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Assignment #1: Planning in Theory

Problem 1:

Prove that a 'bug' executing the Bug2 algorithm to move from a start position to a goal position never travels a distance greater than:

$$d_{tot} = d_{goal} + \frac{1}{2}\sum_{i=1}^{M} n_i \, p_i$$

where:

- $\boldsymbol{d_{goal}}$: Euclidean distance between start to goal (i.e. $\boldsymbol{d(start, goal) = d_{goal}}$)
- $\boldsymbol{p_i}$: perimeter of obstacle $\boldsymbol{i}$, where $\boldsymbol{i} = \{\boldsymbol{1, 2, \dots\ M}\}$ (i.e. M obstacles)
- $\boldsymbol{n_i}$: number of surfaces of obstacle $\boldsymbol{i}$ that cross the line segment from start to goal ***(m-line)***
- $\boldsymbol{q_i^H}$**:** $\boldsymbol{i}$th *hit* point along the m-line where the robot must change course since it hit an obstacle
- $\boldsymbol{q_i^L}$**:** $\boldsymbol{i}$th *leave* point on the perimeter that robot can continue along the m-line towards the goal

Solution:

Assumptions:

1. *For the upper-bound scenario, a finite path exists between start and goal positions*.
   (i.e. No obstacle is completely surrounding the start position such that there is no path to the goal point without colliding with the obstacle)
2. *There are a finite number of obstacles* $\boldsymbol{i}$ *within the finite workspace between goal and start positions.*

Proof:

To prove the upper-bound, we separate the movement of the bug2 robot into 2 types of movement:

(1) Movement on straight line segments along the *m-line* between *start*, *leave* and *hit* points until we reach the *goal*, and
(2) Movement around perimeters between *hit* and *leave* points.

- From the Bug2 algorithm lines 9-14, we know that the robot will move from the start point until it hits an obstacle along the *m-line* at *hit* point $\boldsymbol{q_i^H}$. It then follows the perimeter until re-encountering the *m-line* at point $\boldsymbol{m}$ and if this new point is on the correct side of the obstacle, meaning it satisfies the following two criteria:
  (1) $\boldsymbol{m}$ is closer to the goal than the previous hit point: i.e. $\boldsymbol{d(m, goal) < d(m, q_i^H)}$**,** and
  (2) the robot can continue along the m-line without hitting the obstacle: i.e. $\boldsymbol{q_i^H}$ is not re-encountered,

  then the point $\boldsymbol{m}$ becomes the *leave* point $\boldsymbol{q_i^L}$ and the robot continues until another hit point is encountered at $\boldsymbol{q_{i+1}^H}$**.**

- Based on the above definition of *hit* and *leave* points, that each subsequent hit and leave point must be closer to the goal: i.e.

$$\boldsymbol{d(start, goal) > d(q_i^H, goal) > d(q_i^L, goal) > d(q_{i+1}^H, goal) > d(q_{i+1}^L, goal)} \dots$$

- From this reasoning, we can now say that the total distance of all the line segments along the m-line $< \boldsymbol{d_{goal}} = \boldsymbol{d(start, goal)}$ since we know that each *hit* and *leave* point are on the m-line and because they each get closer to the *goal* than the previous point, they must all fall between the *start* and *goal* points.

Now we focus on the movement along the perimeter of obstacles, which only occurs between various hit and leave points:

- If we have a single obstacle that crosses the m-line only once, there must be two edges of the obstacle that cross the m-line (i.e. $\boldsymbol{n_i = 2}$), since the obstacle has some thickness. The two points that cross the m-line are the pair of *hit* and *leave* point $(\boldsymbol{q_i^H}, \boldsymbol{q_i^L})$. If this obstacle is more complex and crosses the m-line multiple times, we will have an additional hit and leave point for each additional time the obstacle crosses the m-line: $(\boldsymbol{q_i^H}, \boldsymbol{q_i^L}), (\boldsymbol{q_{i+1}^H}, \boldsymbol{q_{i+1}^L}), \dots, (\boldsymbol{q_{n_i}^H}, \boldsymbol{q_{n_i}^L})$, for a total of $\frac{\boldsymbol{n_i}}{\boldsymbol{2}}$ pairs of points. (**Note**. We must have a pair of points because we already specified that there must be a path to the goal and so there cannot be a scenario where we only hit an obstacle and don't leave it.)
- Since we know that each successive hit and leave point must be closer to the goal, we know that the hit point is only considered a hit point, the first time the robot reaches it. If the robot passes through it again while going around the perimeter, it is not "another" hit point because from the algorithm, we know that the robot will continue following the perimeter until the leave point, which is closer to the goal.
- For complex obstacles that cross the m-line multiple times (i.e. $\boldsymbol{n_i > 2}$), each subsequent hit point on the same obstacle may force the robot to go around the perimeter again and potentially cross over the previous hit point. In the worst case, the first hit point ($\boldsymbol{q_i^H}$) will be passed over a total of $\frac{\boldsymbol{n_i}}{\boldsymbol{2}}$ times (i.e. one for each time the same obstacle crosses the m-line). It cannot be greater than this because then that would mean the last *hit* point ($\boldsymbol{q_{n_i}^H}$) on this obstacle will be passed twice, which means that a corresponding last *leave* point was not found ($\boldsymbol{q_{n_i}^L}$). We already said this cannot happen in assumption 1 because it would mean there is no path to the goal.
- Based on this, we can say each additional time that the robot has to go over the perimeter is a cycle and there won't be more than $\frac{\boldsymbol{n_i}}{\boldsymbol{2}}$ cycles for obstacle $\boldsymbol{i}$. Based on this worst case, the robot would have to travel a distance of $\frac{\boldsymbol{n_i}}{\boldsymbol{2}} \cdot \boldsymbol{p_i}$ for obstacle $\boldsymbol{i}$**.** If we have ***M*** obstacles, we would add up this worst case term for each of the ***M*** obstacles.

