Sequences are Functions

A *sequence*, essentially, is a list of things – numbers, points, sets, could be anything.

- An arithmetic progression is a sequence, where the $n-$th term is given by $a + (n-1)d$. This is a countably infinite list of numbers
- Consider the polynomial $x^3 - x$. This has zeros at $-1, 0, 1$. This is a finite list of numbers.
- Consider the lines in the Cartesian plane of slope 4, the general given by $y = 4x + c$ where c is a real number. This is an *uncountable* list of sets (since lines are sets).
- Suppose there is a set, $\mathcal{F}$, that contains functions from a set A to another set B. Fix some $a \in A$. Then we can list out the values a takes across different functions in $\mathcal{F}$, and refer to the value $f(a)$ by a_f instead. This is a list of elements of B.

You should notice, that each list, i.e., each sequence, is written *with respect to* elements of some other set.

- The terms of an arithmetic progression are defined by a parameter $n \in \mathbb{N}$ - so they're listed with respect to $\mathbb{N}$. i.e., each element of the progression is *referred to* by an element of $\mathbb{N}$
- A little subtle, but if we say that the roots of the polynomial are λ_1, λ_2, and λ_3; we implicitly list the roots with respect to the set $\{1, 2, 3\}$. i.e., each of the roots is *referred to* by an element of the set.
- This should be the clearest one. Each line $y = 4x + c$ is *referred to* by a real number c, and thus the sequence is with respect to $\mathbb{R}$
- Each value is *referred to* by some function $f \in \mathcal{F}$, and thus the sequence is with respect to $\mathcal{F}$

If a sequence is written with respect to a set $\mathcal{I}$, we usually say it is *indexed by* $\mathcal{I}$. For example, consider an AP with $a = 2$ and $d = 3$.

Then the set

$$\{2, 5, 8, 11 \dots\}$$

is indexed by $\mathbb{N}$ and

$$a_n = 2 + 3(n-1) = 3n - 1$$

Observe – when we index a set S with respect to another set $\mathcal{I}$ – then for each $\iota \in \mathcal{I}$ there must be some element $s_\iota \in S$. Think of the indexing set as an actual index. For each chapter listed in the index of a book, a chapter exists in the actual content. Similarly, for each element in the index set, an element exists in the set being indexed. So for example, if a set is indexed by the natural numbers, it makes sense to say "the nth element" of that set. Does an $n-$th real number make sense? No. Does the $n-$th term of an AP make sense? Yes.

Thus, intuitively, every sequence is a list with respect to some index set. We've established that if we index some set S with respect to $\mathcal{I}$, then "the ι-th element of S" with $\iota \in \mathcal{I}$ should make sense.

Where have we seen this before? I have a set ($\mathcal{I}$), and for each element of the set (ι), I want to assign it an element (s_ι) of another set (S).

Indeed, a *function* is something that does exactly this!

Therefore, for a set S and an index set $\mathcal{I}$, it makes sense to define a sequence as

Definition 1. An S-valued sequence indexed by $\mathcal{I}$ is a function $f : \mathcal{I} \to S$

And indeed, the informal s_ι can be replaced by the formal $f(\iota)$, although the former can still be used as shorthand. (In fact, you list arithmetic progressions using that shorthand! $(a_1 \ldots a_n)$ etc.)

It is worth noting that this definition is purely formal, and that a *sequence* almost always refers to a sequence indexed by $\mathbb{N}$ or some subset of it. In case of indexing by uncountable sets, the indexing is not usually called a sequence.