Ordered Sets and Upper Bounds

Our discussion of calculus begins at ordered sets. Calculus is done on the real numbers $\mathbb{R}$, and the properties that allow us to do calculus on $\mathbb{R}$ are closely linked with its order properties. Although that will not be a part of our main discussion, it is helpful to start there.

Definition 1. Let S be a set. Let $<$ be a binary relation on S satisfying

- $<$ is antireflexive, i.e for any $a \in S$, $a \not< a$
- $<$ is transitive, i.e if $a < b$ and $b < c$ then $a < c$
- $<$ is total, i.e if $a \neq b$ then either $a < b$ or $b < a$

Under these circumstances, we say S is *totally ordered* and $<$ is an *ordering* on S.

All our familiar sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ are ordered, while a set like $\mathbb{C}$ does not admit a natural ordering. It is, however, possible to define one. Consider two complex numbers $z_1 := x_1 + iy_1$ and $z_2 := x_2 + iy_2$. If $x_1 < x_2$ then declare $z_1 < z_2$, and vice versa. If at all $x_1 = x_2$, then do the same process with y_1 and y_2. If they are also equal, then z_1 and z_2 were already equal. Verify that this is a total ordering on $\mathbb{C}$.

Exercise 1. Show that if $a < b$, then $b \not< a$. (Hint. Suppose for some a and b, both $a < b$ and $b < a$ happen. Use transitivity. Is this allowed?)

Ordered sets are more or less intuitive. We just define a notion of being big or small on a set of objects.

Consider some currency notes. $1, 2, 5, 10$ etc., except 2000. We regard the set of currency notes $S = \{1, 2, 5, 10, 20, 50, 100, 200, 500\}$ as a subset of $\mathbb{N}$, and 2000 as an element of $\mathbb{N}$.

Observe that for every $s \in S$, $s < 2000$ holds. In this case we say that 2000 is an *upper bound* of S. In fact, $501, 502, 503 \ldots$ are all upper bounds of S. A natural question at this stage is that "is 500 an upper bound of S?". The answer is yes, and we will understand why shortly.

For now, familiarise yourself with this definition

Definition 2. Let S be a totally ordered set with ordering $<$, and let $E \subset S$ be a **nonempty** subset of S. Let $s \in S$ be such that for every $x \in E$, $x \leq s$. Then we say s is an *upper bound* of E. We also say E is *bounded above* by s.

Note the use of $\leq$ in the definition. This is what allows 500 to be an upper bound of the set of currency notes except 2000. Why we allow this is something we shall explore later.

Answers to Exercises

The following are brief solutions or hints. You are encouraged to review the exercises before checking the answers.

Answer 1. If $a < b$ and $b < a$, then by transitivity $a < a$, which cannot happen.