As a result, we would add the worst case scenario of the two types of movements for the robot to get the upper bound:

$$\boldsymbol{d_{line-segmentfollowing}} < \boldsymbol{d_{goal}}, \text{ and } \boldsymbol{d_{perimeterfollowing}} < \sum_{\boldsymbol{i=1}}^{\boldsymbol{M}} \left(\frac{\boldsymbol{n_i}}{\boldsymbol{2}} \cdot \boldsymbol{p_i}\right)$$

Therefore: $$\boldsymbol{upper\ bound} = \boldsymbol{d_{tot}} = \boldsymbol{d_{goal}} + \frac{\boldsymbol{1}}{\boldsymbol{2}} \sum_{\boldsymbol{i=1}}^{\boldsymbol{M}} \boldsymbol{n_i}\, \boldsymbol{p_i}$$

QED

Problem 2:

Consider the Z-Y-X Euler angle sequence, used to represent an element of SO(3). What happens when $\boldsymbol{Y} = \frac{\pi}{2}$ rads or $\boldsymbol{Y} = -\frac{\pi}{2}$ rads? Why is this a concern?

Solution:

Using the standard Euler angle Z-Y-X convention, the overall rotation can be represented by the combination of the principle rotations along Z-Y-X directions as follows:

$$\boldsymbol{R}(\alpha,\beta,\gamma) = \boldsymbol{R_Z}(\alpha)\cdot \boldsymbol{R_Y}(\beta)\cdot \boldsymbol{R_X}(\gamma)$$

$$\boldsymbol{R}(\alpha,\beta,\gamma) = \begin{bmatrix} cos(\alpha)cos(\beta) & cos(\alpha)sin(\beta)sin(\gamma) - sin(\alpha)cos(\gamma) & cos(\alpha)sin(\beta)cos(\gamma) + sin(\alpha)sin(\gamma) \\ sin(\alpha)cos(\beta) & sin(\alpha)sin(\beta)sin(\gamma) + cos(\alpha)cos(\gamma) & sin(\alpha)sin(\beta)cos(\gamma) - cos(\alpha)sin(\gamma) \\ -sin(\beta) & cos(\beta)sin(\gamma) & cos(\beta)cos(\gamma) \end{bmatrix}$$

By substituting in $\boldsymbol{\beta} = \frac{\pi}{2}$, the cos and sin terms reduce to: $\boldsymbol{cos}\left(\frac{\pi}{2}\right) = \boldsymbol{0}, \boldsymbol{sin}\left(\frac{\pi}{2}\right) = \boldsymbol{1}$, and therefore:

$$\boldsymbol{R}\left(\alpha,\frac{\pi}{2},\gamma\right) = \begin{bmatrix} 0 & cos(\alpha)(1)sin(\gamma) - sin(\alpha)cos(\gamma) & cos(\alpha)(1)cos(\gamma) + sin(\alpha)sin(\gamma) \\ 0 & sin(\alpha)(1)sin(\gamma) + cos(\alpha)cos(\gamma) & sin(\alpha)(1)cos(\gamma) - cos(\alpha)sin(\gamma) \\ -(1) & 0 & 0 \end{bmatrix}$$

We can convert R using the following four trigonometric identities:

$$sin(u)sin(v) = \frac{1}{2}[cos(u-v) - cos(u+v)], \text{ and } cos(u)cos(v) = \frac{1}{2}[cos(u-v) + cos(u+v)]$$

$$sin(u)cos(v) = \frac{1}{2}[sin(u+v) + sin(u-v)], \text{ and } cos(u)sin(v) = \frac{1}{2}[sin(u+v) - sin(u-v)]$$

