$\mathbb{Q}$ has holes

Prereqs RA-02

It is well known that the square root of 2 is not a rational number. In particular, one can say $\mathbb{Q}$ has a *hole* at $\sqrt{2}$. We will try to capture this idea in mathematical terms. i.e., we will try to rigorously define what exactly constitutes a *hole*.

Consider $S = \{x \in \mathbb{Q} \mid x \geq 0 \text{ and } x^2 < 2\}$. This is clearly bounded above by 2, for $x^2 < 2$ implies $x^2 < 4$, in turn implies $x < 2$. We don't worry about absolute value when taking square roots in the last step because $x \geq 0$.

Recall in the examples in RA-02, we were able to find some number α such that nothing smaller than α was an upper bound of the set. Can we do the same here?

As it turns out, the answer is no, we can't. In fact, we can *prove* that S has no least upper bound in $\mathbb{Q}$

Lemma 1. S has no least upper bound in $\mathbb{Q}$

Proof. Let $p \in \mathbb{Q}$ be so that $p \geq 0$. We will observe a few things about p.

First, $p^2 \neq 2$. In the case that $p^2 < 2$, $p \in S$. In the case that $p^2 > 2$, p is an upper bound of S.

We wish to show p is not a least upper bound of S. We do so by exhibiting another upper bound of S which is smaller than p. Let

$$q = p - \frac{p^2 - 2}{p + 2} = \frac{2p + 2}{p + 2}$$

Since $p^2 - 2 > 0$, we subtract a positive quantity from p and thus $q < p$. q is clearly > 0 because $p > 0$. Now observe

$$\begin{aligned} q^2 - 2 &= \frac{(2p+2)^2}{(p+2)^2} - 2 \\ &= \frac{4p^2 + 8p + 4}{p^2 + 4p + 4} - 2 \\ &= \frac{4p^2 + 8p + 4 - 2p^2 - 8p - 8}{(p+2)^2} \\ &= \frac{2(p^2 - 2)}{(p+2)^2} \end{aligned}$$

Again since $p^2 - 2 > 0$, we get that $q^2 - 2 > 0$, so q is an upper bound of S! $\square$

Here, have some intuition. Consider the interval $[0, \sqrt{2}]$ as a subset of $\mathbb{R}$. I know the least upper bound α has the property that any x smaller than it satisfies $x^2 < 2$ (compare to above) and any x greater than it satisfies $x^2 > 2$. The least upper bound is thus *forced* to satisfy $x^2 = 2$. Since no such number exists in $\mathbb{Q}$, there's a *hole* there.