Expt. No. .... Page No. 1 Date : ....

Name : Syed Abdul Kareem Ahmed

Roll no : 1604-21-748-048 (AI ML) - III Sem

M-III Unit III Assignment

1. Find Variance and MGF & Mean of Uniform distribution.

Ans: Mean : Uniform Distribution is defined as

$$f(x) = \begin{cases} \frac{1}{b-a} & , a \le x \le b \\ 0 & , \text{Elsewhere} \end{cases}$$

where a & b are parameters

$$\text{Mean} = \mu = E[X] = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

Here, $\text{Mean} = \int_a^b x \cdot \frac{1}{b-a}\, dx$

$$= \frac{1}{b-a} \int_a^b x \cdot dx$$

$$= \frac{1}{b-a} \left[\frac{x^2}{2}\right]_a^b = \frac{b^2 - a^2}{2(b-a)}$$

$$= \frac{\cancel{(b-a)}(b+a)}{2\cancel{(b-a)}} = \boxed{\frac{b+a}{2} = \text{Mean}}$$

Roll No. : .... Lecturer's Signature : ....

②

Variance: Uniform Distribution is defined as

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{Elsewhere} \end{cases}$$

where a, b are parameters

$$\text{Variance} = V[X] = \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

then Variance $= \int_a^b x^2 \cdot \frac{1}{b-a}\,dx - \left(\frac{b+a}{2}\right)^2$

$$= \frac{1}{b-a}\int_a^b x^2\,dx - \frac{(b+a)^2}{4}$$

$$= \frac{1}{b-a}\left[\frac{x^3}{3}\right]_a^b - \frac{(b+a)^2}{4}$$

$$= \frac{(b^3-a^3)}{3(b-a)} - \frac{(b+a)^2}{4}$$

$$= \frac{\cancel{(b-a)}(b^2+ab+a^2)}{3\cancel{(b-a)}} - \frac{(b^2+2ab+a^2)}{4}$$

$$= \frac{4(b^2+ab+a^2) - (3)(b^2+2ab+a^2)}{12}$$

$$\frac{b^2+a^2-2ab}{12} = \frac{(b-a)^2}{12}$$

$$\boxed{\text{Variance} = \frac{(b-a)^2}{12}}$$

MGF: Uniform Distribution is defined as

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b \\ 0, & \text{elsewhere} \end{cases}$$

where a & b are parameters

$$MGF = M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\,dx$$

here, $M_x(t) = \int_a^b e^{tx} f(x)\,dx$

$$M_x(t) = \int_a^b e^{tx} \frac{1}{b-a}\,dx = \frac{1}{b-a}\int_a^b e^{tx}\,dx$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b = \frac{e^{bt}-e^{at}}{t(b-a)}$$

$$\boxed{M_x(t) = \frac{e^{bt}-e^{at}}{t(b-a)}}$$

2. Find Mean, Variance and MGF of Exponential Distribution.

Ans: **Mean**: exponential distribution is defined as

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{Elsewhere} \end{cases}$$

provided $\lambda > 0$

$$\text{Mean} = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

Here, $\text{Mean} = \int_0^{\infty} x \cdot \lambda e^{-\lambda x}\, dx$

$$= \lambda \int_0^{\infty} x \cdot e^{-\lambda x}\, dx$$

Integration by parts

$u \to x \quad v_1 \to e^{-\lambda x}$

$u' \to 1 \quad v_1 \to \frac{-e^{-\lambda x}}{\lambda}$

$u'' \to 0 \quad v_2 \to \frac{e^{-\lambda x}}{\lambda^2}$

$$= \lambda [u v_1 - u' v_2]_0^{\infty}$$

$$= \lambda \left[x \cdot \left(\frac{-e^{-\lambda x}}{\lambda}\right) - 1\left(\frac{e^{-\lambda x}}{\lambda^2}\right)\right]_0^{\infty}$$

$$= \lambda \left[\{\infty(0) - (0)\} - \{0 - \frac{1}{\lambda^2}\}\right]$$

$$= \lambda \left[0 - \left(-\frac{1}{\lambda^2}\right)\right] = \lambda \left[\frac{1}{\lambda^2}\right] = \boxed{\frac{1}{\lambda} = \text{Mean}}$$

Expt. No. .... Page No. .... Date : ....

