amenable to engineering analysis by employing several techniques not previously used. This approach solved several existing problems and anticipated the requirements of future design needs.

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A Self-Consistent Iterative Scheme for One-Dimensional Steady State Transistor Calculations

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Summary—**A self-consistent iterative scheme for the numerical calculation of dc potentials and currents in a one-dimensional transistor model is presented. Boundary conditions are applied only at points representing contacts. Input data are: doping profile, parameters governing excess carrier recombination, parameters describing the dependence of mobility on doping and on electric field, applied emitter and collector voltages, and a trial solution for the electrostatic potential. The major limitation of the present approach results from use of Boltzmann rather than Fermi statistics. Convergence of the iteration scheme is good for low and moderate injection levels.**

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I. INTRODUCTION

THE CLASSICAL method of analysis of transistors [1]–[3] is based on dividing the transistor into field-free regions where the carriers move by diffusion and recombine, and space charge regions, where the carriers are ignored and where the space charge is taken to be that of the doping impurities. Neighboring field-free regions are linked by boundary conditions which relate the minority carrier densities on either side of the intervening space charge region to the voltage applied

across the space charge region. This description was very successful, especially for low injection levels in transistors having a base width large compared to the space charge regions and having constant doping in the base.

If a transistor has a graded base [4], [5], and also if high injection exists, the base region is no longer field free and drift of carriers must be included in the description. Mobile carriers may now contribute to space charge in the transition regions [6] and their recombination in the emitter-base transition region may constitute a significant part of the alpha-defect current. Important consequences of these effects for the dc characteristics and high frequency performance of the transistor have been considered by previous authors [6–12], mostly as corrections that are superimposed on the basic transistor model. However, because the base region in modern high frequency transistors is made quite narrow, the dividing up of the transistor into separate regions becomes problematic, and it becomes desirable [13] to give a unified treatment of the transitor.

In this paper, a set of equations for the potential distribution and the flow of carriers under steady state conditions is formulated. An iteration scheme, suitable for solution by digital computers, is presented for a one-dimensional transistor of given doping profile, with boundary conditions applied only at points representing emitter, base, and collector contacts.

II. General Method

For transistor analysis, the state of a region of semiconductor material may be considered specified if the hole and electron concentrations and the electric field are known.[1] Equivalent information is provided by the electrostatic potential (EP) and the hole and electron quasi-Fermi potentials (HQFP, EQFP). These quantities are governed by a set of three second-order, nonlinear differential equations. Methods for the numerical solution of such a system are well established (see, *e.g.* Hamming [14]) provided that initial values are given. Thus, if it were possible to establish a self-consistent set of initial values for the transistor problem, then solutions could be obtained rather readily. The major difficulty is connected just with the establishment of initial values.

In this paper, an alternate approach is taken. Boundary values, rather than initial values, are specified and solutions are obtained by successive iterations. Two sets of boundary conditions for the potentials, described in detail at the end of Section III, are applied well outside the active region of the transistor at points representing the emitter and collector contacts. In addition, the quasi-Fermi potential for the majority carriers in the base is specified at a point in the base region representing the base contact.

The approach used here is based on the fact that for narrow-base transistors, the EP and HQFP[2] are almost unchanged by recombination until it becomes very heavy. Therefore, if recombination is neglected, the hole current is constant through the transistor and the HQFP can be expressed in terms of an integral containing the EP.[3] We shall call this relation between HQFP and EP the modified Moll-Ross relation [7].

This makes feasible an iteration scheme in which

1) A guess at the EP is made.
2) The modified Moll-Ross relation is used to obtain the HQFP, which, with the EP, yields the hole concentration.
3) Poisson's equation, which states that the second derivative of the EP is proportional to the total space charge, is used to obtain a better approximation for the EP.

Steps 2) and 3) are alternately executed until the solution is obtained to the required accuracy.

It is possible to generalize the Moll-Ross relation to include current densities which vary with position as a result of recombination, and to consider the dependence of mobility on doping and electric field. In the presence of recombination, electron currents flow also, and thus the EQFP is not constant and is treated analogously to the HQFP. The scheme presented here incorporates these refinements.

Input data for the calculations are:

1) Doping profile for the transitor.
2) Mobility as function of doping and electric field.[4] See Appendix III.
3) Excess carrier generation-recombination law; *e.g.*, for a single level Hall [15] Shockley-Read [16] recombination center, the limiting hole and electron lifetimes and the energy of the center are specified.
4) Applied emitter and collector voltage.
5) A trial solution for the EP.

Since Boltzmann statistics are used for calculating carrier concentrations, the treatment of this paper is justified only for moderate doping levels. Nevertheless, computer solutions using the scheme presented here may be useful even for transistors that have some heavily doped regions, if the solutions in the heavily doped regions are judiciously adjusted.

Section III describes the basic physical equations, and discusses the boundary conditions. Section IV establishes the generalized Moll-Ross equation in a form suitable for machine calculations. In Section V, the treatment of Poisson's equation is described and the over-all iteration scheme is outlined. The last section contains discussion.

[1] A more complete description would include the occupancy of states in the forbidden gap and specify how, if at all, the distribution of holes in the valence band and of electrons in the conduction band, deviates from the equilibrium distribution. Normally, such detail is not required.

[2] For convenience, a structure having *p-n-p* polarity is considered throughout this paper.

[3] See (18) and following.

[4] Mobility reduction resulting from hole-electron scattering may also be included.

III. Basic Equations

This section describes the basic physical equations and the boundary conditions at the points representing the emitter, base, and collector terminals.

Let p be the hole concentration and n the electron concentration in units of n_i, the intrinsic carrier concentration. The hole and electron concentrations are related in the customary way to ψ, the EP (in units of the Boltzmann voltage kT/q) and φ_p and φ_n, the HQFP and EQFP (also in units of the Boltzmann voltage):

$$p = e^{\varphi_p - \psi} \tag{1}$$

$$n = e^{\psi - \varphi_n}. \tag{2}$$

These equations may be taken as definitions of the quasi-Fermi potentials; they represent the Boltzmann approximation to Fermi statistics[5] and are valid only as long as $\varphi_p - \psi$ and $\psi - \varphi_n$ do not approach half the bandgap. This approximation places a restriction on the doping levels for which this paper is applicable.

