UNIT – III

CONTINOUS PROBABILITY DISTRIBUTION

Uniform Distribution (Rectangular Distribution)

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$

→ The pdf of uniform distribution is defined as $f(x) = \begin{cases} 1/b-a & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$ where a, b are parameter

Mean of Uniform Distribution

$$\text{Mean} = \int_{-\infty}^{\infty} x f(x)\, dx$$

Here in Uniform distribution

$$\text{Mean} = \int_a^b x \frac{1}{b-a}\, dx$$

$$\Rightarrow \frac{1}{b-a} \int_a^b x\, dx$$

$$\Rightarrow \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b \Rightarrow \frac{1}{b-a}\left[\frac{b^2}{2} - \frac{a^2}{2}\right]$$

$$\Rightarrow \frac{b^2-a^2}{2(b-a)} \Rightarrow \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} \Rightarrow \frac{b+a}{2}$$

Variance of Uniform Distribution

$$\text{Variance} = V[X] = \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

$$\text{Variance U.D} = \int_{a}^{b} x^2 \frac{1}{b-a}\,dx - \mu^2$$

$$\Rightarrow \frac{1}{b-a}\int_{a}^{b} x^2\,dx - \left(\frac{b+a}{2}\right)^2$$

$$\Rightarrow \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b - \frac{b^2+a^2+2ab}{4}$$

$$\Rightarrow \frac{b^3-a^3}{3(b-a)} - \frac{b^2+a^2+2ab}{4}$$

$$\Rightarrow \frac{\cancel{(b-a)}(a^2+b^2+ab)}{3\cancel{(b-a)}} - \frac{b^2+a^2+2ab}{4}$$

$$\frac{2a^2+2b^2+4ab-3b^2-3a^2-6ab}{12}$$

$$\frac{+b^2+a^2-2ab}{12}$$

$$\cancel{-(b^2-a^2+2ab)} \quad \frac{(a-b)^2}{12}$$

MGF of Uniform distribution

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

$$M_x(t) = \int_a^b e^{tx} \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \int_a^b e^{tx}\, dx$$

$$= \frac{1}{b-a} \left[\frac{e^{tx}}{t} \right]_a^b$$

$$M(x)t = \frac{e^{bt-at}}{t(b-a)}$$

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A uniform distribution is given by $f(x) = 1$, $0 \le x \le 1$ find the mean & variance

Sol Here $a = 0$

$b = 1$

Mean of U.D $= \frac{b+a}{2} = \frac{1+0}{2} = 1/2$

Variance of U.D $= \frac{(b-a)^2}{12} = \frac{(1-0)^2}{12}$

$= 1/12$

② The C.R.V of uniform D with mean = 1 variance = 3 find $P(x<0)$

$$U.D = f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{elsewhere} \end{cases}$$

given mean $= \frac{b+a}{2} = 1$

$\Rightarrow b + a = 2$ — ①

Variance $\Rightarrow \frac{(b-a)^2}{12} = 3$

$= (b-a)^2 = 36$

$(b-a) = \pm\sqrt{36}$

$b-a=-6$ ✗

$b-a=6$ ✓ — ②

Solving ① & ②

$b+\cancel{a}=2$

$b-\cancel{a}=6$

$2b=8$

$\boxed{b=4}$

$4+a=2$

$\boxed{a=-2}$

$$f(x) = \begin{cases} \frac{1}{4+2} & , -2 \le x \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

$$f(x) = \begin{cases} 1/6 & -2 \le x \le 4 \\ 0 & \text{elsewhere} \end{cases}$$

$$P(x<0) = \int_{-2}^{0} 1/6 \, dx \Rightarrow \frac{1}{6}[x]_{-2}^{0}$$

$$\frac{1}{6}[0-(-2)] \Rightarrow \frac{1}{\cancel{6}_3}(\cancel{2})$$

$$\boxed{P(x<0) = 1/3}$$

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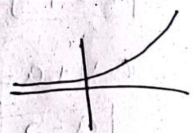
□ Exponential Distribution

→ In an E.D RV x takes values 0 to ∞, the pdf of x is then defined as :

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & x>0,\ \lambda>0 \\ 0 & \text{, otherwise} \end{cases}$$