Variance: Exponential Distribution is defined as

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{Elsewhere} \end{cases} \quad \text{provided } \lambda > 0$$

$$\text{Variance} = \int_{-\infty}^{\infty} x^2 \cdot f(x)\, dx - \mu^2$$

Here, $\text{Variance} = \sigma^2 = \int_0^{\infty} x^2 (\lambda e^{-\lambda x})\, dx - \left(\frac{1}{\lambda}\right)^2$

$$= \lambda \int_0^{\infty} \frac{x^2}{4} \frac{e^{-\lambda}}{\lambda} dx - \frac{1}{\lambda^2}$$

Integration by parts

$u \to x^2 \quad v \to e^{-\lambda x}$

$u' \to 2x \quad v_1 \to \frac{-e^{-\lambda x}}{\lambda}$

$u'' \to 2 \quad v_2 \to \frac{e^{-\lambda x}}{\lambda^2}$

$u''' \to 0 \quad v_3 \to \frac{-e^{-\lambda x}}{\lambda^3}$

$$= \lambda \left[\frac{x^2 \cdot e^{-\lambda x}}{\lambda} - \frac{2x e^{-\lambda x}}{\lambda^2} - \frac{2e^{-\lambda x}}{\lambda^3}\right]_0^{\infty} - \frac{1}{\lambda^2}$$

$$= \lambda \left[0 - \left(-\frac{2}{\lambda^3}\right)\right] - \frac{1}{\lambda^2}$$

$$= \cancel{\lambda}\left(\frac{2}{\lambda^{\cancel{3}}}\right) - \frac{1}{\lambda^2} = \frac{2}{\lambda^2} - \frac{1}{\lambda^2}$$

$$\boxed{\text{Variance} = \frac{1}{\lambda^2}}$$

Roll No. : .... Lecturer's Signature : ....

MGF : Exponential distribution is defined as as

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & \text{elsewhere} \end{cases}$$ provided $\lambda > 0$

$$MGF = M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

Here, $$M_x(t) = \int_0^{\infty} e^{tx} \lambda e^{-\lambda x}\, dx$$

$$= \lambda \int_0^{\infty} e^{-\lambda x} e^{tx}\, dx$$

$$= \lambda \int_0^{\infty} e^{-(\lambda - t)x}\, dx$$

$$M_x(t) = \lambda \left[\frac{e^{-(\lambda-t)x}}{-(\lambda-t)} \right]_0^{\infty}$$

$$= \frac{\lambda}{-(\lambda-t)} [0-1]$$

$$\boxed{M_x(t) = \frac{\lambda}{\lambda - t}}$$

3. Find Mean, Variance and MGF of Normal Distribution.

Ans: **Mean** : Normal Distribution is defined as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$\text{Mean} = \int_{-\infty}^{\infty} x \cdot f(x)\, dx$$

Here, $$\text{Mean} = \int_{-\infty}^{\infty} x \cdot \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} x e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{z^2}{2}} \cancel{\sigma}\, dz$$

Put $z = \frac{x-\mu}{\sigma}$

$\sigma z = x - \mu$

$x = \mu + \sigma z$

$dx = \sigma dz$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\mu + \sigma z) e^{-\frac{z^2}{2}} dz$$

$$= \frac{1}{\sqrt{2\pi}} \left[\mu \int_{-\infty}^{\infty} e^{-\frac{z^2}{2}} dz + \sigma \int_{-\infty}^{\infty} z e^{-\frac{z^2}{2}} dz \right]$$

$$= \frac{1}{\sqrt{2\pi}} \left[\mu \cdot 2 \int_0^{\infty} e^{-\frac{z^2}{2}} dz + \sigma(0) \right]$$

