Boundary Value Problem (BVP)

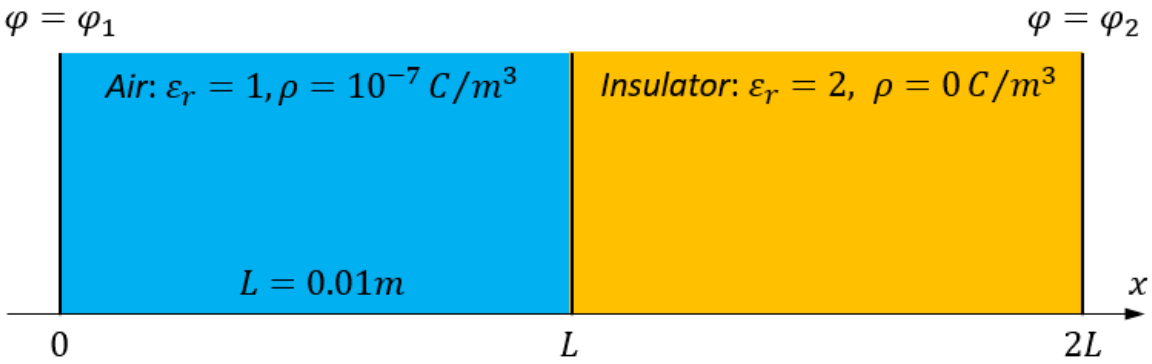

Figure 1. Description of an electrostatic problem to be solved by using 1-D FEM is presented. Two electrodes (at $x = 0$ and $x = 2L$) with the known potentials φ_1 and φ_2 are separated by two dielectric layers.

The electric scalar potential is the solution of the following BVP

$$-\frac{d}{dx}\left(\varepsilon\frac{d\varphi}{dx}\right) = \rho \text{ in } \Omega = \left\{x \middle| x \geq 0 \wedge x \leq 2L\right\} \tag{1}$$

$$\varphi = \varphi_1 \text{ at } x = 0 \tag{2}$$

$$\varphi = \varphi_2 \text{ at } x = 2L \tag{3}$$

In addition to the boundary conditions (BCs) (2) and (3) the following interface conditions should be fulfilled

$$\varphi\left(x^-\right) = \varphi\left(x^+\right) \text{ at } x = L \tag{4}$$

$$\varepsilon\left(x^-\right)\frac{d\varphi}{dx}\left(x^-\right) = \varepsilon\left(x^+\right)\frac{d\varphi}{dx}\left(x^+\right) \text{ at } x = L \tag{5}$$

Weak Form

The electric energy of the system presented in Fig. 1 can be written in the following way

$$W_e = \frac{1}{2}\int_0^{2L}\rho\varphi dx = \frac{1}{2}\int_0^{2L}\varepsilon\left(\frac{d\varphi}{dx}\right)^2 dx \tag{6}$$

Based on the energy (6) it is possible to write the following energy functional

$$V^G = \int_0^{2L}\rho\varphi dx - \frac{1}{2}\int_0^{2L}\varepsilon\left(\frac{d\varphi}{dx}\right)^2 dx \tag{7}$$

that also represents the total energy of the system written in a little unusual form.

If the solution φ minimizes the energy functional (7), the variation of this functional with respect to any variation of the unknown function should be zero

$$\delta V^G = 0, \ \forall \delta\varphi \neq 0 \tag{8}$$

$$\delta V^G = \int_0^{2L} \rho \, \delta\varphi dx - \underbrace{\int_0^{2L} \varepsilon \frac{d\varphi}{dx} \frac{d}{dx}(\delta\varphi) dx}_{Integration\ by\ parts} \tag{9}$$

$$\delta V^G = \int_0^{2L} \rho \, \delta\varphi dx - \varepsilon(2L) \frac{d\varphi}{dx}(2L) \underbrace{\delta\varphi(2L)}_{0} + \varepsilon(0) \frac{d\varphi}{dx}(0) \underbrace{\delta\varphi(0)}_{0} + \int_0^{2L} \delta\varphi \frac{d}{dx}\left(\varepsilon \frac{d\varphi}{dx} \right) dx \tag{10}$$

