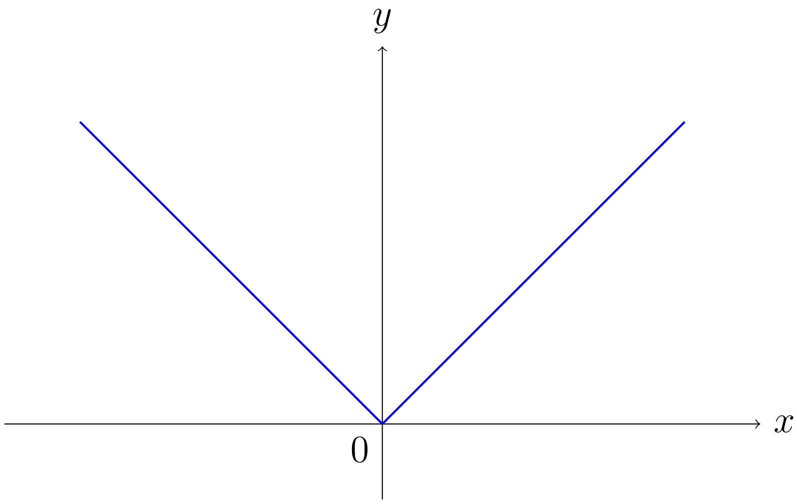
Differentiability of Functions

Prereqs FD-02, RA-11

So far, we've dealt with whether you can draw a function without lifting your pen. Now we ask – can you draw it without snapping your wrist?

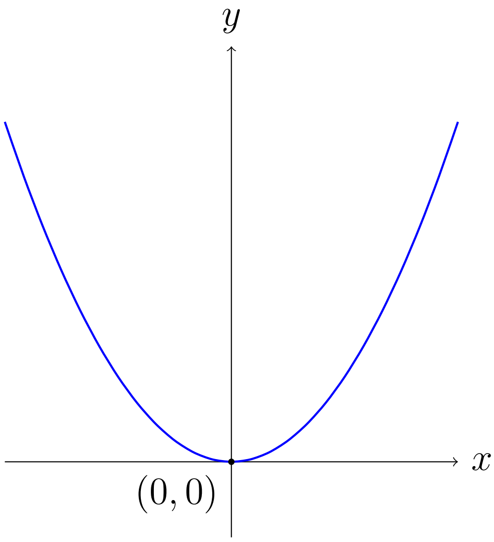
The idea of *differentiability* deals with this question, and is the central idea in understanding velocity, curvature, and continuous optimization. (Differentiability has a *much* different meaning in a several variable context, but is more or less a generalization. Weird how that works, right?).

Consider the graph of $f(x) = |x|$



The pen does not lift at $(0,0)$, but your wrist will snap.

You absolutely can draw this without lifting your pen off the paper, but you'd have to sharply change directions at the origin. Compare that to the curve of $f(x) = x^2$

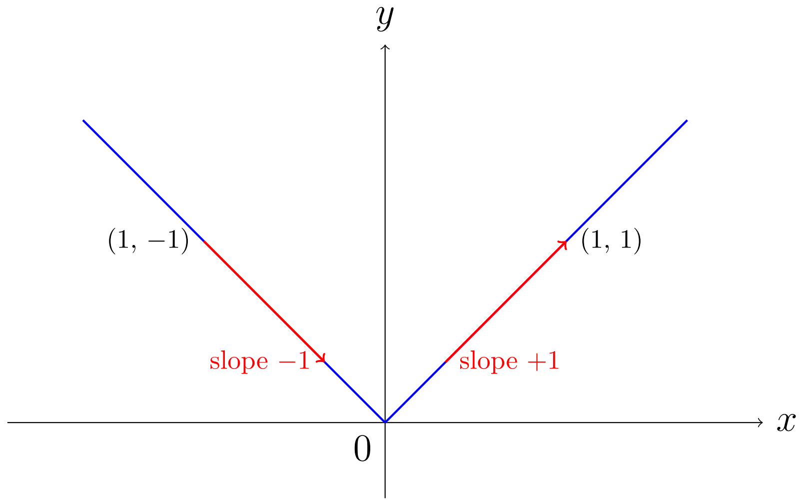


The pen glides smoothly from its downward movement to upward, like butter.

Differentiability, like continuity, is defined at a point. For example, your pen *will* move smoothly when drawing $f(x) = |x|$ at all points other than the origin. So it becomes unfair to say that f is not differentiable, when only one point serves as counterexample.

So how do we quantify this lack of "jerkiness"? We want to capture that at a given point $(x_0, f(x_0))$, the movement of the pen is *smooth*. The way we do this is simple. If the direction in which your pen approaches a point $(x_0, f(x_0))$ is the same as the direction in which it exits, then we consider the movement to be smooth.

Contrast this against $|x|$ at $(0, 0)$. We enter in the direction $(1, -1)$ and exit in the direction $(1, 1)$. This idea is thus intuitively strong enough to form the definition.