Therefore:

$$\boldsymbol{R}\left(\alpha,\frac{\pi}{2},\gamma\right) = \begin{bmatrix} 0 & -sin(\alpha-\gamma) & cos(\alpha-\gamma) \\ 0 & cos(\alpha-\gamma) & sin(\alpha-\gamma) \\ -1 & 0 & 0 \end{bmatrix}$$

Similarly, for $\boldsymbol{\beta} = -\frac{\pi}{2}$:

$$\boldsymbol{R}\left(\alpha,-\frac{\pi}{2},\gamma\right) = \begin{bmatrix} 0 & cos(\alpha)(-1)sin(\gamma) - sin(\alpha)cos(\gamma) & cos(\alpha)(-1)cos(\gamma) + sin(\alpha)sin(\gamma) \\ 0 & sin(\alpha)(-1)sin(\gamma) + cos(\alpha)cos(\gamma) & sin(\alpha)(-1)cos(\gamma) - cos(\alpha)sin(\gamma) \\ (1) & 0 & 0 \end{bmatrix}$$

And using the same trigonometric identities, we obtain:

$$\boldsymbol{R}\left(\alpha,-\frac{\pi}{2},\gamma\right) = \begin{bmatrix} 0 & -sin(\alpha+\gamma) & -cos(\alpha+\gamma) \\ 0 & cos(\alpha+\gamma) & -sin(\alpha+\gamma) \\ 1 & 0 & 0 \end{bmatrix}$$

In conclusion, the result at these particular angles $\left(\boldsymbol{Y} = \frac{\pi}{2}, -\frac{\pi}{2}\right)$ for the Z-Y-X Euler representation is degenerate (or singular) situation where we have lost a degree of freedom. At these particular values of β, the rotation about the Z-axis and X-axis are indistinguishable and it becomes impossible to separate α from γ. The either are in the same $(\boldsymbol{\alpha + \gamma})$ or opposite $(\boldsymbol{\alpha - \gamma})$ direction about the same axis. This is what causes things like “gimbal-lock.”

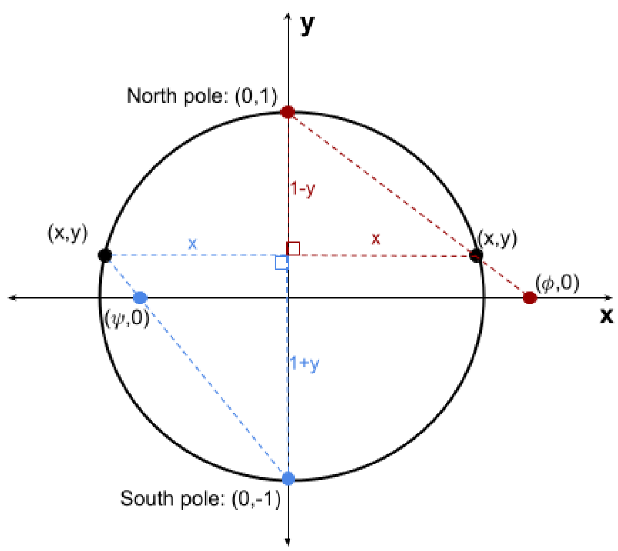
Problem 3: The unit circle S^1: $\boldsymbol{S^1 = \{\,(x, y) \mid x^2 + y^2 = 1\}}$,

cannot be covered by a single chart (mapping to an open set in $\mathbb{R}$). Define two charts, $(\boldsymbol{U}, \boldsymbol{\phi})$, and $(\boldsymbol{V}, \boldsymbol{\psi})$, $s.t.\, \boldsymbol{U} \subset \boldsymbol{S^1}, \boldsymbol{\phi}: \boldsymbol{U} \to \mathbb{R}^1$, and $\boldsymbol{V} \subset \boldsymbol{S^1}, \boldsymbol{\psi}: \boldsymbol{V} \to \mathbb{R}^1$. Show that the two charts cover $\boldsymbol{S^1}$ and that the map $\boldsymbol{\phi} \circ \boldsymbol{\psi}^{-1}$ is a diffeomorphism.

Solution:

In order to map $S^1 \to \mathbb{R}^1$, we use charts that map the stereographic projection of the points in S^1 from the respective "north" and "south" poles. Figure 1 shows the projection of a random point $(x, y) \in S^1$ from both poles onto the $\mathbb{R}^1$ number line (coincident with the x-axis):

Figure 1



In order to use stereographic projections, the two charts are defined as follows:

$$\boldsymbol{U = S^1 \setminus (0, 1)}, \qquad with\!: \ \boldsymbol{\phi(x, y) = \frac{x}{1 - y} = z_1}, \qquad where\!: \boldsymbol{z_1} \in \mathbb{R}^1$$

$$\boldsymbol{V = S^1 \setminus (0, -1)}, \qquad with\!: \ \boldsymbol{\psi(x, y) = \frac{x}{1 + y} = z_2}, \qquad where\!: \boldsymbol{z_2} \in \mathbb{R}^1$$

U is the set of all points in S^1 without the north pole and V is the set of all points in S^1 without the south pole. The functions ϕ, ψ are obtained by using similar triangles as per figure 1 to determine what the value on $\mathbb{R}^1$ are. Since U and V overlap, with the exception of the two poles, and their union: $U \cup V = S^1$, every point in S1 is mapped to $\mathbb{R}^1$ by the two charts so long as the mapping functions ϕ, ψ are diffeomorphisms.

