Definitions of Limits

Here we'll look at the definitions of statements of the form

$$\lim_{x \to x_0} f(x) = L$$

Where x varies over the real numbers (or some subset), and x_0 and L are either real numbers or $\pm\infty$

The idea behind the definitions is simple. We'll essentially say "however close you want $f(x)$ to be to L, that is achievable, provided x is closer to x_0 than some threshold".

In the cases where x_0 and L are ∞, being sufficiently close to ∞ is replaced by being sufficiently large. Let's see.

Definition 1. Let $x_0, L \in \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ with $\Omega \subseteq \mathbb{R}$. Then, we say

$$\lim_{x \to x_0} f(x) = L$$

if for every $\epsilon > 0$, there is some $\delta > 0$ such that whenever

$$x \in (x_0 - \delta, x_0 + \delta)$$

then

$$f(x) \in (L - \epsilon, L + \epsilon)$$

Now let us see what happens when the independent variable is infinite.

Definition 2. Let $L \in \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ with $\Omega \subseteq \mathbb{R}$. Then, we say

$$\lim_{x \to \infty} f(x) = L$$

if for every $\epsilon > 0$, there exists $M > 0$ in $\mathbb{R}$ such that whenever

$$x > M$$

then

$$f(x) \in (L - \epsilon, L + \epsilon)$$

Here, x "getting closer to" ∞ is measured by letting x be larger than an arbitrary number. Of course, $x \to -\infty$ can be similarly defined.

Definition 3. Let $x_0 \in \mathbb{R}$ and $f : \Omega \to \mathbb{R}$ with $\Omega \subseteq \mathbb{R}$. Then, we say

$$\lim_{x \to x_0} f(x) = \infty$$

if for every $N > 0$ there is some $\delta > 0$ such that whenever

$$x \in (x_0 - \delta, x_0 + \delta)$$

then

$$f(x) > N$$

Finally, we state the definition where both variables go to ∞.

Definition 4. Let $f : \Omega \to \mathbb{R}$ with $\Omega \subseteq R$. We say

$$\lim_{x\to\infty} f(x) = \infty$$

if for every $N > 0$, there is some $M > 0$ such that whenever

$$x > M$$

then

$$f(x) > N$$

Observe that every single one of these definitions can be replaced with a sequential counterpart. x_0, L, and f as above, note that the following definitions are respective equivalents of those stated above.

Definition 5. We say that

$$\lim_{x\to x_0} f(x) = L$$

if for every $x_n \to x_0$, $f(x_n) \to L$

Definition 6. We say that

$$\lim_{x\to\infty} f(x) = L$$

if for every $x_n \to \infty$, $f(x_n) \to L$

Definition 7. We say that

$$\lim_{x\to x_0} f(x) = \infty$$

if for every $x_n \to x_0$, $f(x_n) \to \infty$

Definition 8. We say that

$$\lim_{x\to\infty} f(x) = \infty$$

if for every $x_n \to \infty$, $f(x_n) \to \infty$