Taylor Series

Prereqs RA-17, RA-18, RA-20

Let f be an infinitely differentiable function. More formally, suppose f admits an $n-$th derivative near a point a for every $n \in \mathbb{N}$. Then, f admits the $n-$th Taylor Polynomial near a

$$P_n(x) := f(a) + (x-a) \cdot f'(a) + \frac{(x-a)^2}{2} \cdot f''(a) + \cdots + \frac{(x-a)^n}{n!} \cdot f^{(n)}(a)$$

for every $n \in \mathbb{N}$.

In the limit as $n \to \infty$, the Taylor Polynomial P_n becomes a power series around a with $a_n = \frac{f^{(n)}(a)}{n!}$.

$$P_\infty(x) = \sum \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n$$

The Taylor Polynomials P_n were originally framed as polynomial approximations to differentiable functions. It is intuitive that for "nice" enough functions, the power series is an ∞-degree polynomial approximation to f near a, and thus should be accurate.

The question we want to investigate is if the Taylor Series P_∞ converges, and if it converges to f.

Observe that $P_n(x)$ is the sequence of partial sums of $P_\infty(x)$, and therefore by definition $P_\infty(x)$ converges iff the sequence $a_n := P_n(x)$ converges.

We first define the $n-$th error term

$$E_n(x) := f(x) - P_n(x) \tag{1}$$

and expand out $P_n(x)$.

$$E_n(x) = f(x) - f(a) - f'(a) \cdot (x-a) - \cdots - \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n \tag{2}$$

By (1), $P_n(x) \to f(x)$ if and only if $E_n(x) \to 0$. Look at equation (2), and recall Taylor's Theorem. These look eerily familiar.

Theorem 1. (Taylor-Lagrange) Let f be infinitely differentiable in $(a-h, a+h)$ for some $h > 0$. Then for every $n \in \mathbb{N}$ and for every $a \neq x \in (a-h, a+h)$, there is some ξ in (x, a) or (a, x) whichever the case may be such that

$$E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-a)^{n+1}$$

This is precisely the statement of Taylor's Theorem. We also have the following which follows from equation (1).

Theorem 2. Everything as above, $P_n(x) \to f(x)$ if and only if $E_n(x) \to 0$.

Let's look at some examples.

Example 1. Let $f(x) = \frac{1}{1-x}$, verify that $f^{(n)}(x)$ exists on $(-1,1)$ for every n and is given by

$$f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$$

Therefore the derivatives at 0 are

$$f^{(n)}(0) = n!$$

and hence the Taylor Series around 0 is given by

$$\sum \frac{f^{(n)}(0)}{n!} \cdot x^n = \sum x^n$$

The error term for the $n-$th partial sum is

$$E_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \cdot x^n = \frac{1}{(1-\xi)^{n+2}} \cdot x^{n+1}$$

which indeed $\to 0$ whenever $|x| < 1$.

We can similarly show that the following series converge to their respective functions.

$$\sum \frac{x^n}{n!} \to e^x$$
$$\sum (-1)^n \frac{x^{2n+1}}{(2n+1)!} \to \sin x$$
$$\sum (-1)^{n+1} \frac{x^n}{n} \to \ln(1+x)$$

and so on.

Having continuous $n-$th derivatives for all n is not sufficient for convergence. Consider

$$f(x) = \begin{cases} e^{-\frac{1}{x^2}} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Then $f^{(k)}(0) = 0$ for every k and thus the Taylor Series is identically 0. Clearly it doesn't converge to $f(x)$ for any $x > 0$, since $f(x) \neq 0$ here.