To prove diffeomorphism, we need the inverse functions, which we obtain by setting values of ϕ, ψ to points $z_1, z_2 \in \mathbb{R}^1$ respectively, squaring both sides and substituting in the eqn of the circle: $x^2 + y^2 = 1$:

$$z_1^2 = \frac{x^2}{(1 - y)^2} = \frac{1 - y^2}{(1 - y)^2} = \frac{(1 - y)(1 + y)}{(1 - y)^2} = \frac{(1 + y)}{(1 - y)}$$

$$z_1^2 - 1 = y(1 + z_1^2) \Longrightarrow \boldsymbol{y = \frac{(z_1^2 - 1)}{(z_1^2 + 1)}}$$

When we substitute this equation back in for y, we can solve for x:

$$z_1(1-y) = x \Rightarrow x \ = \ z_1\left[\frac{(z_1^2+1)-(z_1^2-1)}{(z_1^2+1)}\right] \Rightarrow \ \boldsymbol{x} = \frac{\mathbf{2}\boldsymbol{z_1}}{(\boldsymbol{z_1^2}+\mathbf{1})}$$

Similarly, we do the same calculations for ψ in terms of z_2 and we obtain:

$$\boldsymbol{y} = \frac{(\mathbf{1}-\boldsymbol{z_2^2})}{(\boldsymbol{z_2^2}+\mathbf{1})}, \qquad \boldsymbol{x} = \frac{\mathbf{2}\boldsymbol{z_2}}{(\boldsymbol{z_2^2}+\mathbf{1})}$$

As a result, we can define the inverse functions as follows:

$$\boldsymbol{\phi^{-1}}(\boldsymbol{z_1}) = \left(\frac{\mathbf{2}\boldsymbol{z_1}}{(\boldsymbol{z_1^2}+\mathbf{1})}, \frac{(\boldsymbol{z_1^2}-\mathbf{1})}{(\boldsymbol{z_1^2}+\mathbf{1})}\right), \qquad \boldsymbol{\psi^{-1}}(\boldsymbol{z_2}) = \left(\frac{\mathbf{2}\boldsymbol{z_2}}{(\boldsymbol{z_2^2}+\mathbf{1})}, \frac{(\mathbf{1}-\boldsymbol{z_2^2})}{(\boldsymbol{z_2^2}+\mathbf{1})}\right)$$

To reiterate, in order to show that these two charts (U, ϕ), and (V, ψ), are valid charts and map to $\mathbb{R}^1$, we need to show that the mapping functions ϕ, and ψ are diffeomorphisms. According to Choset 3.4.2, this is true if ϕ, and ψ are both bijective and smooth, and their respective inverses ϕ^{-1}, and ψ^{-1} are also both smooth (i.e. derivatives/partial derivatives are defined for the entire domain).

We start by showing that ϕ, and ψ are bijective by proving that they are invertible. A function f is invertible if: $f\left(f^{-1}(a)\right) = a$, and $f^{-1}\left(f(b)\right) = b$

$$\boldsymbol{\phi}\left(\boldsymbol{\phi^{-1}}(\boldsymbol{z})\right) = \phi\left(\frac{2z}{(z^2+1)}, \frac{(z^2-1)}{(z^2+1)}\right) = \frac{\frac{2z}{z^2+1}}{1-\frac{z^2-1}{z^2+1}} = \frac{2z}{z^2+1-z^2+1} = \frac{2z}{2} = \boldsymbol{z}$$

$$\boldsymbol{\phi^{-1}}\left(\boldsymbol{\phi}(\boldsymbol{a},\boldsymbol{b})\right) = \phi^{-1}\left(\frac{a}{1-b}\right) = \left(\frac{\frac{2a}{1-b}}{\frac{a^2}{(1-b)^2}+1}, \frac{\frac{a^2}{(1-b)^2}-1}{\frac{a^2}{(1-b)^2}+1}\right) = \left(\frac{2\left(\frac{a}{1-b}\right)}{\frac{a^2+(1-b)^2}{(1-b)^2}}, \frac{\frac{a^2-(1-b)^2}{(1-b)^2}}{\frac{a^2+(1-b)^2}{(1-b)^2}}\right)$$

$$= \left(\frac{2a(1-b)}{a^2+(1-b)^2}, \frac{a^2-(1-b)^2}{a^2+(1-b)^2}\right)$$

We substitute in $a^2 = 1-b^2$ (equation of circle) to simplify:

$$= \left(\frac{2a(1-b)}{(1-b^2)+1-2b+b^2}, \frac{(1-b^2)-(1-2b+b^2)}{(1-b^2)+1-2b+b^2}\right) = \left(\frac{2a(1-b)}{2(1-b)}, \frac{2b(1-b)}{2(1-b)}\right) = (\boldsymbol{a}\,, \boldsymbol{b})$$