The variation of the unknown function over Dirichlet boundaries is equal to zero, as the Dirichlet condition is explicitly enforced.

$$\delta V^G = \int_0^{2L} \rho \, \delta\varphi dx + \int_0^{2L} \delta\varphi \frac{d}{dx}\left(\varepsilon \frac{d\varphi}{dx} \right) dx = 0, \forall \delta\varphi \neq 0 \quad \Rightarrow \quad \frac{d}{dx}\left(\varepsilon \frac{d\varphi}{dx} \right) + \rho = 0 \tag{11}$$

We have just proven the following: finding the unknown function φ that minimizes the functional (7) is equivalent to solving the BVP (1)-(3).

FEM Discretization

The computational domain is discretized by introducing the following set of line elements

$$\Omega = \bigcup_{e=1}^{N_e} L_e, \ L_e = \left\{ x \middle| x \leq x_1^e \wedge x \geq x_2^e \right\} \tag{12}$$

The unknown function φ is approximated over line elements in the following way

$$\varphi(x) = \sum_{j=1}^{Nn} N_j(x) \varphi_j \tag{13}$$

where N_j is the shape function related to the node j of the mesh having a nonzero value only over those line elements attached to the node j, φ_j is the value of the unknown function at the node j, N_n is the number of nodes, and N_e is the number of line elements.
According to (12) and (13) the energy functional (7) can be discretized as follows

$$V^G \approx \sum_{e=1}^{N_e} \int_{(L_e)} \rho \left(\sum_{j=1}^{N_n} N_j \varphi_j \right) dl - \frac{1}{2} \sum_{e=1}^{N_e} \int_{(L_e)} \varepsilon \left[\frac{d}{dx} \left(\sum_{j=1}^{N_n} N_j \varphi_j \right) \right]^2 dx \tag{14}$$

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The stationary condition for finding the minimum

$$\frac{\partial V^G}{\partial \varphi_i} = 0, \quad i = 1, 2, ..., N_n \tag{16}$$

$$\frac{\partial V^G}{\partial \varphi_i} = \sum_{e=1}^{N_e} \int_{(L_e)} \rho \, N_i \, dl - \frac{1}{2} \sum_{e=1}^{N_e} \int_{(L_e)} \varepsilon 2 \left(\sum_{j=1}^{N_n} \frac{dN_j}{dx} \varphi_j \right) \frac{dN_i}{dx} dx = 0 \tag{17}$$

In Equation (17) the following identity has been used

$$\frac{\partial}{\partial \varphi_i} \left(\sum_{j=1}^{N_n} \frac{dN_j}{dx} \varphi_j \right)^2 = 2 \left(\sum_{j=1}^{N_n} \frac{dN_j}{dx} \varphi_j \right) \frac{\partial}{\partial \varphi_i} \left(\sum_{j=1}^{N_n} \frac{dN_j}{dx} \varphi_j \right) = 2 \left(\sum_{j=1}^{N_n} \frac{dN_j}{dx} \varphi_j \right) \frac{dN_i}{dx} \tag{18}$$

The sums over elements and over nodes can exchange the order in Equation (17), as they are mutually independent

$$\sum_{j=1}^{N_n} \left(\varphi_j \sum_{\substack{e \\ i,j \in L_e}} \int_{(L_e)} \varepsilon \frac{dN_i}{dx} \frac{dN_j}{dx} dx \right) = \sum_{\substack{e \\ i \in L_e}} \int_{(L_e)} N_i \, \rho \, dl \, , \; i = 1, 2, ..., N_n \tag{19}$$

