Well Ordering Theorem

Prereqs FD-IND

Here we will have a look at an interesting property of the natural numbers. As an example, consider the sets $[0, 1], (1, 2)$, and $(-\infty, 0)$ in $\mathbb{R}$

The first set admits a minimum element – 0. The second set admits a *greatest lower bound* – 1. Finally the third set is unbounded below.

Take any nonempty subset of the natural numbers, say $\{3n \mid n \in \mathbb{N}\}$. This has a minimum element, 3. Keep trying nonempty subsets until you build an intuition for the following statement.

Theorem 1. Every nonempty subset of $\mathbb{N}$ has a minimum element

We will present a proof in two parts. First, using induction, we will prove the property for finite nonempty subsets. Then we will generalise to all subsets.

Lemma 1. Every nonempty finite subset of $\mathbb{N}$ has a least element

Proof. We proceed by induction on the number of elements of a given finite $S \subset \mathbb{N}$. If S has one element, i.e $S = \{m\}$, then m is the least element of S.

If S has 2 elements, then by totality of $<$ one of them must be minimum in S.

Now suppose for some k, every subset S of size k has a least element, and let S' be a subset of size $k + 1$. Indeed since S' is nonempty, pick some $m' \in S'$ and consider $T := S' \setminus \{m'\}$. T is therefore a subset of size k, and has a minimum element m by induction hypothesis.

Now finally $\{m, m'\}$ is a subset of $\mathbb{N}$ of size 2 and again admits a minimum. This is the required minimum of S. $\square$

Essentially, our argument boiled down to

$$\min(\text{set of size } k + 1) = \min(\text{set of size } k \text{ obtained by removing some } m, m)$$

Now we are in position to prove the infinite case.

Proof. (Of Theorem 1.) Let $S \subset \mathbb{N}$ be nonempty. If S is finite we are done. Otherwise, take $k \in S$ and consider the set

$$\begin{aligned} S' &= \{n \in S \mid n \leq k\} \\ &= S \cap \{n \in \mathbb{N} \mid n \leq k\} \end{aligned}$$

i.e, those natural numbers which are smaller than or equal to k and belong to S. S' is nonempty because $k \in S'$. Further, S' is finite. Thus S' has a minimum element m. This is also the minimum element of S. $\square$

We can in fact go the other way. The principle of mathematical induction states that let $P(k)$ be a statement depending on the natural number k. If $P(0)$ is true and $P(k) \implies P(k+1)$ for every k, then $P(n)$ is true for every n.

Suppose the Well Ordering Theorem is true. Suppose $P(0)$ is true, and $P(k) \implies P(k+1)$. Can you show that $P(n)$ is true for every n?

Exercise 1. Show that Well Ordering Theorem implies Principle of Mathematical Induction (Hint. Use the above outline. Suppose that $P(k)$ is false for some $k > 0$, then the set S of natural numbers for which P is false is nonempty. Now what?)

Indeed, Mathematical Induction and Well Ordering Theorem are **equivalent**. Neither of these can be proved without each other, and for discrete math we take these to be axioms.

Answers to Exercises

The following are brief solutions or hints. You are encouraged to review the exercises before checking the answers.

Answer 1. Let S be the set $\{k \in \mathbb{N} \mid P(k) \text{ is false}\}$. Suppose for contradiction that S is nonempty, so it admits a minimum m. We know $m \neq 0$ since we assumed $P(0)$ to be true. i.e, $m \geq 1$ or $m - 1 \geq 0$ and thus $m - 1$ is also natural. We know m is the smallest k for which $P(k)$ is false, therefore $P(m-1)$ must be true. But we assumed $P(k) \implies P(k+1)$ so $P(m)$ must also be true! S must be an empty set. $\square$