Least Upper Bounds and Completeness

Prereqs RA-01

Consider the set $[0, 1] \subset \mathbb{R}$ and answer these questions

- Is this set bounded above?
- Is 1 an upper bound?
- Is anything greater than 1 an upper bound?
- Is anything smaller than 1 an upper bound?

Answers

- Yes
- Yes, 1 is an upper bound as everything in the subset is ≤ 1
- Yes, in fact every number ≥ 1 is an upper bound. Let y be such that $1 \leq y$. Let $x \in [0, 1]$. Since 1 is an upper bound we get $x \leq 1$. Chaining with $1 \leq y$ we get $x \leq y$ so that y is an upper bound.
- No. In fact, nothing smaller than 1 is an upper bound. Let $x < 1$. But $1 \in S$. Thus the definition that x is an upper bound of S if $x \geq s$ for every element s of S fails at $s = 1$.

If we replace S to be the set $[0, 1)$ instead, you'll find the answers remain the same, just that the reasoning for the last one has to be slightly different, because 1 is no longer in S. Now we'll say let $x < 1$. If at all $x < 0$ then x clearly cannot be an upper bound. If instead $0 \leq x < 1$, then the number $\frac{1+x}{2}$ is larger than x and lies in $[0, 1)$. Thus x is not an upper bound for any $x < 1$.

1 here has a special property - that no number smaller than it is an upper bound of the set S. We'll look at this in more detail with another example.

Revisit the currency notes example, $S = \{1, 2, 5, 10, 20, 50, 100, 200, 500\}$. Answer the questions

- Is S bounded above?
- Is 500 an upper bound?
- Is anything greater than 500 an upper bound?
- Is anything smaller than 500 an upper bound?

Answers

- Yes
- Yes

- Yes, everything greater than 500 is
- No, nothing smaller than 500 is

500 also appears to enjoy this peculiar property.

There is nothing particular about 1 and 500 here. These numbers exist because of an interesting property of the underlying sets ($\mathbb{N}$ and $\mathbb{R}$) themselves. We took a nonempty bounded set, and produced a number α such that nothing smaller than α is an upper bound. It makes sense, then, to call these numbers the *least upper bound* of the respective subsets $[0, 1]$ and the currency denominations.

Definition 1. Let $E \subset S$ be nonempty and bounded above by some s. A number $\alpha \in S$ is called a *least upper bound* of E if the following conditions hold

- α is an upper bound of E
- for every $\beta < \alpha$, β is NOT an upper bound of E

Finally we quantify the property $\mathbb{N}$ and $\mathbb{R}$ enjoy

Definition 2. Let $<$ be an ordering on S. Say S is *complete* if S satisfies the following property:

For *any* nonempty bounded subset $E \subset S$, there is some $x \in S$ such that x is the least upper bound of E

What this property says, essentially, that we can *always* find a least upper bound for a subset E so long as it is nonempty and bounded. Try this yourself. Take bounded subsets of $\mathbb{N}$ and $\mathbb{R}$ and find their least upper bounds. I have demonstrated three examples here already.

But why do we specify complete sets? Isn't every ordered set complete? As we'll soon find out, that is not the case.

Exercise 1. Show that if α and β are two least upper bounds of a set, they are equal. (Hint. If x is an upper bound and y is a least upper bound, can you comment on whether $y \leq x$ holds?)

Answers to Exercises

The following are brief solutions or hints. You are encouraged to review the exercises before checking the answers.

Answer 1. Let α and β be two least upper bounds of a set. Since α is a least upper bound and β is an upper bound, it holds that $\alpha \leq \beta$. Similarly it holds that $\beta \leq \alpha$. If $\alpha \neq \beta$ then we get $\alpha < \beta$ and $\beta < \alpha$ simultaneously which cannot happen. Thus $\alpha = \beta$.