The equation system (19) can be written in its matrix from

$$\left[A \right]_{N_n \times N_n} \left\{ \varphi \right\}_{N_n \times 1} = \left\{ b \right\}_{N_n \times 1} \tag{20}$$

with the matrix- and RHS-entries can be computed as follows

$$A_{ij} = \sum_{\substack{e \\ i,j \in L_e}} \int_{(L_e)} \varepsilon \frac{dN_i}{dx} \frac{dN_j}{dx} dx \tag{21}$$

$$b_i = \sum_{\substack{e \\ i \in L_e}} \int_{(L_e)} N_i \, \rho \, dl \tag{22}$$

After assembling the matrix, the Dirichlet boundary conditions are enforced as follows

$$i = 1 \quad \Rightarrow \quad A_{ij} = 0 \wedge A_{ii} = 1 \wedge b_i = \varphi_1 \tag{23}$$

$$i = N_n \quad \Rightarrow \quad A_{ij} = 0 \wedge A_{ii} = 1 \wedge b_i = \varphi_2 \tag{23}$$

FEM Postprocessing

Once the system of equations (20) is solved, the distribution of the electric scalar potential is obtained. After this step the electric field can be computed as follows

$$E_x = \frac{d\varphi}{dx}(x) = \sum_{j=1}^{Nn} \frac{dN_j}{dx}(x) \varphi_j \tag{24}$$

and the electric flux density

$$D_x = \varepsilon_0 \varepsilon_r E_x \tag{25}$$

The electric energy of the system can be computed as follows

$$W_e = \frac{1}{2}\int_0^{2L} \rho\varphi dx + \frac{1}{2}\sigma_1\varphi_1 + \frac{1}{2}\sigma_2\varphi_2 \tag{26}$$

where σ_1 is the areal charge density of the first electrode, φ_1 is the electric scalar potential of the first electrode, σ_2 is the areal charge density of the second electrode, and φ_2 is the electric scalar potential of the second electrode.

The electric energy (26) can be also written in the following different form

$$W_e = \frac{1}{2}\underbrace{\int_0^{2L} \frac{dD_x}{dx}\varphi dx}_{Integration\ by\ parts} + \frac{1}{2}D_{x1}\varphi_1 - \frac{1}{2}D_{x2}\varphi_2 \tag{27}$$

$$W_e = \frac{1}{2}\int_0^{2L} \frac{d}{dx}\left(D_x\varphi\right)dx - \frac{1}{2}\int_0^{2L} D_x\frac{d\varphi}{dx}dx + \frac{1}{2}D_{x1}\varphi_1 - \frac{1}{2}D_{x2}\varphi_2 \tag{28}$$

$$W_e = \frac{1}{2}D_{x2}\varphi_2 - \frac{1}{2}D_{x1}\varphi_1 + \frac{1}{2}\int_0^{2L} D_x E_x dx + \frac{1}{2}D_{x1}\varphi_1 - \frac{1}{2}D_{x2}\varphi_2 \tag{29}$$

$$W_e = \frac{1}{2}\int_0^{2L} D_x E_x dx = \frac{1}{2}\int_0^{2L} \varepsilon\left(\frac{d\varphi}{dx}\right)^2 dx \tag{30}$$

Numerical Results

The problem presented in Fig. 1, with the following data was solved

$$\varepsilon_{r1} = 1, \varepsilon_{r2} = 2 \tag{31}$$

$$\rho_1 = 10^{-7}\left(C/m^3\right), \rho_2 = 0\left(C/m^3\right) \tag{32}$$

$$\varphi_1 = 1.5(V), \varphi_2 = 0(V) \tag{33}$$

$$L = 0.01(m) \tag{34}$$

The mesh presented in Fig. 2 was used for solving this problem. The corresponding FEM sparse matrix is shown in Fig. 3. Due to the local character of the FEM shape functions the obtained matrix is sparse with the sparsity ratio of 97.5%, which means that 97.5% of the matrix entries is equal to zero.