Mean of Exponential Distribution

$$\text{Mean} = \int_0^\infty x \lambda e^{-\lambda x}\, dx$$

$$\lambda \int_0^\infty x e^{-\lambda x}\, dx$$

$$\lambda \left[\frac{-x e^{-\lambda x}}{\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^\infty$$

x	$e^{-\lambda x}$
1	$\frac{-e^{-\lambda x}}{\lambda}$
0	$\frac{e^{-\lambda x}}{\lambda^2}$

$$\cancel{-xe^{-\lambda x} - \frac{e^{-\lambda x}}{\lambda}}$$

$$\cancel{-x\lambda e^{-\lambda x} - e^{-\lambda x}}$$

$$\lambda \left[\frac{e^{-\lambda x}}{\lambda^2} \right]_0^\infty$$

$$\boxed{\text{Mean} = 1/\lambda}$$

Variance of Exponential Distribution

$$\cancel{V[X] = E[X^2] - [E[X]]^2}$$

$$\text{variance} = \int_{-\infty}^{\infty} x^2 f(x) - \mu^2$$

$$\text{Variance} = \int_0^\infty x^2 \lambda e^{-\lambda x}\, dx - \left(\frac{1}{\lambda}\right)^2$$

$$= \lambda \int_0^\infty x^2 e^{-\lambda x}\, dx - \frac{1}{\lambda^2}$$

$$\Rightarrow \lambda \left[\frac{-x^2 e^{-\lambda x}}{\lambda} - \frac{2x e^{-\lambda x}}{\lambda^2} - \frac{2e^{-\lambda x}}{\lambda^3} \right]_0^\infty$$

x^2	$e^{-\lambda x}$
$2x$	$\frac{-e^{-\lambda x}}{\lambda}$
2	$+\frac{e^{-\lambda x}}{\lambda^2}$
0	$\frac{-e^{-\lambda x}}{\lambda^3}$

$$\Rightarrow \lambda \left[0-0-0-\left(0-0+\frac{2}{\lambda^3}\right) \right] - \frac{1}{\lambda^2}$$

$$2\left(\frac{1}{\lambda^2}\right) - \frac{1}{\lambda^2}$$

$$\boxed{\text{Variance} = \frac{1}{\lambda^2}}$$

M.G.F of Exponential Distribution

$$M_{(x)}t = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

$$= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\,dx$$

$$\lambda \int_0^{\infty} e^{tx} e^{-\lambda x}\,dx$$

$$\lambda \int_0^{\infty} e^{tx-\lambda x}\,dx$$

$$\lambda \int_0^{\infty} e^{x(t-\lambda)}\,dx$$

$$\lambda \left[\frac{e^{-x(\lambda-t)}}{\cancel{x}(\lambda-t)} \right]_0^{\infty}$$

$$\lambda \left[\frac{e^{-x(\lambda-t)}}{\lambda-t} \right]_0^{\infty}$$

$$\lambda \left[\frac{1}{\lambda-t} \right]$$

$$\boxed{M_{(x)}(t) = \frac{\lambda}{\lambda-t}}$$

Q → If X is a C.R.V with p.d.f given by $f(x) = \begin{cases} \frac{1}{4} e^{-x/4} & \text{where } x>0 \\ 0 & \text{otherwise} \end{cases}$

find mean & variance

Mean = 4

Variance = 16

Q → $f(x) = \begin{cases} \frac{1}{b} e^{x/b} & x>0 \\ 0 & \text{otherwise} \end{cases}$

mean = b

variance = b^2

28/12/2022

Mean of Normal Distribution

Mean $= \int_{-\infty}^{\infty} x f(x)\,dx$

Mean $= \int_{-\infty}^{\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

put $\frac{x-\mu}{\sigma} = z$

$x-\mu = \sigma z$

$dx = \sigma dz$

$x = \sigma z + \mu$

$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{1}{2} z^2} \cancel{\sigma} dz$

