第四次作业

（1） 利用角动量算符的对易关系，证明$\left[\hat{J}^2, \hat{J}_\alpha\right] = 0$，其中 $\alpha = x, y, z$。

（2） 证明$\hat{J}_z \widehat{D}_y(\pi)|jm\rangle = -m\hbar \widehat{D}_y(\pi)|jm\rangle$。

提示：由 Baker-Hausdorff 公式可得$\widehat{D}_y(-\pi)\hat{J}_z \widehat{D}_y(\pi) = -\hat{J}_z$

（3） 设在某混合态，电子有 30%几率处在$\hat{S}_x$的本征值为$\frac{\hbar}{2}$的本征态上，有 70%几率处在$\hat{S}_y$的本征值为$-\frac{\hbar}{2}$的本征态上，给出其密度矩阵。

（4）

Consider a sequence of Euler rotations represented by

$$\mathscr{D}^{(1/2)}(\alpha, \beta, \gamma) = \exp\left(\frac{-i\sigma_3\alpha}{2}\right)\exp\left(\frac{-i\sigma_2\beta}{2}\right)\exp\left(\frac{-i\sigma_3\gamma}{2}\right)$$
$$= \begin{pmatrix} e^{-i(\alpha+\gamma)/2}\cos\frac{\beta}{2} & -e^{-i(\alpha-\gamma)/2}\sin\frac{\beta}{2} \\ e^{i(\alpha-\gamma)/2}\sin\frac{\beta}{2} & e^{i(\alpha+\gamma)/2}\cos\frac{\beta}{2} \end{pmatrix}.$$

Because of the group properties of rotations, we expect that this sequence of operations is equivalent to a *single* rotation about some axis by an angle θ. Find θ.

第五次作业

（1） 证明

$$d^{(j=1)}(\beta) =$$

$$\begin{pmatrix} \left(\frac{1}{2}\right)(1+\cos\beta) & -\left(\frac{1}{\sqrt{2}}\right)\sin\beta & \left(\frac{1}{2}\right)(1-\cos\beta) \\ \left(\frac{1}{\sqrt{2}}\right)\sin\beta & \cos\beta & -\left(\frac{1}{\sqrt{2}}\right)\sin\beta \\ \left(\frac{1}{2}\right)(1-\cos\beta) & \left(\frac{1}{\sqrt{2}}\right)\sin\beta & \left(\frac{1}{2}\right)(1+\cos\beta) \end{pmatrix}.$$

提示：在 j=1 子空间先表示$\hat{J}_y$，再证明$\hat{J}_y^3 = \hbar^2 \hat{J}_y$并展开 d 函数。

（2）

The wave function of a particle subjected to a spherically symmetrical potential $V(r)$ is given by

$$\psi(\mathbf{x}) = (x + y + 3z) f(r).$$

a. Is ψ an eigenfunction of $\mathbf{L}^2$? If so, what is the l-value? If not, what are the possible values of l we may obtain when $\mathbf{L}^2$ is measured?
b. What are the probabilities for the particle to be found in various m_l states?
c. Suppose it is known somehow that $\psi(\mathbf{x})$ is an energy eigenfunction with eigenvalue E. Indicate how we may find $V(r)$.

（3）

We are to add angular momenta $j_1 = 1$ and $j_2 = 1$ to form $j = 2$, 1, and 0 states. Using either the ladder operator method or the recursion relation, express all (nine) $\{j, m\}$ eigenkets in terms of $|j_1 j_2; m_1 m_2\rangle$. Write your answer as

$$|j=1, m=1\rangle = \frac{1}{\sqrt{2}}|+,0\rangle - \frac{1}{\sqrt{2}}|0,+\rangle, \ldots,$$

where + and 0 stand for $m_{1,2} = 1, 0$, respectively.

(4)

a. Evaluate

$$\sum_{m=-j}^{j} |d_{mm'}^{(j)}(\beta)|^2 m$$

for *any* j (integer or half-integer); then check your answer for $j = \frac{1}{2}$.

b. Prove, for any j,

$$\sum_{m=-j}^{j} m^2 |d_{m'm}^{(j)}(\beta)|^2 = \frac{1}{2} j(j+1)\sin^2\beta + m'^2 \frac{1}{2}(3\cos^2\beta - 1).$$

[*Hint*: This can be proved in many ways. You may, for instance, examine the rotational properties of J_z^2 using the spherical (irreducible) tensor language.]

第六次作业

（1）

a. Construct a spherical tensor of rank 1 out of two different vectors $\mathbf{U} = (U_x, U_y, U_z)$ and $\mathbf{V} = (V_x, V_y, V_z)$. Explicitly write $T^{(1)}_{\pm 1, 0}$ in terms of $U_{x,y,z}$ and $V_{x,y,z}$.

b. Construct a spherical tensor of rank 2 out of two different vectors $\mathbf{U}$ and $\mathbf{V}$. Write down explicitly $T^{(2)}_{\pm 2, \pm 1, 0}$ in terms of $U_{x,y,z}$ and $V_{x,y,z}$.

提示：利用 CG 系数解 b.

（2）

a. Write xy, xz, and $(x^2 - y^2)$ as components of a spherical (irreducible) tensor of rank 2.

b. The expectation value

$$Q \equiv e\langle \alpha, j, m = j|(3z^2 - r^2)|\alpha, j, m = j\rangle$$

is known as the *quadrupole moment*. Evaluate

$$e\langle \alpha, j, m'|(x^2 - y^2)|\alpha, j, m = j\rangle,$$

(where $m' = j, j-1, j-2, \ldots$) in terms of Q and appropriate Clebsch-Gordan coefficients.

（3）

Let $\mathscr{T}_{\mathbf{d}}$ denote the translation operator (displacement vector $\mathbf{d}$); $\mathscr{D}(\hat{\mathbf{n}}, \phi)$, the rotation operator ($\hat{\mathbf{n}}$ and ϕ are the axis and angle of rotation, respectively); and π the parity operator. Which, if any, of the following pairs commute? Why?

a. $\mathscr{T}_{\mathbf{d}}$ and $\mathscr{T}_{\mathbf{d}'}$ ($\mathbf{d}$ and $\mathbf{d}'$ in different directions).

b. $\mathscr{D}(\hat{\mathbf{n}}, \phi)$ and $\mathscr{D}(\hat{\mathbf{n}}', \phi')$ ($\hat{\mathbf{n}}$ and $\hat{\mathbf{n}}'$ in different directions).

c. $\mathscr{T}_{\mathbf{d}}$ and π.

d. $\mathscr{D}(\hat{\mathbf{n}}, \phi)$ and π.

(4)

A quantum-mechanical state Ψ is known to be a simultaneous eigenstate of two Hermitian operators A and B which *anticommute*,

$$AB + BA = 0.$$

What can you say about the eigenvalues of A and B for state Ψ? Illustrate your point using the parity operator (which can be chosen to satisfy $\pi = \pi^{-1} = \pi^{\dagger}$) and the momentum operator.