Similarly, we show the same with ψ:

$$\boldsymbol{\psi}\left(\boldsymbol{\psi^{-1}}(\boldsymbol{z})\right) = \psi\left(\frac{2z}{(z^2+1)}, \frac{(1-z^2)}{(z^2+1)}\right) = \frac{\frac{2z}{z^2+1}}{1+\frac{1-z^2}{z^2+1}} = \frac{2z}{z^2+1+1-z^2} = \frac{2z}{2} = \boldsymbol{z}$$

$$\boldsymbol{\psi^{-1}}\left(\boldsymbol{\psi}(\boldsymbol{a},\boldsymbol{b})\right) = \psi^{-1}\left(\frac{a}{1+b}\right) = \left(\frac{2\left(\frac{a}{1+b}\right)}{\frac{a^2}{(1+b)^2}+1}, \frac{1-\frac{a^2}{(1+b)^2}}{\frac{a^2}{(1+b)^2}+1}\right) = \left(\frac{2a(1+b)}{a^2+(1+b)^2}, \frac{(1+b)^2-a^2}{a^2+(1+b)^2}\right)$$

We substitute in $a^2 = 1 - b^2$ (equation of circle) to simplify:

$$= \left(\frac{2a(1+b)}{(1-b^2)+1+2b+b^2}, \frac{(1+2b+b^2)-(1-b^2)}{(1-b^2)+1+2b+b^2}\right) = \left(\frac{2a(1+b)}{2(1+b)}, \frac{2b(1+b)}{2(1+b)}\right) = (\boldsymbol{a}\,, \boldsymbol{b})$$

Next, we show that the original mapping functions ϕ, ψ are smooth by taking partial derivatives of each w.r.t. x and y:

$$\frac{\boldsymbol{\partial d}}{\boldsymbol{\partial d\phi}}[\boldsymbol{\phi}(\boldsymbol{x},\boldsymbol{y})] = \left[\frac{\partial d}{\partial dx}\left(\frac{x}{1-y}\right), \frac{\partial d}{\partial dy}\left(\frac{x}{1-y}\right)\right] = \left[\frac{1}{1-y}, \frac{x}{(1-y)^2}\right], \qquad \Rightarrow \boldsymbol{y} \neq \mathbf{1}$$

$$\frac{\boldsymbol{\partial d}}{\boldsymbol{\partial d\psi}}[\boldsymbol{\psi}(\boldsymbol{x},\boldsymbol{y})] = \left[\frac{\partial d}{\partial dx}\left(\frac{x}{1+y}\right), \frac{\partial d}{\partial dy}\left(\frac{x}{1+y}\right)\right] = \left[\frac{1}{1+y}, \frac{-x}{(1+y)^2}\right], \qquad \Rightarrow \boldsymbol{y} \neq -\mathbf{1}$$

Since y = 1 only at the north pole, which is not part of set *U*, the partial derivatives of ϕ are defined for the entire domain of *U*. Similarly, y = -1 only at the south pole, which is not part of set *V*, so the partial derivatives of ψ are defined over the full domain of *V*. Therefore, both functions are smooth.

Finally, we prove that ϕ^{-1}, and ψ^{-1} are both smooth:

$$\begin{aligned}\frac{\boldsymbol{d}}{\boldsymbol{dz_1}}\boldsymbol{\phi}^{-\mathbf{1}}(\boldsymbol{z_1}) &= \frac{d}{dz_1}\left(\frac{2z_1}{(z_1^2+1)}, \frac{(z_1^2-1)}{(z_1^2+1)}\right) \\ &= \left(\frac{\mathbf{2}\left(\boldsymbol{z_1^2}+\mathbf{1}\right)-\mathbf{2}\boldsymbol{z_1}(\mathbf{2}\boldsymbol{z_1})}{\left(\boldsymbol{z_1^2}+\mathbf{1}\right)^{\mathbf{2}}}, \frac{\left(\boldsymbol{z_1^2}+\mathbf{1}\right)(\mathbf{2}\boldsymbol{z_1})-\left(\boldsymbol{z_1^2}-\mathbf{1}\right)(\mathbf{2}\boldsymbol{z_1})}{\left(\boldsymbol{z_1^2}+\mathbf{1}\right)^{\mathbf{2}}}\right)\end{aligned}$$