$= \frac{1}{\sqrt{2\pi}} \left\{ \mu \int_{-\infty}^{\infty} e^{-z^2/2} dz + \sigma \int_{-\infty}^{\infty} z e^{-z^2/2} dz \right\}$

even funcⁿ $\Rightarrow f(-x) = f(x)$

odd funcⁿ $\rightarrow f(-x) = -f(x)$

$\int_{-a}^{a} f(x)dx = 0$ if $f(x)$ is odd

$\int_{-a}^{a} f(x)dx = 2\int_{0}^{a} f(x)dx$ if $f(x)$ is even

$\Rightarrow \frac{1}{\sqrt{2\pi}} \left[2\mu \int_{0}^{\infty} e^{-z^2/2} dz + 0 \right]$

$= \frac{2\mu}{\sqrt{2}\sqrt{\pi}} \int_{0}^{\infty} e^{-z^2/2} dz$

put $\frac{z^2}{2} = p$

$\frac{2z\,dz}{2} = dp$

$dz = \frac{dp}{z}$

$dz = \frac{dp}{\sqrt{2}\sqrt{p}}$

$= \frac{\cancel{2}\mu}{\cancel{\sqrt{2}}\sqrt{\pi}} \int_{0}^{\infty} \frac{dp}{\cancel{\sqrt{2}}\sqrt{p}}$

$= \frac{\mu}{\sqrt{\pi}} \int_{0}^{\infty} e^{-p} p^{-1/2}$

$\sqrt{\pi} = \Gamma_{1/2} = \int_{0}^{\infty} e^{-p} p^{1/2 - 1} dp \Rightarrow \frac{\mu}{\cancel{\sqrt{\pi}}} \cancel{\sqrt{\pi}} = \mu$

Variance of Normal Distribution

$V[X] = \int_{-\infty}^{\infty} x^2 f(x)dx - \mu^2$

also

$V[X] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx = \int_{-\infty}^{\infty} x^2 f(x)dx + \mu^2 \int_{-\infty}^{\infty} f(x)dx - 2\mu \int_{-\infty}^{\infty} x f(x)dx$

$V[X] = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx$

Here $V[X] = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2 z^2 e^{-\frac{1}{2} z^2} \cancel{\sigma} dz$

$= \frac{\sigma^2}{\sqrt{2}\sqrt{\pi}} \int_{-\infty}^{\infty} z^2 e^{-z^2/2} dz$

$= \frac{2\sigma^2}{\sqrt{2}\sqrt{\pi}} \int_{0}^{\infty} z^2 e^{-z^2/2} dz$

$$= \frac{2\sigma^2}{\sqrt{2}\sqrt{\pi}} \int_0^\infty 2pe^{-p} \frac{dp}{\sqrt{2}\sqrt{p}}$$

Put $\frac{z^2}{2} = p \quad z^2 = 2p$

$\frac{2z\,dz}{2} = dp$

$dz = \frac{dp}{z}$

$dz = \frac{dp}{\sqrt{2}\sqrt{p}}$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty e^{-p} p^{1/2} dp$$

$\Gamma 3/2$

$$\text{Variance} = \frac{2\sigma^2}{\sqrt{\pi}} \Gamma 3/2$$

$\Gamma n = \Gamma n+1 = n\Gamma n$

$\Gamma 3/2 = \Gamma 1/2 + 1 = 1/2 \Gamma 1/2$

$= \sqrt{\pi}/2$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = \sigma^2$$

☐ M.G.F of Normal Distribution

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x) dx$$

$$\text{Here } M_x(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{-1/2\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $\frac{x-\mu}{\sigma} = z$

$x - \mu = \sigma z$

$dx = \sigma dz$

$x = \mu + \sigma z$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{(-1/2 z^2)} \cancel{\sigma} dz$$

$$= \frac{1}{\sqrt{2}\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\mu t} \cdot e^{t\sigma z} e^{-z^2/2} dz$$

$$= \frac{1}{\sqrt{2}\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\mu t} \cdot e^{t\sigma z} e^{-z^2/2} \cdot e^{-t^2\sigma^2/2} e^{\frac{t^2}{2}\sigma^2} dz.$$

$$= \frac{e^{\mu t} e^{t^2\sigma^2/2}}{\sqrt{2}\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-z^2/2} e^{-t^2\sigma^2/2} e^{t\sigma z} dz$$

$$= \frac{e^{\mu t + 1/2 t^2\sigma^2}}{\sqrt{2}\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-p^2/2} dp$$

$$M_x(t) = \frac{2e^{\mu t + 1/2 t^2\sigma^2}}{\sqrt{2}\sqrt{\pi}} \int_0^\infty e^{-p^2/2} dp$$

