Convergence is Simple

Consider the following limit

$$L := \lim_{n\to\infty} 1 - \frac{1}{n}$$

Indeed, $L = 1$. Here, we say that *the sequence* $1 - \frac{1}{n}$ *converges to* 1. Intuitively, we know what convergence of a sequence should mean, but how do we define it?

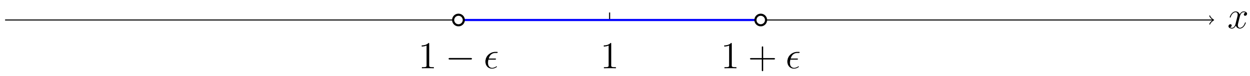
We want to quantify a simple idea. If we say $a_n \to L$ as $n \to \infty$ it should be if and only if a_n gets closer and closer to L as $n \to \infty$.

That in and of itself isn't enough. For starters, the sequence $1 - \frac{1}{n}$ also gets closer and closer to, say, 1.5 as $n \to \infty$. If you further try some shenanigans along the lines of "as $n \to \infty$, a_n should *go towards* L", that doesn't work either, because the sequence $\frac{(-1)^n}{n}$ doesn't go anywhere, it alternates and eventually diminishes.

Let's sit down for a conversation, shall we? You claim that the sequence $1-\frac{1}{n}$ converges to 1. I ask you what that means. You say "well, the sequence gets closer and closer to 1 as $n \to \infty$". I ask you how close to 1 does the sequence get, really?

Now you do some thinking. How close? Eventually, you answer - "as close as you want". I'm slightly taken aback, but I decide to test your claim. Does the sequence ever land within 0.5 of 1? Sure, just take $n > 2$. Within 0.33? Sure, take $n > 3$. Within 0.25? Sure, take $n > 4$, and so on.

Eventually I get frustrated and give you some arbitrary number ϵ and ask you - does your sequence get closer than ϵ to 1?



Does the sequence fall within ϵ of 1?

Now you do some thinking. You want $1 - a_n < \epsilon$, or $\frac{1}{n} < \epsilon$. Can you guarantee this for a large enough n? Yes! We use the Archimedean Property of $\mathbb{R}$ to guarantee that there is some natural number N such that

$$N > \frac{1}{\epsilon}$$

and therefore for every $n \geq N$, we have

$$\epsilon > \frac{1}{n}$$

Thus whenever $n > \frac{1}{\epsilon}$, the term a_n falls within ϵ distance of 1.

A subtle but important feature of this formulation is this – let ϵ be some fixed number. Then, there is some fixed N such that ALL of $a_N, a_{N+1}, a_{N+2} \ldots$ fall within ϵ of 1

It is necessary for us to impose this condition that all values beyond a fixed N fall within the stipulated distance of the limit. This and only this captures in the truest sense what it means for a sequence to get closer and closer to a number.

For a nonexample, consider the sequence

$$-1, 1, -1, 1 \dots$$

where the general term is given by $(-1)^n$. Here, *infinitely* many terms of the sequence are arbitrarily close to 1 (all terms of the form a_{2n} are equal to 1 and hence their distance from n is 0, which in smaller than ϵ for every positive ϵ). However, would you say that $a_n \to 1$? Probably not. Does our intuition capture this?

Remember, we want that given some $\epsilon > 0$, there should be some N such that $a_{N+1}, a_{N+2}, a_{N+3} \dots$ all fall within ϵ of the limit. If 1 was the limit and we take $\epsilon = 0.5$, then the distance between 1 and either a_{N+1} or a_{N+2} wil be 2 (one of them is guaranteed to be -1). Since $\epsilon = 0.5$, the sequence fails to fall within ϵ of 1, therefore it doesn't converge to 1.

We are ready to formalise the definition

Definition 1. Say a sequence a_n *converges to* $L \in \mathbb{R}$ if and only if the following holds:

Given some $\epsilon > 0$, there is some N such that ignoring the first N terms of the sequence, all other terms lie within ϵ distance of L.

i.e., for every $\epsilon > 0$, there is some $N \in \mathbb{N}$, such that $|L - a_n| < \epsilon$ for every $n = N + 1, N + 2, N + 3 \dots$