$\mathbb{R}$ is complete

Prereqs RA-03, RA-02

We are now ready to state a very fundamental property of $\mathbb{R}$.

Theorem 1. $\mathbb{R}$ is complete.

The proof is irrelevant and depends on the construction of $\mathbb{R}$, for which I'll leave an outline. It can also be found in [1]. Let's see what completeness allows us to do.

Theorem 2. Let $x \in \mathbb{R}$ be such that $x > 0$ and let $n \in \mathbb{N}$. Then, there is some $y \in \mathbb{R}$ such that $y^n = x$ and $y > 0$

The theorem states that every positive number admits an $n-$th root, whenever n is natural. This already is something that we cannot say about $\mathbb{Q}$.

Let's try to intuitively understand this. We know that x^n is increasing for $n \in \mathbb{N}$. Suppose I know $y_1^n < x$, so my required y is certainly larger than y_1. Suppose I also know $y_2^n > x$, so my required y is certainly smaller than y_2. Now I can take the average of y_1 and y_2 and repeat this all over again.

In formal terms, let $S := \{y \in \mathbb{R}_{\geq 0} \mid y^n \leq x\}$. Certainly, $x + 1$ is an upper bound of S. Also $0 \in S$. S thus admits a least upper bound α.

Any β smaller than α is already in S and thus $\beta^n < x$. Surprisingly, this goes the other way, if some β satisfies $\beta^n < x$, I can be sure that $\beta < \alpha$ (x^n is an increasing function!). So α^n cannot be smaller than x.

What if $\beta > \alpha$? Any such β is an upper bound of S. Take $\gamma = \frac{\alpha+\beta}{2}$. We know β^n cannot be smaller than x, otherwise $\beta \in S$ and it will contradict that α is an upper bound. If $\beta^n = x$, then γ^n must be smaller than x and γ must lie in S, again contradicting that α is an upper bound of S. We are forced to accept that $\beta > \alpha \iff \beta^n > x$.

What about α^n? We already knew α^n cannot be smaller than x. From the preceding paragraph, $\alpha^n > x$ would mean $\alpha > \alpha$ which cannot happen.

It must be the case that $\alpha^n = x$.

This proof is handwavy, the details can be found in [1]. The overall idea should be clear. We recognise that x^n is an increasing function; so on the number line, our required $n-$th root must be sandwiched between those y for which $y^n < x$, and those y for which $y^n > x$.

On the rational number line, there might be a hole where this sandwiched number is supposed to be. The completeness property ensures no such hole exists.

References

[1] Walter Rudin. *Principles of Mathematical Analysis*, volume 3. McGraw-Hill, 1976.