The current density resulting from the flow of holes is given by the superposition of a diffusion and a drift term.[6]

$$j_p = -qD_p n_i p' - q\mu_p \frac{kT}{q} n_i p \psi' \tag{3a}$$

$$j_p = -qD_p n_i (e^{\varphi_p})' e^{-\psi}. \tag{3b}$$

In the transition from (3a) to (3b), the Einstein relation $D_p = \mu_p kT/q$ between the hole diffusion coefficient D_p and the hole mobility μ_p has been used, and (1) and (2) have been employed. If now $L_D = (kT\epsilon/q^2 n_i)^{1/2}$ is adopted as the unit length, and $qD_0 n_i/L_D$ as unit current, we obtain for the normalized current density

$$J_p = -\frac{D_p}{D_0} (e^{\varphi_p})' e^{-\psi}. \tag{4}$$

Here D_p is the diffusivity, which is considered as a function of doping and electric field; D_0 is a reference diffusivity employed in the definition of the unit current. A corresponding equation for the electron current density is

$$J_n = \frac{D_n}{D_0} (e^{-\varphi_n})' e^{\psi}. \tag{5}$$

If no recombination of minority carriers took place, the current densities J_p and J_n would each be constant. Assume now a recombination process that gives a recombination rate of $u(n, p)$ carriers per sec per cm^3; then the derivatives of the current densities are given by

$$-J_p' = J_n' = L_D^2 \frac{u}{D_0 n_i} \equiv U. \tag{6}$$

If we assume that the recombination is due to a set of single level Hall [15]-Shockley-Read [16] recombination centers, then

$$u = n_i \frac{e^{\varphi_p - \varphi_n} - 1}{\tau_p(e^{\psi - \varphi_n} + e^{\varphi_t}) + \tau_n(e^{\varphi_p - \psi} + e^{-\varphi_t})} \tag{7}$$

where τ_p and τ_n are the lifetimes of excess holes and electrons in heavily doped material and φ_t is the energy level of the center. However, other generation-recombination laws, if known to apply, should be used instead of (7). For example, several types of centers may be present; or it may be of interest to include excess carrier generation due to impact ionization in regions of high electric field.[7]

The EP is governed by Poisson's equation

$$\psi'' = e^{\psi - \varphi_n} - e^{\varphi_p - \psi} - N \tag{8}$$

where N is the ionized impurity concentration in units of n_i. N is positive for donors and negative for acceptors. Eqs. (1) through (8) together with appropriate boundary conditions completely specify a solution.

In the discussion of the boundary values, consider, for convenience, a p-n-p structure as shown in Fig. 1. We take as reference level for the potentials the value of the EQFP at some point M in the base and set $\varphi_n(M) =$ (

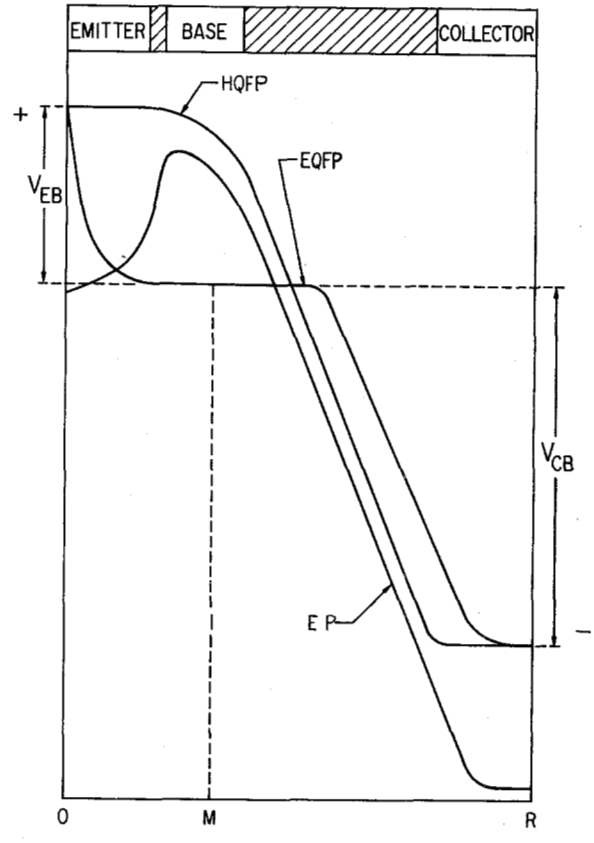


Fig. 1—EP ψ, HQFP φ_p, and EQFP φ_n for p-n-p transistor (schematic).

[5] If Fermi statistics were used, the simple relation (4) below between φ_p, ψ, and J_p would not hold and the approach in the present form of Section IV, could not be taken.

[6] Throughout this paper, ′ denotes differentiation with respect to the distance variable.

[7] For avalanche breakdown, precautions must be taken that the field associated with trial solutions is not excessive; otherwise, convergence might be endangered.

Usually the EQFP has a nearly constant value throughout the base and the choice of the point M is not critical. If φ_n changes considerably across the base, the details of how the base current is brought into the base become important and a one-dimensional calculation is inadequate.

At the end points, 0 and R, we specify that carrier equilibrium ($\varphi_p = \varphi_n$) prevails. This can be enforced, if necessary, with an infinite recombination rate U at the end points. We also assume that space charge neutrality exists at the end points, *i.e.*, that the right-hand side of (8) is zero. This requirement fixes ψ if $\varphi_p = \varphi_n$ is known. We specify $\varphi_p(0) = qV_{EB}/kT$ and $\varphi_p(R) = qV_{CB}/kT$ where V_{EB} and V_{EC} are the applied emitter and collector voltages.

From consideration of the structure as a two-port network, it is plausible that application of two biases defines the state of the structure completely. Section IV shows explicitly how the boundary conditions enter into the solutions for the quantities φ_n, φ_p, J_p and J_n.