$\left[\because f(z) = e^{\frac{-z^2}{2}} \text{ is even} \right.$

and $\left. f(z) = 2e^{\frac{-z^2}{2}} \text{ is odd} \right]$

$\int_{-a}^{a} f(x)\,dx = 0$, if $f(x)$ is odd

$\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$ if $f(x)$ is even

$= \frac{1}{\sqrt{2\pi}}\left[2\mu \int_0^\infty e^{\frac{-z^2}{2}} dz\right]$

$= \frac{1}{\sqrt{2\pi}} 2\mu \int_0^\infty e^{\frac{-z^2}{2}} dz$

$= \frac{2\mu}{\sqrt{2}\sqrt{\pi}} \int_0^\infty e^{-p} \frac{dp}{\sqrt{2}\sqrt{2}}$

$= \frac{\mu}{\sqrt{\pi}} \int_0^\infty e^{-p} p^{\frac{-1}{2}} dp$

$\boxed{\Gamma n = \int_0^\infty e^{-t} t^{n-1} dt}$

gamma function

$= \frac{\mu}{\sqrt{\pi}} \int_0^\infty e^{-p} p^{\frac{1}{2}-1} dp$

$= \frac{\mu}{\sqrt{\pi}} \Gamma\frac{1}{2} \quad \left(\because \Gamma\frac{1}{2} = \sqrt{\pi}\right)$

$= \frac{\mu}{\cancel{\sqrt{\pi}}} \cdot \cancel{\sqrt{\pi}} = \mu$

$\boxed{\text{Mean} = \mu}$

$f(z) = 2e^{\frac{-z^2}{2}}$

$f(-z) = -2e^{-\frac{(-z)^2}{2}}$

$= -2e^{\frac{-z^2}{2}}$

$f(-z) = -f(z)$

$f(z) = e^{\frac{-z^2}{2}}$

$f(-z) = e^{\left(\frac{-z^2}{2}\right)} = e^{\frac{-z^2}{2}}$

$f(-z) = f(z)$

put $\frac{z^2}{2} = p \Rightarrow z^2 = 2p$

$z = \sqrt{2}\sqrt{p} \Rightarrow z^2 = 2p$

$2z\,dz = 2\,dp$

$dz = \frac{dp}{z} = \frac{dp}{\sqrt{2}\sqrt{p}}$

Expt. No. .... Page No. .... Date : ....

Variance: Normal distribution is defined as $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$

$\text{Variance} = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$

Here, $\text{variance} = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^2 e^{\frac{-1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma^2 z^2 e^{\frac{-z^2}{2}} \cancel{\sigma} dz$

$= \frac{\sigma^2}{\sqrt{2}\sqrt{\pi}} \int_{-\infty}^{\infty} z^2 e^{\frac{-z^2}{2}} dz$

$= \frac{\sigma^2}{\sqrt{2}\sqrt{\pi}} 2 \int_0^\infty z^2 e^{\frac{-z^2}{2}} dz$

$= \frac{2\sigma^2}{\cancel{\sqrt{2}}\sqrt{\pi}} \int_0^\infty 2p e^{-p} \frac{dp}{\cancel{\sqrt{2}}\sqrt{p}}$

$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty e^{-p} p \cdot p^{\frac{-1}{2}} dp$

put $\frac{x-\mu}{\sigma} = z$

$\sigma z = x - \mu$

$x = \mu + \sigma z$

$dx = \sigma\,dz$

$\because f(z) = z^2 e^{\frac{-z^2}{2}}$ is a even function

$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$

$\frac{z^2}{2} = p \Rightarrow z^2 = 2p$

$z = \sqrt{2}\sqrt{p} \Rightarrow 2z\,dz = 2\,dp$

$dz = \frac{dp}{z} = \frac{dp}{\sqrt{2}\sqrt{p}}$

Roll No. : .... Lecturer's Signature : ....

