Taylor's Theorem II

Prereqs RA-16

The information about how a function f twists and turns near a point x_0 is stored in the derivatives $f'(x_0), f''(x_0) \dots f^{(k)}(x_0) \dots$ if they exist. Recall our second-order Extended Lagrange approximation

$$f(x) \approx f(a) + (x-a) \cdot f'(a) + \frac{(x-a)^2}{2} \cdot f''(a)$$

And observe that the RHS is a $2-$degree polynomial.

$$P_2(x) = f(a) + (x-a) \cdot f'(a) + \frac{(x-a)^2}{2} \cdot f''(a)$$

Here's what makes P_2 interesting. The polynomial and its derivatives upto order 2 agree with f. Note that

$$\begin{aligned} P_2(x) &= f(a) + (x-a) \cdot f'(a) + \frac{(x-a)^2}{2} \cdot f''(a) \\ P_2'(x) &= f'(a) + (x-a) \cdot f''(a) \\ P_2''(x) &= f''(a) \end{aligned}$$

Hence at the point $x = a$,

$$\begin{aligned} P_2(a) &= f(a) \\ P_2'(a) &= f'(a) \\ P_2''(a) &= f''(a) \end{aligned}$$

i.e., the $0-$th, $1-$st and $2-$nd order curvatures of P_2 at a agree precisely with that of f, so long as f admits those derivatives.

We're ready to ask the general question. Suppose f is $k-$times differentiable. Does f admit a $k-$degree polynomial P_k such that

$$f^{(m)}(a) = P_k^{(m)}(a)$$

for every $0 \leq m \leq k$? The answer is yes, and the construction is outlined below.

Suppose the following form for the required polynomial.

$$P_k(x) = a_0 + a_1 \cdot (x-a) + a_2 \cdot (x-a)^2 + \dots + a_k \cdot (x-a)^k$$

$P_k(a) = f(a)$ immediately forces

$$a_0 = f(a)$$

Now consider some m with $1 \leq m \leq k$ and consider $P_k^{(m)}(x)$. To know what $P_k^{(m)}$ looks like, it suffices to find out what happens to the term $a_n(x-a)^n$ when it is differentiated m times.

It is a rather simple verification that

$$(a_n(x-a)^n)^{(m)} = a_n \cdot n \cdot (n-1) \dots (n+1-m)(x-a)^{n-m}$$

Which simplifies as follows

$$\begin{aligned} n < m &: \; 0 \\ n = m &: \; a_m \cdot m! \\ n > m &: \; \frac{a_n \cdot n!}{(n-m)!}(x-a)^{n-m} \end{aligned}$$

Therefore $P_k^{(m)}$ is given by

$$P_k^{(m)}(x) = a_m \cdot m! + \sum_{n=m+1}^{k} \frac{a_n \cdot n!}{(n-m)!}(x-a)^{n-m}$$

which gives

$$P_k^{(m)}(a) = a_m \cdot m!$$

Here we enforce $f^{(m)}(a) = P_k^{(m)}(a)$. Hence,

$$a_m = \frac{f^{(m)}(a)}{m!}$$

Thus finally we obtain $P_k^{(m)}$

$$\begin{aligned} P_k^{(m)}(x) &= f(a) + (x-a) \cdot f'(a) + \frac{(x-a)^2}{2!} \cdot f''(a) + \dots \\ &= \sum_{n=0}^{k} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n \end{aligned}$$

Do you now see why there was a 2 in the denominator alongside h^2 in RA-16?

Now we ask the main question. Suppose f is differentiable k times near a. Then it admits a natural approximation near a in the form of $P_k(x)$. We know that degree 1 and degree 2 approximations of this kind are in fact precise, if we just adjust the last term. Is this true for a general $k-$degree approximation?

The answer is in the affirmative, and the proof mimics that of Extended Lagrange in RA-16. The precise formulation is provided below.

Theorem 1. (Taylor-Lagrange) Let $f : [a,b] \to \mathbb{R}$ be $k-$times differentiable on (a,b), and let $f, f', f'' \dots f^{(k-1)}$ be continuous on $[a,b]$. Then there is $c \in (a,b)$ such that

$$f(b) = f(a) + (b-a) \cdot f'(a) + \dots + \frac{(b-a)^{k-1}}{(k-1)!} \cdot f^{(k-1)}(a) + \frac{(b-a)^k}{k!} \cdot f^{(k)}(c)$$

Proof. We will mimic the proof provided in RA-16.