In summary, the boundary conditions are:

$$\varphi_p(0) = \varphi_n(0) = \frac{qV_{EB}}{kT} \tag{9}$$

$$\varphi_p(R) = \varphi_n(R) = \frac{qV_{CB}}{kT} \tag{10}$$

$$\varphi_n(M) = 0 \tag{11}$$

$$\psi(0) = \varphi_p(0) - \ln\left(\sqrt{\left(\frac{N(0)}{2}\right)^2 + 1} - \frac{N(0)}{2}\right) \tag{12}$$

$$\psi(R) = \varphi_p(R) - \ln\left(\sqrt{\left(\frac{N(R)}{2}\right)^2 + 1} - \frac{N(R)}{2}\right). \tag{13}$$

Eqs. (12) and (13) are obtained if the right-hand side of (8) is set equal to zero. More general boundary conditions that relate the density of holes and electrons at the boundary to "boundary velocities" are given in Appendix II.

IV. Generalized Moll-Ross Relation

In this section, the HQFP and EQFP will be computed in terms of the EP ψ, the relative diffusion coefficients

$$\gamma_p = \frac{D_0}{D_p} \tag{14}$$

$$\gamma_n = \frac{D_0}{D_n} \tag{15}$$

and the "recombination current" J_r, defined by

$$J_r(x) = \int_0^x U(t)\, dt. \tag{16}$$

Also, the total hole and electron current densities, J_p and J_n, will be calculated.

Eq. (4) can be rewritten

$$(e^{\varphi_p})' = -J_p\gamma_p e^{\psi}. \tag{17}$$

Integration yields

$$e^{\varphi_p(x)} = -\int_0^x J_p(t)\gamma(t)e^{\psi(t)}\, dt + \text{const.} \tag{18}$$

The task now is to find proper values for J_p and the constant of integration so that φ_p has the correct value at the boundaries.

Let us first assume no recombination; then J_p is a constant. We divide $e^{\varphi_p(x)} = B(x)$ into two parts: one part $B_e(x)$ that is controlled by the emitter and vanishes at the collector and a part $B_c(x)$ that is controlled by the collector and vanishes at the emitter. Define

$$F(x) = \int_x^R \gamma_p(t)e^{\psi(t)}\, dt. \tag{19}$$

Then

$$B_e(x) = \frac{e^{\varphi_p(0)}F(x)}{F(0)}, \tag{20}$$

and B_c can be written

$$B_c(x) = e^{\varphi_p(R)} - \frac{e^{\varphi_p(R)}F(x)}{F(0)}. \tag{21}$$

Combining both parts one obtains for the exponential of the HQFP:

$$e^{\varphi_p(x)} = \frac{e^{\varphi_p(0)} - e^{\varphi_p(R)}}{F(0)}F(x) + e^{\varphi_p(R)}. \tag{22}$$

If (22) is differentiated and the result compared with equation (17), one easily identifies the coefficient of $F(x)$ as the current

$$J_p = \frac{e^{\varphi_p(0)} - e^{\varphi_p(R)}}{F(0)}. \tag{23}$$

Eq. (23) contains an important, well-known result. In unnormalized notation, it is

$$I = \frac{Aqn_i^2}{\int_0^R \frac{1}{D_p(x)} n_i e^{\psi(x)/kT}\, dx}\left(e^{qV_{EB}/kT} - e^{qV_{CB}/kT}\right) \tag{24}$$

where A is the area and I the hole current, which, in the recombinationless case, is equal to the emitter and collector current. Remarkable, but well known, is the symmetry in emitter and collector voltage which holds for any base doping profile. This symmetry is manifested in the Ebers-Moll [15] equations by the equality of the two coefficients describing the emitter current dependence on collector voltage and collector dependence on emitter voltage. The integral in the denominator of the first term of (24) contains the reciprocal diffusion coefficient. If $1/D(x)$ inside the integral is replaced by an average value $1/\bar{D}$ outside of it, then

$$I = \frac{A q\bar{D}n_i^2}{\int_R^0 n_i e^{\psi(x)/kT}dx}\left(e^{qV_{EB}/kT} - e^{qV_{CB}/kT}\right). \tag{25}$$

The overwhelming contributions to the integral in the denominator come from regions where ψ is large, *i.e.*, from the base. In this region, the EP under low-injection conditions is controlled by the base doping, and the integrand equals the donor concentration. Thus, we have the well-known result [7], [16] that the collector current

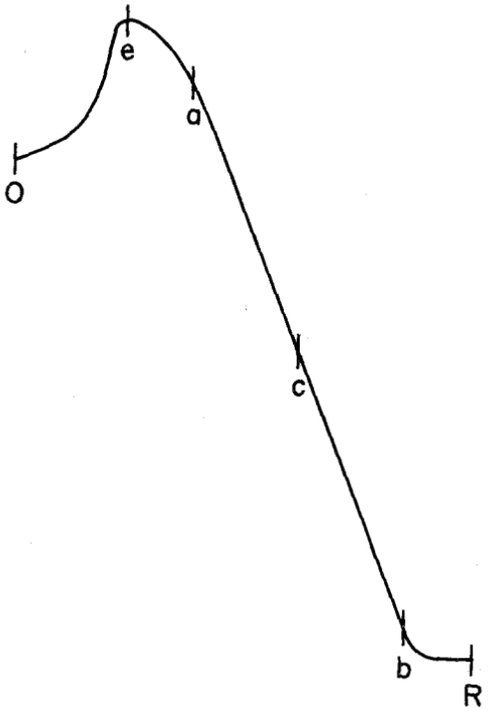


Fig. 2—EP for actively biased transistor (schematic).

for given bias conditions is inversely proportional to the number of impurities per unit area in the base.

Before returning to the general case in which excess carrier generation and recombination take place, we want to discuss the functions B_e and B_c and the hole concentration to which they give rise.

Shown schematically in Fig. 2 is the potential for a transistor under forward emitter and reverse collector bias. Consider the function F, (19). As we integrate from $x = R$ towards $x = 0$, the integral increases strongly until we reach point e. As we proceed further, the integrand is small compared with the value of the integral and the integral remains essentially constant. Therefore, by (22), the value of φ_p is essentially constant from $x = 0$ to $x = e$. This is the well-known result that the HQFP is constant in the emitter region and changes only slightly in the emitter space charge region.