$$\frac{\boldsymbol{d}}{\boldsymbol{dz_1}}\boldsymbol{\phi}^{-\mathbf{1}}(\boldsymbol{z_1}) = \left(\frac{-\mathbf{2}\boldsymbol{z_1^2}+\mathbf{1}}{\left(\boldsymbol{z_1^2}+\mathbf{1}\right)^{\mathbf{2}}}, \frac{\mathbf{4}\boldsymbol{z_1}}{\left(\boldsymbol{z_1^2}+\mathbf{1}\right)^{\mathbf{2}}}\right)$$

$$\begin{aligned}\frac{\boldsymbol{d}}{\boldsymbol{dz_2}}\boldsymbol{\psi}^{-\mathbf{1}}(\boldsymbol{z_2}) &= \frac{d}{dz_2}\left(\frac{2z_2}{(z_2^2+1)}, \frac{(1-z_2^2)}{(z_2^2+1)}\right) \\ &= \left(\frac{\mathbf{2}\left(\boldsymbol{z_2^2}+\mathbf{1}\right)-\mathbf{2}\boldsymbol{z_2}(\mathbf{2}\boldsymbol{z_2})}{\left(\boldsymbol{z_2^2}+\mathbf{1}\right)^{\mathbf{2}}}, \frac{\left(\boldsymbol{z_2^2}+\mathbf{1}\right)(-\mathbf{2}\boldsymbol{z_2})-\left(\mathbf{1}-\boldsymbol{z_2^2}\right)(\mathbf{2}\boldsymbol{z_2})}{\left(\boldsymbol{z_2^2}+\mathbf{1}\right)^{\mathbf{2}}}\right)\end{aligned}$$

$$\frac{\boldsymbol{d}}{\boldsymbol{dz_1}}\boldsymbol{\psi}^{-\mathbf{1}}(\boldsymbol{z_2}) = \left(\frac{-\mathbf{2}\boldsymbol{z_2^2}+\mathbf{1}}{\left(\boldsymbol{z_2^2}+\mathbf{1}\right)^{\mathbf{2}}}, \frac{-\mathbf{4}\boldsymbol{z_2}}{\left(\boldsymbol{z_2^2}+\mathbf{1}\right)^{\mathbf{2}}}\right)$$

The derivative for both inverse functions exist for all real numbers $z_1, z_2 \in \mathbb{R}^1$ except where the denominator = 0. However, we can see that the denominator is only zero if $z_1^2 = -1,\ z_2^2 = -1$, which only occurs at $z_1, z_2 = i$, which is an imaginary number. Therefore, the derivative exists for all $\mathbb{R}^1$.

The last part of this question is to show that the composite transition function $\boldsymbol{\phi} \circ \boldsymbol{\psi}^{-\mathbf{1}}$ is also a diffeomorphism. Using the same methodology, we show that it is $\boldsymbol{\phi} \circ \boldsymbol{\psi}^{-\mathbf{1}}$ is bijective by showing it is invertible and that it and its inverse ($\boldsymbol{\psi} \circ \boldsymbol{\phi}^{-\mathbf{1}}$) are smooth. First we evaluate the composite function:

$$\boldsymbol{\phi} \circ \boldsymbol{\psi}^{-\mathbf{1}} = \phi\left(\psi^{-1}(z_2)\right) = \phi\left(\frac{2z_2}{(z_2^2+1)}, \frac{(1-z_2^2)}{(z_2^2+1)}\right) = \frac{\frac{2z_2}{z_2^2+1}}{1-\frac{1-z_2^2}{z_2^2+1}} = \frac{2z_2}{z_2^2+1-1+z_2^2} = \frac{\mathbf{1}}{\boldsymbol{z_2}}$$

From here, we can see that the function fully map from $\mathbb{R}^1 \to \mathbb{R}^1$, except for the cases where $z_2 = 0$. And we know from the original definition of ψ that $\psi(x, y) = \frac{x}{1+y} = z_2$, which means z_2 can only be 0 when $y = -1$, which only occurs at the south pole: (0, -1). Since we defined $\psi^{-1}: \mathbb{R}^{\mathbf{1}} \to V$, and the set V is all points on S^1 except for the south pole, we can say that all the points in the domain are mapped to a point in $\mathbb{R}^{\mathbf{1}}$.

Similarly, we calculate the inverse composite function as follows:

$$\boldsymbol{\psi} \circ \boldsymbol{\phi}^{-\mathbf{1}} = \psi\big(\phi^{-1}(z_1)\big) = \psi\left(\frac{2z_1}{(z_1^2+1)}, \frac{(z_1^2-1)}{(z_1^2+1)}\right) = \frac{\frac{2z_1}{z_1^2+1}}{1 + \frac{z_1^2-1}{z_1^2+1}} = \frac{2z_1}{z_1^2+1+z_1^2-1} = \frac{\mathbf{1}}{\boldsymbol{z_1}}$$

Using the same logic as above, the function is defined across all the points in the domain of *U* except for z_1 corresponding to the north pole (0,1).