$$\frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty e^{-p} p^{\frac{1}{2}} dp$$

$$= \frac{2\sigma^2}{\sqrt{\pi}} \int_0^\infty e^{-p} p^{\frac{3}{2}-1} dp$$

$\Gamma n = \Gamma n+1 = n\Gamma n$

$\Gamma\frac{3}{2} = \Gamma\frac{1}{2}+1 = \frac{1}{2}\Gamma\frac{1}{2}$

$= \frac{\sqrt{\pi}}{2}$

$$\text{Variance} = \frac{2\sigma^2}{\sqrt{\pi}} \Gamma\frac{3}{2}$$

$$= \frac{\cancel{2}\sigma^2}{\cancel{\sqrt{\pi}}} \cdot \frac{\cancel{\sqrt{\pi}}}{\cancel{2}}$$

$$\text{Variance} = \sigma^2$$

MGF: Normal distribution is defined as

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$M_x(t) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$$

Here, $M_x(t) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{tx} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx$$

Put $z = \frac{x-\mu}{\sigma}$

$\sigma z = x-\mu$

$x = \sigma z + \mu$

$dx = \sigma dz$

$$= \frac{1}{\cancel{\sigma}\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\mu+\sigma z)} e^{\frac{-z^2}{2}} \cancel{\sigma} dz$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\mu t} e^{\sigma z t} e^{\frac{-z^2}{2}} dz$$

$$M_x(t) = \frac{1}{\sqrt{2}\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-ut} . e^{\sigma z t} . e^{\frac{-z^2}{2}} e^{\frac{-1}{2}t^2\sigma^2} e^{\frac{1}{2}t^2\sigma^2} dz$$

$$M_x(t) = \frac{e^{-ut} . e^{\frac{1}{2}\sigma^2+2}}{\sqrt{2}\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}(z-t\sigma)^2} dz$$

$$M_x(t) = \frac{e^{\mu t + \frac{1}{2}t^2\sigma^2}}{\sqrt{2}\sqrt{\pi}} \int_{-\infty}^{\infty} e^{\frac{-1}{2}p^2} dp$$

Put $z = \sigma t\, p$

$dz = dp$

$$= \frac{\cancel{2} e^{\mu t + \frac{1}{2}t^2\sigma^2}}{\cancel{\sqrt{2}}\sqrt{\pi}} \int_0^\infty e^{-q} \frac{dq}{\cancel{\sqrt{2}}\sqrt{q}}$$

$\frac{p^2}{2} = q$

$p^2 = 2q$

$p = \sqrt{2}\sqrt{q}$

$2p\,dp = 2dq$

$dp = \frac{dq}{p} = \frac{dq}{\sqrt{2}\sqrt{q}}$

$f(x) = \frac{1}{e^2} p^2$ is a even function

$$= \frac{e^{\mu t + \frac{1}{2}t^2\sigma^2}}{\sqrt{\pi}} \int_0^\infty e^{-q} q^{\frac{-1}{2}} dq$$

$$M_x(t) = \frac{e^{\mu t + \frac{1}{2}t^2\sigma^2}}{\sqrt{\pi}} \int_0^\infty e^{-q} q^{\frac{1}{2}-1} dq$$

$$= \frac{e^{\mu t + \frac{1}{2}t^2\sigma^2}}{\sqrt{\pi}} \Gamma\frac{1}{2} \qquad \left(\because \Gamma\frac{1}{2} = \int_0^\infty e^{-q} q^{\frac{1}{2}-1} dq\right)$$

$$= \frac{e^{\mu t + \frac{1}{2}t^2\sigma^2}}{\sqrt{\pi}} \sqrt{\pi} \qquad \left(\because \Gamma\frac{1}{2} = \sqrt{\pi}\right)$$

$$\boxed{M_x(t) = e^{\mu t + \frac{1}{2}t^2\sigma^2}}$$