It is interesting to resolve also the hole concentration p into an emitter controlled part p_e and a collector controlled part p_c, *i.e.*,

$$p_e(x) = e^{-\psi(x)}B_e(x) \tag{26}$$

$$p_c(x) = e^{-\psi(x)}B_c(x). \tag{27}$$

In the region to the left of point e,

$$p_e \approx e^{\varphi_p(0)-\psi(x)}.$$

We now consider the value of p_e at point c in the collector depletion region. Assume that ψ' and γ are essentially constant between $x=a$ and $x=b$. We re-express $e^{-\psi(x)}F(x)$:

$$e^{-\psi(x)}F(x) = \int_x^R \gamma_p(t)e^{\psi(t)-\psi(x)}\,dt. \tag{28}$$

The integrand is shown in Fig. 3. At $x = c$, its value is $\gamma_p(c)$ and for larger values of x, it decays exponentially

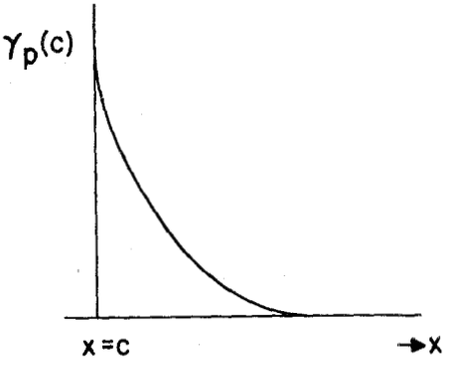


Fig. 3—Integrand in (28).

with a decay constant equal to $\psi'(c)$. The exact values of γ_p and ψ' for x much larger than c are irrelevant because they contribute negligibly to the integral. Thus the integral has the approximate value $\gamma_p(c)/\psi'(c)$. The emitter part of the hole concentration at c is approximately

$$p_e(c) \approx J_p(c)\gamma_p(c)/\psi'(c). \tag{29}$$

If the electric field ψ' is sufficiently large, then γ_p/ψ' is independent of ψ' and is proportional to the scatter limited velocity. Because the electric field ψ' decreases as one approaches the collector from the base, (28) reaches a maximum before it goes to zero at $x = R$.

Finally, for moderate to high reverse collector biases the collector controlled function B_c is almost entirely contained in the first term of (21), and therefore

$$p_c(x) = e^{\varphi_p(R)-\psi(x)}. \tag{30}$$

A schematic representation of p_e and p_c for three current densities in a double diffused epitaxial transistor structure under high reverse bias is shown in Fig. 4.

Now let us return and generalize (22) and (23) so as to include excess carrier generation and recombination. To do this, we add to the right-hand side of (22) the terms

$$\frac{FR(0)}{F(0)}F(x) - FR(x) \tag{31}$$

where

$$FR(x) = \int_x^R J_r(t)\gamma_p(t)e^{\psi(t)}\,dt. \tag{32}$$

Expression (31) vanishes for $x = 0$ and $x = R$; thus the boundary values of e^{φ_p} remain unchanged. The general expression for J_p and e^{φ_p} can be written

$$J_p(x) = \frac{e^{\varphi_p(0)} - e^{\varphi_p(R)} + FR(0)}{F(0)} - J_r(x) \tag{33}$$

$$\equiv J_{p\,\mathrm{const}} - J_r(x) \tag{34}$$

$$e^{\varphi_p(x)} = J_{p\,\mathrm{const}}F(x) - FR(x) + e^{\varphi_p(R)}. \tag{35}$$

Solutions analogous to (35) can be obtained for the EQFP. However, since the EQFP is specified also at $x = M$ in the base, two separate domains, the emitter domain $0 \leq x \leq M$, and the collector domain $M \leq x \leq R$

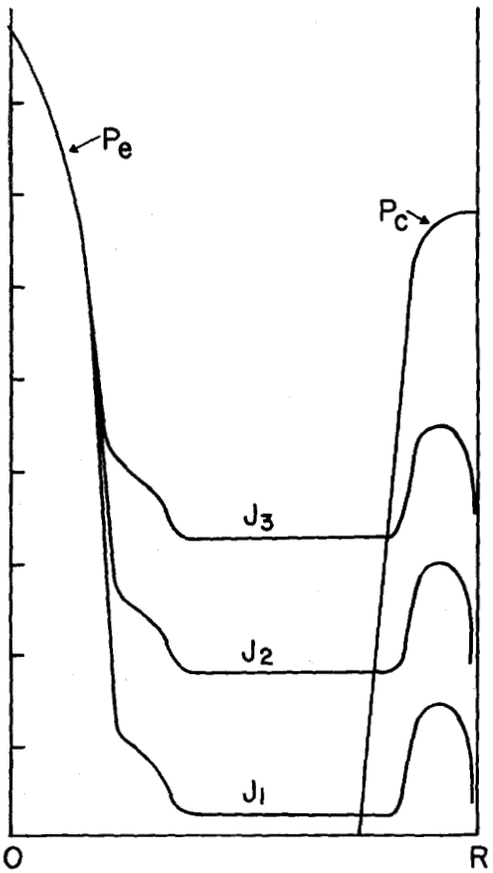


Fig. 4—Logarithm of hole concentrations p_e and p_c for three current densities $J_1 < J_2 < J_3$ (schematic).

must be considered. Because of the boundary conditions on the EQFP, electron current will flow even in the absence of recombination in the interior of the transistor. Such current is not included in the recombination current J_r, even though, of course, the boundary conditions (9) and (10) are equivalent to infinite recombination planes at $x = 0$ and $x = R$. We do the bookkeeping on the currents just inside the boundaries, so that emitter and collector currents are made up of hole and electron contributions. The solutions for the quasi-Fermi levels and the currents are given in Appendix I.

V. Poisson's Equation and Iteration Loop

In the previous section, it was shown how the quasi-Fermi levels are computed if the EP is known. In this section, the quasi-Fermi potentials are assumed to be known, along with a trial solution for the EP and we consider how improved values for the EP can be obtained. However, while in the previous section the result was obtainable through quadratures, we now have to solve the second order differential equation:

$$\psi'' = e^{\psi-\varphi_n} - e^{\varphi_p-\psi} - N. \tag{36}$$

This is Poisson's equation; the right-hand side represents the net space charge. Our approach is to expand the nonlinear equation in terms of the difference $\delta(x)$ between the available trial solution ψ and the exact solution.