We enter in the direction of the vector $(1, -1)$ and exit in $(1, 1)$

Definition 1. A function $f : \Omega \to \mathbb{R}$ with $\Omega \subseteq \mathbb{R}$ is *differentiable at* x_0 if the limits

$$\lim_{h \to 0^+} \frac{f(x_0 + h) - f(x_0)}{h}$$

and

$$\lim_{h \to 0^+} \frac{f(x_0) - f(x_0 - h)}{h}$$

exist and are equal.

Where to quantify direction of exiting (resp. entering) the point, we use the slope of the line between our concerned point and a point slightly ahead (resp. behind) it.

Note that the second limit in the definition is precisely equal to the following limit.

$$\lim_{h \to 0^-} \frac{f(x_0 + h) - f(x_0)}{h}$$

This equality helps us frame a second definition.

Definition 2. A function $f : \Omega \to \mathbb{R}$ with $\Omega \subseteq \mathbb{R}$ is *differentiable at* x_0 if the limit

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

exists.

Of course, we need to justify the restatement.

Proof. (Defn. 1 $\implies$ Defn. 2) Suppose for some f both limits in Defn. 1 exist and are equal at some x_0. Consider the limit in Defn. 2. Take any sequence $h_n \to 0$ with $h_n \neq 0$ for every n and divide it into subsequences h_{n_k} and $h_{n_{k'}}$ with $h_{n_k} > 0$ and $h_{n_{k'}} < 0$. Then by Defn. 1, the sequences

$$\frac{f(x_0 + h_{n_k}) - f(x_0)}{h_{n_k}}$$

and

$$\frac{f(x_0 + h_{n_{k'}}) - f(x_0)}{h_{n_{k'}}}$$

converge and are equal to L (say). Apply the definition of convergence to both these sequences for some ϵ to obtain N_1, N_2.

Now consider the sequence

$$\frac{f(x_0 + h_n) - f(x_0)}{h_n}$$

Whenever $n \geq \max\{N_1, N_2\}$, by convergence of above two sequences (since h_n is either > 0 or < 0, it falls into precisely one of h_{n_k} and $h_{n_{k'}}$) we get that

$$\left| \frac{f(x_0 + h_n) - f(x_0)}{h_n} - L \right| < \epsilon$$

Therefore we obtain

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x)}{h} = L$$

as required. □

(Defn. 2 $\implies$ Defn. 1) We know for any sequence $h_n \to 0$, the limit

$$\lim_{h_n \to 0} \frac{f(x_0 + h_n) - f(x_0)}{h_n} = L$$

Since $h_n \to 0^+$ and $h_n \to 0^-$ both imply $h_n \to 0$, their respective limits of the difference quotient also converge to L. □

Whenever a function f is differentiable at x_0, the limit L of the difference quotient is usually denoted by $f'(x_0)$.

If f' is defined for all $x_0 \in \Omega$, we say f is *differentiable on* Ω and that f' is the *derivative of* f.

Suppose now there is some curve which we can draw smoothly on a set Ω. Does it intuitively make sense that we can draw it without lifting our pen? Of course! Semi-formally, take the contrapositive. If I have to lift my pen to draw it at some point x_0, how can I smoothly draw it at x_0? I can't!

Theorem 1. If $f : \Omega \to \mathbb{R}$ with $\Omega \subseteq \mathbb{R}$ is differentiable at some $x_0 \in \Omega$, then f is continuous at x_0.

Proof. Suppose f is differentiable at x_0. Take any sequence $x_n \to x_0$ with $x_n \neq x_0$ for every n. Then the limit

$$\lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0}$$

exists by definition of differentiability, and is denoted by $f'(x_0)$. Further, we know that $x_n - x_0 \to 0$. Now observe that

$$\begin{aligned}
\lim_{n\to\infty} f(x_n) - f(x_0) &= \lim_{n\to\infty} (x_n - x_0) \cdot \frac{f(x_n) - f(x_0)}{x_n - x_0} && (1) \\
&= \lim_{n\to\infty} (x_n - x_0) \cdot \lim_{n\to\infty} \frac{f(x_n) - f(x_0)}{x_n - x_0} && (2) \\
&= 0 \cdot f'(x_0) && (3) \\
&= 0 && (4)
\end{aligned}$$

Where (1) is valid because $x_n - x_0$ is never 0 by choice of x_n, and we split the limit in (1) into two limits in (2) because both limits exist.

Therefore $f(x_n) \to f(x_0)$ and f is continuous. $\square$

So functions differentiable on some interval (a, b) are also continuous on (a, b). Hence, functions differentiable on (a, b) and continuous on $[a, b]$ have some interesting properties – including the Mean Value Theorems and characterizations of local extrema which we will now look at.

Observe the hesitation in calling a function differentiable on $[a, b]$. We cannot have a function f defined on $[a, b]$ differentiable on a, because the second limit in the original definition (or the "entry slope") will not be defined. Similarly, we cannot have f differentiable on b, because the first limit (or the "exit slope") will not be defined.