The obtained electric scalar potential, electric field, and flux density are shown in Fig. 4. It is important to notice that the electric scalar potential and electric flux density are continuous over the boundary between the two insulation materials. The electric field, however, has a jump crossing this boundary, which is expected as the electric field is perpendicular to the interface between the insulators.

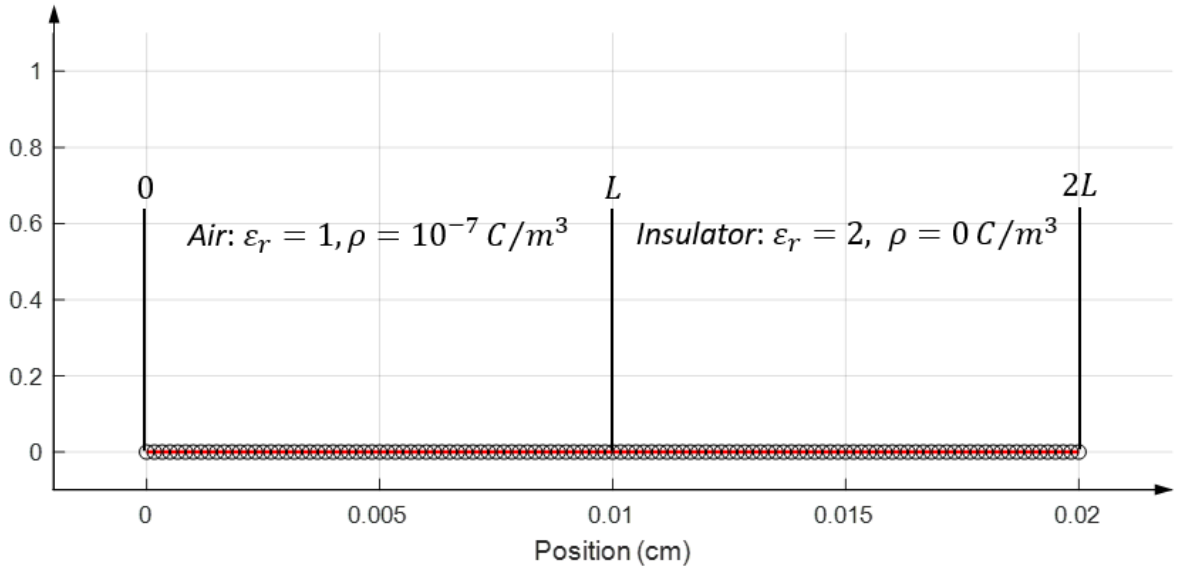

Figure 2. Mesh consisting of 120 line elements and 121 nodes used for solving the problem presented in Fig. 1 is shown. The elements are considered to be linear.