Let

$$\psi_{\text{exact}} = \psi + \delta. \tag{37}$$

Then,

$$\delta'' - \delta(e^{\psi-\varphi_n} + e^{\varphi_p-\psi}) = (-\psi'' + e^{\psi-\varphi_n} - e^{\varphi_p-\psi} - N) + 0(\delta^2). \tag{38}$$

Neglecting terms of second order and higher order, we consider (43) as a linear differential equation for δ. We solve it numerically by considering δ at a sufficiently dense set of points and converting the differential equation into a system of difference equations. In this process, the left-hand side forms a matrix T having nonzero elements in the main diagonal and nearest adjacents only. Our problem then amounts to finding the inverse of this "tridiagonal" matrix

$$T\delta = B \tag{39}$$

where δ and $B = (-\psi'' + e^{\psi-\varphi_n} - e^{\varphi_p-\psi} - N)$ are considered as column vectors. Techniques for such inversions are available [19] and quite suitable for machine calculations. The solution δ is given by

$$\delta = T^{-1}B, \tag{40}$$

and the improved value ψ_{new} for the EP is

$$\psi_{\text{new}} = \psi + T^{-1}B. \tag{41}$$

We will now sketch the over-all iteration scheme. With ψ_{new}, the improved approximation of ψ and the available quasi-Fermi potentials φ_p and φ_n, one computes hole and electron concentrations and the recombination rate U,[8] (7) or equivalent, and (6). With this, the calculations of Section IV (or Appendix I) are done again, from which improved quasi-Fermi potentials are obtained, and the cycle is repeated. The important question arises whether the above iteration scheme converges. It is intuitively plausible that convergence should take place if injection is low, *i.e.*, if the biases are such that the resulting concentration of holes in the base is smaller than the electron concentration, and if excess carrier recombination is small. Preliminary computer calculations demonstrate that this is indeed the case. In this paper, no further discussion of the conditions for convergence is offered. Iteration schemes potentially superior to the one described here are under consideration.

VI. Preliminary Results

This section contains some preliminary results of calculations for a double diffused silicon epitaxial structure. The doping profile is shown in Fig. 5, where $\sinh^{-1}(N_D - N_A)$ is plotted as a function of distance from the emitter contact; N_A is the concentration of acceptors and N_D the concentration of donors. Fig. 5 is essentially a log plot of N_D for positive values and of N_A for negative

[8] When φ_p and φ_n are being computed for the first time, values for U and J_r are not yet available. Normally, a satisfactory starting value is $J_r(x) = 0$, except, perhaps, for cases of heavy recombination.

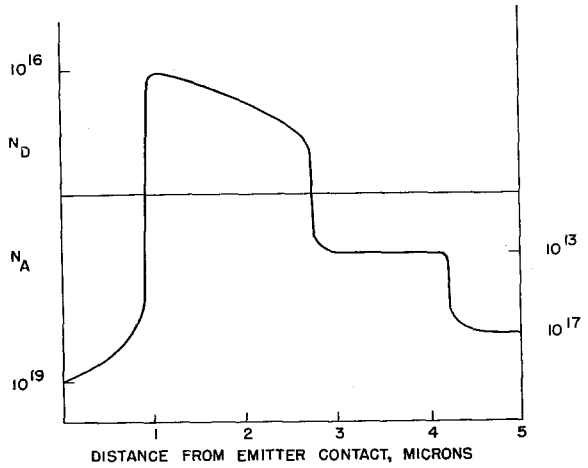


Fig. 5—Doping profile of transistor.

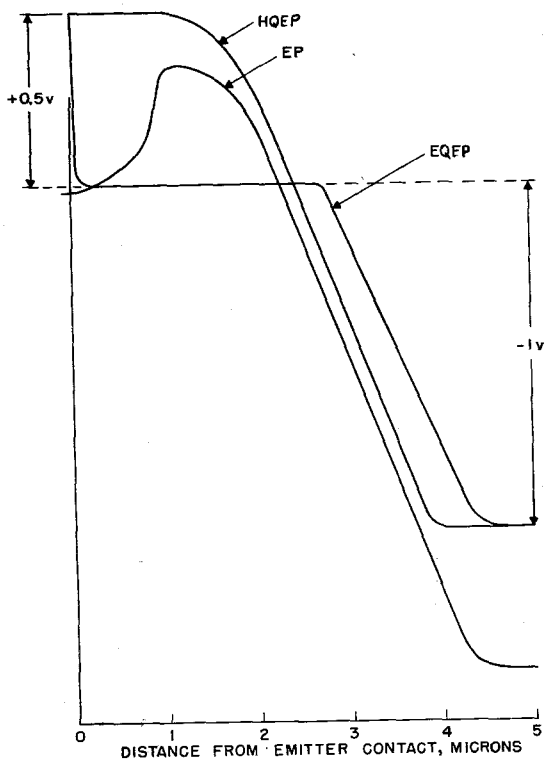


Fig. 6—Potentials for actively biased transistors.

values of the ordinate. Fig. 6 shows the computed EP, HQFP, and EQFP for an emitter bias of 0.5 v and a collector bias of −1 v.

A comparison of the EP for a low emitter bias of 0.3 v and a higher bias of 0.7 v is shown in Fig. 7. It is seen that at the higher bias, a substantial region of low electric field exists in the middle of the transistor, much more so than at the low bias. This is the phenomenon of "base push-out" described by Early [2] and Kirk [11].