Next, we check for invertibility using the definition $f\big(f^{-1}(a)\big) = a$, and $f^{-1}\big(f(b)\big) = b$:

$$\boldsymbol{\phi} \circ \boldsymbol{\psi}^{-\mathbf{1}}\left(\boldsymbol{\psi} \circ \boldsymbol{\phi}^{-\mathbf{1}}(\boldsymbol{z_1})\right) = \phi \circ \psi^{-1}\left(\frac{1}{z_1}\right) = \frac{\mathbf{1}}{\frac{\mathbf{1}}{\boldsymbol{z_1}}} = \boldsymbol{z_1}$$

$$\boldsymbol{\psi} \circ \boldsymbol{\phi}^{-\mathbf{1}}\left(\boldsymbol{\phi} \circ \boldsymbol{\psi}^{-\mathbf{1}}(\boldsymbol{z_2})\right) = \psi \circ \phi^{-1}\left(\frac{1}{z_2}\right) = \frac{1}{\frac{1}{z_2}} = \boldsymbol{z_2}$$

Thus, we can see that both composite functions are invertible and therefore, are bijective. Lastly, we check that the composite function and its inverse are both smooth:

$$\frac{\boldsymbol{d}}{\boldsymbol{dz_1}}\boldsymbol{\phi} \circ \boldsymbol{\psi}^{-\mathbf{1}}(\boldsymbol{z_2}) = \frac{d}{dz_2}\left[\frac{1}{z_2}\right] = \frac{-1}{z_2^2}, \qquad \Rightarrow \boldsymbol{z_2} \neq \mathbf{0}$$

$$\frac{\boldsymbol{d}}{\boldsymbol{dz_1}}\boldsymbol{\psi} \circ \boldsymbol{\phi}^{-\mathbf{1}}(\boldsymbol{z_1}) = \frac{d}{dz_1}\left[\frac{1}{z_1}\right] = \frac{-1}{z_1^2}, \qquad \Rightarrow \boldsymbol{z_1} \neq \mathbf{0}$$

Here we can see that both the composite transition function and its inverse are smooth as their derivatives are defined for all $\mathbb{R}^{\mathbf{1}}$, except for the same point where z_2, z_1 = 0 respectively, which occurs at the south pole (0, -1) and north pole (0, 1), respectively. Those points are not part of the domain of the respective sets *V* and *U* from which z_2 and z_1 are mapped.

In order for a composite transition function to be a diffeomorphism, it needs to satisfy bijectivity, smoothness and it's inverse must also be smooth for all points in the intersection of the two sets U, V, that it is mapping with. We already know that U and V are the entire set S^1 without the respective north and south pole points. Therefore, the intersection of U and V is:

$$U \cap V \ = \ S\text{^}1 \setminus (0, -1) \cup (0, 1)$$

It is all points in S^1 except for the two poles, and so we know that $\boldsymbol{\phi} \circ \boldsymbol{\psi}^{-\mathbf{1}}$ is a diffeomorphism because it is bijective and smooth and it's inverse is smooth over the entire intersection of sets U and V.

Problem 4: Give the dimension of the configuration spaces of the following systems and briefly explain your answers.

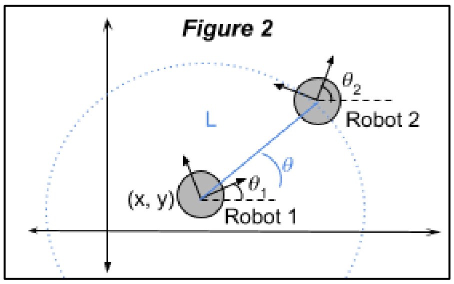
Solution:

(a) Two mobile robots rotating and translating in the plane:

- $\mathbb{R}^4 \times \boldsymbol{T^2}$ or $\boldsymbol{SE(2)} \times \boldsymbol{SE(2)}$
- Since each robot can be anywhere in the workspace, we require separate pose (translation an rotation) for each robot. These can be expressed as either $\mathbb{R}^2 \times S^1$ or SE(2). Note, since S^1 x S^1 = T^2, we can combine for an expression of the configuration of the overall system.

(b) Two mobile robots tied together by a rope rotating and translating in the plane:

- $\mathbb{R}^2 \times \boldsymbol{S^1} \times (\boldsymbol{0}, \boldsymbol{L}) \times \boldsymbol{S^1} \times \boldsymbol{S^1}$ or $\boldsymbol{SE(2)} \times (\boldsymbol{0}, \boldsymbol{L}) \times \boldsymbol{T^2}$



- The position and orientation of a robot (arbitrarily robot 1) is unconstrained, so it can be represented by SE(2). However, the second robot attached by a rope of length L cannot be anywhere in SE(2) since its position must lie within the radius of the rope (see blue circle). This is easiest represented in polar coordinates of radius: (0, L) and angle θ in S^1. The orientation of robot 2 is still unconstrained so also can be represented in S^1 for an overall configuration space of (0, L) x S^1 x S^1.