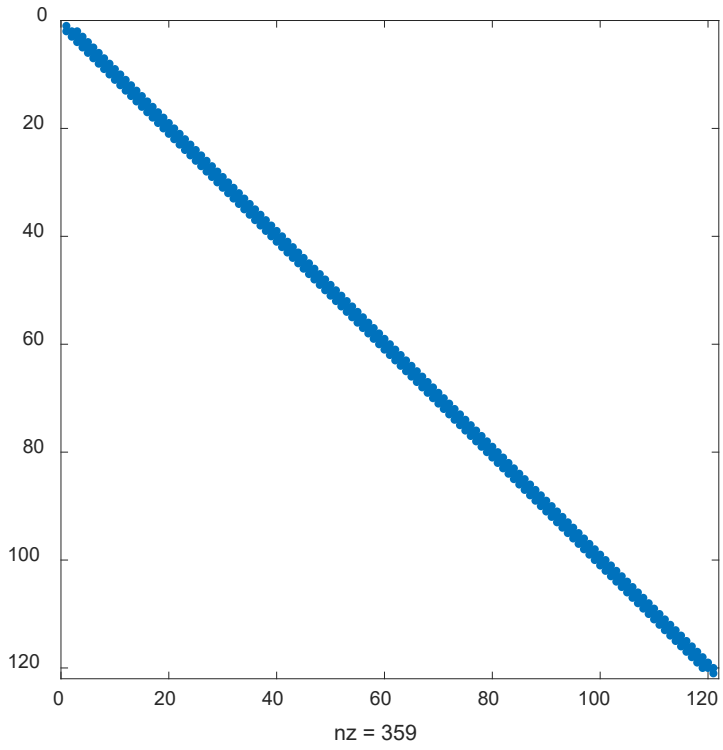

Figure 3. The FEM matrix assembly process resulted in this matrix. The matrix has 359 nonzero elements, which means that only 2.4% matrix entries are filled with nonzero matrix entries ($spasity\ ratio = 1 - N_{nz} / N_n^2 = 0.975$).

In addition to the explained field computation, a convergence study of the FEM model was carried out. The results of the convergence study are presented in Fig. 5. The element size of the mesh was gradually decreased starting with a very coarse mesh (only 2 element per material) and ending with a very fine mesh (100 elements per material). For each mesh the problem was solved and the energy according to two different approaches was computed. Both energy integrals converge from different sides to the same value, if the elements are sufficiently small.

