Real Valued Sequences - I

Prereqs INT-01

When we refer to real-valued sequences, we usually mean sequences indexed by $\mathbb{N}$. Therefore, specifically for real analysis, the following definition is standard.

Definition 1. A *sequence* is a function $f : \mathbb{N} \to \mathbb{R}$

A favorite example is the arithmetic sequences

$$f(n) = a + (n-1)d$$

whose terms are usually denoted by a_n. Observe that it is not necessary that the function f can be written in a nice clean form. As long as it is defined, f forms a sequence. For example, consider the sequence

$$1, 6, 1, 8, 0, 3, 3, 9, 8, 8, 7, 4, 9 \ldots$$

Do you see a pattern? Maybe you do recognise what the sequence is, but I assure you, there is no *pattern* per se. Regardless, the sequence can still be listed via a function $f : \mathbb{N} \to \mathbb{R}$ if we just let

$$f(1) = 1, f(2) = 6, f(3) = 1 \ldots$$

and so on. Any list of real numbers defines a real-valued sequence.

How about sequence of a sequence? say I have an arithmetic progression

$$1, 4, 7, 10, 13, 16, 19 \ldots$$

Let these be $a_1, a_2, a_3 \ldots$ respectively. If we then take only the even-indexed terms $a_2, a_4 \ldots$ we get the sequence

$$4, 10, 16 \ldots$$

which is again an arithmetic progression and therefore a sequence. Here, since the second sequence is taken from the first, we call it a *subsequence.*

Definition 2. Let $f : \mathbb{N} \to \mathbb{R}$ be a sequence, and $g : \mathbb{N} \to \mathbb{N}$ a monotone increasing function. Then, $f \circ g : \mathbb{N} \to \mathbb{R}$ is a *subsequence* of f

In the above example, $f(n) = 3n - 2$ which describes the sequence, and $g(n) = 2n$ which describes the *sequence of indices of f we want to take.* Indeed,

$$f \circ g(n) = f(2n) = 6n - 2$$

which describes the subsequence.