Sequential Properties of R

Prereqs RA-06, RA-07

We will now move forward to some properties of real-valued sequences. It is worth noting these properties are closely related to the completeness axiom, as you can find in RA-XX

We will first look at a useful theorem.

Theorem 1. (Nested Intervals) Let $\{I_n\}_n$ be a sequence of intervals $I_n = [a_n, b_n]$ such that $I_{n+1} \subset I_n$ and $b_n - a_n \to 0$. Then, there is some x such that $\cap I_n = \{x\}$

Proof. Observe that in other words, if we have a decreasing sequence of intervals whose lengths coverge to 0, then the intersection of these intervals has one and only one point. Yet again, in the rationals, where this point is supposed to be might have a hole.

First we show that the intersection is nonempty. Consider the sequences of the left and right endpoints of the intervals, viz. $\{a_n\}_n$ and $\{b_n\}_n$. Notice that $\{a_n\}$ is monotone increasing, and $\{b_n\}$ is monotone decreasing. (that is, $a_1 \leq a_2 \leq a_3 \leq \dots$ and $\cdots \leq b_3 \leq b_2 \leq b_1$)

We also know that $a_k \leq b_k$ for every fixed k. Thus for any m we have

$$a_1 \leq a_m \leq b_m \leq b_1$$

That is, $\{a_n\}$ is bounded above by b_1 and $\{b_n\}$ is bounded below by a_1.

By monotone convergence theorem, $a_n \to A$ and $b_n \to B$. We know $b_n - a_n \to 0$ so $B - A = 0$, i.e. $B = A =: L$ (say). We claim that this L lies in the intersection and is the only point that does.

To show that L lies in the intersection is straightforward. We need to show $a_k \leq L \leq b_k$ for every k.

From the proof of monotone convergence theorem (see RA-07) we know L is the least upper bound of $\{a_n\}$ and thus $a_k \leq L$ for all k. Similarly L is also the greatest lower bound of $\{b_n\}$ and thus $L \leq b_k$ for every k.

Now suppose some $L + \epsilon$ lies in the intersection with $\epsilon > 0$ (we can proceed in a similar manner for $\epsilon < 0$). If it lies in the intersection, then $L + \epsilon \leq b_k$ for every k, but then that would make $L + \epsilon$ a lower bound of $\{b_n\}$, when L was the greatest lower bound. We have a contradiction!$\rightarrow\!\leftarrow$

Therefore

$$\bigcap_{n\in\mathbb{N}} I_n = \{L\}$$

as required. $\square$

We will now have a look at another interesting property of $\mathbb{R}$.

Theorem 2. (Bolzano-Weierstrass) Every bounded sequence has a convergent subsequence

The theorem for now will be treated as a direct consequence of the following theorem, however it admits another proof using Nested Intervals which I really encourage you to read (again, it will be in RA-XX).

Theorem 3. (Monotone Subsequence) Every sequence has a monotone subsequence

Proof. Consider a sequence $\{a_n\}$. We call a term $\{a_k\}$ a *peak* if $a_{k+1}, a_{k+2} \dots$ are all smaller than a_k.

Suppose first that our sequence has infinitely many peaks, $a_{n_1}, a_{n_2} \dots$ and so on. Since a_{n_1} is a peak, we have $a_{n_1} \geq a_{n_2}$. Since a_{n_2} is a peak, we have $a_{n_2} \geq a_{n_3}$. Inductively, we obtain

$$a_{n_1} \geq a_{n_2} \geq a_{n_3} \geq \dots$$

and hence a monotone decreasing subsequence.

Now suppose otherwise that there are finitely many peaks with the last being at a_N (if there are no peaks, consider the last peak to be at a_0)

Consider the term $a_{n_1} := a_{N+1}$. This is not a peak since it is after the last peak. Thus there is some term a_{n_2} after it such that $a_{n_2} \geq a_{n_1}$.

But again, a_{n_2} is not a peak, so there is some a_{n_3} after it such that $a_{n_3} \geq a_{n_2}$. Inductively, we get

$$a_{n_1} \leq a_{n_2} \leq a_{n_3} \leq \dots$$

and hence a monotone increasing subsequence. $\square$

Corollary 1. Bolzano-Weierstrass Theorem

Proof. Let $\{a_n\}$ be bounded. By monotone subsequence theorem, let $\{a_{n_k}\}$ be a monotone subsequence. This is also bounded. By monotone convergence theorem, a_{n_k} converges. $\square$