Power Series

Prereqs RA-19

Definition 1. A series of the form

$$\sum_{n=0}^{\infty} a_n(x-a)^n$$

is called a *power series around* a, or simply a power series.

The number a is colloquially called the "center" of the power series.

The convergence or divergence of a power series clearly depends on the value of x. As an example, consider

$$\sum_{n=0}^{\infty} x^n$$

We know a priori this is the geometric sum and it converges to $\frac{1}{1-x}$ iff $|x| < 1$. i.e., if $x \in (-1, 1)$ then our series converges, otherwise it doesn't converge, and therefore depending on the value of x.

Let's consider another series,

$$\sum_{n=0}^{\infty} n! x^n$$

Using the ratio test, $\left|\frac{a_{n+1}}{a_n}\right| = (n+1) \cdot |x| \to \infty$, except when $x = 0$. Thus the series only converges for $x = 0$ and for no other x.

Finally consider the series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Using the ratio test again, $\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{n+1} \to 0$ for every x. Thus the series converges for every $x \in \mathbb{R}$.

So we have made a few observations. A power series around a always converges for $x = a$. Now except for the extreme cases where the series only converges for $x = a$ or converges on the entire real line, it appears that the series converges inside some open interval around a and doesn't converge outside of it.

Refer to the geometric series example, in which a power series around $a = 0$ converges in $(-1, 1)$ and diverges outside of it, which is an interval of *radius* 1 around a.

It so happens that this is precisely how power series behave.

Lemma 1. Consider a power series

$$f(x) := \sum a_n(x-a)^n$$

1. If $f(x_0)$ converges, and x_1 is such that $|x_1 - a| < |x_0 - a|$ then $f(x_1)$ converges absolutely.
2. If $f(x_0)$ doesn't converge, and x_1 is such that $|x_1 - a| > |x_0 - a|$ then $f(x_1)$ doesn't converge.

Proof. Suppose $\sum a_n(x_0 - a)^n$ converges, thus $\exists\, M > 0$ such that $\left|a_n(x_0 - a)^n\right| \le M$ for every n $(*)$. Now consider

$$\begin{aligned} \left|a_n(x_1 - a)^n\right| &= |a_n||x_1 - a|^n \\ &= \left|\frac{x_1 - a}{x_0 - a}\right|^n \cdot |a_n||x_0 - a|^n \\ &\le M \cdot q^n \end{aligned}$$

where $q < 1$, hence the series

$$\sum |a_n| \cdot |x_1 - a|^n$$

is bounded above by a convergent geometric series therefore converges. As a consequence, the series

$$\sum a_n \cdot (x - a)^n$$

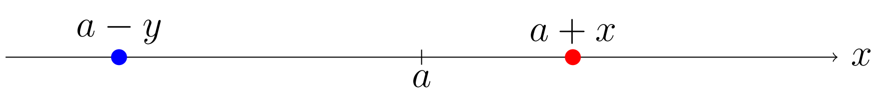
also converges.

Now suppose $\sum a_n(x_0 - a)^n$ does not converge and x_1 is such that $|x_1 - a| > |x_0 - a|$. If at all the series $\sum a_n(x_1 - a)^n$ converges, by the previous part of the theorem, $\sum a_n(x_0 - a)^n$ will converge. Contradiction. Thus $\sum a_n(x_1 - a)^n$ does not converge. $\square$

Where I've used the result $(*)$ – that a convergent series is uniformly bounded. Allow me to make this precise.

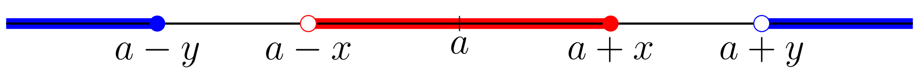
Exercise 1. Suppose $\sum a_n$ converges. Show that for some $M > 0$, $|a_n| < M$ for every n.

Let me make the lemma a little more clear. Suppose for some power series around a point a, I know of its convergence and divergence at two points.



Suppose a power series centered at a converges at $a + x$ and doesn't converge at $a - y$

We just saw that for points closer to a than x, the series necessarily converges, and for those further away than $-y$, it necessarily diverges. Let's have a look.



The power series doesn't converge in the blue region, and converges in the red region

This is what the lemma is saying. The existence of a singular blue or red point directly gives you the existence of blue and red "regions" where the behavior of the power series is known.

Observe – the behavior at $a + x$ is not enough to conclude the behavior at $a - x$. Similarly the behavior at $a - y$ is not enough to conclude the behavior at $a + y$.

