R is Cauchy Complete

Prereqs RA-07

We will now revisit an important theorem left without proof in RA-07. This is important enough to be seperated from other sequential properties despite being one.

Theorem 1. Every Cauchy sequence in $\mathbb{R}$ converges

We will need two lemmas. Once I state them, it should be immediately clear how the proof follows. If it isn't, you need to revisit RA-08.

Lemma 1. Every Cauchy sequence is bounded

Proof. We'll proceed similar to the proof that every convergent sequence is bounded. Let $\{a_n\}$ be Cauchy and fix $\epsilon_0 = 1$. Then some $N \in \mathbb{N}$ is such that whenever $m, n \geq N$ then $|a_m - a_n| < \epsilon_0$, by definition of Cauchy sequence.

If we fix $n = N$, then for every $m \geq N$ we have $|a_m - a_N| < \epsilon_0$. Thus, if $k \geq N$, then $|a_k| \leq \epsilon_0 + |a_N|$ by triangle inequality. i.e, $a_N, a_{N+1}, a_{N+2} \dots$ are all bounded. Since the remaining terms are all finitely many, they are also bounded. Therefore the entire sequence is bounded. $\square$

Lemma 2. Let $\{a_n\}$ be a Cauchy sequence and let $\{a_{n_k}\}$ be a subsequence such that $a_{n_k} \to L$. Then, $a_n \to L$

Proof. We want to show $\{a_n\}$ converges to L. Fix some $\epsilon > 0$. We apply the definition of Cauchy sequence to $\{a_n\}$ and convergent sequence to $\{a_{n_k}\}$ with $\frac{\epsilon}{2}$.

Since $\{a_n\}$ is Cauchy, there is some N_1 such that whenever $m, n \geq N_1$, we get $|a_n - a_m| < \frac{\epsilon}{2}$

Since a_{n_k} converges, there is some N_2 such that whenever $n_k \geq N_2$, we get $\left|a_{n_k} - L\right| < \frac{\epsilon}{2}$

Take $N_3 = \max\{N_1, N_2\}$. Fix $k > N_3$, then certainly $n_k > N_3$, and for any $n > k$ we get

$$\begin{aligned}
|a_n - L| &= \left|a_n - a_{n_k} - (L - a_{n_k})\right| \\
&\leq \left|a_n - a_{n_k}\right| + \left|L - a_{n_k}\right| \ \text{(triangle inequality)} \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \ \text{(since } n, n_k \geq N_1 \text{ and } a_n \text{ is Cauchy; } n_k \geq N_1 \text{ and } a_{n_k} \text{ converges)} \\
&= \epsilon
\end{aligned}$$

Therefore, if a Cauchy sequence has a convergent subsequence, the original sequence also converges to the same limit. $\square$

Finally, let $\{a_n\}$ be Cauchy, thus it is bounded. Thus it has a convergent subsequence. Thus it converges. $\square$