For analysis of small-signal ac operation, the distribution of small-signal stored charge is of interest. We treat here carrier concentration differences corresponding to neighboring steady states. Fig. 8 shows the differential changes in hole and electron concentration per kT/q

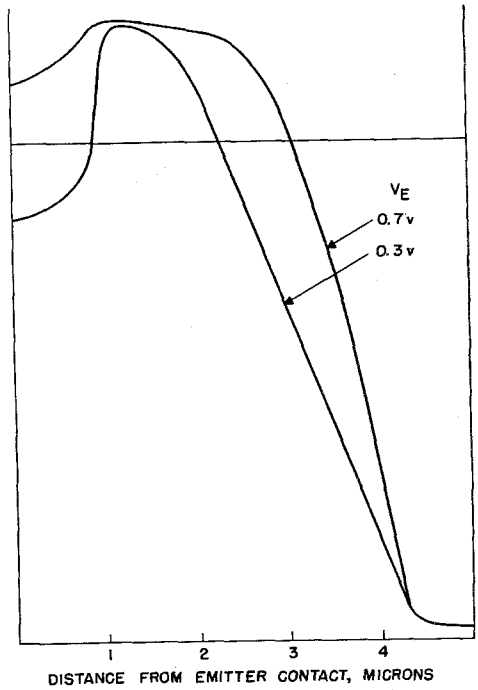


Fig. 7—Comparison of EP for low and higher emitter bias. At the higher bias "base push-out" occurs.

change in emitter voltage for the transistor biased as in Fig. 6. At the emitter junction, localized hole and electron concentration changes take place which represent the charging of the "emitter capacitance." Additional holes are stored in the base to carry the increased current caused by the emitter voltage increment. The hole storage is accompanied by electron storage. However, for the structure shown at the given bias there is only a small region in the base where the concentrations of additional holes and additional electrons are approximately equal.

In the collector depletion region, the additional hole concentration is approximately constant; it is the concentration of charge carriers needed to carry the incremental current at carrier velocities approaching scattering limited velocity. These holes in the collector depletion region constitute a positive space charge which, by Poisson's equation, causes a negative second derivative of the change $\delta\Psi$ of Ψ. The quantity $\delta\Psi$ is shown in Fig. 9. In the heavily doped portions of base and collector, Ψ changes little, *i.e.*, $\delta\Psi$ is small. At the edges of the collector depletion layer, *i.e.*, at points A and B, the second derivative of $\delta\Psi$ is positive and therefore negative space charge is needed. At point A, this space charge is provided by additional electrons and at point B by a decrease in holes (Fig. 8).

Differential stored hole charge for a higher current density is shown in Fig. 10. Here most of the charge is stored in the base, in contrast to the low level case of Fig. 8, where most of the charge is stored in the "emitter capacitance."

In this section, some examples of preliminary calculations were given to demonstrate the method. More complete results of calculations for typical transistor structures will be the subject of another paper.

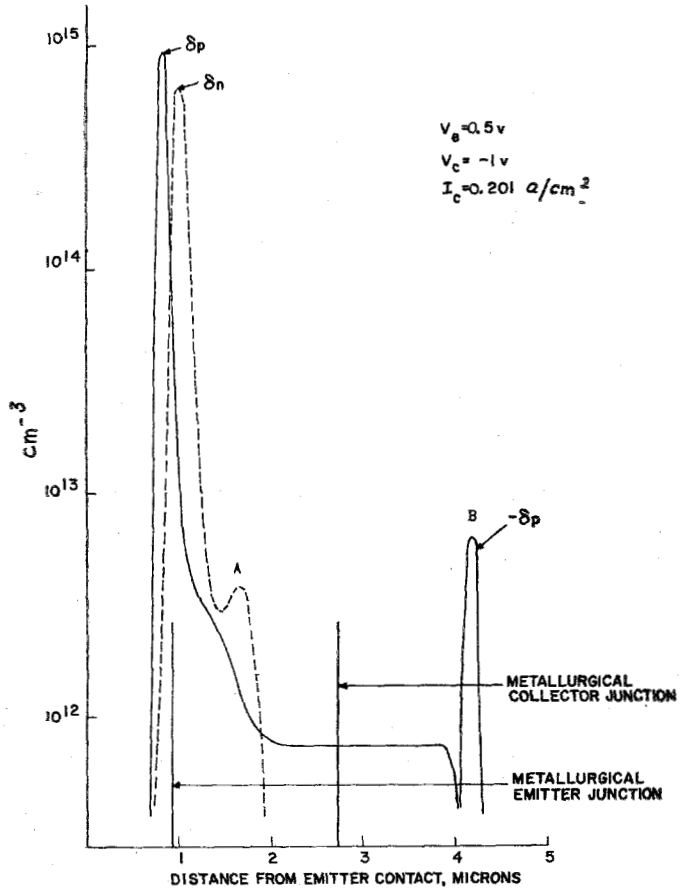


Fig. 8—Change in carrier concentration per kT/q increase of emitter voltage.

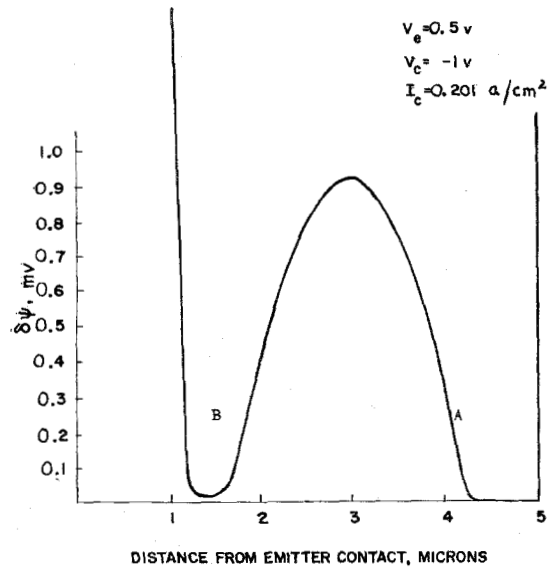


Fig. 9—Change of EP per kT/q increase of emitter voltage.

