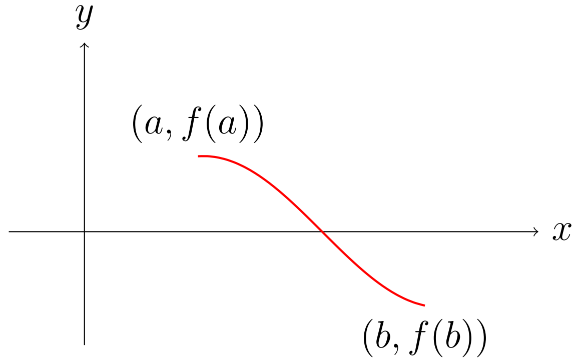
Intermediate Value Theorem

Prereqs RA-11

Let's put our definition of continuity through a litmus test. Suppose f is a continuous function in the pen and paper sense and is defined on a closed interval $[a, b]$.

Suppose that $f(a) > 0$ and $f(b) < 0$. Is it true that $f(c) = 0$ for some $c \in (a, b)$?



Does a continuous curve hit the $x-$axis if it starts above and ends below?

Intuitively – yes! If I have to reach below the $x-$axis from above, can I do that without lifting my pen? Absolutely not. Does our definition capture this?

Theorem 1. (Bolzano's Theorem) If $f : [a, b] \to \mathbb{R}$ is continuous with $f(a) < 0$ and $f(b) > 0$ (or vice versa), then $f(c) = 0$ for some c in (a, b)

Proof. Consider the set

$$S = \{x \in [a, b] \mid f(x) \leq 0\}$$

Observe that $a \in S$ therefore S is nonempty. Also since $S \subset [a, b]$ therefore every $x \in S$ is smaller than b. One can even say S is bounded above by b.

Whenever a nonempty set is bounded above, it has a least upper bound. Let it be α in this case. Intuitively, α is the last possible value of x for which $f(x) \leq 0$. Can we claim that $f(\alpha) = 0$?

Since α is the least upper bound of S, there is a sequence $\{a_n\} \subset S$ such that $a_n \to \alpha$ $(*)$. Since f is continuous, $f(a_n) \to f(\alpha)$. But all of $f(a_n)$ are ≤ 0, therefore $f(\alpha) \leq 0$ $(**)$. Here we use the "least" part of the least upper bound.

Since $f(\alpha) \leq 0$, we conclude that $\alpha \neq b$. Combining with $\alpha \leq b$ gives $\alpha < b$. Now we construct another sequence which approaches α from the right. Take $c_n = \alpha + \dfrac{b - \alpha}{n}$. Here, $c_1 = b$ and the sequence monotonically decreases to α.

Again, $c_n \to \alpha$ and thus $f(c_n) \to f(\alpha)$. Here we will exploit the "upper bound" property. Since $c_n > \alpha$ and α is an upper bound of S, none of the c_n lie in S and hence $f(c_n) > 0$. Further, $f(c_n) \to f(\alpha)$ gives $f(\alpha) \geq 0$. Thus $f(\alpha) = 0$. $\square$

Our definition passes the litmus test!

Corollary 1. (Intermediate Value Theorem) Let $f : [a, b] \to \mathbb{R}$ be continuous and let $k \in \mathbb{R}$ be such that $f(a) < k$ and $f(b) > k$ (or vice versa). Then $f(c) = k$ for some $c \in (a, b)$

Observe that I have used results $(*)$ and $(**)$ in the proof, which I'll properly state and leave out as exercises.

Exercise 1. Let S be a nonempty bounded subset of $\mathbb{R}$ and α be its least upper bound. Show that there is a sequence a_n with all its terms in S and $a_n \to \alpha$.

Exercise 2. Let $\{a_n\}$ be a sequence such that $a_n \to L$ and $a_n > 0$ for every n. Show that $L \geq 0$ and give an example of a sequence where $L = 0$.

Answers to Exercises

The following are brief solutions or hints. You are encouraged to review the exercises before checking the answers.

Answer 1. Since α is the least upper bound, $\alpha - \frac{1}{n}$ cannot be an upper bound of S, therefore there is some $a_n \in S$ such that

$$\alpha - \frac{1}{n} \leq a_n \leq \alpha$$

where the first inequality is because $\alpha - \frac{1}{n}$ is not an upper bound of S, and the second one is because α is an upper bound of S.

Now we need to check if the sequence converges to α. Let $\epsilon > 0$ be given, and take N to be such that $N > \dfrac{1}{\epsilon}$ (such N exists by Archimedean property). Take $n \geq N$, then

$$\begin{aligned} n &\geq \frac{1}{\epsilon} \\ \implies \epsilon &\geq \frac{1}{n} \\ \implies \alpha - \epsilon &\leq \alpha - \frac{1}{n} \end{aligned}$$

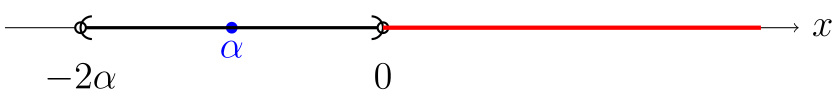
Thus whenever $n \geq N$, we get

$$\alpha - \epsilon \leq \alpha - \frac{1}{N} \leq \alpha - \frac{1}{n} \leq a_n \leq \alpha$$

Therefore $a_n \to \alpha$ $\square$

Answer 2. Let $a_n > 0$ for every n and suppose $a_n \to \alpha$. Suppose for contradiction that $\alpha < 0$. Let the distance between α and 0 be d (i.e, $\alpha = -d$ with $d > 0$).

The idea is, if all $a_n > 0$ and $\alpha < 0$, then α is sufficiently seperated from the sequence. Just take $\epsilon = d$. Can the sequence fall within d distance of α? No!



The sequence lies in the red section and cannot touch the $|\alpha|$ $-$neighborhood of α

Therefore $\alpha \geq 0$. An example where $\alpha = 0$ is the classic $a_n = \frac{1}{n}$