Series I

Prereqs RA-07

A series (or specifically, an infinite series) is essentially a sum of countably many terms. The axioms of $\mathbb{R}$ only define a sum of finitely many terms. Through series, we seek to define a sum of infinitely many terms.

Let a_n be a sequence. Then we define a new sequence, S_n, given by

$$S_n = \sum_{m=1}^{n} a_m$$

and we define the infinite sum as the limit of S_n.

$$\sum_{m=1}^{\infty} a_m := \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{m=1}^{n} a_m$$

In other words, the infinite sum

$$a_1 + a_2 + a_3 + \dots$$

is defined as the limit of the sequence

$$a_1, a_1 + a_2, a_1 + a_2 + a_3 \dots$$

which, at first glance, seems to make sense. And it does! If you want to evaluate an infinite sum by hand, this is the exact thing you will do. You'll take a_1, and then add a_2 (to obtain S_2), and then add a_3 (to obtain S_3) and so on until you get an overall idea of what the sum is "doing".

Definition 1. A series $\sum a_n$ *converges* if $S_n \to S$ for some S. The series *diverges* if $S_n \to \pm\infty$.

Example 1. Let $a_n = \dfrac{1}{2^n}$. Then the series $\sum a_n$ converges to 1. Let $b_n = n$. Then the series $\sum b_n$ diverges. Finally let $c_n = (-1)^n$. Then the series $\sum c_n$ neither converges nor diverges.

When can we say that a series converges? Unfortunately, the answer is not straightforward. It is easy to sniff out when a series *doesn't* converge. Suppose $a_n \not\to 0$ as $n \to \infty$. Can $\sum a_n$ converge?

As a handwavy proof, suppose $\sum a_n$ converges, i.e. the sequence S_n converges to some limit S. Consider a new sequence defined by $T_n = S_{n+1}$.

Exercise 1. Does T_n converge? Where? What is $T_n - S_n$ equal to? Does this converge? Where?

Theorem 1. If a series $\sum a_n$ converges, then $a_n \to 0$.

This condition is only *necessary*, not sufficient. What that means is, even if $a_n \to 0$, it may be the case that S_n doesn't converge.

Example 2. Consider $a_n = \dfrac{1}{n}$. Then $\sum a_n$ doesn't converge. Write the (2^k)-th partial sums as follows

$$\begin{aligned}
\sum \frac{1}{n} &= 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) + \dots \\
&\geq 1 + \left(\frac{1}{2}\right) + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) + \dots \\
&= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \dots \\
&= 1 + \sum_{n=1}^{\infty} \frac{1}{2} \\
&= \infty
\end{aligned}$$

We will revisit this in the Condensation test and the Rearrangement Theorem.

It is in general hard to fully characterise converging series. An important class of examples are the geometric series $a_n = ar^{n-1}$ with first term $a \neq 0$ and common ratio $r \neq 0$. (The identically 0 series trivially converge).

Theorem 2. Let a_n be a geometric series with $a_2 \neq 0$ (neither of a or r is 0). Then, $\sum a_n$ converges if and only if $|r| < 1$.

Proof. ($\rightarrow$) Suppose $\sum a_n$ converges. Then $a_n \to 0$, or $ar^{n-1} \to 0$. Since a and r are both $\neq 0$, we must have $r^{n-1} \to 0$ and therefore $|r| < 1$. For if $|r| \geq 1$ then $|r|^{n-1} \geq 1$ and in this case r^{n-1} cannot converge to 0.

($\leftarrow$) Now suppose for contrapositive that $|r| \geq 1$. As before, in this case $a_n \not\to 0$ and thus $\sum a_n$ doesn't converge. $\square$

So we have our first test for convergence – geometric series with ratios whose absolute value is smaller than 1 converge.

This result is critical in proving the ratio test, a strong sufficient condition for convergence. We will look at that and other tests in RA-19.

Answers to Exercises

The following are brief solutions or hints. You are encouraged to review the exercises before checking the answers.

Answer 1. Yes, T_n also converges to S. $T_n - S_n$ is equal to a_{n+1} by definition. Also, $T_n - S_n$ converges to $S - S = 0$. Thus a_{n+1} (and thus a_n) converges to 0.