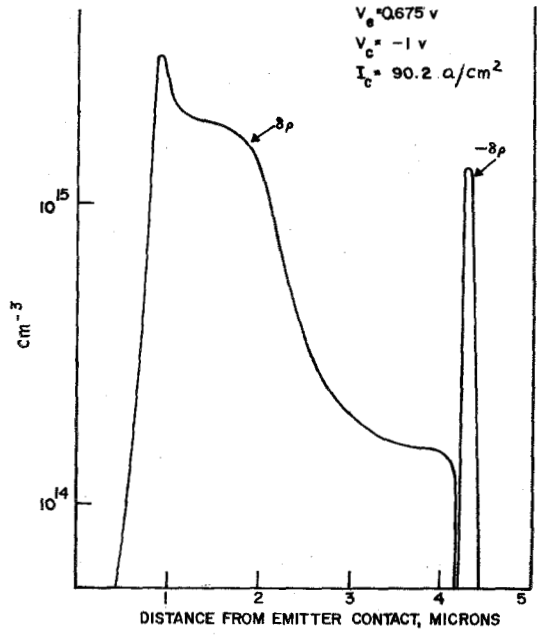


Fig. 10—Change in hole concentration per kT/q increase of emitter voltage.

VII. Discussion

This paper offers a self-consistent scheme for finding electrostatic and quasi-Fermi potentials for given emitter and collector voltage biases in a one-dimensional transistor model. Carrier concentrations and currents are computed concurrently with the potentials.

In real transistors, the base spreading resistance is frequently important and therefore a one-dimensional model cannot represent a transistor realistically. However, solutions for the one-dimensional case can be used as points of departure for calculations of more realistic models. The simplest approach is to represent the base spreading resistance by a fixed resistor at the base terminal. The next level of sophistication might consist in making a lumped representation of the base spreading resistance by interconnecting a number of one-dimensional transistors as in Fig. 11. If surface recombination is important in connection with emitter crowding, the "outside" transistor, *i.e.*, the one closest to the base terminal, should have recombination parameters that represent surface recombination.

The calculations of this paper treat a transistor under steady-state conditions and yield directly the dc characteristics. Since the electric field and carrier concentrations are also available, the results also contain information pertinent to low-frequency ac operation of the transistor, such as stored charge, signal delay time, and capacitances.

Several extensions of the calculations offered here come to mind and are listed below. They are of little mathematical difficulty once the solutions to the potentials considered in this paper are available, but they may be more convenient or furnish useful additional information.

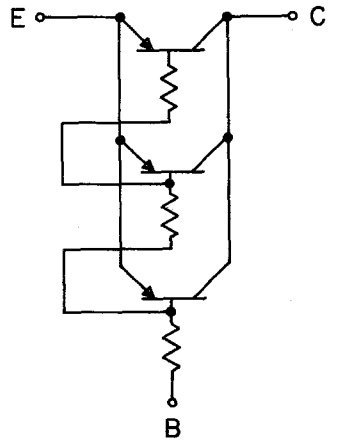


Fig. 11—Lumped representation of transistor in terms of one-dimensional components.

1) At a given bias point, compute not only the hole and electron concentrations, but also the variation of these concentrations per unit change in emitter and collector voltage. From this, low-frequency differential delay time and differential capacitances can be computed.[9] The information is also available if differences of the concentrations for neighboring bias points are taken, but the errors may become excessive if small differences are taken. In the process of computing the variations of carrier concentrations, the variation in EP ψ per unit bias change is also computed and one may use this function to advantage in providing initial trial functions for ψ if the transistor is computed at a number of successive bias points.
2) For switching transistors, it may be of interest to consider collector voltages that depend on the collector current. Thus, instead of specifying the collector voltage, one may specify a supply voltage and a, load resistor and compute, inside the iteration loop, the collector voltage as the supply voltage minus the voltage drop across the load resistor.
3) When power is dissipated in the transistor and when dc characteristics are studied, the variation of parameters with temperature becomes important. Usually, it is sufficient to consider the active region to be at a uniform temperature. Then a thermal resistance can be specified and, inside the iteration loop, the temperature is computed from the dissipated power, and temperature corrections are applied to the parameters.
4) Finally, the mathematical iteration technique that has been applied to transistors here, of course can be applied also to diode problems.

[9] The differential concentration changes in Fig. 8 were obtained by this technique.

Appendix I

Electron Current and EQFP

Emitter Domain, $0 \leq x \leq M$

Define

$$F_e(x) = \int_x^M \gamma_n(t) e^{-\psi(t)}\, dt \tag{42}$$

$$FR_e(x) = \int_x^M J_r(t)\gamma_n(t) e^{-\psi(t)}\, dt \tag{43}$$

$$J_{e\ \mathrm{const}} = \frac{-e^{-\varphi_n(0)} + 1 + FR_e(0)}{F_e(0)}. \tag{44}$$

Then the electron current is

$$J_n(x) = J_{e\ \mathrm{const}} + J_r(x) \tag{45}$$

and the EQFP is given by

$$e^{-\varphi_n(x)} = -J_{e\ \mathrm{const}} F_e(x) + 1 - FR_e(x). \tag{46}$$

Collector Domain, $M \leq x \leq R$

Define

$$F_c(x) = \int_x^R \gamma_n(t) e^{-\psi(t)} \tag{47}$$

$$FR_c(x) = \int_x^R J_r(t)\gamma_n(t) e^{-\psi(t)}\, dt \tag{48}$$

$$J_{c\ \mathrm{const}} = -\frac{1 - e^{-\varphi_n(R)} + FR_c(M)}{F_c(M)}. \tag{49}$$

Then the electron current is

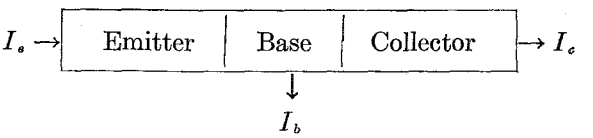
$$J_n(x) = J_{c\ \mathrm{const}} + J_r(x) \tag{50}$$

and the EQFP is given by

$$e^{-\varphi_n(x)} = -J_{c\ \mathrm{const}} F_c(x) + e^{-\varphi_n(R)} - FR_c(x). \tag{51}$$

Terminal Currents

Sign Convention: Currents are considered positive when they flow as in an actively biased *p-n-p* transistor.



