# Continuity of Functions

**Prereqs** RA-10, FD-02

In RA-10 we quantified continuity of functions via sequences, the idea being that as $x_n$ get closer to some $x_0$, $f(x_n)$ should get closer to $f(x_0)$ for $f$ to be continuous.

Here, to define continuity, we impose what appears to be a stronger condition. We start with the value $f(x_0)$ and require that given any $\epsilon > 0$, there *must exist* some $\delta > 0$ such that whenever $x$ is within $\delta$ of $x_0$, $f(x)$ is within $\epsilon$ of $f(x_0)$.

In other words, in the sequential definition, we started on the $x-$axis ("as $x_n$ get closer to $x_0 \dots$") and made conclusions about the $y-$axis ("...$f(x_n)$ get closer to $f(x_0)$").

However, in this "$\epsilon - \delta$" definition, we *start* on the $y-$axis ("given an $\epsilon-$neighborhood around $f(x_0)\dots$") and impose a condition on the $x-$axis ("... whenever $x$ is within $\delta$ of $x_0$, $f(x)$ lands within that neighborhood").

On first glance, the second definition looks stronger. $f(x)$ has no choice BUT to land within $\epsilon$ of $f(x_0)$ on the entire interval $(x_0 - \delta, x_0 + \delta)$. The first definition asks $f(x_n)$ to get $\epsilon-$close to $f(x_0)$ only after a certain $N$. Let's formalise and see if this is true.

**Definition 1.** $f : \Omega \to \mathbb{R}$ with $\Omega \subseteq \mathbb{R}$ is *continuous at* $x_0$ if for every $\epsilon > 0$, there is some $\delta > 0$ such that $x \in (x_0 - \delta, x_0 + \delta) \implies f(x) \in (f(x_0) - \epsilon, f(x_0) + \epsilon)$

That is, you can get $f(x)$ as close to $f(x_0)$ as you want if $x$ is sufficiently close to $x_0$.

We haven't yet justified the fact that both these definitions are for $f$ to be continuous at a fixed point. We must show they are equivalent.

**Theorem 1.** A function $f$ is continuous at $x_0$ if and only if it is sequentially continuous at $x_0$

**Proof.** ($\to$) We want to show continuous implies sequentially continuous. i.e, we're given for every $\epsilon$ some $\delta$ has the property that

$$|x - x_0| < \delta \implies \big|f(x) - f(x_0)\big| < \epsilon$$

and we want to show whenever $x_n \to x_0$ then $f(x_n) \to f(x_0)$.

Let $\epsilon_0 > 0$ be given and choose $\delta_0$ by definition of continuity of $f$ at $x_0$. Suppose $x_n \to x_0$. By definition of convergence, there is some $N$ such that

$$n \geq N \implies |x_n - x_0| < \delta_0$$

But $\delta_0$ was chosen such that

$$|x_n - x_0| < \delta_0 \implies \big|f(x_n) - f(x_0)\big| < \epsilon_0$$

Therefore whenever $n \geq N$, $\big|f(x_n) - f(x_0)\big| < \epsilon_0$. By definition of convergence, $f(x_n) \to f(x_0)$ $\square$

($\leftarrow$) Now we want to show sequentially continuous implies continuous. This is not straightforward, and instead we'll show the contrapositive – that if $f$ is not continuous then it is not sequentially continuous.

Suppose $f$ is not continuous. Recall that $f$ is continuous only when for every $\epsilon$, there is some $\delta$ with a certain property. For $f$ to not be continuous, there must be some $\epsilon_0$ such that every $\delta$ fails to have that property. In all, we're given that there is some $\epsilon_0$, such that for every $\delta$, there is some $x_\delta$ such that

$$x_\delta \in (x_0 - \delta, x_0 + \delta)$$

but

$$f(x_\delta) \notin (f(x_0) - \epsilon_0, f(x_0) + \epsilon_0)$$

And we want to show $f$ is not sequentially continuous – that there is some sequence $x_n \to x_0$ but $f(x_n) \not\to f(x_0)$

Now this is straightforward. Knowing $\epsilon_0$ and that the property fails for every $\delta$, take $\delta_n = \frac{1}{n}$. Then, there is some $x_n := x_{\delta_n}$ such that

$$x_n \in (x_0 - \frac{1}{n}, x_0 + \frac{1}{n})$$

but

$$\left|f(x_n) - f(x_0)\right| \geq \epsilon_0$$

This $x_n$ is our required sequence. Clearly, $x_n \to x_0$ (since every subsequent term lies in smaller and smaller intervals around $x_0$) but $f(x_n) \not\to f(x_0)$ (since $f(x_n)$ fail to get closer to $f(x_0)$ than $\epsilon_0$).

Thus $f$ is not sequentially continuous. $\square$

Continuity and sequential continuity are the same thing! Finally, we make the following definition

**Definition 2.** A function $f : \Omega \to \mathbb{R}$ is *continuous* if it is *continuous at* every point of $\Omega$

Where we generalise from continuity at a point to continuity on a set.

**Exercise 1.** Show that $f(x) = x$ is continuous.

Recall from FD-02 the various definitions of limits. Comparing to the definition of continuity at a point, we can clearly see that the following definition of continuity is equivalent to both our above definitions.

**Definition 3.** A function $f : \Omega \to \mathbb{R}$ with $x_0 \in \Omega \subseteq \mathbb{R}$ is *continuous at* $x_0$ if

$$\lim_{x \to x_0} f(x) = L$$

# Answers to Exercises

*The following are brief solutions or hints. You are encouraged to review the exercises before checking the answers.*

**Answer 1.** Let $x_n$ be a sequence converging to an arbitrary $x_0$, i.e $x_n \to x_0$. But $f(x_n) = x_n$ and $f(x_0) = x_0$, so $f(x_n) \to f(x_0)$. $\square$

**Aliter.** Let $\epsilon > 0$ be given. Take $\delta = \epsilon$ and suppose $|x - x_0| < \delta$. Then,

$$\begin{aligned} \big|f(x) - f(x_0)\big| &= |x - x_0| \\ &< \delta \\ &= \epsilon \end{aligned}$$

Thus $f$ is continuous. $\square$