第一周高等量子力学作业

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1

令

$$
\begin{aligned}
\hat{B}_0 &= \hat{B} \\
\hat{B}_1 &= [\hat{B}_0, \hat{A}] \\
\hat{B}_2 &= [\hat{B}_1, \hat{A}] = [[\hat{B}_0, \hat{A}], \hat{A}] \\
&\cdots
\end{aligned}
$$

则对函数

$$f(\lambda) = e^{-\lambda \hat{A}} \hat{B} e^{\lambda \hat{A}}$$

有

$$
\begin{aligned}
\frac{d}{d\lambda} f(\lambda) &= e^{-\lambda \hat{A}} [\hat{B}_0, \hat{A}] e^{\lambda \hat{A}} = e^{-\lambda \hat{A}} \hat{B}_1 e^{\lambda \hat{A}} \\
\frac{d^2}{d\lambda^2} f(\lambda) &= e^{-\lambda \hat{A}} [\hat{B}_1, \hat{A}] e^{\lambda \hat{A}} = e^{-\lambda \hat{A}} \hat{B}_2 e^{\lambda \hat{A}} \\
\frac{d^3}{d\lambda^3} f(\lambda) &= e^{-\lambda \hat{A}} [\hat{B}_2, \hat{A}] e^{\lambda \hat{A}} = e^{-\lambda \hat{A}} \hat{B}_3 e^{\lambda \hat{A}} \\
&\cdots
\end{aligned}
$$

由 $\hat{B}_2 = 0$ 可知 $f(\lambda)$ 的二阶及更高阶导数全为 0。即函数此时是线性变化。于是

$$f(\lambda) = f(0) + \frac{df}{\mathrm{d}\lambda} \lambda = \hat{B} + \lambda [\hat{B}, \hat{A}]$$

2

由 $[\hat{B}, \hat{C}] = 0$ 可知

$$\hat{C} \hat{B} |\psi\rangle = \hat{B} \hat{C} |\psi\rangle = \lambda \hat{B} |\psi\rangle$$

其中 λ 是对应本征值。因 $|\psi\rangle$ 非简并，可得

$$\hat{B} |\psi\rangle = b |\psi\rangle, \quad \hat{A} |\psi\rangle = a |\psi\rangle$$

a, b 均为复数。故

$$\langle\psi|[\hat{B}, \hat{A}]|\psi\rangle = \langle\psi|[b, a]|\psi\rangle = 0$$

3

对有限维希尔伯特空间，取一完备基矢 $|n\rangle, \quad n = 0, 1, 2...$

$$\sum_n |n\rangle\langle n| = 1$$

则

$$\langle\psi|\phi\rangle = \sum_n \langle\psi|n\rangle\langle n|\phi\rangle = \sum_n \langle n|\phi\rangle\langle\psi|n\rangle = \sum_n (|\phi\rangle\langle\psi|)_{nn} = \mathrm{Tr}(|\phi\rangle\langle\psi|)$$

4

动量空间波函数

$$\begin{aligned}
\varphi(\mathbf{p}) &= \frac{1}{(2\pi\hbar)^{3/2}} \int \psi(\mathbf{r}) e^{-i\mathbf{p}\cdot\mathbf{r}/\hbar} d^3r \\
&= \frac{1}{(2\pi\hbar)^{3/2}} \left(\frac{\lambda}{\pi}\right)^{3/2} \int e^{-\lambda r^2 - i\mathbf{p}\cdot\mathbf{r}/\hbar} d^3r \\
&= \frac{1}{(2\pi\hbar)^{3/2}} e^{-p^2/4\lambda\hbar^2}
\end{aligned}$$