The emitter current density is

$$I_e = J_p(0) + J_e(0) = J_{p\ \mathrm{const}} + J_{e\ \mathrm{const}}.$$

The collector current density is

$$I_c = J_p(R) + J_c(R) = J_{p\ \mathrm{const}} + J_{c\ \mathrm{const}}.$$

The base current density is

$$I_b = I_e - I_c = J_e(M) - J_c(M) = J_{e\ \mathrm{const}} - J_{c\ \mathrm{const}}.$$

Appendix II

Generalized Boundary Conditions

The boundary conditions of Section III can be generalized such that the carrier concentration at the boundary equals the equilibrium concentration plus a term proportional to the particle current density at the boundary. For example, the concentration of electrons at the emitter boundary may be specified

$$n(0) = n_0(0) \pm \frac{J_n(0)}{v_e} \tag{52}$$

where $n_0(0)$ is the equilibrium electron concentration and v_e is a normalized velocity. The sign of the additional term is the same as that of the coefficient v_e, (*i.e.*, carriers are added for positive v_e) if the carriers flow towards the boundary. The converse holds if the carriers flow away from the boundary.

For carriers moving toward the boundary and positive v_e, this condition states that at low current densities the electron concentration at the emitter boundary is the equilibrium concentration, while for high current densities the electron concentration is adjusted such that the electrons move with velocity v_e. Such a boundary condition describes surface recombination where v_e is then the surface recombination velocity. If the boundary represents a contact to material having a low equilibrium electron concentration, this boundary condition should be used with v_e equal to the scatter limited velocity. Then, even for high currents, the carriers are not required to move with average velocities exceeding the scatter-limited velocity.

By the method of this paper, a boundary condition of type (52) can easily be applied for electrons at the emitter and collector and for holes at either the emitter or collector, but not both simultaneously. As will become apparent below, the reason for the latter restriction is that the current density in (52) must be the constant component of the current, *i.e.*, the barrier current plus the average recombination current (see Section IV). For holes and electrons at the emitter, the total current equals the constant current since $J_r(0) = 0$. At the collector, the total hole and electron currents can be made equal to the constant current components if the recombination current is redefined as $J_r(x) - J_r(R)$ and if the average recombination currents are correspondingly decreased by $J_r(R)$. The total currents are invariant to this operation, and now the currents at $x = R$ equal the constant portions. Since the points $x = 0$ and $x = R$ are in different domains for the electrons but in the same domain for holes, velocity boundary conditions can be specified simultaneously at emitter and collector for electrons but not for holes. However, if recombination is zero in a domain then velocity boundary conditions may be specified at both ends of the domain. Velocity boundary conditions for the EQFP can be implemented by slight modifications of some equations of Appendix I. These modifications are shown below. Velocity boundary conditions for the HQFP are implemented analogously; the details are not shown.

Electron Current and EQFP

Emitter Domain $0 \leq x \leq M$

Replace (44), (45) and (46) by the corresponding equations with asterisks.

$$J^*_{e\,\text{const}} = -\frac{e^{-qV_{EB}/kT} - 1 + FR_e(0)}{F_e(0) + e^{-\psi(0)}/v_e} \tag{44*}$$

$$J^*_n(x) = J^*_{e\,\text{const}} + J_r(x) \tag{45*}$$

$$e^{-\varphi_n(x)} = -J^*_{e\,\text{const}}F_e(x) + 1 - FR_e(x). \tag{46*}$$

Then

$$e^{-\varphi_n(0)} = e^{-qV_{EB}/kT} + J^*_n(0)e^{-\psi(0)}/v_e. \tag{53}$$

Multiplication of (53) by $e^{\psi(0)}$ yields

$$n(0) = n_0(0) + J_n(0)/v_e. \tag{54}$$

Collector Domain $M \leq x \leq R$

Replace (48) through (51) by corresponding equations with asterisks

$$FR^*_c(x) = \int_x^R [J_r(t) - J_r(R)]\gamma(t)e^{-\psi(t)}\,dt \tag{48*}$$

$$J^*_{c\,\text{const}} = -\frac{1 - e^{qV_{CB}/kT} + FR^*_c(M)}{F_c(M) + e^{-\psi(R)}/v_c} \tag{49*}$$

$$J^*_n(x) = J^*_{c\,\text{const}} + J_r(x) - J_r(R) \tag{50*}$$

$$e^{-\varphi_n(x)} = -J^*_{c\,\text{const}}(F_c(x) + e^{-\psi(R)}/v_c) + e^{-qV_{CB}/kT} - FR_c(x). \tag{51*}$$

Then

$$e^{-\varphi_n(R)} = e^{-qV_{CB}/kT} - J^*_n(R)e^{-\psi(R)}/v_c. \tag{55}$$

Multiplication by $e^{-\psi(R)}$ yields

$$n(R) = n_0(R) - J^*_n(R)/v_c. \tag{56}$$

Appendix III

Diffusivity as Function of Doping and Field

The following analytical expression for the relative reciprocal diffusivity is suggested:

$$\gamma = \frac{D_0}{D} = \frac{\mu_0}{\mu} = \sqrt{1.0 + \frac{N}{N_C + N/A^2}} + \frac{E\mu_0}{v_{\text{scat}}}. \tag{57}$$

The mobility is reduced below its intrinsic value μ_0 by a doping term and a field term. At low fields the doping term causes a reduction as shown schematically in Fig. 12. Here N_c is the "corner" doping concentration where mobility reduction begins and A is the factor of maximum reduction.

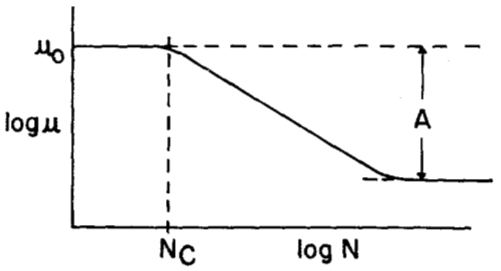


Fig. 12—Dependence of mobility on doping according to (57).

When the electric field E is large, the mobility varies inversely with field and the carriers attain the scatter limited velocity, v_{scat}.

The mobility expression given here is rather crude but it contains the essential features. A more sophisticated model may be adopted if the calculations require it and if available materials data warrant it.

Acknowledgment

The author is indebted to S. P. Morgan and to W. P. Timlake for advice on the inversion of tridiagonal matrices.

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