 > 2$. Moreover, as $d_c^+\nu = 2$, the Griffiths singularities are expected to be dominated by compact rare regions at the lower bound of the disorder distribution. Our theory thus predicts that the (weakly) disordered contact process in $d > 4$ features clean critical behavior, accompanied by power-law Griffiths singularities whose dynamical exponent z' saturates at a finite value at the transition point. We have performed Monte Carlo simulations of the disordered five-dimensional contact process [29] on lattices with up to 51^5 sites. The results (see Fig. 1) are in agreement with these predictions. Similar

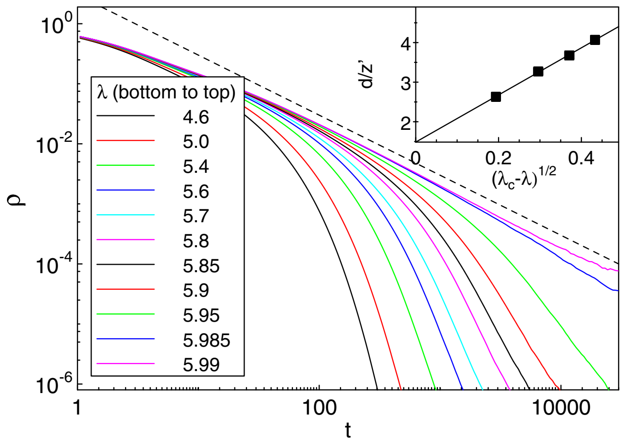

FIG. 1 (color online). Density ρ versus time t for a 5D disordered contact process. The critical behavior is compatible with the clean mean-field result $\rho \sim t^{-1}$ (dashed line). The subcritical curves show Griffiths singularities $\rho \sim t^{-d/z'}$ rather than the exponential decay expected in a clean system. Inset: extrapolation of the Griffiths exponent d/z' to criticality.

behavior has also been observed for the contact process on networks [30].

Our final example is the large-N limit of the quantum Landau-Ginzburg-Wilson theory

$$S = \int dx dy \phi(x)\Gamma(x,y)\phi(y) + (u/2N)\int dx \phi^4(x) \quad (10)$$

in d dimensions. ϕ is an N-component order parameter, $x \equiv (\mathbf{x}, \tau)$ comprises position $\mathbf{x}$ and imaginary time τ, and $\int dx \equiv \int d\mathbf{x} \int_0^{1/T} d\tau$. The Fourier transform of the clean inverse propagator $\Gamma(x,y)$ is given by

$$\Gamma(\mathbf{q}, \omega_n) = (r + \mathbf{q}^2 + \gamma|\omega_n|). \quad (11)$$

This theory describes, among other things, quantum phase transitions in itinerant antiferromagnets [11] and superconducting nanowires [31]. Its upper critical dimension is $d_c^+ = 2$.

As the clean correlation length exponent takes the values $\nu = 1$ in $d = 1$ and $\nu = 1/2$ for $d \geq 2$, the Harris criterion $d\nu > 2$ is violated for dimensions $d < 4$ but fulfilled for $d > 4$. The rare regions in this problem were studied in Ref. [4], giving a characteristic rare region energy $\epsilon = \epsilon_0 \exp[-a'(-r)L_{RR}^d]$ for all dimensions above and below d_c^+. This implies that the FSS above d_c^+ is of q-scaling type. As $d_c^+\nu = 1$, the inequality $d_c^+\nu > 2$ is violated in all dimensions. For $d < 4$, we thus expect weak disorder to be relevant and the Griffiths dynamical exponent z' to diverge at the transition, in agreement with strong-disorder renormalization group studies [32] that yield infinite-randomness criticality. In contrast, for $d > 4$, our theory predicts z' to diverge even though the Harris criterion is fulfilled, suggesting that rare regions change the character of the transition despite the disorder being perturbatively irrelevant.

Note that a similar situation occurs in the transverse-field Ising model in $d > 4$. The Harris criterion is fulfilled, but $d_c^+\nu = 3/2 < 2$, suggesting a diverging z' in all dimensions. Interestingly, recent strong-disorder renormalization group calculations [33] show infinite-randomness criticality even for infinite dimensions.

Let us return to the classification of critical points in weakly disordered systems according to the rare region dimensionality as put forward in Ref. [4] and discussed in the introduction. In the present Letter, we have studied class B of this classification which contains systems whose rare regions are right at the lower critical dimension, $d_{RR} = d_c^-$, leading to power-law Griffiths singularities. According to our results, class B can be subdivided by means of the Harris criterion into class $B1$, where clean critical behavior coexists with subleading Griffiths singularities, and class $B2$, featuring strong or infinite-randomness criticality. Note that this subdivision applies to non-mean-field transitions. As discussed above, further complications may occur above d_c^+.

In summary, we have established a general relation between the Harris criterion and rare region effects in weakly disordered systems. For non-mean-field clean critical points, the scaling of the average disorder strength under coarse graining and the behavior of the quantum Griffiths singularities are governed by the same inequality. For $d\nu > 2$, weak disorder is irrelevant, and the Griffiths dynamical exponent z' remains finite and small at the transition. If $d\nu < 2$, weak disorder is relevant, and z' increases with the renormalized disorder strength upon approaching the transition. Above the upper critical dimension, the situation is more complex. The scaling of the average disorder strength is still governed by the Harris criterion $d\nu > 2$, but the fate of the Griffiths dynamical exponent is controlled by the inequality $d_c^+\nu > 2$. This opens up the exciting possibility that nonperturbative rare region physics can modify the transition even if the Harris criterion is fulfilled.

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[16] Note that a_c vanishes for zero disorder ($r_h = r_l$) and increases with increasing disorder strength.

[17] This follows because the scale dimension of the disorder strength b^2 at the clean critical point is $2/\nu - d$ [34].

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