Taylor's Theorem I

Prereqs RA-15

Recall Lagrange's Theorem

Theorem 1. There is some $c \in (a, b)$ such that

$$f(b) = f(a) + (b - a) \cdot f'(c) \tag{1}$$

and recall the definition of the derivative.

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Observe that if h is "small enough", we can deploy the approximation

$$f(x_0 + h) \approx f(x_0) + h \cdot f'(x_0) \tag{2}$$

by rearranging the definition.

Compare (1) and (2). They look more or less similar. Suppose a and b are *close* to each other and that $b = a + h$. Then, putting (1) and (2) side-by-side with $b - a = h$ and $x_0 = a$, we get

$$\begin{aligned} f(b) &= f(a) + (b - a) \cdot f'(c) \\ f(b) &\approx f(a) + (b - a) \cdot f'(a) \end{aligned}$$

Here's intuitively why these look so similar and how you should make sense of this situation. The approximation

$$f(x) \approx f(a) + (x - a) \cdot f'(a)$$

estimates the value of f near a with a linear function. Namely, if I want to guess the value of $f(a + h)$, I can use a straight line of slope $f'(a)$ passing through $(a, f(a))$

$$f(a + h) \approx f(a) + h \cdot f'(a)$$

Cool. But why did I start with Lagrange's Theorem? Well, Lagrange tells us that *this approximation is in fact precise* if we just adjust the slope term, and that the correct slope is given by some $f'(c)$.

$$f(a + h) = f(a) + h \cdot f'(c)$$

Now, if f' is continuous and h is small, knowing $c \in (a, a + h)$ we can guess

$$f'(a) \approx f'(c)$$

Thus Lagrange's Theorem serves two purposes. It (1) justifies that our approximation is reasonable, and (2) assures us that our approximation can be made precise using some value of f'.

But what if a linear approximation isn't good enough? What if I want a higher order, say, a quadratic approximation?

Theorem 2. (Extended Lagrange) Let f be twice differentiable on (a, b) with f' continuous on $[a, b]$. Then there is some $c \in (a, b)$ such that

$$f(b) = f(a) + (b - a) \cdot f'(a) + \frac{(b - a)^2}{2} \cdot f''(c)$$

Proof. Define

$$F(x) = f(b) - f(x) - (b - x) \cdot f'(x)$$

and observe $F(b) = 0$. Further,

$$\begin{aligned} F'(x) &= -f'(x) - bf''(x) + f'(x) + xf''(x) \\ &= (x - b) \cdot f''(x) \end{aligned}$$

Now define

$$g(x) = F(x) - \left(\frac{b - x}{b - a}\right)^2 F(a)$$

Now observe that $g(a) = g(b) = 0$. Further,

$$g'(x) = F'(x) + 2 \cdot F(a) \cdot \frac{(b - x)}{(b - a)^2}$$

By Rolle's Theorem, there is some $c \in (a, b)$ such that $g'(c) = 0$. Substituting the previously calculated value of $F'(x)$,

$$\begin{aligned} 0 = g'(c) &= F'(c) + 2 \cdot F(a) \cdot \frac{(b - c)}{(b - a)^2} \\ &= (c - b) \cdot f''(c) + 2 \cdot F(a) \cdot \frac{(b - c)}{(b - a)^2} \end{aligned}$$

Transposing, we get

$$\begin{aligned} (b - c) \cdot f''(c) &= 2 \cdot F(a) \cdot \frac{(b - c)}{(b - a)^2} \\ F(a) &= \frac{(b - a)^2}{2} \cdot f''(c) \end{aligned}$$

By definition of $F(x)$ we get

$$\begin{aligned} F(a) &= f(b) - f(a) - (b - a)f'(a) \\ \frac{(b - a)^2}{2} \cdot f''(c) &= f(b) - f(a) - (b - a)f'(a) \end{aligned}$$

And therefore there is some $c \in (a, b)$ such that

$$f(b) = f(a) + (b - a) \cdot f'(a) + \frac{(b - a)^2}{2} \cdot f''(c)$$

$\square$

At its core, this is the same statement as Lagrange's Theorem from an approximation perspective. Suppose I initiate a parabolic approximation at a

$$f(a+h) \approx f(a) + h \cdot f'(a) + \frac{h^2}{2} \cdot f''(a)$$

Then, the Extended Lagrange Theorem tells us that this approximation is good, and adjusting the last term makes it precise.

$$f(a+h) = f(a) + h \cdot f'(a) + \frac{h^2}{2} \cdot f''(c)$$

So we see that linear and quadratic approximations of $f(a+h)$ aren't just educated guesses, they're also *guaranteed* to be accurate up to a final correction term. That correction depends on a higher derivative evaluated somewhere in the interval.

This generalises to $k-$degree polynomial approximations using $k-$th derivatives of f. We will explore a better phrasing of this idea and the proof of the general case, alongside the mysterious 2 in the denominator of the h^2 term in RA-17.

Example 1. Compare the quadratic approximation to the displacement equation

$$s(t) = s(0) + u \cdot t + \frac{1}{2} \cdot a \cdot t^2$$

By rewriting it as

$$s(t) = s(0) + (t-0) \cdot s'(0) + \frac{(t-0)^2}{2} \cdot s''(0)$$