The sharp eyed will notice that a power series *must* fall into one of two categories – a) converges or b) doesn't converge for every fixed x. i.e., the sum $\sum a_n(x-a)^n$ either converges or it doesn't, no third thing can happen.

All in all, the red and blue regions should *completely* cover the number line. The lemma has also given us something special – the red region most definitely exists (every power series around a converges at $x = a$) – but also, it must be *sandwiched* between the blue region.

Formally, this is what we're getting at.

Theorem 1. (Trichotomy) Consider the power series

$$\sum a_n x^n$$

Then the series either only converges at $x = 0$ and nowhere else, converges on the entire number line, or there exists a unique $r > 0$ such that the series absolutely converges when $|x| < r$ and doesn't converge when $|x| > r$.

Proof. We only need to show that if the first two don't happen then the third necessarily does.

Suppose the series converges at some point $x_c \neq 0$ and doesn't converge at some other point x_d. We know from the lemma that $|x_c| \leq |x_d|$. Consider the set

$$S := \{r \in \mathbb{R}_{\geq 0} \mid \text{the series converges for both } -r \text{ and } r\}$$

We know that $\frac{|x_c|}{2}$ is in S (since the series converges for x_c it converges for both $\frac{x_c}{2}$ and $-\frac{x_c}{2}$) and $|x_d|$ is an upper bound of S (by the lemma the series does not converge at any x_0 with $|x_0| > |x_d|$).

Let α be the least upper bound of S. We will show α is the required r.

Let x_1 be such that $|x_1| < \alpha$. Since α is the lub of S, $|x_1|$ is NOT an upper bound of S and thus there is some $r' \in S$ with $|x_1| < r' < \alpha$. But then the series converges for r' and $|x_1| < r'$ therefore the series converges for x_1.

Now suppose x_2 is such that $|x_2| > \alpha$. By whatever theorem you'd like to invoke there is some r' such that $|x_2| > r' > \alpha$. If the series converges for x_2, it must converge for r' and α fails to be an upper bound of S. Thus the series does not converge for x_2. $\square$

This unique r is called the *radius of convergence* of the power series.

Replacing x with $x - a$ keeps all the above arguments intact, allowing us to neatly sum up the following result.

Corollary 1. A power series

$$\sum a_n(x-a)^n$$

1. either converges only at $x = a$
2. or converges for all values of x
3. or converges on some interval $(a - r, a + r)$ and diverge outside it. The behavior at the boundary points $x = a - r$ and $x = a + r$ is **inconclusive** in this case and needs to be seperately investigated.

Example 1. Consider the power series $\sum_{n=1}^{\infty} \frac{x^n}{n}$. Using the root test, we get $\lim |a_n|^{1/n} = |x|$. So whenever $|x| < 1$, the series converges, and whenever $|x| > 1$, the series doesn't converge. Hence it converges for $x \in (0 - 1, 0 + 1)$ and doesn't converge outside of it.

What about when $|x| = 1$? For $x = -1$, we recognise this as the famous series approximation of $\ln \frac{1}{2}$ (neat!). While for $x = 1$, we've already shown in RA-18 that the series diverges to ∞.

Therefore, it is critical that we investigate the boundaries properly. Indeed, the interval of convergence of the series is $[-1, 1)$.

Finally, I'd like you to look at this neat little trick to find the radius of convergence of a power series.

Theorem 2. Consider a power series

$$\sum a_n (x - a)^n$$

1. If $\lim_{n\to\infty} |a_n|^{1/n} = L$, then the radius of convergence of the series is $\frac{1}{L}$, where if $L = \infty$ then the series only converges at a, and if $L = 0$ it converges everywhere.
2. If $\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L$ then the radius of convergence of the series is $\frac{1}{L}$, where if $L = \infty$ then the series only converges at a, and if $L = 0$ it converges everywhere.

One can verify this by using the Trichotomy Theorem and the ratio and root tests for the power series.

Answers to Exercises

The following are brief solutions or hints. You are encouraged to review the exercises before checking the answers.

Answer 1. Suppose $\sum a_n$ converges, then the sequence of partial sums S_n is bounded, i.e there is some $K > 0$ such that $|S_n| < K$ for every n. Now $a_n = S_n - S_{n-1}$ for $n > 1$ and $a_1 = S_1$. Thus,

$$|a_n| \leq |S_n| + |S_{n-1}| < 2K$$

and thus a_n is bounded.