第十一周高等量子力学作业

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当 $i \neq j$ 时

$$\begin{aligned}\hat{a}_i\hat{a}_j^\dagger &= \sum_{\{N\}}\sum_{\{N'\}}\sqrt{N_i(N_j'+1)}|N_1...N_i-1...\rangle\langle N_1...N_i...|N_1'...N_j'+1...\rangle\langle N_1'...N_j'...| \\ &= \sum_{\{N\}}\sqrt{N_iN_j}|N_1...N_i-1...\rangle\langle N_1...N_j-1...|\end{aligned}$$

$$\begin{aligned}\hat{a}_j^\dagger\hat{a}_i &= \sum_{\{N\}}\sum_{\{N'\}}\sqrt{N_i(N_j'+1)}|N_1'...N_j'+1...\rangle\langle N_1'...N_j'...|N_1...N_i-1...\rangle\langle N_1...N_i...| \\ &= \sum_{\{N\}}\sqrt{N_i(N_j+1)}|N_1...N_i-1...N_j+1...\rangle\langle N_1...N_i...N_j...| \\ &= \hat{a}_i\hat{a}_j^\dagger\end{aligned}$$

这里默认 $i<j, i>j$ 的情形完全类似。若 $i=j$，则有

$$\begin{aligned}\hat{a}_i\hat{a}_i^\dagger &= \sum_{\{N\}}\sum_{\{N'\}}\sqrt{N_i(N_i'+1)}|N_1...N_i-1...\rangle\langle N_1...N_i...|N_1'...N_i'+1...\rangle\langle N_1'...N_i'...| \\ &= \sum_{\{N\}}N_i|N_1...N_i-1...\rangle\langle N_1...N_i-1...|\end{aligned}$$

$$\begin{aligned}\hat{a}_i^\dagger\hat{a}_i &= \sum_{\{N\}}\sum_{\{N'\}}\sqrt{N_i(N_i'+1)}|N_1'...N_i'+1...\rangle\langle N_1'...N_i'...|N_1...N_i-1...\rangle\langle N_1...N_i...| \\ &= \sum_{\{N\}}N_i|N_1...N_i...\rangle\langle N_1...N_i...| \\ &= \hat{a}_i\hat{a}_i^\dagger - 1\end{aligned}$$

据此可得

$$[\hat{a}_i, \hat{a}_j^\dagger] = \delta_{ij}$$

而当 $i \neq j$ 时

$$\hat{a}_i\hat{a}_j = \sum_{\{N\}} \sqrt{N_iN_j}|N_1...N_i-1...N_j-1...\rangle\langle N_1...N_i...N_j| = \hat{a}_j\hat{a}_i$$

$$\hat{a}_i^\dagger\hat{a}_j^\dagger = \sum_{\{N\}} \sqrt{(N_i+1)(N_j+1)}|N_1...N_i+1...N_j+1...\rangle\langle N_1...N_i...N_j| = \hat{a}_j^\dagger\hat{a}_i^\dagger$$

于是有 ($i=j$ 时显然成立)

$$[\hat{a}_i, \hat{a}_j] = 0, \quad [\hat{a}_i^\dagger, \hat{a}_j^\dagger] = 0$$

2

根据定义

$$\begin{aligned}
[\widehat{\Phi}(\xi), \widehat{\Phi}^\dagger(\xi')] &= \sum_{ij} \psi_i(\xi)\psi_j^*(\xi')[\hat{a}_i, \hat{a}_j^\dagger] \\
&= \sum_{ij} \psi_i(\xi)\psi_j^*(\xi')\delta_{ij} \\
&= \sum_{i} \psi_i(\xi)\psi_i^*(\xi') \\
&= \sum_{i} \langle\xi|\psi_i\rangle\langle\psi_i|\xi'\rangle \\
&= \langle\xi|\xi'\rangle \\
&= \delta(\xi-\xi')
\end{aligned}$$

