Real Valued Sequences II

Prereqs RA-06, INT-02

In INT-02 we establish a definition of convergence of a sequence which I reiterate here.

Definition 1. A sequence a_n *converges to* L (written $a_n \to L$) if given any $\epsilon > 0$, there is some $N \in \mathbb{N}$ such that $|L - a_k| < \epsilon$ for $k = N+1, N+2, N+3 \ldots$

We will now be tasked with some necessary and sufficient conditions on when a sequence converges.

Definition 2. Say a sequence $\{a_n\}_n$ is *bounded* if there is some $M > 0$ such that $|a_n| < M$ for every n

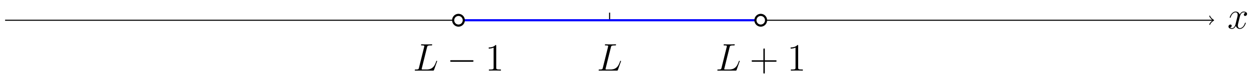
Theorem 1. Every convergent sequence is bounded.

Proof. Let $a_n \to L$. We want to show there is some fixed M such that $|a_n| < M$ for every n. Since $a_n \to L$, by definition for $\epsilon = 1$, there is some N such that all of $a_{N+1}, a_{N+2}, a_{N+3} \ldots$ lie within a distance of 1 from L. Thus, the maximum value of $|a_{N+k}|$ is $|L| + 1$.

So we have dealt with all the terms a_{N+k}, and what remains are the terms $\{a_1, a_2, a_3 \ldots a_N\}$. But this is a finite set. Thus if we let

$$M = \max\{|a_1|, |a_2|, |a_3| \ldots |a_N|, |L| + 1\}$$

Then indeed, $|a_n| < M$ for every n. $\square$



All a_{N+k} lie within the blue region and are therefore bounded
$a_1 \ldots a_N$ are finitely many terms and are therefore bounded
Hence the entire sequence is bounded

In general a bounded sequence does not converge, take for example $a_n = (-1)^n$. However, with one extra condition, we get a sufficient condition for convergence.

Definition 3. A sequence is *monotone increasing* if $a_n \leq a_{n+1}$ for every n. A sequence is *monotone decreasing* if $a_n \geq a_{n+1}$ for every n. A sequence is called *monotone* if it is either monotone decreasing or monotone increasing.

Theorem 2. (Monotone Convergence) A bounded monotone sequence converges.

Proof. WLG we will assume a_n is a monotone increasing sequence, the proof for decreasing sequences is similar. Consider the set $\{a_n \mid n \in \mathbb{N}\}$. The set is both nonempty and bounded, therefore it has a least upper bound, say α.

Let $\epsilon > 0$ be given. Since α is the least upper bound, $\alpha - \epsilon$ can not be an upper bound and therefore, for some n, we have

$$\alpha - \epsilon < a_n \leq \alpha$$

Now use that a_n is monotone and α is an upper bound to observe that

$$\alpha - \epsilon < a_n \leq a_{n+1} \leq a_{n+2} \leq a_{n+3} \leq \cdots \leq \alpha$$

That is, for every $n' \geq n$ we have $|\alpha - a_{n'}| < \epsilon$ and thus $a_n \to \alpha$ □

Definition 4. A sequence a_n is called a *Cauchy sequence* if for every $\epsilon > 0$, there is some $N \in \mathbb{N}$, such thaat whenever $m, n \geq N$ then $|a_m - a_n| < \epsilon$

In other words, a sequence is Cauchy if the points get arbitrarily close to *each other*, rather than just some fixed L as is in convergence. One fact is immediately clear.

Theorem 3. Every convergent sequence is Cauchy

Proof. Let $a_n \to L$, and let $\epsilon > 0$. Then, for some N, $n \geq N \implies |a_n - L| < \frac{\epsilon}{2}$ by definition of convergence. If m, n are greater than N, then

$$\begin{aligned}
|a_m - a_n| &= \big|a_m - L - (a_n - L)\big| \\
&\leq |a_m - L| + |a_n - L| \\
&\leq \frac{\epsilon}{2} + \frac{\epsilon}{2} \\
&= \epsilon
\end{aligned}$$

Therefore a_n is Cauchy. □

It is a subjective matter whether or not you find the following fact surprising, which (for now) I present without proof.

Theorem 4. Every Cauchy sequence converges in $\mathbb{R}$

The proof uses the completeness of $\mathbb{R}$ and thus cannot be recreated for $\mathbb{Q}$. In fact, we know a Cauchy sequence in $\mathbb{Q}$ that does not converge. Consider

$$3, 3.1, 3.14, 3.141, 3.1415, 3.14159, 3.141592, 3.1415926\ldots$$

where a_n is given by truncating π to $n-1$ digits after the decimal. We know the sequence converges in $\mathbb{R}$, therefore it is Cauchy in $\mathbb{R}$, but all its terms belong to $\mathbb{Q}$ so it is also Cauchy in $\mathbb{Q}$; however it does not converge in $\mathbb{Q}$ since π is not in $\mathbb{Q}$.