$\frac{p^2}{2} = q$

$p = \sqrt{2}\sqrt{q}$

$\frac{2p\,dp}{2} = dq$

$dp = \frac{dq}{\sqrt{2}\sqrt{q}}$

$$= \frac{\cancel{2} e^{\mu t + 1/2 t^2\sigma^2}}{\cancel{\sqrt{2}}\sqrt{\pi}} \int_0^\infty e^{-q} \frac{dq}{\cancel{\sqrt{2}}\sqrt{q}}$$

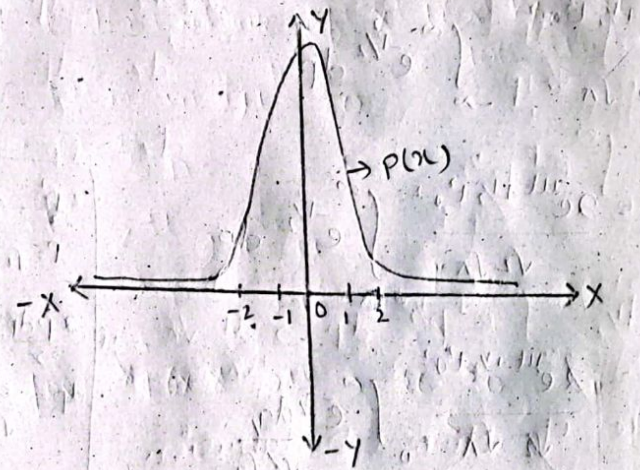
$$= \frac{e^{\mu t + 1/2 t^2\sigma}}{\sqrt{\pi}} \int_0^\infty e^{-q} q^{-1/2} dq$$

$$M_x(t) = \frac{e^{\mu t} + 1/2 t^2\sigma^2 \cancel{\sqrt{\pi}}}{\cancel{\sqrt{\pi}}}$$

$$M_x(t) = e^{\mu t + 1/2 t^2\sigma^2}$$

30/12/2022

□ Properties of Normal Distribution



The above is a famous bell shape curve

→ The top of the bell is directly above the mean

→ The curve is symmetry about the line ($x = \mu$), it has same shape on either side of line

→ Since the distribution is symmetrical, mean, medium, mode coincide

→ No portion of the curve lies below x-axis since P(x) being the probability can never be negetive

Q → A Normal distribution has a mean 20 & std. deviation 4 find out the probabity that the value of x lies b/w 20 & 24

Sol Mean = $\mu = 20$

S.D $= \sigma = 4$

$P(20 \leq x \leq 24)$

w.k.T $z = \dfrac{x-\mu}{\sigma}$ (std. Normal variant)

$$z = \frac{x-20}{4}$$

when $x = 20$

$$\frac{20-20}{4} = 0$$

when $x = 24$

$$\frac{24-20}{4} = 1$$

$\therefore P(20 \leq x \leq 24) = P(0 \leq z \leq 1)$

$= 0.3413$

Q → x is a normal variant with mean 30 & S.D = 5 find out the probability that the value of x lies b/w 26 & 40

mean $= \mu = 30$

$S.D = \sigma = 5$

$$z = \frac{x-\mu}{\sigma}$$

for $x = 26 \Rightarrow \frac{26-30}{5}$

$= -0.8$

for $x = 40 = \frac{10}{5} = 2$

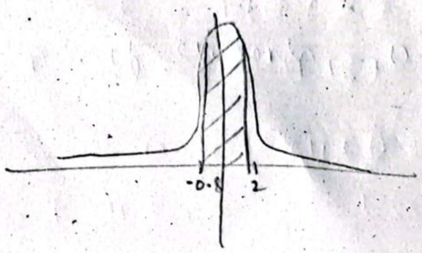
$P(26 \le X \le 40) = P(-0.8 \le Z \le 2)$