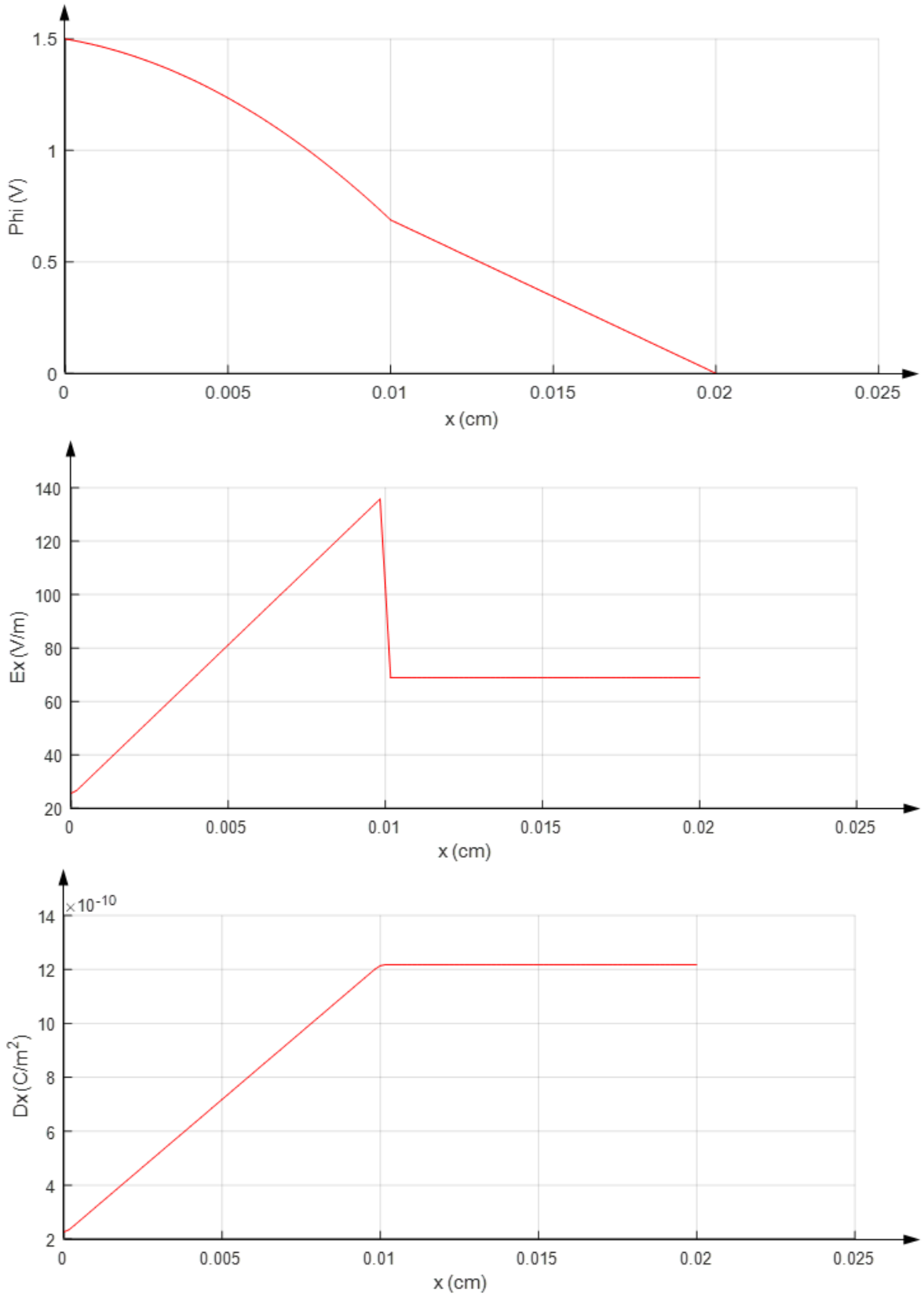

Figure 4. The obtained numerical results in the form of the electric scalar potential (top), electric field (middle), and electric flux density (bottom) are presented. It is important to notice that the electric scalar potential and the electric flux density are continuous at the interface between two materials. On the other hand the electric field has a jump (discontinuity) crossing the border between two insulators. In this arrangement, the electric field has only its normal component to the interface between different materials.

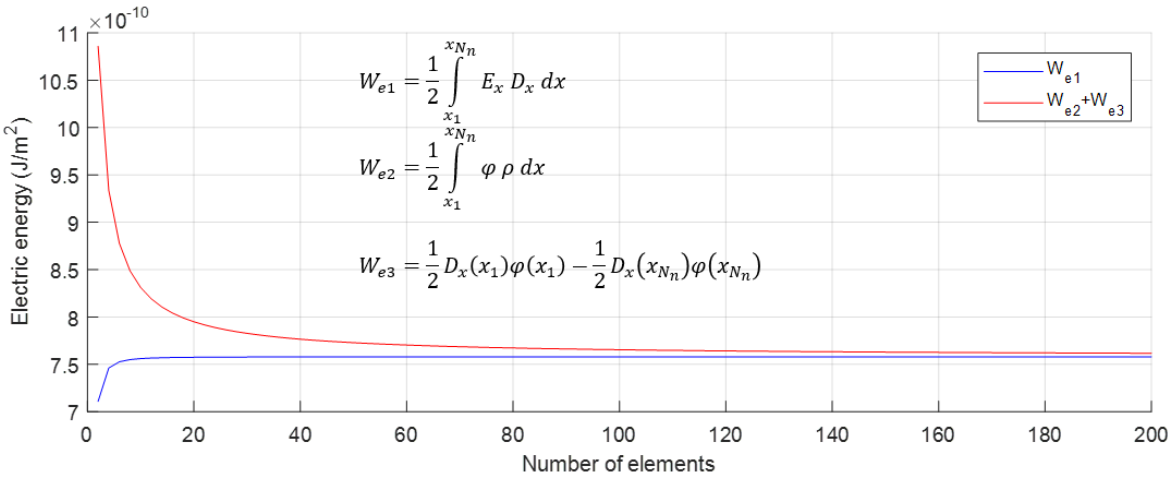

Figure 5. Convergence study of the FEM model presented in Fig. 1 is shown. The electric energy of the system was computed for different meshes, having different number of elements). The electric energy computed with the field integral (the red curve) is compared against the energy computed with the charge and potential integral (the blue curve). Both curves converge evidently to the same value of the elements size is small enough.