Define

$$F(x) = f(b) - f(x) - (b-x)\cdot f'(x) - \ldots - \frac{(b-x)^{k-1}}{(k-1)!}\cdot f^{(k-1)}(x)$$
$$= f(b) - f(x) - \sum_{n=1}^{k-1} \frac{(b-x)^n}{n!}\cdot f^{(n)}(x)$$

Now to take the derivative

$$F'(x) = -f'(x) - \sum_{n=1}^{k-1} \left(\frac{(b-x)^n}{n!}\cdot f^{(n)}(x)\right)'$$

we need to use the product rule for the second term. The $n-$th term breaks into

$$\frac{(b-x)^n}{n!}\cdot f^{(n+1)}(x) - \frac{(b-x)^{n-1}}{(n-1)!}\cdot f^{(n)}(x)$$

where the $-$ sign shows up because of the chain rule on the $(b-x)$ term. The derivative then becomes

$$F'(x) = -f'(x) - \left(\sum_{n=1}^{k-1} \frac{(b-x)^n}{n!}\cdot f^{(n+1)}(x)\right) + \left(\sum_{n=1}^{k-1} \frac{(b-x)^{n-1}}{(n-1)!}\cdot f^{(n)}(x)\right)$$

The sharp eyed will notice that a reindexing on the first sum will simplify the expression.

$$F'(x) = -f'(x) - \left(\sum_{n=2}^{k} \frac{(b-x)^{n-1}}{(n-1)!}\cdot f^{(n)}(x)\right) + \left(\sum_{n=1}^{k-1} \frac{(b-x)^{n-1}}{(n-1)!}\cdot f^{(n)}(x)\right)$$

Only the $n = k$ term in the first sum and the $n = 1$ term in the second sum survive, and the rest get cancelled. Thus

$$F'(x) = -f'(x) - \frac{(b-x)^{k-1}}{(k-1)!}\cdot f^{(k)}(x) + f'(x)$$
$$= -\frac{(b-x)^{k-1}}{(k-1)!}\cdot f^{(k)}(x)$$

As before, $F(b) = 0$. Now if we let

$$g(x) = F(x) - \left(\frac{b-x}{b-a}\right)^k \cdot F(a)$$

Then $g(a) = g(b) = 0$. Thus by Rolle's Theorem $g'(c) = 0$ for some $c \in (a, b)$. Differentiating g we get

$$g'(x) = F'(x) + k\cdot\frac{(b-x)^{k-1}}{(b-a)^k}\cdot F(a)$$

Substituting $g'(c) = 0$ and $F'(x)$ from above,

$$0 = -\frac{(b-c)^{k-1}}{(k-1)!} \cdot f^{(k)}(c) + k \cdot \frac{(b-c)^{k-1}}{(b-a)^k} \cdot F(a)$$

Transposing,

$$\frac{(b-c)^{k-1}}{(k-1)!} \cdot f^{(k)}(c) = k \cdot \frac{(b-c)^{k-1}}{(b-a)^k} \cdot F(a)$$

Since $c \in (a, b)$, $b-c \neq 0$ and we can safely remove it. We also know $k \cdot (k-1)! = k!$. Thus,

$$F(a) = \frac{(b-a)^k}{k!} \cdot f^{(k)}(c)$$

Substituting back $F(a)$ from its definition,

$$f(b) = f(a) + (b-a) \cdot f'(a) + \ldots + \frac{(b-a)^{k-1}}{(k-1)!} \cdot f^{(k-1)}(a) + \frac{(b-a)^k}{k!} \cdot f^{(k)}(c)$$

as stipulated. $\square$

Therefore, *any* k times differentiable function admits a natural $k-$degree polynomial approximation near a point of differentiability

$$f(a+h) \approx f(a) + h \cdot f'(a) + \ldots + \frac{h^k}{k!} f^{(k)}(a)$$

and the approximation is made precise by just adjusting the last term.

$$f(a+h) = f(a) + h \cdot f'(a) + \ldots + \frac{h^k}{k!} f^{(k)}(c)$$