(c) A train on train tracks, including the wheel angles (The wheels roll without slipping):

- $\boldsymbol{x} \in \mathbb{R}^1$
- If we know the radius of the wheels (R) and initial angle (θ_i), we can determine the current angle (θ) based on the current position of the train (x): θ = modulo (x, $R\theta_i$). Therefore, there is still only 1 degree of freedom for the train.

(d) Your legs as you pedal a bicycle (remaining seated with feet fixed to the pedals):

- $\boldsymbol{S^1} \times \boldsymbol{S^1} = \boldsymbol{T^2}$
- Assuming only the upper and lower legs are moving in their respective vertical planes, we have four (1 dof) joints: left hip, right hip, left knee and right knee. Each of the legs can be modeled as 2R (revolute) link manipulators, with 2 degrees of freedom (hip angle and knee angle). Theoretically, we have 4 degrees of freedom but there are two additional constraints imposed because (1) if one foot is driving the pedal, the other foot is constrained to a fixed position by the pedal mechanism, and (2) the knee only flexes in one direction and so there is only one possible configuration for the leg that is being driven and fixed at the hip.

Problem 5: Prove that the union operator propagates from the workspace to the configuration space. That is, the union of two configuration space obstacles is the configuration space obstacle of the union of two workspace obstacles:

$$\boldsymbol{C}(\boldsymbol{WO_i} \cup \boldsymbol{WO_j}) = \boldsymbol{CO_i} \cup \boldsymbol{CO_j}\text{, where: } C \text{ is a configuration space operator.}$$

Notation:

$A(q) \subset \mathbb{R}^{\mathrm{n}}$: set of all points occupied by the robot in workspace ($\mathbb{R}^{\mathrm{n}}$) for a given configuration q

$WO_i \subset \mathbb{R}^{\mathrm{n}}$: set of all points in workspace occupied by obstacle i

Solution:

First, we define what the set of all points in obstacles WO_i and WO_j are in the configuration space:

$$C(\mathrm{W}O_i) = \{q \in C \mid A(q) \cap WO_i \neq 0\} = CO_i$$

$$C(WO_j) = \{q \in C \mid A(q) \cap WO_j \neq 0\} = CO_j$$

From the definition of a configuration space of an obstacle, we can see that each set CO_i and CO_j are the set of all configurations q in the configurations space, where at least one point occupied by the robot in those configurations $A(q)$ intersects the respective obstacles in the workspace.

Therefore, we can define the set of configurations q that have some intersection between points of the robots in that configuration $A(q)$ and either workspace (WO_i or WO_j) as:

$$C(WO_i \cup WO_j) = \{q \in C \mid A(q) \cap (WO_i \cup WO_j) \neq 0\}$$

Since the distributive property of the intersection of the union of two sets is:

$$A(q) \cap (WO_i \cup WO_j) = (A(q) \cap WO_i) \cup (A(q) \cap WO_j)$$

We can apply that to the set of configurations that is the intersection of set A(q) with the union of the two obstacle sets in the workspace:

$$\{q \in C \mid A(q) \cap (WO_i \cup WO_j) \neq 0\} = \{q \in C \mid A(q) \cap WO_i \neq 0\} \bigcup \{q \in C \mid A(q) \cap WO_j \neq 0\}$$

As a result:

$$C(WO_i \cup WO_j) = CO_i \bigcup CO_j$$

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Problem 6: Does the wavefront planner on a discrete grid yield the shortest distance to the goal? Why or why not, briefly? If so, what metric is the wavefront planner using?

Solution:

- Yes, the wavefront planner does yield the shortest distance to the goal. The algorithm starts at the **start cell** with a value of '**2**' and then in each iteration, it propagates to all neighboring cells the current cell value + 1. As a result, when the wave reaches the **goal** cell, it will populate the shortest path + 2 value into the goal cell.
- This shortest path can be found by retracing the path from the goal cell back to the start cell by moving to cells that contain a value one less than the current cell.
- The metric that the wavefront planner is using is a form of distance, which depends on the way a cell neighborhood is defined in the algorithm:
    - For **four-point** neighborhoods, the wave can only move up, down, left and right. As a result, the distance metric contained in each cell is the **Manhattan distance + 2** from the start cell.
    - In **eight-point** neighborhoods, diagonal moves are also allowed. As a result, the distance metric is the **distance in terms of number of cells + 2**. The resulting path will attempt to move in a diagonal as much as possible to the goal unless it reaches an obstacle or an edge, in which case it will make non-diagonal moves until it can either continue moving diagonally or it has reached the goal.