第三周高等量子力学作业

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用粒子数表象展开相干态，并且假设 $\alpha = re^{i\theta}$，有

$$
\begin{aligned}
\iint \mathrm{d}^2\alpha |\alpha\rangle\langle\alpha| &= \iint \mathrm{d}^2\alpha \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{\alpha^n \alpha^{*m}}{\sqrt{n!m!}} e^{-|\alpha|^2} |n\rangle\langle m| \\
&= \iint \mathrm{d}^2\alpha \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{r^{n+m} e^{i(n-m)\theta}}{\sqrt{n!m!}} e^{-r^2} |n\rangle\langle m| \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |n\rangle\langle m| \int \mathrm{d}r \frac{r^{n+m+1}}{\sqrt{n!m!}} e^{-r^2} \int \mathrm{d}\theta e^{i(n-m)\theta} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} |n\rangle\langle m| \int \mathrm{d}r \frac{r^{n+m+1}}{\sqrt{n!m!}} e^{-r^2} 2\pi\delta_{nm} \\
&= 2\pi \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} \int \mathrm{d}r r^{2n+1} e^{-r^2} \\
&= \pi \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} \Gamma(n+1) \\
&= \pi
\end{aligned}
$$

其中用到了 Γ 函数定义

$$
\Gamma(x) = 2\int_0^{\infty} t^{2x-1} e^{-t^2} \mathrm{d}t
$$

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若简并，则有

$$
\hat{H}\psi = E\psi, \quad \hat{H}\phi = E\phi
$$

对一维波函数 $\hat{H} = -\frac{\hbar^2}{2m}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + V(x)$ 有

$$
\psi'' + \frac{2m}{\hbar^2}(E-V)\psi = 0, \quad \phi'' + \frac{2m}{\hbar^2}(E-V)\phi = 0
$$

两式分别乘另式波函数相减得到

$$\psi''\phi = \psi\phi''$$

积分得到

$$\psi'\phi - \psi\phi' = \text{const}$$

由边界条件得到 const=0，于是再积分得到

$$\psi = k\phi$$

其中 k 为常数。因而非简并。

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传播子

$$\begin{aligned}
K(\alpha', t'; \alpha, t) &= \langle\alpha'|U(t', t)|\alpha\rangle \\
&= \sum_n \langle\alpha'|n\rangle\langle n|e^{-iH(t'-t)/\hbar}|\alpha\rangle \\
&= \sum_n \langle\alpha'|n\rangle\langle n|\alpha\rangle e^{-iE_n(t'-t)/\hbar} \\
&= e^{-[|\alpha|^2+|\alpha'|^2+i\omega(t'-t)]/2}\sum_n \frac{1}{n!}(\alpha\alpha'^*)^n e^{-i\omega(t'-t)n} \\
&= \exp\left\{\alpha\alpha'^* e^{-i\omega(t'-t)} - [|\alpha|^2 + |\alpha'|^2 + i\omega(t'-t)]/2\right\}
\end{aligned}$$

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a.

$$\begin{aligned}
[\Pi_x, \Pi_y] &= \left[p_x - \frac{e}{c}A_x, p_y - \frac{e}{c}A_y\right] \\
&= -\frac{e}{c}([p_x, A_y] + [A_x, p_y]) \\
&= \frac{i\hbar e}{c}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) \\
&= \frac{i\hbar eB}{c}
\end{aligned}$$

b. 由磁矢势定义和规范变换条件可知

$$\mathbf{A} = -\frac{1}{2}By\hat{\mathbf{x}} + \frac{1}{2}Bx\hat{\mathbf{y}}$$

令

$$\Pi_y = P, \quad \Pi_x = X, \quad \omega = \frac{|eB|}{mc}$$

则哈密顿量为

$$H = \frac{\Pi^2}{2m} = \frac{p_z^2}{2m} + \frac{P^2}{2m} + \frac{1}{2}m\omega^2 X^2$$

第一项不受其他影响，对应连续本征值；由 $[X, P] = i\hbar$ 关系，后两项完全等同于 $\omega = \frac{|eB|}{mc}$ 的一维谐振子。故能量本征值可立即给出

$$E_{k,n} = \frac{\hbar^2 k^2}{2m} + \frac{\hbar|eB|}{mc}\left(n + \frac{1}{2}\right)$$