其中利用了积分公式

$$\int e^{-\lambda x^2} e^{-ikx} dx = \sqrt{\pi/\lambda} e^{-k^2/4\lambda}$$

第二周高等量子力学作业

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1

对平移算符 $\hat{D}(dr_\beta) = 1 - i\hat{K}_\beta dr_\beta$，当 α 与 β 相异时

$$[\hat{r}_\alpha, \hat{D}(dr_\beta)]|\mathbf{r}\rangle = 0$$

即

$$[\hat{r}_\alpha, \hat{K}_\beta] = 0$$

当 α 与 β 相同时，由

$$[\hat{r}_\alpha, -i\hat{K}_\beta dr_\beta]|\mathbf{r}\rangle = dr_\beta|\mathbf{r} + d\mathbf{r}_\beta\rangle$$

可知当 $dr_\beta \to 0$ 时

$$[\hat{r}_\alpha, \hat{K}_\beta] = i$$

综上有

$$[\hat{r}_\alpha, \hat{K}_\beta] = i\delta_{\alpha\beta}$$

2

根据条件可知

$$\hat{\mathbf{P}}|\psi\rangle = -i\hbar\int dr'(\nabla_{r'}\psi)|\mathbf{r}'\rangle$$

方程两边左乘 $\langle\mathbf{r}|$，方程左侧变为

$$\langle\mathbf{r}|\hat{\mathbf{P}}|\psi\rangle = \int dr'\langle\mathbf{r}|\hat{\mathbf{P}}|\mathbf{r}'\rangle\langle\mathbf{r}'|\psi\rangle = \int dr'\langle\mathbf{r}|\hat{\mathbf{P}}|\mathbf{r}'\rangle\psi(\mathbf{r}')$$

方程右侧分部积分，变为

$$-i\hbar\int dr'(\nabla_{r'}\psi)\langle\mathbf{r}|\mathbf{r}'\rangle = -i\hbar\int dr'(\nabla_{r'}\psi)\delta(\mathbf{r}-\mathbf{r}') = i\hbar\int dr'(\nabla_{r'}\delta(\mathbf{r}-\mathbf{r}'))\psi(\mathbf{r}')$$

对比左右两侧可得

$$\langle\mathbf{r}|\hat{\mathbf{P}}|\mathbf{r}'\rangle = -i\hbar\nabla_r\delta(\mathbf{r}-\mathbf{r}')$$

3

相关振幅

$$
\begin{aligned}
C(t) &= B\exp(-iE_0t/\hbar)\int dE\exp[-(E-E_0)^2/\Delta^2 - i(E-E_0)t/\hbar] \\
&= B\exp(-iE_0t/\hbar - \Delta^2t^2/4\hbar^2)\int dE\exp[-(E-E_0+i\Delta^2t/2\hbar)^2/\Delta^2] \\
&= \Delta B\exp(-iE_0t/\hbar - \Delta^2t^2/4\hbar^2)\int dx\exp(-x^2) \\
&= \sqrt{\pi}\Delta B\exp\left[-\frac{\Delta^2}{4\hbar^2}(t+2iE_0\hbar/\Delta^2)^2 - \frac{E_0^2}{\Delta^2} - i\frac{\pi}{4}\right]
\end{aligned}
$$

由此可见相关振幅的大小主要取决于 Δt 和 $\hbar$ 之间的比较关系。故有

$$\Delta E\Delta t \approx \hbar$$

时间-能量不确定关系和普通的不确定关系的区别在于，后者是由非对易性导致的两个力学量的不确定性，而前者是描述由能态密度导致的系统随时间演化的行为。

4

对于自由粒子

$$H = \frac{p^2}{2m}, \quad [p,H] = 0$$

即动量是守恒量。根据 Heisenberg 方程

$$\frac{d}{dt}x(t) = \frac{1}{i\hbar}[x(t),H] = \frac{1}{i\hbar}e^{iHt/\hbar}\left[x(t),\frac{p^2}{2m}\right]e^{-iHt/\hbar} = \frac{p}{m}$$

则

$$x(t) = x(0) + \frac{p}{m}t$$

$$[x(t),x(0)] = \frac{t}{m}[p,x] = -i\hbar t/m$$