$= P(-0.8 \le Z \le 0) + P(0 \le Z \le 2)$

$= P(0 \le Z \le 0.8) + P(0 \le Z \le 2)$

$= 0.2881 + 0.4772$

$P(26 \le X \le 40) = 0.7653$



③ find the probability b/w 15 & 60 given that $\mu = 40$ & $\sigma = 10$

mean $= \mu = 40$

$S.D = \sigma = 10$

$$z = \frac{x-\mu}{\sigma} = \frac{x-40}{10}$$

for $x = 15 = \frac{-25}{10} = -2.5$

for $x = 60 = \frac{20}{10} = 2$

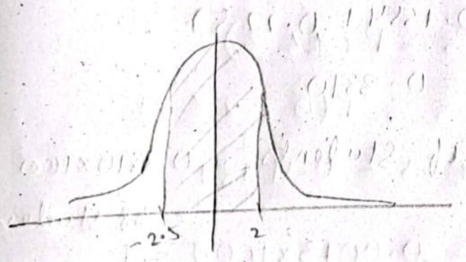
$P(15 \le x \le 60) = P(-2.5 \le Z \le 2)$

$P(-2.5 \le Z \le 0) + P(0 \le Z \le 2)$

$= P(0 \le Z \le 2.5) + P(0 \le Z \le 2)$

$= 0.4938 + 0.4772$

$= 0.971$



Q → Given the mean heights of students is 158 cm with S.D 20 cms find how many students height lies between 150cm & 170cm. If there are 100 students in a class

Sol. $\mu = 158$

$S.D = \sigma = 20$

$$Z = \frac{x-\mu}{\sigma}$$

$$Z = \cancel{100}\ \frac{150-158}{20} = \frac{-8}{20} = -0.4$$

$$Z = \frac{170-188}{20} = 0.6$$

$$\cancel{P(150 \le x \le 158) = P(0.4 \le Z \le 0.6)}$$

$$\cancel{= 0.2389}$$

$0.1554 + 0.2257$

$= 0.3810$

no. of students $= 0.3810 \times 100$

$= 38$ students

Q → 1000 students had written an exam the mean of the test is 35 & SD = 5 assuming the distibution to be normal find (i) How many students marks lies b/w 25 & 40.

(ii) How many students get more than 40

(iii) How many students get below 20

$\mu = 35$

$\sigma = 5$

$$Z = \frac{x+\mu}{5}$$

$$Z\ \cancel{25}\ \frac{25-35}{5} = -2$$

$$Z = \frac{40-35}{5} = +1$$

$$P(25 \le x \le 40) = P(-2 \le Z \le 1)$$

$$P(0 \le Z \le 2) + P(0 \le Z \le 1)$$

$$= 0.4772 + 0.3412$$

$$= 0.8184$$

$0.8184 \times 1000 = 818$ students got marks in b/w 25 & 40

(ii) $P(x > 40)$

$X = 40 = Z = \frac{X - \mu}{\sigma}$

$= \frac{40 - 35}{5}$

$= 1$

$0.5 - P(\cancel{Z} > 1\cancel{00})$

$= 0.5 - P(0 \le Z \le 1)$

$= 0.5 - 0.3413$

$= 0.1587$

$0.1587 \times 1000 \approx 159$ students got more than 40.

(iii) $P(X < 20)$

$X = 20 = Z = \frac{20 - 35}{5}$

$= +3$

$0.5 - P(\cancel{Z} < 3)$

$0.5 - P(0 \le Z \le 3)$

$0.5 - 0.4987$

$= 0.0013 \times 1000$

1 student got less than 20 marks

Q → 66 students wrote M-3 exam:

Mean = 28 S.D = 5

find (i) how many students lies between 25 and 30

(ii) More than 40

(iii) below 20

Sol $\mu = 28$

$\sigma = 5$

(i) $Z = \frac{X - \mu}{\sigma}$

$= \frac{25 - 28}{5}$

$= \frac{-3}{5} = -0.6$

$Z = \frac{30 - 28}{5}$

$= \frac{2}{5} = 0.4$

$P(-0.6 \le Z \le 0.4)$

$P(0 \le Z \le 0.6) + P(0 \le Z \le 0.4)$

$= 0.2257 + 0.1554 \Rightarrow 0.3811 \times 66$

$= 25.15 \sim 25$ students got b/w 25 & 30 marks