3

由于场算符不依赖于单体基函数

$$\widehat{\Phi}(\xi) = \sum_i \psi_i(\xi)\hat{a}_i = \sum_j \phi_j(\xi)\hat{b}_j$$

可以得到选取不同的单体基函数时湮灭算符之间的变换关系

$$\hat{b}_j = \sum_i \langle\phi_j|\psi_i\rangle\hat{a}_i$$

因而有

$$\begin{aligned}
\widehat{\Phi}^\dagger(\xi)|0\rangle &= \sum_i \psi_i^*(\xi)\hat{a}_i^\dagger|0\rangle \\
&= \sum_i \psi_i^*(\xi)|\psi_i\rangle \\
&= \sum_i |\psi_i\rangle\langle\psi_i|\xi\rangle \\
&= |\xi\rangle
\end{aligned}$$

4

当 $i \neq j$ 时

$$
\begin{aligned}
\hat{a}_i \hat{a}_j^\dagger &= \sum_{\{N\}} \sum_{\{N'\}} P_i P_j |N_1 ... N_i = 0 ...\rangle\langle N_1 ... N_i = 1 ... | N_1' ... N_j' = 1 ...\rangle\langle N_1' ... N_j' = 0 ...| \\
&= \sum_{\{N\}} P_i(N_j = 1) P_j(N_i = 1) |N_1 ... N_i = 0 ... N_j = 1 ...\rangle\langle N_1 ... N_i = 1 ... N_j = 0 ...|
\end{aligned}
$$

$$
\begin{aligned}
\hat{a}_j^\dagger \hat{a}_i &= \sum_{\{N\}} \sum_{\{N'\}} P_i P_j |N_1' ... N_j' = 1 ...\rangle\langle N_1' ... N_j' = 0 ... | N_1 ... N_i = 0 ...\rangle\langle N_1 ... N_i = 1 ...| \\
&= \sum_{\{N\}} P_i(N_j = 0) P_j(N_i = 0) |N_1 ... N_i = 0 ... N_j = 1 ...\rangle\langle N_1 ... N_i = 1 ... N_j = 0 ...|
\end{aligned}
$$

这里默认 $i < j, i > j$ 的情形完全类似。其中

$$
P_i = (-)^{\sum_{l=1}^{i-1} N_l}
$$

显然有

$$
P_i(N_j = 1) P_j(N_i = 1) = -P_i(N_j = 0) P_j(N_i = 0)
$$

此时有

$$
\hat{a}_i \hat{a}_j^\dagger + \hat{a}_j^\dagger \hat{a}_i = 0
$$

若 $i = j$，则有

$$
\begin{aligned}
\hat{a}_i \hat{a}_i^\dagger + \hat{a}_i^\dagger \hat{a}_i &= \sum_{\{N\}} P_i^2 (|N_1 ... N_i = 0 ...\rangle\langle N_1 ... N_i = 0 ...| + |N_1 ... N_i = 1 ...\rangle\langle N_1 ... N_i = 1 ...|) \\
&= \sum_{\{N\}} |N_1 ... N_i ...\rangle\langle N_1 ... N_i ...| \\
&= 1
\end{aligned}
$$

据此可得

$$
\{\hat{a}_i, \hat{a}_j^\dagger\} = \delta_{ij}
$$

而当 $i \neq j$ 时

$$
\hat{a}_i \hat{a}_j = \sum_{\{N\}} P_i(N_j = 1) P_j(N_i = 0) |...0...0...\rangle\langle ...1...1...| = -\hat{a}_j \hat{a}_i
$$

$$
\hat{a}_i^\dagger \hat{a}_j^\dagger = \sum_{\{N\}} P_i(N_j = 1) P_j(N_i = 0) |...1...1...\rangle\langle ...0...0...| = -\hat{a}_j^\dagger \hat{a}_i^\dagger
$$

于是有 ($i = j$ 时显然成立)

$$
\{\hat{a}_i, \hat{a}_j\} = 0, \quad \{\hat{a}_i^\dagger, \hat{a}_j^\dagger\} = 0
$$